GATE ASSIGNMENT 4

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https://github.com/Dishank422/EE3900/blob/main/Gate−Assignment4/latex_code.tex

1 EC 1999 Q.2.1

The Fourier representation of an impulse train represented by $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$ is given by

(a) $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{j2\pi nt}{T_0}\right)$
(b) $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{j\pi nt}{T_0}\right)$
(c) $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j\pi nt}{T_0}\right)$
(d) $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j2\pi nt}{T_0}\right)$

2 Solution

Lemma 2.1. *Any periodic signal x(t) with period T_0 can be written as*

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{j2\pi nt}{T_0}\right) \tag{2.0.1}$$

where, a_n is given by

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \tag{2.0.2}$$

Proof. We shall verify equation (2.0.2).

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \tag{2.0.3}$$

$$= \frac{1}{T_0} \int_{T_0} \left(\sum_{m=-\infty}^{\infty} a_m \exp\left(\frac{j2\pi mt}{T_0}\right)\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \tag{2.0.4}$$

$$= \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \int_{T_0} a_m \left(\exp \frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \tag{2.0.5}$$

When $m = n$,

$$\int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt = \int_{T_0} a_n \tag{2.0.6}$$

$$= T_0 a_n \tag{2.0.7}$$

When $m \neq n$,

$$\int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \tag{2.0.8}$$

$$= a_m \int_{T_0} \exp\left(\frac{j2\pi(m-n)t}{T_0}\right) dt \tag{2.0.9}$$

Since $\exp\left(\frac{j2\pi(m-n)t}{T_0}\right)$ is periodic with period $\frac{T_0}{m-n}$, it's integral over any time interval of length $\frac{T_0}{m-n}$ or any integral multiple of $\frac{T_0}{m-n}$ will be 0. Therefore, when $m \neq n$,

$$\int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt = 0 \tag{2.0.10}$$

Continuing from equation (2.0.5),

$$a_n = \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \tag{2.0.11}$$

$$= \frac{1}{T_0}(T_0 a_n) = a_n \tag{2.0.12}$$

□

We observe that $s(t)$ is periodic with period T_0. Thus it's Fourier representation as a sum of complex exponents is given by

$$s(t) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{j2\pi nt}{T_0}\right) \tag{2.0.13}$$

where, a_n can be calculated as

$$a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \tag{2.0.14}$$

Between $-\frac{T_0}{2}$ and $\frac{T_0}{2}$, we can say that $s(t) = \delta(t)$.

$$\implies a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.15)$$

$$= \frac{1}{T_0} \quad (2.0.16)$$

$$\implies s(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j2\pi nt}{T_0}\right) \quad (2.0.17)$$

Therefore option (d) is the correct option.

Lemma 2.2. *The Fourier transform of* $\exp(j2\pi f_0 t)$ *is* $\delta(f - f_0)$.

Proof. We verify the lemma by finding the inverse Fourier transform of $\delta(f - f_0)$.

$$\int_{-\infty}^{\infty} \delta(f - f_0) \exp(j2\pi f t)\, df = \exp(j2\pi f_0 t) \quad (2.0.18)$$

□

Let $S(f)$ be the Fourier transform of s(t). Then using the above lemma and equation (2.0.17)

$$S(f) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) \quad (2.0.19)$$

We can also write equation (2.0.19) by substituting $f_0 = \frac{1}{T_0}$ as

$$S(f) = \sum_{n=-\infty}^{\infty} f_0 \delta(f - n f_0) \quad (2.0.20)$$

Thus we can observe that the Fourier transform of an impulse train is an impulse train in the frequency domain.