Assignment 4+5

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Questions

Q. 4) A point moves so that it's distance from the y-axis is equal to the distance from the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find the equation of the locus.

Q. 5) Find the normal at the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the curve $2y + x^2 = 3$.

Equation of line

If $\mathbf{m}$ is the vector perpendicular to a given line and $\mathbf{q}$ is a point on the line, then the equation of the line is given by

$$\mathbf{m}^\top(\mathbf{x} - \mathbf{q}) = 0 \tag{1}$$

Equation of conic section

Equation 2 represents a conic section in 2D. Here V and u are 2×2 and 2×1 matrices respectively and f is a scalar.

$$\mathrm{x}^\top \mathrm{V}\mathrm{x} + 2\mathrm{u}^\top \mathrm{x} + f = 0 \tag{2}$$

Equation 2 represents a parabola when $|\mathrm{V}| = 0$

Affine transform

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \tag{3}$$

Here P is the eigenvector matrix of V. For a parabola, c is given by equation 4.

$$\begin{pmatrix} \mathbf{u}^\top + \eta \mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \tag{4}$$

For a parabola, since $|V| = 0$, one eigenvalue must be 0. $\mathbf{p}_1$ is the eigenvector corresponding to that eigenvalue. η is given by equation 5.

$$\eta = \mathbf{u}^\top \mathbf{p}_1 \tag{5}$$

Under such a transformation, the equation of a parabola is given by

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \tag{6}$$

where D is the eigenvalue matrix of V.

Equation of normal

If $\mathbf{q}$ is a point on a conic, the direction ($\mathbf{n}$) of the normal at $\mathbf{q}$ is given by

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \tag{7}$$

Therefore the direction ($\mathbf{m}$) perpendicular to the normal is given by

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{8}$$

Therefore, the equation of the normal is given by

$$\mathbf{m}^\top(\mathbf{x} - \mathbf{q}) = 0 \tag{9}$$

Question 4 solution

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ be the point. The equation of y-axis is given by

$$\mathbf{R} = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

xR is perpendicular to y-axis.

$$\implies (\mathbf{R} - \mathbf{x})^\top \mathbf{R} = 0 \tag{11}$$

$$\implies \mathbf{x}^\top \mathbf{R} = \|\mathbf{R}\|^2 \tag{12}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix} \|\mathbf{R}\| = \|\mathbf{R}\|^2 \tag{13}$$

$$\implies \|\mathbf{R}\| = \mathbf{x}^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14}$$

Solution Contd.

Let $C = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then

$$xC = \|x - C\| \tag{15}$$
$$xR = \|x - R\| \tag{16}$$

We are given $xR = xC$.

$$\implies \|x - C\|^2 = \|x - R\|^2 \tag{17}$$
$$\implies \|x\|^2 + \|C\|^2 - 2x^\top C = \|x\|^2 + \|R\|^2 - 2x^\top R \tag{18}$$

Solution Contd.

Subtracting $\|x\|^2$ on both sides and using 12,

$$\|C\|^2 - 2x^\top C = \|R\|^2 - 2\|R\|^2 \tag{19}$$

$$\implies 2x^\top C = \|C\|^2 + \|R\|^2 \tag{20}$$

$$\implies 2C^\top x = \|C\|^2 + \left(x^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^2 \tag{21}$$

$$\implies x^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} x - 2C^\top x + \|C\|^2 = 0 \tag{22}$$

$$\implies x^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x + 2(-C)^\top x + \|C\|^2 = 0 \tag{23}$$

$$\implies x^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x + 2\begin{pmatrix} -2 & -1 \end{pmatrix} x + 5 = 0 \tag{24}$$

$$\implies V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ u = \begin{pmatrix} -2 \\ -1 \end{pmatrix},\ f = 5 \tag{25}$$

Solution Contd.

For obtaining the affine transformation, we use

$$\mathrm{x} = \mathrm{P}\mathrm{y} + \mathrm{c} \tag{26}$$

The corresponding eigenvalues of V are

$$\lambda_1 = 0,\ \lambda_2 = 1 \tag{27}$$

$$\implies \mathrm{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{28}$$

The corresponding eigenvectors are

$$\mathrm{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathrm{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{29}$$

$$\implies \mathrm{P} = \begin{pmatrix} \mathrm{p}_1 & \mathrm{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{30}$$

Solution Contd.

Since $|\mathrm{V}| = 0$,

$$\begin{pmatrix} \mathrm{u}^\top + \eta \mathrm{p}_1^\top \\ \mathrm{V} \end{pmatrix} \mathrm{c} = \begin{pmatrix} -f \\ \eta \mathrm{p}_1 - \mathrm{u} \end{pmatrix} \tag{31}$$

$$\eta = \mathrm{u}^\top \mathrm{p}_1 \tag{32}$$

$$\implies \eta = -2 \tag{33}$$

$$\implies \begin{pmatrix} -4 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{c} = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \tag{34}$$

$$\implies \mathrm{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{35}$$

Therefore, the locus of x is given by

$$\mathrm{y}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{y} = 4 \begin{pmatrix} 1 & 0 \end{pmatrix} \mathrm{y} \tag{36}$$

Solution Contd.

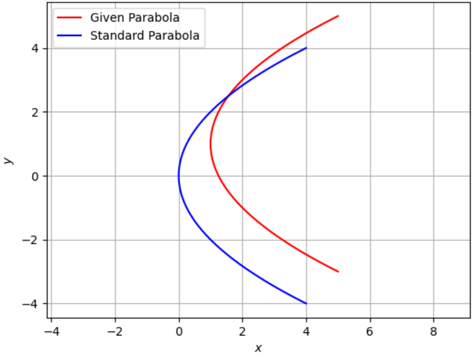

Figure: Plot of the locus

Question 5 solution

The given curve can be expressed as

$$x^2 + 2y - 3 = 0 \tag{37}$$

$$\implies \mathrm{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathrm{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathrm{f} = -3 \tag{38}$$

Since $|\mathrm{V}| = 0$, the given curve represents a parabola. To evaluate the direction vector m,

$$\mathrm{m}^\top (\mathrm{V}\mathrm{q} + \mathrm{u}) = 0 \tag{39}$$

$$\implies \mathrm{m}^\top \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 0 \tag{40}$$

$$\implies \mathrm{m}^\top \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \tag{41}$$

$$\implies \mathrm{m} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{42}$$

Solution Contd.

The normal is obtained as

$$\mathbf{m}^{\top}(\mathbf{x}-\mathbf{q}) = 0 \tag{43}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix}\left(\mathbf{x}-\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 0 \tag{44}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{45}$$

Solution Contd.

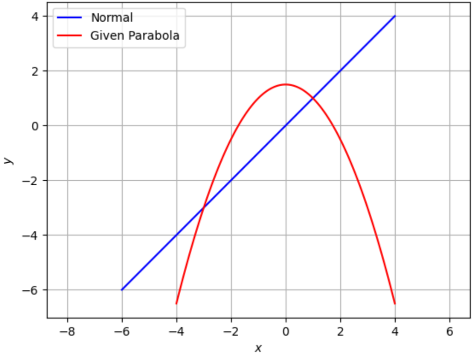

Figure: Plot of the normal