Gate Assignment 2

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Question

The DFT of a vector $\begin{pmatrix} a & b & c & d \end{pmatrix}$ is the vector $\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}$. Consider the product

$$\begin{pmatrix} p & q & r & s \end{pmatrix} = \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \tag{1}$$

The DFT of the vector $\begin{pmatrix} p & q & r & s \end{pmatrix}$ is a scaled version of

A. $\begin{pmatrix} \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \end{pmatrix}$
B. $\begin{pmatrix} \sqrt{\alpha} & \sqrt{\beta} & \sqrt{\gamma} & \sqrt{\delta} \end{pmatrix}$
C. $\begin{pmatrix} \alpha+\beta & \beta+\delta & \delta+\gamma & \gamma+\alpha \end{pmatrix}$
D. $\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}$

Lemma

If T is a circulant matrix, then the eigenvector matrix of T is the same as the DFT matrix W and the eigenvalues are the DFT of the first column of T.

Proof

The i^{th} column of the $n \times n$ DFT matrix is given by

$$p_i = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^{i} \\ \omega^{2i} \\ \vdots \\ \vdots \\ \omega^{(n-1)i} \end{pmatrix} \tag{2}$$

where ω is the n^{th} root of 1. We shall show that this p_i is the eigenvector of T. Observe that the kth component of Tp_i is given by

$$y_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \mathrm{T}_{kj}\omega^{ij} \tag{3}$$

$$= \frac{\omega^{ki}}{\sqrt{n}} \sum_{j=0}^{n-1} \mathrm{T}_{kj}\omega^{(j-k)i} \tag{4}$$

Proof contd.

$$\implies y_k = \frac{\omega^{ki}}{\sqrt{n}} \sum_{j=0}^{n-1} \mathrm{T}_{(j-k)mod(n)1}\omega^{(j-k)i} \tag{5}$$

$$= \frac{\omega^{ki}}{\sqrt{n}} \sum_{m=0}^{n-1} \mathrm{T}_{m1}\omega^{mi} \tag{6}$$

Therefore

$$\mathrm{Tp}_i = \frac{\sum_{m=0}^{n-1} \mathrm{T}_{m1}\omega^{mi}}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^{i} \\ \omega^{2i} \\ \vdots \\ \vdots \\ \omega^{(n-1)i} \end{pmatrix} \tag{7}$$

But $\sum_{m=0}^{n-1} \mathrm{T}_{m1}\omega^{mi}$ is nothing but the i^{th} element of the DFT of the first column of T. Therefore p_i is an eigenvector of T with eigenvalue as i^{th} element of the DFT of the first column of T.

Solution

First we express the equations in a more convenient form.

$$\begin{pmatrix} p & q & r & s \end{pmatrix} = \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \tag{8}$$

$$\implies \begin{pmatrix} p & q & r & s \end{pmatrix}^\top = \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}^\top \begin{pmatrix} a & b & c & d \end{pmatrix}^\top \tag{9}$$

$$\implies \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{10}$$

Solution contd.

$$Let\ \mathrm{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix};\ \mathrm{X} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix};\ \mathrm{y} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \tag{11}$$

$$\mathrm{T} = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix} \tag{12}$$

Then we have to find Y the DFT of y. We know

$$\mathrm{X} = \mathrm{W}\mathrm{x} \tag{13}$$

$$\implies \mathrm{x} = \mathrm{W}^{-1}\mathrm{X} \tag{14}$$

$$\mathrm{y} = \mathrm{T}\mathrm{x} \tag{15}$$

$$\mathrm{Y} = \mathrm{W}\mathrm{y} \tag{16}$$

$$\implies \mathrm{Y} = \mathrm{W}\mathrm{T}\mathrm{x} \tag{17}$$

$$\implies \mathrm{Y} = \mathrm{W}\mathrm{T}\mathrm{W}^{-1}\mathrm{X} \tag{18}$$

Solution contd.

But T is a circulant matrix with first column as $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. Therefore the eigenvalues are $\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$. Using eigen decomposition

$$\mathrm{T} = \mathrm{W} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathrm{W}^{-1} \tag{19}$$

$$\implies \mathrm{Y} = \mathrm{W}\mathrm{W} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathrm{W}^{-1}\mathrm{W}^{-1}\mathrm{X} \tag{20}$$

Solution contd.

Using properties of DFT

$$\mathrm{WW} = \mathrm{I};\ \mathrm{W}^{-1}\mathrm{W}^{-1} = \mathrm{I} \tag{21}$$

$$\implies \mathrm{Y} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathrm{X} \tag{22}$$

$$\implies \mathrm{Y} = \begin{pmatrix} \alpha^2 \\ \beta^2 \\ \gamma^2 \\ \delta^2 \end{pmatrix} \tag{23}$$

Therefore option (A) is the correct option.