GATE ASSIGNMENT 3

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Download all python codes from

https://github.com/Dishank422/EE3900/blob/main/Gate−Assignment3/codes

and latex-tikz codes from

https://github.com/Dishank422/EE3900/blob/main/Gate−Assignment3/latex_code.tex

1 EC 2005 Q.6

The region of convergence of Z-transform of the sequence $\left(\frac{5}{6}\right)^n u(n) - \left(\frac{6}{5}\right)^n u(-n-1)$ must be

(A) $|Z| < \frac{5}{6}$

(B) $|Z| > \frac{6}{5}$

(C) $\frac{5}{6} < |Z| < \frac{6}{5}$

(D) $\frac{6}{5} < |Z| < \infty$

2 Solution

Lemma 2.1. *The Z-transform of the sequence $a^n u(n)$ is $\frac{1}{1 - az^{-1}}$ with a region of convergence $|z| > |a|$.*

Proof.

$$\mathcal{Z}(a^n u(n)) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}} \tag{2.0.1}$$

ROC is given by $|\frac{a}{z}| < 1$, i.e. $|z| > |a|$. □

Lemma 2.2. *The Z-transform of the sequence $a^n u(-n)$ is $\frac{az^{-1}}{az^{-1} - 1}$ with a region of convergence $|z| < |a|$.*

Proof.

$$\mathcal{Z}(a^n u(-n)) = \sum_{n=-\infty}^{0} a^n z^{-n} = \sum_{n=0}^{\infty} a^{-n} z^{n} = \frac{az^{-1}}{az^{-1} - 1} \tag{2.0.2}$$

ROC is given by $|\frac{z}{a}| < 1$, i.e. $|z| < |a|$. □

Therefore,

$$\mathcal{Z}\left(\left(\frac{5}{6}\right)^n u(n)\right) = \frac{1}{1 - \frac{5}{6}z^{-1}} \tag{2.0.3}$$

with ROC $|z| > \frac{5}{6}$. We shall call this ROC_1.

Moving ahead,

$$\mathcal{Z}\left(\left(\frac{6}{5}\right)^n u(-n-1)\right) = z\left(\frac{6}{5}\right)^{-1} \mathcal{Z}\left(\left(\frac{6}{5}\right)^n u(-n)\right) \tag{2.0.4}$$

$$= \frac{1}{\frac{6}{5}z^{-1} - 1} \tag{2.0.5}$$

with ROC $|z| < \frac{6}{5}$. We shall call this ROC_2. The ROC of the given expression will be the intersection of ROC_1 and ROC_2. Therefore, the ROC of the given sequence is $\frac{5}{6} < |z| < \frac{6}{5}$. Therefore, option (C) is the correct option.

The Z-transform of the given sequence is

$$\frac{1}{1 - \frac{5}{6}z^{-1}} + \frac{1}{1 - \frac{6}{5}z^{-1}} \tag{2.0.6}$$

$$= \frac{2 - \frac{61}{30}z^{-1}}{\left(1 - \frac{5}{6}z^{-1}\right)\left(1 - \frac{6}{5}z^{-1}\right)} \tag{2.0.7}$$

Therefore the poles of the Z-transform are given by

$$z = \frac{5}{6}, \frac{6}{5} \tag{2.0.8}$$

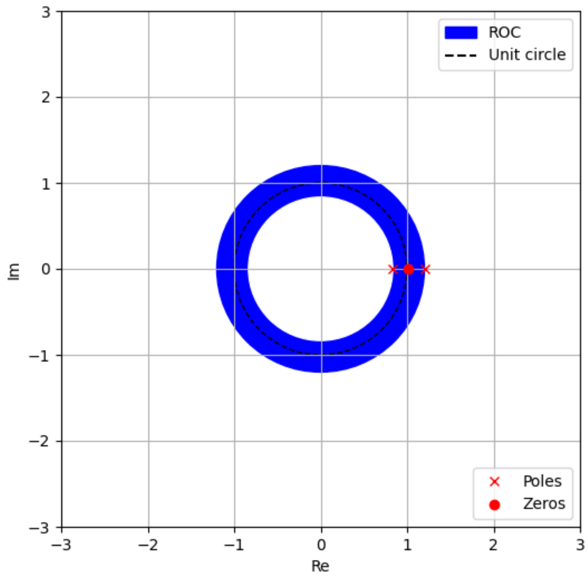

Fig. 4: Pole-zero plot of the system

The zero of the Z-transform is given by

$$z = \frac{61}{60} \tag{2.0.9}$$

Lemma 2.3. *A system is causal if and only if the ROC of the Z-transform of the impulse response of the system lies outside the outermost pole.*

Lemma 2.4. *A system is stable if and only if the ROC of the Z-tranform of the impulse response of the system includes the unit circle.*

Assuming that the given sequence is the impulse response of some system, then using lemmas 2.3 and 2.4, the system is not causal but stable. The above two lemmas further result in lemma 2.5.

Lemma 2.5. *A causal system is stable if and only if all the poles of the Z-transform of the impulse response of the system lie inside the unit circle.*

The DTFT of the sequence can be found by substituting $z = e^{j\omega}$ in 2.0.6 as

$$\mathbf{H}(e^{j\omega}) = \frac{1}{1 - \frac{5}{6}e^{-j\omega}} + \frac{1}{1 - \frac{6}{5}e^{-j\omega}} \tag{2.0.10}$$

The plot of magnitude of this DTFT is given in figure 4. From this plot, we can observe that the given filter is a high-pass filter.

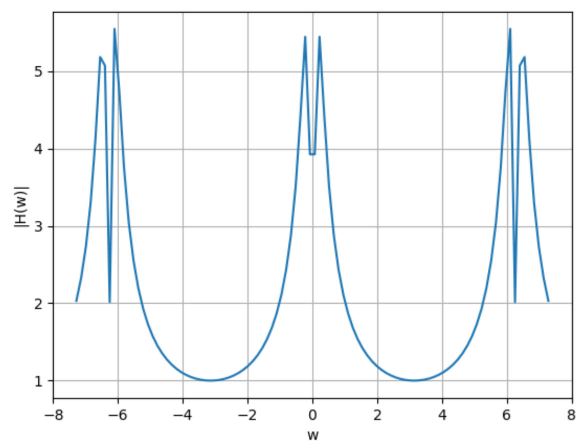

Fig. 4: DTFT of the filter