

GATE ASSIGNMENT 4

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https://github.com/Dishank422/EE3900/blob/main/Gate-Assignment4/latex_code.tex

1 EC 1999 Q.2.1

The Fourier representation of an impulse train represented by $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$ is given by

- (a) $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{j2\pi nt}{T_0}\right)$
- (b) $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{j\pi nt}{T_0}\right)$
- (c) $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j\pi nt}{T_0}\right)$
- (d) $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j2\pi nt}{T_0}\right)$

2 SOLUTION

Lemma 2.1. Any periodic signal $x(t)$ with period T_0 can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{j2\pi nt}{T_0}\right) \quad (2.0.1)$$

where, a_n is given by

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.2)$$

Proof. We shall verify equation 2.0.2.

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.3)$$

$$= \frac{1}{T_0} \int_{T_0} \left(\sum_{m=-\infty}^{\infty} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.4)$$

$$= \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \int_{T_0} a_m \left(\exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) \right) dt \quad (2.0.5)$$

When $m = n$,

$$\begin{aligned} \int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt &= \int_{T_0} a_n \quad (2.0.6) \\ &= T_0 a_n \quad (2.0.7) \end{aligned}$$

When $m \neq n$,

$$\int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.8)$$

$$= a_m \int_{T_0} \exp\left(\frac{j2\pi(m-n)t}{T_0}\right) dt \quad (2.0.9)$$

Since $\exp\left(\frac{j2\pi(m-n)t}{T_0}\right)$ is periodic with period $\frac{T_0}{m-n}$, it's integral over any time interval of length $\frac{T_0}{m-n}$ or any integral multiple of $\frac{T_0}{m-n}$ will be 0. Therefore, when $m \neq n$,

$$\int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt = 0 \quad (2.0.10)$$

Continuing from equation 2.0.5,

$$a_n = \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.11)$$

$$= \frac{1}{T_0} (T_0 a_n) = a_n \quad (2.0.12)$$

□

We observe that $s(t)$ is periodic with period T_0 . Thus it's Fourier representation as a sum of complex exponents is given by

$$s(t) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{j2\pi nt}{T_0}\right) \quad (2.0.13)$$

where, a_n can be calculated as

$$a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.14)$$

Between $-\frac{T_0}{2}$ and $\frac{T_0}{2}$, we can say that $s(t) = \delta(t)$.

$$\Rightarrow a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.15)$$

$$= \frac{1}{T_0} \quad (2.0.16)$$

$$\Rightarrow s(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j2\pi nt}{T_0}\right) \quad (2.0.17)$$

Therefore option (d) is the correct option.

Lemma 2.2. *The Fourier transform of $\exp(j2\pi f_0 t)$ is $2\pi\delta(f - f_0)$.*

Proof. We verify the lemma by finding the inverse Fourier transform of $2\pi\delta(f - f_0)$.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(f - f_0) \exp(j2\pi ft) df = \exp(j2\pi f_0 t) \quad (2.0.18)$$

□

Let $S(f)$ be the Fourier transform of $s(t)$. Then using the above lemma and equation 2.0.17

$$S(f) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) \quad (2.0.19)$$

We can also write the above by substituting $\omega_0 = \frac{1}{T_0}$ as

$$S(f) = \sum_{n=-\infty}^{\infty} 2\pi f_0 \delta(f - n f_0) \quad (2.0.20)$$

Thus we can observe that the Fourier transform of an impulse train is an impulse train in the frequency domain.