

# GATE ASSIGNMENT 4

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[https://github.com/Dishank422/EE3900/blob/main/Gate-Assignment4/latex\\_code.tex](https://github.com/Dishank422/EE3900/blob/main/Gate-Assignment4/latex_code.tex)

1 EC 1999 Q.2.1

The Fourier representation of an impulse train represented by  $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$  is given by

- (a)  $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{j2\pi nt}{T_0}\right)$
- (b)  $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{j\pi nt}{T_0}\right)$
- (c)  $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j\pi nt}{T_0}\right)$
- (d)  $\frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j2\pi nt}{T_0}\right)$

2 SOLUTION

**Lemma 2.1.** Any periodic signal  $x(t)$  with period  $T_0$  can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{j2\pi nt}{T_0}\right) \quad (2.0.1)$$

where,  $a_n$  is given by

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.2)$$

*Proof.* We shall verify equation 2.0.2.

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.3)$$

$$= \frac{1}{T_0} \int_{T_0} \left( \sum_{m=-\infty}^{\infty} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.4)$$

$$= \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \int_{T_0} a_m \left( \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) \right) dt \quad (2.0.5)$$

When  $m = n$ ,

$$\begin{aligned} \int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt &= \int_{T_0} a_n \quad (2.0.6) \\ &= T_0 a_n \quad (2.0.7) \end{aligned}$$

When  $m \neq n$ ,

$$\int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.8)$$

$$= a_m \int_{T_0} \exp\left(\frac{j2\pi(m-n)t}{T_0}\right) dt \quad (2.0.9)$$

Since  $\exp\left(\frac{j2\pi(m-n)t}{T_0}\right)$  is periodic with period  $\frac{T_0}{m-n}$ , it's integral over any time interval of length  $\frac{T_0}{m-n}$  or any integral multiple of  $\frac{T_0}{m-n}$  will be 0. Therefore, when  $m \neq n$ ,

$$\int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt = 0 \quad (2.0.10)$$

Continuing from equation 2.0.5,

$$a_n = \frac{1}{T_0} \sum_{m=-\infty}^{\infty} \int_{T_0} a_m \exp\left(\frac{j2\pi mt}{T_0}\right) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.11)$$

$$= \frac{1}{T_0} (T_0 a_n) = a_n \quad (2.0.12)$$

□

We observe that  $s(t)$  is periodic with period  $T_0$ . Thus it's Fourier representation as a sum of complex exponents is given by

$$s(t) = \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{j2\pi nt}{T_0}\right) \quad (2.0.13)$$

where,  $a_n$  can be calculated as

$$a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} s(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.14)$$

Between  $-\frac{T_0}{2}$  and  $\frac{T_0}{2}$ , we can say that  $s(t) = \delta(t)$ .

$$\Rightarrow a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \quad (2.0.15)$$

$$= \frac{1}{T_0} \quad (2.0.16)$$

$$\Rightarrow s(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j2\pi nt}{T_0}\right) \quad (2.0.17)$$

Therefore option (d) is the correct option.

**Lemma 2.2.** *The Fourier transform of  $\exp(j2\pi f_0 t)$  is  $\delta(f - f_0)$ .*

*Proof.* We verify the lemma by finding the inverse Fourier transform of  $\delta(f - f_0)$ .

$$\int_{-\infty}^{\infty} \delta(f - f_0) \exp(j2\pi ft) df = \exp(j2\pi f_0 t) \quad (2.0.18)$$

□

Let  $S(f)$  be the Fourier transform of  $s(t)$ . Then using the above lemma and equation 2.0.17

$$S(f) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) \quad (2.0.19)$$

We can also write the above by substituting  $\omega_0 = \frac{1}{T_0}$  as

$$S(f) = \sum_{n=-\infty}^{\infty} f_0 \delta(f - nf_0) \quad (2.0.20)$$

Thus we can observe that the Fourier transform of an impulse train is an impulse train in the frequency domain.