GATE ASSIGNMENT 2

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Download all python codes from

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https://github.com/Dishank422/EE3900/blob/main/
    Gate-Assignment2/codes
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and latex-tikz codes from

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https://github.com/Dishank422/EE3900/blob/main/
    Gate-Assignment2/latex_code.tex
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1 EC 2019 Q.33

The DFT of a vector $\begin{pmatrix} a & b & c & d \end{pmatrix}$ is the vector $\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}$. Consider the product

$$\begin{pmatrix} p & q & r & s \end{pmatrix} = \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \tag{1.0.1}$$

The DFT of the vector $\begin{pmatrix} p & q & r & s \end{pmatrix}$ is a scaled version of

(A) $\begin{pmatrix} \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \end{pmatrix}$
(B) $\begin{pmatrix} \sqrt{\alpha} & \sqrt{\beta} & \sqrt{\gamma} & \sqrt{\delta} \end{pmatrix}$
(C) $\begin{pmatrix} \alpha+\beta & \beta+\delta & \delta+\gamma & \gamma+\alpha \end{pmatrix}$
(D) $\begin{pmatrix} \alpha & \beta & \gamma & \delta \end{pmatrix}$

2 Solution

Lemma 2.1. *If* $\mathbf{T}$ *is a circulant matrix, then the eigenvector matrix of* $\mathbf{T}$ *is the same as the DFT matrix* $\mathbf{W}$ *and the eigenvalues are the DFT of the first column of* $\mathbf{T}$.

Proof. The i^{th} column of the $n \times n$ DFT matrix is given by

$$p_i = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^i \\ \omega^{2i} \\ \vdots \\ \vdots \\ \omega^{(n-1)i} \end{pmatrix} \tag{2.0.1}$$

where ω is the n^{th} root of 1. We shall show that this p_i is the eigenvector of $\mathbf{T}$. Observe that the kth component of $\mathbf{Tp}_i$ is given by

$$y_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \mathbf{T}_{kj} \omega^{ij} \tag{2.0.2}$$

$$= \frac{\omega^{ki}}{\sqrt{n}} \sum_{j=0}^{n-1} \mathbf{T}_{kj} \omega^{(j-k)i} \tag{2.0.3}$$

$$= \frac{\omega^{ki}}{\sqrt{n}} \sum_{j=0}^{n-1} \mathbf{T}_{(j-k)mod(n)1} \omega^{(j-k)i} \tag{2.0.4}$$

$$= \frac{\omega^{ki}}{\sqrt{n}} \sum_{m=0}^{n-1} \mathbf{T}_{m1} \omega^{mi} \tag{2.0.5}$$

Therefore

$$\mathbf{Tp}_i = \frac{\sum_{m=0}^{n-1} \mathbf{T}_{m1} \omega^{mi}}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^i \\ \omega^{2i} \\ \vdots \\ \vdots \\ \omega^{(n-1)i} \end{pmatrix} \tag{2.0.6}$$

But $\sum_{m=0}^{n-1} \mathbf{T}_{m1} \omega^{mi}$ is nothing but the ith element of the DFT of the first column of $\mathbf{T}$. Therefore $\mathbf{p}_i$ is an eigenvector of $\mathbf{T}$ with eigenvalue as ith element of the DFT of the first column of $\mathbf{T}$. □

Now we start with the solution. First we express the equations in a more convenient form.

$$\begin{pmatrix} p & q & r & s \end{pmatrix} = \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} \tag{2.0.7}$$

$$\implies \begin{pmatrix} p & q & r & s \end{pmatrix}^\top = \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}^\top \begin{pmatrix} a & b & c & d \end{pmatrix}^\top \tag{2.0.8}$$

$$\implies \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{2.0.9}$$

$$Let\ \mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix};\ \mathbf{X} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix};\ \mathbf{y} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \tag{2.0.10}$$

$$\mathbf{T} = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix} \tag{2.0.11}$$

Then we have to find $\mathbf{Y}$ the DFT of $\mathbf{y}$. We know

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{2.0.12}$$
$$\implies \mathbf{x} = \mathbf{W}^{-1}\mathbf{X} \tag{2.0.13}$$
$$\mathbf{y} = \mathbf{T}\mathbf{x} \tag{2.0.14}$$
$$\mathbf{Y} = \mathbf{W}\mathbf{y} \tag{2.0.15}$$
$$\implies \mathbf{Y} = \mathbf{W}\mathbf{T}\mathbf{x} \tag{2.0.16}$$
$$\implies \mathbf{Y} = \mathbf{W}\mathbf{T}\mathbf{W}^{-1}\mathbf{X} \tag{2.0.17}$$

But $\mathbf{T}$ is a circulant matrix with first column as $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. Therefore the eigenvalues are $\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$. Using eigen decomposition

$$\mathbf{T} = \mathbf{W} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathbf{W}^{-1} \tag{2.0.18}$$

$$\implies \mathbf{Y} = \mathbf{W}\mathbf{W} \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathbf{W}^{-1}\mathbf{W}^{-1}\mathbf{X} \tag{2.0.19}$$

Using properties of DFT

$$\mathbf{W}\mathbf{W} = \mathbf{I};\ \mathbf{W}^{-1}\mathbf{W}^{-1} = \mathbf{I} \tag{2.0.20}$$

$$\implies \mathbf{Y} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \mathbf{X} \tag{2.0.21}$$

$$\implies \mathbf{Y} = \begin{pmatrix} \alpha^2 \\ \beta^2 \\ \gamma^2 \\ \delta^2 \end{pmatrix} \tag{2.0.22}$$

Therefore option (A) is the correct option.