Quiz 2

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AI20BTECH11011

Download python codes from

```
https://github.com/Dishank422/EE3900/blob/main/quiz2/codes
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and latex-tikz codes from

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https://github.com/Dishank422/EE3900/blob/main/quiz2/latex_code.tex
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1 Discrete Time Signal Processing 3.7(b,c)

The input to a causal linear time-invariant system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n] \tag{1.0.1}$$

The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})} \tag{1.0.2}$$

(b) What is the region of convergence for Y(z)?
(c) Determine y[n].

2 Solution

Lemma 2.1. *The Z-transform of the sequence* $a^n u(n)$ *is* $\frac{1}{1-az^{-1}}$ *with a region of convergence* $|z| > |a|$.

Proof.

$$\mathcal{Z}(a^n u(n)) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}} \tag{2.0.1}$$

ROC is given by $|\frac{a}{z}| < 1$, i.e. $|z| > |a|$. □

Lemma 2.2. *The Z-transform of the sequence* $u(-n-1)$ *is* $\frac{-1}{1-z^{-1}}$ *with a region of convergence* $|z| < 1$.

Proof.

$$\mathcal{Z}(u(-n-1)) = \sum_{n=-\infty}^{-1} z^{-n} = \sum_{n=1}^{\infty} z^n = \frac{-1}{1-z^{-1}} \tag{2.0.2}$$

ROC is given by $|z| < 1$. □

(b) Using lemmas 2.1 and 2.2,

$$X(z) = \mathcal{Z}(x[n]) \tag{2.0.3}$$
$$= \mathcal{Z}(u[-n-1]) + \mathcal{Z}\left(\left(\frac{1}{2}\right)^n u[n]\right) \tag{2.0.4}$$
$$= \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \tag{2.0.5}$$
$$= \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})} \tag{2.0.6}$$

The ROC of X(z) is the intersection $|z| < 1$ and $|z| > \frac{1}{2}$. Thus the ROC of X(z) is $\frac{1}{2} < |z| < 1$. Since the system is causal, the ROC of H(z) is $|z| > 1$. We also have

$$H(z) = \frac{Y(z)}{X(z)} \tag{2.0.7}$$

Region	$\frac{1}{X(z)}$	H(z)	Y(z)
$\lvert z\rvert < \frac{1}{2}$	Converges	Doesn't converge	Must not converge
$\frac{1}{2} < \lvert z\rvert < 1$	Converges	Doesn't converge	Must not converge
$\lvert z\rvert > 1$	Converges	Converges	Must converge

TABLE (b): ROC analysis for Y(z)

From table (b), it is clear that ROC of Y(z) is $|z| > 1$.
Note that in $\frac{1}{2} < |z| < 1$, $\frac{1}{X(z)}$ converges because the only zero of X(z) is $|z| = 0$ which doesn't lie in this region.

(c)

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})} \tag{2.0.8}$$
$$= \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1+z^{-1}} \tag{2.0.9}$$

Using the fact that ROC of Y(z) is $|z| > 1$ and using converse of lemma 2.1,

$$y[n] = -\frac{1}{3}\left(\frac{1}{3}\right)^n u[n] + \frac{1}{3}(-1)^n u[n] \quad (2.0.10)$$

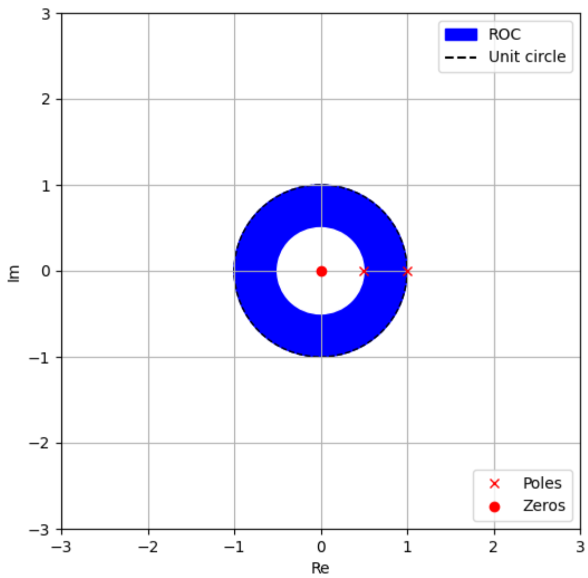

Fig. 3: ROC of X(z)

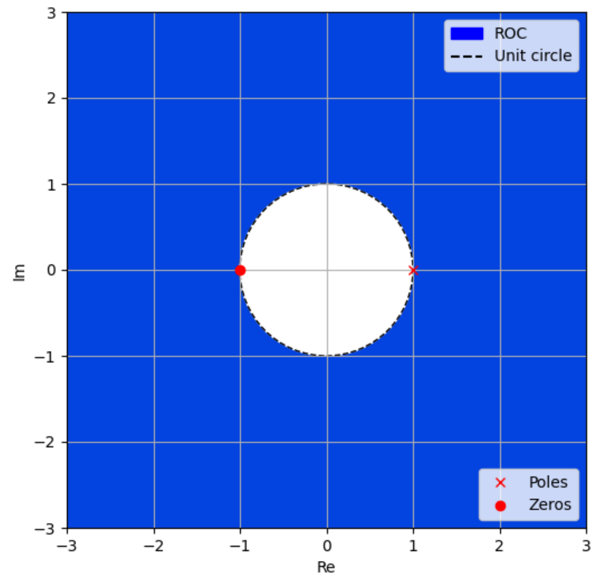

Fig. 3: ROC of H(z)

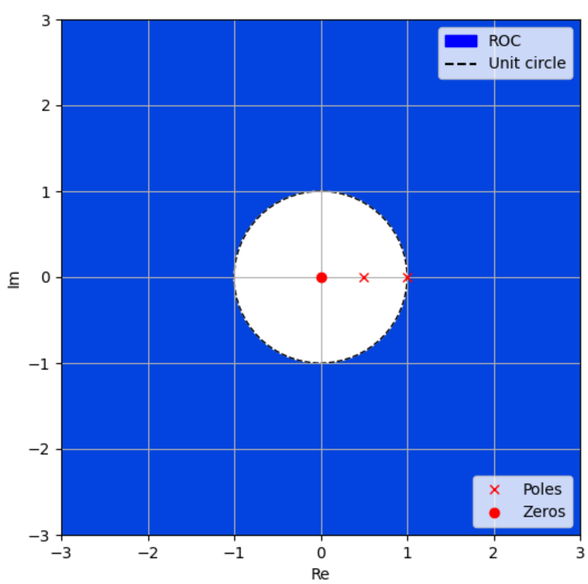

Fig. 3: ROC of Y(z)

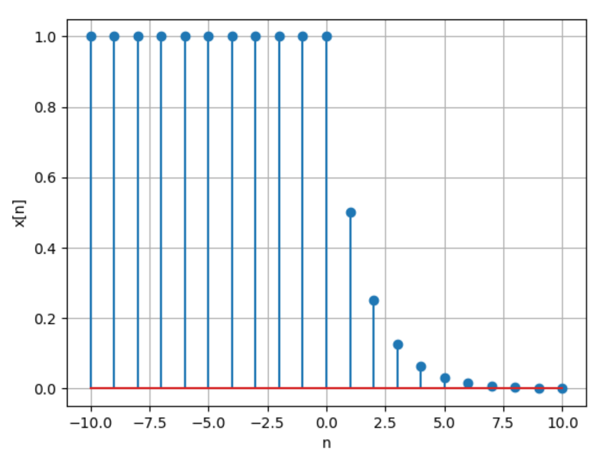

Fig. 3: x[n]

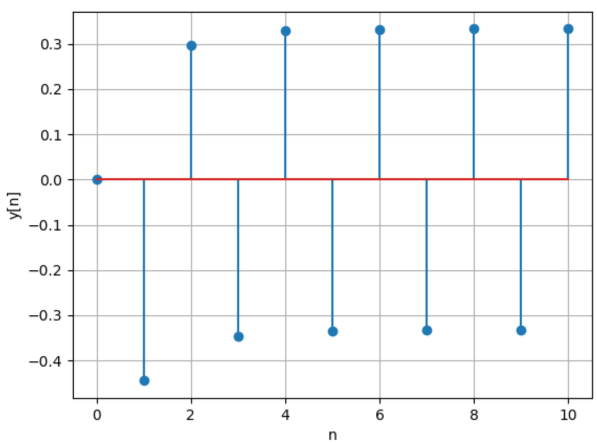

Fig. 3: y[n]