HW 2

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Question 1

[Normal to Lognormal Transformation] Show that if a random variable X ~ Normal(μ, σ), then exp(X) has the following pdf:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right],\ x > 0.$$

If X is normally distributed with μ mean and σ variance then exp(X) said to be in Lognormal distribution. Which means that it is not symmetric, and it cannot take negative values.

Let's define few things first

$$y = e^x$$

$$FY(y) = Fx(ln\ y)\ and\ Fy(y) = 1/y\ fx(ln\ y)$$

And we also know that $X \sim (\mu, \sigma)$

So, $fx(x) = \frac{1}{\sigma\sqrt{2\pi}}$ exp($-\frac{(x-\mu)^\wedge 2}{2\sigma}$)

then,

$\boldsymbol{fy(y) = \frac{1}{y} \frac{1}{\sqrt{2\pi\,\sigma^\wedge 2}}}$ **exp(**$\boldsymbol{-\frac{(\ln(y)-\mu)^\wedge 2}{2\sigma^\wedge 2}}$**)** $\boldsymbol{, y > 0}$

Question 2

In the bank example in page 44-45 of the slides, the joint pdf of X, Y is given by

$$f_{X,Y}(x,y) = \frac{6}{5}(x + y^2), \qquad 0 \le x \le 1, 0 \le y \le 1$$

a) What is the marginal pdf of fy(y)?

$fy(y) = \int_0^1 \frac{6}{5}(x + y^2)\, dy$

= $\int_0^1 \frac{6}{5}\, x\, dy$ + $\int_0^1 \frac{6}{5}\, y^\wedge 2\, dy$

= $\frac{6}{5}\, x$ + $\int_0^1 \frac{6}{5}\, y^\wedge 2\, dy$

= $\frac{6}{5}\, x$ + $\frac{6}{5}\frac{y^\wedge 3}{3}\ |^1{}_0$

= $\frac{6}{5}\, x$ + $\frac{6}{5}\, \frac{1}{3}$

= $\frac{6}{5}\, x$ + $\frac{6}{15}$

= $\boldsymbol{\frac{6}{5}\,(x + \frac{1}{3})}$

b) What is the conditional pdf of fX|Y (x|y = 0.5)?

$fX|Y\ (x|y) = \frac{f(x,y)}{fy(y)}$

= $\frac{\frac{6}{5}(x+y^2)}{\frac{6}{5}(x+\frac{1}{3})}$

= $\frac{(x+y^2)}{x+1/3}$

So,

$fX|Y\ (x \mid y = 0.5) = \frac{x+0.5^2}{x+1/3}$

$= \frac{\boldsymbol{x+0.25}}{\boldsymbol{x+1/3}}$

Question 3

Exponential Distribution has a memoryless property. Intuitively, it means that the probability of customer service answering you call (assuming waiting time is exponential) in the next 10 mins is the same, no matter if you have waited an hour on the line or just picked up the phone. Formally, if X ~ exponential(λ), f(x) = λ exp(-λx), and t and s are two positive numbers, use the definition of conditional probability to show that
P(X > t + s | X > t) = P(X > s).
Hint: Find the cdf of X first, and note that P(X > t + s Ç X > t) = P(X > t + s)

Now we know that CDF of $f(x)$ is,

$$F(x) = 1 - exp(-\lambda x)\,, x >= 0$$
$$0\,, Otherwise$$

Now according to conditional probability rule,

$p(X > t + s \mid X > t) = \frac{p\ (X> t+s \,\cap\, X>t)}{p(X > t)}$ ------ [1]

Since $s > 0$. If $X > t + s$ then $X > s$. So,

$p(X > t + s \mid X > t) = p(X > t + s)$

And also, s and t both are positive, $p(X > t + s) = exp(-\lambda(t + s))$ and $p(X > s) = exp(-\lambda t)$

So [1] can be written as

$$= \frac{\frac{\exp(-\lambda(t+s))}{2}}{\frac{\exp(-\lambda t)}{2}}$$

$$= exp(-\lambda s)$$

Hence proved.

Question 4

Roll a fair die (uniform 1,2,3,4,5,6) repeatedly. You and Peter are betting on the number shown on each roll. If the number is 4 or less, you win $1; otherwise, you pay Peter $2.5.

a) What is the expected value of the payoff for you?

$$p(Pay\ off\ for\ you) = 4/6 = 2/3$$

$$E[Pay\ off\ for\ you] = 2/3(1) + 2/3(2) + 2/3(3) + 2/3(4)$$
$$= 2/3 + 4/3 + 6/3 + 8/3$$
$$= 20/3$$
$$= \mathbf{6.66}$$

b) What is the variance of your payoff?

$$Var[Pay\ off\ for\ you] = (1 - 6.66)\char`^2 * 0.16 + (2 - 6.66)\char`^2 * 0.16 + (3 - 6.66)\char`^2 * 0.16 + (4 - 6.66)\char`^2 * 0.16$$
$$= 5.12 + 3.37 + 2.14 + 1.13$$
$$= \mathbf{11.86}$$

Question 5

Let X be uniformly distributed (continuous) on the interval [1,2]. Find E(1/X).

We know that,
$E[X] \; = \int_{-\infty}^{\infty} x f(x) \; dx$

So,
$E[1/X] \; = \; \int_{1}^{2} \frac{1}{x} f(x) \, dx$

Here $f(x) = \frac{1}{2-1} = \; 1 \, , \quad 1 < \; x < \; 2$

$E[1/x] \; = \int_{1}^{2} \frac{1}{x} \; dx$

= ln |x| $|^2{}_1$

= $ln(2) - \; ln(1)$

= **0.6931**

Question 6

Let us assume that the lifetime of light bulbs follows an exponential distribution with the pdf:
f(x) = λ exp(-λx)
We test 10 bulbs and their lifetimes are 5, 3, 5, 1, 7, 3, 4, 8, 2, 3 years, respectively. What is the MLE for λ? What is the method of moments estimator for λ?

Here following information is given,

$$n \; = \; 10$$
$$\sum xi = 41$$

Now,

$$f(x) \; = \; (\lambda e^{-\lambda xi})$$

$$f(x1, x2, \ldots, x10) = \; \left(\lambda e^{-\lambda x1}\right) * \; \left(\lambda e^{-\lambda x2}\right) * \; \left(\lambda e^{-\lambda x3}\right) * \ldots \ldots \; * \; (\lambda e^{-\lambda x10})$$
$$= (\lambda\text{^}{}^{10} \; * \; e^{-\lambda \sum xi})$$
$$= (\lambda\text{^}{}^{10} \; * \; e^{-\lambda 41})$$

Taking log on both sides,

$$\ln \left[f(x1, x2, \ldots, x10)\right] = \; \ln \left[(\lambda\text{^}{}^{10} \; * \; e^{-\lambda 41})\right]$$

$$= \; 10 * \ln[n(\lambda) \; - \; 41\lambda]$$

Let's take MLE,

$$\frac{d(ll)}{d\lambda} = 0$$

$$\frac{10}{\lambda} - 41 = 0$$

$$\frac{10}{\lambda} = 41$$

$$\boldsymbol{\lambda}\text{^} = \frac{\mathbf{10}}{\mathbf{41}}$$

Question 7

Let x_1, x_2,….,x_n be an iid samples with pdf

$$\boldsymbol{f(x|\theta) = \theta x^{\theta} - 1 \ , 0 \ \leq x \ \leq 1, \qquad \theta > 0}$$

Find MLE of $\boldsymbol{\theta}$

$L(xi\ \theta) \ = \ \prod_{i=1}^{n} x\theta i^{\theta-1}$

Taking ln

$$ln[L(xi\ \theta)] \ = \ n\ ln(\theta) \ + \ \sum_{1}^{n} ln\left(xi^{\theta-1}\right)$$

$$= \ n\ ln(\theta) \ + \ (\theta - 1) \ * \sum_{1}^{n} ln(xi)$$

Taking derivative,

$$\frac{d\ ln[L(xi\ \theta)]}{d\theta} = \frac{n}{\theta} + \sum_{1}^{n} \ln{(xi)}$$

Setting $d/d\theta \ = \ 0$,

$$0 = \frac{n}{\theta} + \sum_{1}^{n} \ln{(xi)}$$

$$0 = \ n + \ \theta * \sum_{1}^{n} \ln(xi) \ \ [Multiply\ by\ \theta\ on\ both\ sied\ ]$$

$$\boldsymbol{\theta = \frac{-n}{\sum_{1}^{n} \ln{(xi)}}\ , 0 \leq x \leq 1}$$