An example of a Lax pair

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1 1 dimension

Consider the $\mathfrak{su}(2)$ algebra generated by $\{H, E, F\}$ with the relations

$$
\begin{aligned}
[H, E] &= 2E \\
[H, F] &= -2F \\
[E, F] &= H
\end{aligned}
$$

Let us re-write this by introducing $X = E + F$, $Y = E - F$, which now have the commutation relations

$$
\begin{aligned}
[H, X] &= 2Y \\
[H, Y] &= 2X \\
[X, Y] &= -2H
\end{aligned}
$$

Consider the Hamiltonian system with phase space $q, p \in \mathbb{R}$ and Hamiltonian

$$\mathcal{H} = \frac{1}{2}p^2 + V(q)$$

Proposition 1.1. *The algebra elements*

$$
\begin{aligned}
L(\zeta) &= pH + W(q)X + \zeta Y, \quad \zeta \in \mathbb{C} \\
M &= \frac{1}{2}W_q Y
\end{aligned}
$$

for a Lax pair for the system if $V_q = WW_q$ (i.e. $V = \frac{1}{2}W^2 + c$) and $\cdot_q = \partial_q \cdot$.

Proof. Hamilton's equations for this system are

$$
\begin{aligned}
\dot{q} &= p \\
\dot{p} &= -V_q
\end{aligned}
$$

so

$$\dot{L} = -V_q H + pW_q X$$

whereas

$$[L, M] = pW_q X - WW_q H$$

□

Now we want to think of generalisations of this Lax pair. A generic Hamiltonian for a $2d$ phase space can be written in the form

$$\mathcal{H} = \frac{p^2}{2\lambda(q)} + V(q)$$

Proposition 1.2. *The algebra elements*

$$
\begin{aligned}
L(\zeta) &= \frac{1}{\sqrt{\lambda}}pH + W(q)X + \zeta Y, \quad \zeta \in \mathbb{C} \\
M &= \frac{1}{2\sqrt{\lambda}}W_q Y
\end{aligned}
$$

form a Lax pair for the system.

Proof. We repeat a similar calculation:

$$\dot{q} = \frac{p}{\lambda}$$
$$\dot{p} = -V_q + \frac{p^2 \lambda_q}{2\lambda^2}$$

so

$$\begin{aligned}\dot{L} &= \frac{1}{\sqrt{\lambda}}\left[\left(-V_q + \frac{p^2\lambda_q}{2\lambda^2}\right) - \frac{p^2\lambda_q}{2\lambda^2}\right]H + \frac{pW_q}{\lambda}X \\ [L, M] &= \frac{pW_q}{\lambda}X - \frac{WW_q}{\sqrt{\lambda}}H\end{aligned}$$

□

Now if, like me, you do not see the spitting obvious thing that scaling $p \to \frac{p}{\sqrt{\lambda}}$ is a sensible thing to do, how might you approach this? Start by supposing a more general form related to our original pair

$$\begin{aligned}L(\zeta) &= f(q,p)H + g(q,p)X + \zeta Y \\ M &= h(q,p)Y\end{aligned}$$

We then get that for L, M to be a Lax pair we have the equations

$$\begin{aligned}f_q\left(\frac{p}{\lambda}\right) + f_p\left(-V_q + \frac{p^2\lambda_q}{2\lambda^2}\right) &= -2gh \\ g_q\left(\frac{p}{\lambda}\right) + g_p\left(-V_q + \frac{p^2\lambda_q}{2\lambda^2}\right) &= 2fh\end{aligned}$$

Let's make the ansatz that g, h are functions of q only. Then we get from the second equation

$$g_q \cdot \frac{p}{\lambda} = 2fh$$

Equation the order of p on each side we must get $f(q,p) = pF(q)$ and then

$$g_q = 2Fh\lambda$$

Substituting our new form of f into the first equation gives

$$\frac{p^2F_q}{\lambda} + F\left(-V_q + \frac{p^2\lambda_q}{2\lambda^2}\right) = -2gh$$

Again equating orders of p we have

$$\begin{aligned}\frac{F_q}{\lambda} + \frac{F\lambda_q}{2\lambda^2} = 0 &\Rightarrow F_q\lambda^{\frac{1}{2}} + \frac{1}{2}F\lambda^{-\frac{1}{2}}\lambda_q = 0 \\ &\Rightarrow \left(F\lambda^{\frac{1}{2}}\right)_q = 0 \\ &\Rightarrow F = \frac{\alpha}{\sqrt{\lambda}}, \quad \alpha \in \mathbb{R}\end{aligned}$$

Subbing this back into the first equation gives

$$-\alpha\lambda^{-\frac{1}{2}}V_q = -2gh$$

so

$$gg_q = \frac{\alpha\lambda^{-\frac{1}{2}}V_q}{2h}2\alpha h\lambda^{\frac{1}{2}} = \alpha^2 V_q$$

and

$$h = \frac{\lambda^{-\frac{1}{2}}g_q}{2\alpha}$$

We recognise taking $g = W$, $\alpha = 1$, this is the solution from above.

2 2 dimensions

We want to now try to see if we can expand upon this result. Note that our first systems has a natural generalisation of the following form:

Proposition 2.1. *The algebra elements*

$$\begin{aligned} L(\boldsymbol{\zeta}) &= \sum_{i=1}^{n} p_i H_i + W_i X_i + \zeta_i Y_i \\ M &= \frac{1}{2} W_i' Y_i \end{aligned}$$

form a Lax pair for the evolution of the Hamiltonian

$$\mathcal{H} = \sum_{i=}^{n} \frac{1}{2} p_i^2 + V_i(q^i)$$

where $\langle H_i, X_i, Y_i \rangle$ are distinct copies of the previous algebra that commute with each other.

This, however, covers only a small class of Hamiltonians, so we might try the next simplest non-trivial case; that of the 2d in Liouville form.

Definition 2.2. *An n-dimensional **Liouville system** is one whose Hamiltonian is of the form*

$$\mathcal{H} = \frac{1}{\lambda} \left[\sum_{i=1}^{n} \frac{1}{2} \sigma_i p_i^2 + V_i \right]$$

where $\lambda = \sum_{i=1}^{n} \lambda_i$ and λ_i, σ_i, V_i are functions of q^i only.

We have the following theorem that says that in 2d, considering Liouville form is sufficiently general:

Theorem 2.3. *On a 2d Riemannian manifold, any separable metric can be written locally in Liouville form.*

Proof. Historically, this predates Stäckel's theorem, but using Stäckel the proof becomes very easy. Write the Stäckel matrix as

$$\begin{aligned} U &= \begin{pmatrix} \lambda_1/\sigma_1 & -1/\sigma_1 \\ \lambda_2/\sigma_2 & 1/\sigma_2 \end{pmatrix} \\ \Rightarrow U^{-1} &= \frac{1}{\lambda} \begin{pmatrix} \sigma_1 & \sigma_2 \\ -\lambda_2 \sigma_1 & \lambda_1 \sigma_2 \end{pmatrix} \end{aligned}$$

We can then read off the top row. □

Hamilton's equations for a Liouville system can be read off as

$$\begin{aligned} \dot{q}^i &= \frac{\sigma_i p_i}{\lambda} \\ \dot{p}_i &= \frac{\lambda_i'}{\lambda} \mathcal{H} - \frac{1}{\lambda} \left[\frac{1}{2} \sigma_i' p_i^2 + V_i' \right] \end{aligned}$$

Suppose we naively tried to port our Lax pair from the 1d system using the rough scaling argument, that is have

$$\begin{aligned} L(\boldsymbol{\zeta}) &= \sum_{i=1}^{2} \sqrt{\frac{\sigma_i}{\lambda}} p_i H_i + \frac{1}{\sqrt{\lambda}} W_i X_i + \zeta_i Y_i \\ M &= \sum_{i=1}^{2} \frac{1}{2} \frac{\sqrt{\sigma_i}}{\lambda} \left(W_i' - \frac{\lambda_i'}{2\lambda} W_i \right) Y_i \end{aligned}$$

To calculate the terms in $\dot{L}$ it is necessary to calcualte

$$\begin{aligned}
\frac{d}{dt}\sqrt{\frac{\sigma_i}{\lambda}} &= \sqrt{\frac{\sigma_i}{\lambda}}\left\{\frac{\sigma_i p_i}{\lambda}\left[\frac{\sigma_i'}{2\sigma_i} - \frac{\lambda_i'}{2\lambda}\right] - \sum_{j\neq i}\frac{\sigma_j p_j \lambda_j'}{2\lambda^2}\right\} \\
\Rightarrow \frac{d}{dt}p_i\sqrt{\frac{\sigma_i}{\lambda}} &= \sqrt{\frac{\sigma_i}{\lambda}}\left\{\frac{\sigma_i p_i^2}{\lambda}\left[\frac{\sigma_i'}{2\sigma_i} - \frac{\lambda_i'}{2\lambda}\right] - p_i\sum_{j\neq i}\frac{\sigma_j p_j \lambda_j'}{2\lambda^2} + \dot{p}_i\right\} \\
&= \frac{1}{\lambda}\sqrt{\frac{\sigma_i}{\lambda}}\left\{\sigma_i p_i^2\left[\frac{\sigma_i'}{2\sigma_i} - \frac{\lambda_i'}{2\lambda}\right] - p_i\sum_{j\neq i}\frac{\sigma_j p_j \lambda_j'}{2\lambda} + \lambda_i'\mathcal{H} - \left[\frac{1}{2}\sigma_i' p_i^2 + V_i'\right]\right\} \\
&= \frac{1}{\lambda}\sqrt{\frac{\sigma_i}{\lambda}}\left\{\left[\frac{1}{2\lambda}\sum_{j\neq i}\sigma_j p_j(\lambda_i' p_j - \lambda_j' p_i)\right] + \frac{\lambda_i'}{\lambda}\sum_j V_j - V_i'\right\}
\end{aligned}$$