

Gauge Theory Notes

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1 Introduction

These are lecture notes taken from "Topics in Mathematical Physics". It may go poorly trying to type this live but let's see. The course will be covering essentially differential geometric aspects of gauge theory

1.1 Historical Overview

1.1.1 Physics

Starting with Maxwell's equations in the 19th century for describing classical electromagnetism, these are differential equations for some fields. These were reformulated by Weyl in the 1920-30s, and the term gauge theory was coined. Namely, electromagnetism was reformulated as a classical $U(1)$ gauge theory. The Aharonov-Bohm theory gave that this extra degree of freedom corresponding to the vector potential has a physical significance for quantum mechanics. In the 1950s, Yang & Mills gave the equations for gauge theories with arbitrary non-abelian Lie groups. The physical relevance of the commutativity is that non-abelian groups - when quantised - lead to particles which self interact. Note that this is not the case for photons (which correspond to $U(1)$). In the 1960-70s it was shown that non-abelian gauge theories can be renormalised, so correctly quantised, and this allows the standard model to be developed. This is a quantum gauge theory with group $U(1) \times SU(2) \times SU(3)$.

1.1.2 Maths

In the 1930-60s the theory of the principal bundle was developed, and this included the concept of connections. In the 1970s it was realised that these are the same constructs as being developed in physics. In the late 70s the concept of moduli space of instantons was being developed, and in the 80s the Donaldson invariant theory was being worked on. These are invariants of differential manifolds, and led to the discovery of exotic \mathbb{R}^4 . This continued to develop ideas of Seiberg-Witten invariants in the 90s.

1.2 Course direction

The course will be slightly dictated by the will of the class, but the basic structure will be as follows:

- Review of manifolds and De Rham cohomology
- Riemannian and symplectic geometry, including symplectic reduction
- Lie groups, Lie algebras, and basic rep theory
- Principal bundles, associated vector bundles, fibre bundles
- Connections from different points of view
- Chern-Weil theory, characteristic classes, and classifying spaces
- Yang Mills functional, electromagnetism as a gauge theory, and the Aharonov-Bohm experiment
- Gauge theory in dimensions 2,3 and 4, instantons, Chern-Simons theory

1.3 Suggested Reading

Possible texts to look at include:

Mathematically sound, though written for physicists -

- Nakahara, "Geometry, topology & physics"
- Nash, "Topology and geometry for physicists"
- Frankel, "Geometry of physics"

Written for mathematicians -

- Marcathe & Martucci, "The mathematical formulation of gauge theory"
- Naber, "Topology, geometry, and gauge fields"

Other options:

- Morgan, "Intro to gauge theory" in Gauge theory and the topology of 4-manifolds
- Nicolaescu, "Lectures on the geometry of manifolds".

1.4 Prelim Tools

1.4.1 Quaternions

Here we will cover just some separate material that will be necessary for some comprehension, but is very separate.

Definition 1.1. The **quaternions**, denoted \mathbb{H} , is the real associative algebra spanned by $\{1, i, j, k\}$ subject to

$$i^2 = -1 = j^2, \quad ij = k = -ji$$

For $q = xi + yj + zk + w \in \mathbb{H}$ we define its **real part** as $\text{Re}(q) = w$ and as such its **quaternionic conjugate** by

$$\bar{q} = -xi - yj - zk + w$$

Lemma 1.2. We can deduce the following relations:

- $k^2 = -1$
- $jk = i = -kj$
- $ki = j = -ik$
- $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$.

Proof. These will all be calculations:

- $k^2 = -ij^2i = i^2 = -1$
- $jk = -j^2i = i$ and likewise for kj
- $ki = -ji^2 = j$ and likewise for ik .
- If we let $q_a = x_ai + y_aj + z_ak + w_a$ then

$$\begin{aligned} q_1 q_2 &= x_1 x_2 i^2 + y_1 y_2 j^2 + z_1 z_2 k^2 + w_1 w_2 + ij(x_1 y_2 - x_2 y_1) + jk(y_1 z_2 - y_2 z_1) + ki(z_1 x_2 - z_2 x_1) \\ &\quad + i(x_1 w_2 + x_2 w_1) + j(y_1 w_2 + y_2 w_1) + k(z_1 w_2 + z_2 w_1) \\ &= w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 + i[(x_1 w_2 + x_2 w_1) + (y_1 z_2 - y_2 z_1)] \\ &\quad + j[(y_1 w_2 + y_2 w_1) + (z_1 x_2 - z_2 x_1)] + k[(z_1 w_2 + z_2 w_1) + (x_1 y_2 - x_2 y_1)] \end{aligned}$$

We can see that taking the negative of the i, j, k coefficients is equivalent to taking the negative of the x_a, y_a, z_a and swapping 1 \leftrightarrow 2.

□

Definition 1.3. Define an inner product on the quaternion space by $\langle q_1, q_2 \rangle = \text{Re}(\bar{q}_1 q_2)$ and the norm induced of the quaternions as $\|q\| = \sqrt{\langle q, q \rangle} = \sqrt{\bar{q}q}$

Lemma 1.4. $\langle q_1, q_2 \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2 + q_1 w_2$.

Proof. Using the above calculation of q_1, q_2 the answer is immediate. □

Lemma 1.5. If $q = xi + yj + zk + w$ then $\|q\| = \sqrt{x^2 + y^2 + z^2 + w^2}$. Further, $\|q_1 q_2\| = \|q_1\| \|q_2\|$.

Proof. Using the proof of the above lemma the first result is immediate. To prove the latter note $\|q_1 q_2\| = \sqrt{\bar{q}_2 \bar{q}_1 q_1 q_2}$, and as $\bar{q}_1 q_1$ is a positive real it will pull out of the square root. □

2 Review of Manifold Theory

You will definitely need this knowledge to carry on into the connection material. Some possible reading if this is not sufficient (as is possible)

- Taubes, "Differential geometry"
- Warner, "Foundations of differential manifolds and Lie groups"
- Lee, "Introduction to smooth manifolds"

2.1 Initial definitions

2.1.1 The Starting Point

Definition 2.1. A **topological manifold** of dimension n , M , is a topological space that is Hausdorff, second countable, paracompact, and locally homeomorphic to \mathbb{R}^n . i.e. $\forall m \in M, \exists U \ni m$ such that U open and homeomorphic to \mathbb{R}^n .

Recall that topological manifolds can have further properties such as **compact** or **orientable** (Note that orientability is a well defined topological property, but for manifolds not general topological spaces).

Now we want to be able to do calculus on manifolds, but the standard definition of derivative of a real valued function f does not immediately parse. Namely the concept of $f(x + \epsilon)$ leads us to want a vector space structure, and so we are led to think locally, where we know our space looks like \mathbb{R}^n , which is a vector space. Now we have a new problem, that our answer for the derivative may be ambiguous depending on which homeomorphism we chose

Example 2.2. This manifests itself in dimension 1. Let $M = \mathbb{R}$, but take the homeomorphisms id or x^3 in a neighbourhood of $m = 0$. Take the function $f(x) = x^{\frac{1}{3}}$. If we ask whether this function is differentiable, that depends on which homeomorphism we are using.

To solve this issue, we require extra information, and this turns a topological manifold into a smooth manifold:

Definition 2.3. A **smooth manifold** is a topological manifold with a preferred set of homeomorphisms $\{\varphi_\alpha : U_\alpha \xrightarrow{\cong} \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n\}$ such that

- $\cup_\alpha U_\alpha = M$
- $\varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$ is a diffeomorphism

This is called a **smooth atlas**, and the φ_α are called **charts**.

2.1.2 Other viewpoints

We can define $C^0(M)$, the \mathbb{R} -algebra of real valued continuous functions on a topological manifold naturally, but we require the smooth structure of transition functions in order to define $C^\infty(M)$, the subalgebra of smooth functions. An atlas is said to be compatible with a local homeomorphism φ if $\{\varphi\} \cup \{\varphi_\alpha\}$ is still an atlas. This allows us to talk of **maximal atlases**, which are those which contain all possible compatible charts. Each atlas has a unique compatible maximal atlas, and the corresponding equivalence classes of atlases are called **smooth structure**

Example 2.4. Consider again $M = \mathbb{R}$. The atlases given by id and x^3 are incompatible as atlases, and so correspond to different smooth structures. However, the homeomorphism $x^{\frac{1}{3}} : M \rightarrow M$ becomes a diffeomorphism wrt the corresponding smooth structure.

We can, from this example, begin to ask the question of what smooth structures are possible up to diffeomorphism. This is a question answered by learning from gauge theory, which leads to the concept of exotic \mathbb{R}^4 .

2.2 Objects on manifolds

2.2.1 Vector Fields

We can take different viewpoints towards what is a vector field. Suppose we have a chart φ_α with coordinates x^1, \dots, x^n .

A physicist approach would be to define a vector field as such:

Definition 2.5 (vector field - physicist way). A **vector field** is an assignment of n smooth functions (X^1, \dots, X^n) to each chart s.t. when switching to a chart with coordinates \tilde{x}^i the smooth functions \tilde{X}^i are given by

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^j$$

A mathematical definition would be

Definition 2.6 (Vector field - mathematician way). A **vector field** is a derivation of $C^\infty(M)$. Recall a derivation is a map $X : C^\infty(M) \rightarrow C^\infty(M)$ s.t. X is \mathbb{R} linear and that $X(fg) = X(f)g + fX(g)$.

The connection between the two definitions is that $X^i \frac{\partial}{\partial x^i}$ is a derivation of $C^\infty(U_\alpha)$

$\forall m \in M$, we have a **tangent space** to M at m , written $T_m M$. This requires a smooth structure in order to define. Given the tangent spaces we can define

$$TM = \bigsqcup_m T_m M$$

the **tangent bundle**. It is a manifold in its own right, and moreover it comes with the projection

$$\begin{aligned} \pi : TM &\rightarrow M \\ v \in T_m M &\mapsto m \end{aligned}$$

With this we can then say

Definition 2.7. A **vector field** is a section of π , i.e. a smooth map $X : M \rightarrow TM$ s.t. $\pi \circ X = id_M$.

Given a vector field $X \in \mathfrak{X}(M)$, \exists a 1-parameter group of diffeomorphisms $\phi_t^X : M \rightarrow M$ for $t \in \mathbb{R}$ (we will sometimes denote this as $\exp(tX)$) such that

- $\phi_t^X \circ \phi_s^X = \phi_{t+s}^X$
- $\left. \frac{d}{dt} \phi_t^X(m) \right|_{t=0} = X_m$

If M is not compact, ϕ_t^X will only exist 'locally'. If they do exist $\forall t$ we say that X is **complete**.

Example 2.8. Take $M = (0, 1) \times \mathbb{R}$, we can have $X = \partial_x$ a horizontal vector field and this is not complete.

Definition 2.9. If X is a (complete) vector field, the **Lie derivative** of anything which can pullback under the diffeomorphisms ϕ_t^X by

$$\mathcal{L}_X T = \frac{d}{dt} (\phi_t^X)^* T \Big|_{t=0}$$

Example 2.10. Some simple cases are

1. for $f \in C^\infty(M)$, $\mathcal{L}_X f = X(f)$
2. for $\alpha \in \Omega^*(M)$, $\mathcal{L}_X \alpha = i_X d\alpha + d(i_X \alpha)$
3. for $Y \in \mathfrak{X}(M)$, $\mathcal{L}_X Y = [X, Y]$. This bracket is the **Lie bracket**

Lemma 2.11. We can calculate $\phi_t^{[X,Y]} = \phi_{-\sqrt{t}}^X \circ \phi_{-\sqrt{t}}^Y \circ \phi_{\sqrt{t}}^X \circ \phi_{\sqrt{t}}^Y$

2.2.2 One forms

Each tangent space is a vector space, and so has dual vector space $T_m^* M$. We can hence similarly define the **cotangent bundle**

$$T^* M = \bigsqcup_m T_m^* M$$

Definition 2.12. A **one form** is a section of $T^* M \rightarrow M$

we can also apply operations from multilinear algebra, e.g. $TM \oplus TM, \dots$

Definition 2.13 (Tensor field - physicist way). A **tensor** is a section of a bundle of the form (p, q)

$$\underbrace{TM \otimes \dots \otimes TM}_{\times p} \otimes \underbrace{T^* M \otimes \dots \otimes T^* M}_{\times q} \rightarrow M$$

Definition 2.14. A **k-form** is an element of

$$\Omega^k(M) = \Gamma(\Lambda^k(T^* M))$$

where Γ is the space of sections. These are \mathbb{R} -vector spaces, and further $C^\infty(M)$ -modules.

2.2.3 Exterior algebra

We can define a **wedge operator**

$$\begin{aligned} \wedge : \Omega^k(M) \times \Omega^l(M) &\rightarrow \Omega^{k+l}(M) \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta \end{aligned}$$

We can thus get the graded algebra

$$\Omega^*(M) = \bigoplus_{i=0}^n \Omega^i(M)$$

We can further define an **exterior derivative** by

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

Whereas the wedge operation was done pointwise, the exterior derivative requires local information. It has the property

$$d^2 = 0$$

This gives a natural motivation to

Definition 2.15. The i^{th} *De Rham Cohomology class* of M is

$$H_{dR}^i(M) = \ker(d : \Omega^i(M) \rightarrow \Omega^{i+1}(M)) / \text{Im}(d : \Omega^{i-1}(M) \rightarrow \Omega^i(M))$$

Often (e.g when M is compact) this is finite dimensional. Note that, despite the fact that we needed smooth structure in order to define this, it is in fact a topological property.

2.2.4 The Hodge star operator

The Hodge star is an operator fraught with minus sign problems. Check nlab to try be consistent.

Definition 2.16. A *volume form* of a differentiable manifold M is a top-degree form (i.e. form of degree $\dim M$).

Definition 2.17. An *oriented* manifold is one equipped with a nowhere-vanishing volume form.

Lemma 2.18. Every Riemannian manifold (M, g) has a canonical choice of volume form given in local coordinates x^1, \dots, x^n by

$$\omega_{vol} = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^n$$

Lemma 2.19. If V is a vector space with bilinear $\langle \cdot, \cdot \rangle$ then $\bigwedge^k V$ inherits a bilinear by the Grammian determinant.

$$\langle \alpha_1 \wedge \dots \wedge \alpha_k, \beta_1 \wedge \dots \wedge \beta_k \rangle = \det \left(\langle \alpha_i, \beta_j \rangle_{i,j=1}^k \right)$$

Definition 2.20. If M is a n -dimensional Riemannian manifold, the **Hodge star** is the operator $\star : \Omega^r(M) \rightarrow \Omega^{n-r}(M)$ given by

$$\alpha \wedge (\star \beta) = \langle \alpha, \beta \rangle \omega_{vol}$$

for $\alpha, \beta \in \Omega^r(M)$, where $\langle \cdot, \cdot \rangle$ is the Euclidean structure on $\Omega^r(M)$.

Lemma 2.21. This property defines \star completely.

Proof. exercise □

Proposition 2.22. If M has determinantal sign s of the inner product, and $\alpha \in \Omega^r(M)$ then

$$(\star)^2 \alpha = (-1)^{r(n-r)} s \alpha$$

Corollary 2.23. On Euclidean \mathbb{R}^4 , $(\star)^2 = 1$, so \star has eigenvalues ± 1 .

Proposition 2.24. Take coordinates $\{x^\mu\}$ on an n -dimensional manifold M . Let g be a metric with $g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$. Then

$$\begin{aligned}\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) &= \frac{\sqrt{|g|}}{(n-p)!} g^{\mu_1\nu_1} \cdots g^{\mu_p\nu_p} \epsilon_{\nu_1 \dots \nu_n} dx^{\nu_{p+1}} \wedge \cdots \wedge dx^{\nu_n} \\ &= \frac{1}{(n-p)!} \epsilon^{\mu_1 \dots \mu_p}_{\mu_{p+1} \dots \mu_n} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n}\end{aligned}$$

Here we are using the Levi-Civita tensor, which means that, for example, in Minkowski

$$\epsilon_{\mu_1 \dots \mu_p \mu_{p+1} \dots \mu_n} \epsilon^{\nu_1 \dots \nu_p \mu_{p+1} \dots \mu_n} = -(n-p)! \delta^{\nu_1 \dots \nu_p}_{\mu_1 \dots \mu_p}$$

2.3 Lie Theory

2.3.1 Lie Group Basics

Definition 2.25. A **Lie group** G is a group object in the category of manifolds, i.e.

- G is a manifold
- $\mu : G \times G \rightarrow G$ multiplication is smooth
- $i : G \rightarrow G$ inverse is smooth

s.t. multiplication is associative, inverses commute, etc. We may write the Lie group as a **pointed** manifold (G, e) to make the identity explicit.

Example 2.26. We have $GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$

Lemma 2.27. $GL_n(k) \subset M_n(k)$ is open if $k = \mathbb{R}, \mathbb{C}$.

Proof. Defining $\det : M_n(k) \rightarrow k$ we have $GL_n(k) = \det^{-1}(k \setminus 0)$. □

Remark. We cannot do the above proof for quaternions as there is no quaternionic determinant.

Example 2.28. We can define $Sp(1) = \{q \in \mathbb{H} \mid \bar{q}q = 1\}$, where \mathbb{H} are the **quaternions**.

Theorem 2.29 (Cartan). Any closed subgroup of a Lie group is a Lie group itself.

Example 2.30. A non-example of the above is given by

$$H = \{(e^{it}, e^{iat}) \mid t \in \mathbb{R}, a \in \mathbb{R} \setminus \mathbb{Q}\} \subset U(1) \times U(1)$$

Example 2.31. $SL_n(\mathbb{R}) = \det^{-1}(1) \subset GL_n(\mathbb{R})$. By differentiable manifold theory we know $\dim SL_n(\mathbb{R}) = n^2 - 1$.

Remark. All of these are matrix groups. Not all Lie groups are matrix groups but most are.

2.3.2 Lie Group Homomorphisms

Definition 2.32. A **Lie group homomorphism** $\phi : G \rightarrow H$ is a pointed smooth map obeying $\forall a, b \in G, \phi(ab) = \phi(a)\phi(b)$.

Example 2.33. $\det : GL_n(\mathbb{R}) \rightarrow \mathbb{R}^\times$ is a Lie group hom.

Definition 2.34. Given $g \in G$ we have two diffeomorphisms $L_g, R_g : G \rightarrow G$ given by

$$L_g(h) = gh, \quad R_g(h) = hg$$

$L : G \rightarrow \text{Diff}(G), g \mapsto L_g$ is a group homomorphism, and R is an antihomomorphism respectively.

Lemma 2.35. L_g and R_h commute.

2.3.3 Compact Lie Groups

In standard applications of gauge theory to particle physics phenomenology, it turns out that we typically want compact Lie groups. We will define some compact Lie groups as subgroups of $GL_n(k)$ preserving a positive definite inner product:

Definition 2.36. The **classical Lie groups** are the subgroups of $LG_n(k)$ ($k = \mathbb{R}, \mathbb{C}, \mathbb{H}$) preserving:

- (\mathbb{R}) the Euclidean inner product, this is the **orthogonal group**
- (\mathbb{C}) the Hermitian inner product, this is the **unitary group**
- (\mathbb{H}) quaternionic Hermitian inner product, this is the **quaternionic unitary group**.

Example 2.37. $O(n) = \{A \in GL(n, \mathbb{R}) \mid A^T A = I\}$

Example 2.38. $Sp(2n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) \mid A^T J A = J\}$ where J is block diagonal with blocks $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

Example 2.39. We have $GL(n, \mathbb{C}) = \{A \in GL(2n, \mathbb{R}) \mid AJ = JA\}$

Example 2.40. $U(n) = \{A \in GL(n, \mathbb{C}) \mid A^\dagger A = I\}$

Remark. $U(1)$ will turn out to be the gauge group for electromagnetism.

Lemma 2.41.

$$\begin{aligned} U(n) &= GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R}) \\ &= GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{R}) \\ &= GL(n, \mathbb{C}) \cap O(2n, \mathbb{R}) \\ &= Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R}) \end{aligned}$$

Lemma 2.42. $SO(2) \cong U(1) \cong S^1$.

Proof. View $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \rightarrow e^{i\theta}$. □

Lemma 2.43. $SO(3) \cong \mathbb{RP}^3 \cong S^3/\sim$, and $SU(3) \cong S^3$.

Remark. We actually have a double cover $SU(2) \rightarrow SO(3)$

Lemma 2.44. $SO(4) \cong (S^3 \times S^3)/\sim$.

2.3.4 Lie Algebra

Definition 2.45. A (real) **Lie algebra** is a \mathbb{R} -vector space \mathfrak{g} with a bilinear, antisymmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ that satisfies the Jacobi identity.

Definition 2.46. We say $X \in \mathfrak{X}(G)$ is **left invariant** if $\forall h \in G, (dL_g)_h X_h = X_{gh}$.

Lemma 2.47. LIFs are determined by $X_e \in T_e G = \mathfrak{g}$.

Proof. Simply $X_g = (dL_g)_e X_e$ □

Lemma 2.48. If X, Y are LI then $[X, Y]$ is LI, where $[\cdot, \cdot]$ is the usual Lie bracket of vector fields.

Remark. Given any Lie group (G, e) we have an associated Lie algebra $\mathfrak{g} = \text{Lie}(G) = T_e G$. Note $\dim \mathfrak{g} = \dim G$. Given the above results, we can give \mathfrak{g} a bracket by

$$[X_e, Y_e] = [X, Y]_e$$

Now recall we have the exponential map $\exp : \mathfrak{g} \rightarrow G$, and for matrix groups this is given by $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$.

Remark. All the LIFs on G are complete, and so we can define the exponential map by

$$\begin{aligned} \exp : \mathfrak{g} &\rightarrow G \\ X &\mapsto \phi_1^X(e) \end{aligned}$$

We can find X from this by using $X = \left. \frac{d}{dt} \exp(tX) \right|_{t=0}$

Example 2.49. One can ask what the Lie algebras for the corresponding classical matrix groups are.

- $GL(n, \mathbb{R})$ is a vector space, and so $T_e GL(n, \mathbb{R}) = GL(n, \mathbb{R})$. The bracket turns out to be the standard matrix commutator.
- As $SL(n) = \{A \mid \det A = 1\}$ we find $\mathfrak{sl}_n = \{B \mid \det(\exp(tB)) = 1\} = \{B \mid \text{Tr}(B) = 0\}$
- $\mathfrak{o}_n = \{B \mid \left. \frac{d}{dt} \exp(tB)^T \exp(tB) \right|_{t=0} = 0\} = \{B \mid B^T + B = 0\}$

This process can be continued as similarly.

Definition 2.50. We have the **adjoint representation** of a group G acting on \mathfrak{g} by

$$\text{Ad}_g(X) = \left. \frac{d}{dt} g \exp(tX)^{-1} g^{-1} \right|_{t=0}$$

Proposition 2.51. An equivalent definition of Ad is that $\text{Ad}_g = (L_g \circ R_{g^{-1}})_*$.

Proof. We start by verifying that this other definition could make sense. $(L_g)_* : T_e G \rightarrow T_g G$ and $(R_{g^{-1}})_* : T_g G \rightarrow T_e G$ so

$$(R_{g^{-1}})_* \circ (L_g)_* = (L_g \circ R_{g^{-1}})_* : T_e G \rightarrow T_e G$$

finish this off. □

Proposition 2.52. $\text{Ad} : G \rightarrow GL(T_e G)$ is a Lie group hom

Proof. If we use the definition of $\text{Ad}_g = d(L_g \circ R_{g^{-1}}) = (L_g \circ R_{g^{-1}})_*$ we have

$$\begin{aligned} \text{Ad}_{gh} &= (L_{gh} \circ R_{(gh)^{-1}})_* \\ &= (L_g \circ R_{g^{-1}} \circ L_h \circ R_{h^{-1}})_* \\ &= (L_g \circ R_{g^{-1}})_* \circ (L_h \circ R_{h^{-1}})_* \\ &= \text{Ad}_g \circ \text{Ad}_h \end{aligned}$$
□

Lemma 2.53. For a matrix group, $\text{Ad}_A(B) = ABA^{-1}$.

Definition 2.54. We also have an adjoint rep of \mathfrak{g} acting on \mathfrak{g} by

$$\text{ad}_X(Y) = \left. \frac{d}{dt} \text{Ad}_{\exp(tx)}(Y) \right|_{t=0} = [X, Y]$$

Exercise 2.55. Show that if G is a matrix Lie group, the Lie bracket is indeed the commutator of the matrices.

Definition 2.56. A group rep ρ is **faithful** if $\ker \rho = e$

Fact 2.57. Ad need not be a faithful rep.

Definition 2.58. Given a basis $\{t_i\}$ of a Lie algebra, the **structure constants** are the coefficients c_{ij}^k defined by

$$[t_i, t_j] = \sum_k c_{ij}^k t_k$$

Proposition 2.59. Let $\phi : G \rightarrow H$ be a Lie group hom. Then $(\phi_*)_e : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra hom

Proof. See

$$\begin{aligned}\phi(ab) = \phi(a)\phi(b) &\Leftrightarrow (\phi \circ L_a)(b) = (L_{\phi(a)} \circ \phi)(b) \\ &\Leftrightarrow \phi \circ L_a = L_{\phi(a)} \circ \phi \\ &\Rightarrow (\phi \circ L_a)_* = (L_{\phi(a)} \circ \phi)_* \\ &\Rightarrow \phi_* \circ (L_a)_* = (L_{\phi(a)})_* \circ \phi_* \\ &\Rightarrow \forall X \in \mathfrak{g}, (X)_a =\end{aligned}$$

□

Example 2.60. We have $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ so there is an associated $\text{Ad}_* : \mathfrak{g} \rightarrow gl(\mathfrak{g})$. This coincides with $\text{ad} : X \mapsto \text{ad}_X$, as should be clear.

2.3.5 Maurer Cartan Forms

Definition 2.61. The *Maurer Cartan one form* is $\theta \in \Omega^1(G; \mathfrak{g})$ given by

$$\theta_g = (L_{g^{-1}})_* : T_g G \rightarrow T_e G = \mathfrak{g}$$

Lemma 2.62. θ is left invariant, i.e. $L_g^* \theta = \theta$.

Proof. See

$$\begin{aligned}(L_g^* \theta)_h &= \theta_{gh} \circ (L_g)_* \\ &= (L_{(gh)^{-1}})_* \circ (L_g)_* \\ &= (L_{(gh)^{-1}} \circ L_g)_* \\ &= (L_{h^{-1}})_* = \theta_h\end{aligned}$$

□

Proposition 2.63. θ obey the structure equation

$$d\theta + \frac{1}{2} [\theta, \theta] = 0$$

i.e. $\forall \xi, \eta \in \mathfrak{X}(G), d\theta(\xi, \eta) + [\theta(\xi), \theta(\eta)] = 0$.

Proof. Note $\theta(\xi) = \xi_e$ as

$$\theta_g(\xi_g) = (L_{g^{-1}})_*(L_g)_* \xi_e = \xi_e$$

and then

$$d\theta(\xi, \eta) = \xi\theta(\eta) - \eta\theta(\xi) - \theta([\xi, \eta])$$

□

Exercise 2.64. Show that for matrix groups $\theta_g = g^{-1}dg$.

2.3.6 Invariant inner products on \mathfrak{g}

Definition 2.65. For a \mathbb{R} LA \mathfrak{g} , a bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is said to be **invariant** if

$$\forall X, Y, Z \in \mathfrak{g}, \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0$$

Definition 2.66. The **Killing form** is $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ given by

$$\kappa(X, Y) = \text{Tr}(\text{ad}_X \circ \text{ad}_Y)$$

Lemma 2.67. κ is a symmetric, invariant, bilinear inner product.

Fact 2.68 (Cartan's Criterion). κ is non-degenerate $\Leftrightarrow \mathfrak{g}$ is semisimple.

Fact 2.69. κ is negative definite $\Leftrightarrow \mathfrak{g}$ is the LA of a compact semisimple G .

Fact 2.70. \mathfrak{g} admits a positive definite invariant inner product only when \mathfrak{g} is the LA of a compact G .

2.4 Symplectic Geometry

2.4.1 Symplectic Manifolds

Definition 2.71. If M is an even-dimensional manifold we say M is symplectic if it is equipped with $\omega \in \Omega^2(M)$ which is

- closed: $d\omega = 0$
- non-degenerate: $\forall m \in M, i. : T_m M \xrightarrow{\cong} T_m^* M$ is an isomorphism.

Definition 2.72. If (M, ω) is symplectic and $f \in C^\infty(M)$, then $df \in \Omega^1(M)$ and $\exists! X^f \in \mathfrak{X}(M)$ s.t $i_{X^f}\omega = df$. X^f is called the **Hamiltonian function associated to f** .

Lemma 2.73. The Hamiltonian function is constant along the flow lines of its vector field, i.e

$$t \mapsto (f \circ \phi_t^{X^f}(m))$$

is constant.

Given a smooth manifold N , we have the following result:

Theorem 2.74. The cotangent bundle $M = T^*N$ has a canonical symplectic structure. $\omega = -d\theta$ where θ is the tautological one form

Definition 2.75. The **tautological one form** θ on the cotangent bundle is given by noting

$$\begin{aligned} \pi : T^*N &\rightarrow N \\ \Rightarrow d\pi : T(T^*N) &\rightarrow TN \end{aligned}$$

so if $m \in M$, $m \in T_{\pi(m)}^*N$ and so we have

$$m : T_{\pi(m)}N \rightarrow \mathbb{R}$$

and

$$\theta_m \equiv m \circ d\pi : T_mM \rightarrow \mathbb{R}$$

Lemma 2.76. $\omega = -d\theta$ is (obviously) closed and moreover non-degenerate, hence symplectic.

Proof. If q^1, \dots, q^n are coordinates on N , then $\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}$ are vector fields on N . Letting p_i be the one form associated with $\frac{\partial}{\partial q^i}$ we get coordinates (q^i, p_i) on M in which

$$\begin{aligned}\theta &= p_i dq^i \\ \omega &= dq^i \wedge dp_i\end{aligned}$$

□

Aside. If $f \in C^\infty(M)$ we get associated Hamiltonian vector fields X^f . We then define the **Poisson bracket** of f and $g \in C^\infty(M)$ by

$$\{f, g\} \equiv X^f(g) = \omega(X^f, X^g)$$

Theorem 2.77 (Darboux). On any symplectic manifold there exist local coordinates (q, p) s.t. $\omega = dq \wedge dp$.

Corollary 2.78. There is no local invariant in symplectic geometry.

Remark. When we think of symplectic geometry vs complex geometry vs Riemannian geometry what we are really doing is thinking of the frame bundle of the tangent bundle of M , which when $\dim M = 2n$ is a $GL(2n, \mathbb{R})$ bundle, and then asking about reduction of structure group to either $SP(n)$, $GL(n, \mathbb{C})$, or $O(2n, \mathbb{R})$.

2.4.2 Symmetries and Moment Maps

Definition 2.79. A group action of G on a symplectic manifold (M, ω) is **symplectic** if $\forall g \in G, g^*\omega = \omega$.

Remark. There is a mismatch of minus signs in the convention of the literature, namely that given a group action of G on (M, ω) constructing a map

$$\begin{aligned}\mathfrak{g} &\rightarrow \mathfrak{X}(M) \\ X &\mapsto \tilde{X}\end{aligned}$$

can be done in one of two ways

1. Set $\tilde{X}_m = \frac{d}{dt} \exp(tX) \cdot m|_{t=0}$ as is natural, but then the map is a Lie algebra **antihomomorphism**
2. Set $\tilde{X}_m = \frac{d}{dt} \exp(-tX) \cdot m|_{t=0}$, and then the map is Lie algebra homomorphism.

For the purpose of this course the second convention is used.

Now let us think about

$$\begin{array}{ccccc}
C^\infty(M) & \xrightarrow{d} & \Omega^1(M) & \longrightarrow & \mathfrak{X}(M) \\
\uparrow \mu^* & & & \nearrow & \\
\mathfrak{g} & & & &
\end{array}$$

Definition 2.80. If such a map μ^* exists, we say the action is **Hamiltonian** and call the map the **co-moment map**.

Definition 2.81. Given a co-moment map μ^* the corresponding **moment map** is

$$\begin{aligned}
\mu : M &\rightarrow \mathfrak{g}^* \\
\mu(m)(X) &= \mu^*(X)(m)
\end{aligned}$$

Example 2.82. Let $U(1)$ act on T^2 by rotation around the central axis. If we view $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ then we can take the 2-form $dx \wedge dy$ on \mathbb{R}^2 which descends to T^2 . The vector field corresponding to this action is nowhere 0. This action then cannot be Hamiltonian, as any smooth map on a compact manifold must have critical points (see Morse theory) and so 1-form must have critical points, and hence the associated Hamiltonian vector field would be somewhere 0.

Example 2.83. If G is action on coadjoint orbits of \mathfrak{g}^* , μ is inclusion.

Proposition 2.84. If G acting on M is semi-simple, moment maps always exist.

Proposition 2.85. If G acts on (M, ω) and 0 is a regular value of μ , then $\mu^{-1}(0)/G$ is a new symplectic manifold with natural symplectic form ω_{red} characterised by $i^*\omega = \pi^*\omega_{red}$ as below

$$\begin{array}{ccc}
\mu^{-1}(0) & \xhookrightarrow{i} & (M, \omega) \\
\pi \downarrow & & \\
(\mu^{-1}(0)/G, \omega_{red}) & &
\end{array}$$

This process is **symplectic reduction**.

3 Bundles Theory

3.1 Bundles

3.1.1 Principle Bundles

Definition 3.1. A group action $G \curvearrowright M$ is **free** if $\forall m \in M, g \in G, e \neq g \Rightarrow g \cdot m \neq m$

Definition 3.2. A group action $G \curvearrowright M$ is **proper** if the map between topological spaces

$$\begin{aligned}
G \times M &\rightarrow M \times M \\
(g, m) &\mapsto (m, g \cdot m)
\end{aligned}$$

has compact preimages of compact subsets.

Proposition 3.3. *If G is proper, any continuous group action is proper.*

Lemma 3.4. *If $G \curvearrowright M$ is smooth, free, and proper. Then M/G exists and is a smooth manifold*

Definition 3.5. *If G is a Lie group, M a smooth manifold, a **G principal bundle over M** is given by*

- P a smooth manifold with projection $\pi : P \rightarrow M$
- a smooth right action of G , $P \times G \rightarrow P$
- a G -equivariant local trivialisation, i.e $\{U_\alpha\}$ an open covering s.t $\Psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$, $\Psi_\alpha(p) = (\pi(p), \psi_\alpha(p))$ and $(m, g) \cdot h = (m, gh)$

This is often noted as

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

Remark. *By G -equivariance $\pi^{-1}(U_\alpha)$ are all G -orbits, and moreover the action is free and proper. This means we have $M \cong P/G$. As such a G -principal bundle is given by a smooth, free, and proper action $P \curvearrowright G$.*

If U_α, U_β are part of the open cover s.t $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ then we can ask about the diagram

$$\begin{array}{ccc} & \pi^{-1}(U_{\alpha\beta}) & \\ \Psi_\alpha \swarrow & & \searrow \Psi_\beta \\ U_{\alpha\beta} \times G & \longrightarrow & U_{\alpha\beta} \times G \end{array}$$

which sends $(m, g) \mapsto (m, \phi_{\beta\alpha}(m)g)$ where the map $\phi_{\beta\alpha}$ satisfies

- $\phi_{\alpha\beta}(m) = \phi_{\beta\alpha}(m)^{-1}$
- $\phi_{\alpha\alpha} : U_\alpha \rightarrow \{e\} \leq G$
- on triple intersects $\phi_{\gamma\beta}\phi_{\beta\alpha} = \phi_{\gamma\alpha}$.

Remark. *If a covering $M = \cup_\alpha U_\alpha$ is given with transition function $\phi_{\beta\alpha} : U_{\alpha\beta} \rightarrow G$ satisfying the above conditions, then we can recover P . This is done as*

$$P = \left(\bigsqcup_\alpha U_\alpha \times G \right) / \sim$$

where $(m, g) \sim (m, \phi_{\beta\alpha}(m)g)$

Example 3.6 (Trivial Bundle). *Given $P = M \times G$ we get a G action by right multiplication.*

Example 3.7 (Hopf Fibration). *Take $P = S^{2n-1} \subset \mathbb{C}^n$ and $G = U(1)$. Let G act on \mathbb{C}^n by $v \cdot g = g^{-1}v$. We then get $M = \mathbb{CP}^{n-1}$*

Example 3.8. The edge of a Mobius strip is a \mathbb{Z}_2 -principal bundle over S^1 .

Remark. All of the fibres are isomorphic to G as manifolds, but they do not have an intrinsic group structure. As such the fibres are G -torsors.

Definition 3.9. A **section** of $P \xrightarrow{\pi} M$ is a right inverse to π , i.e $s : M \rightarrow P$ s.t. $\pi \circ s = \text{id}_M$.

Lemma 3.10. A PB has a global section iff it is trivial.

Definition 3.11. A **vector bundle** $E \rightarrow M$ is a (smooth) morphism s.t. all the fibres are vector spaces of the same dimension $r \equiv \text{rank } E$.

Definition 3.12. If $E \rightarrow M$ is a vector bundle, the associated **frame bundle** is

$$P \equiv \text{Fr}_E = \{\text{bases of fibres of } E\}$$

This is a $GL(r, \mathbb{R})$ -bundle. The action is given by

$$(\sigma_1, \dots, \sigma_r) \mapsto (\sigma_1, \dots, \sigma_r) \cdot A = (\sigma_1 A, \dots, \sigma_r A)$$

3.1.2 Bundles from Bundles

Definition 3.13. Given $P \rightarrow M$ a ppal G -bundle and $f : N \rightarrow M$ smooth we define the **pullback bundle** to be $f^*P : f^*P \rightarrow N$ the fibered product

$$f^*P = \{(y, p) \in N \times P \mid f(y) = \pi(p)\}$$

If we are given a principal bundle $G \rightarrow P \xrightarrow{\pi} M$ and a left action $G \curvearrowright F$ we can get the action $(P \times F) \curvearrowright G$ by $(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f)$. This action is always smooth, free, and proper.

Definition 3.14. The **associated fibre bundle** to $G \rightarrow P \xrightarrow{\pi} M$ with $G \curvearrowright F$ is

$$P_F \equiv (P \times F) /_G \rightarrow M$$

quotienting by orbits under the group action. All the fibres are diffeomorphic to F .

If $G \curvearrowright F$ preserves extra structure (e.g. if $F = V$ is a vector space and the action is linear) then all the fibres have this structure canonically.

Example 3.15. Suppose $F = V$ is a vector space and $G \curvearrowright V$ is linear (i.e. V is a rep space for G). Then P_V is a vector bundle over M .

Proposition 3.16. The assignment $P \times V \rightarrow P_V$ is the "inverse" of taking the frame bundle.

Proof. Using the defining rep of $GL(r, k) \curvearrowright k^r$ we get $(\text{Fr}_E)_{k^r} \cong E$ □

Example 3.17. If $G \curvearrowright \mathfrak{g}$ via the adjoint action Ad , we get $\text{ad}(P) = (P \times \mathfrak{g}) /_G$

Example 3.18. Let $G \curvearrowright G$ by conjugation, i.e. $C_g(h) = ghg^{-1}$. This preserves group structure, so we have $C : G \rightarrow \text{Aut}(G)$ and

$$\text{Ad}(G) \equiv P_{G,C} = (P \times G)/_G$$

where $(p, g) \sim (ph, h^{-1}gh)$.

Definition 3.19. Given $G \rightarrow P \rightarrow M$ and a Lie group hom $\rho : G \rightarrow H$, letting $G \curvearrowright H$ by $g \cdot h = \rho(g)h$ and noting $H \curvearrowright H$ by right multiplication, making it into a H -torsor, is preserved by the action of G and so we have

$$P_H = (P \times H)/_G$$

a H -torsor bundle, or a principal H bundle. This provides an **extension of structure group**.

Definition 3.20. Given a H -principal bundle $H \rightarrow \tilde{P} \rightarrow M$ and a Lie group hom $G \rightarrow H$, if $\exists G \rightarrow P \rightarrow M$ s.t. $\tilde{P} \cong P$ as H -principal bundle, we say that it is a **reduction of structure group** of \tilde{P} .

Example 3.21. If $E \rightarrow M$ is a Euclidean vector bundle of rank r , we can look at Fr_E as a $GL(r)$ -principal bundle, or at $\text{Fr}_E^\perp = \{\text{bundle of orthonormal frames}\}$ which is a $O(r)$ -principal bundle. Fr_E^\perp is a reduction of structure group of Fr_E using $O(r) \hookrightarrow GL(r)$

Proposition 3.22. There is a 1:1 correspondence

$$\{\text{reduction of structure groups of } \text{Fr}_E \text{ using } O(r) \hookrightarrow GL(r)\} \leftrightarrow \{\text{Choice of Euclidean structure on } E\}$$

Example 3.23. If E is a rank r bundle, and $\Lambda^2 E$ is a line bundle. Then $\Lambda^2 E$ is the vector bundle associated to Fr_E with $GL(2, k) \xrightarrow{\det} k^\times$. Hence we get a k^\times -principal bundle by extension.

Example 3.24. There is a 1:1 correspondence

$$\{\text{reduction of structure group of } \text{Fr}_E \text{ using } GL(r) \rightarrow SL(r)\} \leftrightarrow \{\text{trivialisations } \Lambda^2 E \cong M \times \mathbb{R}\}$$

If we do not have $\Lambda^2 E \cong M \times \mathbb{R}$ then reduction might not always be possible.

Example 3.25. If (M, g) is an oriented Riemannian Manifold of dimension n , then Fr_{TM} has a reduction of structure group from $GL_n(\mathbb{R}) \rightarrow SO(n)$. $n \geq 3 \Rightarrow \pi_1(SO(n)) = \mathbb{Z}_2$ and so there is a 2:1 "universal cover" $\text{Spin}(n) \rightarrow SO(n)$

Definition 3.26. A *Spin structure* on M is a reduction of structure group from $SO(n)$ to $\text{Spin}(n)$.

If a spin structure is given, then you can look at irreducible representations of the spin group that do not come from $SO(n)$.

3.1.3 Gauge transformations

Definition 3.27. Given $G \rightarrow P \rightarrow M$, a ***gauge transformation*** for P is a G -equivariant diffeomorphism ψ s.t.

$$\begin{array}{ccc}
P & \xrightarrow{\psi} & P \\
& \searrow \pi & \swarrow \pi \\
& M &
\end{array}$$

commutes. As the ψ compose, we get a group $\mathcal{G}(P)$

Remark. Note both G and $\mathcal{G}(P)$ act on P by right and left action respectively.

Theorem 3.28. There are canonical group isomorphisms

$$\mathcal{G}(P) \cong \Gamma(\text{Ad}(P)) \cong \{f : P \rightarrow G \mid f \text{ smooth}, f(p \cdot g) = g^{-1}f(p)g\}$$

Proof. We have

$$\begin{array}{ccc}
\mathcal{G}(P) & \xleftarrow{C} & \Gamma(\text{Ad}(P)) \\
& \searrow A & \swarrow B \\
& \{f : P \rightarrow G\} &
\end{array}$$

with

$$A : \psi \mapsto (f : p \mapsto g \text{ if } \psi(p) = pg)$$

we may check

$$\psi(p \cdot \tilde{g}) = \psi(p) \cdot \tilde{g} = pg\tilde{g} = (p\tilde{g})(\tilde{g}^{-1}g\tilde{g})$$

so f obeys the necessary condition. We then have

$$B : f \mapsto (s : m \mapsto [p, f(p)] \in (P \times G)/_G \text{ for any } p \in \pi^{-1}(m))$$

Again we can check if $\tilde{p} \in \pi^{-1}(m)$, then

$$[p \cdot g, f(p \cdot g)] = [p \cdot g, g^{-1}f(p)g] = [p, f(p)]$$

as this is the action we are quotienting out by. Finally we have

$$C : s \mapsto (\psi : p \mapsto p \cdot g \text{ where } s(\pi(p)) = [p, g])$$

and we can see $s(\pi(p \cdot \tilde{g})) = s(\pi(p)) = [p, g] = [p\tilde{g}, \tilde{g}^{-1}g\tilde{g}]$ so

$$\psi(p \cdot \tilde{g}) = (p \cdot \tilde{g}) \cdot (\tilde{g}^{-1}g\tilde{g}) = p \cdot g \cdot \tilde{g} = \psi(p) \cdot \tilde{g}$$

It is necessary to check that A, B, C are group isomorphisms now. □

Proposition 3.29. $\text{Lie}(\mathcal{G}(P)) = \Gamma(\text{ad}(P))$.

3.2 Connections

3.2.1 Kozul connections

Let $E \rightarrow M$ be a vector bundle. We want to be able to take directional derivatives of sections of E

Definition 3.30 (Kozul connection). A **Kozul connection** on E is a map

$$\nabla : \Gamma(E) \rightarrow \Gamma(E \otimes T^*M)$$

For $s \in \Gamma(E)$, $X \in \mathfrak{X}(M) = \Gamma(TM)$, $(\nabla s)(X) \in \Gamma(E)$ is denoted as $\nabla_X s$.

We want this to satisfy a product rule that for $f \in C^\infty(M)$

$$\nabla_X(fs) = X(f)s + f\nabla_X s$$

and linearity in $\Gamma(E)$, $\mathfrak{X}(M)$, i.e. $\forall c_i \in \mathbb{R}, f_i \in C^\infty(M)$

$$\begin{aligned}\nabla_X(c_1 s_1 + c_2 s_2) &= c_1 \nabla_X s_1 + c_2 \nabla_X s_2 \\ \nabla_{(f_1 X_1 + f_2 X_2)} s &= f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s\end{aligned}$$

If E is a Euclidean vector bundle with inner product $\langle \cdot, \cdot \rangle$, we may ask that a connection respects this additional structure.

Definition 3.31. ∇ respects $\langle \cdot, \cdot \rangle$ if

$$X(\langle s_1, s_2 \rangle) = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle$$

3.2.2 Ehresmann connections

Notation. Let $G \rightarrow P \xrightarrow{\pi} M$ be a G -principal bundle. Denote the fibre through $p \in P$ by $F_p = p \cdot G$.

Definition 3.32. We call $T_p F_p = \ker d\pi_p$ the **subspace of vertical tangent vectors at $p \in P$** and denote it by V_p . The collection of V_p gives a smooth distribution.

Definition 3.33. A differential form on P is called **horizontal** if it is zero on V .

Definition 3.34. A form on P is called **basic** if it is horizontal and G -invariant

Definition 3.35. An **Ehresmann connection** on P , H , is a smooth choice of complement $H_p \hookrightarrow T_p P$ s.t.

- $\forall p \in P, \dim H_p = \dim M$
- $\forall p \in P, T_p P = T_p F_p \oplus H_p$
- H is G -equivariant, i.e. $dR_g H_p = H_{p \cdot g}$

The distribution is called **horizontal**.

Remark. $d\pi_p : T_p P \rightarrow T_{\pi(p)} M$ gives a isomorphism $T_{\pi(p)} M \xrightarrow{\cong} H_p$, or equivalently a splitting of the SES

$$0 \rightarrow T_p F_p \rightarrow T_p P \rightarrow T_{\pi(p)} M \rightarrow 0$$

Remark. Smooth connections always exists. This is as they certainly exist on local trivialisations, and then they can be glued with partitions of unity.

3.2.3 Holonomy

Lemma 3.36. Given $G \rightarrow P \xrightarrow{\pi} M$, a smooth connection H , and a smooth closed curve in the base $C : I \rightarrow M$, we can lift tangent vectors to $\pi^{-1}(C(I)) \subset M$ to tangent vectors to P to get a vector field on $\pi^{-1}(p)$, and there is a unique such lift that is horizontal.

We can then flow along this vector field starting at $\pi^{-1}(C(0))$. This will give a curve $\tilde{C} : I \rightarrow P$, where it is not necessary that $\tilde{C}(0) = \tilde{C}(1)$. However, it will be true that $\tilde{C}(1) \in \pi^{-1}(C(0))$ so $\exists g \in G$, $\tilde{C}(1) = p \cdot g$.

Definition 3.37. $g = g(p)$ is the **holonomy** of C w.r.t H starting at p .

Definition 3.38. For $X \in \mathfrak{g}$ we have $\tilde{X} \in T_p P$ s.t.

$$\tilde{X}(f) = \left. \frac{d}{dt} f(p \cdot \exp(tX)) \right|_{t=0}$$

Definition 3.39. A **principal connection** is a connection form $\omega \in \Gamma(T^*P \otimes \mathfrak{g}) = \Gamma(\text{Hom}(TP, \mathfrak{g}))$ s.t.

- $\omega(\tilde{X}) = X$
- $(R_g)^* \omega = \text{Ad}_{g^{-1}} \omega$

Example 3.40. On the Hopf bundle, $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \rightarrow \mathbb{CP}^1$, $(z, w) \mapsto [z : w]$, as a $U(1)$ -bundle acting by multiplication, we can have $\omega = i\Im \bar{z} dx + \bar{w} dw$.

3.2.4 Relations between viewpoints

Let V be a vector space and $S \subset V$ a subspace.

Definition 3.41. A **complement** to S is a choice \tilde{S} s.t. $V = S \oplus \tilde{S}$.

Lemma 3.42. A complement to S is equivalent to a projection $p : V \rightarrow S$ s.t. $p|_S = \text{id}_S$

Corollary 3.43. $\tilde{S} = \ker p$

Now, noting that ω gives a projection $\omega_p : T_p P \rightarrow \mathfrak{g}$ for each $p \in P$, we get the following relation:

Proposition 3.44. ω gives an Ehresmann connection by letting $H_p = \ker \omega_p$.

We now recall some facts from the workshops: Given a vector bundle $E \rightarrow M$, $\forall U \subset M$ open, we have a short exact sequence

$$0 \rightarrow \text{End}(E)(U) \hookrightarrow \mathcal{D}^{\leq 1}(E)(U) \xrightarrow{\sigma} [TM \otimes \text{End}(E)](U) \rightarrow 0$$

where $\mathcal{D}^{\leq 1}(E)(U)$ are the first order differential operators on $E|_U$. Now for $D \in \mathcal{D}^{\leq 1}(E)(U)$, $s \in \Gamma(U, E)$, $f \in C^\infty(U)$, we have

$$D(fs) = \sigma(D)(f)(s)$$

Putting $\mathcal{D}_{\text{diag}}^{\leq 1}(E)(U) = \sigma^{-1}(TM \otimes \text{id}_E)$ we have SES

$$0 \rightarrow \text{End}(E)(U) \rightarrow \mathcal{D}_{\text{diag}}^{\leq 1}(E)(U) \rightarrow TM|_U \rightarrow 0$$

A Kozul connection is now a splitting of this SES, $\nabla : TM|_U \rightarrow \mathcal{D}_{\text{diag}}^{\leq 1}(E)(U)$, i.e. given $X \in \mathfrak{X}(U)$, ∇_X is a 1st order differential operator satisfying $\sigma(\nabla_X) = X$.

Now, if we have a principal bundle $G \rightarrow P \rightarrow M$, the **Atiyah sequence** associated with it is a SES

$$0 \rightarrow \text{ad}(P) \rightarrow TP/G \rightarrow TM \rightarrow 0.$$

This is split by the map $\left(d\pi|_{H_p}\right)^{-1} : TM \rightarrow TP/G$, given by an Ehresmann connection. Now with $G \curvearrowright V$ via a representation, we have the following result:

Proposition 3.45. $\forall U \subset M$ we have

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{ad}(P)(U) & \longrightarrow & TP/G(U) & \longrightarrow & TM|_U \longrightarrow 0 \\
 & & \downarrow & & \downarrow & \swarrow \text{red} & \downarrow \text{id} \\
 0 & \longrightarrow & \text{End}(E)(U) & \longrightarrow & \mathcal{D}_{\text{diag}}^{\leq 1}(E)(U) & \xrightarrow{\sigma} & TM|_U \longrightarrow 0 \\
 & & & & & \nwarrow \text{red} &
 \end{array}$$

commutes.

Proof. $G \curvearrowright V$ so we get $\mathfrak{g} \rightarrow \text{End}(V)$, a morphism of Lie algebras that is G -equivariant. Then we have

$$(P \times \mathfrak{g})/G \rightarrow (P \times \text{End}(V))/G = \text{End}(E)$$

Further, the space of sections of E is the space of G -equivariant functions $P \rightarrow V$. This means we can say that G -equivariant vector fields on P also act on G -equivariant functions $P \rightarrow V$, i.e. sections of E . It can then be checked that these all commute. \square

Corollary 3.46. A splitting of the Atiyah sequence gives a Kozul connection.

Remark (On the Atiyah sequence). We can send $P \times \mathfrak{g} \rightarrow TP$ by sending (p, ξ) to the vector field whose value at p is given by that generated by the action of G on P . As this action is fibre-wise, the vector field in the image must lie in the vertical component $\ker d\pi$. This gives the SES

$$0 \rightarrow P \times \mathfrak{g} \rightarrow TP \rightarrow TM \rightarrow 0$$

As all these maps are G invariant we can quotient out by the action of G to get the Atiyah sequence.

3.2.5 The space of all connections

Definition 3.47. If $P \rightarrow M$ is a G -principal bundle we can denote the **space of all connections on P** as \mathcal{C}_P

Remark. Some will denote this space as \mathcal{A}_P .

Proposition 3.48. $\mathcal{C}_P \neq \emptyset$, i.e. connections always exist.

Proof. First note

- A convex combination of connection 1-forms is a 1-form
- We can get trivial connection 1-forms on each local trivialisation.

With these two facts, we can make a connection by using a partition of unity subordinate to the open cover giving the trivialisation. \square

Proposition 3.49. \mathcal{C}_P is an affine space modelled on the vector space $\Gamma(T^*M \otimes \text{ad}(P))$.

Proof. Given $P \xrightarrow{\pi} M$ and $\text{ad } P \rightarrow M$ we can form the pullback bundle $\pi^* \text{ad } P$ and

$$\begin{aligned}
 \pi^* \text{ad } P &\cong P \times \mathfrak{g} \\
 (p, [p, X]) &\mapsto (p, X)
 \end{aligned}$$

We can check though that

$$\text{Im}(\pi^* : \Omega^1(M; \mathfrak{g}) \rightarrow \Omega^1(P; \mathfrak{g})) = \left\{ \alpha \mid \alpha_p(\tilde{X}) = 0 \right\}$$

i.e the image is zero on V_p

□

Remark. \mathcal{C}_P is not a vector space, but this is the next best thing. This means that $\Gamma(T^*M \otimes \text{ad}(P))$ acts as an additive group on \mathcal{C}_P freely and transitively, i.e. \mathcal{C}_P is a torsor for the additive group. Alternatively this can be seen as saying that the difference between two connections gives an element of $\Gamma(T^*M \otimes \text{ad}(P))$.

Let us try and understand this result in terms of the three ways to view a connection.

1. **Splitting of the Atiyah sequence:** We can think of a section of $T^*M \otimes \text{ad}(P)$ as a $C^\infty(M)$ -linear map from TM to $\text{ad}(P)$.

$$0 \longrightarrow \text{ad}(P) \xrightarrow{\quad} TP/G \xrightarrow{\quad} TM \longrightarrow 0$$

$\text{ad}(P)$ is a vector sub-bundle of TP/G , and as it is exactly the kernel of $TP/G \rightarrow TM$ we can add it to any splitting of such to get a new splitting. Similarly, given any two splittings of $TP/G \rightarrow TM$ the difference has to be in the kernel of the map, i.e. takes values in $\text{ad}(P)$, so gives a well defined map $TM \rightarrow \text{ad}(P)$.

2. **Distribution on TP :** Take a vector space $V = V_1 \oplus V_2$. Then any linear map $f : V_1 \rightarrow V_2$ gives a linear subspace of V

$$\text{graph}(f) = \{v \oplus f(v) \mid v \in V_1\}$$

with $\dim \text{graph}(f) = \dim V_1$. This subspace lets us write

$$V = \text{graph}(f) \oplus V_2$$

Now suppose we have a distribution $H \subset TP$ and a section of $T^*M \otimes \text{ad}(P)$. The latter can be pulled back to P and using H and using H it gives a G -invariant linear map from H to the vertical tangent space. This means we have $TP = H \oplus V$ and a linear map $H \rightarrow V$. As such we get a new distribution from $\text{graph}(H \rightarrow V)$.

3. **\mathfrak{g} -valued one-form on P :** Let ω be the connection one-form and $\gamma : TP \rightarrow \text{ad}(P)$ be the section of $T^*M \otimes \text{ad}(P)$. From γ we can build $\tilde{\gamma} = \gamma \circ d\pi : TP \rightarrow \mathfrak{g}$. $\tilde{\gamma} + \omega$ is a new connection one-form.

3.2.6 Curvature

To any connection on P we can associate a curvature - a section of $\wedge^2 T^*M \otimes \text{ad}(P)$. To aid in the definition we need the following lemma:

Lemma 3.50. *If $E \rightarrow M$ is a vector bundle then a rule that assigns a section of E to each r -tuple of vector fields X^1, \dots, X^r which is alternating in the X^i and $C^\infty(M)$ -linear corresponds to a section of $\wedge^r T^*M \otimes E$.*

Now suppose we have a Kozul connection ∇ on E

Lemma 3.51. *Given $s \in \Gamma(E)$, $X, Y \in \Gamma(TM)$, the expression*

$$\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s$$

is

- C^∞ linear in s, X, Y
- alternating in X, Y

Corollary 3.52. *The expression corresponds to $F_{\nabla} \in \Gamma(\wedge^2 T^*M \otimes \text{End}(E))$ given by*

$$F_{\nabla}(X, Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s$$

We can again interpret this three ways via the different ways of thinking about a connection. We will need the following definition:

Definition 3.53. *Given a connection on P given by a distribution $H \subset TP$, for $X \in \Gamma(TP)$ write X^H for the horizontal component. Now for $\phi \in \Omega^r(P) \otimes V$, where V is a vector space, define the **covariant derivative** of ϕ to be the V -valued $(r+1)$ -form $D\phi$ given by*

$$D\phi(X_0, \dots, X_r) = d\phi(X_0^H, \dots, X_r^H)$$

1. **Splitting of the Atiyah sequence:** Given $U \subset M$ open we can form the exact sequence

$$0 \longrightarrow \Gamma(U, \text{ad}(P)) \longrightarrow \Gamma\left(U, \frac{TP}{G}\right) \longrightarrow TU \longrightarrow 0$$

This is in fact an exact sequence of Lie algebra - bundles where the maps in the sequence respect the bracket structure. If we let $\gamma : TM \rightarrow TP/G$ be the map that splits the Atiyah sequence then we define F by

$$F(X, Y) = [\gamma(X), \gamma(Y)] - \gamma([X, Y])$$

As the Lie algebra structure is respected by the exact sequence we must have that $F(X, Y) \mapsto 0 \in TM$, and we have by exactness that it must take values in $\text{ad}(P)$.

2. **Distribution on TP :** Given a choice of horizontal subspace $H_p \subset T_p P$ we can lift $X \in \Gamma(TM)$ to a horizontal $\tilde{X} \in \Gamma(TP)$. Given $Y \in \Gamma(TM)$ we then get \tilde{F} defined by

$$\tilde{F}(X, Y) = [\tilde{X}, \tilde{Y}] - [\tilde{X}, Y]$$

$\tilde{F}(X, Y) \in \Gamma(TP)$, and moreover $\tilde{F}(X, Y) \in \ker d\pi$, so it is vertical. Further G -invariance under the adjoint action means that \tilde{F} to $F \in \Gamma(\wedge^2 T^*M \otimes \text{ad}(P))$ a section on M (it is clear that F is alternating in X, Y and is $C^\infty(M)$ -linear).

3. **\mathfrak{g} -valued one-form on P :** Let ω be the connection one-form, with corresponding distribution H . We then get the curvature $\tilde{F} \in \Gamma(\wedge^2 T^*P \otimes \mathfrak{g})$ by

$$\tilde{F} = D\omega$$

One can check (**exercise**) that \tilde{F} is G -equivariant in the sense that

$$R_g^* \tilde{F} = \text{Ad}_{g^{-1}} \tilde{F}$$

Because of the equivariance this descends to a section $F \in \Gamma(\wedge^2 T^*M \otimes \text{ad}(P))$.

Proposition 3.54. *In terms of just the connection one-form we can write*

$$\tilde{F}(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$$

We often write this as the short hand

$$\tilde{F} = d\omega + \omega \wedge \omega$$

which is **Cartan's structure equation**.

Theorem 3.55 (Bianchi identity). $D\tilde{F} = 0$

Proof. We have

$$\begin{aligned} D\tilde{F}(X_1, X_2, X_3) &= d\tilde{F}(X_1^H, X_2^H, X_3^H) \\ &= d^2\omega(X_1^H, X_2^H, X_3^H) + d\omega \wedge \omega(X_1^H, X_2^H, X_3^H) - \omega \wedge d\omega(X_1^H, X_2^H, X_3^H) \\ &= 0 \end{aligned}$$

as $d^2 = 0$ and $H = \ker \omega$ so $\omega(X^H) = 0$. □

Definition 3.56. A connection is **flat** (or **integrable**) if its curvature is 0.

Theorem 3.57. If P is a G -principal bundle equipped with a flat connection then there exist local trivialisations such that all transition functions are constant.

Proof. **exercise** □

3.2.7 Local expressions

Recall that a bundle is trivial iff it has a global section $\sigma : M \rightarrow P$. This gives a canonical choice of connection given by

$$H = \ker(p_2 : M \times G \rightarrow G)$$

We refer to this induced connection as d^σ .

Now any two connections differ by a section of $T^*M \otimes \text{ad}(P)$, and $\text{ad}(P)$ is also trivialised as $\text{ad}(P) \cong M \times \mathfrak{g}$ if P is, we can write any other connection as

$$d^\sigma + A$$

If the connection is given by one-form ω we have $A = \sigma^*\omega$.

Suppose we have expressed our G -principal bundle P in terms of local trivialisations - that is we have written $M = \cup_\alpha U_\alpha$, taken $\Psi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times G$ with transition maps $\phi_{\alpha\beta} : U_{\alpha\beta} \rightarrow G$, and we have expressed this data as a collection of local sections $\sigma_\alpha : U_\alpha \rightarrow P$. In this case we may carry out the procedure as before to write a connection locally as

$$d^{\sigma_\alpha} + A^\alpha$$

These are related according to the following results:

Theorem 3.58 (compatibility conditions). Let θ be the LI MC one-form. Then

$$A^\beta = \text{Ad}_{\phi_{\beta\alpha}}(A^\alpha) + \phi_{\alpha\beta}^*\theta$$

Proof. **exercise** □

Corollary 3.59. If G is a matrix group, we have

$$A^\beta = \phi_{\beta\alpha} A^\alpha \phi_{\beta\alpha}^{-1} + \phi_{\beta\alpha} d\phi_{\beta\alpha}$$

Remark. Physicists often refer to the A^α as **local gauge potentials**.

Proposition 3.60. *In terms of the local gauge potentials the curvature is given by*

$$F = dA^\alpha + A^\alpha \wedge A^\alpha$$

Recall we know there is a group isomorphism between $\mathcal{G}(P)$, the group of gauge transformations, and $\Gamma(\text{Ad}(P))$, the group of section of the bundle $\text{Ad}(P)$. If we trivialise P we trivialise $\text{Ad}(P)$ making the sections into G -valued functions. The local trivialisations, G -valued functions ψ_α , are related by

$$\psi_\beta = \phi_{\beta\alpha} \psi_\alpha \phi_{\beta\alpha}^{-1}$$

Proposition 3.61. $\mathcal{G}(P)$ then acts locally by the compatibility condition on the space of all connections as

$$A^\alpha \mapsto \text{Ad}_{\psi_\alpha} A^\alpha + \psi_\alpha^* \theta$$

3.3 Characteristic Classes in Chern-Weil Theory

Lemma 3.62. *Suppose we have*

$$\phi : \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{\times k} \rightarrow \mathbb{R}$$

multilinear, symmetric, and ad-invariant. Think of $\phi(\tilde{F}, \dots, \tilde{F}) \in \Omega^{2k}(M, \mathbb{R})$, then if ω, ω' are two connections

$$\exists \alpha \in \Omega^{2k-1}, \phi(\tilde{F}_\omega, \dots, \tilde{F}_\omega) - \phi(\tilde{F}_{\omega'}, \dots, \tilde{F}_{\omega'}) = d\alpha$$

Corollary 3.63. $[\phi(\tilde{F}, \dots, \tilde{F})] \in H_{dR}^{2k}$ is a cohomology class.

Example 3.64. If $G = U(n)$, define c_k by

$$\sum_{k=0}^n c_k(X) t^k = \det \left(I - \frac{t}{2\pi i} X \right)$$

for $X \in \mathfrak{u}(n)$ and then

$$c_k(P) = c_k(\tilde{F}, \dots, \tilde{F}) \in H^{2k}$$

for some G -bundle p , the **Chern class**.

Remark. Here we are using the polarisation identity to define c_k on multiple elements.

Example 3.65. If $G = O(n), SO(n)$ then have

$$\sum_k p_k(X) t^k = \det \left(I - \frac{t}{2\pi} X \right)$$

for $X \in \mathfrak{g}$, and

$$p_k(P) = p_k(\tilde{F}, \dots, \tilde{F}) \in H^{4k}$$

is the **Pontryagin class**

Example 3.66. If $G = O(2n)$, define

$$\Delta(X) = \left(\frac{-1}{2\pi}\right)^m \text{Pfaff}(X)$$

The *Euler class*

4 The Yang-Mills Theory

4.1 Implicitly used results

Maybe move this earlier We will first start by spelling out explicitly some results that we have been using but not making clear:

Lemma 4.1. If $P \rightarrow M$ is a G -principal bundle, $V \rightarrow P$ is a vector bundle equipped with a linear lift of the action $P \curvearrowright G$ then V/G is a vector bundle over M of the same rank as V over P .

Lemma 4.2. Sections of the bundle $V/G \rightarrow M$ correspond exactly to G -equivariant sections of $V \rightarrow P$, i.e. sections $s : V \rightarrow P$ s.t.

$$\begin{array}{ccc} V & \xrightarrow{g} & V \\ s \downarrow & & \downarrow s \\ P & \xrightarrow{g} & P \end{array}$$

Lemma 4.3. pull-backs of sections of $\wedge^2 T^*M \otimes (P \times V)/G$ are exactly basic V -valued forms on P .

The reason we have made these explicit is as it's important to realise it is the case with $V = \mathfrak{g}$ that is necessary for us, as it means we can descend $\tilde{F} \in \Gamma(\wedge^2 T^*P \otimes \mathfrak{g})$ (a \mathfrak{g} -valued form on P) to $F \in \Gamma(\wedge^2 T^*M \otimes \text{ad}(P))$.

4.2 Theory

4.2.1 Yang-Mills functional

We are now going to restrict our mathematical picture. We will make the following assumptions:

- M is an oriented (pseudo-)Riemannian manifold
- \mathfrak{g} has a Euclidean inner product invariant under the adjoint action $G \curvearrowright \mathfrak{g}$. (This gives a Euclidean structure on the bundle $\text{ad}(P)$).

Remark. For compact, semi-simple Lie groups an ad-invariant Euclidean inner product on \mathfrak{g} is given by the Killing form. For other compact Lie groups we can take a faithful representation $\rho : G \rightarrow GL(V)$ and then have

$$\langle X, Y \rangle = -\text{Tr}(d\rho(X) \circ d\rho(Y))$$

For simplicity we will notate this as

$$\langle X, Y \rangle = -\text{Tr}(XY)$$

What we have gained is that the bundle which the curvature takes values in has a Euclidean structure, as such we can make the following definition:

Definition 4.4. The *Yang-Mills function* is the functional $S_{YM} : \mathcal{C}_P \rightarrow \mathbb{R}$ given by

$$S_{YM} = \int_M |F|^2 \omega_{vol}$$

where F is the curvature associated to the connection the functional is evaluated at.

Lemma 4.5. S_{YM} is invariant under the action of the gauge group $\mathcal{G}(P)$.

Corollary 4.6. The functional becomes a well-defined function $S_{YM} : \mathcal{C}_P / \mathcal{G}(P) \rightarrow \mathbb{R}$.

Lemma 4.7. The Yang-Mills functional can be written as

$$S_{YM} \propto \int_M \text{Tr}(F \wedge \star F)$$

Proof. The inner product on \mathfrak{g} -valued forms is the tensor product of the inner products on \mathfrak{g} and on Ω^\cdot . Hence by the definition of the Hodge star

$$|F|^2 \omega_{vol} = \langle F, F \rangle \omega_{vol} = \langle F \wedge \star F \rangle$$

We often denote the inner product on \mathfrak{g} as proportional to trace to highlight the fact that ad-invariant implies cyclicity of trace on a matrix Lie algebra. \square

Remark. If M is compact, the integration over M is always fine. If not, we need to restrict to connections for which S_{YM} is well-defined.

4.2.2 The Yang-Mills equations

We now want to derive equations that give us the stationary points for the action (as is standard for an action theory). We will work with a local trivialisation, and then for notation simplicity omit the index α . For some generic $\tau \in \Gamma(T^*M \otimes \text{ad}(P))$ we want to find A s.t.

$$\left. \frac{d}{dt} S_{YM}(A + t\tau) \right|_{=0}$$

We can calculate the associated curvature to this new connection is

$$\begin{aligned} F_t &= d(A + t\tau) + (A + t\tau) \wedge (A + t\tau) \\ &= F + t(d\tau + A \wedge \tau + \tau \wedge A) + t^2 \tau \wedge \tau \end{aligned}$$

We now need the following lemma

Lemma 4.8. The covariant derivative descends to an operator

$$d^A : \Gamma(\wedge^r T^*M \otimes \text{ad}(P)) \rightarrow \Gamma(\wedge^{r+1} T^*M \otimes \text{ad}(P))$$

Proof. By the definition that for $\phi \in \Gamma(\wedge^r T^*P \otimes V)$

$$D\phi(X_0, \dots, X_r) = d\phi(X_0^H, \dots, X_r^H)$$

we can see that $D\phi$ is horizontal. It is also G -invariant if ϕ is, and as such D maps basic forms to basic forms. As such it descends as previously discussed. \square

Corollary 4.9. The Bianchi identity descends to $d^A F = 0$.

Remark. We will want to make use of the fact throughout that for forms

$$d^A \tau = d\tau + [A \wedge \tau]$$

where $[\omega \wedge \eta] = \omega \wedge \eta - (-1)^{pq} \eta \wedge \omega$ where the wedge without square brackets of forms should just take the multiplication of their algebra element coefficients. Be aware that the covariant derivative acts differently on connections and this is why we still have

$$F = d^A A = dA + \frac{1}{2} [A \wedge A] = dA + A \wedge A$$

To see more on this check out Lie algebra-valued differential forms on Wikipedia and the 2006 gauge theory notes by Jose (pg 18).

Now using that $d^A \tau = d\tau + A \wedge \tau$ we get

$$\begin{aligned} F_t &= F + td^A \tau + t^2 \tau \wedge \tau \\ \Rightarrow |F_t|^2 &= |F|^2 + t \langle d^A \tau, F \rangle + \mathcal{O}(t^2) \end{aligned}$$

Hence the Yang-Mills equation is equivalent to

$$\int_M \langle d^A \tau, F \rangle \omega_{vol} = 0$$

Using the formal adjoint $(d^A)^*$ and realising τ is generic we get

$$(d^A)^* F = 0$$

We need some lemmas:

Lemma 4.10. $d \langle \beta \wedge \star F \rangle = \langle d^A \beta \wedge \star F \rangle - \langle \beta \wedge d^A(\star F) \rangle$

Proof. We will show this in coordinates, by writing $\beta = \beta_\mu dx^\mu$, $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$. Then

$$\star F = \frac{1}{4} F_{\mu\nu} \epsilon^{\mu\nu}{}_{\rho\sigma} dx^\rho \wedge dx^\sigma$$

and we get

$$\begin{aligned} d \langle \beta \wedge \star F \rangle &= \frac{1}{4} \partial_\mu \langle \beta_\nu, F_{\rho\sigma} \rangle \epsilon^{\rho\sigma}{}_{\tau\lambda} dx^\mu \wedge dx^\nu \wedge dx^\tau \wedge dx^\lambda \\ \langle d^A \beta \wedge \star F \rangle &= \frac{1}{4} \langle \partial_\mu \beta_\nu + [A_\mu, \beta_\nu], F_{\rho\sigma} \rangle \epsilon^{\rho\sigma}{}_{\tau\lambda} dx^\mu \wedge dx^\nu \wedge dx^\tau \wedge dx^\lambda \\ \langle \beta \wedge d^A(\star F) \rangle &= \frac{1}{4} \langle \beta_\nu, \partial_\mu F_{\rho\sigma} + [A_\mu, F_{\rho\sigma}] \rangle \epsilon^{\rho\sigma}{}_{\tau\lambda} dx^\nu \wedge dx^\mu \wedge dx^\tau \wedge dx^\lambda \end{aligned}$$

which means

$$d \langle \beta \wedge \star F \rangle - \langle d^A \beta \wedge \star F \rangle + \langle \beta \wedge d^A(\star F) \rangle = \frac{-1}{4} [\langle \beta_\nu, [A_\mu, F_{\rho\sigma}] \rangle + \langle [A_\mu, \beta_\nu], F_{\rho\sigma} \rangle] \epsilon^{\rho\sigma}{}_{\tau\lambda} dx^\mu \wedge dx^\nu \wedge dx^\tau \wedge dx^\lambda$$

and we have asked as the inner product on \mathfrak{g} is ad-invariant, which precisely says that $\forall X, Y, Z \in \mathfrak{g}$,

$$\langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle = 0$$

□

Remark. No part of the above calculation required properties of β, F , other than that they were \mathfrak{g} -valued. This means that this result holds for general forms with the modification that if $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$, then

$$d \langle \alpha \wedge \beta \rangle = \langle d^A \alpha \wedge \beta \rangle + (-1)^p \langle \alpha \wedge d^A \beta \rangle$$

Lemma 4.11. Let d^* be the formal adjoint of d (the **codifferential**) wrt the inner product on the subset of $\Omega^k(M)$ that decays sufficiently if M is non-compact given by

$$\langle\langle \eta, \zeta \rangle\rangle = \int_M \langle \eta, \zeta \rangle \omega_{vol}$$

that is, if $\eta \in \Omega^{k-1}(M)$, $\zeta \in \Omega^k(M)$, $\langle\langle d\eta, \zeta \rangle\rangle = \langle\langle \eta, d^*\zeta \rangle\rangle$. We can express $d^* : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ as

$$d^* = (-1)^{n(k-1)+1} s \star d \star$$

where $n = \dim M$, s is the parity of the signature of M ,

Proof. We prove this more generally for the covariant derivative. Using the previous lemma means

$$\begin{aligned} \langle\langle d^A \eta, \zeta \rangle\rangle &= \int_M \langle d^A \eta, \zeta \rangle \omega_{vol} = \int_M \langle d^A \eta \wedge \star \zeta \rangle \\ &= \int_M d \langle \eta \wedge \star \zeta \rangle - (-1)^{k-1} \langle \eta \wedge d^A \star \zeta \rangle \\ &= \int_M (-1)^{(n-k+1)[n-(n-k+1)]+k} s \langle \eta, \star d^A \star \zeta \rangle \omega_{vol} \\ &= \langle\langle \eta, (-1)^{(n-k+2)(k-1)+1} s \star d^A \star \zeta \rangle\rangle \\ &= \langle\langle \eta, (-1)^{n(k-1)+1} s \star d^A \star \zeta \rangle\rangle \end{aligned}$$

where we have used the (abelian) Stokes' theorem and sufficient decay at infinity to set the boundary term to 0, and that as $d^A \star \zeta$ is a $(n-k+1)$ -form. \square

Remark. This carefulness with signs is not important in the case of pure Yang-Mills, but if you consider Yang-Mills-Higgs this becomes necessary. Note another route of proof is possible if we use the non-abelian Stokes' theorem (see [1] and nlab).

As such we get the **Yang-Mills equation**

$$d^A(\star F) = 0$$

4.3 Gauge theory in 4d

4.3.1 Abelian gauge theory and Electromagnetism

Recall Maxwell's equations for electromagnetism

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \partial_t \mathbf{E} \\ \nabla \times \mathbf{E} &= -\frac{1}{c} \partial_t \mathbf{B} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

We are going to see how this can be derived as a Yang-Mills gauge theory.

Restricting to the vacuum Maxwell equations where $\rho = 0$, $\mathbf{J} = 0$, we can build a two form

$$F = \tilde{B} - c dt \wedge E$$

where

$$\begin{aligned} E &= E_x dx + E_y dy + E_z dz \\ B &= B_x dx + B_y dy + B_z dz \\ \tilde{B} &= \star_3 B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \end{aligned}$$

are the natural one and two forms associated to a vector in 3d (using the 3d Hodge star to get the two form) written in cartesian coordinates. Now consider a $U(1)$ -principal bundle P over \mathbb{R}^4 with Lorentzian signature. As the Lie group is abelian, the commutator in the Lie algebra is 0 and $d^A = d$ is the standard differential for any connection. We now calculate

$$\begin{aligned} dF &= d\tilde{B} + cdt \wedge dE \\ &= (\nabla \cdot \mathbf{B})dx \wedge dy \wedge dz + [\partial_t B_x + c(\partial_y E_z - \partial_z E_y)] dt \wedge dy \wedge dz \\ &\quad + [\partial_t B_y + c(\partial_z E_x - \partial_x E_z)] dt \wedge dz \wedge dx + [\partial_t B_z + c(\partial_x E_y - \partial_y E_x)] dt \wedge dx \wedge dy \end{aligned}$$

so this gives us two of the Maxwell equations, namely $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{B}$. Then

$$\star_4 F = c\tilde{E} + dt \wedge B$$

We can then see the duality between E and B and we get the other two equations from $d(\star F) = 0$.

As the base manifold is simply connected it has trivial cohomology, and any closed two-form is exact. Hence $\exists \mathcal{A}$ s.t. $F = d\mathcal{A}$. Write

$$\mathcal{A} = -\phi dt + A$$

where A is the one form that corresponds to a vector \mathbf{A} in 3d. We then recover the formula from electromagnetism that

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla \phi - \partial_t \mathbf{A} \end{aligned}$$

\mathcal{A} is the connection for which F is the curvature, and so we know we are free to apply a gauge transformation, under which (if $\phi_{\beta\alpha} = e^{i\psi}$) we get

$$\mathcal{A} \mapsto \mathcal{A} - i d\psi$$

4.3.2 Instantons

We now want to consider $M = \mathbb{R}^4$ with Euclidean signature (in practice this might be achieved though a Wick rotation of a Lorentzian metric if necessary). We will, as previously stated, be considering configurations which decay sufficiently quickly to infinity. This means we can add the point at infinity to M to obtain S^4 .

Remark. Although \mathbb{R}^4 does not have an isometric embedding into S^4 , it does have a conformal embedding, and this will turn out to be sufficient, because the defining conditions (namely the self duality ones) are conformally invariant.

Let us now use some useful results in 4d:

Definition 4.12. Decompose, as eigenspaces of \star , $\Omega^2(M) = \Omega_+^2(M) \oplus \Omega_-^2(M)$. We call these the **self-dual** and **anti-self-dual** parts.

Lemma 4.13. Elements of $\Omega_+^2(M)$, $\Omega_-^2(M)$ are orthogonal.

Proof. We have

$$\begin{aligned} \langle \omega_+, \omega_- \rangle \omega_{vol} &= \omega_+ \wedge \star \omega_- = -\omega_+ \wedge \omega_- \\ \langle \omega_-, \omega_+ \rangle \omega_{vol} &= \omega_- \wedge \star \omega_+ = \omega_+ \wedge \omega_- \end{aligned}$$

but $\langle \omega_+, \omega_- \rangle = \langle \omega_-, \omega_+ \rangle$ so must be 0. □

As such we can decompose the Yang-Mills functional as

$$S_{YM} = \int_M |F|^2 \omega_{vol} = \int_M |F_+|^2 \omega_{vol} + \int_M |F_-|^2 \omega_{vol}$$

Now we make the following definition

Definition 4.14. We call

$$c = \int_M \text{Tr}(F \wedge \star F)$$

the *second Chern number*

Proposition 4.15. c depends only on the principal bundle and not the choice of connection.

Lemma 4.16. $c = \int_M |F_+|^2 \omega_{vol} - \int_M |F_-|^2 \omega_{vol}$

Proof. Expanding $F = F_+ + F_-$ and using $\text{Tr}(F \wedge \star F) = |F|^2 \omega_{vol}$ gives the result. \square

Hence we get the inequality

$$S_{YM} \geq |c|$$

with equality iff either $F = F_+$ or $F = F_-$. Note in either case the Yang-Mills equations are trivially satisfied as

$$d^A(\star F) = \pm d^A F = 0 \quad \text{by the Bianchi identity}$$

Definition 4.17. An instanton is a solution to the (anti-)self-dual equations which can be extended to S^4 .

We want to understand instantons up to gauge transformation, and the space parameterising these is the **moduli space of instantons**.

5 Chern-Simons Theory

Let Σ be a compact oriented surface, G a compact Lie group, P a principal G -bundle on Σ , $\langle \cdot, \cdot \rangle$ a G -invariant inner product on \mathfrak{g} .

Proposition 5.1. \mathcal{C}_P is an ∞ -dimensional symplectic manifold.

Proof. As \mathcal{C}_P is an affine space modelled on $\Gamma(T^*\Sigma \otimes \text{ad}(P))$, we have that all the tangent spaces to \mathcal{C}_P are canonically isomorphic to $\Gamma(T^*\Sigma \otimes \text{ad}(P))$. Here we can take

$$\omega(A, B) = \int_\Sigma \langle A \wedge B \rangle$$

Claim: ω is a well-defined symplectic form. \square

Theorem 5.2 (Atiyah, Bott). The action of $\mathcal{G}(P)$ on \mathcal{C}_P is Hamiltonian with moment map given by curvature

$$\nabla \mapsto F_\nabla \in \Gamma(\wedge^2 T^*\Sigma \otimes \text{ad}(P)) \cong \Gamma(\text{ad}(P))^*$$

Definition 5.3. The moduli space of flat connections is

$$\mathcal{M}_P = F^{-1}(0)/\mathcal{G}(P)$$

Remark. \mathcal{M}_P is not usually smooth everywhere, but smooth on an open and dense locus.

Proposition 5.4. \mathcal{M}_P is finite dimensional.

Example 5.5. If G is semisimple, then $\dim \mathcal{M}_P = (2g - 2) \dim G$, where g is the genus of Σ .

Remark. \mathcal{M}_P also arises as

- the moduli space of representations of $\pi_1(\Sigma)$ into G , i.e

$$\mathcal{M}_P = \text{Hom}(\pi_1(\Sigma), G) / G$$

- the moduli space of semi-stable holomorphic $G_{\mathbb{C}}$ -bundle on Σ , now as a Riemann surface.

References

- [1] Urs Schreiber, Konrad Waldorf. Smooth functors vs. differential forms. *Homology Homotopy Appl.*, 13(1):pp. 143–203, 2011.