Prolongation Structures

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1 Introduction

These will be notes I have written to further my understanding in a geometric manner of prolongations. These will start with a review of the original papers by Walqhuist and Estabrook [2, 1].

2 The Korteweg-de Vries equation

2.1 Definition and Existence of solutions

We start with a reminder of the golden child of non-linear pdes for integrability - the KdV equation

Definition 2.1. *The* ***Korteweg-de Vries (KdV) equation*** *is the non-linear pde for* $u : \mathbb{R}^{1+1} \to \mathbb{R}$ *given by*

$$u_t + u_{xxx} + 12uu_x = 0$$

Remark. *The coefficient of* 12 *here reflects a choice for convenience. It can be changed by rescaling* u.

If we make the definitions

$$z = u_x$$
$$p = z_x = u_{xx}$$

this turns the KdV eqn into the first order pde

$$u_t + p_x + 12uz = 0\,.$$

Remark. *When given an ODE, it is standard to complete this process, after which we can guarantee a local solution by the Picard-Lindelöf theorem. We are going to do a similar thing, by imposing the conditions of Frobenius' theorem to get an integrable distribution.*

Let us now consdier these new variables as independent coordinates on the $5d$ manifold $M = \mathbb{R}^5$, and let us define

$$\alpha^1 = du \wedge dt - z dx \wedge dt$$
$$\alpha^2 = dz \wedge dt - p dx \wedge dt$$
$$\alpha^3 = -du \wedge dx + dp \wedge dt + 12uz dx \wedge dt$$

Proposition 2.2. *On any 2d submanfold $S_2 \subset M$ on which x, t are good coordinates and $u_x = z$, $z_x = p$ we have $\alpha^i\big|_{S_2} = 0$.*

Proof. We can check each case separately. E.g. by requiring $u_x = z \Rightarrow du = zdx$, so

$$\alpha^1 = zdx \wedge dt - zdx \wedge dt = 0$$

□

Proposition 2.3. *Let $I = (\alpha^1, \alpha^2, \alpha^3) \triangleleft \Omega(M)$. Then $dI \subset I$.*

Proof. Again this is merely checking. E.g.

$$d\alpha^1 = -dz \wedge dx \wedge dt = dx \wedge \alpha^2$$

□

Corollary 2.4. *The KdV equation has a solution locally.*

Proof. This is just Frobenius' theorem for differential forms. □

Remark. *[2] tells us that by a Cartan's theorem this solution patches together to give us a global one. I do not know this theorem, so find it.*

3 General Theory

3.1 Conserved quantites

We prove now an important proposition in developing our geometric theory. We will use summation notation unless otherwise stated, and take $u : \mathbb{R}^{1+1} \to \mathbb{R}$ to obey some pde. As with KdV, we let α_i be forms whose null set S_2 characterises the solution to the pde.

Proposition 3.1. *Exact 2-forms give quantities conserved through evolution of the pde.*

Proof. Let $\beta = f_i\alpha^i \in \Omega^2(M)$ be exact with $\beta = d\omega$. Recall that by Stokes' theorem

$$\int_{\partial S} \omega = \int_S d\omega$$

where $S \subset M$ is a $2d$ submanifold. If we choose $S \subset S_2$ to correspond to $(x, t) \in [R, S] \times [0, T]$ we get

$$\int_{\partial S} \omega = 0$$

and writing $\omega_{(x,t)} = F(x, t)dx + G(x, t)dt$ in S we see

$$0 = \int_R^S F(x, 0)\, dx + \int_0^T G(R, t)\, dt + \int_R^S F(x, T)\, dx + \int_0^T G(S, t)\, dt$$

If we assume sufficient decay conditions s.t.

$$\lim_{R \to -\infty} \int_0^T G(R, t)\, dt = 0$$

$$\lim_{S \to \infty} \int_0^T G(S, t)\, dt = 0$$

then we have

$$\int_{-\infty}^{\infty} F(x,0)\,dx = \int_{-\infty}^{\infty} F(x,T)\,dx$$

i.e. $\int_{-\infty}^{\infty} F(x,t)\,dx$ is a conserved quantity. □

Example 3.2. *Take KdV, and consider $\beta = -\alpha^3 - 12u\alpha^1$. One can check $d\beta = 0$, and as $H^2(M) = 0$ we know β is exact. It can be worked out that*

$$\omega = udx - (p + 6u^2)dt$$

is suitable. From this we then find the conserved quantity

$$\int_{\mathbb{R}} u(x,t)\,dx$$

We may verify this through standard techniques, as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}} u(x,t)\,dx &= \int_{\mathbb{R}} u_t\,dx \\ &= -\int_{\mathbb{R}} u_{xxx} + 12uu_x\,dx \\ &= -\int_{\mathbb{R}} \frac{\partial}{\partial x}\left(u_{xx} + 6u^2\right)\,dx \\ &= -\left[u_{xx} + 6u^2\right]_{-\infty}^{\infty} = 0\end{aligned}$$

Now note for any ω s.t. $\beta = d\omega$, we have freedom in the choice of ω by adding on any element of $H^1(M)$. i.e. we can make the change

$$\omega \mapsto \omega + dy$$

where y is some scalar function.

3.2 Prolongation

From our previous section, we had an additional degree of freedom on ω arising from the choice of scalar function y. We view y as a new independent variable to extend our space $M \mapsto \tilde{M}$. We have also extended the ideal $I \mapsto \tilde{I} = (\alpha^1, \ldots, \alpha^n, \omega)$. We again know $d\tilde{I} \subset \tilde{I}$.

Definition 3.3. *The process of generating a new independent variable and larger closed ideal is called the* ***prolongation*** *of the orginial set.*

If we now define S_2 to be the $2d$ submanifold of $\tilde{M}$ parameterised by x, t that nulls $\alpha^1, \ldots \alpha^3, \omega$ we must get

$$(y_x + F)dx + (y_t + G)dt = 0 \Rightarrow \begin{cases} y_x = -F \\ y_t = -G \end{cases}$$

If we can eliminate p, u, z from these equations we get a pde for y. We call y a **potential function**.

Example 3.4. *Using ω as in the previous example, we have the equations for y*

$$\begin{aligned} y_x &= -u \\ y_t &= p + 6u^2 \end{aligned}$$

From these we get

$$w_t + w_{xxx} - 6w_x^2 = 0$$

We can now come up with a procedure to try and find such conservation laws. Expanding out the condition $\beta = d\omega$ we get

$$(F_{,\mu} dz^\mu \wedge dx + G_{,\mu} dz^\mu \wedge dt) - f_i \alpha^i = 0$$

where we have used the notation $F_{,\mu} = \frac{\partial F}{\partial z^\mu}$ for $z^\mu = (x, t, u, z = x_x, p = u_{xx}, \dots)$. This is called the **closure equation**. These are a set of overdetermined coupled linear first-order pdes, and each solution gives rise to a conservation law, as well as a prolongation.

Remark. *Note that a solution to the above equation is not unique, we are free to add any constant onto F, G, and to scale F, G, f_i all by the same scale factor.*

Example 3.5. *For KdV, we get*

$$\begin{aligned}
-F_{,t} + G_{,x} &= -f_1 z - p f_2 + 12uz f_3 && (dx \wedge dt)\\
F_{,u} &= -f_3 && (du \wedge dx)\\
F_{,z} &= 0 && (dz \wedge dx)\\
F_{,p} &= 0 && (dp \wedge dx)\\
G_{,u} &= f_1 && (du \wedge dt)\\
G_{,z} &= f_2 && (dz \wedge dt)\\
G_{,p} &= f_3 && (dp \wedge dt)
\end{aligned}$$

If we made an ansatz that F, G are independent of x, t we get that $F = F(u)$. Then $f_3 = -F'$ and

$$G(u, z, p) = p f_3(u) + g(u, z)$$

Substituting in

$$\begin{aligned}
0 &= -z\,(p f_3' + g_{,u}) - p g_{,z} + 12uz f_3\\
&= -p\,(z f_3' + g_{,z}) + z\,(12u f_3 - g_{,u})
\end{aligned}$$

Setting each of these two brackets to 0 gives

$$g(z, u) = -\frac{1}{2} z^2 f_3'(u) + h(u)$$

where h satisfies

$$12u f_3(u) + \frac{1}{2} z^2 f_3''(u) - h'(u) = 0$$

Looking at this second equation we must have $f_3'' = 0 \Rightarrow f_3(u) = cu + d$ for $c, d \in \mathbb{R}$. This then gives

$$h(u) = \int^u 12v(cv + d)\, dv = 4cu^3 + 6du^2 + const$$

Within this set of solutions we get our previous example taking $c = 0, d = -1$.

Once we have found a prolongation, we may repeat the process, and so on forming a sequence of ω^k s.t.

$$\omega^k = dy^k + F^k dx + G^k dt$$

where $F^k = F^k(z^\mu, y^i)$, $G^k = G^k(z^\mu, y^i)$. We can then generalise the prolongation process to allow a closure equation

$$d\omega^k - f_i^k \alpha^i - \eta_i^k \wedge \omega^i = 0$$

where η_i^k are some one-forms. Note that the nullity condition for ω^k means that 'on-shell'

$$(y_x^k + F^k) dx + (y_t^k + G^k) dt = 0 \Rightarrow \begin{cases} y_x^k = -F^k \\ y_t^k = -G^k \end{cases}$$

This means the closure equation will now have non-linearity in the $dx \wedge dt$ term where we will get contributions of the form

$$-F^k_{,y^i} y^i_{,t} + G^k_{,y^i} y^i_{,x} = G^i F^k_{,y^i} - F^i G^k_{,y^i}$$

This is a term of the form $[\boldsymbol{G}, \boldsymbol{F}]^k$, leading to an underlying algebraic structure to the prolongation.

3.3 Extended example - prolongation structure of KdV

We have previously worked out the closure equation for the first prolongation of the KdV equation, and found a class of solutions to it. We can go further, and work out the more general prolongation sequence now. We again make the ansatz that F, G are independent of x, t explicitly.

Remark. *[2] notes that this ansatz is taken primarily for simplicity, and while it can be midly motivated by noting that the KdV equation does not depend on x, t explicitly, it is noted that this may lose possible solutions.*

Now the general closure equation, can be written as

$$\begin{aligned} F^k_{,z} &= 0 \\ F^k_{,p} &= 0 \\ F^k_{,u} + G^k_{,p} &= 0 \\ zG^k_{,u} + pG^k_{,z} - 12uzG^k_{,p} + G^i F^k_{,y^i} - F^i G^k_{,y^i} &= 0 \end{aligned}$$

References

[1] F. B. Estabrook, H. D. Wahlquist. Prolongation structures of nonlinear evolution equations. II. *Journal of Mathematical Physics*, 17(7):pp. 1293–1297, 1976. ISSN 0022-2488. doi:10.1063/1.523056.

[2] H. D. Wahlquist, F. B. Estabrook. Prolongation structures of nonlinear evolution equations. *Journal of Mathematical Physics*, 16(1):pp. 1–7, 1975. ISSN 0022-2488. doi:10.1063/1.522396.