Algebraic Topology Example Sheet 2

Linden Disney-Hogg

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Contents

1 Question 1

1.1 (i) and (ii)

Proposition 1.1.

$$H_n(X \sqcup Y) \cong H_n(X) \oplus H_n(Y)$$

Proof. Recall that we have universal properties defining the direct sum of A, B abelian groups and the disjoint union of X, Y topological spaces described by the diagrams:

$$\begin{tikzcd} A \arrow[r, hook, "i_A"] \arrow[dr, "\phi"'] & A \oplus B \arrow[d, dashed, "\phi\oplus\psi"] & B \arrow[l, hook', "i_B"'] \arrow[dl, "\psi"] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "i_X"] \arrow[dr, "f"'] & X \sqcup Y \arrow[d, dashed, "f\sqcup g"] & Y \arrow[l, hook', "i_Y"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Applying the map H_n, known to be a functor, to the right diagram gives

$$\begin{tikzcd}
H_n(X) \arrow[r, hook, "(i_X)_*"] \arrow[dr, "f_*"'] & H_n(X \sqcup Y) \arrow[d, dashed, "(f \sqcup g)_*"] & H_n(Y) \arrow[l, hook', "(i_Y)_*"'] \arrow[dl, "g_*"] \\
& H_n(Z) &
\end{tikzcd}$$

As homology groups are abelian groups, by the universal property we are done. $\square$

Moreover, as a result of the uniqueness of the maps in the universal property we have by setting $\phi = f_*, \psi = g_*$,

$$(f \sqcup g)_* = f_* \oplus g_*$$

1.2 (iii)

Now with the notation, recall

$$\ker(\phi \oplus \psi) = \{(a,b) \in A \oplus B \,|\, \phi(a) + \psi(b) = 0_C\}$$

Suppose ψ is an isomorphism, then we may write

$$\ker(\phi \oplus \psi) = \left\{(a, \psi^{-1}(-\phi(a)))\right\}$$

which given an immediate isomorphism between A and $\ker(\phi \oplus \psi)$.

1.3 (iv)

Let $X_+ = X \sqcup \{+\}$.

Proposition 1.2.

$$\tilde{H}_n(X_+) = H_n(X)$$

Proof. We have now developed the machinery to calculate this simply. We define

$$\tilde{H}_n(X_+) = \ker(\gamma_*^{X_+} : H_n(X_+) \to H_n(+)$$

where $\gamma^{X_+} : X_+ \to \{+\}$ is the unique such continuous map. By our universal property we must have

$$\begin{aligned} \gamma^{X_+} &= \gamma^X \sqcup \gamma^+ \\ \Rightarrow \gamma_*^{X_+} &= \gamma_*^X \oplus \gamma_*^+ \end{aligned}$$

Now $\gamma^+ = \mathrm{id}_+ \Rightarrow \gamma_*^+ = id_{H_n(+)}$, an isomorphism, so

$$\tilde{H}_n(X_+) \cong H_n(X)$$

$\square$

2 Question 2

Note the definition of a Moore space

2.1 (i)

Example 2.1. *S^n is a Moore space of type $(\mathbb{Z}, n)$*

Example 2.2. *$S^n \cup e^{n+1}$, a CW complex where the $n+1$-cell is attached along a map of degree k is a Moore space of type $(\mathbb{Z}_k, n)$.*

2.2 (ii)

To construct a general Moore space $M(G, n)$ for finitely generated G, we may use the following construction:

1. Take $\{g_1, \dots g_i, g_{i+1}, \dots, g_{i+j}\}$ a generating set for G such that

$$ord(g_k) = \begin{cases} \infty & k = 1, \dots, i \\ d_{k-i} & k = i+1, \dots, i+j \end{cases}$$

   That is,

$$G \cong \mathbb{Z}^i \oplus \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_j}$$

   Note this is always possible of finitely generated abelian groups (with maybe i of j being zero).

2. Recall $\tilde{H}_n(X \vee Y) = \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$

3. Hence construct

$$X = \left(\vee_{a=1}^{i} S^n\right) \vee (S^n \cup_{f_1} e^{n+1}) \vee \dots \vee (S^n \cup_{f_j} e^{n+1})$$

   where $f_k : S^n \to S^n$ are attaching maps with $\deg f_k = d_k$

The constructed X is Moore of type (G, n).

2.3 (iii)

For this question I do not have a correct answer, but effectively I know I somehow want to find a exact sequence like

$$\dots \longrightarrow \tilde{H}_n(M(K,n)) \xrightarrow{\phi_*} \tilde{H}_n(M(G,n)) \longrightarrow \tilde{H}_n(C_\phi) \longrightarrow \tilde{H}_{n-1}(M(K,n)) \longrightarrow \dots$$

3 Question 3

3.1 (i)

We want to calculate $H_n(\mathbb{RP}^2)$, which we will do in two ways:

3.1.1 a)

Note that $\mathbb{RP}^2 = S^1 \cup_f e^2$ where the attaching map $f : S^1 \to S^1$ has degree 2. We immediately see from previous results in cellular homology that $\mathbb{RP}^2$ is a Moore space of type $(\mathbb{Z}_2, 1)$. As such

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

3.1.2 b)

Using the decomposition provided of $\mathbb{RP}^2 = U \cup V$, let $A = \overline{U}, B = \overline{V}$ and consider the Mayer Vietoris sequence for reduced homology. $C = A \cap B$ is the dashed circle, so $\tilde{H}_n(C) = \tilde{H}_n(S^1)$, A deformation retracts onto S^1, so $\tilde{H}_n(A) = \tilde{H}_n(S^1)$, and finally B is contractible so $\tilde{H}_n(B) = 0$. Hence the sequence looks like

$$\dots \longrightarrow \tilde{H}_n(S^1) \xrightarrow{i_*} \tilde{H}_n(S^1) \xrightarrow{j_*} \tilde{H}_n(\mathbb{RP}^2) \xrightarrow{\partial} \tilde{H}_{n-1}(S^1) \longrightarrow \dots$$

Where $i : C \to A \to S^1, j : S^1 \to A \to \mathbb{RP}^2$ are both inclusion composed with deformation retraction (or its inverse). This sequence is only interesting in the case $n = 1$ in which we have

$$\mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \xrightarrow{j_*} \tilde{H}_1(\mathbb{RP}^2) \xrightarrow{\partial} 0$$

Then we have

$$\begin{aligned} \operatorname{Im} j_* &= \ker \partial = \tilde{H}_n(\mathbb{RP}^2) \\ \ker j_* &= \operatorname{Im} i_* = 2\mathbb{Z} \end{aligned}$$

where the second follows from the fact that our i ends up being degree 2, as the deformation retract onto a is degree 2. We then have the same result as before (after 1st iso thm application) that

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

3.2 (ii)

We want to calculate $H_n(K)$, which we will do in two ways:

3.2.1 a)

Note that with the given construction we can write

$$K = S^1 \vee (S^1 \cup_f e^2)$$

as the attaching map to a has degree 0, and f is the attaching map corresponding to f, which is of degree 2. As such K is a Moore space of type $(\mathbb{Z} \oplus \mathbb{Z}_2, 1)$, so

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

3.2.2 b)

By following similar procedure as in part i), we get the only interesting part of the Mayer Vietoris sequence to be

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \tilde{H}_1(K) \longrightarrow 0$$

This replicates our result from above.