Standard Model Notes

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Contents

1 Introduction

The Standard Model (SM) describes three fundamental forces (EM, weak, and strong). The forces are mediated by gauge bosons (spin = 1).

- EM (QED) : photon γ, (massless)
- Weak : W and Z bosons
- Strong : gluons g (massless)

Matter content (spin $-\frac{1}{2}$ fermions) comes in three generations

- Neutrinos $\nu_e, \nu_\mu \nu_\tau$ weak
- Charged leptons e, μ, τ weak and EM
- Quarks u, dc, s, t, b, weak, EM, and strong.

There is also the Higgs boson H (scalar, spin 0), responsible for generating the mass of W and Z bosons.

Gauge bosons are manifestations of local symmetries. In SM the gauge group is

$$\underbrace{SU(3)_c}_{\text{color}} \times \underbrace{SU(2)_L}_{\text{chiral left}} \times \underbrace{U(1)_\gamma}_{\text{hypercharge}}$$

2 Chiral and Gauge symmetries

2.1 Chiral symmetry

Spin 1/2 particles are Dirac fermions with spinor field ψ that satisfy the Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$. The Dirac adjoint $\overline{\psi} = \psi^\dagger \gamma^0$ satisfies $\overline{\psi}(i\overleftarrow{\partial\!\!\!/} - m) = 0$ where $\overleftarrow{\partial\!\!\!/}$ acts to the left. Dirac matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$$

where $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Also define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

which satisfies $(\gamma^5)^2 = I$ and $\{\gamma^\mu, \gamma^5\} = 0$. In general the *Chiral* or *Weyl* basis is used, where

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$
$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

In this representation

$$\gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$$

Consider the massless limit of the Dirac equation

$$\partial\!\!\!/\psi = 0 \Rightarrow \partial\!\!\!/(\gamma^5\psi) = 0.$$

Define $P_{R,L} = \frac{1}{2}(1 \pm \gamma^5)$, projection operators. Then correspondingly let $\psi_{R,L} = P_{R,L}\psi$. Then $\gamma^5\psi_{R,L} = \pm\psi_{R,L}$, so the projections are eigenstates of chirality. In the chiral basis, $\psi_{R,L}$ only contain lower/upper 2-component spinor degrees of freedom (d.o.f). As a result $\psi_{R,L}$ annihilates left/right handed chiral particles respectively. In addition

$$\overline{\psi}_{R,L} = (P_{R,L}\psi)^\dagger\gamma^0 = \psi^\dagger\frac{1}{2}(1 \pm \gamma^5)\gamma^0 = \overline{\psi}P_{L,R}$$

A massless Dirac fermion has a *global* $U(1)_L \times U(1)_R$ chiral symmetry. Under $U(1)_{R,L}$, $\psi_{R,L} \mapsto e^{i\alpha_{R,L}}\psi_{R,L}$, as can be seen from the Dirac Lagrangian

$$\mathcal{L}_D = \overline{\psi}(i\partial\!\!\!/ - m)\psi = \overline{\psi}_L i\partial\!\!\!/\psi_L + \overline{\psi}_R i\partial\!\!\!/\psi_R - m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R)$$

The mass term explicitly breaks the chiral symmetry to a remaining vector symmetry where $\alpha_L = \alpha_R = \alpha$, so $\psi \mapsto e^{i\alpha}\psi$, $U(1)_L \times U(1)_R \to U(1)_V$.

2.2 Review of Dirac Field

Quantise

$$\psi(x) = \sum_{p,s}\left[b^s(p)u^s(p)e^{-ip\cdot x} + d^{s\dagger}(p)v^s(p)e^{ip\cdot x}\right]$$

$s = \pm\frac{1}{2}$, and $\sum_p = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}}$, $\langle p|q\rangle = (2\pi)^2(2E_p)\delta^3(\boldsymbol{p} - \boldsymbol{q})$ with $|p\rangle = b^\dagger(p)\,|0\rangle$. u and v are solutions of

$$(p\!\!\!/ - m)u = 0, \quad (p\!\!\!/ + m)v = 0$$

In the chiral basis these become

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix}$$
$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ -\sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix}$$

with $\sigma^\mu = (1, \boldsymbol{\sigma})$, $\overline{\sigma}^\mu = (1, -\boldsymbol{\sigma})$.

Definition 2.1 (Helicity)**.** ***Helicity*** *is defined as the projection of angular momentum onto the linear momentum direction.*

$$h = \boldsymbol{J} \cdot \hat{\boldsymbol{p}} = \boldsymbol{s} \cdot \hat{\boldsymbol{p}}$$

as $\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{s}$*, and*

$$s_i = \frac{i}{4}\epsilon_{ijk}\gamma^i\gamma^j = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

A massless spinor satisfies $\not{p}u = 0$ so

$$hu(p) = \frac{\gamma^5}{2}u(p)$$

$$\Rightarrow hu_{R,L} = \frac{\gamma^5}{2}u_{R,L} = \pm\frac{1}{2}u_{R,L}$$

Note

- Chiral states are only eigenstates of the Dirac equation when $m = 0$
- Helicity is defined for $m = 0$ and $m \neq 0$, but it's not Lorentz Invariant when $m \neq 0$.
- There is only a 1-1 correspondence between helicity and chirality when $m = 0$.

2.3 Gauge Symmetry

Promoting α to a function of x, $\alpha(x)$, i.e. gauging the symmetry $\psi \to e^{i\alpha(x)}\psi$, the kinetic term is no longer invariant.

$$\overline{\psi}i\not{\partial}\psi \to \overline{\psi}i\not{\partial}\psi - (\overline{\psi}\gamma^m u\psi)(\partial_\mu\alpha(x))$$

Introduce a gauge covariant derivative D_μ such that

$$D_\mu\psi(x) \to e^{i\alpha(x)}D_\mu\psi(x)$$

To do this introduce a gauge field $A_\mu(x)$ so

$$D_\mu\psi = (\partial_\mu + igA_\mu)\psi$$

where $A_\mu \to A_\mu - \frac{1}{g}\partial_\mu\alpha$ so $\overline{\psi}i\not{D}\psi$ is invariant.
Introduce a kinetic term for the gauge fields

$$\begin{aligned} \mathcal{L}_{gauge} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ igF_{\mu\nu} &= [D_\mu, D_\nu] \end{aligned}$$

QED has a $U(1)$ gauge symmetry that treats LH and RH fields equivalently. The weak gauge bosons only couple to LH fields, but $U(1)$ is not the appropriate symmetry, we need $SU(2)$. (We will review non-abelian gauge symmetries later.)

2.4 Types of Symmetry

Symmetries may manifest themselves in a variety of ways:

- Symmetry is intact e.g. $U(1)_{EM}$, and $SU(3)_c$ gauge symmetries.
- Symmetry of $\mathcal{L}$ is broken by an **anomaly** (holds classically but is broken by quantum loop effects). Not actually a true symmetry. E.g. global axial $U(1)$ symmetry in the SM.
- Symmetries can hold for some terms in $\mathcal{L}$ but not others. This is called "broken explicitly". It may be an approximate symmetry if the breaking terms are small. E.g. global 'isospin' symmetry relating u and d quarks in QCD.
- Hidden symmetries - respected by $\mathcal{L}$ but *not* the vacuum. These can be: a) **spontaneously broken symmetries**, vacuum expectation value from one or more scalar fields non-zero, e.g $SU(2)_L \times U(1)_\gamma \to U(1)_{EM}$, or b) Even without scalar fields get **dynamical breaking** from quantum effects. e.g. $SU(2)_L \times SU(2)_R$ global symmetry in QCD (massless quarks).

3 Discrete Symmetries

The discrete symmetries are

- Parity $P : (t, \boldsymbol{x}) \to (t, -\boldsymbol{x})$
- Time reversal $T : (t, \boldsymbol{x}) \to (-t, \boldsymbol{x})$
- Charge conjugation C : particles $\leftrightarrow$ antiparticles

Parity and Time reversal are spacetime symmetries. These have properties such as

- $\overline{\psi}\gamma^\mu\psi$ couplings between gauge bosons and fermions, e.g. QED and QCD, are invariant under P and C separately.
- $\overline{\psi}\gamma^\mu(1-\gamma^5)\psi$ couplings to fermions, e.g. weak interactions, are not.
- Weak interaction violates CP, which leads to T violation from the CPT theorem.

We will first investigate the consequences of C, P, T symmetries in order to understand the above statements.

3.1 Symmetry Operators

Theorem 3.1 (Winger). *If physics is invariant under $\Psi \to \Psi'$ (where $\Psi, \Psi' \in \mathcal{H}$ some Hilbert space), then $\exists W$ an operator such that $\Psi' = W\Psi$ where either W i) linear and unitary or ii) antilinear and antiunitary.*

Proof. Consider a Poincare transform

$$x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu$$

Then for a parity transform

$$\Lambda^\mu_\nu = \mathbb{P}^\mu_\nu = \text{diag}(1, -1, -1, -1)$$

and for time reversal

$$\mathbb{T}^{\mu}_{\nu} = \text{diag}(-1,1,1,1)$$

The corresponding operator can be expanded as

$$W(\Lambda, a) = W(1+1, \epsilon) = 1 + \frac{i}{2} w_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu p^\mu$$

where the J are the generators of boosts and rotations, and p the generators of translation i.e $p^0 = H$ =Hamiltonian and p^i =momentum. Then

$$\begin{aligned}\hat{P} &= W(\mathbb{P}, 0)\\ \hat{T} &= W(\mathbb{T}, 0)\end{aligned}$$

Now from general composition rules

$$\hat{P} W(\Lambda, a) \hat{P}^{-1} = W(\mathbb{P}\Lambda\mathbb{P}^{-1}, \mathbb{P}a)$$

Insert expansion of W and compare coefficients of $-\epsilon_0$ to get

$$\hat{P} i H \hat{P}^{-1} = iH$$

and doing likewise for $\hat{T}$ gives

$$\hat{T} i H \hat{T}^{-1} = -iH \tag{3.1.1}$$

Suppose Ψ is an energy eigenstate

$$(\Psi, iH\Psi) = (\Psi, iE\Psi) = iE$$

If $\hat{P}, \hat{T}$ are symmetries then $\hat{P}\Psi, \hat{T}\Psi$ should also be energy eigenstates with the same energy. Suppose $\hat{P}$ is linear. Then

$$\begin{aligned}(\hat{P}\Psi, iH\hat{P}\Psi) &= (\hat{P}\Psi, \hat{P}iH\Psi) \text{ from 3.1.1}\\ &= (\hat{P}\Psi, \hat{P}iE\Psi)\\ &= iE(\hat{P}\Psi, \hat{P}\Psi) \text{ as } \hat{P} \text{ linear}\\ &= iE\end{aligned}$$

Now suppose $\hat{T}$ linear and complete as above giving

$$(\hat{T}\Psi, iH\hat{T}\Psi) = -iE(\hat{T}\Psi, \hat{T}\Psi)$$

Contradiction. Supposing instead $\hat{T}$ antilinear gives

$$(\hat{T}\Psi, iH\hat{T}\Psi) = iE(\hat{T}\Psi, \hat{T}\Psi)$$

Hence must have $\hat{P}$ linear and unitary, whereas $\hat{T}$ antilinear and antiunitary. $\square$

3.2 Parity

Scalar fields

Definition 3.2 (Scalar Fields)**.** *A complex scalar field is*

$$\phi(x) = \sum_p \left[a(p)e^{-ip\cdot x} + c^\dagger(p)e^{ip\cdot x}\right]$$

$\hat{P} : |p\rangle \to \eta^{a*}|p_P\rangle$ where $p_P = (p^0, -\boldsymbol{p})$ and η^{a*} is a complex phase. For later define in analogy $x_P = (x^0, -\boldsymbol{x})$

$$\Rightarrow \hat{P}a^\dagger(p)|0\rangle = \eta^{a*}a^\dagger(p_P)|0\rangle$$

since $\hat{P}\hat{P}^{-1} = I$ and assuming $\hat{P}|0\rangle = |0\rangle$

$$\hat{P}a^\dagger(p)\hat{P}^{-1} = \eta^{a*}a^\dagger(p_P)$$

To conserve normalisation, $\hat{P}a(p)\hat{P}^{-1} = \eta^a a(p_P)$. Similarly $\hat{P}c^\dagger(p)\hat{P}^{-1} = \eta^{c*}c^\dagger(p_P)$. Thus

$$\begin{aligned}
\hat{P}\phi(x)\hat{P}^{-1} &= \sum_p \left[\hat{P}a(p)\hat{P}^{-1}e^{-ip.x} + \hat{P}c^\dagger(p)\hat{P}^{-1}e^{ip\cdot x}\right] \\
&= \sum_p \left[\eta^a a(p_P)e^{-ip\cdot x} + \eta^{c*}c^\dagger(p_P)e^{ip\cdot x}\right] \\
\text{Relabelling } p \leftrightarrow p_P \ &= \sum_{p_P} \left[\eta^a a(p)e^{-ip_P\cdot x} + \eta^{c*}c^\dagger(p)e^{ip_P\cdot x}\right] \\
[p_P \cdot x = p \cdot x_P] \Rightarrow &= \sum_{p_P} \left[\eta^a a(p)e^{-ip\cdot x_P} + \eta^{c*}c^\dagger(p)e^{ip\cdot x_P}\right] \\
[\sum_p = \sum_{p_P}] \Rightarrow &= \sum_p \left[\eta^a a(p)e^{-ip\cdot x_P} + \eta^{c*}c^\dagger(p)e^{ip\cdot x_P}\right]
\end{aligned}$$

This does not look like $\phi(x_P)$ unless $\eta^a = \eta^{c*} \equiv \eta_P$ and otherwise would note in general find $\left[\phi(x), \hat{P}\phi^\dagger(y)\hat{P}^{-1}\right]$ vanishes for spacelike $x - y_p$. Hence

$$\hat{P}\phi(x)\hat{P}^{-1} = \eta_P\phi(x_P)$$

Note that for a real scalar field $a = c$ and so $\eta^a = \eta^{a*} \Rightarrow \eta_P = \pm 1$. For a complex field, we may not have real η_P, but if there is some conserved charge we can redefine $\hat{P}$ such that $\eta_P = \pm 1$

Vecotr fields

Definition 3.3 (Vector Fields)**.** *A vector field is*

$$V^\mu(x) = \sum_{p,\lambda} \left[\epsilon^\mu(\lambda, p)a^\lambda(p)e^{-ip\cdot x} + \epsilon^{\mu*}(\lambda, p){c^\lambda}^\dagger(p)e^{ip\cdot x}\right]$$

$\lambda = 0, \pm 1$ *is helicity,* ϵ^μ *polarisation vectors.*

Use $\epsilon^\mu(\lambda, p_P) = -\mathbb{P}^\mu_\nu \epsilon^\nu(\lambda, p)$. In analogy to the above treatment we find

$$\hat{P} V^\mu(x) \hat{P}^{-1} = -\eta_P \mathbb{P}^\mu_\nu V^\nu(x_P)$$

Vectos have $\eta_P = -1$, axial vectors have $\eta_P = 1$.

Dirac Field

For a Dirac field, creation/annihilation operators should behave like those for bosons. The 3-momentum reverses direction, the spin component s is changed.

$$\hat{P} b^s(p) \hat{P}^{-1} = \eta^b b^s(p_P)$$
$$\hat{P} d^{s\dagger}(p) \hat{P}^{-1} = \eta^{d^\dagger} d^{s\dagger}(p_P)$$

Then

$$\begin{aligned}
\hat{P}\psi(x)\hat{P}^{-1} &= \sum_{p,s} \left[\eta^b b^s(p_P) u^s(p) e^{-ip\cdot x} + \eta^{d^*} d^{s\dagger}(p_P) v^s(p) e^{ip\cdot x} \right] \\
&= \sum_{p,s} \left[\eta^b b^s(p) u^s(p_P) e^{-ip\cdot x_P} + \eta^{d^*} d^{s\dagger}(p) v^s(p_P) e^{ip\cdot x_P} \right]
\end{aligned}$$

One can verify

$$u^s(p_P) = \gamma^0 u^s(p)$$
$$v^s(p_P) = -\gamma^0 v^s(p)$$

so

$$\hat{P}\psi(x)\hat{P}^{-1} = \sum_{p,s} \left[\eta^b b^s(p) \gamma^0 u^s(p) e^{-ip\cdot x_P} - \eta^{d^*} d^{s\dagger}(p) \gamma^0 u^s(p) e^{ip\cdot x_P} \right]$$

Require $\eta^b = -\eta^{d^*} = \eta_P$ so that

$$\hat{P}\psi(X)\hat{P}^{-1} = \eta_P \gamma^0 \psi(x_P) \equiv \psi^P(x)$$

Then

$$\overline{\psi}^P(x) = \hat{P}\overline{\psi}(x)\hat{P}^{-1} = \eta_P^* \overline{\psi}(x_P)\gamma^0$$

Note

- $\hat{P}\psi_L(x)\hat{P}^{-1} = \eta_P \gamma^0 \psi_R(x_P)$
- It can be checked that ψ satisfies the Dirac equation $\Rightarrow$ ψ^P satisfies the Dirac equation.

From the above we can determine the transformation properties of fermion bilinears.

$$\begin{aligned}
\overline{\psi}(x)\psi(x) &\to \hat{P}\overline{\psi}(x)\hat{P}^{-1}\hat{P}\overline{\psi}(x)\hat{P}^{-1} = \overline{\psi}(x_P)\psi(x_P) && \text{(scalar)} \\
\overline{\psi}(x)\gamma^5\psi(x) &\to -\overline{\psi}(x_P)\gamma^5\psi(x_P) && \text{(pseudoscalar)} \\
\overline{\psi}(x)\gamma^\mu\psi(x) &\to \mathbb{P}^\mu_\nu \overline{\psi}(x_P)\gamma^\nu\psi(x_P) && \text{(vector)} \\
\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x) &\to -\mathbb{P}^\mu_\nu \overline{\psi}(x_P)\gamma^\nu\gamma^5\psi(x_P) && \text{(axial vector)}
\end{aligned}$$

3.3 Charge conjugation

$\hat{C}$ is a linear and unitary operator sending particles $\leftrightarrow$ antiparticles

Scalar Field

Lorentz symmetry constrains the phases

$$\begin{aligned}\hat{C}a(p)\hat{C}^{-1} &= \eta_C c(p)\\ \hat{C}c(p)\hat{C}^{-1} &= \eta_C{}^* a(p)\end{aligned}$$

Then

$$\begin{aligned}\hat{C}\left|particle, p\right\rangle &= \hat{C}a^\dagger(p)\left|0\right\rangle\\ &= \eta_C{}^* c^\dagger(p)\left|0\right\rangle\\ &= \eta_C{}^*\left|antiparticle, p\right\rangle\end{aligned}$$

From the decomposition

$$\begin{aligned}\hat{C}\phi(x)\hat{C}^{-1} &= \eta_C \phi^\dagger(x)\\ \hat{C}\phi^\dagger(x)\hat{C}^{-1} &= \eta_C{}^* \phi(x)\end{aligned}$$

For a real scalar field $\phi = \phi^\dagger$ and so $\eta_C = \pm 1$.

Vector field

Photon field must obey $\hat{C}A_\mu(x)\hat{C}^{-1} = -A_\mu(x)$. Hence a π^0 meson can decay to $2\gamma \Rightarrow \eta_C^{\pi^0} = (-1)^2 = 1$. For a complex field, η_C is arbitrary. Say $\eta_C = e^{2i\beta}$, we can do a global $U(1)$ transform st $\phi \to \phi' = e^{-i\beta}\phi$ such that $\eta'_C = 1$.

Dirac Field

Define the matrix C such that $(\gamma^\mu C)^T = \gamma^\mu C$. In the chiral basis where

$$\begin{aligned}\gamma^{0T} &= \gamma^0\\ \gamma^{1T} &= -\gamma^1\\ \gamma^{2T} &= \gamma^2\\ \gamma^{3T} &= -\gamma^3\end{aligned}$$

a suitable choice is

$$C == i\gamma^0\gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}$$

giving

$$C = -C^T = -C^\dagger = -C^{-1}$$

Then

$$(\gamma^\mu)^T = -C^{-1}\gamma^\mu C$$
$$\gamma^{5T} = C^{-1}\gamma^5 C$$

similarly to bosons

$$\hat{C}b^s(p)\hat{C}^{-1} = \eta_C d^s(p)$$
$$\underbrace{\hat{C}d^{s\dagger}(p)\hat{C}^{-1}}_{\text{in } \psi} = \underbrace{\eta_C b^{s\dagger}(p)}_{\text{in } \overline{\psi}}$$

Now consider

$$\hat{C}\psi(x)\hat{C}^{-1} = \eta_C \sum_{p,s} \left[d^s(p)u^s(p)e^{-ip\cdot x} + b^{s\dagger}v^s(p)e^{ip\cdot x} \right]$$

and compare with

$$\overline{\psi}^T(x) = \sum_{p,s} \left[b^{s\dagger(p)}(\overline{u}^s)^T(p)e^{ip\cdot x} + d^s(\overline{v}^s)^T(p)e^{-ip\cdot x} \right]$$

Consider the spinors and take $\eta^s = i\sigma^2\xi^{s*}$ (choosing a basis for the spinors) we can write

$$v^s(p) = C(\overline{u}^s)^T$$
$$u^s(p) = C(\overline{v}^s)^T$$

and so

$$\psi^C(x) = \hat{C}\psi(x)\hat{C}^{-1} = \eta_C C\overline{\psi}^T(x)$$

similarly

$$\overline{\psi}^C(x) = \hat{C}\overline{\psi}(x)\hat{C}^{-1} = \eta_C{}^*\psi^T(x)C = -\eta_C{}^*\psi^T(x)C^{-1}$$

Note that $\psi(x)$ satsifes the Dirac eqn $\Rightarrow$ $\psi^C(x)$ does.

- Majorana fermions have $b^s(p) = d^s(p) \Rightarrow$ particle is its own anti particle. In this case $\psi = \psi^C$.
- Is is note known whether the only neutral fermions in the SM (neutrinos) are Majorana (c.f. neutrinoless double beta decay).

Fermion Bilinears

Note it is important to keep track of what's an operator ($\hat{C}$) nad what is a matrix in spinor space (C).

Example 3.4.

$$
\begin{aligned}
j^\mu(x) &= \overline{\psi}(x)\gamma^\mu\psi(x) \\
\Rightarrow \hat{C}j^\mu\hat{C}^{-1} &= \hat{C}\overline{\psi}\hat{C}^{-1}\gamma^\mu\hat{C}\psi\hat{C}^{-1} \\
&= -\eta_C^*\eta_C\psi^T C^{-1}\gamma^\mu C\overline{\psi}^T \\
&= -\psi_\alpha(C^{-1}\gamma^\mu C)_{\alpha\beta}\overline{\psi}_\beta \\
&= \overline{\psi}_\beta(C^{-1}\gamma^\mu C)_{\alpha\beta}\psi_\alpha \quad \textit{(fermions anticommute)} \\
&= \overline{\psi}_\beta(C^{-1}\gamma^\mu C)^T_{\beta\alpha}\psi_\alpha \\
&= \overline{\psi}(C^{-1}\gamma^\mu C)^T\psi \\
&= -\overline{\psi}\gamma^\mu\psi = -j^\mu
\end{aligned}
$$

Similarly

$$\hat{C}j^{\mu 5}\hat{C}^{-1} = j^{\mu 5}$$

where $j^{\mu 5} = \overline{\psi}\gamma^\mu\gamma^5\psi$.

3.4 Time Reversal

Take the notation $x_T^\mu = (-x^0, \boldsymbol{x})$, $p_T^\mu = (p^0, -\boldsymbol{p})$. In T-symmetric theories, the physics is unchanged if time runs backwards.

Boson Field

$$
\begin{aligned}
\hat{T}a(p)\hat{T}^{-1} &= \eta_T a(p_T) \\
\hat{T}c^\dagger(p)\hat{T}^{-1} &= \eta_T c^\dagger(p_T)
\end{aligned}
$$

From the decomposition, recalling $\hat{T}$ antihermitian,

$$\hat{T}\phi(x)\hat{T}^{-1} = \sum_p \left[\hat{T}a(p)\hat{T}^{-1}e^{ip\cdot x} + \hat{T}c^\dagger(p)\hat{T}^{-1}e^{-ip\cdot x}\right]$$

So using $p_T \cdot x = -p \cdot x_T$

$$\hat{T}\phi(x)\hat{T}^{-1} = \eta_T \sum_p \left[a(p)e^{-p\cdot x_T} + c^T(p)e^{ip\cdot x_T}\right]$$

Dirac Field

$\hat{T}$ flips sign of any momentum. The creation/annihilation ops can be taken to transform as

$$
\begin{aligned}
\hat{T}b^s(p)\hat{T}^{-1} &= \eta_T(-1)^{\frac{1}{2}-s}b^{-s}(p_T) \\
\hat{T}d^{s\dagger}(p)\hat{T}^{-1} &= \eta_T(-1)^{\frac{1}{2}-s}d^{-s\dagger}(p_T)
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
(-1)^{\frac{1}{2}-s} u^{-s*}(p_T) &= -B u^s(p) \\
(-1)^{\frac{1}{2}-s} v^{-s*}(p_T) &= -B v^s(p)
\end{aligned}
$$

where

$$
B = C^{-1}\gamma^5 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}
$$

Then

$$
\begin{aligned}
\hat{T}\psi(x)\hat{T}^{-1} &= \eta_T \sum_{p,s} (-1)^{\frac{1}{2}-s} \left[b^{-s}(p_T) u^{s*}(p) e^{ip\cdot x} + d^{-s\dagger}(p_T) v^{s*}(p) e^{-ip\cdot x} \right] \\
&= \eta_T \sum_{p,s} (-1)^{\frac{1}{2}-s+1} \left[b^{s}(p) u^{-s*}(p_T) e^{ip\cdot x_T} + d^{s\dagger}(p) v^{-s*}(p_T) e^{ip\cdot x_T} \right] \\
&= \eta_T B \psi(x_T)
\end{aligned}
$$

Similarly

$$
\hat{T}\overline{\psi}(x)\hat{T}^{-1} = \eta_T^* \overline{\psi}(x_T) B^{-1}
$$

Then some bilinears we have

$$
\begin{aligned}
\hat{T}\overline{\psi}(x)\psi(x)\hat{T}^{-1} &= \overline{\psi}(x_T)\psi(x_T) \\
\hat{T}\overline{\psi}(x)\gamma^\mu\psi(x)\hat{T}^{-1} &= \overline{\psi}(x_T) B^{-1}\gamma^{\mu*} B \psi(x_T)
\end{aligned}
$$

Now we can check

$$
\begin{aligned}
B^{-1}\gamma^{0*} B &= \gamma^0 \\
B^{-1}\gamma^0 i^* B &= -\gamma^i \\
\Rightarrow B^{-1}\gamma^{\mu*} B &= -\mathbb{T}^\mu_\nu \gamma^\nu
\end{aligned}
$$

3.5 Scattering S-Matrix

Define

$$
\begin{aligned}
\langle p_1, p_2, \ldots | S | k_A, k_B, \ldots \rangle &= \ {}_o\langle p_1, p_2, \ldots | k_A, k_B, \ldots \rangle_i \\
&= \lim_{T\to\infty} \langle p_1, p_2, \ldots | T e^{-i\int_{-T}^{T} V(t)dt} | k_A, k_B, \ldots \rangle
\end{aligned}
$$

with

$$
V(t) = -\int d^3x \mathcal{L}_I
$$

the potential energy term.

Example 3.5. *In QED*

$$\mathcal{L}_I = -e\overline{\psi}\gamma^\mu A_\mu \psi$$

Now we have the table of transformations

Quantity	$\hat{P}\cdot\hat{P}^{-1}$	$\hat{C}\cdot\hat{C}^{-1}$	$\hat{T}\cdot\hat{T}^{-1}$
$\mathcal{L}_I(x)$	$\mathcal{L}_I(x_P)$	$\mathcal{L}_I(x)$	$\mathcal{L}_I(x_T)$
$V(t)$	$V(t)$	$V(t)$	$V(-t)$
S	?	?	?

Now write

$$S = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n V(t_1)V(t_2)\ldots V(t_n)$$

$$\Rightarrow S_T\ \hat{T}S\hat{T}^{-1} = \sum_{n=0}^{\infty}(i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n V(-t_1)V(-t_2)\ldots V(-t_n)$$

Substituting $\tau = -t_{n+1-i}$

$$\begin{aligned}
S_T &= \sum_{n=0}^{\infty}(i)^n \int_{\infty}^{-\infty} (-d\tau_n) \int_{\infty}^{-t_1} (-d\tau_{n-1}) \cdots \int_{\infty}^{-t_{n-1}} (-d\tau_1) V(\tau_n)V(\tau_{n-1})\ldots V(\tau_1) \\
&= \sum_{n=0}^{\infty}(i)^n \int_{-\infty}^{\infty} d\tau_n \int_{-t_1}^{\infty} d\tau_{n-1} \cdots \int_{-t_{n-1}}^{\infty} d\tau_1 V(\tau_n)V(\tau_{n-1})\ldots V(\tau_1) \\
&= \sum_{n=0}^{\infty}(i)^n \int_{-\infty}^{\infty} d\tau_n \int_{\tau_n}^{\infty} d\tau_{n-1} \cdots \int_{\tau_2}^{\infty} d\tau_1 V(\tau_n)V(\tau_{n-1})\ldots V(\tau_1)
\end{aligned}$$

Geometrically it can be seen

$$\int_{-\infty}^{\infty} d\tau_n \int_{\tau_n}^{\infty} d\tau_{n-1} = \int_{-\infty}^{\infty} d\tau_{n-1} \int_{-\infty}^{\tau_{n-1}} d\tau_n$$

so successively swapping

$$S_T = \sum_{n=0}^{\infty}(i)^n \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{n-1}} d\tau_n V(\tau_n)V(\tau_{n-1})\ldots V(\tau_1)$$

Now consider

$$\begin{aligned}
S^\dagger &= \sum_{n=0}^{\infty}(i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n [V(t_1)V(t_2)\ldots V(t_n)]^\dagger \\
&= \sum_{n=0}^{\infty}(i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n V(t_n)V(t_{n-1})\ldots V(t_1)
\end{aligned}$$

So we have shown

$$S_T = S^\dagger$$

Hence note ${S_T}^\dagger = S$. Now consider $|\xi\rangle, |\eta\rangle$ with

$$\begin{aligned}|\xi_T\rangle &= \hat{T}\,|\xi\rangle \\ |\eta_T\rangle &= \hat{T}\,|\eta\rangle\end{aligned}$$

Then

$$\begin{aligned}\langle\eta_T|S\xi_T\rangle &= (\hat{T}\eta, S_T^\dagger \hat{T}\xi) \\ &= (\hat{T}\eta, \hat{T}S^\dagger\xi) \\ &= (\eta, S^\dagger\xi)^* \quad (\ \hat{T} \text{ antiunitary}) \\ &= (S^\dagger\xi, \eta) \\ &= (\xi, S\eta) \\ &= \langle\xi|S|\eta\rangle\end{aligned}$$

Hence if $\hat{T}\mathcal{L}_I(x)\hat{T}^{-1} = \mathcal{L}_I(x_T)$, S-matrix elements are equal form time reversed processes where initial and final states are swapped.

3.6 CPT theorem

Theorem 3.6. *Any Lorentz invariant $\mathcal{L}$ with a Hermitian Hamiltonian should be invariant under the product of P,C, and T.*

Proof. See Streater and Wightman "PCT, spin and statistics, and all that" (1989). □

All observations suggest that CPT is respected in nature. This means a particle (positive charge, spin up) propagating forward in time cannot be distinguished from an antiparticle (negative charge, spin down) propagating backwards in time.

3.7 Baryogenesis

Definition 3.7 (Baryogenesis)**.** ***Baryogenesis*** *is the generation of matter-antimatter asymmetry in the universe.*

There are three necessary conditions for Baryongenesis, the **Sakarov conditions**

- Baryon number violation: $X \to Y + B$, B excess baryons (or leptogenesis, i.e lepton number violation giving baryon number asymmetry through B+L violation).
- Non-equilibrium : Otherwise $\Gamma(Y + B \to X) = \Gamma(X \to Y + B)$
- C and CP violation: Otherwise

  $$\frac{dB}{dt} \propto \Gamma(X \to Y + B) - \Gamma(\overline{X} \to \overline{Y} + \overline{B}) = 0$$

  with C-symmetry, or

  $$\Gamma(x \to nq_L) + \Gamma(x \to nq_R) = \Gamma(\overline{x} \to n\overline{q}_R) + \Gamma(\overline{x} \to n\overline{q}_L)$$

  with CP-symmetry.

4 Spontaneous Symmetry Breaking (SSB)

There are hidden symmetries present in $\mathcal{L}$ but not in observable.

4.1 SSB of discrete symmetry

Consider a real scalar field $\phi(x)$ with symmetric $V(\phi)$ and $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu - V(\phi)$, e.g. $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$, $\lambda > 0$
We have either

- the typical case to analyze, $m^2 > 0$, where $V(\phi)$ has a minimum at $\phi = 0$.
- $m^2 < 0$, then $V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$ up to a constant, where $v = \sqrt{\frac{-m^2}{\lambda}}$. Now $\phi = 0$ is an unstable vacuum and there are two degenerate vacua at $\phi = \pm v$.

In the second case ϕ has a acquired a non zero **Vacuum Expectation Value (vev)**. Wlog we may study small excitations about $\phi = v$

$$\phi(x) = v + f(x)$$
$$\mathcal{L} = \frac{1}{2}\partial_\mu f\partial^\mu f - \lambda(v^2 f + vf^3 + \frac{1}{4}f^4) + \text{ constant}$$

Hence f is a scalar field with mass $m_f = \sqrt{2\lambda v^2}$. This $\mathcal{L}$ is *not* invariant under $f \to -f$. The symmetry of the original $\mathcal{L}$ is broken by the VEV of ϕ.

4.2 SSB of continuous (global) symmetry

Consider a real N- component scalar field $\phi = (\phi_1, \ldots, \phi_N)^T$, with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)\cdot(\partial^\mu\phi) - V(\phi)$$
$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4} \quad \phi^2 = \phi\cdot\phi, \phi^4 = (\phi^2)^2, \lambda > 0$$

invariant under a global $O(N)$ symmetry. We're interested in $m^2 < 0$ again. In this case

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

Up to an irrelevant constant where

$$v^2 = -\frac{m^2}{\lambda}$$

This is the "Mexican hat" potential. There is then a continuum of vacua with $\phi^2 = v^2$. Wlog choose $\phi_0 = (0, \ldots, 0, v)^T$ and study small fluctuations about this

$$\phi(x) = (\pi_1(x), \ldots, \pi_{N-1}(x), v + \sigma(x))^T$$

then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)\cdot(\partial^\mu \pi) + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - V(\pi,\sigma)$$
$$V(\pi,\sigma) = \frac{1}{2}m_\sigma^2\sigma^2 + \lambda v(\sigma^2+\pi^2)\sigma + \frac{\lambda}{4}(\sigma^2)+\pi^2)^2$$

The σ field, which is a radial excitation in the potential, has mass $m_\sigma^2 = 2\lambda v^2$, but the $N-1$ π fields, which are azimuthal excitations that see flat potential, are massless

Generalise to a symmetry group G of $\mathcal{L}$ which is broken to a subgroup $H \subset G$ by the vacuum (we'll generally be considering normal subgroups). The transform is $\phi \to g\phi$ with $g \in G$ in some representation, and $\mathcal{L}(\phi) = \mathcal{L}(g\phi)$. Assume G is spontaneously broken and hence the vacuum is not unique but a manifold[1].

$$\Phi_0 = \{\, \phi_0 : V(\phi_0) = V_{min} \,\}$$

The invariant subgroup (or stability group) $H \subset G$ is

$$H = \{\, h \in G : h\phi_0 = \phi_0 \,\}$$

Different vacua are related by $\phi_0' = g\phi_0$ for some $g \in G$. Stability groups for different vacua are isomorphic. For ϕ_0' the stability group is $H' = gHg^{-1}$. Group elements that map one vacuum to another are in the coset space ${}^{G}\!/_{H}$ and fall into equivalence classes

$$g_1 \sim g_2 \Leftrightarrow g_2^{-1}g_1 \in H$$

which are the left cosets. Hence there's one equivalence class for each $\phi_0' \in \Phi_0 \Rightarrow \Phi_0 \cong {}^{G}\!/_{H}$. If H is a normal subgroup the this is a group[2]. Now let's consider infinitesimal transforms $g\phi = \phi + \delta\phi$, $\delta\phi = i\alpha^a t^a \phi$, where $a = 1,\dots,\dim G$ and t^a are the generators of the Lie algebra of G in the representation acting on ϕ, and α^a are 'small' parameters. G invariance means that $V(\phi) = V(\phi+\delta\phi)$, or

$$V(\phi+\delta\phi) - V(\phi) = i\alpha^a (t^a\phi)_r \left(\frac{\partial V}{\partial \phi}\right)_r = 0 \quad \text{to first order} \tag{4.2.1}$$

where $r = 1,\dots,N$ are indices of the components of ϕ. If ϕ_0 is a min of V,

$$V(\phi_0+\delta\phi) - V(\phi_0) = \frac{1}{2}\delta\phi_r \underbrace{\frac{\partial^2 V}{\partial\phi_r\partial\phi_s}}_{=M_{rs}\ \text{(mass matrix)}} \delta\phi_s + \dots$$

Differentiate 4.2.1 and evaluate at ϕ_0 to get

$$\frac{\partial}{\partial\phi_s}\left[(t^a\phi))_r \frac{\partial V}{\partial\phi_r}\right] = \frac{\partial}{\partial\phi_s}(t^a\phi)_r \frac{\partial V}{\partial\phi_r}|_{\phi_0} + (t^a\phi_0)_r M_{sr}^2 = 0$$

Two cases

[1] I suspect that this is necessarily a manifold as our configuration space is assume to be a manifold (in this case $\mathbb{R}^N$) and then Φ_0 is a closed subgroup for continuous V, so the closed subgroup theorem applies

[2] I may prove this for fun if I find the time

- Unbroken symmetry : $\forall g \in G\; g\phi_0 = \phi_0 \Rightarrow \delta\phi = 0 \Rightarrow \forall a\; t^a\phi_0 = 0$
- Brojen symmetry : $\exists g \in G\, s.t. \exists a\; t^a\phi_0 \neq 0 \Rightarrow t^a\phi_0$ is an eigenstate of M^2_{rs} with eigenvalue 0. Generatros of $H \subset G$ are $\tilde{t}^i$ $i = 1, \ldots, \dim H$ and $\tilde{t}^i\phi_0 = 0$

Now a fact from SFP, for a compact semi-simple lie algebra of G we can define a group invariant inner product and orthogonality. Choose a basis of the Lie algebra $t^a = \{\, \tilde{t}^i, \theta^{\tilde{a}} \,\}$ where $\theta^{\tilde{a}}$ are orthogonal to $\tilde{t}^i$ (i.e. $\operatorname{Tr} \tilde{t}^i\theta^{\tilde{a}} = 0$. Then $\theta^{\tilde{a}}\phi_0$ is a unique zero eigenvector of M^2_{sr} for $\tilde{a} = 1, \ldots, \dim G - \dim H \Rightarrow \dim G - \dim H$ massless modes exists (**Goldstone Bosons**) and in general $N - (\dim G - \dim H)$ massive modes exist.
This is the *classical* proof of Goldstone's theorem

Example 4.1. *For $O(N)$ model, $O(N) \to O(N-1)$ as $\Phi_0 = S^{N-1}$, so we expect*

$$\frac{1}{N}(N-1) - \frac{1}{2}(N-1)(N-2) = N - 1$$

massless modes, and this is what was found.

Insert on Group Theory

Suppose a $\mathcal{L}$ written in terms of a complex $N \times N$ matrix field M is invariant under $M \to AMB^{-1}$ where $A, B \in U(N)$. There should be only one identity element in the group, $(I_A, I_B) \in U(N) \times U(N)$ s.t. This is true when $M = I \Rightarrow I = I_A I_B^{-1} \Rightarrow I_A = I_B$. Hence

$$I_A M = M I_A \text{ for arbitrary } M$$

Lemma 4.2 (Schur's lemma)**.** *If $\forall g \in G\; SD(g) = D(g)S$ for D some irreducible rep of G then $S \propto I$.*

Schur's Lemma gives $I_A \propto I \Rightarrow I_A = e^{i\theta}I$ for $\theta \in \mathbb{R}$. Thus these I_A form a $U(1)$ normal subgroup. Hence the symmetry group is $U(N) \times U(N) \big/ U(1)$

4.3 Goldstone's Theorem

Now consider SSB in a fully quantum way. Suppose the symmetry group G of $\mathcal{L}$ is psontaneously broken to $H \subset G$, i.e. ϕ gets a non-zero VEE $\langle 0|\phi|0\rangle = \phi_0 \neq 0$. The VEV is invariant under $h \in H$,but not under $g' \in G \setminus H$. Let

- Lie algebra of G be $\{\, t^a : a = 1, \ldots, \dim G \,\}$
- Lie algebra of H be $\{\, \tilde{t}^i : i = 1, \ldots, \dim H \,\}$

G is a symmetry of $\mathcal{L} \Rightarrow$ conserved currents from Noether's theorem

$$j^{a\mu}(x) = i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} t_a \phi$$

and charges

$$Q^a = \int d^3x j^{a0}(x) = \int d^3x \pi(x) t_a \phi(x)$$

These induce a representation on the Lie algebra

$$\delta\phi(0) = i\alpha^a t^a \phi(0) = i\left[Q^a, \phi(0)\right]\alpha^a$$

Consider now

$$\begin{aligned}
C^{a\mu} &= \langle 0| \left[j^{a\mu}(x), \phi(0)\right] |0\rangle \\
&= \sum_n \left[\langle 0|j^{a\mu}(x)|n\rangle \langle n|\phi(0)|0\rangle - \langle 0|\phi(0)|n\rangle \langle n|j^{a\mu}(x)|0\rangle\right] \\
&= i\int \frac{d^4k}{(2\pi)^3}\left[\rho^{a\mu}(k)e^{-ik\cdot x} - \tilde{\rho}^{a\mu}(k)e^{ik\cdot x}\right]
\end{aligned}$$

where

$$\begin{aligned}
i\rho^{a\mu}(k) &= (2\pi)^3 \sum_n \delta^{(4)}(k-p_n)\langle 0|j^{a\mu}(0)|n\rangle \langle n|\phi(0)|0\rangle \\
i\tilde{\rho}^{a\mu}(k) &= (2\pi)^3 \sum_n \delta^{(4)}(k-p_n)\langle 0|\phi(0)|n\rangle \langle n|j^{a\mu}(0)|0\rangle
\end{aligned}$$

and recall

$$j^{a\mu}(x) = e^{ip\cdot x} j^{a\mu}(0) e^{-ip\cdot x}$$

This is the **Källen Lehmann spectral representation**. Lorentz covariance gives $\rho^{a\mu} \propto k^\mu \propto \tilde{\rho}^{a\mu}$, physical states with $k^0 > 0$. Hence

$$\begin{aligned}
\rho^{a\mu}(k) &= k^\mu \Theta(k^0)\rho^a(k^2) \\
\tilde{\rho}^{a\mu}(k) &= k^\mu \Theta(k^0)\tilde{\rho}^a(k^2)
\end{aligned}$$

So

$$C^{a\mu} = -\partial^\mu \int \frac{d^4k}{(2\pi)^3}\Theta(k^0)\left[\rho^a(k^2)e^{-ik\cdot x} + \tilde{\rho}^a e^{ik\cdot x}\right]$$

Now consider the propagator

$$\begin{aligned}
D(z-y;\sigma) &= \langle 0|\phi(z)\phi(y)|0\rangle \\
&= \int \frac{d^4p}{(2\pi)^3}\,\Theta(p^0)\delta(p^2-\sigma)e^{-ip\cdot(z-y)}
\end{aligned}$$

and recognise

$$\rho(k^2) = \int d\sigma\, \rho(\sigma)\delta(k^2-\sigma)$$

so

$$C^{a\mu} = -\partial^\mu \int d\sigma\, \left[\rho^a(\sigma)D(x;\sigma) + \tilde{\rho}^a D(-x,\sigma)\right]$$

For $x^2 < 0$ $D(x,\sigma) = D(-x,\sigma)$. The requiring $x^2 < 0 \Rightarrow C^{a\mu} = 0$, i.e. causality, yields

$$\rho^a(\sigma) = -\tilde{\rho}^a(\sigma)$$

$$\Rightarrow C^{a\mu} = -\partial^\mu \int d\sigma\, \rho^a(\sigma) i\Delta(x,\sigma) \tag{4.3.1}$$

where

$$\begin{aligned} i\Delta(x,\sigma) &= D(x,\sigma) - D(-x,\sigma) \\ &= \int \frac{d^4k}{(2\pi)^3}\delta(k^2-\sigma)\epsilon(k^0)e^{ik\cdot x} \\ \epsilon(k^0) &= \begin{cases} 1 & k^0 > 0 \\ -1 & k^0 < 0 \end{cases} \end{aligned}$$

Now

$$\partial_\mu j^{a\mu} = 0 \Rightarrow -\partial^2 \int d\sigma\, \rho^a(\sigma) i\Delta(x,\sigma) = 0$$

and the Klein Gordon equation gives

$$(\partial^2 + \sigma)\Delta(x,\sigma) = 0 \Rightarrow \int d\sigma\, \sigma\rho^a(\sigma) i\Delta(x,\sigma) = 0$$

For this second equation to hold $\forall x$, using that the norm of the states is positive definites so $\rho > 0$ gives

$$\sigma\rho(\sigma) = 0$$

This gives two possibilites

- $\forall\sigma\ \rho(\sigma) = 0 \Rightarrow C^{a\mu} = 0 \Rightarrow t^a\phi = 0$ (unbroken generator)
- $\rho^a(\sigma) = N^a\delta(\sigma)$ where N^a is a dimensionful non-zero constant.

In the second case substitute into 4.3.1 to get

$$\begin{aligned} C^{a\mu} &= -iN^a\partial^\mu\Delta(x,\sigma) \\ \Rightarrow \langle 0|\,[Q^a,\phi(0)]\,|0\rangle &= -iN^a \int d^3x\, \partial^0\Delta(x,0) = iN^a \\ \Rightarrow t^a\phi &= [,0|\,[Q^a,\phi(0)]\,|0] = iN^a \end{aligned}$$

Now some states in $\rho^{a\mu}, \tilde{\rho}^{a\mu}$ must be non zero. Label these $B(p)$ s.t.

$$\begin{aligned} \langle 0|j^{a\mu}(0)|B(p)\rangle &= iF^a_B p^\mu \quad F^a_B \text{ a dim 1 constant} \\ \langle B(p)|\phi(0)|0\rangle &= Z^B \quad Z^B \text{ dim 0 constant} \end{aligned}$$

$B(p)$ are spin 0 and massless as $\sigma = p^2 = 0$. Now

$$\begin{aligned}
i\rho^{a\mu}(k) &= ik^\mu \Theta(k^0) N^a \delta(k^2) \\
&= \sum_B \int \frac{d^3p}{2|\boldsymbol{p}|}\, \delta^{(4)}(k-p) \langle 0|j^{a\mu}(0)|B(p)\rangle \langle B(p)|\phi(0)|0\rangle \\
\Rightarrow \int \frac{d^3p}{2|\boldsymbol{p}|} \delta^{(4)}(k-p) ik^\mu N^a &= \int \frac{d^3p}{2|\boldsymbol{p}|} \delta^{(4)}(k-p) ip^\mu \sum_B F^a_B Z^B \\
\Rightarrow N^a &= \sum_B F^a_B Z^B
\end{aligned}$$

Hence we have n ρ^a which have non-zero contribution at $\sigma = 0$, so F^a_B is a rank n matrix gives we have n **Goldstone bosons**.

Note we've assumed Lorentz invariance in our theory with > 2 spacetime dimensions, and also that states have positive definite norm.

4.4 Abelian Higgs Mechanism

Gauge theories can violate this theorem, e.g. in QED, imposing a Lorentz invariance gauge condition (Lorentz gauge) can lead to states with a negative norm. Hence a gauge with no negative norm states breaks Lorentz invariance.

Consider scalar electrodynamics with complex scalar $\phi(x)$ and photon $A_\mu(x)$

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi) - V(\phi^* \phi) \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
D_\mu &= \partial_\mu + iqA_\mu
\end{aligned}$$

With $U(1)$ gauge invariance, $\phi(x) \to e^{i\alpha(x)}\phi(x)$, $\alpha \in \mathbb{R}$, and $A_\mu(x) \to A_\mu(x) - \frac{1}{q}\partial_\mu \alpha(x)$. Take

$$V(\phi^*\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4 \quad \lambda > 0$$

Then

- $\mu^2 > 0 \Rightarrow |\phi|^2$ is usual mass term for ϕ and there is a unique vacuum at $\phi = 0$.
- $\mu^2 < 0 \Rightarrow$ minima at $|\phi_0|^2 = -\frac{\mu^2}{2\lambda} = \frac{v^2}{2}$.

Wlog expand around real ϕ_0

$$\begin{aligned}
&\phi(x) = \frac{1}{\sqrt{2}} e^{i\frac{\theta(x)}{v}} (v + \eta(x)) \\
&\Rightarrow \mathcal{L} = \frac{1}{2}\left(\partial_\mu \eta \partial^\mu \eta - 2\lambda v^2 \eta^2\right) + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + qvA_\mu \partial^\mu \theta + \frac{q^2 v^2}{2} A_\mu A^\mu + \mathcal{L}_{int}
\end{aligned}$$

where $\mathcal{L}_{int}$ are terms with > 2 fields. Appear to have mass for η, A_μ but not θ. Transform to unitary gauge $\alpha(x) = -\frac{1}{v}\theta(x)$

$$\begin{aligned}
\phi \to e^{-i\frac{\theta}{v}}\phi = \frac{1}{\sqrt{2}}(v+\eta) \\
A_\mu \to A_\mu + \frac{1}{vq}\partial_\mu \theta \\
\mathcal{L} = \frac{1}{2}(\partial_\mu \eta \partial^\mu \eta - 2\lambda v^2 \eta^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^2 v^2}{2}A_\mu A^\mu + \mathcal{L}_{int}
\end{aligned}$$

Hence

- Photon has mass $m_A^2 = q^2 v^2$
- Scalar η has mass $m_\eta^2 = 2\lambda v^2 = -2\mu^2$
- Goldstone modes θ has been 'eaten' to become longitudinal polarisation of A_μ.

Now

$$\mathcal{L}_{int} = \frac{q^2}{2}A_\mu A^\mu \eta^2 q m_A A_\mu A^\mu \eta - \frac{\lambda}{4}\eta^4 - m_\eta \sqrt{\frac{\lambda}{2}}\eta^3$$

4.5 Non Abelian Gauge Theories (SU(N))

Consider the transform

$$\psi_i(x) \to U_{ij}(x)\psi_j(x) = \exp\left(it^a\theta^a(x)\right)_{ij}\psi_j(x)$$

where the U are matrices for an n-dimensional representation R of a unitary Lie group, and t^a are the hermitian generators of R forming a Lie algebra. Then

$$\overline{\psi}_i(x) \to \overline{\psi}_j(x)(U^\dagger(x))_{ji} = \overline{\psi}_j(x)\exp\left(-it^a\theta^a(x)\right)_{ji}$$

Let the Lie algebra be defined by

$$\left[t^a, t^b\right] = if^{abc}t^c$$

with

$$\mathrm{Tr}(t^a t^b) = T(R)\delta^{ab}$$

as the normalisation, $T(R)$ the **Dynkin index** of the representation. (Note for the fundamental rep of SU(N) $T(R) = \frac{1}{2}$). The covariant derivative is

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + ig(t^a A_\mu^a)_{ij}$$

we want

$$\begin{aligned}
(D_\mu\psi)_i \to (UD_\mu\psi)_i \\
\text{s.t} \quad \mathcal{L} = \overline{\psi}_i(i\not{D}_{ij} - m\delta_{ij})\psi_j
\end{aligned}$$

Hence the gauge field transformation

$$t^a A^a_\mu \to U t^a A^a_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$$

The infinitesimal transform is

$$\delta A^a_\mu = -\frac{1}{g}\partial_\mu \theta^a - f^{abc}\theta^b A^c_\mu$$

Then

$$\begin{aligned}
[D_\mu, D_\nu] &= igt^a F^a_{\mu\nu} \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu \\
F_{\mu\nu} &= F^a_{\mu\nu}t^a \\
\mathcal{L}_{gauge} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} &= -\frac{1}{2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}
\end{aligned}$$

5 Electroweak Theory

Electroweak theory uses Higgs mechanics to break $SU(2)_L \times U(1)_Y \to U(1)_{EM}$ to give gauge bosons mass.

5.1 EW Gauge Theory (Gauge and Higgs part)

The gauge symmetry is $SU(2)_L \times U(1)_Y$. The complex scalar (Higgs) field is the fundamental (doublet) representation of $SU(2)_L$, and hypercharge $Y = \frac{1}{2}$. Under gauge transform

$$\begin{aligned}
\phi(x) &\to e^{i\alpha^a(x)\tau^a} e^{i\frac{\beta(x)}{2}}\phi(x) \\
\tau^a &= \frac{1}{2}\sigma^a \quad \sigma^a \text{ the Pauli matrices} \\
\mathcal{L}_{gauge} = -\frac{1}{2}\operatorname{Tr} F^W_{\mu\nu}F^{W\mu\nu} - \frac{1}{4}F^B_{\mu\nu}F^{B\mu\nu} &+ (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2|\phi|^2 - \lambda|\phi|^4 \quad \lambda > 0 \\
D_\mu\phi &= (\partial_\mu + ig \underbrace{W^a_\mu}_{SU(2)\text{ gauge bosons}} \tau^a + ig'\frac{1}{2}\underbrace{B_\mu}_{U(1)\text{ gauge bosons}})\phi \\
F^{Wa}_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\epsilon^{abc}W^b_\mu W^c_\nu \\
F^B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu
\end{aligned}$$

SSB occurs when $\mu^2 = -\lambda v^2 < 0$, and the scalar acquires a VEV, wlog $\phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$. Hence we get $U(1)_{EM}$ ($\alpha^1 = 0 = \alpha^2, \alpha^3 = \beta$). Some gauge fields get mass: $(D_\mu\phi)^\dagger(D^\mu\phi)$ contains

$$\frac{1}{2}\frac{v^2}{4}\left[g^2(W')^2 + g^2(W^2)^2 + (-gW^3 + g'B)^2\right]$$

Define $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$, then these get mass $m_Q = \frac{vg}{2}$

$$\begin{pmatrix} Z_\mu^0 \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$$
$$\Rightarrow \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}$$
$$\Rightarrow \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}$$

giving

$$m_2 = \frac{v}{2}\sqrt{g^2 + g'^2}$$
$$m_\gamma = 0 \quad \text{(massless photon)}$$

We see $m_W = m_2 \cos\theta_W$, where we call θ_W the **Weinberg angle**. Experimentally we find

$$m_W \approx 80GeV$$
$$m_2 \approx 91GeV$$
$$m_\gamma < 10^{-18}GeV$$

The Higgs boson get mass $m_H = \sqrt{2\lambda v^2}$ (λ and so m_H not predicted), but experimentally $m_H \approx 125GeV$. There are $W^\pm, Z$- Higgs interaction, but there are no Higgs - photon interactions (i.e. Higgs chargeless)

5.2 Coupling to matter (fermions) - leptons

Leptons to start with (quarks similar but some complications).

$$\begin{aligned}
D_\mu &= \partial_\mu + igW_\mu^a T^a + ig'YB_\mu \\
&= \partial_\mu + \frac{ig}{\sqrt{2}}(W_\mu^- T^+ + W_\mu^- T^-) + \frac{igZ_\mu}{\cos\theta_W}(\cos^2\theta_W T^3 - \sin^2\theta_W Y) + \underbrace{ig\sin\theta_W}_{=e} A_\mu \underbrace{(T^3 + Y)}_{Q} \\
&= \partial_\mu + \frac{ig}{\sqrt{2}}(W_\mu^- T^+ + W_\mu^- T^-) + \frac{igZ_\mu}{\cos\theta_W}(T63 - \sin^2\theta_W Q) + ig\sin\theta_W A_\mu (T^3 + Y)
\end{aligned}$$

with $T^\pm = T^1 \pm iT^2$.

note

- Experimentally $W^\pm$ only conuple to LH leptons and quarks, hence RH fermions are in the trivial/scalar representation of $SU(2)$ where $T^a = 0$, e.g. $R(X) = e_R(X)$ where

$$e_{R/L} = \frac{1}{2}(1 + \pm\gamma^5)e(X)$$

  and LH fermions in the fundamental representation of $SU(2)$ where $T^a = \tau^a = \frac{\sigma^a}{2}$, e.g.

$$L(X) = \begin{pmatrix} \nu_{eL}(X) \\ e_L(X) \end{pmatrix}$$

  where $\nu_{eL}(X), e_L(X)$ are 4-component Dirac spinors.

- Assuming (for now) neutrinos are massless and LH only, then for $R(X)$

$$Q = Y = -1$$

and for $L(X)$

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \; Q = \tau^3 + YI \Rightarrow Y = -\frac{1}{2}$$

$$\mathcal{L}_{lept}^{EW} = \overline{L} i \not{D} L + \overline{R} i \not{D} R$$

fermion mass terms explicitly break $SU(2)_L \times U(1)_Y$ gauge invariance, but consider the Higgs-fermion interactions

$$\mathcal{L}_{lept,\phi} = -\sqrt{2}\lambda_e(\overline{L}\phi R + \overline{R}\phi^\dagger L) \quad \lambda_e = \text{ Yukawa coupling}$$

We may check $\sum Y = 0$ for each term, and $SU(2)$ gauge invariance for each term. Working in the unitary gauge and expanding

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

$$\Rightarrow \mathcal{L}_{lept,\phi} = -\lambda_e(v+h)(\overline{e}_L e_R + \overline{e}_R e_L) = -\underbrace{m_e}_{=\lambda_e v} \overline{e}e - \lambda_e h \overline{e}e$$

The second term is a fermion Higgs coupling α mass of fermion. The gauge-fermion interactions are

$$\begin{aligned} \mathcal{L}_{lept}^{EM,int} &= -g\overline{L}\gamma^\mu T^a W_\mu^a L - g'(-\frac{1}{2}\overline{L}\gamma^\mu L - \overline{R}\gamma^\mu R)B_\mu \\ &= -\frac{g}{2\sqrt{2}}(J^\mu W_\mu^+ + J^{\mu\dagger} W_\mu^-) - eJ_{EM}^\mu A_\mu - \frac{g}{2\cos\theta_W} J_n^\mu Z_\mu \end{aligned}$$

where

$$J_{EM}^\mu = \frac{1}{2}\overline{L}\gamma^\mu(\sigma^3 - I)L - \overline{R}\gamma^\mu R = -\overline{e}\gamma^\mu e \quad \text{(EM current)}$$

$$J^\mu = \overline{\nu}_{eL}\gamma^\mu(1-\gamma^5)e \quad \text{(charged weak current)}$$

$$J_n^\mu = \frac{1}{2}\left[\overline{\nu}_{eL}\gamma^\mu(1-\gamma^5)\nu_{eL} - \overline{e}\gamma^\mu(1-\gamma^5 - 4\sin^2\theta_W)e\right] \quad \text{(neutral weake current)}$$

The standard model has *three* generations of leptons, e, μ, τ.

$$\begin{aligned} L^1 &= \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ L^2 &= \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \\ L^3 &= \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \\ R^1 &= e_R \\ R^2 &= \mu_R \\ R^3 &= \tau_R \end{aligned}$$

$$\mathcal{L}_{lept,\phi} = -\sqrt{2}(\lambda^{ij}\overline{L}^i\phi R^j - (\lambda^\dagger)^{ij}\overline{R}^i\phi^\dagger L^j)$$

Note λ^{ij} (3x3 matrices) are *not* predicted by SM. Now $\lambda\lambda^\dagger$ is a Hermitian matrix with non-negative eigenvalues, so $\exists K$ a unitary matrix s.t

$$\lambda\lambda^\dagger = K\Lambda^2 K^\dagger$$

where Λ^2 is diagonal with non negative eigenvalues. Choose

$$\begin{aligned} S &= \lambda^\dagger K \Lambda^{-1} \quad \text{unitary} \\ &\Rightarrow \lambda^\dagger\lambda = S\Lambda^2 S^\dagger \\ &\Rightarrow \lambda = K\Lambda S^\dagger \end{aligned}$$

If $L^i \to K^{ij}L^j, R^i \to S^{ij}R^j \Rightarrow \mathcal{L}_{lept,\phi}$ diagonalised and $\mathcal{L}^{EW}_{lept}$ unchanged. Simultaneous diagonaliseability implies mass eigenstates are also weak eigenstates.

5.3 Quark Flavour

There are 6 flavours of quark in nature (as we know it).

- RH are in $SU(2)$ singlets : $u^i_R = (u_R, c_R, t_R)$, $Y = Q = \frac{2}{3}$, $d^i_R = (d_R, s_R, b_R)$, $Y = Q = -\frac{1}{3}$.
- LH are in $SU(2)$ doublets :

$$Q^i_L = \begin{pmatrix} u^i_L \\ d^i_L \end{pmatrix} = \left(\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right)$$

$Y = \frac{1}{6}$ and $Q = T_3 + Y$.

Then

$$\begin{aligned} \mathcal{L}^{EW}_{quark} &= \overline{Q}^i_L i\not{D} Q^i_L + \overline{u}^i_R i\not{D} u^i_R + \overline{d}^i_R i\not{D} d^i_R \\ \mathcal{L}_{quark,\phi} &= -\sqrt{2}\left[\lambda^{ij}_d \overline{Q}^i_L \phi d^i_R + \lambda^{ij}_u \overline{Q}^i_L \phi^c u^j_R + \text{hermitian conjugate}\right] \end{aligned}$$

where

$$(\phi^c)^\alpha = \epsilon^{\alpha\beta}{\phi^\dagger}^\beta$$

transforms in the fundamental representation of $SU(2)$. This term is needed to ensure gauge invariance. Note each term has $\sum Y = 0$.
Diagonalising λ_u and λ_d as for letptons

$$\begin{aligned} \lambda_u &= K_u \Lambda_u S^\dagger_u \\ \lambda_d &= K_d \Lambda_d S^\dagger_d \end{aligned}$$

with Λ diagonal, K, S unitary. Then the quark fields transform as

$$\begin{aligned} u_i &\to K_u u_i \\ d_i &\to K_d d_L \\ u_R &\to S_u u_R \\ d_R &\to S_d d_R \end{aligned}$$

$$\lambda_d^{ij} Q_L^i \phi d_R^i \to \overline{Q}_L^i \phi \Lambda_d^{ij} d_R^j$$

(recall $\phi = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 \\ v+h(x) \end{smallmatrix}\right))$ and the $\phi = \phi_0$ term gives

$$-\sum_i (m_d^i \overline{d}_L^i d_R^i + m_i^i \overline{u}_L^i u_R^i + \text{hermitian conjugate})$$

where $m_q^i = \Lambda_q^{ii} v$ (no sum).
In this basis, $\mathcal{L}_{quark,phi}$ is C, P, T invariant, as is $\mathcal{L}_{gauge,\phi}$. As originally written, $\mathcal{L}_{quark}^{EW}$ violates C and P but conserves CP and T. *However*, this basis transform has an effect on $|mcL_{quark}^{EW}$, $\overline{u}_R^i i\not{D} u_R^i$ and $\overline{d}_R^i i\not{D} d_R^i$ are unchanged but the $W_\mu^\pm$ piece in $\overline{Q}_L^i i\not{D} Q_L^i$ is transformed (in $-\frac{g}{2\sqrt{2}} J^{\mu\pm} W_\mu^\pm$

$$J^{\mu\pm} \propto \overline{u}_L^i \gamma^\mu d_L^i \to \overline{u}_L^i \gamma^\mu \underbrace{(K_u^\dagger K_d)^{ij}} d_L^i$$

Interactions with $W^\pm$ lead to inter-generational quark couplings, weak eigenstates are linear combinations of mass eigenstates.
The **Cabibbo-Kobyashi-Maskawa** (CKM) matrix is

$$V_{CKM} = K_u^\dagger K_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}.$$

V_{CKM} is unitary

- For 2 generations (Cabibbo mixing) there are 4 parameters, one angle and three phases. However, redefining each of the 4 quark fields (u,d,s,c) with a global $U(1)$ transform we can remove 3 phases (relative phases) so we end up with one angle θ_c = Cabibbo angle. Then

  $$V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$$

  Experimentally $\sin\theta_c \approx 0.22$. That V is real implies CP conservation.

  $$\frac{1}{2} J^{\mu\pm} = \cos\theta_c \overline{u}_L \gamma^\mu d_L + \sin\theta_c \overline{u}_L \gamma^\mu s_L - \sin\theta_c \overline{c}_L \gamma^\mu d_L + \cos\theta_c \overline{c}_L \gamma^\mu s_L$$

- For 3 generations, there are 9 parameters, 3 angles and 6 phases. We can remove 5 phases through $U(1)$ transform giving V_{CKM} in terms of 3 angles and 1 phases, so in general V_{CKM} is no real, giving CP and T violation.

5.4 Neutrino oscillations and mass

We have known since the start of the Millennium that weak and mass eigenstates for neutrinos are not equivalent, so at least some neutrinos has mass. The analogous mixing matrix is U_{PMNS}. If neutrinos are

- Dirac fermions : 3 angles and 1 phase (CP violation)
- Majorana fermions : 3 angles and 3 phases (CP violation)

Dirac Fermions

$$N^i = \mathcal{V}_R^i = (\mathcal{V}_{eR}, \mathcal{V}_{\mu R}, \mathcal{V}_{\tau R})$$

must occur and modify

$$\mathcal{L}_{lept,\phi} = -\sqrt{2}(\lambda^{ij}\overline{L}^i \phi P^j + \lambda_\nu^{ij}\overline{L}^i \phi^c N^j + \text{ hermitian conjugate})$$

This has the same structure as for quarks, so neutrinos would get mass terms

$$-\sum_i m_\nu^i (\overline{\mathcal{V}}_R^i \mathcal{V}_L^i + \overline{\mathcal{V}}_L^i \mathcal{V}_R^i)$$

Majorana fermions

Because $\mathcal{V}$ are neutral, they could be their own antiparticle, i.e. $d^s(p) = b^s(p)$

$$\mathcal{V} = \sum_{p,s} \left[b^s(p) u^s(p) e^{-ip\cdot x} + b^{s\dagger}(p) v^s(p) e^{ip\cdot x} \right]$$

Taking intrinsic c-partiy to be 1 wlog,

$$\mathcal{V}^c = C\overline{\mathcal{V}}^T = C(C^{-1}\mathcal{V}) = \mathcal{V}$$

and so

$$\mathcal{V}_R = \frac{1}{2}(1+\gamma^5)\mathcal{V} = \frac{1}{2}(1+\gamma^5)\mathcal{V}^c = \mathcal{V}_L^c$$

Hence the RH neutrino field is not an independent field.

Mass terms are

$$-\frac{1}{2}\sum_i m_\nu^i (\overline{\mathcal{V}}_L^{i\ c} \mathcal{V}_L^i + \overline{\mathcal{V}}_L^i \mathcal{V}_L^{i\ c})$$

To arise form a Higgs VEV, need term

$$\sim -\frac{Y^{ij}}{M} \left(L^{i^T} \tilde{\phi} \right) C \left(\tilde{\phi}^T L^j \right) + \text{ Hermitian conjugate}$$

where $\tilde{\phi}_a = \epsilon_{ab}\phi_b$. The 5-dimensional operator is non renormalisable, so we need M to have dimensions of mass. This is still ok as long as we think of SM as an effective field theory, valid for physics below some energy cutoff

5.5 Summary of EW theory

- $\mathcal{L}_{gauge,\phi}$ - Masses for $W^\pm, Z$ and Higgs bosons, as well as W, Z-Higgs, and Higgs-Higgs interactions.
- $\mathcal{L}_{lept,\phi}$ - Lepton masses and lepton-Higgs interactions

- $\mathcal{L}^{EW}_{lept}$ - Lepton interactions with W, Z, γ bosons (PMNS matrix : $\mathcal{V}$ mixing and CP violation possibly)
- $\mathcal{L}_{quark,\phi}$ - Quark masses and quark-Higgs interactions
- $\mathcal{L}^{EW}_{quark}$ - Quark interactions with W, Z, γ boson (CKM matrix : flavour mixing and CP violation)

6 Weak Interactions

6.1 Effective Lagrangian

We'll consider some processes due to weak interactions where energies, momenta $\ll m_W, m_Z$ so we can use an effective field theory (Fermi weak $\mathcal{L}$)
The weak interaction part of the EW theory $\mathcal{L}$ is

$$\mathcal{L}_W = -\frac{g}{2\sqrt{2}}(J^\mu W^+_\mu J^{\mu\dagger} W^-_\mu) - \frac{g}{2\cos\theta_W} J^\mu_n Z_\mu$$

and the S-matrix is

$$S = \mathcal{T} \exp\left[i\int d^4x\, \mathcal{L}_W(x)\right]$$

For small g we can expand in a Taylor series

$$\langle f|S|i\rangle = \langle f|\mathcal{T}\left\{1 - \frac{g^2}{8}\int d^4x\, d^4x' \left[J^{\mu\dagger}(x) D^W_{\mu\nu}(x-x') J^\nu(x') + \frac{1}{\cos^2\theta_W} J^{\mu\dagger}_n(x) D^2_{\mu\nu}(x-x') J^\mu_n(x')\right] + \mathcal{O}(g^4)\right\}|i\rangle$$

where we've assumed that W,Z are not in the initial final state, used wicks theorem and

$$D^W_{\mu\nu}(x-x') = \langle 0|T\left\{W^-_\mu(x) W^+_\nu(x')\right\}|0\rangle$$
$$D^Z_{\mu\nu}(x-x') = \langle 0|T\left\{Z_\mu(x) Z_\nu(x')\right\}|0\rangle$$

In terms of an integral, we have:

$$D^{Z/W}_{\mu\nu}(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \tilde{D}^{Z/W}_{\mu\nu}(p)$$

where

$$\tilde{D}^{Z/W}_{\mu\nu}(p) = \frac{i}{p^2 - m^2_{Z/W} + i\epsilon}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2_{Z/W}}\right)$$

At low energies (for example decays of leptons or quarks, except for the top quark which has mass comparable to $m_{Z/W}$) $m^2_{Z/W} \gg p^2$ where p is any combination of initial and final state momenta. In this regime, we can approximate the propagators by

$$\tilde{D}^{Z/W}_{\mu\nu}(p) \approx \frac{ig_{\mu\nu}}{m^2_{Z/W}}$$

$$\Rightarrow D^{Z,W}_{\mu\nu}(x-x') \approx \frac{ig_{\mu\nu}}{m^2_{Z/W}} \delta^4(x-x')$$

Therefore we can descirbe the interactions in the Lagrangian by a 4-fermion interaction called a **contact interaction**

Example 6.1.

$$-\frac{g^2}{8}J^{\mu\dagger}(x)D^{Z,W}_{\mu\nu}(x-x')J^\nu(x') \to -\frac{ig^2 g_{\mu\nu}}{8m_W^2}J^{\mu\dagger}(x)J^\nu(x')\delta^4(x-x')$$

Similarly for the neutral current J_n^μ part.

The effective weak Lagrangian is

$$i\mathcal{L}_W^{eff}(x) = -\frac{iG_F}{\sqrt{2}}(J^{\mu\dagger}(x)J_\mu(x) + \rho J_n^{\mu\dagger}(x)J_{n\mu}(x))$$

where $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$ is the **Fermi constant** and $\rho = \frac{m_W^2}{m_Z^2\cos^2\theta_W}$[3]. This theory descibres Fermi nuclear beta decay, and was discovered long before the standard model.

We can write

$$\rho = 1 + \underbrace{\Delta\rho}_{\substack{\text{quantum}\\ \text{loop effects}}}$$

Re exponentiating the S matrix we get that

$$\begin{aligned}\langle f|S|i\rangle &\approx \langle f|T\left\{1 + i\int d^4x\,\mathcal{L}_W^{eff}(x)\right\}|i\rangle \\ &= \langle f|T\exp\left(i\int d^4x\,\mathcal{L}_W^{eff}(x)\right)|i\rangle\end{aligned}$$

- Note the dimensions, $[G_f] = -2$ to compensate for having a (mass)^6 operator. So this theory is non-renormalisable. In particular, we can't think of this theory as valid to arbitrary high energy scales. This is fine at energies much less than m_W. The $\frac{1}{m_W}$ in G_F indicates that Fermi theory breaks down at scales $\sim m_W$, which is what we'd expect from how we constructed the theory.

6.2 Decay Rates and Cross Section

The questions we can ask of particle physics experiments boil down to

- How frequently does X decay to $A_1 + A_2 + \dots$
- Given N collision between two things A, B how many times do we produce $X_1 + X_2 + \dots$

Definition 6.2 (Decay Rate)**.** *The* ***decay rate*** Γ_X *is the number of decays of* X *per unit time in the rest frame of* X*, divided by the number of* X *present.*

Definition 6.3 (Lifetime)**.** *The* ***lifetime*** *is defined by* $\tau_X = \frac{1}{\Gamma_X}$.

[3] Tree level value of this is 1, loop effects correct this

We can write $\Gamma_X = \sum_i \Gamma_{X\to f_i}$ where f_i are the possible final states and $\Gamma_{X\to f_i}$ is the **partial decay rate** to final state f_i.

We need $\langle f|S|i\rangle$ with initial state $|i\rangle = |X\rangle$. Removing the boring 1 from the S matrix, corresponding to the case where nothing happens, we define the **invariant amplitude**

$$\langle f|S-I|i\rangle = (2\pi)^4\delta^4(p_f - p_i)i \underbrace{\mathcal{M}_{fi}}_{\substack{\text{invariant}\\ \text{amplitude}}}$$

The probability of decay is given by

$$P(i\to f) = \frac{|\langle f|S-I|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}$$

In an infinite volume, $|i\rangle, |f\rangle$ are typically not normalisable. It is possible to take $|i\rangle, |f\rangle$ as wavepackets but we'll opt instead to work in a finite spatial volume V with finite temporal extent T (to avoid the subtleties of non-renormalisable states). This means we must turn delta functions into finite numbers

$$\begin{aligned}(2\pi)^3\delta^3(0) &\to V\\ (2\pi)^4\delta^4(0) &\to VT\end{aligned}$$

We normalise the initial state as

$$\langle i|i\rangle = (2\pi)^3 \cdot 2p_i^0\delta^3(0) \to 2p_i^0 V$$

The final state is normalised as

$$\langle f|f\rangle = \prod_r (2p_r^0 V)$$

Then

$$P(i\to f) = \frac{|\mathcal{M}_{f,i}|^2(2\pi)^4\delta^4(p_f-p_i)VT}{2\underbrace{m_i}_{\substack{p_i^0=m_i\\ \text{in rest frame}}} V\cdot\prod_r(2p_r^0V)}$$

We never measure with infinite precision,

$$\Gamma_{i\to f} = \frac{1}{T}\int P(i\to f)\prod_r\left(\frac{V}{(2\pi)^3}d^3p_r\right)$$

This is the answer, but it is not manifestly Lorentz invariant. Recall that a Lorentz invariant integration measure is

$$d\rho_f = (2\pi)^4\delta^4\left(p_i - \sum_r p_f\right)\prod_r\left(\frac{d^3p_r}{(2\pi)^3\cdot 2p_r^0}\right)$$

Thus

$$\Gamma_{i\to f} = \frac{1}{2m_i}\int |\mathcal{M}_{fi}|^2 d\rho_f$$

the total decay rate is then

$$\Gamma_i = \frac{1}{2m_i}\sum_f \int |\mathcal{M}_{fi}|^2 d\rho_f$$

The $\frac{1}{2m_i}$ means that this is defined *only* in the rest frame of the decaying particle.

6.3 Cross Section

Let n be the number of scattering events per unit time per unit particle. The **incident flux** is F, which is the number of incoming particles per unit area per unit time

$$F = \underbrace{|\boldsymbol{v}_a - \boldsymbol{v}_b|}_{\substack{\text{relative velocity}\\\text{of incident}\\\text{beam compared}\\\text{to target}}} \rho_a$$

The **cross section** is define to be

$$\sigma = \frac{n}{F}$$

This indeed has units of area. The total number of scattering events per unit time is then

$$\begin{aligned} N &= n\rho_b B \\ &= F\sigma\rho_b V \\ &= |\boldsymbol{v}_a - \boldsymbol{v}_b|\rho_a\rho_b V\sigma \end{aligned}$$

Our normalisation corresponds to having 1 particle per unit volume, so $\rho_a = \rho_b = \frac{1}{V}$ so

$$N = \frac{|\boldsymbol{v}_a - \boldsymbol{v}_b|\sigma}{V}$$

Normally we think about **differential cross sections**

$$dN = \frac{|\boldsymbol{v}_a - \boldsymbol{v}_b|}{V}d\sigma$$

We now go through the same calculation as with the decay rate. The difference is the probability

$$\frac{|\langle f|S-1|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}$$

this will give a differential contribution since $|i\rangle$ now contains two particles, hence we get an extra factor $2E_bV$ in the denominator

$$|i|i\rangle \to (2E_aV)(2E_bV)$$

So, using the decay rate derivation, we find

$$dN = \frac{1}{2E_a 2E_b V}|\mathcal{M}_{fi}|^2 d\rho_f$$

$$\Rightarrow d\sigma|\boldsymbol{v}_a - \boldsymbol{v}_b| = \frac{1}{2E_a 2E_b}|\mathcal{M}_{fi}|^2 d\rho_f$$

$$\Rightarrow d\sigma = \frac{|\mathcal{M}_{fi}|^2}{4E_a E_b|\boldsymbol{v}_a - \boldsymbol{v}_b|}d\rho_f$$

Remark. *Cross sections are often measure in* ***barns****. We have* $1barn = 10^{-28}m^2$

6.4 Muon Decay

Consider the decay of a muon into an electron, neutrino, and anti-neutrino: $\mu^- \to e^-\overline{\nu}_e\nu_\mu$. In Fermi effective theory this is described by the tree level diagram

diagram

Recallthe Lagrangian is

$$\mathcal{L}_W^{eff} = -\frac{G_F}{\sqrt{2}}(J^{\alpha\dagger}J_\alpha + \rho J_N^{\alpha\dagger}J_{n\alpha})$$

Note that since this is a flavour changing interaction, no Z bosons are involved, hence the entire interaction comes from the $J^{\alpha\dagger}J_\alpha$ term.[4].

We assume the neutrinos are massless, $m_{\nu_e} = m_{\nu_\mu} = 0$. Then recalls

$$J^\alpha = \overline{\nu}_e\gamma^\alpha(1-\gamma^5)e + \overline{\nu}_\mu\gamma^\alpha(1-\gamma^5)\mu + \overline{\nu}_\tau\gamma^\alpha(1-\gamma^5)\tau$$

We should also check we can validly apply Fermi theory here. Note $m_\mu \approx 106MeV$ and $m_W \approx 80GeV \Rightarrow m_W \gg m_\mu$. Thus Fermi theory indeed applies. The amplitude is given by

$$\mathcal{M} = \langle e^-(k)\overline{\nu}_e(q)\nu_\mu(q')|\mathcal{L}_W^{eff}|\mu^-(p)\rangle$$

We have

$$\begin{aligned}
\mathcal{M} &= -\frac{G_F}{\sqrt{2}}\langle e^-(k)\overline{\nu}_e(q)|\overline{e}\gamma^\alpha(1-\gamma^5)\nu_e|0\rangle \cdot \langle\nu_\mu(q')|\overline{\nu}_\mu\gamma_\alpha(1-\gamma^5)\mu|\mu^-(p)\rangle \\
&= \dots \\
&= -\frac{G_F}{\sqrt{2}}\left[\overline{u}_e(k)\gamma^\alpha(1-\gamma^5)v_{\nu_e}(q)\right]\left[\overline{u}_{\nu_\mu}(q')\gamma_\alpha(1-\gamma^5)u_\mu(p)\right]
\end{aligned}$$

We're not interested in particular final state spins, so we're going to sum over these. We also assume that we don't know the spin of the μ^- particle, but we will assume that $\pm\frac{1}{2}$ polarisations are produced with equal probability. So we average over the initial state spins. Hence we want

$$\begin{aligned}
\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{G_F^2}{4}\sum_{\text{spins}}\left[\overline{u}_e(k)\gamma^\alpha(1-\gamma^5)v_{\nu_e}(q)\overline{v}_{\nu_e}\gamma^\beta(1-\gamma^5)u_e(k)\right]\left[\overline{u}_{\nu_\mu}(q')\gamma_\alpha(1-\gamma^5)u_\mu(p)\overline{u}_\mu(p)\gamma_\beta(1-\gamma^5)u_{\nu_\mu}(q')\right] \\
&= \frac{G_F^2}{4}S_1^{\alpha\beta}S_{2\alpha\beta}
\end{aligned}$$

[4]More clearly, the Feynmann diagram we use in the full, non-Fermi theory is

To calculate we use

$$\sum_s u^s(p)\overline{u}^s(p) = \not{p} + m$$
$$\sum_s v^s(p)\overline{v}^s(p) = \not{p} - m$$

Then

$$S_1^{\alpha\beta} = \mathrm{Tr}\left[(\not{k} + m_e)\gamma^\alpha(1-\gamma^5)\not{q}\gamma^\beta(1-\gamma^5)\right]$$
$$S_{2\alpha\beta} = \mathrm{Tr}\left[\not{q}'\gamma_\alpha(1-\gamma^5)(\not{p} + m_\mu)\gamma_\beta(1-\gamma^5)\right]$$

we now use the results of tracing over gamma matrices.

$$\mathrm{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n} = 0 \quad (nodd)$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$
$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$$

Then get

$$S_1^{\alpha\beta} = 8[k^\alpha q^\beta + k^\beta q^\alpha - (k\cdot q)g^{\alpha\beta} - i\epsilon^{\alpha\beta\mu\rho}k_\mu q_\rho]$$

and similarly

$$S_{2,\alpha\beta} = 8[q'_\alpha p_\beta + q'_\beta p_\alpha - (p.q')g_{\alpha\beta} - i\epsilon_{\alpha\beta\mu\rho}q'^\mu p^\rho]$$

so

$$\frac{1}{2}\sum_{\text{spins}}|M|^2 = G_F^2 \cdot 64(p\cdot q)(k\cdot q')$$

Consider the case where θ^-, ν_μ go out along $+z$, and $\overline{\nu}_e$ along $-z$, $k\cdot q' = \sqrt{m_e^2 + k_z^2}q'_z - k'_z \to 0$ as $m_e \to 0$. Weak interactions only couples to LH parts, so in a colinear decay $|S_Z| = \frac{3}{2} > S_\mu$, ie. can't conserve angular momentum, so this particular final state can't happen if $m_e = m_{\nu_\mu} = m_{\nu_e} = 0$. If $m_e \neq 0$ then LH and RH components are coupled ($-\frac{1}{2}$ helicity not the same as LH chirality) so the decay may occur $|S_z| = \frac{1}{2} = S_\mu$
Now

$$\Gamma = \frac{1}{2m_\mu}\int\frac{d^3k}{(2\pi)^3 2k^0}\int\frac{d^3q}{(2\pi)^3 2q^0}\int\frac{d^3q'}{(2\pi)^3 2q'^0}(2\pi)^4\delta^{(4)}(p-k-q-q')\frac{1}{2}\sum_{\text{spins}}|M|^2$$
$$= \frac{G_F^2}{8\pi^5 m_\mu}\int\frac{d^3k}{k^0}\int\frac{d^3q}{|\boldsymbol{q}|}\int\frac{d^3q'}{|\boldsymbol{q}'|}\delta^{(4)}(p-k-q-q')(p\cdot q)(k\cdot q')$$

$$I_{\mu\nu}(p-k) = \int\frac{d^3q}{|\boldsymbol{q}|}\int\frac{d^3q'}{|\boldsymbol{q}'|}\delta^{(4)}(p-k-q-q')q_\mu q'_\nu$$
$$= a(p-k)_\mu(p-k)_\nu + bg_{\mu\nu}(p-k)\cdot(p-k)$$

where a,b, depends on (p-k). Then

$$g^{\mu\nu} I_{\mu\nu} = \int \ldots q \cdot q' = a(p-k)\cdot(p-k) + 4b(p-k)\cdot(p-k)$$

$$\Rightarrow a + 4b = \frac{I}{2}$$

where

$$I = \int \frac{d^3q}{|\boldsymbol{q}|} \frac{d^3q'}{|\boldsymbol{q}'|} \delta^{(4)}(p-k-q-q')$$

Further

$$(p-k)^\mu (p-k)^\nu I_{\mu\nu} = a(p-k)^4 + b(p-k)^4$$

$$\Rightarrow a + b = \frac{I}{4}$$

so

$$a = \frac{I}{6},\ b = \frac{I}{12}$$

I is a Lorentz scalar, so evaluate it in a frame with $\boldsymbol{p} - \boldsymbol{k} = 0 \Rightarrow \boldsymbol{q} = -\boldsymbol{q}'$. Then

$$I = \int \frac{d^3q}{|\boldsymbol{q}|^2} \delta(p^0 - l^0 - 2|\boldsymbol{q}|) = 4\pi \int d|\boldsymbol{q}|\, \delta(p^0 - k^0 - 2|\boldsymbol{q}|) = 2\pi$$

so $a = \frac{\pi}{3}, b = \frac{\pi}{6}$. Back to Γ:

$$\Gamma = \frac{G_F^2}{(2\pi)^4 3m_\mu} \int \frac{d^3k}{k^0} \left[2p\cdot(p-k)k\cdot(p-k) + (p\cdot k)(p-k)^2\right]$$

Recall that Γ is given in the rest frame of the decaying particle, so $p \cdot k = m_\mu E(E - k^0), p \cdot p = m_\mu^2, k \cdot k = m_e^2$. Note $\frac{m_e}{m_\mu} \approx 0.0048 \ll 1$, so approximating $m_e = 0$ is reasonable.
Hence

$$\Gamma = \frac{G_F^2 m_\mu}{3(2\pi)^4} \int d^3k\,(3m_\mu - 4E) = \frac{4\pi G_F^2 m_\mu}{3(2\pi)^4} \int_0^{\frac{m_\mu}{2}} dE\, E^2(3m_\mu - 4E)$$

(Note: we get the integration limits by saying at E_{min}, e is at rest, so $E = m_e = 0$, and at E_{max}, $\nu_\mu, \overline{\nu}_e$ are in the same direction, but opposite to e, so

$$E_{max} + (E_{\nu_\mu} + E_{\overline{\nu}_e}) = m_\mu \quad \text{(conservation of energy)}$$
$$E_{max} - (E_{\nu_\mu} + E_{\overline{\nu}_e}) = 0 \quad \text{(conservation of momentum)}$$
$$\Rightarrow E_{max} = \frac{m_\mu}{2}$$

Finally.

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

is the decay rate of $\mu^- \to e\overline{\nu}_e\nu_\mu$. This is the only allowed decay of μ, so no more calculations. The lifetime of a muon is

$$\begin{aligned}\tau_\mu &= \frac{1}{\Gamma} \approx 2.1970 \times 10^{-6} s \\ \Rightarrow G_F &= 1.164 \times 10^{-5} GeV^2\end{aligned}$$

$$\Gamma = \frac{G_F^2 m_\mu^2}{192\pi^3}$$

- One loop corrections are $\sim 10^{-6}$ level
- Experimnetlaly, G_F is consisten from $\tau \to e\overline{\nu}_e\nu_\tau, \mu\overline{\nu}_\mu\nu_\tau \Rightarrow$ lepton universality,

  Weak decays violate P as, in the massless limit, RH ν_μ don't couple to W bosons.

6.5 Pion Decay

Consider

$$\pi^-(\overline{u}d) \to e^-\overline{\nu}_e$$

with $m_\nu = 0$. The d and $\overline{u}$ do not propagate freely as they are bound in the π^-. Relevant currents are

$$\begin{aligned}J^\alpha_{lept} &= \overline{\nu}_e\gamma^\alpha(1-\gamma^5)e \\ J^\alpha_{had} &= \overline{u}\gamma^\alpha(1-\gamma^5)(V_{ud}d + V_{us}s + V_{ub}b) \\ &= V^\alpha_{had} - A^\alpha_{had}\end{aligned}$$

Amplitude is

$$\begin{aligned}M &= \langle e^-(k)\overline{\nu}_e(q)|\mathcal{L}_W^{eff}|\pi^-(p)\rangle \\ &= -\frac{G_F}{\sqrt{2}}\langle e^-(k)\overline{\nu}_e(q)|\overline{e}\gamma_\alpha(1-\gamma^5)\nu_e|0\rangle \langle 0|J^\alpha_{had}|\pi^-(p)\rangle \\ &= -\frac{G_F}{\sqrt{2}}\overline{u}_e(k)\gamma_\alpha(1-\gamma^5)V_{\nu_e}(q)\langle 0|\underbrace{V^\alpha_{had}}_{\gamma^\alpha part} - \underbrace{A^\alpha_{had}}_{\gamma^\alpha\gamma^5 part}|\pi^-(p)\rangle\end{aligned}$$

Parametrise unknown non-perturbative QCD part in the pion decay constant F_π.

$$\langle 0|\overline{u}\gamma^\alpha\gamma^5 d|\pi^-(p)\rangle = i\sqrt{2}F_\pi p^\alpha$$

However

$$\langle 0|\overline{u}\gamma^\alpha d|\pi^-(p)\rangle = -\mathbb{P}^\alpha{}_\beta \langle 0|\overline{u}\gamma^\beta d|\pi^-(p_P)\rangle$$

so if

$$\begin{aligned}\langle 0|\overline{u}\gamma^\alpha d|\pi^-(p)\rangle = Ap^\alpha \Rightarrow -\mathbb{P}^\alpha{}_\beta \langle 0|\overline{u}\gamma^\beta d|\pi^-(p_P)\rangle &= Ap^\alpha \\ \Rightarrow -\langle 0|\overline{u}\gamma^\alpha d|\pi^-(p_P)\rangle &= Ap^\alpha_P \\ \Rightarrow A &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
M &= iG_F F_\pi V_{ud} \overline{u}_e(k) \not{p} (1-\gamma^5) V_{\nu_e}(q) \\
&= iG_F F_\pi m_e V_{ud} \overline{u}_e(k)(1-\gamma^5) V_{\nu_e}(q)
\end{aligned}
$$

as $p = q + k, \not{q} V_{\nu_e} = 0, \overline{u}_e \not{k} = m_e \overline{u}_e$. This again shows a **helicity suppression**. Spin 0 π decay conserves angular momentum. π^- decays to positive spin $\overline{\nu}_e$ and positive helicity e^-. If $m_e = 0$, this is Rh chirality of e^-, which is forbidden.
Sum over final spin states gives

$$
\begin{aligned}
\sum_{\text{spins}} |M|^2 &= \sum_{\text{spins}} |G_F F_\pi m_e V_{ud}|^2 \left[\overline{u}_e(k)(1-\gamma^5)V_{\nu_e}(q) \times \overline{V}_{\nu_e}(q)(1-\gamma^5)u_e(k)\right] \\
&= 8|G_F F_\pi m_e V_{ud}|^2 (k \cdot q) \\
\Rightarrow \Gamma_{\pi \to e \overline{\nu}_e} &= \frac{1}{2m_\pi} \int \frac{d^3k}{(2\pi)^3 2k^0} \int \frac{d^3q}{(2\pi)^3 2q^0} (2\pi)^4 \delta^{(4)}(p-k-q) 8 |G_F F_\pi m_e V_{ud}|^2 (k \cdot q) \\
&= \frac{|G_F F_\pi m_e V_{ud}|^2}{4 m_\pi \pi^2} \int \frac{d^3k}{E|\boldsymbol{k}|} \delta(m_\pi - E - |bmk|) \left(E|\boldsymbol{k}| + |\boldsymbol{k}|^2\right)
\end{aligned}
$$

where $E = k^0 =$ energy of e^- and $q^0 = |\boldsymbol{q}| = |\boldsymbol{k}|$. So

$$
\Gamma = \frac{|G_F F_\pi m_e V_{ud}|^2}{4 m_\pi \pi^2} \int \frac{4\pi |\boldsymbol{k}|^2 \, d|\boldsymbol{k}|}{E\left(1 + \frac{k^0}{E}\right)} (E + |\boldsymbol{k}|) \delta(|\boldsymbol{k}| - k^0)
$$

where $k^0 = \frac{m_\pi^2 - m_e^2}{2m_\pi}$. Then

$$
\Gamma = \frac{|G_F F_\pi V_{ud}|^2 m_e^2}{\pi m_\pi} \left(\frac{m_\pi^2 - m_e^2}{2m_\pi}\right)^2
$$

Recall we were studying

$$
\pi^-(\overline{u}d)(p) \to e^-(k) \overline{\nu}_e(q)
$$

We found that

$$
\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}|^2 = \cdots = \sum_{\text{spins}} | \ldots |^2 \left[\overline{u}_e(k)(1-\gamma^5) v_{\nu_e}(q) \overline{v}_{\nu_e}(q)(1+\gamma^5) u_e(k)\right] \\
\Rightarrow \Gamma_{\pi^- \to e^- \overline{\nu}_e} = \frac{|G_F F_\pi V_{ud}|^2}{4\pi} m_e^2 m_\pi \left(1 - \frac{m_e^2}{m_\pi^2}\right)^2
\end{aligned}
$$

The expression for $\pi^- \to \mu^- \overline{\nu}_\mu$ is the same, but with $m_e \to m_\mu$. We see

$$
r = \frac{\Gamma_{\pi^- \to e^- \overline{\nu}_e}}{\Gamma_{\pi^- \to \mu^- \overline{\nu}_\mu}} = \frac{m_e^2}{m_\mu^2} \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \approx 1.28 \times 10^{-4}
$$

Experimentally we find

$$
r = 1.230(4) \times 10^{-4}
$$

This is reasonable agreement. We need quantum loop effects to make out prediction more accurate.
o

Remark. *The ratio $r \ll 1$. This is because $m_\mu \gg m_e \Rightarrow$ it has much less helicity suppressed.*

6.6 $K^0 - \overline{K}^0$ Mixing

A **Kaon** contains a strange quark or antiquark. The lightest kaon flavour eigenstates are

$$K^0(\overline{s}d),\ \overline{K}^0(\overline{d}s),\ k^+(\overline{s}u),\ K^-(\overline{u}s)$$

These are all the possible mesons made an (anti)strange quark and a light quark (u or d). We'll mainly consider $K^0, \overline{K}^0$. These particles are pseudo scalars like pions. They have $J^p = 0^-$ (J spin, p intrinsic parity).
For kaons at rest, we can take relative phases such that

$$\hat{C}\hat{P}\,|K^0\rangle = -\,|\overline{K}^0\rangle \quad \hat{C}\hat{P}\,|\overline{K}^0\rangle = -\,|K^0\rangle$$

The CP eigenstates are

$$|K^0_+\rangle = \tfrac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K}^0\rangle) \quad |K^0_-\rangle = \tfrac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K}^0\rangle)$$

Consider $K^0 \to \pi^0\pi^0$ and $\pi^+\pi^-$ (weak decays). We have the Feynmann diagrams (...) with mixing from the V_{CKM} matrix. From conservation of angular momentum (recall pions have $J^p = 0^-$ the total angular momentum of $\pi\pi$ must be spin 0. Hence the orbital angular momentum between the pions must be $L = 0$. Hence

$$\begin{aligned}\hat{C}\hat{P}\,|\pi^+\pi^-\rangle = \hat{C}\,|\pi^-\pi^+\rangle = |\pi^+\pi^-\rangle\\ \hat{C}\hat{P}\,|\pi^0\pi^0\rangle = \hat{C}(-1)^L\,|\pi^0\pi^0\rangle = (-1)^L\,|\pi^0\pi^0\rangle = |\pi^0\pi^0\rangle\end{aligned}$$

Therefor both final states have $\hat{C}\hat{P} = 1$. If $\hat{C}\hat{P}$ is conserved in this interaction, we expect $|K^0_+\rangle$ to decay to $\pi^0\pi^0, \pi^+\pi^-$, but not to see $|K^0_-\rangle$ to do this. We expect $|K^0_+\rangle$ to be short lived since there is a large phase space available for decay, and we expect $|K^0_-\rangle$ to be long lived. $|K^0_-\rangle$ must decay in the same way, so there is a smaller phase space available. Experimentally, we find K^0_S has a *short* lifetime ($\tau \approx 9 \times 10^{-11}s$) and K^0_L as a *long* lifetime ($\tau =\approx 5 \times 10^{-8}s$). Defining the ratios

$$\begin{aligned}\eta_{+-} &= \frac{\left|\langle\pi^+\pi^-|H|K^0_L\rangle\right|}{|\langle\pi^+\pi^-|H|K^0_S\rangle|}\\ \eta_{00} &= \frac{\left|\langle\pi^0\pi^0|H|K^0_L\rangle\right|}{|\langle\pi^0\pi^0|H|K^0_S\rangle|}\end{aligned}$$

Experimentally,

$$\eta_{+-} = \eta_{00} \approx 10^{-3} \neq 0$$

so we have CP violation in the weak interaction.

Two possible ways for CP violation

Write CP violation as $\not{C}P$.

- Direct : $\not{C}P$ of $s \to u$ due to a phase in V_{CKM}

- Indirect : $\not{C}P$ due to $K^0 \leftrightarrow \overline{K}^0$ mixing, then decay. This ultimately comes from a phase in the V_{CKM} matrix.

It turns out that indirect $\not{C}P$ is mainly responsible for this process. The dominant contribution to this are loops called **box diagrams**. THey have Δstrangeness = 2. $\overline{s}$ has strangeness 1, s has strangeness -1. This is the next-to-leading order set of diagrams in perturbation theory. In the standard model there are *no* tree level diagrams for this process. Therefore

$$
\begin{aligned}
|K_S^0\rangle &= \frac{1}{\sqrt{1+|\epsilon_1|^2}}\left(|K_+^0\rangle + \epsilon_1\,|K_-^0\rangle\right) \approx |K_+^0\rangle \quad \text{at tree level} \\
|K_L^0\rangle &= \frac{1}{\sqrt{1+|\epsilon_2|^2}}\left(|K_-^0\rangle + \epsilon_2\,|K_+^0\rangle\right) \approx |K_-^0\rangle \quad \text{at tree level}
\end{aligned}
$$

Here, $\epsilon_i \in \mathbb{C}$ are small loop contributions.

Assume two state mixing and we can ignor details of the strong interactions

$$
\begin{aligned}
|K_S^0(t)\rangle &= a_S(t)\,|K^0\rangle + b_s(t)\,|\overline{K}^0\rangle \\
|K_L^0(t)\rangle &= a_L(t)\,|K^0\rangle + b_L(t)\,|\overline{K}^0\rangle
\end{aligned}
$$

The Schrödinger equation gives

$$
i\frac{d}{dt}\,|\psi(t)\rangle = H'\,|\psi(t)\rangle
$$

where H' is the next-to-leading order weak Hamiltonian. So

$$
i\frac{d}{dt}\begin{pmatrix} a_i(t) \\ b_i(t) \end{pmatrix} = \begin{pmatrix} \langle K^0|H'|K^0\rangle & \langle K^0|H'|\overline{K}^0\rangle \\ \langle\overline{K}^0|H'|K^0\rangle & \langle\overline{K}^0|H'|\overline{K}^0\rangle \end{pmatrix}\begin{pmatrix} a_i(t) \\ b_i(t) \end{pmatrix}
$$

(Winger Weisskopf approximation) This off diagonal matrix elements are responsible for the mixing. Let

$$
R = \begin{pmatrix} \langle K^0|H'|K^0\rangle & \langle K^0|H'|\overline{K}^0\rangle \\ \langle\overline{K}^0|H'|K^0\rangle & \langle\overline{K}^0|H'|\overline{K}^0\rangle \end{pmatrix}
$$

Because K decay, R is not Hermitian, so can be written $R = M - \frac{i}{2}\Gamma$ where M is the dispersive mass matrix and Γ are Hermitian absorptive decay matrices.
If $\hat{\Theta} = \hat{C}\hat{P}\hat{T}$, $\Theta H6\!\!/\Theta^{-1} = H'^{\dagger}$. Then

$$
\begin{aligned}
\Theta\,|K^0\rangle &= -\,|\overline{K}^0\rangle \\
\Theta\,|\overline{K}^0\rangle &- |K^0\rangle \\
\Rightarrow R_{11} &= \langle K^0|H'K^0\rangle \\
&= (\Theta^{-1}\Theta K^0, H'\Theta^{-1}\Theta K^0) \\
&= (\Theta^{-1}\overline{K}^0, H'\Theta^{-1}\overline{K}^0) \\
&= (\overline{K}^0, \Theta H'\Theta^{-1}\overline{K}^0)^*
\end{aligned}
$$

i.e. Θ is antiunitary. Further

$$R_{11} = (\overline{K}^0, H'^{\dagger}\overline{K}^0)^* = (\overline{K}^0, H'\overline{K}^0) = \langle \overline{K}^0 | H' | \overline{K}^0 \rangle = R_{22}$$

CPT symmetry gives $R_{11} = R_{22}$.

If CP a valid symmetry

If CP is a good symmetry, T is a good symmetry.

$$\begin{aligned} R_{12} = \langle K^0 | H' \overline{K}^0 \rangle &= (\hat{T}^{-1}\hat{T}K^0, H'\hat{T}^{-1}\hat{T}\overline{K}^0) \\ &= (\hat{T}^{-1}K^0, H'\hat{T}^{-1}\overline{K}^0) \\ &= (K^0, H'^{\dagger}\overline{K}^0)^* \\ &= (\overline{K}^0, H'K^0) = R_{21} \end{aligned}$$

Hence CP symmetry gives $R_{12} = R_{21}$. Hence

$$\epsilon_1 = \epsilon_2 = \epsilon = \frac{\sqrt{R_{12}} - \sqrt{R_{21}}}{\sqrt{R_{12}} + \sqrt{R_{21}}}$$

so CP symmetry gives $\epsilon_1 = \epsilon_2 = 0$ Can show that

$$\begin{aligned} \eta_{+-} &= \epsilon + \epsilon' \\ \eta_{00} &= \epsilon - 2\epsilon' \end{aligned}$$

ϵ' a direct measure of CP violation. Other decyas can be use to probe $K^0_{i,s}$

Example 6.4 (Semileptonic decay)**.**

$$\begin{aligned} K^0 &\to \pi^- e^+ \nu_e \\ K^0 &\not\to \pi^+ e^- \overline{\nu}_e \\ \overline{K}^0 &\to \pi^+ e^- \overline{\nu}_e \\ \overline{K}^0 &\not\to \pi^- e^+ \nu_e \end{aligned}$$

If CP valid, then se expect

$$\Gamma(K_{L,S} \to \pi^- e^+ \nu_e) = \Gamma(K_{L,S} \to \pi^+ e^- \overline{\nu}_e)$$

If we define

$$A_{L,S} = \Gamma(K_{L,S} \to \pi^- e^+ \nu_e) - \Gamma(K_{L,S} \to \pi^+ e^- \overline{\nu}_e)$$

we find $A_L \approx 3.32 \pm 0.06 \times 10^{-3}$

7 Quantum Chromodynamics (QCD)

- Protons, Neutrons (Lightest baryons) have similar mas, hence have isospin $I = \frac{1}{2}$ in a doublet.
- $\pi^{0,\pm}$ are the lightest mesons, have $I = 1$ in a triplet.

Postulate a global symmetry $SU(2)_I$. $m_{neutron} \neq m_{proton}$, so symmetry is broken, but there is a good approximation.

- Strange Hadrons discovered later, so the symmetry extended to $SU(3)_F$ a global symmetry. This is not the gauge symmetry $SU(3)_c$. This is badly broken but still useful for classifying hadrons. Lightest mesons are pseudoscalars in $\mathbf{8} + \mathbf{1}$. Lightest baryons in $\mathbf{8} + \mathbf{10}$
- Quark model : constituent quarks, spin-$\frac{1}{2}$ fermions u,d,s, form $\mathbf{3}$ rep. $m_u \approx m_d < m_s$. Baryons are qqq, mesons $\overline{q}q \Rightarrow \mathbf{3} \times \overline{\mathbf{3}} = \mathbf{8} + \mathbf{1}$. $\frac{1}{2} \times \frac{1}{2} = \underbrace{0}_{\eta} + \underbrace{1}_{\pi}$.
- Δ^{++} is uuu, spin-$\frac{3}{2}$, so w/f appears sym violation Fermi stats. Hence we need extra 'colour' quantum number (rgb).