Linearising Flows and a Cohomological Interpretation of Lax Equations - Unpacking the Paper

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1 Introduction

The purpose of this document is to facilitate the understanding of [1] by discussing the terms and how they fit into the wider picture of algebraic geometry.

2 The Preliminaries

2.1 Divisors

Definition 2.1. *A* ***divisor*** *on* C *is a formal finite sum of points, i.e.* $D = \sum_i n_i p_i$ *for* $n_i \in \mathbb{Z}$, $p_i \in C$. *The group of divisors under addition is denoted* $\mathrm{Div}(C)$.

Definition 2.2. *The* ***degree*** *of a divisor* $D = \sum_i n_i p_i$ $\deg D = \sum_i n_i$

Definition 2.3. *Given a meromorphic function* $f : C \to \mathbb{C}$ *we define* $(f) \in \mathrm{Div}(C)$ *by*

$$(f) = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p$$

For $D \in \mathrm{Div}(C)$, *if* $\exists f$ *s.t.* $D = (f)$ *we say* D *is a* ***principal divisor***.

Lemma 2.4. $(fg) = (f) + (g)$

Corollary 2.5. *Principal divisors form a subgroup* $\mathrm{Lin}(C) \leq \mathrm{Div}(C)$.

Lemma 2.6. *If X is a compact Riemann surface and $f : X \to \mathbb{C}$ meromorphic then* $\deg(f) = 0$.

Proposition 2.7. *Let C be compact. Then* $\mathrm{Lin}(C) = \{D \in \mathrm{Div}(C) \,|\, \deg(D) = 0\}$.

Definition 2.8. *The **divisor class group** of C is* $\mathrm{Cl}(C) = {}^{\mathrm{Div}(C)}\!\diagup_{\mathrm{Lin}(C)}$.

Remark. $\deg : \mathrm{Div}(C) \to \mathbb{Z}$ *is a group homomorphism and as the kernel is* $\mathrm{Lin}(C)$ *we see* $\mathrm{Cl}(C) \cong \mathrm{Im}\deg$

Corollary 2.9. $\mathrm{Cl}(\mathbb{CP}^n) \cong \mathbb{Z}$.

Definition 2.10. *Two divisors D, E are **linearly equivalent** if $D - E$ is a principal.*

Lemma 2.11. *Linear equivalence of divisors is an equivalence relation.*

Lemma 2.12. *$f : X \to Y$ induces a group morphism $f : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ by*

$$f\left(\sum_i n_i p_i\right) = \sum_i n_i f(p_i)$$

Proposition 2.13. *If $f : X \to Y$ is a map of Riemann surfaces and $D \in \mathrm{Div}(X)$, then* $\deg(f(D)) = \deg f \cdot \deg D$.

Definition 2.14. *A divisor $D = \sum_i n_i p_i$ is **effective** if each $n_i \geq 0$.*

Proposition 2.15. *We have a partial ordering on $\mathrm{Div}(C)$ by saying $D \geq D'$ id $D - D'$ is effective.*

2.2 Abel-Jacobi

Suppose C has genus g, then we know that $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ where the generators are the loops $\{\gamma_i\}_{i=1}^{2g}$. There is an alternative way to say this condition:

Definition 2.16. *The **canonical bundle** on a space X with $\dim X = n$ is the line bundle of exterior n-forms on X.*

Remark. *Note we know the canonical bundle is a line bundle as there is only 1 basis element of n-forms on an n-dimensional space.*

Proposition 2.17. *If $X = C$ is a Riemann surface of genus g then $H^0(C, K) \cong \mathbb{C}^g$.*

Proof. Find this □

Corollary 2.18. *We can take a basis $\{\omega_i\}_{i=1}^{g}$ of 1-forms on C.*

Definition 2.19. *The **Jacobian** of C is defined to be*

$$J(C) = {}^{\mathbb{C}^g}\!\diagup_{\Lambda}$$

where Λ is the lattice generated over $\mathbb{R}$ by the vectors

$$\Omega_j = \left(\int_{\gamma_j} \omega_1, \ldots, \int_{\gamma_j} \omega_g\right), \quad 1 \leq j \leq 2g$$

Definition 2.20. *The* ***Abel-Jacobi map*** *for $p_0 \in C$ is*

$$\begin{aligned} u : C &\to J(C) \\ p &\mapsto \left(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g \right) \mod \Lambda \end{aligned}$$

This is independent of the path of integration as we have quotiented by Λ.

Theorem 2.21 (Abel's Theorem)**.** *Let u be the Abel-Jacobi map and D, E effective divisors. Then $u(D) = u(E) \Leftrightarrow D \sim E$.*

Theorem 2.22 (Jacobi's Theorem)**.** *The map Abel-Jacobi map is surjective.*

Corollary 2.23. *There is an isomorphism from the space of degree-0 divisors to the Jacobian.*

2.3 Bundles and Sheaves

We recall a few necessary bundle definitions and results:

Definition 2.24. *The tensor product of vector bundles $E, F \to M$ is $E \otimes F \to M$ s.t. $(E \otimes F)_m = E_m \otimes F_m$ for $m \in M$.*

Lemma 2.25. *If O is the trivial line bundle thn $E \otimes O = E$.*

Lemma 2.26. *Line bundles have tensor inverses, i.e given L, $\exists L^{-1}$ s.t. $L \otimes L^{-1} \cong O$ the trivial bundle.*

These results motivate the definition of the **Picard group** which we will cover now:

Definition 2.27. *A* ***ringed space*** *is a pair (X, O_X) where X is a topological space and O_X is a sheaf of rings on X.*

Example 2.28. *Given a topological space X, if we take O_X to be $\mathbb{R}$-valued continuous functions on open subsets of X then (X, O_X) is a ringed space.*

Definition 2.29. *The* ***Picard group*** *of a locally ringed space X is $\mathrm{Pic}(X)$ the group of isomorphism classes of line bundles on X with the group operation being $\otimes$.*

Remark. *In place of line bundles we can actually say* ***invertible sheaves***

Theorem 2.30. $\mathrm{Cl}(C) \cong \mathrm{Pic}(C)$ *naturally.*

Corollary 2.31. *We get a map $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ giving the degree of the corresponding divisor in $\mathrm{Cl}(C)$.*

Corollary 2.32. $\mathrm{Pic}(\mathbb{CP}^1) \cong \mathbb{Z}$.

Notation. *We denote the isomorphism class of line bundles degree d as $\mathrm{Pic}^d(C)$*

Proposition 2.33. *There is a canonical isomorphism $\mathrm{Pic}(X) \cong H^1(X, O_X^*)$.*

Let us now consider a specific class of bundles:

Definition 2.34. *The* ***hyperplane bundle on*** $\mathbb{CP}^n$ *is the bundle* $\mathbb{C}^{n+1} \to \mathbb{CP}^n$ *given by the standard projection* $(z_0, \ldots, z_n) \to [z_0 : \cdots : z_n]$*. It is often denoted* $\mathcal{O}(1)$*. We denote* $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$*.*

Definition 2.35. *The* ***dual bundle*** *of a vector bundle* $E \to M$ *is* $E^* \to M$ *where the fibres of* E^* *are the dual spaces of the fibres of* E*, with the transition functions* $g_{ij}^* = \left(g_{ij}^T\right)^{-1}$*.*

Example 2.36. *The dual bundle to the tangent bundle is the cotangent bundle, i.e.* $(TM)^* = T^*M$

Lemma 2.37. $E \otimes E^* \cong \mathrm{End}(E)$*.*

Definition 2.38. *The* ***tautological line bundle*** *on projective space is* $\mathcal{O}(-1) = \mathcal{O}(1)^*$*. We denote* $\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n}$*.*

Proposition 2.39. *The canonical bundle on the projective space is* $K = \mathcal{O}(-n-1)$*.*

Lemma 2.40. $\mathcal{O}(1)^{-1} = \mathcal{O}(-1)$*.*

Corollary 2.41. $\mathrm{Pic}(\mathbb{CP}^n)$ *is generated by* $\mathcal{O}(\pm 1)$*.*

2.4 Lax Pairs and Spectral Curves

Definition 2.42. *A Lax pair is a pair of* $\mathfrak{g} \subset \mathfrak{gl}_n$*-valued matrices* A, B*, functions of a spectral parameter* ξ *and time* t *satisfying* $\dot{A} = [A, B]$*.*

Definition 2.43. *The* ***spectral curve*** *is* C *given by the solution in* $\mathbb{P}^1$ *of*

$$\det \left[\eta I - A(\xi, t)\right] = 0$$

Proposition 2.44. *The flow* $t \mapsto A(\xi, t)$ *is isospectral.*

It will be the understanding of this isospectral flow that we want to gain. We formulate this flow as the family of holomorphic map gained by the eigenvectors

$$f_t : C \to \mathbb{CP}^{n-1}$$

References

[1] Phillip A. Griffiths. Linearizing flows and a cohomological interpretation of lax equations. *American Journal of Mathematics*, 107(6):pp. 1445–1484, 1985. ISSN 00029327, 10806377. doi: 10.2307/2374412.