

The Eisenhart Lift

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1 The Eisenhart Lift

1.1 The metric and equations of motion

Consider the $(d+2)$ -dimensional line element,

$$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = h_{ij} dx^i dx^j + 2dt (dv - \Phi dt + N_i dx^i), \quad (1.1.1)$$

where $i, j = 1, \dots, d$, $x^{d+1} = t$, $x^{d+2} = v$ and Φ , N_i and h_{ij} are independent of the coordinate v . Then $\xi = \partial_v$ is a Killing vector. We have

$$\hat{g} = \begin{pmatrix} h_{ij} & N_i & 0 \\ N_j & -2\Phi & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{g}^{-1} = \begin{pmatrix} h^{ij} & 0 & -h^{ik} N_k \\ 0 & 0 & 1 \\ -h^{jk} N_k & 1 & 2\Phi + N_i h^{ij} N_j \end{pmatrix},$$

where h^{ij} is the inverse of h_{ij} . The geodesic Lagrangian is

$$\mathcal{L} = \frac{1}{2} \hat{g}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j + \dot{t} \dot{v} - \Phi \dot{t}^2 + N_i \dot{x}^i \dot{t} := \tilde{L} + \dot{t} \dot{v},$$

where $\dot{x}^\mu = dx^\mu/d\lambda$ for an affine geodesic parameter λ (\tilde{L} is defined below). Calculating the equations of motion from \mathcal{L} enables a simple determination of (appropriate combinations of) the Christoffel symbols for \hat{g} . Recall

$$\hat{\Gamma}_{\nu\rho}^\mu = \frac{1}{2} \hat{g}^{\mu\delta} (\hat{g}_{\delta\nu,\rho} + \hat{g}_{\delta\rho,\nu} - \hat{g}_{\nu\rho,\delta}) := \hat{g}^{\mu\delta} [\nu\rho, \delta]_{\hat{g}}.$$

and the equations of motion are

$$0 = \ddot{x}^\mu + \hat{\Gamma}_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho.$$

Setting

$$A := A_\mu dx^\mu = N_i dx^i - \Phi dt, \quad F = dA = \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

we get

$$\begin{aligned} F_{ij} &= \partial_i N_j - \partial_j N_i = -F_{ji} \\ F_{it} &= -(\partial_t N_i + \partial_i \Phi) = -F_{ti} \end{aligned}$$

and using that the equations of motion for v , x^i and t (from $\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu}$) yield

$$0 = \frac{d}{d\lambda} \dot{t} = \ddot{t}$$

(for v) then

$$\begin{aligned} \frac{1}{2}(\partial_i h_{jk}) \dot{x}^j \dot{x}^k - (\partial_i \Phi) \dot{t}^2 + (\partial_i N_j) \dot{x}^j \dot{t} &= \frac{d}{d\lambda} (h_{ij} \dot{x}^j + N_i \dot{t}) \\ &= h_{ij} \ddot{x}^j + (\partial_k h_{ij}) \dot{x}^j \dot{x}^k + (\partial_t h_{ij}) \dot{x}^j \dot{t} + (\partial_j N_i) \dot{x}^j \dot{t} + (\partial_t N_i) \dot{t}^2 \end{aligned}$$

(for \dot{x}^i) and

$$\begin{aligned} \frac{1}{2}(\partial_t h_{ij}) \dot{x}^i \dot{x}^j - (\partial_t \Phi) \dot{t}^2 + (\partial_t N_i) \dot{x}^i \dot{t} &= \frac{d}{d\lambda} (\dot{v} - 2\Phi \dot{t} + N_i \dot{x}^i) \\ &= \ddot{v} - 2(\partial_i \Phi) \dot{t} \dot{x}^i - 2(\partial_t \Phi) \dot{t}^2 - 2\Phi \ddot{t} + (\partial_j N_i) \dot{x}^i \dot{x}^j + (\partial_t N_i) \dot{x}^i \dot{t} + N_i \ddot{x}^i \end{aligned}$$

we get (collating them together)

$$\begin{aligned} 0 &= \ddot{t}, \\ 0 &= h_{ij} \ddot{x}^j + [jk, i]_h \dot{x}^j \dot{x}^k + (\partial_t h_{ij} + \partial_j N_i - \partial_i N_j) \dot{x}^j \dot{t} + (\partial_i \Phi + \partial_t N_i) \dot{t}^2, \\ &= h_{ij} \ddot{x}^j + [jk, i]_h \dot{x}^j \dot{x}^k + (\partial_t h_{ij} - F_{ij}) \dot{x}^j \dot{t} + F_{ti} \dot{t}^2, \\ 0 &= \ddot{v} + N_i \ddot{x}^i + \left[\frac{1}{2} (\partial_j N_i + \partial_i N_j) - \frac{1}{2} \partial_t h_{ij} \right] \dot{x}^i \dot{x}^j - 2\partial_i \Phi \dot{t} \dot{x}^i - \partial_t \Phi \dot{t}^2, \\ &= \ddot{v} + \left[\frac{1}{2} (\partial_j N_i + \partial_i N_j) - N_k \Gamma_{ij}^k - \frac{1}{2} \partial_t h_{ij} \right] \dot{x}^i \dot{x}^j + [-N^k (\partial_t h_{ki} - F_{ki}) - 2\partial_i \Phi] \dot{t} \dot{x}^i + (-\partial_t \Phi + N^i F_{it}) \dot{t}^2 \end{aligned}$$

where we have substituted F in the latter, and usde the notation

$$[jk, i]_h = \frac{1}{2} (\partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk}) .$$

Recall

$$\Gamma_{jk}^i = h^{il} [jk, l]_h .$$

Note that to raise the index of N has required we recognise that

$$N^i = \hat{g}^{ij} N_j = h^{ij} N_j$$

1.2 Equivalence of equations of motion

The canonical momenta are given by $p_\mu = \partial \mathcal{L} / \partial \dot{x}^\mu = \hat{g}_{\mu\nu} \dot{x}^\nu$ giving

$$p_v = \dot{t}, \quad p_i = h_{ij} \dot{x}^j + N_i \dot{t}, \quad p_t = \dot{v} - 2\Phi \dot{t} + N_i \dot{x}^i,$$

and so

$$\dot{t} = p_v, \quad \dot{x}^i = h^{ij} (p_j - N_j p_v), \quad \dot{v} = p_t - N^i p_i + [2\Phi + N^2] p_v.$$

Likewise, the geodesic Hamiltonian is

$$\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2} \hat{g}^{\mu\nu} p_\mu p_\nu = \frac{1}{2} h^{ij} (p_i - N_i p_v)(p_j - N_j p_v) + p_t p_v + \Phi p_v^2.$$

The equations of motion are

$$\begin{aligned} \frac{dt}{d\lambda} &= \frac{\partial \mathcal{H}}{\partial p_t} = p_v, & \frac{dv}{d\lambda} &= \frac{\partial \mathcal{H}}{\partial p_v}, & \frac{dx^i}{d\lambda} &= \frac{\partial \mathcal{H}}{\partial p_i} = h^{ij} (p_j - N_j p_v), \\ \frac{dp_t}{d\lambda} &= -\frac{\partial \mathcal{H}}{\partial t}, & \frac{dp_v}{d\lambda} &= -\frac{\partial \mathcal{H}}{\partial v} = 0, & \frac{dp_i}{d\lambda} &= -\frac{\partial \mathcal{H}}{\partial x^i}. \end{aligned}$$

Because v is a cyclic coordinate its conjugate momentum p_v is conserved along geodesics: thus $p_v = m$ is a constant and we may write

$$\mathcal{H} := H + m p_t, \quad H := \frac{1}{2} h^{ij} (p_i - m N_i)(p_j - m N_j) + m^2 \Phi.$$

We observe that we have the geodesics have the conserved quantities,

$$\begin{aligned} \frac{1}{2} \hat{g}^{\mu\nu} p_\mu p_\nu &= m \left[\frac{p^i p_i}{2m} - N^i p_i + m N^i N_i + p_t + m \Phi \right] := -m E_0, \\ \hat{g}^{\mu\nu} p_\mu \xi_\nu &= p_v = m. \end{aligned}$$

Following the identifications of [3] we view $p_v = m$ as the mass, $-p_t = E$ as the energy, E_0 as the internal energy, and $m\Phi = V$ as the potential energy. Taking the internal energy to vanish in the nonrelativistic limit the null geodesics of \hat{g} may be identified with the motion in the d -dimensional space with potential energy V . We note that two conformally related metrics have the same null geodesics, and so the d -dimensional world lines will be the same. For $m \neq 0$ the equations of motion for t then give $dt/d\lambda = m$, whence $dt = m d\lambda$ and we may eliminate the affine geodesic parameter λ for t . The equations of motion are then precisely those coming from the standard mechanical system

$$\tilde{L} = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j + N_i \dot{x}^i - \Phi$$

where \dot{x}^i is now the standard dx^i/dt (and $\dot{t} = 1$). Now

- (a) in the case of a non-null geodesic, if we parameterised the curve by arc length, $\lambda = s$ and $t = ms$, then from (1.1.1) we have

$$\frac{dv}{dt} = \frac{1}{2m^2} - \tilde{L}.$$

The equations of motion for v follow from this and

$$v = \frac{t}{2m^2} - \int \tilde{L} dt + b.$$

- (b) in the case of a null geodesics we have

$$\frac{dv}{dt} = -\tilde{L}, \quad v = - \int \tilde{L} dt + b.$$

Thus we have for each $m \neq 0$ and b a bijection between the geodesics of \hat{g} and the equations of motion of \tilde{L} .

1.3 Connection and Curvature

From the equations of motion we read that the nonvanishing Christoffel symbols for \hat{g} are

$$\begin{aligned}\hat{\Gamma}_{jk}^i &= \Gamma_{jk}^i, & \hat{\Gamma}_{jt}^i &= -\frac{1}{2}F_j^i + \frac{1}{2}h^{ik}\partial_t h_{kj}, & \hat{\Gamma}_{tt}^i &= h^{ik}(\partial_t N_k + \partial_k \Phi) = -F_t^i, \\ \hat{\Gamma}_{tt}^v &= -\partial_t \Phi + N^k F_{kt}, & \hat{\Gamma}_{ij}^v &= \frac{1}{2}[\nabla_i N_j + \nabla_j N_i - \partial_t h_{ij}], & \hat{\Gamma}_{ti}^v &= -\frac{1}{2}N^k(\partial_t h_{ki} - F_{ki}) - \partial_i \Phi.\end{aligned}$$

Here we have used ∇_i for the Levi-Civita connection from the metric h . Recall now the equation for the Riemann tensor

$$\hat{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho \hat{\Gamma}_{\nu\sigma}^\mu - \partial_\sigma \hat{\Gamma}_{\nu\rho}^\mu + \hat{\Gamma}_{\rho\lambda}^\mu \hat{\Gamma}_{\nu\sigma}^\lambda - \hat{\Gamma}_{\sigma\lambda}^\mu \hat{\Gamma}_{\nu\rho}^\lambda$$

We immediately notice

$$\hat{R}^i{}_{jkl} = R^i{}_{jkl} + \hat{\Gamma}_{kt}^i \hat{\Gamma}_{jl}^t - \hat{\Gamma}_{lt}^i \hat{\Gamma}_{jk}^t + \hat{\Gamma}_{kv}^i \hat{\Gamma}_{jl}^v - \hat{\Gamma}_{lv}^i \hat{\Gamma}_{jk}^v = R^i{}_{jkl}$$

as there are non-vanishing Christoffel symbols with v as lower index, or t as an upper index. Further, as all Christoffel symbols are independent of v (as the metric is) we can then say that $\hat{R}^\mu{}_{\nu v \sigma} = 0$. As such $\hat{R}^\mu{}_{\nu\rho\sigma} = 0$ if any of $\nu, \rho, \sigma = v$. We can also see that $\hat{R}^t{}_{\nu\rho\sigma} = 0$ by the formula. so we now need only determine

- | | |
|------------------------|------------------------|
| 1. $\hat{R}^i{}_{jtl}$ | 4. $\hat{R}^v{}_{jkl}$ |
| 2. $\hat{R}^i{}_{tkl}$ | 5. $\hat{R}^v{}_{jtl}$ |
| 3. $\hat{R}^i{}_{ttl}$ | 6. $\hat{R}^v{}_{tkl}$ |
| | 7. $\hat{R}^v{}_{ttl}$ |

Making the observation

$$\hat{R}^v{}_{\nu\rho\sigma} = -h^{ik}N_k R_{i\nu\rho\sigma} + R_{t\nu\rho\sigma}$$

and seeing that

$$\begin{aligned}\hat{R}_{i\nu\rho\sigma} &= \hat{g}_{i\mu} \hat{R}^\mu{}_{\nu\rho\sigma} \\ &= h_{ij} \hat{R}^j{}_{\nu\rho\sigma}\end{aligned}$$

we can simplify

$$\hat{R}^v{}_{j\rho\sigma} = -N_i \hat{R}^i{}_{j\rho\sigma} - h_{ji} \hat{R}^i{}_{t\rho\sigma}$$

and

$$\hat{R}^v{}_{t\rho\sigma} = -N_i \hat{R}^i{}_{t\rho\sigma}$$

This lets us get the second column of terms immediately after we have the first. The Bianchi identity also tells us that

$$\hat{R}^\mu{}_{\nu\rho\sigma} = -\hat{R}^\mu{}_{\rho\sigma\nu} - \hat{R}^\mu{}_{\sigma\nu\rho}$$

This means

$$\begin{aligned}\hat{R}^i_{tkl} &= -\hat{R}^t_{klt} - \hat{R}^i_{lkt} \\ &= 2\hat{R}^i_{[k|t|l]}\end{aligned}$$

and so we just need to work out 1 and 3.

Let us begin the slog:

$$\begin{aligned}\hat{R}^i_{jtl} &= \partial_t \hat{\Gamma}^i_{jl} - \partial_l \hat{\Gamma}^i_{jt} + \hat{\Gamma}^i_{t\mu} \hat{\Gamma}^\mu_{jl} - \hat{\Gamma}^i_{l\mu} \hat{\Gamma}^\mu_{jt} \\ &= \Gamma^i_{jl,t} - \nabla_l \hat{\Gamma}^i_{jt} \\ &= \Gamma^i_{jl,t} + \frac{1}{2} \nabla_l [h^{ik} (F_{kj} - h_{kj,t})] \\ &= \Gamma^i_{jl,t} - \frac{1}{2} h^{ik} \nabla_l h_{kj,t} + \frac{1}{2} \nabla_l F^i_j\end{aligned}$$

(This form will be useful to give coherence with [3]). Now note

$$\begin{aligned}h^{ik} \nabla_l (\partial_t h_{kj}) &= h^{ik} [h_{kj,tl} - \Gamma^m_{lk} h_{mj,t} - \Gamma^m_{lj} h_{km,t}] \\ &= h^{ik} [h_{kj,lt} - \partial_t (h_{mj} \Gamma^m_{lk}) + h_{mj} \Gamma^m_{lk,t} - \partial_t (h_{km} \Gamma^m_{lj}) + h_{mk} \Gamma^m_{lj,t}] \\ &= h^{ik} [\partial_t (h_{kj,l} - [lk, j]_h - [lj, k]_h) + h_{mj} \Gamma^m_{lk,t}] + \Gamma^i_{lj,t}\end{aligned}$$

Calculating

$$h_{kj,l} - [lk, j]_h - [lj, k]_h = h_{kj,l} - \frac{1}{2} (h_{lj,k} + h_{kj,l} - h_{lk,j}) - \frac{1}{2} (h_{lk,j} + h_{jk,l} - h_{jl,k}) = 0$$

we have

$$\hat{R}^i_{jtl} = \frac{1}{2} [\nabla_l F^i_j + \Gamma^i_{jl,t} - h^{ik} h_{jm} \Gamma^m_{lk,t}]$$

Hence

$$\begin{aligned}\hat{R}^i_{tkl} &= 2\hat{R}^i_{[k|t|l]} = \nabla_{[l} F^i_{k]} - h^{ij} h_{m[k} \Gamma^m_{l]j,t} \\ \text{or } &= \nabla_{[l} F^i_{k]} - \nabla_{[l} h^{ij} h_{k]j,t}\end{aligned}$$

Further

$$\begin{aligned}\hat{R}^i_{ttl} &= \partial_t \hat{\Gamma}^i_{tl} - \partial_l \hat{\Gamma}^i_{tt} + \hat{\Gamma}^i_{t\mu} \hat{\Gamma}^\mu_{tl} - \hat{\Gamma}^i_{l\mu} \hat{\Gamma}^\mu_{tt} \\ &= -\frac{1}{2} \partial_t (F^i_l - h^{ij} h_{jl,t}) + \frac{1}{4} (F^i_j - h^{ik} h_{kj,t}) (F^j_l - h^{jm} h_{ml,t}) + \nabla_l F^i_t \\ &= -\frac{1}{2} \partial_t (F^i_l - h^{ij} h_{jl,t}) + \frac{1}{4} (F^i_j + h^{ik} h_{kj,t}) (F^j_l - h^{jm} h_{ml,t}) + \nabla_l F^i_t \\ &= -\frac{1}{2} \left[\partial_t (F^i_l - h^{ij} h_{jl,t}) - \frac{1}{2} (F^{ij} + h^{ij} h_{j,t}) (F_{jl} - h_{jl,t}) - 2 \nabla_l F^i_t \right] \quad (\text{useful to give coherence with [3]}) \\ &= -\frac{1}{2} (F^i_{l,t} - h^{ij} h_{jl,tt}) + \frac{1}{4} (F^i_j F^j_l - F^{ij} h_{jl,t} + F_{jl} h^{ij} h_{j,t} + h^{ij} h_{jl,t}) + \nabla_l F^i_t\end{aligned}$$

With these three we can read off

$$\begin{aligned}
\hat{R}^v_{jkl} &= -N_i R^i_{jkl} - h_{ji} \left[h^{ia} h_{m[l} \Gamma^m_{k]a,t} - \nabla_{[k} F^i_{l]} \right] \\
&= -N_i R^i_{jkl} - h_{m[l} \Gamma^m_{k]j,t} + \nabla_{[k} F_{j]l} \\
&= -N_i R^i_{jkl} - h_{m[l} \Gamma^m_{k]j,t} + \frac{1}{2} \nabla_j F_{kl}
\end{aligned}$$

using

$$\begin{aligned}
\nabla_{[k} F_{j]l} &= F_{j[l,k]} - F_{jm} \Gamma^m_{[kl]} - F_{m[l} \Gamma^m_{k]j} \\
&= N_{[l,[j|k]} - N_{j,[lk]} - F_{m[l} \Gamma^m_{k]j} \\
&= \frac{1}{2} (F_{kl,j} - \Gamma^m_{kj} F_{ml} + \Gamma^m_{jl} F_{mk}) \\
&= \frac{1}{2} \nabla_j F_{kl}
\end{aligned}$$

$$\begin{aligned}
\hat{R}^v_{jtl} &= -\frac{1}{2} N_i [\nabla_l F^i_j + \Gamma^i_{jl,t} - h^{ik} h_{jm} \Gamma^m_{lk,t}] - h_{ji} \left\{ -\frac{1}{2} (F^i_{l,t} - h^{ik} h_{kl,tt}) \right. \\
&\quad \left. + \frac{1}{4} (F^i_k F^k_l - F^{ik} h_{kl,t} + F_{kl} h^{ik}_{,t} + h^{ik}_{,t} h_{kl,t}) + \nabla_l F^i_t \right\} \\
&= -\frac{1}{2} N_i [\nabla_l F^i_j + \Gamma^i_{jl,t} - h^{ik} h_{jm} \Gamma^m_{lk,t}] + \frac{1}{2} (h_{ji} F^i_{l,t} + h_{jl,tt}) \\
&\quad + \frac{1}{4} (F_{jk} F^k_l - F_j{}^k h_{kl,t} + F_{kl} h_{ji} h^{ik}_{,t} + h_{ji} h^{ik}_{,t} h_{kl,t}) + \nabla_l F_{jt} \\
&= -\frac{1}{2} N_i [\nabla_l F^i_j + \Gamma^i_{jl,t} - h^{ik} h_{jm} \Gamma^m_{lk,t}] + \frac{1}{2} (h_{ji} F^i_{l,t} + h_{jl,tt}) \\
&\quad + \frac{1}{4} (F_{jk} F^k_l - F_j{}^k h_{kl,t} - F_l^i h_{ji,t} + h_{ji} h^{ik}_{,t} h_{kl,t}) + \nabla_l F_{jt}
\end{aligned}$$

Now we have constructed Riemann curvature tensors, we can go on to calculate the Ricci tensor given by

$$\hat{R}_{\nu\sigma} = \hat{R}^\mu_{\nu\mu\sigma}$$

As before, we said that we can never have t as the first upper index, or a v as the lower indices, so we know that

$$\hat{R}_{v\mu} = 0$$

and that this formula reduces to

$$\hat{R}_{\nu\sigma} = \hat{R}^i_{\nu i\sigma}$$

We straight away recognise that the spatial part is the same as for the spatial manifold, i.e.

$$\hat{R}_{ij} = R_{ij}$$

All that's left to calculate is $\hat{R}_{ti} = \hat{R}_{it}$, \hat{R}_{tt} , which are given by

$$\begin{aligned}
\hat{R}_{tl} &= \hat{R}_{til} \\
&= \nabla_{[l} F_{k]}^i - \nabla_{[l} h^{ij} h_{i]j,t} \\
&= \frac{1}{2} [\nabla_l F_i^i - \nabla_i F_l^i - \partial_l (h^{ij} h_{ij,t}) + \nabla_i h^{ij} h_{jl,t}] \\
&= -\frac{1}{2} [\nabla^i (F_{il} - h_{il,t}) + \partial_l (h^{ij} h_{ij,t})] \quad (\text{compare to [3]})
\end{aligned}$$

$$\begin{aligned}
\hat{R}_{tt} &= \hat{R}_{tit} \\
&= \frac{1}{2} \left[\partial_t (F_i^i - h^{ij} h_{ji,t}) - \frac{1}{2} (F^{ij} + h^{ij},_t) (F_{ji} - h_{ji,t}) - 2\nabla_i F_t^i \right] \\
&= -\frac{1}{2} \left[\partial_t (h^{ij} h_{ij,t}) - \frac{1}{2} (F^{ji} - h^{ji},_t) (F_{ji} - h_{ji,t}) + 2\nabla_i F_t^i \right] \quad (\text{compare to [3]}) \\
&= -\frac{1}{2} \left[\partial_t (h^{ij} h_{ij,t}) - \frac{1}{2} (F^{ij} F_{ij} + F^{ij} h_{ij,t} + h^{ij},_t F_{ij} + h^{ij},_t h_{ij,t}) + 2\nabla^i F_{it} \right] \\
&= -\frac{1}{2} \left[\partial_t (h^{ij} h_{ij,t}) - \frac{1}{2} (F^{ij} F_{ij} - 2F_{ij,t} h^{ij} + h^{ij},_t h_{ij,t}) + 2\nabla^i F_{it} \right]
\end{aligned}$$

1.4 The Frame

An alternative way to get this result is using the frame field formulation.

Given the metric (1.1.1) we define the frame $\{\hat{e}^A\}$,

$$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = h_{ij} dx^i dx^j + 2dt (dv - \Phi dt + N_i dx^i) = \hat{\eta}_{AB} \hat{e}^A \hat{e}^B = \eta_{ab} e^a e^b + \hat{e}^+ \hat{e}^- + \hat{e}^- \hat{e}^+.$$

Here $A \in \{+, -, a, b, \dots\}$, $\hat{\eta}_{+-} = \hat{\eta}_{-+} = 1$, and we take

$$\hat{e}^+ := dt, \quad \hat{e}^- := dv - \Phi dt + N_i dx^i, \quad \hat{e}^a := \hat{e}_\mu^a dx^\mu = e_i^a dx^i = e^a,$$

and

$$e_i^a \eta_{ab} e_j^b = h_{ij}.$$

The coframe $\{\hat{E}_A\}$ with $\hat{e}^A(\hat{E}_B) = \delta_B^A$ is given by

$$\hat{E}_+ := \partial_t + \Phi \partial_v, \quad \hat{E}_- := \partial_v, \quad \hat{E}_a := E_a - N_a \partial_v,$$

where $N_a = N_i E_a^i$ and similarly

$$e^a(E_b) = \delta_b^a, \quad E_b = E_b^i \partial_i.$$

We emphasise that N , ϕ and e^a may depend on x^i and t .

Denoting the structure constants $[\hat{E}_B, \hat{E}_C] = c_{BC}^A \hat{E}_A$ we have from

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

for a one-form α , then for the torsion free connection

$$d\hat{e}^A = -\hat{\omega}_B^A \wedge e^B = \hat{\omega}_{BC}^A e^B \wedge e^C$$

we have

$$d\hat{e}^A(\hat{E}_B, \hat{E}_C) = \hat{\omega}_{BC}^A - \hat{\omega}_{CB}^A = -\hat{e}^A([\hat{E}_B, \hat{E}_C]) = -c_{BC}^A,$$

from which

$$\hat{\omega}_{BC}^A = \frac{1}{2}\hat{\eta}^{AF}(c_{CFB} + c_{BFC} - c_{FBC}).$$

The v -independence of the metric means that

$$[\hat{E}_-, \hat{E}_B] = 0, \quad c_{-B}^A = 0$$

while

$$\begin{aligned} [\hat{E}_+, \hat{E}_a] &= \partial_t E_a - (\partial_t N_a) \partial_v - (E_a \Phi) \partial_v \\ &= (\partial_t E_a^j) [\partial_j - N_j \partial_v] - E_a^j [\partial_t N_j + \partial_j \Phi] \partial_v \\ &= (\partial_t E_a^j e_j^b) \hat{E}_b + F_{at} \hat{E}_- \end{aligned}$$

and

$$\begin{aligned} [\hat{E}_a, \hat{E}_b] &= [E_a, E_b] - (E_a N_b - E_b N_a) \partial_v \\ &= c_{ab}^f \hat{E}_f - F_{ab} \hat{E}_- \end{aligned}$$

giving the (possibly) non-vanishing structure constants as

$$c_{ab}^f, \quad c_{ab}^- = -F_{ab}, \quad c_{+a}^b = (\partial_t E_a^j e_j^b), \quad c_{+a}^- = F_{at}.$$

Now

$$\begin{aligned} d\hat{e}^+ &= 0, \\ d\hat{e}^- &= \frac{1}{2} F_{ab} e^a \wedge e^b + F_{it} dx^i \wedge dt = \frac{1}{2} F_{ab} \hat{e}^a \wedge \hat{e}^b + F_{at} \hat{e}^a \wedge \hat{e}^+, \\ d\hat{e}^a &= d(\hat{e}_\mu^a dx^\mu) = e_i^a dx^i = e^a = (\partial_j e_i^a) dx^j \wedge dx^i + (\partial_t e_i^a) dt \wedge dx^i \\ &= \omega_{bc}^a e^b \wedge e^c - (E_b^i \partial_t e_i^a) dt \wedge e^b = \omega_{bc}^a e^b \wedge e^c + (\partial_t E_b^i e_i^a) dt \wedge e^b, \end{aligned}$$

from which we see

$$\begin{aligned} \hat{\omega}_{BC}^a \hat{e}^B \wedge \hat{e}^C &= \omega_{bc}^a e^b \wedge e^c + (\partial_t E_b^i e_i^a) \hat{e}^+ \wedge \hat{e}^b, \\ \hat{\omega}_{BC}^- \hat{e}^B \wedge \hat{e}^C &= \frac{1}{2} F_{bc} \hat{e}^b \wedge \hat{e}^c + F_{at} \hat{e}^a \wedge \hat{e}^+. \end{aligned}$$

Set

$$\alpha_{-b}^a := e_i^a \partial_t E_b^i = c_{+b}^a, \quad \alpha_{ab} = -\alpha_{ba},$$

Using the antisymmetry of the connection then $0 = \hat{\omega}_{++A} = \hat{\omega}_{+A}^-$ and so

$$\hat{\omega}_{a+}^- = F_{at}, \quad \hat{\omega}_{ab}^- = \frac{1}{2}F_{ab}, \quad \hat{\omega}_{bc}^a = \omega_{bc}^a, \quad \hat{\omega}_{ab+} = -\frac{1}{2}F_{ab} - \frac{1}{2}[\partial_t E_a^i E_{ib} - \partial_t E_b^i E_{ia}] = -\frac{1}{2}F_{ab} + \alpha_{ab}.$$

From the structure of the connection we see that $\hat{R}_B^A \in \{\hat{R}_a^-, \hat{R}_+^a, \hat{R}_b^a\}$ and so $\widehat{\text{Ric}}_{AB} = \hat{R}_{AFB}^F = \hat{R}_{AfB}^f$. Thus the (possible) non-vanishing components of the Ricci tensor are

$$\widehat{\text{Ric}}_{++} = \hat{R}_{+f+}^f, \quad \widehat{\text{Ric}}_{a+} = \hat{R}_{af+}^f, \quad \widehat{\text{Ric}}_{ab} = \hat{R}_{afb}^f.$$

1.5 Einstein metrics

Now because $\hat{g}_{tv} = 1$ and $\hat{R}_{tv} = 0$, then (B, \hat{g}) is Einstein if and only if it is Ricci flat. With this we can follow the paper [1] to try and derive conditions on the functions N_i, Φ .

Insert lead in - left out to get calculation of after (43) done quicker

Define

$$\begin{aligned} H_{ij} &= h_{ij,t} \\ H &= h^{ij} H_{ij} \\ f_j &= \nabla_i H_j^i - \nabla_j H, \end{aligned}$$

then

$$\begin{aligned} f^i &= \nabla_j H^{ij} - h^{ij} \nabla_j H \\ &= (h^{ik} h^{jl} - h^{ij} h^{kl}) \nabla_j H_{kl}. \end{aligned}$$

We have shown previously

$$\begin{aligned} \nabla_j H_{kl} &= h_{mk} \Gamma_{lj,t}^m + h_{ml} \Gamma_{kj,t}^m \\ \Rightarrow f^i &= h^{jl} \Gamma_{lj,t}^i + h^{ik} \Gamma_{kj,t}^j - 2h^{ij} \Gamma_{jl,t}^l \\ &= h^{jk} \Gamma_{jk,t}^i - h^{ij} \Gamma_{jk,t}^k. \end{aligned}$$

Hence

$$\begin{aligned} \nabla_i f^i &= h^{jk} \nabla_i \Gamma_{jk,t}^i - h^{ij} \nabla_i \Gamma_{jk,t}^k \\ &= h^{ij} [\nabla_k \Gamma_{ij,t}^k - \nabla_i \Gamma_{jk,t}^k] \\ &= h^{ij} [\Gamma_{ij,tk}^k + \Gamma_{kl}^k \Gamma_{ij,t}^l - \Gamma_{ki}^l \Gamma_{lj,t}^k - \Gamma_{kj}^l \Gamma_{il,t}^k - \Gamma_{jk,ti}^k + \Gamma_{ij}^l \Gamma_{lk,t}^k] \\ &= h^{ij} [\partial_t (\Gamma_{ij,k}^k - \Gamma_{jk,i}^k + \Gamma_{kl}^k \Gamma_{ij}^l - \Gamma_{il}^k \Gamma_{jk}^l) + \Gamma_{il}^k \Gamma_{jk,t}^l - \Gamma_{ki}^l \Gamma_{lj,t}^k] \\ &= h^{ij} R_{jki,t}^k = h^{ij} R_{ij,t} \end{aligned}$$

1.6 Bargman Structures

A Bargmann structure (B, \hat{g}, ξ) is a principal bundle $\pi : B \rightarrow M$, where $\dim B = \dim M + 1$, equipped with a Lorentzian metric \hat{g} and nowhere vanishing null vector field ξ such that with

respect to the usual Levi-Civita connection $\hat{\nabla}\xi = 0$. Then $M := B/\mathbb{R}\xi$ is equipped with a Newton-Cartan geometry (M, K, θ, ∇) where

$$K = \pi_* \hat{g}^{-1}, \quad \hat{g}(\xi) = \pi^* \theta,$$

K is degenerate and $\pi^* \theta$ generates $\ker K$.

In our setting we have a metric of Brinkmann form

$$\hat{g} = h + dt \otimes \omega + \omega \otimes dt, \quad \omega = dv - \Phi(x, t) dt + N_i(x, t) dx^i, \quad h = h_{ij}(x, t) dx^i \otimes dx^j.$$

Then $\xi = \partial_v$, $\theta = dt$.

2 Introduction

Let us start with a bit of back story, so we can develop and go further. This will be built off of [2].

2.1 Galilei and Newton Structures

We start with some more classical work.

Definition 2.1 (Galilei group). *The **Galilei group** is the matrix group*

$$G = \left\{ \begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \mid R \in SO(d), \quad b, c \in \mathbb{R}^n, \quad e \in \mathbb{R} \right\} \leq GL_{d+2}(\mathbb{R})$$

We think of G as acting on $(\mathbf{x}, t, 1)$ s.t.

$$\begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Rx + tb + c \\ t + e \\ 1 \end{pmatrix}$$

with this action we see:

1. R are rotations in space
2. b are boosts
3. c, e are translations in space and time respectively

With this interpretation we have

Definition 2.2. *The **Homogeneous Galilei group/Euclidean group** H is the group of Galilean transformations that preserve the spatio-temporal origin $(\mathbf{0}, 0, 1)$.*

Proposition 2.3. *H consists of matrices of the form*

$$\begin{pmatrix} R & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover $H \cong SO(d) \ltimes \mathbb{R}^d$ as a Lie group (*not a as a Lie transformation group* [4]) is faithfully represented by matrices of the form

$$\begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \in GL_{d+1}.$$

Proof. See my CQIS notes for a more built up discussion of this fact. □

We now recall the following def:

Definition 2.4. The **frame bundle** of a k -dimensional smooth manifold M is $GL(M)$, the GL_k -principal fibre bundle with fibres at $x \in M$ given by the space of ordered bases of $T_x M$.

Definition 2.5. A **proper Galilei structure** $H(M)$ is a reduction of structure group of the frame bundle of a $(d+1)$ -dimensional M via $H \hookrightarrow GL_{d+1}$.

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