Monopoles

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November 2020

Contents

1 Introduction

1.1 Preamble

I already have notes on Gauge Theory, Algebraic Geometry, Solitons, and Algebraic Topology, but I have yet to actually make any notes on Monopoles. The purpose of these notes is to be a comprehensive cover of the knowledge required to understand [4]. This will include previous works by Atiyah, Donaldson, Hitchin, Nahm, and more.

2 Preliminaries

As with all my projects, the preliminaries will undoubtedly end up being too long, but I will try keep this minimal this time:

Definition 2.1. *The **annihilator** of $U \leq V$ is*

$$U^0 = \{f \in V^* \,|\, \forall u \in U,\ f(u) = 0\} \leq V^*$$

If V has bilinear $\langle \cdot, \cdot \rangle$ we can use the isomorphism of $V^* \cong V$ to understand

$$U^0 = \{v \in V \,|\, \forall u \in U,\ \langle u, v \rangle = 0\} \leq V$$

Lemma 2.2. *THe annihilator is a subspace,* $\dim U^0 = \dim V - \dim U$.

Definition 2.3. *A subspace U is called **isotropic** if $U \subset U^0$.*

2.1 The Dirac Monopole

The standard maxwell equations prohibit monopoles, by which we mean point magnetic field sources, as $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{0}$. Dirac showed in [5] that it is possible to escape this conclusion by giving non-trivial topology to the space by allowing $\boldsymbol{B} = \frac{g}{4\pi r^2}\hat{\boldsymbol{x}}$ to have a singularity at $\boldsymbol{x} = \boldsymbol{0}$. We can calculate $\boldsymbol{\nabla} \cdot \boldsymbol{B} = g\delta(\boldsymbol{x})$. Removing this circle gives $\mathbb{R}^3 \setminus 0$, homotopic to S^2, and the corresponding magnetic two form on this sphere is

$$f = \frac{g}{4\pi} \sin\theta d\theta \wedge d\phi$$

and so the flux through a 2-sphere enclosing the origin is $\int_{S_R^2} f = g$. For $g \neq 0$, we know $f \neq da$ for a global $a \in \Omega^1(S^2)$ by Stokes' theorem, but if we take a cover of the sphere $U_{N/S}$ (north/south) and define gauge potentials

$$a_N = \frac{g}{4\pi}(1 - \cos\theta)d\phi \in \Omega^1(U_N)$$
$$a_S = \frac{g}{4\pi}(-1 - \cos\theta)d\phi \in \Omega^1(U_S)$$

On the intersect $U_N \cap U_S$ we have $da_N = f = da_S$ and $a_N = a_S + \frac{g}{2\pi}d\phi$.
Now taking $A = ia$, $F = if$, we have that $g_{NS}(\theta, \phi) = e^{-i\frac{g\phi}{2\pi}}$. Requiring that this is a well-defined transition function gives $g \in \mathbb{Z}$. This is equivalent to the integrality of the Chern number.
We will not want to consider this as this solution is not solitonic (it has infinite mass), but for further discussion see [10].

2.2 Pauli Matrices

Definition 2.4 (Pauli Matrices)**.** *The* ***Pauli matrices*** *are*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note they are all Hermitian and traceless.

Fact 2.5. $\sigma_i\sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k \Rightarrow \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$

2.3 $SU(2)$

We can write

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

This can be expressed as, for $A \in SU(2)$

$$A = a_0 I + i\boldsymbol{a} \cdot \boldsymbol{\sigma}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, and $a_0^2 + |\boldsymbol{a}|^2 = 1$. Hence $SU(2) \cong S^3$. In addition, by parametrising $SU(2)$ by the a_i, it can be seen that $\{ i\sigma_i \}$ forms a basis of $\mathfrak{su}(2)$. It is typical to normalise this basis to $\{ T^a = -\frac{1}{2} i\sigma_a \}$.

Lemma 2.6. *The structure constants in this basis* $\{ T^a \}$ *are* $f_c^{ab} = \epsilon_{abc}$*.*

Corollary 2.7. *The Killing form is given by* $\kappa\left(T^a, T^b\right) = \kappa^{ab} = -2\delta^{ab} = 4\,\mathrm{Tr}(T^a T^b)$*. Hence* $\kappa(X, Y) = 4\,\mathrm{Tr}(XY)$

2.4 Degree of a Map

We will want to consider continuous maps $f : X \to Y$ between connected oriented n-dimensional manifold. We state the following lemma:

Lemma 2.8. X *orientable iff* $H_n(X) = \mathbb{Z}$*.*

Proof. See [7]. □

Denote $[X], [Y]$ to be the generators of $H_n(X), H_n(Y)$ respectively. Recalling that we get an induced homomorphism $f_* : H_n(X) \to H_n(Y)$ we make the following definition:

Definition 2.9. *The* ***degree*** *of* f *is defined s.t.*

$$f_*([X]) = \deg(f)[Y]$$

Now recall we have a pairing between k-forms and k-chains on a manifold given by

$$\langle c, \omega \rangle = \int_c \omega$$

and this descends to a pairing between cohomology and homology. With this we have

$$\langle f_*[c], \omega \rangle = \langle [c], f^*[\omega] \rangle$$

Meaning that we can also express the degree via

$$\deg(f) \int_Y \omega = \int_X f^* \omega$$

for some top form on Y. This reformulation gives a convenient way to interpret the degree:

Proposition 2.10. *Take $y \in Y$ s.t. $f^{-1}(y) \subset X$ is a set of isolated points, say $\{x_i\}_{i=1}^m$. Then*

$$\deg f = \sum_{i=1}^{m} \operatorname{sgn}\left(J(x_i)\right)$$

Proof. See [10], the idea is to choose a volume form on Y localised around y, which pulls back to a volume form on X localised around the x_i, with the ± 1 coming from the change of orientation. □

2.5 Properties of Curves

Here we want to take an example curve and consider the kind of constructs on it we will be looking at. To this we will effectively go through Miranda [12] and apply the relevant sections.
Define the degree-6 homogeneous polynomial $P : \mathbb{C}^3 \to \mathbb{C}$ by

$$P(x, y, z) = y^6 - x^6 + z^2 x^4$$

Define the corresponding projective plane curve

$$X = \left\{ [x : y : z] \in \mathbb{P}^2 \,|\, P(x, y, z) = 0 \right\}$$

Throughout we will want to compare our by-hand calculations with numerics in Sage, so we start by initialising X.

```
sage: x,y,z = QQ['x,y,z'].gens()
sage: X = Curve(y^6-x^6+z^2*x^4)
```

2.5.1 Singularities

The corresponding affine plane curve on the intersection with the open set $U_x = \{x \neq 0\}$ is $y^6 = 1 - z^2$. Hence

$$X = \left\{ [1 : y : z] \,|\, y^6 = 1 - z^2 \right\} \cup \{[0 : 0 : 1]\}$$

Proposition 2.11. *X is singular at $[0 : 0 : 1]$ only.*

Proof. We have

$$\frac{\partial P}{\partial x} = 2x^3(2z^2 - 3x^2), \qquad \frac{\partial P}{\partial y} = 6y^5, \qquad \frac{\partial P}{\partial z} = 2zx^4$$

It can be seen that a common solution to these must have $y = 0$ from $\partial_y P = 0$, and then either $z = 0$ or $x = 0$ from $\partial_z P = 0$. If $z = 0$, $\partial_z P = 0$ enforces $x = 0$ which isn't in $\mathbb{P}^2$. If $x = 0$, z is arbitrary and we find the point $[0 : 0 : 1] \in \mathbb{P}^2$. This is calculated in Sage as follows:

```
sage: X.singular_points()
[(0 : 0 : 1)]
```

□

We need to get an understanding of this singularity. If we take the neighbourhood $U_z = \{z \neq 0\}$ the corresponding affine plane curve on the intersection is

$$p(x, y) = y^6 - x^6 + x^4 = 0$$

as

$$X = \left\{[x : y : 1] \,|\, y^6 - x^6 + x^4 = 0\right\} \cup \left\{[x : y : 0] \,|\, y^6 - x^6 = 0\right\}$$

Proposition 2.12. *The only singular point on the affine plane curve $p(x, y) = 0$ is $(0, 0)$ and it is 2-monomial.*

Proof. $\partial_x p(x, y) = 2x^3(2-3x^2)$, $\partial_y p(x, y) = 6y^5$. It can be seen that the only simultaneous solution to these is $(0, 0)$. Here we write $p(x, y) = g(x, y)^4 - h(x, y)^6$ where

$$\begin{aligned} g(x, y) &= x\left[1 - \frac{1}{4}x^2 + \dots\right] = x\left[1 - x^2\right]^{\frac{1}{4}} \\ h(x, y) &= iy \end{aligned}$$

We then note $\gcd(4, 6) = 2$. □

As such by general theory we know that we can resolve this singularity to make X into a compact Riemann surface. This process involves removing the singularity at $[0 : 0 : 1]$ and patching the two holes created.

2.5.2 Genus

We now check some topology using Riemann-Hurwitz.

Proposition 2.13. $g(X) = 2$.

Proof. As standard we consider the map $X \to \mathbb{P}^1$ taking the coordinate y. Then z is generically double valued so the degree of the map is 2. This is ramified at the the 6 roots of $1 - y^6 = 0$, so using $g(\mathbb{P}^1)$

$$g(X) = 1 + 2(0 - 1) + \frac{1}{2} \times 6 \times (2 - 1) = 1 - 2 + 3 = 2$$

This is calculated in Sage as follows:

```
sage: X.genus()
2
```

□

Remark. *This is to be expected, as here we are dealing with a hyperelliptic curve, of the form $z^2 = h(y)$ where h is a polynomial of degree $2g + 2$.*

2.5.3 Symmetries

From the above prop we have the following corollary.

Corollary 2.14. *Through Hurwitz' theorem we know* $|\mathrm{Aut}(X)| \le 84$.

This is a very lax bound, we know many groups of order ≤ 84. We now try to calculate $\mathrm{Aut}(X)$ more systematically. Firstly we note that as a hyperelliptic curve it comes with the hyperelliptic involution $(y,z) \mapsto (y,-z)$ generating a C_2 symmetry. We also have a D_6 dihedral group of automorphisms generated by $r : (y,z) \mapsto (\zeta y, z)$ and $s : (y,z) \mapsto (1/y, iz/y^3)$ where $\zeta = e^{\frac{i\pi}{3}}$. We can check that as $\zeta^3 = -1$

$$srs : (y,z) \mapsto (1/y, iz/y^3) \mapsto (\zeta/y, iz/y^3) \mapsto (\zeta^{-1}y, i^2\zeta^{-3}z) = (\zeta^{-1}y, z)$$

giving $srs = r^{-1}$ as required from the dihedral group. The hyperelliptic involution commutes with this dihedral action, so in total we have found a $\mathrm{Aut}(X) \ge C_2 \times D_6$. As $|C_2 \times D_6| = 24$, by Hurwitz' theorem we know that we could have at most one other automorphism of order either 2 or 3. This turns out not to be the case, and to see a general classification see for example [13].
To do the calculation in Sage, not we can use a result in [9] (translated here):

Lemma 2.15. *If X is a hyperelliptic curve then* $\mathrm{Aut}(J(X), a) \cong \mathrm{Aut}(X)$ *where a is the canonical principal polarisation of the Jacobian.*

This allows us to calculate in Sage:

```
sage: A.<u,v> = QQ[]
sage: Mod = sage.schemes.riemann_surfaces.riemann_surface
sage: S = Mod.RiemannSurface(u^2-1+v^6)
sage: G = S.symplectic_automorphism_group()
sage: G.structure_description()
'(C6␣x␣C2)␣:␣C2'
```

Remark. *Note that above Sage is acting as a GAP wrapper. The formatting translates to say* $G \cong (C_6 \times C_2) \rtimes C_2$

2.5.4 Riemann's Bilinears and Period Matrices

We now want to consider the pairing of homology and cohomology on X. This can lead to information about the Abel-Jacobi map, but will also lead to an invariant of a Riemann surface. Relevant references include Eynard's Lectures on Compact Riemann Surfaces, Miranda, these notes.

Definition 2.16. *A homology basis* $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b})$ *is* ***canonical*** *if* $\boldsymbol{c} \cdot \boldsymbol{c} = J$ *where we are taking the intersection number and* $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

Remark. *Despite the term canonical, there is no unique choice of such a basis. Given any homology basis* $\boldsymbol{c}$ *we can make a new basis by taking* $M\boldsymbol{c}$ *for* $M \in GL_{2g}(\mathbb{Z})$*. If the original basis was canonical then so is the transformed basis iff* $MJM^T = J$*, which is the condition for* M^T *to be a symplectic matrix, giving* M *symplectic.*

The object we want to calculate is explicitly the period matrix, defined as follows:

Definition 2.17. *Let $\{a_i, b_i\}_{i=1}^g$ be a canonical choice of homology basis, and $\{\omega_i\}_{i=1}^g$ a basis of holomorphic differentials. Then the* ***matrix of periods*** *is $\Omega = (A, B)$ where we calculate the $g \times g$ matrices A, B by*

$$A_{ij} = \int_{a_j} \omega_i, \qquad B_{ij} = \int_{b_j} \omega_i$$

In order to say something about the form of Ω, we need the following result:

Proposition 2.18 (Riemann's Bilinear Identities)**.** $\Omega J \Omega^T = 0$ *and* $-i\overline{\Omega} J \Omega^T > 0$

Corollary 2.19. *For fixed differential basis $\{\omega_i\}$, $\exists \boldsymbol{c}$ a canonical basis s.t $A_{ij} = \delta_{ij}$.*

Definition 2.20. *The* ***Riemann matrix*** *corresponding to Ω is $\tau = A^{-1}B$.*

Definition 2.21. *We define the* ***Siegel upper half space*** *to be*

$$\mathbb{H}_g = \left\{ M \in M_g(\mathbb{C}) \,|\, M^T = M, \operatorname{Im} M > 0 \right\}$$

Lemma 2.22. $\tau \in \mathbb{H}_g$.

Example 2.23. *$\mathbb{H}_1 = \mathbb{H} \subset \mathbb{C}$ is the upper half plane, explaining the name and notation convention. The case $g = 1$ corresponds to elliptic curves, and we are recovering the usual τ we assign which tells us about the period lattice of the torus. This is not a coincidence, and we shall later related τ to the Jacobian of the curve.*

Remark. *There was a choice made in arranging this matrix as $g \times 2g$. Had we chosen instead Ω to be $2g \times g$ as*

$$\tilde{\Omega}_{ij} = \int_{c_i} \omega_j$$

where $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b})$ is the tuple of cycles, then we get that $\tilde{\Omega} = \Omega^T \Leftrightarrow \tilde{A} = A^T, \tilde{B} = B^T$. In order to continue this correspondence to get $\tilde{\tau} = \tau^T$ we need $\tilde{\tau} = \tilde{B}\tilde{A}^{-1}$. As τ is symmetric, we know that $\tilde{\tau} = \tau$, so our value of Riemann matrix is not dependent on convention.

Lemma 2.24. *If we change canonical homology basis by $\boldsymbol{c}' = M\boldsymbol{c}$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(2g, \mathbb{Z})$, then Ω transforms as $\tilde{\Omega}' = M\tilde{\Omega}$ and hence $\tilde{\tau}' = (d\tilde{\tau} + c)(b\tilde{\tau} + a)^{-1}$.*
If we change cohomology basis by $\boldsymbol{\omega}' = G\boldsymbol{\omega}$ for $G \in GL_g(\mathbb{C})$, then Ω transforms as $\Omega' = G\Omega$ and hence $\tau' = \tau$.

Remark. *It is for this reason that, although taking Ω to be $g \times 2g$ is seemingly a more prevalent convention, the latter is more sensible because of the corresponding actions. Moreover, if conjugate M with the outer automorphism $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ then we get $\tilde{\tau}' = (a\tilde{\tau} + b)(c\tilde{\tau} + d)^{-1}$, which recovers the fractional linear transformation action of $SL_2(\mathbb{Z}) \cong Sp_2(\mathbb{Z})$ known in the case of elliptic curves. The reason for the conjugation is as with elliptic curves, we typically normalise the bottom element to 1 instead.*

We now want to return from the theory realm to calculate τ explicitly for our curve. To do this concretely we recall the definition of monodromy:

Definition 2.25. *Given a n-fold covering $f : X \to Y$ and a closed path γ based at $p \in Y$, then γ lift to a path $\hat{\gamma}$ in X s.t. $f \circ \hat{\gamma} = \gamma$. Then γ induces a bijection $\sigma_\gamma : f^{-1}(p) \to f^{-1}(p)$ called the* ***monodromy associated to*** *γ. By fixing an ordering of $f^{-1}(p)$ we identify $\sigma \in S_n$.*

Remark. *The monodromy is fixed under basepoint preserving homotopies of paths, and actually we get a functor $\Pi_1(Y) \to Set$. See nlab for more.*

We now view our curve X as a 6-fold covering of $\mathbb{P}^1$ with branch points at $z = \pm 1$. We want to first calculate the monodromy associated to encircling these points, so let $\gamma_\pm$ be a curve in $\mathbb{P}^1$ that encircles ± 1 once anticlockwise (explicitly choose a circle of small radius ϵ). On $\gamma_\pm$ take $z = \pm 1 + \epsilon e^{i\theta}$.
As we recall $y^6 = 1 - z^2$, on γ_+ we see

$$y^6 = -\epsilon e^{i\theta}\left(2 + \epsilon e^{i\theta}\right)$$

and on γ_-

$$y^6 = \epsilon e^{i\theta}\left(2 - \epsilon e^{i\theta}\right)$$

To both of these the associated monodromy can be written as ζ the fundamental solution to $\zeta^6 = 1$ as before, where we mean that multiplying by ζ acts as the cycle of sheets that corresponds to the paths $\gamma_\pm$. Observe then that the curve $\gamma = \gamma_+ - \gamma_-$ (a figure of 8 if we deform so the curves intersect at $z = 0$) has associated monodromy 0, but is non-contractible, and so corresponds to cycles on X. Moreover, by changing the sheet the lift starts on we get actually get 6 distinct cycles, and we can choose them to have the right intersection numbers we would want of a canonical homology basis. Explicitly we take

$$\begin{aligned}
a_1 &= \gamma : (1 \to 2 \to 1) \\
b_1 &= \gamma : (2 \to 3 \to 2) \\
a_2 &= \gamma : (4 \to 5 \to 4) \\
b_2 &= \gamma : (5 \to 6 \to 5)
\end{aligned}$$

where hopefully the notation is self-explanatory.

Remark. *One might ask, how do we know these have the right orientation to get the correct intersection?*

To determine what forms we need to integrate to get the period matrix we recall a statement in Miranda (pg 112)

Lemma 2.26. *Suppose that X is a projective plane curve of degree d $F(x, y, z) = 0$ where $F(x, y, 1) = f(x, y)$ then if $p(x, y)$ is a polynomial of degree at most $d - 3$ then*

$$p(x, y)\frac{dx}{\left(\frac{\partial f}{\partial y}\right)}$$

is a holomorphic differential.

This applies to tell us that we can get a basis of holomorphic on differentials on X with $\left\{\frac{dz}{y^5}, \frac{dz}{y^4}\right\}$.

Remark. *The polynomials p we are taking here are actually $6, 6y$. We know that we only need $g = 2$ of them so be sufficient.*

Now we can explicitly calculate. We fix the 1st sheet by asking that $y(z = 0) = 1$. We then set a normalisation our basis called ω_1, ω_2 respectively by asking that

$$\int_{a_1} \omega_i = 1$$

.

Remark. *This can be done as over our cycles the forms will not integrate to zero, as can be seen for example with*

$$\int_{a_1} \frac{dz}{y^5} = (1 - \zeta^{-5}) \int_{-1}^{1} \frac{dz}{(1 - z^2)^{5/6}} = (1 - \zeta) B\left(\frac{1}{2}, \frac{1}{6}\right)$$
$$\int_{a_1} \frac{dz}{y^4} = (1 - \zeta^{-4}) \int_{-1}^{1} \frac{dz}{(1 - z^2)^{4/6}} = (1 - \zeta^2) B\left(\frac{1}{2}, \frac{1}{3}\right)$$

where to do the above calculation we have used

$$\int_{-1}^{1} \frac{dx}{(1 - x^2)^{\alpha}} = 2 \int_{0}^{1} \frac{dx}{(1 - x^2)^{\alpha}} = \int_{0}^{1} t^{-\frac{1}{2}} (1 - t)^{-\alpha} dt = B\left(\frac{1}{2}, 1 - \alpha\right)$$

One might wonder is there a way to use the Pochhammer contour as part of the homology basis to recreate this result slightly more easily?

We can get onto other sheets just by multiplying y by powers of ζ, so for example we can calculate that

$$\begin{aligned} \int_{b_1} \omega_1 &= \zeta^{2 \times -5} \left[(0 \mapsto 1) + \zeta^{-1 \times -5} (1 \mapsto -1) + (-1 \mapsto 0) \right] \\ &= \zeta^{-10} (1 - \zeta^5)(-1 \mapsto 1) \\ &= -\zeta^{-5} (1 - \zeta^{-5})(-1 \mapsto 1) \end{aligned}$$

and following this through we get that the period matrix is

$$\Omega = \begin{pmatrix} 1 & \zeta^{-5(4-1)} & \zeta^{-5(2-1)} & \zeta^{-5(5-1)} \\ 1 & \zeta^{-4(4-1)} & \zeta^{-4(2-1)} & -\zeta^{-4(5-1)} \end{pmatrix} = \begin{pmatrix} 1 & -1 & \zeta & -\zeta \\ 1 & 1 & \zeta^2 & \zeta^2 \end{pmatrix}$$

and so

$$\tau = \frac{\zeta}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ \zeta & -\zeta \end{pmatrix} = \frac{\zeta}{2} \begin{pmatrix} 1 + \zeta & -(1 - \zeta) \\ -(1 - \zeta) & 1 + \zeta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{3}i & -1 \\ -1 & \sqrt{3}i \end{pmatrix}$$

This can be calculated numerically in Sage as below (output formatted to fit in the document):

```
sage: S.period_matrix()
[ -0.6-1.1*I  0.6+1.1*I 1.2+0.0*I -1.2+0.0*I]
[ -1.1-0.6*I -1.1-0.6*I 0.0-1.2*I  0.0-1.2*I]
sage: S.riemann_matrix()
[ 0.000+0.866*I 0.500+0.000*I]
[ 0.500+0.000*I 0.000+0.866*I]
```

Remark. *These numerical results seem to suggest the analytic answer is wrong, and that I should really be finding*

$$\Omega \sim \begin{pmatrix} 1 & -1 & \zeta^2 & -\zeta^2 \\ 1 & 1 & \zeta & \zeta \end{pmatrix}$$

giving

$$\tau = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \zeta^2 & -\zeta^2 \\ \zeta & \zeta \end{pmatrix} = \frac{\zeta}{2}\begin{pmatrix} 1+\zeta & 1-\zeta \\ 1-\zeta & 1+\zeta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3}i & 1 \\ 1 & \sqrt{3}i \end{pmatrix}$$

Remark. *Because of the action of both the modular group and the general linear group on Ω, it is difficult to inspect the computed answer to see if it is correct. This is less true of τ, for which the induced action of the general linear group is trivial, but we retain a the action of the modular group now through matrix-valued fractional linear transforms.*

Remark. *We could also try this calculation viewing X as a standard hyperelliptic. General theory says that we have a basis of $\Omega^1(X)$ given by*

$$\left\{\omega_j = \frac{y^j\, dy}{z}\right\}_{j=0}^{g-1}$$

and a generating set of $H_1(X, \mathbb{Z})$ given by a suitable choice of independent closed paths $c = \gamma_1 - \gamma_2$ where γ_i are the two lifts of a path in $\mathbb{P}^1$ between branch points of the map $X \to \mathbb{P}^1$. In order to find the period matrices explicitly we need to calculate

$$\int_c \omega_j$$

for these loops c. To be explicit for the paths in the base that we choose we take

$$\begin{aligned} a_1 &= \left\{e^{\frac{it\pi}{3}} \,|\, t \in [0,1]\right\} \\ b_1 &= \left\{e^{\frac{i(t+1)\pi}{3}} \,|\, t \in [0,1]\right\} \\ a_2 &= \left\{e^{\frac{i(t+3)\pi}{3}} \,|\, t \in [0,1]\right\} \\ b_2 &= \left\{e^{\frac{i(t+5)\pi}{3}} \,|\, t \in [0,1]\right\} \end{aligned}$$

2.5.5 Invariants

Write up some notes on the j-invariant, Igusa invariants for genus 2, and the larger conversation. Should include some notes on modular forms.

2.6 Spinors

2.6.1 Spinor Bundles

Definition 2.27. *The **complexification** of a vector space V is the vector space $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$.*

Proposition 2.28. *$V_{\mathbb{C}}$ comes with a well defined map of conjugation $v \otimes z \to v \otimes \bar{z}$. This is a $\mathbb{C}$-linear isomorphism onto $\overline{V_{\mathbb{C}}}$.*

Lemma 2.29. $(V^*)_{\mathbb{C}} = (V_{\mathbb{C}})^*$.

We will now state a few results about $SL_2(\mathbb{C})$ bundles:

Lemma 2.30. *A rank-2 complex vector bundle $S \to M$ has a reduction of structure group to a $SL_2(\mathbb{C})$-bundle iff $\exists \varepsilon \in \Gamma(M, \wedge^2 S)$ a non-degenerate form.*

Remark. *Such an ϵ will be a symplectic form on S.*

Corollary 2.31. *The cotangent bundle on a 2-dimensional manifold M reduces to a $SL_2(\mathbb{C})$ bundle iff the canonical bundle is trivial.*

Definition 2.32. *Suppose M is a 4-dimensional manifold s.t $\exists S \to M$ a $SL_2(\mathbb{C})$-bundle s.t.*

$$TM_{\mathbb{C}} \cong S \otimes \overline{S}$$

*Then we call S a **spinor bundle** on M. A section $s \in \Gamma(S)$ is called a **spinor**.*

Remark. *This is a quite specific definition, so to see this done in more generality check out [11].*

Notation. *Given a spinor bundle $S \to M$ we will use the notation*

$$S^A = S^- = S \qquad S^{A'} = S^+ = \overline{S} \qquad S_A = S_- = S^* \qquad S_{A'} = S_+ = \overline{S}^*$$

Each of the above comes with an associated non-degenerate two form $\epsilon^{\pm} \in \Gamma(M; \wedge^2 S^{\pm})$, $\epsilon_{\pm} \in \Gamma(M; \wedge^2 S_{\pm})$, and these provide isomorphisms to the dual as follows:

$$\epsilon^{A'B'} = \epsilon^+ : S_+ \to S^+ \quad \epsilon^{AB} = \epsilon^- : S_- \to S^- \quad \epsilon_{A'B'} = \epsilon_+ : S^+ \to S_+ \quad \epsilon_{AB} = \epsilon_- : S_- \to S_-$$

If we take an open $U \subset M$ and trivialise s.t $S|_U \cong \mathbb{C}^2$, we have the standard action of $SL_2(\mathbb{C})$ given by $v \mapsto gv$ for $g \in SL_2(\mathbb{C})$. We think of this as saying that a spinor in S transforms in the fundamental representation of $SL_2(\mathbb{C})$. We can then build the following table:

Bundle	Representation	Actions
S^-	Fundamental	$v \mapsto gv$
S_-	Dual	$v \mapsto v^T g^T$
S^+	Congujate	$v \mapsto \overline{g}v$
S_+	Conjugate Dual	$v \mapsto v^T \overline{g}^T$

Definition 2.33. *A **spinor frame** is a choice of frame of S^- wrt which*

$$\epsilon^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

2.6.2 Spinors on Minkowski Space

One important example of Spinors will be those on Minkowski space, which we will now denote as M^4, so we shall cover it in slightly more detail.

Remark. *Here, as in the twistor section, we shall follow Ward & Wells [16]. They remark that we should really consider Minkwoski space as an affine space in order to not distinguish the origin, and cite [15] for a further discussion. The result of this is that if we choose an origin $0 \in M^4$ we have $M^4 \cong T_0M^4$.*
Note also, [16] uses the mostly-minus Minkowski metric, so initially this section should be assumed to be using this convention unless stated otherwise.

We first introduce a useful coordinate system via the following map to $H_2 = \{2 \times 2 \text{ Hermitian matrices}\}$:

$$M^4 \overset{\cong}{\to} H_2$$
$$x = (x^\mu) \mapsto \tilde{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

Note the correspondence can be written as $\tilde{x} = x^\mu \sigma_\mu$ where $\sigma_\mu = (I, \boldsymbol{\sigma})$ corresponds to the Pauli matrices.

Proposition 2.34. $\det \tilde{x} = \frac{1}{2} |x|^2$.

Proof. It is simple to calculate

$$\det \tilde{x} = \frac{1}{2} \left[(x^0 + x^3)(x^0 - x^3) - (x^1 - ix^2)(x^1 + ix^2) \right] = \frac{1}{2} \left[(x^0)^2 - (x^3)^2 - (x^1)^2 - (x^2)^2 \right]$$

□

Note that under the same isomorphism, when we complexify, we get that $M^4_{\mathbb{C}} \cong M_2(\mathbb{C})$. This is as every complex can be decomposed into a Hermitian and anti-Hermitian part.
Now let $S = \mathbb{C}^2$, where we consider an element $s \in S$ as a column vector. Then $S^* = \mathbb{C}^2$ with elements $\tilde{s}$ considered as row vectors.

Proposition 2.35. *S is a spinor bundle on M^4.*

Proof.

$$T_0M^4 \otimes \mathbb{C} \cong M^4 \otimes \mathbb{C} \cong M_2(\mathbb{C}) \cong S \otimes S^*$$

We are then done as $\overline{S} = S^*$ here. □

Under this correspondence, we get the following result

Lemma 2.36. *$v \in T_0M^4_{\mathbb{C}} \otimes \mathbb{C}^4$ is null iff $\exists s \in S, \tilde{s} \in S^*$ s.t. $v = s\tilde{s}$. Moreover v corresponds to a real vector iff $\tilde{s} = rs$ for $r \in \mathbb{R}$.*

Remark. *By the above a vector $v \in T_0M^4_{\mathbb{C}}$ is real iff $\exists s \in S$ s.t. $v = ss^\dagger$. In this sense we can think of spinors as the square root of real null vectors.*

2.7 Twistors

2.7.1 Grassmannians and Flag Manifolds

Definition 2.37. *Let V be an n-dimensional vector space over $\mathbb{F}$. Define the* ***Grassmannian manifold of k-dimensional subspaces*** *as $G_k(V)$.*

Lemma 2.38. *If $\mathbb{F} = \mathbb{R}$, $G_k(V) \cong {}^{O(n)}\!/\!_{[O(k) \times O(n-k)]}$*

Proof. We can view the Grassmannian as the quotient of $GL(V)$ by the stabiliser of a k-dim subspace. □

Remark. *Likewise if $\mathbb{F} = \mathbb{C}$, $G_k(V) \cong {}^{U(n)}\!/\!_{[U(k) \times U(n-k)]}$,*

Corollary 2.39. $\dim G_k(V) = k(n-k)$.

Example 2.40. $G_1(\mathbb{C}) \cong \mathbb{CP}^1$.

Notation. *If we pick a basis of complex vector space V to make $V \cong \mathbb{C}^n$, we identify $G_k(V) \cong G_k(\mathbb{C}^n)$ which we will notation $G_{k,n}$.*

If we denote $M^*_{n\times k}(\mathbb{C})$ to be the full rank $m \times k$ matrices we can construct G via the projection

$$\begin{aligned} M^*_{n\times k}(\mathbb{C}) &\to G_{k,n} \\ M &\mapsto \text{Span}\{\text{columns of } M\} \end{aligned}$$

$M^*_{n\times k}(\mathbb{C})$ carries a transitive left action via multiplication that doesn't affect span, and so we get a bundle structure. We can then consider coordinates on $G_{k,n}$ as $Z \in M_{(n-k)\times k}(\mathbb{C}) \cong \mathbb{C}^{k(n-k)}$ via the map

$$\begin{aligned} \phi : M_{(n-k)\times k}(\mathbb{C}) &\to G_{k,n} \\ Z &\mapsto \left[\begin{pmatrix} Z \\ I_k \end{pmatrix}\right] \end{aligned}$$

It will now be necessary to generalise this concept slightly:

Definition 2.41. *Given a fixed complex n-dimensional vector space V and $\{d_i\}_{i=1}^m$ integers satisfying $1 \leq d_1 < d_2 < \cdots < d_m < n$ we define a* ***flag manifold of type*** *$(d_1, \ldots, d_m)$ as*

$$F_{d_1 \ldots d_m}(V) = \{(S_1, \ldots, S_m)\,|S_i \text{ a } d_i\text{-dimensional subspace of } V,\ S_1 \subset S_2 \subset \cdots \subset S_m\}$$

2.7.2 Twistor Space

Now given $\mathbb{T}$ a 4d $\mathbb{C}$ vector space we can get a **double fibration**

$$F_1(\mathbb{T}) \overset{\pi_1}{\twoheadleftarrow} F_{12}(\mathbb{T}) \overset{\pi_2}{\twoheadrightarrow} F_2(\mathbb{T})$$

from the projections of (S_1, S_2)

Proposition 2.42. *The π_i are holomorphic.*

Definition 2.43. *Given seta A, B a **correspondence** is a map $f : A \to \mathcal{P}(B)$ sending each $a \in A$ to $f(a) \subset B$.*

The double fibration induces natural correspondences $c = \pi_2 \circ \pi_1^{-1}$ and $c^{-1} = \pi_1 \circ \pi_2^{-1}$.

Notation. *We shall call $\mathbb{T}$ the **twistor space**, and denote*

- $\mathbb{PT} = F_1(\mathbb{T}) \cong \mathbb{CP}^3$ ***projective twistor space***
- $M_{\mathbb{C}} = F_2(\mathbb{T}) \cong G_{2,4}(\mathbb{C})$ ***compactified complexified Minkowski space***
- $F = F_{12}(\mathbb{T})$ *the **correspondence** between $\mathbb{PT}$ and $M_{\mathbb{C}}$.*

For $A \subset \mathbb{PT}$ we shall let $c(A) = \tilde{A}$, and for $B \subset M_{\mathbb{C}}$ let $\hat{B} = c^{-1}(A)$.

Idea. *The point of twistor geometry is understanding how to use the correspondence*

$$\mathbb{PT} \leftarrow F \to M_{\mathbb{C}}$$

to transfer information about (subsets of) $\mathbb{PT}$ to (subsets of) $M_{\mathbb{C}}$.

Remark. *F is sometimes denoted as $\mathbb{PS}$ and called the **projective spinor bundle**. We will hopefully return to this later.*

Proposition 2.44. *For $p \in \mathbb{PT}$, $q \in M_{\mathbb{C}}$, $\tilde{p} \cong \mathbb{P}^2$ and $\hat{q} \cong \mathbb{P}^1$*

We now wish to understand the correspondence in coordinates. We consider the map ϕ as described before, though now we send

$$z \mapsto \left[\begin{pmatrix} iz \\ I_2 \end{pmatrix}\right]$$

for convenience, let $M_{\mathbb{C}}^I = \phi(M_2(\mathbb{C}))$, $\mathbb{PT}^I = c^{-1}(M_{\mathbb{C}}^I)$ and $F^I = \pi_2^{-1}(M^I)$.

Proposition 2.45. *$F^I \cong M^I \times \mathbb{P}^1$.*

Proof. The isomorphism is given as

$$\begin{aligned} M_{\mathbb{C}}^I \times \mathbb{P}^1 &\to F^I \\ (z, [v]) &\mapsto \left(\left[\begin{pmatrix} izv \\ v \end{pmatrix}\right], \left[\begin{pmatrix} iz \\ I_k \end{pmatrix}\right] \right) \end{aligned}$$

$\square$

Corollary 2.46. *In coordinates we can see the double fibration as*

$$\begin{aligned} \mathbb{PT}^I \leftarrow &\quad F^I \quad \to M_{\mathbb{C}}^I \\ [izv, v] \leftarrow &(z, [v]) \to z \end{aligned}$$

2.7.3 Actions on Twistor Space

We will often want to equip $\mathbb{T}$ with a Hermitian form Φ. We state a quick lemma:

Lemma 2.47. *Φ determines a volume form $\Omega \in \wedge^4 \mathbb{T}^*$ on $\mathbb{T}$ by*

$$\Omega = \operatorname{Im}\Phi \wedge \operatorname{Im}\Phi$$

Definition 2.48. *We define $SU(\mathbb{T}, \Phi) \subset GL(\mathbb{T})$ to be the subset preserving Φ and Ω.*

Example 2.49. *If Φ has signature $(2,2)$ then $SU(\mathbb{T}, \Phi) \cong SU(2,2)$.*

Given a $(2,2)$-signature Hermitian form on $\mathbb{T}$ $\exists (Z^\alpha)$ coordinates wrt which we have

$$\Phi = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \Leftrightarrow \Phi(Z,Z) = Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1}$$

Notation. *We denote the vector space $\mathbb{T}$ with the above coordinates adapted to Φ as $\mathbb{T}^\alpha$ and its dual as $\mathbb{T}_\alpha$. We call the corresponding dual coordinates W_α.*

$SL_2(\mathbb{C}) \leq SU(2,2)$ will act reducibly on $\mathbb{T}^\alpha$ meaning we have $\mathbb{T}^\alpha = S_1 \oplus S_2$, and so we write our coordinates as $Z^\alpha = (\omega^A, \pi_{A'})$.

2.7.4 Complexified Minkowski Space

It is at this point useful to discuss slightly more concretely what is meant by complexified Minkowski space. The resource to use here is [1].

Definition 2.50. *Let (M, g) be a smooth real manifold, and take a coordinate system x^a s.t. $g = g_{ab}dx^a dx^b$. The* ***complexification*** *is defined by allowing the x^a to take complex values and extending $g_{ab}(x)$ holomorphically.*

3 The Monopole Equations

3.1 Yang-Mills-Higgs equations

Definition 3.1. *Take a principal G-bundle $P \to M$, ω_{vol} an orientation on M, and $\langle \cdot, \cdot \rangle$ to be an* ad*-invariant inner product on $\mathfrak{g}$. Then the* ***Yang-Mills-Higgs actions*** *on M is*

$$S_{YMH}[A, \phi] = \int_M \Big[-|F|^2 - |D\phi|^2 - V(\phi) \Big] \omega_{vol}$$

where $F = dA + A \wedge A$ is the curvature associated to a section $A \in \Gamma(T^*M \otimes \mathrm{ad}(P))$, $D = d + A$ is the associated covariant derivative, and $\phi \in \Gamma(\mathrm{ad}(P))$.

To connect with physical theory we want our Lagrangian to be of the form kinetic-potential. This will manifest itself in our choice of signs by requiring that

$$-|D\phi|^2 = (\partial_0 \phi)^2 + \dots$$
$$-|F|^2 = E_i^2 + \dots$$

This is the reason for the somewhat strange looking sign choice at this point. Obviously in the end it will be equivalent to take

$$S_{YMH} = \int |F|^2 + |D\phi|^2 + V$$

when it comes to the variational equations, but this will not be true when we consider the energy functional.

Remark. *A common choice of potential function V is $V(\phi) = \lambda \left(1 - |\phi|^2\right)^2$, the ϕ^4**-potential**. By our choice of signs, we want $\lambda > 0$. We can check $V'(\phi) = -4\lambda(1 - |\phi|^2)\,|\phi|$.*

Definition 3.2. *A **monopoles** will be a soliton-like solution to the Yang-Mills-Higgs equations when $G = SU(2)$, $M = \mathbb{R}^4$ with the Minkowski metric, the principal bundle is $P = M \times G$, and the potential is ϕ^4.*

Proposition 3.3. *The variational equations corresponding to S_{YMH} in Minkowski $\mathbb{R}^4$ are the **Yang-Mills-Higgs equations***

$$\begin{aligned} DF &= 0 \quad \textit{(Bianchi)} \\ \star D \star F &= -\left[\phi, D\phi\right] \\ \star D \star D\phi &= -\frac{1}{2\,|\phi|} V'(\phi)\phi \end{aligned}$$

Proof. We first consider the equation that comes from the variation of A. Let $A_t = A + t\beta$, then $F_t = F + t\,(d\beta + \beta \wedge A + A \wedge \beta) + \mathcal{O}(t^2)$ and $D_t\phi = D\phi + t\,[\beta, \phi]$. Hence

$$S_t = S + 2t \int_M \left[-\langle F, D\beta\rangle - \langle D\phi, [\beta, \phi]\rangle\right] \omega_{vol} + \mathcal{O}(t^2)$$

Hence to be at a stationary point of the action variation we want

$$\int_M \left[-\langle F, D\beta\rangle - \langle D\phi, [\beta, \phi]\rangle\right] \omega_{vol} = 0$$

Using the fact that inner product is ad-invariant and letting D^* be the formal adjoint of D wrt to inner product $\langle\langle \eta, \omega\rangle\rangle = \int_M \langle \eta, \omega\rangle\, \omega_{vol}$ we can rewrite this as

$$\int_M \langle -D^*F + \left[D\phi, \phi\right], \beta\rangle\, \omega_{vol} = 0$$

Using results on the dual of the covariant derivative we can say that

$$D^*F = (-1)^{4(2-1)+1}(-1) \star D \star F = \star D \star F$$

Hence as β was a generic variation we must have $\star D \star F - [D\phi, \phi] = 0$.
We now consider a ϕ variation so $\phi_t = \phi + t\psi$. Note

$$|\phi_t| = \sqrt{\langle \phi_t, \phi_t\rangle} = \sqrt{|\phi|^2 + 2t\,\langle \phi, \psi\rangle + \mathcal{O}(t^2)} = |\phi| \sqrt{1 + \frac{2t\,\langle \phi, \psi\rangle}{|\phi|^2} + \mathcal{O}(t^2)} = |\phi| + t\frac{\langle \phi, \psi\rangle}{|\phi|} + \mathcal{O}(t^2)$$

so if we consider V as $V(\phi) = V(|\phi|)$ (i.e. as a function of a real variable) then

$$V(\phi_t) = V(\phi) + t\,|\phi|^{-1}\,V'(\phi)\,\langle\phi,\psi\rangle + \mathcal{O}(t^2)$$

Then a variational argument as before means that we need to set

$$\begin{aligned}
& & \int_M \left[-2\,\langle D\phi, D\psi\rangle - |\phi|^{-1}\,V'(\phi)\,\langle\phi,\psi\rangle\right]\omega_{vol} = 0\\
\Rightarrow & & -2D^*D\phi - |\phi|^{-1}\,V'(\phi)\phi = 0\\
\Rightarrow & & (-1)^{4(1-1)+1}(-1)(-1)\star D\star D\phi - \frac{1}{2\,|\phi|}V'(\phi)\phi = 0\\
\Rightarrow & & \star D\star D\phi + \frac{1}{2\,|\phi|}V'(\phi)\phi = 0
\end{aligned}$$

□

Remark. *The Dirac monopole is a solution in the case of $G = U(1)$.*

We may make these equations explicit in coordinates. The first approach is to try and substitute in coordinate expressions into the YMH equations. Taking coordinates x^μ on $\mathbb{R}^4$ and writing $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$

$$\begin{aligned}
\star F &= \frac{1}{4}F_{\mu\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}dx^\rho \wedge dx^\sigma\\
\Rightarrow D\star F &= \frac{1}{4}D_\tau F_{\mu\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}dx^\tau \wedge dx^\rho \wedge dx^\sigma\\
\Rightarrow \star D\star F &= \frac{1}{4}D_\tau F_{\mu\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}\epsilon^{\tau\rho\sigma}{}_\lambda dx^\lambda\\
&= \frac{1}{4}D_\tau F_{\mu\nu}\epsilon_{\rho\sigma}{}^{\mu\nu}\epsilon^{\rho\sigma\tau}{}_\lambda dx^\lambda\\
&= -D_\tau F^\mu{}_\nu\,\delta^\tau_{[\mu}\delta^\nu_{\lambda]}dx^\lambda\\
&= -D_\tau F^\tau{}_\lambda\,dx^\lambda
\end{aligned}$$

Hence the first monopole equation reads

$$D_\mu F^{\mu\nu} + [D^\nu\phi, \phi] = 0$$

Next we have

$$\begin{aligned}
\star D\phi &= \frac{1}{6}(D_\mu\phi)\epsilon^\mu{}_{\nu\rho\sigma}dx^\nu \wedge dx^\rho \wedge dx^\sigma\\
\Rightarrow D\star D\phi &= \frac{1}{6}(D_\tau D_\mu\phi)\epsilon^\mu{}_{\nu\rho\sigma}dx^\tau \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma\\
\Rightarrow \star D\star D\phi &= \frac{1}{6}(D_\tau D_\mu\phi)\epsilon^\mu{}_{\nu\rho\sigma}\epsilon^{\tau\nu\rho\sigma}\\
&= -(D_\tau D^\mu\phi)\delta^\tau_\mu = -D_\mu D^\mu\phi
\end{aligned}$$

which yields (taking the ϕ^4 potential)

$$D_\mu D^\mu\phi + 2\lambda(1 - |\phi|^2)\phi = 0$$

Collecting this with the Bianchi identity (taking it as $\star DF = 0$) we have

$$\begin{aligned}
\epsilon^{\rho\mu\nu\tau} D_\rho F_{\mu\nu} &= 0 \\
D_\mu F^{\mu\nu} + [D^\nu \phi, \phi] &= 0 \\
D_\mu D^\mu \phi + 2\lambda(1 - |\phi|^2)\phi &= 0
\end{aligned}$$

An alternative approach to deriving these equations is to first write the Lagrangian in coordinate form and then derive the variational equations. We take the inner product on $\mathfrak{g} = \mathfrak{su}(2)$ to be $\langle X, Y\rangle = -\frac{1}{2}\kappa(X, Y) = -2\operatorname{Tr}(XY)$ for concreteness, which gives $\langle t^a, t^b\rangle = \delta^{ab}$, and as such we need the mostly-positive Minkowski metric.

Remark. *It is possible to change around the signs here in order to assure that we use the mostly-minus metric, as Manton & Sutcliffe do. We will want to avoid this as the mostly positive makes more sense when reducing from Minkowski $\mathbb{R}^{n+1}$ to Euclidean $\mathbb{R}^n$.*

We have

$$\begin{aligned}
\langle F, F\rangle &= \frac{1}{4}\langle F_{\mu\nu}, F_{\rho\sigma}\rangle \langle dx^\mu \wedge dx^\nu, dx^\rho \wedge dx^\sigma\rangle \\
&= \frac{1}{4}\langle F_{\mu\nu}, F_{\rho\sigma}\rangle (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\
&= \frac{1}{2}\langle F_{\mu\nu}, F^{\mu\nu}\rangle \\
&= -\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) \\
\langle D\phi, D\phi\rangle &= \langle D_\mu\phi, D_\nu\phi\rangle \langle dx^\mu, dx^\nu\rangle \\
&= \langle D_\mu\phi, D_\nu\phi\rangle \eta^{\mu\nu} \\
&= -2\operatorname{Tr}(D_\mu\phi D^\mu\phi)
\end{aligned}$$

Hence the corresponding Lagrangian density is

$$\mathcal{L} = \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) + 2\operatorname{Tr}(D_\mu\phi D^\mu\phi) - V(\phi) = \sum_a \left[-\frac{1}{2}F^{(a)}_{\mu\nu}F^{(a)\mu\nu} - (D_\mu\phi)^{(a)}(D^\mu\phi)^{(a)}\right] - \lambda\left[1 - \sum_a (\phi^{(a)})^2\right]^2$$

To check that we have the correct signs in this Lagrangian we verify that it takes the form kinetic-potenial with

$$\begin{aligned}
\text{kinetic} &= -2\operatorname{Tr}(E_iE_i) - 2\operatorname{Tr}(D_0\phi D_0\phi) \\
\text{potential} &= -\operatorname{Tr}(F_{ij}F_{ij}) - 2\operatorname{Tr}(D_i\phi D_i\phi) + V(\phi)
\end{aligned}$$

where $E_i = F_{0i}$.

Remark. *Something we can immediately recognise is that in order to get finite energy solutions when $\lambda \neq 0$, we need $|\phi| \to 1$.*

We now recall the Euler-Lagrange equations for Lagrangian with field ψ

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = 0$$

The fields here are really the coefficients in $\mathfrak{su}(2)$ of A_μ, ϕ, that is they are $A_\mu^{(a)}, \phi^{(a)}$, so we expand

$$\begin{aligned} F_{\mu\nu}^{(a)} &= \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + A_\mu^{(b)} A_\nu^{(c)} \epsilon_{abc} \\ (D_\mu \phi)^{(a)} &= \partial_\mu \phi^{(a)} + A_\mu^{(b)} \phi^{(c)} \epsilon_{abc} \end{aligned}$$

giving

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^{(a)})} &= -2F^{(a)\mu\nu} \\ \frac{\partial \mathcal{L}}{\partial A_\mu^{(a)}} &= -2\epsilon_{bac} A_\nu^{(c)} F^{(b)\mu\nu} - 2\phi^{(c)} \epsilon_{bac} (D^\mu \phi)^{(b)} \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(a)})} &= -2(D^\mu \phi)^{(a)} \\ \frac{\partial \mathcal{L}}{\partial \phi^{(a)}} &= -2\epsilon_{cba} A_\mu^{(b)} (D^\mu \phi)^{(c)} + 4\lambda \phi^{(a)} \left[1 - \sum_b (\phi^{(b)})^2\right] \end{aligned}$$

We can now write the Euler-Lagrange equations

$$\begin{aligned} 0 &= -\partial_\nu F^{(a)\nu\mu} - \left[-\epsilon_{bac} A_\nu^{(c)} F^{(b)\mu\nu} - \phi^{(c)} \epsilon_{bac} (D^\mu \phi)^{(b)}\right] \\ &= -\left[\partial_\nu F^{(a)\nu\mu} + A_\nu^{(c)} F^{(b)\nu\mu} \epsilon_{cba}\right] - (D^\mu \phi)^{(b)} \phi^{(c)} \epsilon_{bca} \\ \Rightarrow 0 &= D_\nu F^{\nu\mu} + [D^\mu \phi, \phi] \end{aligned}$$

and

$$\begin{aligned} 0 &= -\partial_\mu (D^\mu \phi)^{(a)} - \left[-\epsilon_{cba} A_\mu^{(b)} (D^\mu \phi)^{(c)} + 2\lambda \phi^{(a)} \left[1 - \sum_b (\phi^{(b)})^2\right]\right] \\ &= -\left[\partial_\mu (D^\mu \phi)^{(a)} + A_\mu^{(b)} (D^\mu \phi)^{(c)} \epsilon_{bca}\right] - \lambda \phi^{(a)} \left[1 - \sum_b (\phi^{(b)})^2\right] \\ \Rightarrow 0 &= D_\mu D^\mu \phi + 2\lambda (1 - |\phi|^2) \phi \end{aligned}$$

We happily see that these two approaches agree, and we should see that these are indeed the sort of equations we want (e.g the ϕ equation looks like Klein-Gordon if we linearise around $|\phi| = 1$.)

3.2 BPS limit

The monopole equations we have found so far are second order, but we want to apply the classic strategy when working with topological solitons: write the energy functional of a static configuration as the integral of a square term plus a topological term, and then we locally must have a minimising solution by setting the squared term to 0. This will be possible if we set $\lambda = 0$ but retain that

$|\phi| = 1$ at infinity. More specifically we take the conditions

$$|\phi| = 1 - \frac{m}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right)$$
$$\frac{\partial\,|\phi|}{\partial\Omega} = \mathcal{O}\left(\frac{1}{r^2}\right)$$
$$|D\phi| = \mathcal{O}\left(\frac{1}{r^2}\right)$$

With $\lambda = 0$ we can rewrite the energy functional of static configurations at any time as

$$E[A,\phi] = \int_M [-\operatorname{Tr}(F_{ij}F_{ij}) - 2\operatorname{Tr}(D_i\phi D_i\phi)]\, d^3x$$

If we keep the same inner product on $\mathfrak{su}(2)$, but now take Euclidean $\mathbb{R}^3$ we can express

$$E[A,\phi] = \int_{\mathbb{R}^3} \left[|F|^2 + |D\phi|^2\right] d^3x$$

As we are in Euclidean $\mathbb{R}^3$ we can write

$$|\star D\phi|^2\,\omega_{vol} = \left\langle \star D\phi \wedge \star^2 D\phi\right\rangle = (-1)^{1(3-1)}\left\langle \star D\phi \wedge D\phi\right\rangle = (-1)^{2\times 1}\left\langle D\phi \wedge \star D\phi\right\rangle = |D\phi|^2\,\omega_{vol}$$

and, using the Bianchi identity,

$$\langle F, \star D\phi\rangle\,\omega_{vol} = \left\langle F \wedge \star^2 D\phi\right\rangle = (-1)^{1(3-1)}\langle F \wedge D\phi\rangle = (-1)^2\left[d\langle F\wedge\phi\rangle - \langle DF \wedge \phi\rangle\right] = d\langle F \wedge \phi\rangle$$

so

$$\begin{aligned} E[A,\phi] &= \int_{\mathbb{R}^3} \left[\langle F \mp \star D\phi, F \mp \star D\phi\rangle \pm 2\langle F, \star D\phi\rangle\right] d^3x \\ &= \int_{\mathbb{R}^3} |F \mp \star D\phi|^2\, d^3x \pm 2\lim_{R\to\infty}\int_{S_R^2}\langle F \wedge \phi\rangle \end{aligned}$$

Remark. *In the above discussion, make sure to check that as this the 3d Hodge star, our inner products are well defined in the sense that the are the inner product of a k-form with a k-form.*

This boundary term turns out to be a topological contribution, and we can see this in two ways. We first need to prove the following lemma

Lemma 3.4. *In the $\mathbb{R}^3$ bulk*

$$\langle \phi \wedge (D\phi \wedge D\phi)\rangle = \langle \phi \wedge (d\phi \wedge d\phi)\rangle + |\phi|^2\left[\langle F \wedge \phi\rangle - d\langle A \wedge \phi\rangle\right] - \frac{1}{2}\langle A \wedge \phi\rangle \wedge d\,|\phi|^2$$

Proof. Exercise. □

Corollary 3.5. *Given the decay conditions, on S^2_∞ we have*

$$\langle F \wedge \phi\rangle - d\langle A \wedge \phi\rangle = -\langle \phi \wedge (d\phi \wedge d\phi)\rangle$$

and hence the topological term is $\mp 2\int_{S^2_\infty}\langle \phi \wedge (d\phi \wedge d\phi)\rangle$.

The two interpretations are then as follows:

1. Degree of a map of spheres - We will want to recall some standard vector calculus results stated in the language of forms. Firstly recall that if we have a manifold M with volume form ω_M and orientable submanifold Σ with normal N we get a volume form on Σ given by

$$\omega_\Sigma = i_N \omega_M$$

If we parametrise Σ on an open patch U, that is find $\psi : \mathbb{R}^d \to U \subset \Sigma$, and do an area integral over $U \subset \Sigma$ in these coordinates, that is equivalent to pulling back ω_Σ by ψ.
By noting that, on S^2_∞, $\phi : S^2_\infty \to S^2_1 \subset \mathfrak{su}(2)$, where on this S^2 ϕ is also the normal, and that we have chosen the metric on $\mathfrak{su}(2)$ to be Euclidean, we have that

$$\int_{S^2_\infty} \langle \phi \wedge (d\phi \wedge d\phi) \rangle = \int_{S^2_\infty} \boldsymbol{\phi} \cdot (\partial_u \boldsymbol{\phi} \times \partial_v \boldsymbol{\phi})\, du dv = \int_{S^2_\infty} \phi^* \omega_{S^2} = 4\pi \deg \phi$$

where we have the used the notation $\phi = \boldsymbol{\phi} \cdot \boldsymbol{t}$, and the degree here refers to of ϕ as a map of spheres. This degree is a topological invariant.

2. Chern class of a line bundle - Consider the matrix $\boldsymbol{\phi} \cdot \boldsymbol{\sigma}$. We give the following lemma

Lemma 3.6. *Let $P = \frac{1}{2}(I + \boldsymbol{\phi} \cdot \boldsymbol{\sigma})$. Then P is a projection operator, i.e. $P^2 = P$.*

Proof. Do the multiplication □

Corollary 3.7. *$\boldsymbol{\phi} \cdot \boldsymbol{\sigma}$ has eigenvalues ± 1.*

Proof. Projection operators have eigenvalues $0, 1$. Hence we get the result, and as we can never have $\boldsymbol{\phi} \cdot \boldsymbol{\sigma} = \pm I$, both eigenvalues must occur. □

Now we can consider the eigenvector bundle

$$L = \left\{ (\boldsymbol{x}, \psi) \in S^2_\infty \times \mathbb{C}^2 \mid [\boldsymbol{\phi}(\boldsymbol{x}) \cdot \boldsymbol{\sigma}]\, \psi = \psi \right\}$$

Remark. *This will be a complex line bundle, so the only possible Chern class we can relate to it will be c_1.*

We can get a connection on the bundle by viewing it as a subbundle of $S^2_\infty \times \mathbb{C}^2$, and getting covariant derivative on vector fields by projection, that is $DX = PdX$.

Remark. *Note that here the variations in ψ that preserve the the eigenvector condition are exactly those that scale ψ by some element of $\mathbb{C}^\times$. This really makes the bundle look like the tangent bundle to the sphere S^2_∞ in the sense that there is a plane attached at every point of S^2.*

Now at every point $\boldsymbol{x} \in S^2_\infty$ if $\hat{\psi}(\boldsymbol{x})$ is a normalised eigenvector we have that a local section $\sigma : U \subset S^2_\infty \to E$ is given by $\sigma = h\hat{\psi}$ where $h : U \to \mathbb{C}$ is some scale. Then as we are dealing with 2×2 non-trivial projection matrices we can write $P = \hat{\psi}\hat{\psi}^\dagger$ giving

$$\begin{aligned} D(h\hat{\psi}) &= \hat{\psi}\hat{\psi}^\dagger \left[(dh)\hat{\psi} + h(d\hat{\psi}) \right] \\ &= (dh)\hat{\psi} + h\hat{\psi}\hat{\psi}^\dagger (d\hat{\psi}) \end{aligned}$$

What we have done by choosing a local section $\hat{\psi}$ is given a local trivialisation of the bundle. To find the connection locally we need to see how the covariant derivaitve acts locally on $\mathbb{C}$, that is how it acts on h. By writing

$$D(h\hat{\psi}) = \hat{\psi}\left[dh + h\hat{\psi}^\dagger d\hat{\psi}\right]$$

We can read off that $A = \hat{\psi}^\dagger d\hat{\psi}$ locally, giving $F_L = d\hat{\psi}^\dagger \wedge d\hat{\psi}$.
We now want to relate this to our map ϕ. As we are dealing with 2×2 non-trivial projection matrices we can write

$$P = \hat{\psi}\hat{\psi}^\dagger \Rightarrow \phi = -i\hat{\psi}\hat{\psi}^\dagger + \frac{i}{2}I \Rightarrow d\phi = -i\left[(d\hat{\psi})\hat{\psi}^\dagger + \hat{\psi}(d\hat{\psi}^\dagger)\right]$$

We can see that, as the inner product corresponds to the trace, $\langle d\phi \wedge d\phi\rangle = \langle(\partial_\mu\phi)(\partial_\nu\phi)\rangle\, dx^\mu \wedge dx^\nu = \frac{1}{2}\langle[\partial_\mu\phi, \partial_\nu\phi]\rangle\, dx^\mu \wedge dx^\nu = 0$ which means

$$\begin{aligned}\langle\phi \wedge (d\phi \wedge d\phi)\rangle &= -i\left\langle\left(\hat{\psi}\hat{\psi}^\dagger - \frac{1}{2}I\right)(d\phi \wedge d\phi)\right\rangle \\ &= -i\left\langle\hat{\psi}\hat{\psi}^\dagger\left[(d\hat{\psi})\hat{\psi}^\dagger + \hat{\psi}(d\hat{\psi}^\dagger)\right]^{\wedge 2}\right\rangle\end{aligned}$$

We expand out the latter, using the cyclicity of the trace and Leibniz' rule with $\hat{\psi}^\dagger\hat{\psi} = 1$, for example

$$\begin{aligned}\left\langle\hat{\psi}\hat{\psi}^\dagger(d\hat{\psi})\hat{\psi}^\dagger \wedge (d\hat{\psi})\hat{\psi}^\dagger\right\rangle &= \left\langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}^\dagger(d\hat{\psi})\hat{\psi}^\dagger \wedge (d\hat{\psi})\right\rangle \\ &= \left\langle\hat{\psi}^\dagger(d\hat{\psi})\hat{\psi}^\dagger \wedge (d\hat{\psi})\right\rangle \\ &= \left\langle\left[d(\hat{\psi}^\dagger\hat{\psi}) - (d\hat{\psi}^\dagger)\hat{\psi}\right]\hat{\psi}^\dagger \wedge (d\hat{\psi})\right\rangle \\ &= -\left\langle(d\hat{\psi}^\dagger)\hat{\psi} \wedge \hat{\psi}^\dagger(d\hat{\psi})\right\rangle \\ &= -\left\langle\hat{\psi}\hat{\psi}^\dagger\hat{\psi}(d\hat{\psi}^\dagger) \wedge \hat{\psi}(d\hat{\psi}^\dagger)\right\rangle\end{aligned}$$

$$\begin{aligned}\left\langle\hat{\psi}\hat{\psi}^\dagger(d\hat{\psi}) \wedge (d\hat{\psi}^\dagger)\right\rangle &= -\left\langle\hat{\psi}(d\hat{\psi}^\dagger)\hat{\psi} \wedge (d\hat{\psi}^\dagger)\right\rangle \\ &= \left\langle\hat{\psi}(d\hat{\psi}^\dagger) \wedge (d\hat{\psi})\hat{\psi}^\dagger\right\rangle \\ &= \left\langle d\hat{\psi}^\dagger \wedge d\hat{\psi}\right\rangle \\ &= \left\langle\hat{\psi}\hat{\psi}^\dagger\hat{\psi}(d\hat{\psi}^\dagger) \wedge (d\hat{\psi})\hat{\psi}^\dagger\right\rangle\end{aligned}$$

and as a result we get

$$\langle\phi \wedge (d\phi \wedge d\phi)\rangle = -2i\left\langle d\hat{\psi}^\dagger \wedge d\hat{\psi}\right\rangle = -4\pi\left(\frac{i}{2\pi}\langle F_L\rangle\right)$$

We can then see that

$$\int_{S^2_\infty}\langle\phi \wedge (d\phi \wedge d\phi)\rangle = -4\pi c_1(L)$$

Remark. *A point to be made about the above is that, a priori, the connection is not related to the Higgs field on S^2_∞. It is the decay condition on $D\phi$ which enforces that on S^2_∞ we have $\partial_\mu \phi = [\phi, A_\mu]$.*

Through either discussion we have $E \geq \pm 8\pi k$ for some $k \in \mathbb{Z}$ with equality iff $F = \mp \star D\phi$ where we choose the sign to make the bound positive. This is the **BPS equation**.

Remark. *We might wonder what is the connection between these two viewpoints, where we have that* $\deg \phi = -c_1(L)$. *To make this connection we recall a result stated in [3]:*

Proposition 3.8. *If L is a complex line bundle on a Riemann surface and f is a meromorphic section then*

$$\deg(D(f)) = c_1(L)$$

Proof. See the Riemann Surfaces notes by Joel Robbin, University of Wisconsin, which in turn reference [6]. ☐

We can think of L as a complex line bundle over $\mathbb{P}^1$ viewed as the Riemann sphere. If we take a section $\sigma = h\hat{\psi}$ where we choose $h = \hat{\psi}_2$, this section has poles exactly where the eigenvector is $(1,0) \Rightarrow \boldsymbol{\phi} = (1,0,0)$, *and has no zeros. Hence the Chern class is counting the number of preimages of* $(1,0,0)$, *that is*

$$-c_1(L) = |\{\textit{preimages of } (1,0,0)\}|$$

This gives $c_1(L) = \deg \phi$, *provided we have an argument to tell us that these all should be counted with sign 1.*

3.3 Self-Dual Reduction

Suppose now we consider pure Yang-Mills on $\mathbb{R}^{n+1}$ with contant diagonal metric g (i.e we are considering it to be either Minkowski or Euclidean). Take coordinates x^μ and ask that the connection $A = A_\mu dx^\mu$ is x^0-independent. Then writing $A = \phi dx^0 + A_i dx^i$ we have

$$\begin{aligned} F &= \frac{1}{2} F_{ij} dx^i \wedge dx^j + (D_i \phi) dx^i \wedge dx^0 \\ &= {}^3F + D\phi \wedge dx^0 \end{aligned}$$

as $F_{i0} = \partial_i \phi + [A_i, \phi] = D_i \phi$ and denoting ${}^3F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$. We calculate

$$\begin{aligned} \left\langle {}^3F, D\phi \wedge dx^0 \right\rangle &= \frac{1}{2} \langle F_{ij}, D_k \phi \rangle \underbrace{\left\langle dx^i \wedge dx^j, dx^k \wedge dx^0 \right\rangle}_{=0} \\ \left\langle D\phi \wedge dx^0, D\phi \wedge dx^0 \right\rangle &= \langle D_i \phi, D_j \phi \rangle \underbrace{\left\langle dx^i \wedge dx^0, dx^j \wedge dx^0 \right\rangle}_{=g^{ij} g^{00}} \end{aligned}$$

and so on Euclidean $\mathbb{R}^{n+1}$, $|F|^2 = \left|{}^3F\right|^2 + |D\phi|^2$. This means we recover the action for Yang-Mills-Higgs with 0 potential from this reduction.

Remark. *It is not absurd to consider Euclidean $\mathbb{R}^4$, as we should view this as performing a Wick rotation from Minkowski space, which is natural when quantising the theory as it means that the path integral is now well defined (see [16] for more of a discussion on this).*

The Yang-Mills equation for F reads

$$D \star F = 0 \Rightarrow g^{\mu\nu} D_\mu F_{\nu\rho} = 0$$

This splits to give

$$\begin{aligned} g^{\mu\nu} D_\mu F_{\nu 0} = 0 &\Rightarrow g^{ij} D_i D_j \phi = D_i D^i \phi = 0 \\ g^{\mu\nu} D_\mu F_{\nu k} = 0 &\Rightarrow -g^{00} D_0 D_k \phi + g^{ij} D_i F_{jk} = 0 \\ &\quad - g^{00} \left[\phi, D_k \phi\right] + D_i F^i{}_k = 0 \end{aligned}$$

Moreover we have the stronger result:

Proposition 3.9. *F is (anti-)self-dual iff $({}^3F, \phi)$ satisfy the Bogomolny equations.*

Proof. The (anti-)self-duality equations for F say $\star_4 F = (-)F$, where we are now making explicit the dimension wrt which $\star$ is acting. We can calculate

$$\begin{aligned} \star_4 F &= \frac{1}{4} F_{ij} \epsilon^{ij}{}_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{2}(D_i \phi)\epsilon^{i0}{}_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= -\frac{1}{2}(D_k \phi)\epsilon^k{}_{ij} dx^i \wedge dx^j - \frac{1}{2} F_{ij} \epsilon^{ij}{}_k dx^k \wedge dx^0 \\ &= - \star_3 D\phi - \star_3 F \wedge dx^0 \end{aligned}$$

which we see to mean, using $\star_3^2 = 1$.

$$\star_4 F = \pm F \Leftrightarrow {}^3F \pm \star_3 D\phi = 0$$

□

4 Constructions

4.1 Nahm's Construction

Relevant reading for this section is Manton & Sutcliffe [10] and Hitchin [8]. The original paper is [14].

4.2 The ADHM construction

This section follows the work first laid out in [2]. Suppose we have the following information:

- W a k-dimensional vector space
- V a $2k+2$-dimensional vector space with skew, non-degenerate bilinear form $(\cdot,\cdot) : \wedge^2 V \to \mathbb{C}$.
- $z = (z_i) \in \mathbb{C}^4$
- $A(z) = \sum_i A_i z_i \in \mathrm{End}(W, V)$ s.t.

$$\forall z \neq 0,\ U_z \equiv A(z)W \subset V \text{ is isotropic and } k\text{-dimensional}$$

We now state some important properties:

Lemma 4.1. *Let* $E_z = {}^{U_z^0}\!/_{U_z}$, *then*

- $\dim E_z = 2$
- E_z *inherits a non-degenerate skew bilinear*
- $\forall \lambda \in \mathbb{C}^\times$, $E_z = E_{\lambda z}$.

Proof. We go point by point:

- $\dim E_z = \dim U_z^0 - \dim U_z = (\dim V - \dim U_z) - \dim U_z = 2k + 2 - 2k = 2.$
- The bilinear on W is only degenerate in U_z^0 on U_z, so by quotienting by this it descends directly to E_z.
- $A(\lambda z) = \lambda A(z)$, so $A(\lambda z)(\lambda^{-1}\boldsymbol{w}) = A(z)(\boldsymbol{w})$. Hence we can see $U_{\lambda z} = U_z$ and so result.

□

Corollary 4.2. *We get a vector bundle* $E \to \mathbb{CP}^3$ *with group* $SL(2, \mathbb{C})$.

A break to introduce Bernd's notes

4.2.1 Connection from Projection

Given an rank-n vector bundle $E \to M$ a subbundle of $\mathbb{R}^{n+k} \times M$. We can decompose $E_x + E_x^\perp = \mathbb{R}^{n+k}$ and define projectors P, Q onto $E, E^\perp$ respectively. Then

$$E = \left\{(x, v) \in M \times \mathbb{R}^{n+k} \mid P_x(v) = v\right\}$$

and sections are maps $x \mapsto (x, f(x))$ s,t $P_x f(x) = f(x)$. We define a connection on E via

$$Df = Pdf$$

Now we want to pick a gauge via $u_x : \mathbb{R}^n \to \mathbb{R}^{n+k}$, i.e $u^\dagger u = I$, and so we can write $P = uu^\dagger$. Then if we write our section as $f = ug$ for $g : U \subset M \to \mathbb{R}^n$. Then

$$Df = uu^\dagger\left[(du)g + u(dg)\right] = u\left[dg + u^\dagger(du)g\right]$$

and we can read off that the connection must be $A = u^\dagger du$. and so

$$D = du^\dagger \wedge du + u^\dagger du \wedge u^\dagger du$$

Now in terms of the other projection, we have the following results:

Lemma 4.3. $dQ = Q(dQ) + (dQ)Q$ *and* $(dQ)f = -Q(df)$.

If we now define $B = QdQ$ we get for $f : M \to \mathbb{R}^{n+k}$

$$D_B f = df + Bf = df + (dQ)f - (dQ)Qf$$

On sections of E we have $Qf = 0$ and so $D_B f = Df$. As we can calculate $F_b = dQ \wedge dQ$ we must get

$$F = P(dQ) \wedge (dQ)P$$

Choose a gauge fo $E^\perp$ $v : \mathbb{R}^k \to \mathbb{R}^{n+k}$ with $u^\dagger v = 0$. Letting $\rho = v^\dagger v$ we get

$$Q = v\rho^{-2}v^\dagger$$

giving

$$F = \cdots = P(dv)\rho^{-2} \wedge (dv^\dagger)P$$

4.2.2 Quaternionic Bundles

We now take $M = S^4 = \mathbb{HP}^1$ with matrix multiplication form the left but scalar $\mathbb{H}$ action on the right. Our plan will be to try find good v s.t. we can get F (anti)-self dual, and $\overline{F} = -F$ to ensure F is $\mathfrak{su}(2)$-valued.
The ADHM idea is

1. take $v(x, y) = Cx + Dy$ for $x, y \in \mathbb{H}$, $C, D \in \mathbb{H}^{(k+n)k}$.
2. Assume maximal rank for $(x, y) \neq 0$. Then image of v is a k-dimensional subspace of $\mathbb{H}^{n+k}$ depending on xy^{-1}.
3. Project to $\mathbb{R}^4 \subset S^4$ where $y \neq 0$ and take affine coordinate $(x, 1)$
4. assume $\rho^2 = (\overline{x}C^\dagger + D^\dagger)(Cx + d)$ is real for $x \in \mathbb{H}$.

We now write L for the tautological quaternionic line bundle over $\mathbb{HP}^1$ with $c_2 = -1$. Take

$$E^\perp = \underbrace{L \oplus \cdots \oplus L}_{\times k}$$

Then since $E \oplus E^\perp$ is trivial and Chern number adds we have $c_2(E) = k$. Thus taking $n = 1$,

$$F = PC(dx)\rho^{-2} \wedge (d\overline{x})C^\dagger P$$

is an $SU(2)$-instanton with charge k.

Remark. *We needed $n = 1$ as we actually get an $Sp(n)$ bundle*

Now we can pick a gauge where

$$v(x) = \begin{pmatrix} \Lambda \\ B - xI_k \end{pmatrix}$$

with $n \times k$ quaternionic matrix Λ and $k \times k$ constant quaternionic matrix B. We have to solve the non-linear condition

$$\overline{\Lambda^\dagger\Lambda + B^\dagger B} = \Lambda^\dagger\Lambda + B^\dagger B$$

for symmetric B. We then solve the linear, x-dependent matrix equation $v^\dagger u = 0$ for u to find an explicit formula for A, F.

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