Affine Toda

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1 Introduction

These will be a set of notes dedicated to a project looking at the affine toda lattice, but in situ we will cover some theory from Lie algebras and representations. See my notes on Kac-Moody algebras and Symmetries, Fields, and Particles for additional background which I will omit here as it is covered there.

2 Lie Algebra Conventions

Let $\mathfrak{g}$ be a simple Lie algebra of rank r and $\mathfrak{h} \subset \mathfrak{g}$ a fixed Cartan subalgebra with a inner product $(\, , \,) := (\, , \,)_{\mathfrak{h}^*}$. Let Φ denote the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$ and W the associated Weyl group. By averaging we may always take $(\, , \,)$ to be Weyl-invariant. We begin with

(i) the linearly independent set $\Delta := \{\alpha_1, \dots, \alpha_r\} \subset \Phi \subset \mathfrak{h}^*$, the simple roots. To each $\alpha \in \Phi$ set

$$\epsilon_\alpha := \frac{2}{(\alpha, \alpha)}, \quad \alpha^\vee := \epsilon_\alpha \alpha := \frac{2\alpha}{(\alpha, \alpha)}.$$

Here $\alpha^\vee \in \mathfrak{h}^*$ are the **coroots** (or **dual** roots) and $\Phi^\vee := \{\alpha^\vee \,|\, \alpha \in \Phi\}$[1]. We write $\epsilon_i := 2/(\alpha_i, \alpha_i)$ for $\alpha_i \in \Delta$.

(ii) The Cartan matrix is $A := (a_{ij})$ with $a_{ij} := (\alpha_i^\vee, \alpha_j)$. Then $A = DB$ where $D = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$ and $B := (b_{ij})$, $b_{ij} = (\alpha_i, \alpha_j)$ is symmetric; A is symmetrizable. Then

$$(\alpha_i^\vee, \alpha_j^\vee) = \epsilon_i(\alpha_i, \alpha_j)\epsilon_j = \epsilon_i\, \alpha_i(\alpha_j^\vee).$$

The choice of ϵ_α is so as to make the Cartan matrix have two's along the diagonal,

(iii) Let $\{H_a\}$ $(a = 1, \ldots, r)$ be a basis of $\mathfrak{h}$. The Cartan-Weyl basis $\{H_a\}$ and $\{E_\alpha\}$, $\alpha \in \Phi$ satisfies

$$[H_a, H_b] = 0, \quad [H_a, E_\alpha] = \alpha_a\, E_\alpha, \quad \alpha_a := \alpha(H_a).$$

The Jacobi identity then yields for $\alpha, \beta \in \Phi$ that

$$[H_a, [E_\alpha, E_\beta]] = (\alpha + \beta)_a\, [E_\alpha, E_\beta]$$

and so

$$[E_\alpha, E_\beta] = \begin{cases} c_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \Phi. \end{cases}$$

Finally, using the fact that the $\mathrm{centraliser}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ we see that $[E_\alpha, E_{-\alpha}] \in \mathfrak{h}$.

(iv) Denote the Killing form by

$$\kappa(x, y) := \mathrm{Tr}\, \mathrm{ad}_x \circ \mathrm{ad}_y, \qquad x, y \in \mathfrak{g}. \tag{2.0.1}$$

Then

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

The non-degeneracy of the Killing form means we get an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ such that $\kappa(h_1, h_2)_{\mathfrak{h}} = \nu(h_1)(h_2)$. For each $\alpha \in \Phi$ define $t_\alpha \in \mathfrak{h}$ by $\nu(t_\alpha) = \alpha$. Thus $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha)$. Then for all $h \in \mathfrak{h}$

$$\begin{aligned} \kappa(h, [E_\alpha, E_{-\alpha}]) &= \kappa([h, E_\alpha], E_{-\alpha}]) = \alpha(h)\kappa(E_\alpha, E_{-\alpha}) = \kappa(t_\alpha, h)\kappa(E_\alpha, E_{-\alpha}) \\ &= \kappa(\kappa(E_\alpha, E_{-\alpha})\, t_\alpha, h). \end{aligned}$$

and the non-degeneracy of the Killing form now yields that

$$[E_\alpha, E_{-\alpha}] = \kappa(E_\alpha, E_{-\alpha})\, t_\alpha.$$

(v) Upon noting that

$$\begin{aligned} \mathrm{ad}_{H_a} \circ \mathrm{ad}_{H_b}(h) &= 0 \\ \mathrm{ad}_{H_a} \circ \mathrm{ad}_{H_b}(E_\alpha) &= \alpha_a \alpha_b\, E_\alpha \end{aligned}$$

we find

$$\kappa(H_a, H_b) = \sum_{\alpha \in \Phi} \alpha_a \alpha_b.$$

[1] Caution: Kac's notation has $\alpha^\vee \in \mathfrak{h}$

(vi) The Weyl group acts irreducibly on the vector space $\mathfrak{h}^*$. If we write the W-invariant metric as $(\alpha, \beta) = \alpha_a g^{ab} \beta_b$ then

$$\sum_{w \in W} (w\alpha)_a (w\alpha)_b = \frac{(\alpha, \alpha)}{r} \, |\mathcal{O}(\alpha)| \, g_{ab}.$$

Now a root system Φ consists of at most root vectors of two lengths two (long L and short S), and those vectors of the same length form a single orbit. Then

$$\sum_{\alpha \in \Phi} \alpha_a \alpha_b = \left((\alpha_L, \alpha_L) \, |\mathcal{O}(\alpha_L)| + (\alpha_S, \alpha_S) \, |\mathcal{O}(\alpha_S)| \right) g_{ab} = 2g \, \frac{(\alpha_L, \alpha_L)}{2} \, g_{ab}.$$

Here g is the **dual Coxeter** number. Therefore

$$\kappa(H_a, H_b) = 2g \, \frac{(\alpha_L, \alpha_L)}{2} \, g_{ab}.$$

(vii) Let us set $c := 2g \, (\alpha_L, \alpha_L)/2$ so that $\kappa_{ab} := \kappa(H_a, H_b) = c \, g_{ab}$. We wish to express t_α in terms of the basis $\{H_a\}$. Now

$$\kappa(t_\alpha, H_a) = \nu(t_\alpha)(H_a) = \alpha(H_a) = \alpha_a.$$

If $t_\alpha = x^b H_b$ then $x^b \kappa_{ba} = \alpha_a$ and so $x^b = \alpha_a g^{ab}/c = \alpha^b/c$ and

$$t_\alpha = \frac{1}{c} \alpha^a H_a = \frac{1}{c} \alpha \cdot H.$$

Note that

$$\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = \frac{\alpha^a}{c} \, \kappa(H_a, H_b) \, \frac{\alpha^b}{c} = \frac{\alpha^a}{c} \, c \, g_{ab} \, \frac{\alpha^b}{c} = \frac{(\alpha, \alpha)}{c}.$$

(viii) Set

$$H_\alpha := \frac{2 \, t_\alpha}{\kappa(t_\alpha, t_\alpha)} = \frac{2 \, \alpha \cdot H}{(\alpha, \alpha)} = \alpha^\vee \cdot H.$$

Upon noting that $[t_\alpha, E_\alpha] = \alpha(t_\alpha) E_\alpha = (\alpha, \alpha) E_\alpha / c$ then for all $\alpha \in \Phi$,

$$[H_\alpha, E_\alpha] = 2 \, E_\alpha.$$

Now

$$[E_\alpha, E_{-\alpha}] = \kappa(E_\alpha, E_{-\alpha}) \, t_\alpha = \left[\frac{1}{2} \kappa(E_\alpha, E_{-\alpha}) \kappa(t_\alpha, t_\alpha) \right] H_\alpha.$$

Setting

$$E_\alpha^{Ch} := E_\alpha \Big/ \sqrt{\frac{1}{2} \kappa(E_\alpha, E_{-\alpha}) \kappa(t_\alpha, t_\alpha)}$$

we then have for all $\alpha \in \Phi$ the standard sl_2 relations

$$[H_\alpha, E_\alpha^{Ch}] = 2 \, E_\alpha^{Ch}, \quad [E_\alpha^{Ch}, E_{-\alpha}^{Ch}] = H_\alpha.$$

Further

$$[H_\alpha, E_\beta^{Ch}] = \epsilon_\alpha \alpha^a \beta(H_a) E_\beta^{Ch} = (\alpha^\vee, \beta) \, E_\beta^{Ch}$$

and

$$\kappa(H_\alpha, H_\beta) = c \, (\alpha^\vee, \beta^\vee), \quad \kappa(E_\alpha^{Ch}, E_{-\alpha}^{Ch}) = c \, \epsilon_\alpha.$$

(ix) The Chevalley basis consists of $\{H_\alpha\}$ for $\alpha \in \Delta$ and $\{E^{Ch}_\beta\}_{\beta\in\Phi}$, where

$$[H_\alpha, E^{Ch}_\beta] = (\alpha^\vee, \beta)\, E^{Ch}_\beta,$$
$$[E^{Ch}_\alpha, E^{Ch}_\beta] = \begin{cases} H_\alpha & \text{if } \alpha+\beta = 0, \\ N_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ 0 & \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Phi. \end{cases}$$

with

$$\kappa(H_\alpha, H_\beta) = c\,(\alpha^\vee, \beta^\vee), \quad \kappa(E^{Ch}_\alpha, E^{Ch}_{-\alpha}) = c\,\epsilon_\alpha, \quad c = 2g\,\frac{(\alpha_L, \alpha_L)}{2}.$$

(x) There is a unique maximal root, which we denote as $\Theta = \sum_{\alpha\in\Delta} n_\alpha\, \alpha$ be the highest root. Set $\overline{\Delta} = \Delta \cup \{-\Theta\}$

3 Background Theory

3.1 Lie Algebras and Representation Theory

We start with a recap of Chapters II and III of [3]. Denote the base of simple roots as Δ.

Proposition 3.1. *There exists a unique root of highest weight $\theta = \sum_{\alpha\in\Delta} n_\alpha \alpha \in \mathfrak{h}^*$.*

Proposition 3.2. *Let A be the cartan matrix corresponding to $\mathfrak{g}$ of finite type, rank n, and let $h_\theta = \sum_i n_i h_i \in \mathfrak{h}$ be the element corresponding to θ under the natural iso $\mathfrak{h} \cong \mathfrak{h}^*$. Define $\hat{A}$ by*

$$\begin{aligned} \hat{A}_{ij} &= A_{ij},\ 1 \le i, j \le n \\ \hat{A}_{00} &= 2 \\ \hat{A}_{i0} &= -\sum_j m_j A_{ij} \\ \hat{A}_{0j} &= -\sum_i n_i A_{ij} \end{aligned}$$

*Then $\hat{A}$ is an affine generalised Cartan matrix corresponding to an **untwisted affine Dynkin diagram**.*

Proposition 3.3. *The Lie algebra corresponding to $\hat{A}$ is isomorphic to the affine Kac-Moody Lie algebra $\mathcal{L}\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$*

3.2 Monopoles

Nahm established an equivalence between non-singular charge-n $SU(2)$-monopoles and $\{T_i(s)\,|\, T_i \in M_n(\mathbb{C}),\ s \in [0, 2]\}$ subject to

1. Nahm's equation:

$$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}\,[T_j, T_k]$$

2. $T_i(s)$ is regular for $s \in (0,2)$ and has simple poles at $s = 0, 2$ with residues that form an irreducible, n-dimensional rep of $\mathfrak{su}(2)$

3. $T_i(2) = -T_i^\dagger(s)$, $T_i(s) = T_i^T(2-s)$.

Remark. *We could have here included another matrix $T_4 = -T_4^\dagger$ by modifying Nahm's equations to*

$$\dot{T}_i = [T_4, T_i] + \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j(z), T_k(z)]$$

but this can always be gauged away by the transform

$$\begin{aligned} T_i &\mapsto uT_iu^{-1} \\ T_4 &\mapsto uT_0u^{-1} - \frac{du}{ds}u^{-1} \end{aligned}$$

for $u : (0,2) \to U(n)$ satisfying $u(2-s) = \left(u^T(s)\right)^{-1}$.

Proposition 3.4. *Nahm's equation is equivalent to the Lax equation*

$$\left[\frac{d}{ds} + M, A\right] = 0$$

where

$$\begin{aligned} A(\zeta) &= (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2 \\ M(\zeta) &= -iT_3 + (T_1 - iT_2)\zeta \end{aligned}$$

If we make the definitions

$$\begin{aligned} \alpha &= iT_3 \\ \beta &= T_1 + iT_2 \end{aligned}$$

this Lax pair can be rewritten as

$$\begin{aligned} A(\zeta) &= \beta - (\alpha + \alpha^\dagger)\zeta - \beta^\dagger\zeta^2 \\ M(\zeta) &= -\alpha - \beta^\dagger\zeta \end{aligned}$$

4 Affine Toda

We start by introducing affine Toda from a field theory perspective, following [2]:

4.1 Independent Definition

Definition 4.1. *Let $\mathfrak{g}$ be a rank-r Lie algebra with simple roots α_i, taking a particular realisation of these as vectors in $\mathbb{R}^r$. The* ***Toda field theory*** *is that with $\mathbb{R}^r$-valued field $\mathbf{\Phi} = (\phi^a)$ and Lagrangian*

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{\lambda}{\beta^2}\sum_{i=1}^{r} e^{\beta\alpha_i\cdot\mathbf{\Phi}}$$

for parameters λ, β.

Proposition 4.2. *The corresponding classical equations of motion are*

$$\partial^2 \phi_j = -\frac{\lambda}{\beta} \sum_{i=1}^{r} C_{ji} e^{\beta \phi_i}$$

where $\phi_j = \alpha_j \cdot \mathbf{\Phi}$ *and*

$$C_{ij} = \alpha_i \cdot \alpha_j$$

Proof. The e.o.m are

$$\frac{\partial \mathcal{L}}{\partial \phi^a} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}$$
$$\Rightarrow -\frac{\lambda}{\beta} \sum_{i=1}^{r} (\alpha_i)^a \, e^{\beta \phi_i} = \partial^2 \phi^a$$

and the result follow from contracting with α_j. □

Remark. *If we shift* $\phi_i \mapsto \phi_i + \frac{1}{\beta} \log\left(\frac{2}{\alpha_i^2}\right)$ *the matrix* C *is replaced with*

$$A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}$$

which we recognise to be the Cartan matrix.

This field theory does not have a unique minimum for us to consider as the classical vacuum, so we will want to deform it in some way. The following result motivates how do this deformation:

Proposition 4.3. $1+1$*-dimensional Toda field theory has a zero-curvature representation*

Proof. We follow [4]. Define light-cone coordinates

$$u = \frac{1}{2}(x+t)$$
$$v = \frac{1}{2}(x-t)$$

s.t.

$$\partial_u \partial_v = -\partial_t^2 + \partial_x^2 = -\partial_\mu \partial^\mu$$

and a gauge potential with

$$A_u = \sum_{i=1}^{r} \left(\frac{1}{2}\right)$$

finish this off... □

This is thus integrable, and so when we want to generalise this system, we look for integrable deformations.

Definition 4.4. *The field theory obtained by perturbing Toda by*

$$\delta V(\mathbf{\Phi}) = \frac{\epsilon\lambda}{\beta^2} e^{\beta\alpha_0\cdot\mathbf{\Phi}}$$

s.t. $\{\alpha_0, \alpha_j\}$ *are roots of an affine Lie algebra is called* ***affine Toda field theory****.*

Affine Toda has a minimum $\mathbf{\Phi}^{(0)}$ satisfying

$$\sum_i \alpha_i e^{\beta\alpha_i\cdot\mathbf{\Phi}^{(0)}} = -\epsilon\alpha_0 e^{\beta\alpha_0\cdot\mathbf{\Phi}^{(0)}}$$

If we centre around this by letting $\mathbf{\Phi} = \mathbf{\Phi}^{(0)} + \boldsymbol{\phi}$ we have

$$\begin{aligned} V(\boldsymbol{\phi}) &= \frac{\epsilon\lambda}{\beta^2} e^{\beta\alpha_0\cdot\mathbf{\Phi}^{(0)}} \left[e^{\beta\alpha_0\cdot\boldsymbol{\phi}} - \sum_{i,j} e^{\beta\alpha_i\cdot\boldsymbol{\phi}} \left(C^{-1}\right)_{ij} \alpha_j \cdot \alpha_0 \right] \\ &= \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta\alpha_i\cdot\boldsymbol{\phi}} \end{aligned}$$

where we have let $m^2 = \epsilon\lambda e^{\beta\alpha_0\cdot\mathbf{\Phi}^{(0)}}$ and written $\alpha_0 = \sum_{i=1}^r n_i\alpha_i$, $n_0 = 1$. The simplest deformation is to take $\alpha_0 = -\Theta$, the maximal root.

4.2 Braden Approach

We will now obtain the affine Toda Field theory through a different lens.
Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r with a fixed Cartan subalgebra $\mathfrak{h}$. Let $\{X_\mu\} = \{H_i, E_\alpha\}$ be a Cartan-Weyl basis where $\{H_i\}$ is a basis of $\mathfrak{h}$ and $\{E_\alpha\}$ the set of step operators (labelled by the root system Φ of $\mathfrak{g}$) and

$$[H_i, E_\alpha] = \alpha_i\, E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha\cdot H, \quad [E_\alpha, E_\beta] = N_{\alpha,\beta}\, E_{\alpha+\beta} \quad \text{if } \alpha+\beta\in\Phi.$$

Recalling we have the maximal root $\Theta = \sum_{\alpha\in\Delta} n_\alpha\alpha$, and extend $\overline{\Delta} = \Delta\cup\{-\Theta\}$, $n_{-\Theta} = 1$ consider

$$E = \sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, E_\alpha, \quad E^\dagger = \sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, E_{-\alpha}.$$

Then

$$[E, E^\dagger] = \sum_{\alpha\in\Delta} n_\alpha\, [E_\alpha, E_{-\alpha}] + [E_{-\Theta}, E_\Theta] = 0$$

Consider a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\kappa\left(\partial_\mu\phi, \partial^\mu\phi\right) - \kappa\left(e^{b\phi}Ee^{-b\phi}, E^\dagger\right)$$

Proposition 4.5. *The corresponding field equations are*

$$\partial_\mu\partial^\mu\phi + b\,[e^{b\phi}Ee^{-b\phi}, E^\dagger] = 0.$$

These are the affine Toda field equations if we make the normalisation $\kappa(E_\alpha, E_{-\alpha}) = \frac{m^2}{\lambda^2}$.

Proposition 4.6. $1+1$-*dimsensional Toda field theory has a zero curvature representation.*

Proof. With $ds^2 = dt^2 - dx^2 = -dx^+dx^-$, $x^\pm = x \pm t$, $\partial_x = \partial_+ + \partial_-$, $\partial_t = \partial_+ - \partial_-$ these become

$$-\partial_{+-}\phi + \frac{b}{4}\,[e^{b\phi}Ee^{-b\phi}, E^\dagger] = 0$$

which are the consistency of

$$0 = [\partial_+ + A_+, \partial_- + A_-], \quad A_+ = \frac{b}{2}e^{b\phi/2}\,E\,e^{-b\phi/2} + \frac{b}{2}\partial_+\phi, \quad A_- = \frac{b}{2}e^{-b\phi/2}\,E^\dagger\,e^{b\phi/2} - \frac{b}{2}\partial_-\phi.$$

To see this observe that

$$\begin{aligned} e^{\phi}E_\alpha e^{-\phi} &= \mathrm{Ad}_{e^\phi}\,E_\alpha = (1+\phi+\frac{1}{2}\phi^2+\ldots)E_\alpha(1-\phi+\frac{1}{2}\phi^2-\ldots) \\ &= E_\alpha + [\phi, E_\alpha] + \frac{1}{2}[\phi,[\phi,E_\alpha]] + \ldots = e^{\alpha(\phi)}E_\alpha \end{aligned}$$

giving

$$A_+ = \frac{b}{2}\sum_{\alpha\in\overline{\Delta}}\sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}E_\alpha + \frac{b}{2}\partial_+\phi, \quad A_- = \frac{b}{2}\sum_{\alpha\in\overline{\Delta}}\sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}E_{-\alpha} - \frac{b}{2}\partial_-\phi.$$

Then

$$\begin{aligned} A_1 = A_+ + A_- &= \frac{b}{2}\partial_0\phi + b\sum_{\alpha\in\overline{\Delta}}\sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}X^+_\alpha \\ A_0 = A_+ - A_- &= \frac{b}{2}\partial_1\phi + b\sum_{\alpha\in\overline{\Delta}}\sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}X^-_\alpha \end{aligned}$$

where $X^\pm_\alpha = (E_\alpha \pm E_{-\alpha})/2$, and

$$\begin{aligned} 0 = [\partial_0 + A_0, \partial_1 + A_1] &= \frac{b}{2}(\partial_0^2 - \partial_1^2)\phi + \frac{b^2}{2}\sum_{\alpha\in\overline{\Delta}}\sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}\left(\alpha\,(\partial_0\phi)\;X^+_\alpha - \alpha\,(\partial_1\phi)\;X^-_\alpha\right) \\ &+ \frac{b^2}{2}\sum_{\alpha\in\overline{\Delta}}\sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}\left([\partial_1\phi, X^+_\alpha] - [\partial_0\phi, X^-_\alpha]\right) + \frac{b^2}{2}\sum_{\alpha\in\overline{\Delta}}n_\alpha\,e^{b\alpha(\phi)}[E_\alpha, E_{-\alpha}] \end{aligned}$$

so giving

$$0 = \partial_\mu\partial^\mu\phi + b\sum_{\alpha\in\overline{\Delta}}n_\alpha\,e^{b\alpha(\phi)}[E_\alpha, E_{-\alpha}] = \partial_\mu\partial^\mu\phi + b\,[e^{b\phi}Ee^{-b\phi}, E^\dagger].$$

□

To make contact with perturbative affine Toda theory we note the expansion

$$\begin{aligned}\operatorname{Tr} e^{b\phi} E e^{-b\phi} E^\dagger &= \operatorname{Tr}(1 + b\phi + \frac{b^2}{2}\phi^2 + \frac{b^3}{6}\phi^3 + \ldots)E(1 - b\phi + \frac{b^2}{2}\phi^2 - \frac{b^3}{6}\phi^3 + \ldots)E^\dagger \\ &= \operatorname{Tr}\left(EE^\dagger + b\phi[E, E^\dagger] + \frac{b^2}{2}\phi[E,[E^\dagger,\phi]] + \frac{b^3}{6}\phi[[\phi,E^\dagger,][\phi,E]] + \ldots\right) \\ &= \operatorname{Tr} EE^\dagger + \frac{b^2}{2}\operatorname{Tr}\phi[E,[E^\dagger,\phi]] + \frac{b^3}{6}\operatorname{Tr}\phi[[\phi,E^\dagger,][\phi,E]] + \ldots\end{aligned}$$

which is further simplified upon specifying the normalisations $\operatorname{Tr} E_\alpha E_{-\alpha}$. This form of the affine Toda equation has been chosen so that $\phi = 0$ is a classical solution. If we work with

$$\operatorname{Tr} E_\alpha E_{-\alpha} = \epsilon_\alpha := \frac{2}{(\alpha,\alpha)}$$

then

$$\operatorname{Tr} EE^\dagger = \sum_{\alpha\in\overline{\Delta}} n_\alpha^\vee = g, \qquad n_\alpha^\vee := n_\alpha/\epsilon_\alpha,$$

where g is the dual Coxeter number. If we work with the (unshifted) Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \sum_{\alpha\in\overline{\Delta}} \epsilon_\alpha e^{(\alpha,\psi)}$$

and expand $\psi = \psi^i\epsilon_i\lambda_i$ with $(\alpha_i^\vee, \lambda_j) = \delta_{ij}$ for the simple roots, then we obtain equations of motion

$$\epsilon_i(\lambda_i,\lambda_j)\epsilon_j\partial_\mu\partial^\mu\psi^j = -\sum_{\alpha\in\overline{\Delta}} \epsilon_\alpha(\alpha,\epsilon_i\lambda_i)\, e^{(\alpha,\psi)} = -\epsilon_i\, e^{\psi^i} + n_i\epsilon_{-\Theta}\, e^{-(\Theta,\psi)}.$$

Then with $K_{ij} = (\alpha_i^\vee, \alpha_j) = \epsilon_i(\alpha_i,\alpha_j) := \epsilon_i b_{ij}$ and $(\lambda_i,\lambda_j) = G_{ij} = \epsilon_i^{-1} b_{ij}^{-1}\epsilon_j^{-1} = \epsilon_i^{-1}K_{ij}^{-1}$ we obtain

$$-\partial_\mu\partial^\mu\psi^j = b_{ji}\epsilon_i\, e^{\psi^i} - b_{ji}n_i\epsilon_{-\Theta}\, e^{-(\Theta,\psi)} = \overline{K}_{ji}^T\, e^{\psi^i} + \overline{K}_{ji}^T\, e^{-(\Theta,\psi)} = \overline{K}_{ja}^T\, e^{\psi^a}$$

and $\psi^0 := -(\Theta,\psi)$.

In the zero curvature equation there so far has been no appearance of a spectral parameter. We see that taking

$$X_\alpha^\pm = \frac{1}{2}\left(\zeta^{r_\alpha}E_\alpha \pm \zeta^{-r_\alpha}E_{-\alpha}\right)$$

will result in the same equations of motion. Two common choices in the literature are

1. $r_\alpha = 1$ for all $\alpha \in \overline{\Delta}$,
2. $r_{-\Theta} = 1$ and $r_\alpha = 0$ for all $\alpha \in \Delta$.

Observe that the Lax matrix for the monopoles may be written

$$
\begin{aligned}
L/\zeta &= -\dot{\phi} + e^{\phi/2}\, E e^{-\phi/2}/\zeta - e^{-\phi/2}\, E^\dagger e^{\phi/2}\zeta = -\dot{\phi} + \sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, e^{b\alpha(\phi)/2}\left(\zeta^{-1}E_\alpha - \zeta E_{-\alpha}\right) \\
&= -2A_0^\dagger \\
M &= -\frac{1}{2}\dot{\phi} - e^{-\phi/2}\, E^\dagger e^{\phi/2}\zeta = \frac{1}{2}\frac{L}{\zeta} - \frac{1}{2}e^{\phi/2}\,\frac{E}{\zeta}e^{-\phi/2} - \frac{1}{2}e^{-\phi/2}\, E^\dagger e^{\phi/2}\zeta \\
&= \frac{1}{2}\frac{L}{\zeta} - \frac{1}{2}\sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, e^{b\alpha(\phi)/2}\left(\zeta^{-1}E_\alpha + \zeta E_{-\alpha}\right) \\
&= -A_0^\dagger - A_1^\dagger
\end{aligned}
$$

and where $\partial_0\phi = 0$ and $\partial_1\phi = \dot{\phi}$ in the previous section. Then the independence from the 0-coordinate gives $0 = [\partial_1 + A_1, \partial_0 + A_0] = \partial_1 A_0 + [A_1, A_0]$ and $0 = \partial_1 A_0^\dagger - [A_1^\dagger, A_0^\dagger] = [\partial_1 - A_1^\dagger, A_0^\dagger]$ and hence the Lax equation $0 = [\partial_1 + M, L]$.

5 Monopoles and Toda

Upon setting (with ${T_i}^\dagger = -T_i$, $T_4^\dagger = -T_4$)

$$\alpha = T_4 + iT_3, \quad \beta = T_1 + iT_2, \quad L = L(\zeta) := \beta - (\alpha + \alpha^\dagger)\zeta - \beta^\dagger\zeta^2, \quad M = M(\zeta) := -\alpha - \beta^\dagger\zeta,$$

one finds

$$
\begin{aligned}
\dot{T}_i = [T_4, T_i] + \frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}[T_j(z), T_k(z)] &\Longleftrightarrow \dot{L} = [L, M] \\
&\Longleftrightarrow \quad \begin{cases} \left[\dfrac{d}{dz} - \alpha, \beta\right] = 0, \\ \dfrac{d(\alpha+\alpha^\dagger)}{dz} = [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]. \end{cases}
\end{aligned}
\tag{5.0.1}
$$

Let

$$\phi = \phi^\dagger, \quad h = e^\phi, \quad \beta = T_1 + \Im T_2 = e^{\phi/2}Ee^{-\phi/2}, \quad \beta^\dagger = -T_1 + \Im T_2 = e^{-\phi/2}E^\dagger e^{\phi/2}, \quad \alpha + \alpha^\dagger = 2\Im T_3 = \dot{\phi}.$$

$$[\beta, \beta^\dagger] = e^{-\phi/2}[e^\phi E e^{-\phi}, E^\dagger]^\dagger e^{\phi/2} = 2\Im\dot{T}_3 = \ddot{\phi},$$

and Nahm's equations are the Toda equations

$$\ddot{\phi} = [e^\phi E e^{-\phi}, E^\dagger] \Longleftrightarrow \frac{d}{dz}\left(\dot{h}h^{-1}\right) = \left[hEh^{-1}, E^\dagger\right]$$

This coincides with the notation of *Cyclic Monopoles, Affine Toda and Spectral Curves* [1]

$$T_1 + iT_2 = \begin{pmatrix} 0 & e^{(q_1-q_2)/2} & 0 & \dots & 0 \\ 0 & 0 & e^{(q_2-q_3)/2} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{(q_{n-1}-q_n)/2} \\ e^{(q_n-q_1)/2} & 0 & 0 & \dots & 0 \end{pmatrix} \tag{5.0.2}$$

$$T_1 - iT_2 = -\begin{pmatrix} 0 & 0 & \dots & 0 & e^{(q_n-q_1)/2} \\ e^{(q_1-q_2)/2} & 0 & \dots & 0 & 0 \\ 0 & e^{(q_2-q_3)/2} & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & e^{(q_{n-1}-q_n)/2} & 0 \end{pmatrix} \tag{5.0.3}$$

$$T_3 = -\frac{i}{2}\begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & p_n \end{pmatrix} \tag{5.0.4}$$

where p_i, q_i are real.

Upon using $0 = \operatorname{Tr} E^2 = \operatorname{Tr} \dot{\phi}(\beta - \beta^\dagger)$

$$\frac{1}{2}\operatorname{Tr} L^2 = \frac{1}{2}\operatorname{Tr}\left[\beta - (\alpha + \alpha^\dagger)\zeta - \beta^\dagger\zeta^2\right]^2 = \zeta^2 \operatorname{Tr}\left(\frac{1}{2}\dot{\phi}^2 - e^{\phi} E e^{-\phi} E^\dagger\right) := \zeta^2 H$$

and this Hamiltonian is not bounded below[2]. This is necessary as the monopole boundary conditions require $T_a \sim \rho_a/s$ as $s \sim 0$ (and similarly at $s \sim 1$), where ρ_a is an irreducible n-dimensional representation of $su(2)$, thus the momenta are unbounded for $s \sim 0$ and so the potential must also be unbounded below.

Observe that the Lax matrix for the monopoles may be written

$$L/\zeta = -\dot{\phi} + e^{\phi/2}\, E e^{-\phi/2}/\zeta - e^{-\phi/2}\, E^\dagger e^{\phi/2}\zeta = -\dot{\phi} + \sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, e^{b\alpha(\phi)/2}\left(\zeta^{-1}E_\alpha - \zeta E_{-\alpha}\right),$$

$$M = -\frac{1}{2}\dot{\phi} - e^{-\phi/2}\, E^\dagger e^{\phi/2}\zeta = \frac{1}{2}\frac{L}{\zeta} - \frac{1}{2}\sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, e^{b\alpha(\phi)/2}\left(\zeta^{-1}E_\alpha + \zeta E_{-\alpha}\right).$$

Note may change the dependence of a spectral parameter by taking the arbitrary combinations

$$\zeta^{r_\alpha} E_\alpha \pm \zeta^{-r_\alpha} E_{-\alpha}$$

and these will result in the same equations of motion. Two common choices in the literature are

1. $r_\alpha = 1$ for all $\alpha \in \overline{\Delta}$,
2. $r_{-\Theta} = 1$ and $r_\alpha = 0$ for all $\alpha \in \Delta$.

[2]Here the Lagrangian is $\mathfrak{L} := \operatorname{Tr}\left(\frac{1}{2}\dot{\phi}^2 + e^{\phi} E e^{-\phi} E^\dagger\right)$ corresponding to a potential of the wrong sign (see the expansion below).

Questions:

1. What is the effect on the spectral curve of the different scalings r_α? Are the curves birational?
2. What is the analogue of the characteristic polynomial and determinant for the matrices

$$a \cdot H + \sum_{\alpha \in \overline{\Delta}} (b_\alpha E_\alpha + c_\alpha E_{-\alpha})?$$

(We may view these as generalizations of tridiagonal matrices.)

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