

# Linearising Flows and a Cohomological Interpretation of Lax Equations - Unpacking the Paper

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## 1 Introduction

The purpose of this document is to facilitate the understanding of [1] by discussing the terms and how they fit into the wider picture of algebraic geometry.

## 2 The Preliminaries

### 2.1 Divisors

**Definition 2.1.** A *(Weil) divisor* on  $C$  is a formal finite sum of points, i.e.  $D = \sum_i n_i p_i$  for  $n_i \in \mathbb{Z}$ ,  $p_i \in C$ . The group of divisors under addition is denoted  $\text{Div}(C)$ .

**Definition 2.2.** The *degree* of a divisor  $D = \sum_i n_i p_i$   $\deg D = \sum_i n_i$

**Definition 2.3.** Given a meromorphic function  $f : C \rightarrow \mathbb{C}$  we define  $(f) \in \text{Div}(C)$  by

$$(f) = \sum_{p \in X} \text{ord}_p(f) \cdot p$$

For  $D \in \text{Div}(C)$ , if  $\exists f$  s.t.  $D = (f)$  we say  $D$  is a **principal divisor**.

**Lemma 2.4.**  $(fg) = (f) + (g)$

**Corollary 2.5.** Principal divisors form a subgroup  $\text{Lin}(C) \leq \text{Div}(C)$ .

**Lemma 2.6.** If  $X$  is a compact Riemann surface and  $f : X \rightarrow \mathbb{C}$  meromorphic then  $\deg(f) = 0$ .

**Proposition 2.7.** Let  $C$  be compact. Then  $\text{Lin}(C) = \{D \in \text{Div}(C) \mid \deg(D) = 0\}$ .

**Definition 2.8.** The **divisor class group** of  $C$  is  $\text{Cl}(C) = \text{Div}(C) / \text{Lin}(C)$ .

**Remark.**  $\deg : \text{Div}(C) \rightarrow \mathbb{Z}$  is a group homomorphism and as the kernel is  $\text{Lin}(C)$  we see  $\text{Cl}(C) \cong \text{Im } \deg$

**Corollary 2.9.**  $\text{Cl}(\mathbb{CP}^n) \cong \mathbb{Z}$ .

**Definition 2.10.** Two divisors  $D, E$  are **linearly equivalent** if  $D - E$  is a principal.

**Lemma 2.11.** Linear equivalence of divisors is an equivalence relation.

**Lemma 2.12.**  $f : X \rightarrow Y$  induces a group morphism  $f : \text{Div}(X) \rightarrow \text{Div}(Y)$  by

$$f \left( \sum_i n_i p_i \right) = \sum_i n_i f(p_i)$$

**Proposition 2.13.** If  $f : X \rightarrow Y$  is a map of Riemann surfaces and  $D \in \text{Div}(X)$ , then  $\deg(f(D)) = \deg f \cdot \deg D$ .

**Definition 2.14.** A divisor  $D = \sum_i n_i p_i$  is **effective** if each  $n_i \geq 0$ .

**Proposition 2.15.** We have a partial ordering on  $\text{Div}(C)$  by saying  $D \geq D'$  if  $D - D'$  is effective.

**Definition 2.16.** A Weil divisor on  $C$  defines a **coherent** sheaf  $\mathcal{O}_C(D)$  as meromorphic functions  $f$  s.t.  $(f) + D \geq 0$ .

## 2.2 Abel-Jacobi

Suppose  $C$  has genus  $g$ , then we know that  $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$  where the generators are the loops  $\{\gamma_i\}_{i=1}^{2g}$ . There is an alternative way to say this condition:

**Definition 2.17.** The **canonical bundle** on a space  $X$  with  $\dim X = n$  is the line bundle of exterior  $n$ -forms on  $X$ .

**Remark.** Note we know the canonical bundle is a line bundle as there is only 1 basis element of  $n$ -forms on an  $n$ -dimensional space.

**Proposition 2.18.** If  $X = C$  is a Riemann surface of genus  $g$  then  $H^0(C, K) \cong \mathbb{C}^g$ .

*Proof.* **Find this** □

**Corollary 2.19.** We can take a basis  $\{\omega_i\}_{i=1}^g$  of 1-forms on  $C$ .

**Definition 2.20.** The **Jacobian** of  $C$  is defined to be

$$J(C) = \mathbb{C}^g / \Lambda$$

where  $\Lambda$  is the lattice generated over  $\mathbb{R}$  by the vectors

$$\Omega_j = \left( \int_{\gamma_j} \omega_1, \dots, \int_{\gamma_j} \omega_g \right), \quad 1 \leq j \leq 2g$$

**Definition 2.21.** The **Abel-Jacobi map** for  $p_0 \in C$  is

$$u : C \rightarrow J(C)$$

$$p \mapsto \left( \int_{p_0}^p \omega_1, \dots, \int_{p_0}^p \omega_g \right) \mod \Lambda$$

This is independent of the path of integration as we have quotiented by  $\Lambda$ .

**Theorem 2.22** (Abel's Theorem). Let  $u$  be the Abel-Jacobi map and  $D, E$  effective divisors. Then  $u(D) = u(E) \Leftrightarrow D \sim E$ .

**Theorem 2.23** (Jacobi's Theorem). The map Abel-Jacobi map is surjective.

**Corollary 2.24.** There is an isomorphism from the space of degree-0 divisors to the Jacobian.

## 2.3 Bundles and Sheaves

We recall a few necessary bundle definitions and results:

**Definition 2.25.** The tensor product of vector bundles  $E, F \rightarrow M$  is  $E \otimes F \rightarrow M$  s.t.  $(E \otimes F)_m = E_m \otimes F_m$  for  $m \in M$ .

**Lemma 2.26.** If  $O$  is the trivial line bundle then  $E \otimes O = E$ .

**Definition 2.27.** The **dual bundle** of a vector bundle  $E \rightarrow M$  is  $E^* \rightarrow M$  where the fibres of  $E^*$  are the dual spaces of the fibres of  $E$ , with the transition functions  $g_{ij}^* = (g_{ij}^T)^{-1}$ .

**Remark.** We can check the cocycle condition here as

$$g_{kj}^* g_{ji}^* = (g_{kj}^T)^{-1} (g_{ji}^T)^{-1} = (g_{ji}^T g_{kj}^T)^{-1} = ([g_{kj} g_{ji}]^T)^{-1} = (g_{ki}^T)^{-1} = g_{ki}^*$$

**Example 2.28.** The dual bundle to the tangent bundle is the cotangent bundle, i.e.  $(TM)^* = T^*M$

**Lemma 2.29.**  $E \otimes E^* \cong \text{End}(E)$ .

**Lemma 2.30.** Line bundles have tensor inverses, i.e given  $L$ ,  $\exists L^{-1}$  s.t.  $L \otimes L^{-1} \cong O$  the trivial bundle.

*Proof.* We will show this by showing  $L^{-1} = L^*$ . To trivialise  $\text{End}(L)$  we note here the transition maps are  $g_{ij} \otimes g_{ij}^{-1} = 1 \otimes 1$  as  $g_{ij}, g_{ij}^* \in \mathbb{F}$ . Hence any section is globally defined.  $\square$

**Remark.** Why is the identity section not global on any other vector bundle.

These results motivate the definition of the **Picard group** which we will cover now:

**Definition 2.31.** A **ringed space** is a pair  $(X, O_X)$  where  $X$  is a topological space and  $O_X$  is a sheaf of rings on  $X$ .  $O_X$  is called the **structure sheaf**.

**Example 2.32.** Given a topological space  $X$ , if we take  $O_X$  to be  $\mathbb{R}$ -valued continuous functions on open subsets of  $X$  then  $(X, O_X)$  is a ringed space.

**Example 2.33.** An example that will be relevant for later discussions is that an affine variety  $X$  with sheaf  $\mathcal{O}_X$  given by  $\mathcal{O}_X(U)$  being the regular functions on  $U$ , regular functions being those given locally by polynomials.

**Definition 2.34.** The **Picard group** of a locally ringed space  $X$  is  $\text{Pic}(X)$  the group of isomorphism classes of line bundles on  $X$  with the group operation being  $\otimes$ .

**Remark.** In place of line bundles we can actually say **invertible sheaves**

**Theorem 2.35.**  $\text{Cl}(C) \cong \text{Pic}(C)$  naturally.

**Corollary 2.36.** We get a group homomorphism  $\deg : \text{Pic}(C) \rightarrow \mathbb{Z}$  giving the degree of the corresponding divisor in  $\text{Cl}(C)$ .

**Corollary 2.37.**  $\text{Pic}(\mathbb{CP}^1) \cong \mathbb{Z}$ .

**Notation.** We denote the isomorphism class of line bundles degree  $d$  as  $\text{Pic}^d(C)$

**Remark.** With this new notation we may rephrase the corollary of the Abel-Jacobi theorem to say  $J(C) \cong \text{Pic}^0(C)$ .

**Proposition 2.38.** There is a canonical isomorphism  $\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$ .

**Corollary 2.39.**  $T_L(\text{Pic}^d(X)) \cong H^1(X, \mathcal{O}_X)$

*Proof.* You need to use the **exponential sheaf sequence**. □

Let us now consider a specific class of bundles:

**Definition 2.40.** The **hyperplane bundle on  $\mathbb{CP}^n$**  is the bundle  $\mathbb{C}^{n+1} \setminus 0 \rightarrow \mathbb{CP}^n$  given by the standard projection  $(z_0, \dots, z_n) \rightarrow [z_0 : \dots : z_n]$ . It is often denoted  $\mathcal{O}(1)$ . We denote  $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ .

**Definition 2.41.** The **tautological line bundle** on projective space is  $\mathcal{O}(-1) = \mathcal{O}(1)^*$ . We denote  $\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n}$ .

**Proposition 2.42.** The canonical bundle on the projective space is  $K = \mathcal{O}(-n-1)$ .

**Proposition 2.43.**  $\text{Pic}(\mathbb{CP}^n)$  is generated by  $\mathcal{O}(\pm 1)$ .

We make a few more useful definitions.

**Definition 2.44.** Let  $X$  be an algebraic surface and  $\pi : L \rightarrow X$  a line bundle. Then the **tautological section** of  $\pi^*L$  as a bundle over  $L$  is given by  $\sigma(l) = (l, l)$ .

**Remark.** Note that the tautological section is indeed valid as we have

$$\pi^*L = \{(l, l') \in L \times L \mid \pi(l) = \pi(l')\}$$

so certainly  $(l, l) \in \pi^*L$ .

## 2.4 Lax Pairs and Spectral Curves

**Notation.** We start by laying out some notation that will be necessary for the following section. Let:

- $P = \mathbb{CP}^1$  with coordinates  $[\xi_0 : \xi_1]$ . We take  $\xi = \frac{\xi_1}{\xi_0}$ .
- $\mathcal{O}_P$  be the natural structure sheaf on the variety  $P$
- $V$  be a  $m$ -dimensional vector space,  $\mathcal{V} = V \otimes \mathcal{O}_P$ ,  $\mathcal{V}(k) = V \otimes \mathcal{O}_P(k)$  where we view  $V$  as either the constant sheaf or trivial bundle over  $P$ .
- $A(t, \xi) = \sum_{k=0}^n A_k(t) \xi^k \in H^0(P, \text{Hom}(\mathcal{V}, \mathcal{V}(n)))$  for some  $n$ , where we see  $A_i(t) \in \text{End}(V)$  as a time dependent  $m \times m$  matrix and  $\xi^k \in H^0(P, \mathcal{O}(n))$  as

$$[\xi_0 : \xi_1]^k = \underbrace{\xi_0 \otimes \cdots \otimes \xi_0}_{\times (n-k)} \otimes \underbrace{\xi_1 \otimes \cdots \otimes \xi_1}_{\times k}$$

This is homogeneous of degree  $n$ , so we allow  $A$  to not have a scale?

- $B(\xi, t) \in H^0(P, \text{Hom}(\mathcal{V}, \mathcal{V}(N)))$  for some  $N$  likewise .
- $Q(\xi, \eta) = \det[\eta I - A(\xi, t)]$  be the characteristic polynomial of  $A$ .
- $\sigma$  be the tautological section of  $\mathcal{O}_P(n)$ .

**Lemma 2.45.**  $Q(\xi, \sigma) \in H^0(\mathcal{O}_P(n), \pi^* \mathcal{O}_P(mn))$

**Definition 2.46.** The pair  $A, B$  is a Lax pair if  $\dot{A} = [A, B]$ .

**Proposition 2.47.** The Lax equation is invariant under the substitution

$$B \mapsto B + p(A, \xi)$$

for polynomial  $p(x, \xi) \in \mathbb{C}[x, \xi]$ .

**Definition 2.48.** The **spectral curve** is  $C$  given by the solution in  $P$  of

$$Q(\xi, \eta) = 0$$

**Proposition 2.49.** The flow  $t \mapsto A(\xi, t)$  is isospectral.

It will be the understanding of this isospectral flow that we want to gain. We formulate this flow as the family of holomorphic map gained by the eigenvectors

$$f_t : C \rightarrow \mathbb{CP}^{m-1}$$

Suppose that  $C$  has degree  $d$ , then we know we can define

$$L_t = f_t^*(\mathcal{O}(1)) \in \text{Pic}^d(C)$$

Lets choose a reference bundle  $L_0 \in \text{Pic}^d(X)$

**Lemma 2.50.** *The map*

$$\begin{aligned}\mathrm{Pic}^d(C) &\rightarrow J(C) \\ L &\mapsto L \otimes L_0^{-1}\end{aligned}$$

*is an isomorphism.*

Now knowing our result about the tangent space to the Picard group we can say  $\frac{dL_t}{dt} \in H^1(C, \mathcal{O}_C)$ .

## References

- [1] Phillip A. Griffiths. Linearizing flows and a cohomological interpretation of lax equations. *American Journal of Mathematics*, 107(6):pp. 1445–1484, 1985. ISSN 00029327, 10806377. doi: 10.2307/2374412.