

Topics in Rings and Representation Theory - Kac Moody Algebras

Linden Disney-Hogg

January 2020

Contents

1	Introduction	2
2	Groups, Algebras, and their Representations	2
2.1	Algebras	2
2.2	Representations	3
3	Lie algebras	5
3.1	preliminaries	5
3.2	Universal Enveloping Algebras	5
3.2.1	Construction	6
3.3	Representations	7
3.3.1	Simple Lie algebra	7
3.3.2	Classification of complex f.d simple Lie algebras	7
3.4	$\mathfrak{sl}(n)$	8
4	Kac-Moody Algebras	9
4.1	Bilinear forms on $\mathfrak{g}(A)$	13
4.2	Weyl group	16
4.3	Finite vs Affine	18
4.3.1	Central extension of the loop algebra	20
5	Witt and Virasoro Algebras	21
5.1	The Witt algebra	21
5.2	Central extension	22
5.3	Heisenberg algebra	23
5.4	Connection to affine Lie algebras	25
6	Highest weight representations	26
6.1	Verma modules	27
6.2	Unitary reps	28
6.3	Kac Determinant formula	30

7	Lie algebra of infinite matrices	30
7.1	Central extension	30
7.2	Shift operators	31
8	Fermionic Fock space	32
9	Representations of GL_∞, \mathfrak{gl}_∞, on $F^{(m)}$	34
9.1	Actions on tensor products	34

1 Introduction

A set of lecture notes on a masters course on Kac moody algebras

2 Groups, Algebras, and their Representations

2.1 Algebras

Throughout this course we will take k to be a field.

Definition 2.1. An *algebra* is a triple (A, m, i) of

- a k -vector space A
- a linear map $m : A \otimes A \rightarrow A$
- an element $i : k \rightarrow A$

satisfying associativity and unitality.

Notation. For $a, b \in A$ we will denote $m(a, b) = a \cdot b$.

Remark. Linearity of m gives distributivity of the multiplication over k .

Proposition 2.2. If a unit exists for (A, \cdot) , it is unique

Proof. Let $1, 1' \in A$ be the units. Then

$$1 = 1 \cdot 1' = 1'$$

□

Example 2.3. Some examples of algebras are

- The base field k
- polynomials over k , $k[X]$.
- $\text{End}(V)$ where V is a vector space, with multiplication given by composition

Example 2.4. The **free algebra** $k\langle x_1, \dots, x_n \rangle$ is the vector space consisting formally of all possible combinations of the x_i in order to make it a vector space, namely

$$k\langle x_1, \dots, x_n \rangle = \bigoplus_{m=0}^{\infty} k \cdot \prod_{1 \leq j_i \leq n} x_{j_1} \cdots x_{j_m}$$

Example 2.5. Given a group G we have the **group algebra** $A \equiv kG$ with

- basis $\{x_g \mid g \in G\}$
- multiplication $x_g \cdot x_h = x_{gh}$
- unit x_{e_G}

Definition 2.6. (A, \cdot) is **commutative** if $\forall a, b \in A, a \cdot b = b \cdot a$.

Example 2.7. kG is abelian iff G is abelian.

Definition 2.8. A homomorphism of algebras $f : A \rightarrow B$ is a linear map of vector spaces compatible with \cdot s.t.

- $\forall a, b \in A, f(a \cdot b) = f(a) \cdot f(b)$
- $(f(1_A) = 1_B)$

2.2 Representations

Definition 2.9. A **representation** of (A, \cdot) is a vector space V with $\rho : A \rightarrow \text{End}(V)$ a homomorphism of algebras.

Notation. We will often, for simplicity, abuse notation and write for $a \in A, v \in V$

$$\rho(a)(v) = a \cdot v$$

Remark. A representation is also call a **left A-module**. A right A -module has $\sigma : A \rightarrow \text{End}(V)$ an antihomomorphism. We define an algebra (A^{op}, m^{op}) s.t. $A^{op} = A, \forall a, b \in A, m^{op}(a, b) = m(b, a)$. We can then say that a right A -module is a representation of A^{op} .

Example 2.10. We have a few standard examples of reps:

- $V = 0$
- $V = A$ and for $a \in A, \rho : a \mapsto \rho(a)$ s.t. $\forall b \in A, \rho(a)(b) = m(a, b)$. This is called the **regular rep**.
- $A = k$, then any rep is just a vector space over k
- If $A = k\langle x_1, \dots, x_n \rangle$ then a rep is a vector space with $\rho(x_i) \in \text{End}(V)$ specified.

Definition 2.11. Given two representations V_1, V_2 , the **direct sum** representation $V_1 \oplus V_2$ is given with

$$\rho_{V_1 \oplus V_2}(a)(v_1 + v_2) = \rho_{V_1}(a)(v_1) + \rho_{V_2}(a)(v_2)$$

for $a \in A, v_i \in V_i$.

Definition 2.12. A **subrepresentation** is subspace $W \subset V$ s.t. $\forall a \in A, \rho(a)(W) \subset W$.

Definition 2.13. Given $v \in V$ the **minimal subrep** containing v is

$$A \cdot v = \{w \in V \mid \exists a \in A, w = a \cdot v\}$$

Definition 2.14. A rep is **irreducible** if the only subreps are $W = 0, V$

Definition 2.15. Let V_1, V_2 be reps of A . Then a homomorphism of reps (an **intertwiner**) is linear map $\phi : V_1 \rightarrow V_2$ s.t. $\forall v \in V_1, a \in A$, with $\rho_i : A \rightarrow \text{End}(V_i)$ we have

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_2 \\ \rho_1(a) \downarrow & & \downarrow \rho_2(a) \\ V_1 & \xrightarrow{\phi} & V_2 \end{array}$$

commutes, i.e. $\phi(a \cdot v) = a \cdot \phi(v)$

Proposition 2.16. Let $f : V \rightarrow W$ be an intertwiner. Then

- $\ker f \subset V$ is a subrep
- $\text{Im } f \subset W$ is a subrep

Lemma 2.17 (Schur). Let V_1, V_2 be two A -reps and let $f : V_1 \rightarrow V_2$ be a non-zero intertwiner. Then

- V_1 irreducible $\Rightarrow f$ is injective
- V_2 irreducible $\Rightarrow f$ is surjective

Definition 2.18. A representation is **indecomposable** is when $V = V_1 \oplus V_2$, either $V_1 = 0$ or $V_2 = 0$

Proposition 2.19. Any irreducible rep is indecomposable

Proof. V_1, V_2 are subreps of $V_1 \oplus V_2$. □

Remark. The converse to the above is not true,

Aside. Coming from the workshop, we have some points that we want to have made clear in our mind:

1. Rep theory is linear algebra. If $f : V \rightarrow W$ is an intertwiner, the condition that for $a \in A, v \in V$ $f(a \cdot v) = a \cdot f(v)$, is essentially saying that f is A -linear. We now have a correspondence

A -linear	k -linear
Irreducible reps	eigenspaces
indecomposable reps	generalised eigenspaces

3 Lie algebras

3.1 preliminaries

Definition 3.1. A *Lie algebra* is a vector space \mathfrak{g} endowed with a bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying

- *antisymmetry*: $\forall x \in \mathfrak{g}, [x, x] = 0$
- *Jacobi identity*: $\forall x, y, z \in \mathfrak{g}, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Example 3.2. Given an associative algebra A , we can make A a Lie algebra using

$$[a, b] = ab - ba$$

for $a, b \in A$.

Example 3.3. Let $\mathfrak{sl}_n(k) = \{X \in M_n(k) \mid \text{Tr}(X) = 0\}$. Then as $\text{Tr}(XY) = \text{Tr}(YX)$ we have a bracket given by the commutator

$$[X, Y] = XY - YX$$

This gives a Lie algebra structure to $\mathfrak{sl}_n(k)$ which is not inherited from matrix multiplication, as $\mathfrak{sl}_n(k)$ is not closed under multiplication, so is not an associative algebra. To motivate looking at such a vector space, note that

$$\text{Tr}(X) = 0 \Rightarrow \det \exp(X) = e^{\text{Tr}(X)} = 1$$

and $SL_n(k) = \{Y \in M_n(k) \mid \det(Y) = 1\}$ has a natural operation of matrix multiplication, as $\det(XY) = \det(X)\det(Y)$.

3.2 Universal Enveloping Algebras

Definition 3.4. Let $\{x_i\}$ be a basis of \mathfrak{g} a Lie algebra and suppose the bracket is specified by

$$[x_i, x_j] = \sum_k c_{ij}^k x_k$$

for some **structure constants** c_{ij}^k . Then the **universal enveloping algebra** $\mathcal{U}_{\mathfrak{g}}$ is the associative algebra generated by the x_i with the relations

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k$$

i.e.

$$\mathcal{U}_{\mathfrak{g}} = k \langle x_1, \dots, x_n \rangle / \left\langle x_i x_j - x_j x_i - \sum_k c_{ij}^k x_k \right\rangle$$

We get the map

$$\begin{aligned} \iota : \mathfrak{g} &\rightarrow \mathcal{U}_{\mathfrak{g}} \\ x_i &\mapsto x_i \end{aligned}$$

The above definition involves a choice of basis of \mathfrak{g} . In general we want to remove this to give a universal property of $\mathcal{U}_{\mathfrak{g}}$.

Proposition 3.5. *For any associative algebra A , and Lie algebra map $\mathfrak{g} \xrightarrow{f} A^{Lie}$, there exists a unique map of associative algebras $\mathcal{U}_{\mathfrak{g}} \xrightarrow{\mathcal{U}(f)} A$ s.t.*

$$\begin{array}{ccc} & \mathcal{U}_{\mathfrak{g}} & \\ \nearrow \iota & \downarrow \mathcal{U}(f) & \\ \mathfrak{g} & \xrightarrow{f} & A \end{array}$$

commutes

Exercise 3.6. *Prove that $\mathcal{U}_{\mathfrak{g}}$ is uniquely defined (up to isomorphism) by this universal property.*

Proof. Consider

$$\mathcal{U}_{\mathfrak{g}} = \left[\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} \right] / \langle x \otimes y - y \otimes x - [x, y] \rangle$$

□

Proposition 3.7. *If $\mathcal{U}_{\mathfrak{g}}$ satisfies the universal property then there is a bijection*

$$\{\text{rep of Lie algebra } \mathfrak{g}\} \leftrightarrow \{\text{reps of associative algebra } \mathcal{U}_{\mathfrak{g}}\}$$

3.2.1 Construction

Definition 3.8. *Recall that for a vector space V over k , the **tensor algebra** is*

$$T(V) \equiv \bigoplus_{n \geq 0} V^{\otimes n}$$

where $V^{\otimes 0} = k$. It comes with the map

$$\begin{aligned} V^{\otimes n} \times V^{\otimes m} &\rightarrow V^{\otimes(n+m)} \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

Definition 3.9. *The **symmetric algebra** is*

$$S(V) \equiv T(V) / \langle v \otimes w - w \otimes v \mid v, w \in V \rangle$$

Definition 3.10. *Given a Lie algebra \mathfrak{g} , the **universal enveloping algebra** is*

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$$

Theorem 3.11 (Poincare-Birkhoff). *We have the following two properties:*

1. $\mathcal{U}(\mathfrak{g})$ satisfies the universal property

2. As a vector space, $\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$.

Remark. If $\{x_i \mid i = 1, \dots, n\} \subset \mathfrak{g}$ is a basis, then the set of ordered monomials $x_1^{i_1} \cdots x_n^{i_n} = x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$ is a basis of $\mathcal{U}(\mathfrak{g})$.

Remark. Note that in $\mathcal{U}(\mathfrak{g})$, by our quotient it must be that

$$x_i x_j - x_j x_i = [x_i, x_j]$$

which is what we wanted to see.

Example 3.12. If \mathfrak{g} is abelian, then $\forall x, y \in \mathfrak{g}$, $[x, y] = 0$ and we see by definition $\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$.

3.3 Representations

3.3.1 Simple Lie algebra

Definition 3.13. An *ideal* of a Lie algebra \mathfrak{g} is a subspace \mathfrak{g}' s.t. $[\mathfrak{g}', \mathfrak{g}] \subset \mathfrak{g}'$

Definition 3.14. \mathfrak{g} is *simple* if the only ideals of \mathfrak{g} are $0, \mathfrak{g}$. \mathfrak{g} is *semi-simple* if it is a direct sum of simple Lie algebras.

Remark. $\mathfrak{g} \curvearrowright \mathfrak{g}$ by $x \cdot y = [x, y]$ This is the **adjoint rep**. Then we have ideals of \mathfrak{g} correspond to subreps of the adjoint rep.

Example 3.15. Consider $\mathfrak{gl}_n(k) = M_n(k)^{\text{Lie}}$. Recall $\mathfrak{sl}_n(k) \subset \mathfrak{gl}_n(k)$ is the set of traceless matrices. This is an ideal so $\mathfrak{gl}_n(k)$ is non-simple.

Exercise 3.16. Prove $\mathfrak{sl}_n(k)$ is simple.

Theorem 3.17 (Weyl complete reducibility). Let \mathfrak{g} be a finite dimensional semi-simple Lie algebra and $V \in \text{Rep}(\mathfrak{g})$. If $W \subseteq V$ is a subrepresentation, then $\exists W' \subseteq V$ s.t $V \cong W \oplus W'$ as representations.

3.3.2 Classification of complex f.d simple Lie algebras

- $(A_n, n \geq 1)$: $\mathfrak{sl}_{n+1}(\mathbb{C}) = \{\text{Tr}(X) = 0\} \subset \mathfrak{gl}_n$
- $(B_n, n \geq 2)$: $\mathfrak{so}_{2n+1}(\mathbb{C}) = \{\text{Tr}(X) = 0, X^T + X = 0\} \subset \mathfrak{gl}_{2n+1}$
- $(C_n, n \geq 3)$: $\mathfrak{sp}_n(\mathbb{C}) = \{J_n X = X^T J_n\} \subset \mathfrak{gl}_{2n}$
- $(D_n, n \geq 4)$: $\mathfrak{so}_{2n}(\mathbb{C}) = \{\text{Tr}(X) = 0, X^T + X = 0\} \subset \mathfrak{gl}_{2n}$
- Exceptionals, E_6, E_7, E_8, F_4, G_2 , dimensions 52, 133, 248, 52, 14.

Remark. Suppose \mathfrak{g} is simple. Take $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ a morphism of Lie algebras. If $\pi \neq 0$, then as $\ker \pi \subsetneq \mathfrak{g}$ is an ideal it must be the case that $\ker \pi = 0$

Now if we define $\mathfrak{gl}_n = \text{End}_i(k^n)^{\text{Lie}}$, this has a basis $\{E_{ab} = (\delta_{ia} \delta_{jb})_{i,j=1}^n\}$ called the **elementary matrices**. These obey

$$\begin{aligned} E_{ij} E_{kl} &= \delta_{jk} E_{il} \\ \Rightarrow [E_{ij}, E_{kl}] &= \delta_{jk} E_{il} - \delta_{il} E_{jk} \end{aligned}$$

3.4 $\mathfrak{sl}(n)$

Example 3.18. Consider $\mathfrak{sl}(2)$. Taking $n = 2$, we have the basis

$$\begin{aligned} e &= E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f &= E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &= E_{11} - E_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

These get the commutation relations

$$\begin{aligned} [h, e] &= 2e \\ [h, f] &= -2f \\ [e, f] &= h \end{aligned}$$

More generally, we see $\dim_{\mathbb{C}}(\mathfrak{sl}(n)) = n^2 - 1$ by considering the trace condition, but has only $3(n - 1)$ generators

$$\begin{aligned} e_i &= E_{i,i+1} \\ f_i &= E_{i+1,i} \\ h_i &= E_{ii} - E_{i+1,i+1} \end{aligned}$$

for $i = 1, \dots, n - 1$.

Exercise 3.19. Moreover we have

$$\begin{aligned} [e_i, e_{i+2}] &= 0 \\ [e_i, [e_i, e_{i+1}]] &= 0 \end{aligned}$$

These generate as

$$[e_i, e_{i+1}] = E_{i,i+2}$$

and this can be iterated to get all upper triangular matrices, likewise for lower triangular with f and diagonal with all. Explicitly

$$\begin{aligned} [h_i, e_j] &= a_{ji} e_j \\ [h_i, f_j] &= -a_{ji} f_j \\ [e_i, f_j] &= \delta_{ij} h_i \end{aligned}$$

where

$$a_{ij} = \begin{cases} 2 & |i - j| = 0 \\ -1 & |i - j| = 1 \\ 0 & |i - j| > 1 \end{cases}$$

We call $A = (a_{ij})$ the **Cartan matrix**.

Theorem 3.20 (Serre). *If \mathfrak{g} is a finite dimensional simple Lie algebra over \mathbb{C} then \mathfrak{g} has a similar presentation.*

Proposition 3.21. *A has the following properties:*

- $A \in M_n(\mathbb{Z})$
- $\forall i \neq j, a_{ii} = 2 \text{ and } a_{ij} \leq 0$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$
- A is indecomposable
- $\det(A) \neq 0$ and A is positive definite.

4 Kac-Moody Algebras

Idea. *We can try to reverse engineer the Cartan matrix, to generalise it and then assign a Lie algebra $\mathfrak{g}(A)$ to the resulting matrix A .*

Definition 4.1. *A **realisation** of A is a triple $(\mathfrak{h}, \Pi^\vee, \Pi)$ where*

- \mathfrak{h} is a vector space
- $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subseteq \mathfrak{h}$
- $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$

s.t.

- Π^\vee is a linearly independent set
- Π is a linearly independent set
- $\alpha_i(\alpha_j^\vee) = a_{ji}$

*We call Π the **simple roots**, and Π^\vee the **simple coroots***

Exercise 4.2. *Show the following results:*

- *If $(\mathfrak{h}, \Pi^\vee, \Pi)$ is a realisation, $\dim \mathfrak{h} \geq 2n - \text{rank}(A)$*
- *A **minimal realisation** (i.e $\dim \mathfrak{h} = 2n - \text{rank}(A)$) always exists*

Definition 4.3. *A morphism $(\mathfrak{h}, \Pi^\vee, \Pi) \rightarrow (\mathfrak{h}', (\Pi')^\vee, \Pi')$ is*

- $\phi : \mathfrak{h} \rightarrow \mathfrak{h}'$
- $\phi(\Pi^\vee) = (\Pi')^\vee$
- $\phi(\Pi) = \Pi'$

Proposition 4.4. *For any A , $\exists!$ minimal realisation Σ up to isomorphism.*

Definition 4.5. Let $(\mathfrak{h}, \Pi^\vee, \Pi)$ be a realisation of A . Then $\tilde{\mathfrak{g}}(A)$ is the Lie algebra with generators e_i, f_i for $i = 1, \dots, n$ containing \mathfrak{h} s.t.

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} \alpha_i^\vee \in \mathfrak{h} \\ \forall h \in \mathfrak{h}, [h, e_i] &= \alpha_i(h) e_i \\ \forall h \in \mathfrak{h}, [h, f_i] &= -\alpha_i(h) f_i \\ \forall h, h' \in \mathfrak{h}, [h, h'] &= 0 \end{aligned}$$

Example 4.6. If $A = [2]$, we can follow the procedure and find $\tilde{\mathfrak{g}}(A) = \mathfrak{sl}(2)$. To see we see $\dim \mathfrak{h} = 1$ for a minimal realisation, so we only need $\alpha^\vee \in \mathfrak{h}, \alpha \in \mathfrak{h}^*$ s.t. $\alpha(\alpha^\vee) = 2$. With this $\mathfrak{h} = \text{Span}\{\alpha^\vee\}$ We then need e, f to satisfy

$$\begin{aligned} [e, f] &= \alpha^\vee \\ [\alpha^\vee, e] &= \alpha(\alpha^\vee) e = 2e \\ [\alpha^\vee, f] &= -\alpha(\alpha^\vee) f = -2f \end{aligned}$$

This is just \mathfrak{sl}_2 if we relabel $h = \alpha^\vee$.

Idea. $\tilde{\mathfrak{g}}(A)$ is still currently too large, for example as the e_i are not yet related, and we want to try make it look like \mathfrak{sl}_n , i.e maybe simple. Hence we want to consider all ideals of the form

$$\text{trivial ideals} = \{r \subset \tilde{\mathfrak{g}}(A) \text{ ideals} \mid r \cap \mathfrak{h} = 0\}$$

and let

$$r_{\max} = \sum_{r \in \text{trivial}} r$$

and quotient by this to go

$$A \rightarrow (\mathfrak{h}, \Pi^\vee, \Pi) \rightarrow \tilde{\mathfrak{g}}(A) \rightarrow \mathfrak{g}(A) = \tilde{\mathfrak{g}}(A) / r_{\max}$$

Definition 4.7. We define

- $Q = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i$ the **root lattice**
- $Q \supset Q_+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$ the **positive root lattice**
- $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i^\vee$ the **coroot lattice**
- $Q^\vee \supset Q_+^\vee = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i^\vee$ the **positive coroot lattice**

Note that the relations required for $\tilde{\mathfrak{g}}(A)$ give us all the commutators we need, e.g

$$\begin{aligned} [h, [e_1, e_2]] &= -[e_2, [h, e_1]] - [e_1, [e_2, h]] \\ &= \alpha_1(h) [e_2, e_1] + \alpha_2(h) [e_1, e_2] \\ &= (\alpha_1 + \alpha_2)(h) [e_1, e_2] \end{aligned}$$

Definition 4.8. Now for $\alpha \in Q$ we can define

$$\tilde{\mathfrak{g}}_\alpha \equiv \{x \in \tilde{\mathfrak{g}}(\Sigma) \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$$

Example 4.9. We can consider examples of this:

- $\mathfrak{h} \supset \tilde{\mathfrak{g}}_0 = \mathfrak{h}$.
- $\tilde{\mathfrak{g}}_{\alpha_i} = \mathbb{C}e_i$
- $\tilde{\mathfrak{g}}_{-\alpha_i} = \mathbb{C}f_i$
- $\tilde{\mathfrak{g}}_{\alpha_1 - \alpha_2} = 0$

We can then also state the following

Theorem 4.10. We have the following

1. As a vector space, $\tilde{\mathfrak{g}}(\Sigma) = \tilde{n}_+ \oplus \mathfrak{h} \oplus \tilde{n}_-$ where \tilde{n}_+ is the free Lie algebra generated by the e_i , and \tilde{n}_- by the f_i .
2. We have

$$\begin{aligned}\tilde{n}_+ &= \bigoplus_{\alpha \in Q_+ \setminus 0} \tilde{\mathfrak{g}}_\alpha \\ \tilde{n}_- &= \bigoplus_{\alpha \in Q_+ \setminus 0} \tilde{\mathfrak{g}}_{-\alpha}\end{aligned}$$

3. $[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] \subset \tilde{\mathfrak{g}}_{\alpha+\beta} \Rightarrow \tilde{\mathfrak{g}}(\Sigma)$ is Q -graded.

Now

Lemma 4.11. 1. $I \subset \tilde{\mathfrak{g}}(\Sigma)$ is an ideal then $I = \bigoplus_{\alpha \in Q} (I \cap \tilde{\mathfrak{g}}_\alpha)$

2. $\exists!$ maximal ideal $r \subseteq \tilde{\mathfrak{g}}(\Sigma)$ s.t. $r \cap \mathfrak{h} = 0$

3. $r = r_+ \oplus r_-$ with $r_\pm = r \cap \tilde{n}_\pm$

Definition 4.12. The **Kac-Moody** Lie algebra associated to Σ is $\mathfrak{g}(\Sigma) \equiv \tilde{\mathfrak{g}}(\Sigma)/_r$

Remark. As the minimal realisation is unique (up to iso) we have no problem notationally using either $\mathfrak{g}(A)$ or $\mathfrak{g}(\Sigma)$.

Example 4.13. $\mathfrak{sl}(2)$ is simple and so when $A = [2]$, $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A) = \mathfrak{sl}(2)$.

Exercise 4.14. Show that if $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ we get $\mathfrak{g}(A) = \mathfrak{sl}_3$.

Proof. $\text{rank}(A) = 2$, so the dimension of the minimal realisation is $2 \times 2 - 2 = 2$. Hence $\mathfrak{h} = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$. The corresponding $\tilde{\mathfrak{g}}(A)$ gives us all the generators we want letting $h_i = \alpha_i^\vee$. \square

Definition 4.15. Let $\tilde{w} : \tilde{\mathfrak{g}}(A) \rightarrow \tilde{\mathfrak{g}}(A)$ given by

$$\begin{aligned}\tilde{w}(e_i) &= f_i \\ \tilde{w}(f_i) &= e_i \\ \tilde{w}(h) &= -h\end{aligned}$$

This descends to a map $w : \mathfrak{g}(A) \rightarrow \mathfrak{g}(A)$ called the **Chevalley involution**.

The next result is useful for making calculations in $\mathfrak{g}(\Sigma)$:

Lemma 4.16. *We have the following two classifications*

1. For $x \in n_+$, $x = 0 \Leftrightarrow \forall i, [f_i, x] = 0$
2. For $x \in n_-$, $x = 0 \Leftrightarrow \forall i, [e_i, x] = 0$

Proof. Set $\mathfrak{g}_1 = \bigoplus_{i=1}^n \mathbb{C}e_i \subset n_+$. Then define the vector space

$$J_x \equiv \sum_{k \geq 0} \text{ad}(\mathfrak{g}_1)^k(x) \ni x$$

Then we can note

- $[n_+, J_x] \subset J_x$
- $[\mathfrak{h}, J_x] \subset J_x$

and further

Claim: $[f_i, \text{ad}(\mathfrak{g}_1)^k(x)] \subset J_x$. We can show this by induction. Certainly if we have assumed $[f_i, x] = 0$, then $[f_i, x] \in J_x$. Now for $a \in \mathfrak{g}_1, b \in \text{ad}(\mathfrak{g}_1)^{k-1}(x)$ we have

$$[f_i, [a, b]] = \left[\underbrace{[f_i, a]}_{\in \mathfrak{h}}, b \right] + \left[a, \underbrace{[f_i, b]}_{\in J_x} \right]$$

So we have that J_x is an ideal and that $J_x \cap \mathfrak{h} = 0$, so $J_x = 0$. □

Definition 4.17. Define $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(\Sigma) \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$ in analogy to $\tilde{\mathfrak{g}}_\alpha$.

Remark. We have the $\forall \alpha \in Q_+ \setminus 0$,

- $\tilde{\mathfrak{g}}_{\pm\alpha} \neq 0$
- $\dim \tilde{\mathfrak{g}}_{\pm\alpha} < \infty$
- If we define $ht(\alpha) = \sum_i k_i$ for $\alpha = \sum_i k_i \alpha_i$ then $\dim \mathfrak{g}_{\pm\alpha} \leq n^{|ht(\alpha)|}$

Definition 4.18. We call $R = \{\alpha \in Q \setminus 0 \mid \mathfrak{g}_\alpha \neq 0\} \subset Q$ the **set of roots**

Proposition 4.19. We have

1. $\mathfrak{g}(\Sigma) = \bigoplus_{\alpha \in Q_+ \setminus 0} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in Q_+ \setminus 0} \mathfrak{g}_{-\alpha}$.
2. $R = R_+ \cup R_-$ where $R_\pm = R \cap (\pm Q_\pm)$.

Exercise 4.20. We can show

$$\begin{aligned} A = (2) & \Rightarrow R = \{\alpha_1\} \cup \{-\alpha_1\} \\ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \Rightarrow R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, R_- = -R_+ \\ A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} & \Rightarrow |R| = \infty \end{aligned}$$

4.1 Bilinear forms on $\mathfrak{g}(A)$

Sometimes $\mathfrak{g}(\Sigma)$ has a non-degenerate, symmetric, invariant bilinear form

$$(\cdot, \cdot) : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$$

i.e.

- $\ker(\cdot, \cdot) = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g}, (x, y) = 0\} = 0$
- $\forall x, y \in \mathfrak{g}, (x, y) = (y, x)$
- $\forall x, y, z \in \mathfrak{g}, ([x, y], z) = (x, [y, z])$

This will turn out to be the analogue of the Killing form.

Theorem 4.21. *If A is symmetrisable, i.e. $\exists D = \text{diag}(d_1, \dots, d_n)$ s.t. $\det D \neq 0$ and $B = DA$ is symmetric, then \exists such a form on $\mathfrak{g}(\Sigma)$.*

Note that in the above theorem, we have fixed a choice by asking for D . It is then natural to ask how many choices we have.

Example 4.22. $\mathfrak{g}(A) \cong \mathfrak{g}(DA)$, as this simply scales generators $e_i \mapsto d_i e_i$.

We can define a non-degenerate symmetric bilinear form on \mathfrak{h} by

- $\forall h \in \mathfrak{h}, (\alpha_i^\vee, h) = d_i \alpha_i(h)$
- $\forall h_1, h_2 \in \mathfrak{h}'', (h_1, h_2) = 0$

where we have defined $\mathfrak{h}' \equiv \langle \Pi^\vee \rangle = \bigoplus_{i=1}^N \mathbb{C} \alpha_i^\vee$ and then required $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$. It is the coupling to D that gives the symmetry e.g

$$\begin{aligned} (\alpha_i^\vee, \alpha_j^\vee) &= d_i \alpha_i(\alpha_j^\vee) \\ &= d_i a_{ji} \\ &= d_j a_{ij} \\ &= (\alpha_j^\vee, \alpha_i^\vee) \end{aligned}$$

Theorem 4.23. (\cdot, \cdot) extends to a non-degenerate symmetric bilinear invariant form on $\mathfrak{g}(A)$ by setting

- $(e_i, f_j) = \delta_{ij}$
- $(e_i, e_j) = 0 = (f_i, f_j)$
- $(e_i, h) = 0 = (f_i, h)$

Remark. Note that these conditions are imposed on us in order to have invariance, e.g

$$([e_1, e_2], f_1) = (e_1, [e_2, f_1]) = 0$$

or

$$([e_1, e_2], f_1) = -(e_2, \underbrace{[e_1, f_1]}_{\in \mathfrak{h}}) = 0$$

Corollary 4.24. *Let $\alpha \in Q$, and recall $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$. Then*

$$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) \neq 0 \Leftrightarrow \alpha + \beta = 0$$

Then identifying $\mathfrak{g} \cong \mathfrak{g}^$ by $x \mapsto (x, \cdot)$, we have*

$$\mathfrak{g}_\alpha \cong \mathfrak{g}_{-\alpha}^*$$

Now let us make the prop:

Proposition 4.25. *Let $\nu : \mathfrak{h} \xrightarrow{\cong} \mathfrak{h}^*$ be the function*

$$\begin{aligned} \nu(h) &= (h, \cdot) \\ \nu(\alpha_i^\vee) &= d_i \alpha_i \end{aligned}$$

Then for $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$:

$$[x, y] = (x, y) \cdot \nu^{-1}(\alpha)$$

Theorem 4.26 (Serre). *Suppose we have $A \in M_n(\mathbb{Z})$ satisfying $a_{ii} = 2, a_{ij} \leq 0$. Then in $\mathfrak{g}(A)$*

$$\begin{aligned} \text{ad}(e_i)^{1-a_{ij}}(e_j) &= 0 \\ \text{ad}(f_i)^{1-a_{ij}}(f_j) &= 0 \end{aligned}$$

*These are called the **Serre relations**.*

Proof.

□

Theorem 4.27 (Gabber - Kac). *If we have $A \in M_n(\mathbb{Z})$ satisfying $a_{ii} = 2, a_{ij} \leq 0$ **and** A is symmetrisable, then the only relations on $\mathfrak{g}(A)$ are the Serre relations.*

Remark. *If we have the conditions of the above theorem, then we know $\mathfrak{g}(A)$ is generated by e_i, f_i, \mathfrak{h} s.t.*

- $[h, h'] = 0$
- $[h, e_i] = \alpha_i(h)e_i$
- $[h, f_i] = -\alpha_i(h)f_i$
- $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$
- $\text{ad}(e_i)^{1-a_{ij}}(e_j) = 0$
- $\text{ad}(f_i)^{1-a_{ij}}(f_j) = 0$

and this entirely determines $\mathfrak{g}(A)$.

Example 4.28. *Consider $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, which has $\mathfrak{g}(A) = \mathfrak{sl}_3$. Then we found*

$$\text{ad}(e)^{1-(-1)}(e) = [e_1, [e_1, e_2]] = 0$$

Now we have a copy of \mathfrak{sl}_2 , called $\mathfrak{sl}_2^{(i)} = \langle e_i, f_i, \alpha_i^\vee \rangle$ sitting in $\mathfrak{g}(A)$. We prove a lemma about it's adjoint action:

Lemma 4.29. *Let $V \in \text{Rep}(\mathfrak{sl}_2)$ and $v \in V$ s.t. $h \cdot v = \lambda v$, $e \cdot v = 0$. Defining $v_m = \frac{f^m}{m!} \cdot v$ we have*

$$\begin{aligned} h \cdot v_m &= (\lambda - 2m)v_m \\ e \cdot v_m &= (\lambda - m + 1)v_{m-1} \end{aligned}$$

Proof. Note

$$\begin{aligned} hf^m &= ([h, f] + fh)f^{m-1} \\ &= -2f^m + fhf^{m-1}. \end{aligned}$$

and so so repeating this procedure we get

$$hf^m = f^m(h - 2m) \Rightarrow hf^m \cdot v = (\lambda - 2m) \cdot v$$

We do the same with e seeing

$$\begin{aligned} ef^m &= (h + fe)f^{m-1} = hf^{m-1} + fhf^{m-2} + \dots + f^{m-1}h + f^me \\ &= f^{m-1}[(h - 2m + 2) + (h - 2m + 4) + \dots + (h)] + f^me \\ &= f^{m-1}[m(h - m + 1)] + f^me \end{aligned}$$

yielding the result □

Definition 4.30. *Given $V \in \text{Rep}(\mathfrak{g})$, $x \in \mathfrak{g}$ we say x acts **locally nilpotently** if $\forall v \in V, \exists N$ s.t. $x^N \cdot v = 0$.*

Definition 4.31. *We say $V \in \text{Rep}(\mathfrak{g}(A))$ is **integrable** if*

1. $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where $V_\lambda = \{v \in V \mid \forall h \in \mathfrak{h}, h \cdot v = \lambda(h)v\}$
2. e_i, f_i acts locally nilpotently on V .

Definition 4.32. *If $V_\lambda \neq 0$, we call it a **weight space of weight λ***

Example 4.33. $\mathfrak{g}(A)$ is integrable over itself as the adjoint rep because of the Serre relations. Here the weight spaces are \mathfrak{g}_α . Note if $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$ we will have

$$\begin{aligned} h \cdot x &= \alpha(h)x \\ \Rightarrow \exp(h) \cdot x &= e^{\alpha(h)}x \end{aligned}$$

Example 4.34. *Ever finite dimensional representation is integrable. This is as automatically $V = \bigoplus V_\lambda$, and then as eigenspaces cannot mix, there is no way to keep acting.*

Proposition 4.35. *Let $V \in \text{Rep}(\mathfrak{g}(A))^{Int}$ and let $\mathfrak{g}_{(i)} = \langle e_i, f_i, \alpha_i^\vee \rangle \subset \mathfrak{g}(A)$. Then*

1. As $\mathfrak{g}_{(i)}$ modules,

$$V = \bigoplus_{d \geq 0} V_d^{\oplus m_d}$$

where V_d is a irreducible rep of \mathfrak{sl}_2 , $\dim V_d = d + 1$, $m_d \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

2. Take $\lambda \in \text{wt}(V) \equiv \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$ and fix an α_i -string through λ , $M = \{t \in \mathbb{Z} \mid \lambda + t\alpha_i \in \text{wt}(V)\}$. Then

- (a) $\exists p, q \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ s.t. $M = [-p, q]$
- (b) $\text{mult}_V(\lambda) = \dim V_\lambda < \infty \Rightarrow p, q < \infty$.
- (c) $p, q < \infty \Rightarrow p - q = \lambda(\alpha_i^\vee)$
- (d) $t \mapsto m(t) \equiv \dim V_{\lambda+t\alpha_i}$ is symmetric at $t = -\frac{1}{2}\lambda(\alpha_i^\vee)$

Proof. Take $v_\lambda \in V_\lambda$, and define

$$U_{v_\lambda} = \sum_{k,l \geq 0} \mathbb{C} \cdot f_i^k \cdot e_i^l \cdot v_\lambda$$

Now we must have $\dim U_{v_\lambda} \leq 0$ from nilpotency, and we have an action $\mathfrak{g}_{(i)} \curvearrowright U_{v_\lambda}$. By Weyl reducibility

$$U_{v_\lambda} = \bigoplus_{d \geq 0} V_d^{\oplus m_d}$$

Do this for all $v \in V$.

Now let $U = \sum_{t \in M} V_{\lambda+t\alpha_i} \curvearrowright \mathfrak{g}_{(i)}$, and define $p = -\inf M$, $q = \sup M$. As $0 \in M$ it must be the case $p, q \geq 0$. Now we can calculate

$$(\lambda + t\alpha_i)(\alpha_i^\vee) = \lambda(\alpha_i^\vee) + 2t$$

and so we have

$$(\lambda + t\alpha_i)(\alpha_i^\vee) = 0 \Leftrightarrow t = -\frac{1}{2}\lambda(\alpha_i^\vee)$$

□

Corollary 4.36. $\lambda \in \text{wt}(V) \Rightarrow \lambda - \lambda(\alpha_i^\vee)\alpha_i \in \text{wt}(V)$

Example 4.37. Take $\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$. Then $\text{wt}(\mathfrak{g}(A)) = \{\text{roots}\} \cup \{0\}$

4.2 Weyl group

Definition 4.38 (Fundamental reflections). We define the **fundamental reflections** $r_i \in GL(\mathfrak{h}^*)$ by $r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i$

Proposition 4.39. r_i are reflections with fixed points $\ker(\alpha_i^\vee) = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) = 0\}$. Moreover $r_i(\alpha_i) = -\alpha_i$.

Definition 4.40 (Weyl group). We define the **Weyl group** to be

$$W = \langle r_i \rangle \subseteq GL(\mathfrak{h}^*)$$

Proposition 4.41. Take $V \in \text{Rep}(\mathfrak{g}(A))^{Int}$, $\lambda \in \mathfrak{h}^*$, $w \in W$, then

1. $\text{mult}_V(\lambda) = \text{mult}_V(w(\lambda))$

$$2. W \circ R \equiv \{\alpha \in Q \setminus 0 \mid \mathfrak{g}_\alpha \neq 0\}$$

$$3. \dim \mathfrak{g}_\alpha = \text{mult}(\alpha) = \text{mult}(w(\alpha)) = \dim \mathfrak{g}_{w(\alpha)}$$

Remark. $W \circ Q$

Exercise 4.42. $W \cong \langle r_i^\vee \rangle \subseteq GL(\mathfrak{h})$ where $r_i^\vee(h) = h - \alpha_i(h)\alpha_i^\vee$

Now assume we have $x, y : V \rightarrow V$ locally nilpotent s.t $\text{ad}(x)^N(y) = 0$ for some $N \gg 0$. Then

$$\exp(x) \cdot y \cdot \exp(-x) = \text{Ad}(\exp(x))(y) = \exp(\text{ad}(x))(y)$$

Theorem 4.43. Take $V \in \text{Rep}(\mathfrak{g}(A))^{Int}$ with $\pi : \mathfrak{g}(A) \rightarrow \mathfrak{gl}(V)$ the rep. We define $r_i^\pi : \exp(f_i)\exp(-e_i)\exp(f_i)$. Then

$$1. r_i^\pi(V_\lambda) = V_{r_i(\lambda)}$$

$$2. r_i^{\text{ad}} : \mathfrak{g}(A) \rightarrow \mathfrak{g}(A), r_i^{\text{ad}}|_{\mathfrak{h}} = r_i^\vee.$$

Example 4.44. Take $\mathfrak{g}(A) = \mathfrak{sl}_2$, then $W = \langle r_1 \mid r_1^2 = 1 \rangle \cong C_2$

Example 4.45. $\mathfrak{g}(A) = \mathfrak{sl}_3$. Then we have

	α_1	α_2
r_1	$-\alpha_1$	$\alpha_1 + \alpha_2$
r_2	$\alpha_1 + \alpha_2$	$-\alpha_2$
$r_1 r_2$	α_2	$-(\alpha_1 + \alpha_2)$
$r_2 r_1$	$-(\alpha_1 + \alpha_2)$	α_1
$r_1 r_2 r_1$	$-\alpha_2$	$-\alpha_1$
$r_2 r_1 r_2$	$-\alpha_2$	$-\alpha_1$

Hence we have

$$W = \langle r_1, r_2 \mid r_1^2 = 1 = r_2^2, r_1 r_2 r_1 = r_2 r_1 r_2 \rangle \cong S_3$$

We get this rep by taking $r_1 \mapsto (12), r_2 \mapsto (23)$

Remark. We can see that the above would satisfy what we want by using the braid relations.

Now we can decompose

$$\mathfrak{sl}_3 = \underbrace{\mathfrak{g}_{\alpha_1}}_{e_1} \oplus \underbrace{\mathfrak{g}_{\alpha_2}}_{e_2} \oplus \underbrace{\mathfrak{g}_{\alpha_1 + \alpha_2}}_{[e_1, e_2]} \oplus \dots$$

and then

$$R = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \cup \{-\alpha_1, -\alpha_2, -(\alpha_1 + \alpha_2)\}$$

Proposition 4.46. W is generated by the r_i with the relations

$$\bullet r_i^2 = 1$$

- $(r_i r_j)^{m_{ij}} = 1$ or equivalently $\underbrace{r_j r_i r_j \dots}_{m_{ij}} = \underbrace{r_i r_j r_i \dots}_{m_{ij}}$

where

$$\frac{m_{ij}}{a_{ij}a_{ji}} \parallel \begin{array}{c|c|c|c|c} 2 & 3 & 4 & 6 & \infty \\ \hline 0 & 1 & 2 & 3 & \geq 4 \end{array}$$

Remark. A group of this form is called a **Coxeter group**.

Example 4.47. We have correspondences

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &\mapsto \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} &\mapsto \mathfrak{sl}_3 \\ \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} &\mapsto \mathfrak{so}_4 \\ \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} &\mapsto G_2 \\ \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} &\mapsto \widehat{\mathfrak{sl}_2} = a * b / (a^2 = 1 = b^2) \quad (\text{affine } \mathfrak{sl}_2) \end{aligned}$$

Now we could consider working with realisations, and we would get the following results:

- $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$
- $Q^{\vee} = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^{\vee} \subset \mathfrak{h}_{\mathbb{R}} \circ W$

Definition 4.48 (Fundamental chamber). The **fundamental chamber** is $\mathcal{C} \equiv \{h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(h) \geq 0\}$. Correspondingly the **Tits cone** is $X = \bigcup_{w \in W} w(\mathcal{C})$

Proposition 4.49. TFAE:

- $|W| < \infty$
- $|R| < \infty$
- $X = \mathfrak{h}_{\mathbb{R}}$.

4.3 Finite vs Affine

We aim now to construct completely $\mathfrak{g}(\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}) \equiv \hat{g}$. We start by finding a realisation. We need

- $\hat{\mathfrak{h}}$ s.t. $\dim_{\mathbb{C}} \hat{\mathfrak{h}} = 3$
- $\{\alpha_0, \alpha_2\} \subset \hat{\mathfrak{h}}^*$, $\{\alpha_0^{\vee}, \alpha_1^{\vee}\} \subset \hat{\mathfrak{h}}$ linearly indep s.t. $\alpha_i(\alpha_j^{\vee}) = \pm 2$ if $i = j$ or $i \neq j$.

We can take

$$\begin{aligned}\hat{\mathfrak{h}} &= \mathbb{C}\alpha_0^\vee \oplus \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}d \\ \hat{\mathfrak{h}}^* &= \mathbb{C}\alpha_0 \oplus \mathbb{C}\alpha_1 \oplus \mathbb{C}\Lambda\end{aligned}$$

with

$$\begin{aligned}\alpha_0(d) &= 1 \\ \alpha_1(d) &= 0 \\ \Lambda(\alpha_0^\vee) &= 1 \\ \Lambda(\alpha_1^\vee) &= 0 \\ \Lambda(d) &= 0\end{aligned}$$

Now we know we will also have a bilinear form given by

$$\begin{aligned}(\alpha_i^\vee, \alpha_j^\vee) &= a_{ij} \\ (\alpha_0^\vee, d) &= 1 \\ (\alpha_1^\vee, d) &= 0 \\ (d, d) &= 0 \\ \text{also} \\ (\alpha_i, \alpha_j) &= a_{ij} \\ (\alpha_0, \Lambda) &= 1 \\ (\alpha_1, \Lambda) &= 0 \\ (\Lambda, \Lambda) &= 0\end{aligned}$$

Now we recognise this gives a special element $c \equiv \alpha_0^\vee + \alpha_1^\vee$, $\delta = \alpha_0 + \alpha_1$. This gives a map

$$\begin{aligned}\nu : \hat{\mathfrak{h}} &\rightarrow \hat{\mathfrak{h}}^* \\ \alpha_i^\vee &\mapsto \alpha_i \\ d &\mapsto \Lambda \\ c &\mapsto \delta\end{aligned}$$

These now have inner product

$$\begin{aligned}(\delta, \alpha_0) &= 0 = (\delta, \alpha_1) \\ \Rightarrow (\delta, \delta) &= 0 \quad \text{but} \quad (\delta, \Lambda) = 1 \\ (c, \alpha_0^\vee) &= 0 = (c, \alpha_1^\vee) \\ \Rightarrow (c, c) &= 0 \quad \text{but} \quad (c, d) = 1\end{aligned}$$

Now we have orthogonal decompositions

$$\begin{aligned}\hat{\mathfrak{h}} &= \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}c \oplus \mathbb{C}d \\ \hat{\mathfrak{h}}^* &= \mathbb{C}\alpha_1 \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda\end{aligned}$$

We want to relate $\hat{\mathfrak{g}}$ to $\mathfrak{g} = \mathfrak{sl}_2$. In $\hat{\mathfrak{g}}$

$$\begin{aligned} [d, e_1] &= 0 = [d, f_1] \\ [d, e_0] &= e_0 \\ [d, f_0] &= -f_0 \\ [c, e_i] &= 0 = [c, f_i] \end{aligned}$$

Hence $c \in Z(\hat{\mathfrak{g}})$.

4.3.1 Central extension of the loop algebra

Definition 4.50. Define $\mathcal{L} = \mathbb{C}[t, t^{-1}] = \{\text{Laurent polynomials}\}$ and $\text{res} : \mathcal{L} \rightarrow \mathbb{C}$, $\text{res}(\sum_k c_k t^k) = c_{-1}$.

Lemma 4.51. $\text{res} : \mathcal{L} \rightarrow \mathbb{C}$ is actually a linear functional with

- $\text{res}(t^{-1}) = 1$
- $\text{res} \frac{dP}{dt} = 0$

Corollary 4.52. We can construct $\varphi : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{C}$ by

$$\varphi(P, Q) = \text{res} \left(Q \frac{dP}{dt} \right)$$

which has the properties

- $\varphi(P, Q) = -\varphi(Q, P)$
- $\varphi(PQ, R) + \varphi(QR, P) + \varphi(RP, Q) = 0$

Definition 4.53. The *loop algebra* of \mathfrak{g} is

$$\mathcal{L}\mathfrak{g} \equiv \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{L} = \text{Maps}(\mathbb{C}^\times, \mathfrak{g})$$

We give it the Lie bracket

$$[x \otimes P, y \otimes Q]_0 = [x, y] \otimes PQ$$

Definition 4.54. A *central extension* of the loop algebra is $\mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$ with the Lie bracket where $\forall a \in \mathcal{L}\mathfrak{g}$;

$$\begin{aligned} [c, a] &= 0 \\ [a, b] &= [a, b]_0 + \psi(a, b)c \end{aligned}$$

for some antisymmetric bilinear map ψ which makes that Jacobi identity hold.

Restricting back to our example of $\mathfrak{g} = \mathfrak{sl}_2$ we have a bilinear form, so we can set

$$\psi : \mathcal{L}\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{L}\mathfrak{g} \rightarrow \mathbb{C}$$

with

$$\psi(x \otimes P, y \otimes Q) = (x, y) \cdot \varphi(P, Q)$$

This ψ satisfies

- $\forall a, b, \psi(a, b) = -\psi(b, a)$
- $\psi([a, b]_0, c) + \psi([b, c]_0, a) + \psi([c, a]_0, b) = 0$

Now set $\tilde{\mathcal{L}}\mathfrak{g} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$ with the bracket as above to get a central extension of the loop algebra of \mathfrak{sl}_2 .

So in $\tilde{\mathcal{L}}\mathfrak{g}$,

- $Z(\tilde{\mathcal{L}}\mathfrak{g}) = \mathbb{C}c$
- $\alpha_1^\vee = \alpha_1^\vee \otimes 1 \in \tilde{\mathcal{L}}\mathfrak{g}$ satisfies

$$\begin{aligned} [\alpha_1^\vee, y \otimes Q]_0 &= [\alpha_1^\vee, y] \otimes Q = \text{wt}(y)(\alpha_1^\vee)(y \otimes Q) \\ \psi(\alpha_1^\vee, y \otimes Q) &= (\alpha_1^\vee, y) \underbrace{\varphi(1, Q)}_{\text{res}(0)} = 0 \end{aligned}$$

This gives us a well defined subalgebra $\mathbb{C}\alpha_1^\vee \oplus \mathbb{C}c \subset \tilde{\mathcal{L}}\mathfrak{g}$.
Finally we can define

$$\hat{g} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

s.t.

- $[a, b] = [a, b]_0 + \psi(a, b)c$
- $[d, c] = 0$
- $[d, x \otimes P] = x \otimes t \frac{dP}{dt}$

Theorem 4.55. *With the above definition we have*

$$\begin{aligned} \mathfrak{g}\left(\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\right) &\rightarrow \hat{\mathfrak{g}} \\ e_1, \alpha_1^\vee, f_1 &\mapsto e_1, \alpha_1^\vee, f_1 \\ c &\mapsto c \\ d &\mapsto t \frac{d}{dt} \\ e_0 &\mapsto f_1 \otimes t \\ f_0 &\mapsto e_1 \otimes t^{-1} \end{aligned}$$

5 Witt and Virasoro Algebras

5.1 The Witt algebra

Let $A = \mathbb{C}[z, z^{-1}]$ be Laurent polynomials, and then define

$$\text{Der}(A) = \{\phi : A \rightarrow A \mid \phi \text{ } \mathbb{C}\text{-linear, } \phi(fg) = \phi(f)g + f\phi(g)\}$$

Proposition 5.1. *Der(A) is a Lie algebra with bracket*

$$[\phi, \psi](f) = \phi(\psi(f)) - \psi(\phi(f))$$

Proposition 5.2. *The operators $\{L_n = -z^{n+1} \frac{d}{dz} \mid n \in \mathbb{Z}\}$ are a basis for $\text{Der}(A)$*

Proof. They are obviously independent. Write $\phi(z) = -\sum_n a_n z^{n+1}$ (with all but finitely many $a_n \neq 0$). Now the Leibniz rule gives

$$\phi(z^k) = k z^{k-1} \phi(z)$$

and so

$$\begin{aligned} \phi(f)(z) \frac{df}{dz} \phi(z) &= -\sum_n a_n z^{n+1} \frac{d}{dz} f(z) \\ &= \sum_n a_n L_n(f)(z) \end{aligned}$$

□

Definition 5.3. *The **Witt algebra** Witt is the Lie algebra with basis $\{L_n\}$ and bracket as above.*

5.2 Central extension

Definition 5.4. *Let \mathfrak{a} be a Lie algebra. A **central extension** of \mathfrak{a} is a pair $(\tilde{\mathfrak{a}}, \pi)$ s.t.*

- $\tilde{\mathfrak{a}}$ is a Lie algebra
- $\pi : \tilde{\mathfrak{a}} \rightarrow \mathfrak{a}$ is a surjective LA hom
- $\dim_{\mathbb{C}} \ker \pi = 1$
- $\forall x \in \tilde{\mathfrak{a}}, y \in \ker \pi, [x, y] = 0$

Definition 5.5. *Two central extensions $(\tilde{\mathfrak{a}}, \pi), (\tilde{\mathfrak{a}}', \pi')$ are **equivalent** if $\exists \phi : \tilde{\mathfrak{a}} \rightarrow \tilde{\mathfrak{a}}'$ a LA iso s.t. $\pi' \circ \phi = \pi$.*

Example 5.6. *The trivial extension is $\tilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathbb{C}K$ where K is the centre of \mathfrak{a} , and we take the same bracket for $\tilde{\mathfrak{a}}$*

Proposition 5.7. *Up to equivalence, $\exists!$ non trivial central extension of the Witt algebra, the **Virasoro algebra** Vir , written as*

$$0 \rightarrow \underbrace{\mathbb{C}c}_{\ker(\pi_{\text{Vir}})} \rightarrow \text{Vir} \xrightarrow{\pi_{\text{Vir}}} \text{Witt} \rightarrow 0$$

Explicitly we may say that Vir has basis $\{L_n, c\}$ with bracket

$$\begin{aligned} [c, \text{Vir}] &= 0 \\ [L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12} \delta_{m,-n} c \end{aligned}$$

The map to Witt is $L_n \mapsto L_n, c \mapsto 0$

Proof. We check it exists, and this is done simply by observing that the relations given above give a central extension. We can check it is not trivial, as in the trivial extension we would have $[L_2, L_{-2}] = 2[L_1, L_{-1}]$.

For uniqueness, let (\mathfrak{b}, π) be another central extension. Choose a splitting $i : \text{Witt} \rightarrow \mathfrak{b}$ with $\pi \circ i = \text{id}$. We then have $\mathfrak{b} = \mathbb{C}k \oplus i(\text{Witt})$. The bracket is given by

$$\begin{aligned} [i(\text{Witt}), k] &= 0 \\ [i(L_m), i(L_n)] &= (m-n)i(L_{m+n}) + a(m, n)k \end{aligned}$$

for some antisymmetric $a : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$. Define a new splitting i' by

$$i'(L_n) = \begin{cases} i(L_0) & n = 0 \\ i(L_n) - \frac{a(0, n)}{n}k & n \neq 0 \end{cases}$$

Then $[i'(L_0), i'(L_n)] = -ni'(L_n)$. so wlog we may assume $a(0, n) = 0$. Applying the Jacobi identity we get

$$\begin{aligned} 0 &= [[i(L_0), i(L_m)], i(L_n)] + [[i(L_n), i(L_0)], i(L_m)] + [[i(L_m), i(L_n)], i(L_0)] \\ &= (m+n)a(m, n)k \end{aligned}$$

Hence $a(m, n) = a(m)\delta_{m, -n}$ for some odd function $a : \mathbb{Z} \rightarrow \mathbb{C}$. Applying Jacobi again for the triple $i(L_0), i(L_n), i(L_{-n-1})$ gives

$$(n-1)a(n+1) = (n+2)a(n) - (2n+1)a(1)$$

This is a linear recurrence completely determined by $a(1), a(2)$, so the space of solutions is 2-dimensional. It can be found that $a(n) = n, a(n) = n^3$ are both solutions, so the general solution is $a(n) = \alpha n + \beta n^3$ for $\alpha, \beta \in \mathbb{C}$.

If $\beta = 0$, we have a map

$$\begin{aligned} \text{Witt} \oplus \mathbb{C}k &\rightarrow \mathfrak{b} \\ L_n &\mapsto i(L_n) + \frac{1}{2}\alpha\delta_{0n}k \\ k &\mapsto k \end{aligned}$$

which is a LA iso, and so (\mathfrak{b}, π) is trivial.

If $\beta \neq 0$, we have the LA iso

$$\begin{aligned} \text{Vir} &\rightarrow \mathfrak{b} \\ L_n &\mapsto i(L_n) + (\alpha + \beta)\delta_{0n}k \\ c &\mapsto 12\beta k \end{aligned}$$

□

5.3 Heisenberg algebra

Definition 5.8. The *Heisenberg algebra*, *Heis*, has basis $\{\hbar, a_n\}$ and bracket

$$\begin{aligned} [a_m, a_n] &= m\delta_{m, -n}\hbar \\ [\hbar, a_n] &= 0 \end{aligned}$$

Example 5.9 (Natural reps of Heis). *Fixing $\mu, h \in \mathbb{C}$, define*

$$B(\mu, h) = \mathbb{C}[x_1, x_2, \dots]$$

with rep $\rho : \text{Heis} \rightarrow \mathfrak{gl}(B(\mu, h))$ given by

$$\begin{aligned} \rho(\hbar) &= h \\ \rho(a_n) &= \begin{cases} \frac{\partial}{\partial x_n} & n > 0 \\ \mu & n = 0 \\ -hnx_{-n} & n < 0 \end{cases} \end{aligned}$$

*This is called the **Bosonic Fock space** or **oscillator reps**.*

Remark. *The reps $V = B(\mu, h)$ satisfy*

$$\forall v \in V, \exists N \text{ s.t. } \forall n > N, a_nv = 0$$

Let V be any rep satisfying the above condition. For any $n \in \mathbb{Z}$, define

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_{-k} a_{n+k} :$$

where $: \cdot :$ is the normal ordered product, i.e.

$$: a_i a_j := \begin{cases} a_i a_j & i < j \\ a_j a_i & i > j \end{cases}$$

By our requirement on V , we have ensured that $\forall v \in V$, L_nv is well defined as the sum has only finitely many non-zero terms.

We will work towards proving a big theorem now, so we will need some results:

Lemma 5.10. $\forall k, n \in \mathbb{Z}, [a_k, L_n] = ka_{k+n}$

Proof. Define $\psi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\psi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

Choose $\epsilon > 0$ and then set

$$L_n(\epsilon) = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{j+n} : \psi(\epsilon j)$$

Note that $\forall v \in V, \exists \delta > 0$ s.t. $\forall \epsilon < \delta, L_n(\epsilon)v = L_nv$. Now $L_n(\epsilon) = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j} a_{j+n} \psi(\epsilon j)$ acts by a \mathbb{C} -scalar as

$$\begin{aligned} [a_k, L_n(\epsilon)] &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [a_k, a_{-j} a_{j+n}] \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [[a_k, a_{-j}] a_{j+n} \psi(\epsilon j) + a_{-j} [a_k, a_{j+n}] \psi(\epsilon j)] \\ &= \frac{1}{2} [ka_{k+n} \psi(\epsilon k) + ka_{k+n} \psi(\epsilon(-n-k))] \\ &= ka_{k+n} \quad (\text{for } \epsilon \text{ small}) \end{aligned}$$

□

With the above we can state and prove the following:

Theorem 5.11. *Let V be as above and assume $\forall v \in V, \hbar v = v$. (e.g. $V = B(\mu, 1)$). Then*

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12}$$

Proof. Using notation from before calculate

$$\begin{aligned} [L_m(\epsilon), L_n] &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [a_{-j} a_{j+m}, L_n] \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [[a_{-j}, L_n] a_{j+m} + a_{-j} [a_{j+m}, L_n]] \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [(-j)a_{n+j}a_{j+m} + (j+m)a_{-j}a_{j+m+n}] \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [(-j) : a_{n+j}a_{j+m} : + (j+m) : a_{-j}a_{j+m+n} :] \psi(\epsilon j) \\ &\quad - \frac{1}{2} \delta_{m,-n} \sum_{j < -m} (-m-j)j \psi(\epsilon j) + \frac{1}{2} \delta_{m,-n} \sum_{j < 0} (-j)(j+m) \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [(-j) : a_{n+j}a_{j+m} : + (j+m) : a_{-j}a_{j+m+n} :] \psi(\epsilon j) \\ &\quad + \frac{1}{2} \delta_{m,-n} \sum_{j=-1}^{-m} j(j+m) \psi(\epsilon j) \end{aligned}$$

The first sum telescopes (reindexing) giving a finite sum, so we can then take the limit $\epsilon \rightarrow 0$, and we get

$$[L_m, L_n] = \frac{1}{2} \sum_{j \in \mathbb{Z}} (m-n) : a_{-j}a_{j+m+n} : + \frac{1}{2} \delta_{m,-n} \sum_{j=-1}^{-m} j(j+m)$$

and answer follows. □

Definition 5.12. *A Vir rep V has **central charge** $c \in \mathbb{C}$ if*

$$\forall v \in V, c_{\text{Vir}} \cdot v = cv$$

where c_{Vir} is the central charge of Vir.

Remark. *The theorem says that the Heis rep V has central charge 1. If the central charge was 0, these reps would be reps of Witt, and moreover this is a bijection.*

5.4 Connection to affine Lie algebras

Recall that if \mathfrak{g} is a finite-dimensional Lie algebra then $\mathcal{L}\mathfrak{g} = \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with bracket $[xf, yg] = [x, y]fg$, then the affine Lie algebra $\hat{\mathfrak{g}}$ is a natural central extension with SES

$$0 \rightarrow \mathbb{C}k \rightarrow \tilde{\mathcal{L}}\mathfrak{g} \rightarrow \mathcal{L}\mathfrak{g} \rightarrow 0$$

If \mathfrak{g} is simple, then $\hat{\mathfrak{g}} = \tilde{\mathcal{L}}\mathfrak{g} \oplus \mathbb{C}d$ is Kac-Moody.

Example 5.13. If $\mathfrak{g} = \mathfrak{a} = \mathbb{C}\mathfrak{a}$ is a 1-dimensional abelian Lie algebra then

$$\begin{aligned}\mathcal{L}\mathfrak{a} &\rightarrow \text{Heis} \\ at^n &\mapsto a_n \\ k &\mapsto \hbar\end{aligned}$$

is LA hom.

6 Highest weight representations

Definition 6.1. Let V be a rep space of Vir . We say $v \in V$ is **singular** of weight $(h, c) \in \mathbb{C}^2$ if

- $L_0 v = h v$
- $c_{\text{Vir}} v = c v$
- $\forall n > 0, L_n v = 0$

Definition 6.2. Let $v \in V$ be singular. We say it is a **highest weight vector** if

$$V = \text{Span} \{L_{-n_1} \dots L_{-n_k} v \mid k, n_1, \dots, n_k > 0\}$$

Remark. There is a similar definition for Kac-Moody algebras.

Example 6.3. $v = 1 \in B(\mu, 1)$ is a singular vector of weight $(\frac{1}{2}\mu^2, 1)$.

Proposition 6.4. Let V be a highest weight rep of Vir with highest weight (h, c) . Then

- The module V has central charge c , i.e. $\forall w \in V, c_{\text{Vir}} w = c w$
- We have

$$V = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} V_{h+k}$$

where $V_\lambda = \{w \in V \mid L_0 w = \lambda w\}$.

- Each V_{h+k} is finite dimensional
- $\dim V_h = 1$.

Proposition 6.5. Let V be a highest weight module. Then there is a unique maximal proper submodule $V'' \subseteq V$. Hence $V' = V/V''$ is an irreducible highest weight rep with the same highest weight as V .

Proof. Let V'' be the sum of all proper submodules of V . It remains to be shown that $V'' \neq V$. Assume $U \subsetneq V$ is a submodule, and then we know $U \cap V_h = \{0\}$ as the intersection is either 0 (as V_h is 1 dimensional) or contains the highest weight vector, and in the latter case we would have $U = V$. So

$$U = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (U \cap V_{h+k}) \subseteq \bigoplus_{k \in \mathbb{Z}_{> 0}} V_{h+k}$$

and then

$$V'' = \sum U = \bigoplus_{k \in \mathbb{Z}_{>0}} V_{h+k}$$

so $V'' \neq V$. To get the second part, note that $W \subsetneq V'$ has preimage under the quotient which must lie in $V'' \Rightarrow W = 0$. \square

6.1 Verma modules

Proposition 6.6. *Let $(h, c) \in \mathbb{C}^2$. Then \exists a highest weight Vir-module $M(h, c)$ with highest weight (h, c) and highest weight vector v_M s.t.*

- $\forall V$ another rep of highest weight (h, c) with h.w.v $v \in V$ $\exists!$ Vir-module hom

$$\begin{aligned} M(h, c) &\rightarrow V \\ v_M &\mapsto v \end{aligned}$$

- V is isomorphic to a quotient of $M(h, c)$

Proof. As a vector space

$$\text{Vir} = \text{Vir}_+ \oplus \mathfrak{h} \oplus \text{Vir}_-$$

where $\text{Vir}_{\pm} = \text{Span}\{L_n \mid n \gtrless 0\}$ and $\mathfrak{h} = \text{Span}\{L_0, c_{\text{Vir}}\}$. Let $\text{Vir}_{\geq 0} = \text{Vir}_+ \oplus \mathfrak{h}$. Then we have

$$\begin{aligned} \rho : \text{Vir}_{\geq 0} &\rightarrow \mathfrak{gl}_1 = \mathfrak{gl}(\mathbb{C}_{(h,c)}) \\ \forall n > 0, L_n &\mapsto 0 \\ L_0 &\mapsto h \\ c_{\text{Vir}} &\mapsto c \end{aligned}$$

Hence

$$\rho : U(\text{Vir}_{\geq 0}) \rightarrow \mathfrak{gl}_1$$

is an extension to an associative algebra hom from the UEA. We can then construct $M(h, c)$ by

$$\begin{aligned} M(h, c) &= U(\text{Vir}) \otimes_{U(\text{Vir}_{\geq 0})} \mathbb{C}_{(h,c)} \\ &\cong U(\text{Vir}) / U(\text{Vir})(x, -\rho(x), x \in \text{Vir}_+) \end{aligned}$$

Exercise 6.7. *Show that this constructed $M(h, c)$ has the properties required, with h.w.v 1.* \square

Exercise 6.8. *Show that if M, M' are two modules satisfying the above, then $M \cong M'$*

Definition 6.9. $M(h, c)$ is called the **Verma module** of highest weight (h, c)

Corollary 6.10. $\forall (h, c) \in \mathbb{C}^2$, there is a unique irreducible Vir-module $V(h, c)$ of highest weight (h, c)

Proof. Let $V(h, c) = M(h, c)/J(h, c)$ be the unique irreducible quotient of $M(h, c)$. Then by def any other such V irreducible of highest weight is isomorphic to a quotient so $V \cong V(h, c)$ \square

Proposition 6.11. *The Verma module $M(h, c)$ has basis*

$$\{L_{-n_k} \dots L_{-n_1} v_M \mid k \geq 0, 0 < n_1 \leq \dots \leq n_k\}$$

Proof. By the Poincare-Birkhoff-Witt theorem we know

$$\{L_{-n_k} \dots L_{-n_1} c^i h^j L_{m_1} \dots L_{m_l} \mid i, j > 0, k, l \geq 0, 0 < n_1 \leq \dots \leq n_k, 0 < m_1 \leq \dots \leq m_l\}$$

Then as the latter part is a basis for $U(\text{Vir}_{\geq 0})$, it gets cancelled in the quotient. \square

6.2 Unitary reps

Recall we knew that 1 is a singular vector of weight $(\frac{1}{2}\mu^2, 1)$ for the rep $B(\mu, 1)$. If we define

$$B'(\mu, 1) = \text{Span} \{L_{-n_k} \dots L_{-n_1} 1 \mid k \geq 0, n_i > 0\} \subseteq B(\mu, 1)$$

then $B'(\mu, 1)$ is now a highest weight rep with highest weight $(\frac{1}{2}\mu^2, 1)$.

Definition 6.12. *Let \mathfrak{a} be a complex Lie algebra. An **anti-involution** on \mathfrak{a} is a function $\omega : \mathfrak{a} \rightarrow \mathfrak{a}$ s.t.*

- $\omega^2 = \text{id}$
- $\omega(ax + by) = \bar{a}x + \bar{b}y$
- $\omega([x, y]) = -[\omega(x), \omega(y)]$

Definition 6.13. *If V is an \mathfrak{a} -rep, then a Hermitian form on V is **contravariant** if*

$$\forall u, v \in V, x \in \mathfrak{a}, \langle x \cdot u, v \rangle = \langle u, \omega(x) \cdot v \rangle$$

Definition 6.14. *A rep V is **unitary** if it admits a contravariant inner product.*

Example 6.15. *Anti-involutions on Heis and Vir are given by*

1. $\omega_{\text{Heis}}(a_n) = a_{-n}, \omega_{\text{Heis}}(\hbar) = \hbar$
2. $\omega_{\text{Vir}}(L_n) = L_{-n}, \omega_{\text{Vir}}(c) = c$

Note

$$\omega_{\text{Heis}}(L_n) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \omega_{\text{Heis}}(: a_{-j} a_{j+n} :) = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j-n} a_j := L_{-n} = \omega_{\text{Vir}}(L_n)$$

Proposition 6.16. *Assume $\mu \in \mathbb{R}$, then the Heis rep $B(\mu, 1)$ has a unique contravariant inner product s.t. $\langle 1, 1 \rangle = 1$. Explicitly*

$$\langle P, Q \rangle = \langle \omega(P)Q \rangle$$

where $\langle \cdot \rangle = \text{take constant term and}$

$$\omega : \mathbb{C}[x_1, \dots,] \rightarrow \text{Heis}$$

is the complex anti-linear ring hom given by $\omega(x_n) = \frac{1}{n} a_n$

Proof.

Exercise 6.17. *Do this*

□

Corollary 6.18. $B(\mu, 1)$ is a unitary Vir rep.

Lemma 6.19. *Let V be a unitary Vir-module such that $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ is a direct sum of L_0 -eigenspaces and $\dim V_\lambda < \infty$.*

If $U \subseteq V$ is a submodule, then $\exists U^\perp \subseteq V$ another submodule s.t. $V = U \oplus U^\perp$.

Proof. Let

$$U^\perp = \{v \in V \mid \forall u \in U, \langle u, v \rangle = 0\}$$

It is simple to check $U^\perp \subseteq V$ is a submodule, and $U \cap U^\perp = 0$. To show $V = U + U^\perp$, note we can decompose $v \in V$ into eigenvectors of L_0 , so it is sufficient to show $V_\lambda \subseteq U + U^\perp$. But

$$V_\lambda = (V_\lambda \cap U) \oplus (V_\lambda \cap U^\perp)$$

as $\dim V_\lambda < \infty$

□

Lemma 6.20. *Let V be a unitary highest weight rep. Then V is irreducible.*

Proof. Let $V'' \subseteq V$ be the unique maximal proper submodule. Then $(V'')^\perp \subseteq V$ is a submodule and $V'' \cap (V'')^\perp = 0$. Then either

1. $(V'')^\perp = V \Rightarrow$ done
2. $(V'')^\perp = 0 \Rightarrow V = V''$ contradiction.

□

Proposition 6.21. *Assume $\mu \in \mathbb{R}$. Then the highest weight module $B'(\mu, 1)$ is irreducible.*

Proof. Use that B' is unitary. then done by lemma.

□

Proposition 6.22. *Assume $h, c \in \mathbb{R}$. Then*

1. *If $M(h, c)$ is unitary, then $h, c > 0$*
2. *If $h \geq 0, c \geq 1$, then the irreducible representation $V(h, c) = M(h, c)/J(h, c)$ is unitary*
3. *If $h > 0, c > 1$ then $M(h, c) = V(h, c)$.*

Proposition 6.23. *Assume $h, c \in \mathbb{R}$. Let $v \in M(h, c)$ be the highest weight vector. Then*

1. $\exists!$ *contravariant Hermitian form on M s.t. $\langle v, v \rangle = 1$*
2. *The eigenspaces of L_0 are pairwise orthogonal*
3. $\ker \langle \cdot, \cdot \rangle = J(h, c)$ *is a maximal proper submodule*

Hence $V(h, c)$ carries a non-degenerate Hermitian form s.t. $\langle v, v \rangle = 1$

Proof. See notes

□

6.3 Kac Determinant formula

Recall we have a basis for $M(h, c)$. Kac found a formula for the determinant of $\langle \cdot, \cdot \rangle|_{M(h, c)_{h+n}}$

7 Lie algebra of infinite matrices

Definition 7.1. *Define*

$$\mathfrak{gl}_\infty = \{(a_{ij}) \mid a_{ij} \in \mathbb{C}, \text{ almost all entries } 0\}$$

It has basis $\{E_{ij}\}$, the natural extension of that for finite \mathfrak{gl}_n .

Proposition 7.2. \mathfrak{gl}_∞ is a Lie algebra with bracket given by matrix commutation.

Recall the definition of a grading:

Definition 7.3. A **graded Lie algebra** is a Lie algebra \mathfrak{g} with decomposition $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$ s.t.

$$[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l}$$

We write $\forall X \in \mathfrak{g}_k, \deg X = k$

Proposition 7.4. We can make \mathfrak{gl}_∞ into a graded Lie algebra with grading

$$(\mathfrak{gl}_\infty)_k = \text{Span}\{E_{ij} \mid i - j = k\}$$

Definition 7.5. We define the associated group to be

$$GL_\infty = \{(A_{ij}) \mid A_{ij} \in \mathbb{C}, \text{ invertible, almost all } A_{ij} = \delta_{ij} \ 0\}$$

with operation given by matrix multiplication

Proposition 7.6. GL_∞ is a Lie group with Lie algebra \mathfrak{gl}_∞

It turns out we need a bigger Lie algebra

Definition 7.7. Let

$$\mathfrak{gl}_\infty^\Delta = \{(a_{ij}) \mid \forall |i - j| \gg 0, a_{ij} = 0\}$$

Proposition 7.8. $\mathfrak{gl}_\infty \subset \mathfrak{gl}_\infty^\Delta$

7.1 Central extension

Definition 7.9. Consider the central extension $\hat{\mathfrak{gl}}_\infty^\Delta$ defined by

$$0 \rightarrow \mathbb{C}c \rightarrow \hat{\mathfrak{gl}}_\infty^\Delta \rightarrow \mathfrak{gl}_\infty^\Delta \rightarrow 0$$

with bracket given by, for $a, b \in \mathfrak{gl}_\infty^\Delta$,

$$[a, b] = ab - ba + \gamma(a, b)c$$

γ is called the **cocycle** and satisfies

$$\gamma(E_{ij}, E_{ji}) = 1 = -\gamma(E_{j-i}, E_{ij}) \quad \text{for } i \leq 0, j \geq 1$$

and is 0 otherwise.

We have representations of Heis and Vir inside $\hat{\mathfrak{gl}}_\infty^\Delta$ given by

$$V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j$$

There is then a natural action on V by multiplication.

7.2 Shift operators

Definition 7.10. Define the *shift operator* $\Delta_k : V \rightarrow V$, $v_j \mapsto v_{j-k}$. We can write explicitly

$$\Lambda_k = \sum_{i \in \mathbb{Z}} E_{i, i+k}$$

Proposition 7.11. $[\Delta_k, \Delta_j] = 0$

Let $\eta = \bigoplus_k \mathbb{C} \Lambda_k$ be the subalgebra of $\mathfrak{gl}_\infty^\Delta$. We can then let $\hat{\eta}$ be the central extension given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \hat{\mathfrak{gl}}_\infty^\Delta & \longrightarrow & \mathfrak{gl}_\infty^\Delta \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \hat{\eta} & \longrightarrow & \eta \longrightarrow 0 \end{array}$$

Proposition 7.12. We have

- $\gamma(\Lambda_n, \Lambda_k) = n\delta_{n, -k}$
- $\hat{\eta} = \text{Heis}$

Proof. The first point is a calculation, Secondly, there is an explicit isomorphism, and the relations are the same in each. \square

Proposition 7.13. For the Witt algebra we can say

- \exists a family of embeddings depending on $\alpha, \beta \in \mathbb{C}$ given by

$$\begin{aligned} i_{\alpha, \beta} : \text{Witt} &\hookrightarrow \hat{\mathfrak{gl}}_\infty^\Delta \\ L_n &\mapsto \sum_{k \in \mathbb{Z}} [k - \alpha - \beta(n+1)] E_{k+n, k} \end{aligned}$$

- Let $\hat{\text{Witt}} \subset \hat{\mathfrak{gl}}_\infty^\Delta$ be the central extension. Then

$$\gamma(L_i, L_j) = \delta_{i, -j} \left(\frac{i^3 - i}{12} c_\beta + 2ih_0 \right)$$

where $c_\beta = -12\beta^2 + 12\beta - 2$ and $h_0 = \frac{1}{2}\alpha(\alpha + 2\beta - 1)$

- Let $\hat{L}_n = L_n + \delta_{n,0} h_0 c$. Then

$$[\hat{L}_n, \hat{L}_m] = (n - m) \hat{L}_{n+m} + \delta_{n, -m} \left(\frac{n^3 - n}{12} \right) c_\beta c$$

8 Fermionic Fock space

We can then consider $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$, let $I = \langle x \otimes x \rangle$, and then get

$$\Lambda(V) = T(V)/I$$

This comes equipped with a projection map $p : T(V) \rightarrow \Lambda(V)$. Letting $p(T_k(V)) = \Lambda^k(V)$, we get the decomposition

$$\Lambda(V) = \bigoplus_{k \geq 0} \Lambda^k(V)$$

There are also linear maps

$$\begin{aligned} \phi_{s,k}^{(m)} : \Lambda^k(V) &\rightarrow \Lambda^s(V) \\ u &\mapsto u \wedge (v_{-k+m} \wedge \cdots \wedge v_{-s+k+m}) \end{aligned}$$

for fixed $m \in \mathbb{Z}$, $k \leq s$. These maps obey

$$\begin{aligned} \phi_{r,s}^{(m)} \circ \phi_{s,k}^{(m)} &= \phi_{r,k}^{(m)} \\ \phi_{k,k}^{(m)} &= \text{id} \end{aligned}$$

Hence $(\Lambda^k(V), \phi_{r,k}^{(m)})$ for a **direct system** for each $m \in \mathbb{Z}$. I can then take the direct limit to get

Definition 8.1. *The **Fermionic Fock space of charge m** is*

$$F^{(m)} = \Lambda_{(m)}^\infty(V) = \lim_{\rightarrow} \Lambda^k(V)$$

The construction works as

$$\lim_{\rightarrow} \Lambda^k(V) = \bigsqcup_k \Lambda^k(V) / \sim$$

where the equivalence relation is given by

$$x_i \in \Lambda^i(V) \sim x_j \in \Lambda^j(V) \Leftrightarrow \exists h, i, j \leq h, \phi_{h,i}^{(m)}(x_i) = \phi_{h,j}^{(m)}(x_j)$$

We have a basis given by

$$\psi = v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$$

called the **semi-infinite monomials**, requiring the conditions

- $i_0 > i_{-1} > \cdots$
- $i_k = k + m$ for $k \ll 0$.

This generalises naturally to get

$$\psi_m = v_m \wedge v_{m-1} \wedge \dots$$

which is the **vacuum vector of charge m**.

The FFS comes with a grading given by

$$\deg \psi = \sum_{s \geq 0} i_{-s} + s - m$$

which is forced to be finite by our condition on i_k for $k \ll 0$. We then have

$$F_k^{(m)} = \text{Span} \{ \psi \mid \deg \psi = k \}$$

Now let $\lambda = (\lambda_0, \dots, \lambda_{n-1}) + k$ be a partition if h , that is

- $\lambda_0 + \dots + \lambda_{n-1} + k = h$
- $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1}$

We can then get semi-infinite monomials ψ_λ from partitions by saying $j_{-i} = \lambda_i - i + m$ for $i = 0, \dots, n-1$ and then $j_{-n-i} = -n + m - i$

Example 8.2. Take $\lambda = (5, 3, 3, 1) + 12$ is a partition of 24. Take $m = 0$. We then find

$$\begin{aligned} j_0 &= 5j_{-1} & &= 2 \\ j_{-2} &= 1 \\ j_{-3} &= -2 \end{aligned}$$

and so we get

$$\psi_\lambda = (v_5 \wedge v_2 \wedge v_1 \wedge v_{-2}) \wedge v_{-4} \wedge v_{-5} \wedge \dots$$

Proposition 8.3. We have

- $F^{(m)} = \bigoplus_{k \geq 0} F_k^{(m)}$, $F_0^{(m)} = \mathbb{C}\psi_m$
- $\dim F_k^{(m)} = p(k) = \text{number of partitions of } k$
- $\dim_q F^{(m)} \equiv \sum_{k \geq 0} (\dim F_k^{(m)}) q^k = \prod_{l \geq 1} (1 - q^l)^{-1}$

Proof. • Clear

- $\{\psi_\lambda \mid \lambda \text{ partition of } h\}$ is a basis of $F_h^{(m)}$
- The first part is the definition. Then we have

$$\dim_q F^{(m)} = \sum_{k \geq 0} p(k) q^k$$

□

9 Representations of GL_∞ , \mathfrak{gl}_∞ , on $F^{(m)}$

9.1 Actions on tensor products

Let \mathfrak{g} be a Lie algebra and M, N \mathfrak{g} -reps. Then $M \otimes_{\mathbb{C}} N$ is a representation space of \mathfrak{g} given by

$$x \in \mathfrak{g}, m, n \in M, N, x \cdot (m \otimes n) = (x \cdot m) \otimes n + m \otimes (x \cdot n)$$

If G is a group, then we get a rep on $M \otimes_{\mathbb{C}} N$ by

$$g \cdot (m \otimes n) = (gm) \otimes (gn)$$

Remark.