Monopoles

Linden Disney-Hogg

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1 Introduction

1.1 Preamble

I already have notes on Gauge Theory, Algebraic Geometry, Solitons, and Algebraic Topology, but I have yet to actually make any notes on Monopoles. The purpose of these notes is to be a comprehensive cover of the knowledge required to understand [2]. This will include previous works by Atiyah, Donaldson, Hitchin, Nahm, and more.

1.2 Preliminaries

As with all my projects, the preliminaries will undoubtably end up being too long, but I will try keep this minimal this time:

Definition 1.1. *The* ***annihilator*** *of $U \leq V$ is*

$$U^0 = \{f \in V^* \,|\, \forall u \in U,\ f(u) = 0\} \leq V^*$$

If V has bilinear $\langle \cdot, \cdot \rangle$ we can use the isomorphism of $V^* \cong V$ to understand

$$U^0 = \{v \in V \,|\, \forall u \in U,\ \langle u, v \rangle = 0\} \leq V$$

Lemma 1.2. *THe annihilator is a subspace,* $\dim U^0 = \dim V - \dim U$.

Definition 1.3. *A subspace U is called* ***isotropic*** *if $U \subset U^0$.*

2 The Monopole Equations

Definition 2.1. *Take a principal G-bundle $P \to M$, ω_{vol} an orientation on M, and $\langle \cdot, \cdot \rangle$ to be an* ad*-invariant inner product on $\mathfrak{g}$. Then the* ***Yang-Mills-Higgs actions*** *on M is*

$$S_{YMH}[A, \phi] = \int_M \left[|F|^2 + |D\phi|^2 + V(\phi) \right] \omega_{vol}$$

where $F = dA + A \wedge A$ is the curvature associated to a section $A \in \Gamma(T^*M \otimes \mathrm{ad}(P))$, $D = d + A$ is the associated covariant derivative, and $\phi \in \Gamma(\mathrm{ad}(P))$.

Remark. *A common choice of potential function V is $V(\phi) = \lambda \left(1 - |\phi|^2\right)^2$, the* ***$\phi^4$-potential****.*

Proposition 2.2. *The variational equations corresponding to S_{YMH} are the* ***Yang-Mills-Higgs equations***

$$\begin{aligned} DF &= 0 \quad \textit{(Bianchi)} \\ D \star F &= -[\phi, D\phi] \\ D \star D\phi &= -V'(\phi) \end{aligned}$$

3 The ADHM construction

This section follows the work first laid out in [1]. Suppose we have the following information:

- W a k-dimensional vector space
- V a $2k+2$-dimensional vector space with skew, non-degenerate bilinear form $(\cdot, \cdot) : \wedge^2 V \to \mathbb{C}$.
- $z = (z_i) \in \mathbb{C}^4$
- $A(z) = \sum_i A_i z_i \in \mathrm{End}(W, V)$ s.t.

$$\forall z \neq 0,\ U_z \equiv A(z)W \subset V \text{ is isotropic and } k\text{-dimensional}$$

We now state some important properties:

Lemma 3.1. *Let $E_z = {U_z^0}/{U_z}$, then*

- $\dim E_z = 2$
- *E_z inherits a non-degenerate skew bilinear*
- *$\forall \lambda \in \mathbb{C}^\times$, $E_z = E_{\lambda z}$.*

Proof. We go point by point:

- $\dim E_z = \dim U_z^0 - \dim U_z = (\dim V - \dim U_z) - \dim U_z = 2k + 2 - 2k = 2$.
- The bilinear on W is only degenerate in U_z^0 on U_z, so by quotienting by this it descends directly to E_z.
- $A(\lambda z) = \lambda A(z)$, so $A(\lambda z)(\lambda^{-1}\boldsymbol{w}) = A(z)(\boldsymbol{w})$. Hence we can see $U_{\lambda z} = U_z$ and so result.

$\square$

Corollary 3.2. *We get a vector bundle $E \to \mathbb{CP}^3$ with group $SL(2, \mathbb{C})$.*

References

[1] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, Yu. I. Manin. Construction of instantons. *Physics Letters A*, 65(3):pp. 185–187, 1978. ISSN 0375-9601. doi:10.1016/0375-9601(78)90141-X.

[2] H. W. Braden, V. Z. Enolski. The construction of monopoles. *Communications in Mathematical Physics*, 362(2):p. 547–570, 2018. ISSN 1432-0916. doi:10.1007/s00220-018-3199-4.