Monopoles

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1 Introduction

1.1 Preamble

I already have notes on Gauge Theory, Algebraic Geometry, Solitons, and Algebraic Topology, but I have yet to actually make any notes on Monopoles. The purpose of these notes is to be a comprehensive cover of the knowledge required to understand [2]. This will include previous works by Atiyah, Donaldson, Hitchin, Nahm, and more.

1.2 Preliminaries

As with all my projects, the preliminaries will undoubtably end up being too long, but I will try keep this minimal this time:

Definition 1.1. *The* ***annihilator*** *of* $U \leq V$ *is*

$$U^0 = \{f \in V^* \,|\, \forall u \in U,\ f(u) = 0\} \leq V^*$$

If V has bilinear $\langle \cdot, \cdot \rangle$ we can use the isomorphism of $V^* \cong V$ to understand

$$U^0 = \{v \in V \,|\, \forall u \in U,\, \langle u, v\rangle = 0\} \leq V$$

Lemma 1.2. *THe annihilator is a subspace,* $\dim U^0 = \dim V - \dim U$.

Definition 1.3. *A subspace U is called* ***isotropic*** *if $U \subset U^0$.*

1.3 The Dirac Monopole

The standard maxwell equations prohibit monopoles, by which we mean point magnetic field sources, as $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{0}$. Dirac showed in [3] that it is possible to escape this conclusion by giving non-trivial topology to the space by allowing $\boldsymbol{B} = \frac{g}{4\pi r^2}\hat{\boldsymbol{x}}$ to have a singularity at $\boldsymbol{x} = \boldsymbol{0}$. We can calculate $\boldsymbol{\nabla} \cdot \boldsymbol{B} = g\delta(\boldsymbol{x})$. Removing this circle gives $\mathbb{R}^3 \setminus 0$, homotopic to S^2, and the corresponding magnetic two form on this sphere is

$$f = \frac{g}{4\pi} \sin\theta d\theta \wedge d\phi$$

and so the flux through a 2-sphere enclosing the origin is $\int_{S^2_R} f = g$. For $g \neq 0$, we know $f \neq da$ for a global $a \in \Omega^1(S^2)$ by Stokes' theorem, but if we take a cover of the sphere $U_{N/S}$ (north/south) and define gauge potentials

$$a_N = \frac{g}{4\pi}(1 - \cos\theta) d\phi \in \Omega^1(U_N)$$
$$a_S = \frac{g}{4\pi}(-1 - \cos\theta) d\phi \in \Omega^1(U_S)$$

On the intersect $U_N \cap U_S$ we have $da_N = f = da_S$ and $a_N = a_S + \frac{g}{2\pi} d\phi$.
Now taking $A = ia$, $F = if$, we have that $g_{NS}(\theta, \phi) = e^{-i\frac{g\phi}{2\pi}}$. Requiring that this is a well-defined transition function gives $g \in \mathbb{Z}$. This is equivalent to the integrality of the Chern number.
We will not want to consider this as this solution is not solitonic (it has infinite mass), but for further discussion see [4].

1.4 Pauli Matrices

Definition 1.4 (Pauli Matrices)**.** *The* ***Pauli matrices*** *are*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note they are all Hermitian and traceless.

Fact 1.5. $\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k$

1.5 $SU(2)$

We can write

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

This can be expressed as, for $A \in SU(2)$

$$A = a_0 I + i\boldsymbol{a} \cdot \boldsymbol{\sigma}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, and $a_0^2 + |\boldsymbol{a}|^2 = 1$. Hence $SU(2) \cong S^3$. In addition, by parametrising $SU(2)$ by the a_i, it can be seen that $\{ i\sigma_i \}$ forms a basis of $\mathfrak{su}(2)$. It is typical to normalise this basis to $\{ T^a = -\frac{1}{2} i\sigma_a \}$.

Lemma 1.6. *The structure constants in this basis $\{ T^a \}$ are $f_c^{ab} = \epsilon_{abc}$.*

Corollary 1.7. *The Killing form is given by $\kappa\left(T^a, T^b\right) = \kappa^{ab} = -2\delta^{ab}$. Hence $\kappa(X, Y) = 4\,\mathrm{Tr}(XY)$*

1.6 Degree of a Map

Consider a smooth map $n : S_s^2 \to S_t^2$. The number of preimages of $p \in S_t^2$ computed with sign is calculate via pullback to be

$$\deg(n) = \frac{1}{\int_{S_t^2} \omega} \int_{S_s^2} n^* \omega$$

for some normalisable volume form on S_t. Then

2 The Monopole Equations

2.1 Yang-Mills-Higgs equations

Definition 2.1. *Take a principal G-bundle $P \to M$, ω_{vol} an orientation on M, and $\langle \cdot, \cdot \rangle$ to be an* ad*-invariant inner product on $\mathfrak{g}$. Then the **Yang-Mills-Higgs actions** on M is*

$$S_{YMH}[A, \phi] = \int_M \left[|F|^2 + |D\phi|^2 + V(\phi) \right] \omega_{vol}$$

where $F = dA + A \wedge A$ is the curvature associated to a section $A \in \Gamma(T^*M \otimes \mathrm{ad}(P))$, $D = d + A$ is the associated covariant derivative, and $\phi \in \Gamma(\mathrm{ad}(P))$.

Remark. *A common choice of potential function V is $V(\phi) = \lambda \left(1 - |\phi|^2\right)^2$, the ϕ^4**-potential**.*

Proposition 2.2. *The variational equations corresponding to S_{YMH} are the**Yang-Mills-Higgs equations***

$$\begin{aligned} DF &= 0 \quad \textit{(Bianchi)} \\ \star D \star F &= -\left[\phi, D\phi\right] \\ \star D \star D\phi &= -V'(\phi) \end{aligned}$$

Proof. We first consider the equation that comes from the variation of A. Let $A_t = A + t\beta$, then $F_t = F + t\left(d\beta + \beta \wedge A + A \wedge \beta\right) + \mathcal{O}(t^2)$ and $D_t\phi = D\phi + t\left[\beta, \phi\right]$.

$$S_t = S + 2t \int_M \left[\langle F, \rangle\right]$$

□

Definition 2.3. *A **monopoles** will be a soliton-like solution to the Yang-Mills-Higgs equations when $G = SU(2)$ and, the principal bundle is $P = \mathbb{R}^4 \times SU(2)$, and the potential is ϕ^4.*

Remark. *The Dirac monopole is a solution in the case of $G = U(1)$.*

We may make these equations explicit in coordinates. We take the inner product on $\mathfrak{g} = \mathfrak{su}(2)$ to be $\langle X, Y \rangle = -\frac{1}{2}\kappa(X, Y) = -2\operatorname{Tr}(XY)$. As such taking coordinates x^μ on $\mathbb{R}^r$ and writing $F_{\mu\nu} dx^\mu \wedge dx^\nu$ we have

$$
\begin{aligned}
\langle F, F \rangle &= \langle F_{\mu\nu}, F_{\rho\sigma} \rangle \langle dx^\mu \wedge dx^\nu, dx^\rho \wedge dx^\sigma \rangle \\
&= \langle F_{\mu\nu}, F_{\rho\sigma} \rangle (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\
&= 2 \langle F_{\mu\nu}, F^{\mu\nu} \rangle \\
&= -4 \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) \\
\langle D\phi, D\phi \rangle &= \langle D_\mu\phi, D_\nu\phi \rangle \langle dx^\mu, dx^\nu \rangle \\
&= \langle D_\mu\phi, D_\nu\phi \rangle \eta^{\mu\nu} \\
&= -2 \operatorname{Tr}(D_\mu\phi D^\mu\phi)
\end{aligned}
$$

Hence the corresponding Lagrangian density is

$$
\mathcal{L} = -4 \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})
$$

continue

2.2 BPS limit

The idea is now to set $\lambda = 0$ but retain that $|\phi| = 1$ at infinity. More specifically we take the conditions

$$
\begin{aligned}
|\phi| &= 1 - \frac{m}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right) \\
\frac{\partial |\phi|}{\partial \Omega} &= \mathcal{O}\left(\frac{1}{r^2}\right) \\
|D\phi| &= \mathcal{O}\left(\frac{1}{r^2}\right)
\end{aligned}
$$

With $\lambda = 0$ we can rewrite the energy of the configuration as

$$
\begin{aligned}
E[A, \phi] &= \int_M \langle F \mp \star D\phi, F \mp \star D\phi \rangle \pm 2 \langle F, D\phi \rangle \\
&= \int_M |F - \star D\phi|^2 + \int_{\partial M} \langle F, \phi \rangle
\end{aligned}
$$

using that $\langle F, D\phi \rangle = d \langle F, \phi \rangle$. Then as

$$
\int_{\partial M} \langle F, \phi \rangle = -\int_{\partial M} \langle \phi, d\phi \wedge d\phi \rangle
$$

by our discussion of degree we have $E \geq \pm 4\pi k$ for some $K \in \mathbb{Z}$ with equality iff $F = \mp \star D\phi$ where we choose the sign to make the bound positive. This is the **BPS equation**.

3 Constructions

4 The ADHM construction

This section follows the work first laid out in [1]. Suppose we have the following information:

- W a k-dimensional vector space
- V a $2k+2$-dimensional vector space with skew, non-degenerate bilinear form $(\cdot,\cdot) : \wedge^2 V \to \mathbb{C}$.
- $z = (z_i) \in \mathbb{C}^4$
- $A(z) = \sum_i A_i z_i \in \mathrm{End}(W, V)$ s.t.

$$\forall z \neq 0,\ U_z \equiv A(z)W \subset V \text{ is isotropic and } k\text{-dimensional}$$

We now state some important properties:

Lemma 4.1. *Let* $E_z = {}^{U_z^0}\!/_{U_z}$*, then*

- $\dim E_z = 2$
- E_z *inherits a non-degenerate skew bilinear*
- $\forall \lambda \in \mathbb{C}^\times,\ E_z = E_{\lambda z}$.

Proof. We go point by point:

- $\dim E_z = \dim U_z^0 - \dim U_z = (\dim V - \dim U_z) - \dim U_z = 2k + 2 - 2k = 2$.
- The bilinear on W is only degenerate in U_z^0 on U_z, so by quotienting by this it descends directly to E_z.
- $A(\lambda z) = \lambda A(z)$, so $A(\lambda z)(\lambda^{-1}\boldsymbol{w}) = A(z)(\boldsymbol{w})$. Hence we can see $U_{\lambda z} = U_z$ and so result.

$\square$

Corollary 4.2. *We get a vector bundle* $E \to \mathbb{CP}^3$ *with group* $SL(2, \mathbb{C})$.

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