

# Ehresmann, Kozul, and Cartan connections

Linden Disney-Hogg

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## 1 Introduction

These are typset notes based on a small graduate lecture course given by Professor José Figueroa-O'Farrill at the University of Edinburgh in Autumn 2019.

## 2 Fibre bundles

**Definition 2.1** (Fibre Bundle). A **fibre bundle** consists of a smooth surjection  $\pi : E \rightarrow M$  between manifolds  $E$  (the **total space**) and  $M$  (the **base space**) and such that  $\forall a \in M$  there exists a neighbourhood  $U \ni a$  and a diffeomorphism  $\varphi : \pi^{-1}(U) \rightarrow U \times F$  (a **local trivialisation**) for some manifold  $F$  (the **typical fibre**) such that the following triangle commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \downarrow \pi & \swarrow pr_2 & \\ U & & \end{array}$$

We often write  $F \rightarrow E \xrightarrow{\pi} M$ . If we can take  $U = M$  we say that  $E$  is a **trivial bundle**. Now suppose that  $(U, \varphi), (V, \psi)$  are local trivialisations with  $U \cap V \neq \emptyset$ . Then we have two ways to view  $\pi^{-1}(U \cap V)$  as a product.

$$\begin{array}{ccccc} (U \cap V) \times F & \xleftarrow{\psi} & \pi^{-1}(U \cap V) & \xrightarrow{\varphi} & (U \cap V) \times F \\ & \searrow pr_2 & \downarrow \pi & \swarrow pr_2 & \\ & & U \cap V & & \end{array}$$

and hence

$$\begin{aligned}\psi \circ \varphi^{-1} : (U \cap V) \times F &\rightarrow (U \cap V) \times F \\ (a, p) &\mapsto (a, \Phi(a, p))\end{aligned}$$

where  $\Phi(a, \cdot) : F \rightarrow F$  is a diffeomorphism, and hence it defines a **transition function**  $g : U \cap V \rightarrow \text{Diff}(F)$ .

**Definition 2.2.** Let  $F \rightarrow E \xrightarrow{\pi} M$ . A collection  $\{(U_\alpha, \varphi_\alpha)\}$  of local trivialisations where  $M = \cup_\alpha U_\alpha$  is called a **trivialising atlas** for  $E \xrightarrow{\pi} M$ .

Let us introduce the notation  $U_{\alpha\beta} = U_\alpha \cap U_\beta$ , etc. and  $g_{\alpha\beta}$  the transition function defined by  $\varphi_\alpha \circ \varphi_\beta^{-1}$ .

**Fact 2.3.** The transition functions satisfy the **cocycle conditions**

- $\forall a \in U_\alpha, g_{\alpha\alpha}(a) = \text{id}_F$
- $\forall a \in U_{\alpha\beta}, g_{\alpha\beta}(a)g_{\beta\alpha}(a) = \text{id}_F$
- $\forall a \in U_{\alpha\beta\gamma}, g_{\alpha\beta}(a)g_{\beta\gamma}(a) = g_{\alpha\gamma}(a)$ .

**Definition 2.4.** Let  $E \xrightarrow{\pi} M, E' \xrightarrow{\pi'} N$  be fibre bundles. A bundle map is a pair  $(\Phi, \phi)$  of smooth maps  $\Phi : E \rightarrow E', \phi : M \rightarrow N$  such that the following commutes

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{\phi} & N \end{array}$$

Since  $\pi$  is surjective,  $\phi$  is uniquely determined by  $\Phi$ , which is said to **cover**  $\phi$ . Notice that  $\Phi$  is **fibre preserving**.

**Definition 2.5.** Let  $f : M \rightarrow N$  be smooth and  $E \xrightarrow{\pi_E} N$  a fibre bundle. Then we can define the **pullback bundle**  $f^*E \rightarrow M$  as the categorical pullback, i.e.

$$f^*E \equiv \{(a, e) \in M \times E \mid \pi_E(e) = f(a)\}$$

Restricting the canonical projections from  $M \times E$  we get maps  $\pi : f^*E \rightarrow M, \Phi : f^*E \rightarrow E$  making the following commute

$$\begin{array}{ccc} f^*E & \xrightarrow{\Phi} & E \\ \pi \downarrow & & \downarrow \pi_E \\ M & \xrightarrow{f} & N \end{array}$$

Taking  $a \in M$ , and  $(V, \psi)$  a local trivialisation for  $E \rightarrow N$  with  $f(a) \in V$ , then  $(f^{-1}(V), \varphi)$  with  $\varphi : \pi^{-1}(f^{-1}(V)) \rightarrow f^{-1}(V) \times F$  defined by  $\varphi(b, e) = (b, \text{pr}_2(\psi(e)))$  is a local trivialisation for  $f^*E \rightarrow M$ . This shows that  $f^*E \rightarrow M$  is a fibre bundle, and it has fibres  $(f^*E)_a = E_{f(a)}$ .

**Definition 2.6.** A **section** of a fibre bundle  $F \rightarrow E \xrightarrow{\pi} M$  is a smooth map  $s : M \rightarrow E$  such that  $\pi \circ s = \text{id}_M$ .

Sections *may not exist*, but if the fibre bundle is trivial, then any smooth map  $\sigma : M \rightarrow F$  defines a sections by  $s(a) = (a, \sigma(a))$ . Since fibres are locally trivial, they admit local sections  $s_\alpha : U_\alpha \rightarrow \pi^{-1}(U_\alpha)$  via local smooth maps  $\sigma_\alpha : U_\alpha \rightarrow F$ . A section  $s : N \rightarrow E$  can be pulled back via  $f : M \rightarrow N$  to give a section  $f^*s : M \rightarrow f^*E$  via  $(f^*s)(a) = (a, s(f(a)))$ .

**Definition 2.7.** Consider  $F \rightarrow E \xrightarrow{\pi} M$ . Then the fibres  $E_a = \pi^{-1}(a) \subset E$  are submanifolds of  $E$ . The tangent space at  $e \in E_a$  is  $\mathcal{V}_e = \ker((\pi_*)_e : T_e E \rightarrow T_e M)$  and is called the **vertical subspace** of  $T_e E$

In the absence of any additional structure, there is no preferred complementary subspace of  $T_e E$ .

**Definition 2.8.** A **connection** on  $E \rightarrow M$  is a smooth choice of complementary subspace  $\mathcal{H}_e \subset T_e E$  i.e.  $T_e E = \mathcal{V}_e \oplus \mathcal{H}_e$ . That is, a connection is a distribution  $\mathcal{H} \subset TE$

Note  $(\pi_*)_e|_{\mathcal{H}_e} : \mathcal{H}_e \xrightarrow{\cong} T_{\pi(e)}M$ , so  $\mathcal{H}$  gives a choice of how to lift tangent vectors, and so curves, from  $M$  to  $E$ .

Given a distribution one can ask whether it is integrable (in the sense of Frobenius), i.e. is  $E$  foliated by submanifolds whose tangent spaces are  $\mathcal{H}$ . We shall see that the obstruction to the integrability of  $\mathcal{H}$  can be interpreted as the 'curvature' of the connection.

### 3 Principal fibre bundles

We now specialise to principal fibre bundles, so called because the typical fibre is a principally homogeneous space for a lie group.

**Definition 3.1.** A **Lie group** consists of a manifold  $G$  which is also a group such that group multiplication  $G \times G \rightarrow G$ ,  $(g, h) \mapsto gh$ , and group inversion  $G \rightarrow G$ ,  $g \mapsto g^{-1}$ , are smooth maps

For  $g \in G$  a Lie group, we define diffeomorphisms  $L_g : G \rightarrow G$ ,  $L_g(h) = gh$ , and  $R_g : G \rightarrow G$ ,  $R_g(h) = hg$ , call **left** & **right** multiplication.

**Definition 3.2.** Recall that given a diffeomorphism  $F : M \rightarrow N$  we define the **pushforward**  $F_* : \mathfrak{X}(M) \rightarrow \mathfrak{X}(N)$  by, for  $\xi \in \mathfrak{X}(M)$ ,  $f \in C^\infty(N)$ ,  $(F_*\xi)(f) = \xi(f \circ F)$ .

**Remark.** Note that given any smooth map of manifolds  $F : M \rightarrow N$ , the derivative  $dF : TM \rightarrow TN$  gives a map  $\forall a \in M$ ,  $dF_a : T_a M \rightarrow T_{F(a)}N$  which for  $\xi \in T_a M$ ,  $f \in C^\infty(N)$  acts as  $(dF_a(\xi))(f) = \xi(f \circ F)$ . This is often written as  $F_*$ , but the two concepts are subtly different.

**Definition 3.3.** A vector field  $\xi \in \mathfrak{X}(G)$  is **left invariant** if  $\forall g \in G$ ,  $(L_g)_*\xi = \xi$ . Similarly we define **right invariant**.

**Lemma 3.4.** If  $\xi$  is a LIVF,  $\xi_g = (L_g)_*\xi_e$ , where  $e \in G$  is the identity.

*Proof.* Let  $f \in C^\infty(G)$ . Then

$$(L_g)_*\xi = \xi \Rightarrow \xi(f \circ L_g) = \xi(f)$$

Now evaluating at  $g \in G$ ,  $\xi_g \in T_g G$  so  $\xi_g(f \circ L_g) = ((L_g)_*\xi_e)(f)$ . Result follows.  $\square$

It can be shown that the lie bracket of two left invariant vector fields is also left invariant.

**Definition 3.5.** The vector space of left invariant vector fields is the **Lie algebra**  $\mathfrak{g}$  of  $G$ .

Since a LIVF is uniquely determined by its value at the identity, we have that  $\mathfrak{g} \cong T_e G$  as a vector space, but we can also transport the Lie bracket from  $\mathfrak{g}$  to  $T_e G$  so they are isomorphic as algebras.

**Definition 3.6.** The maps  $(L_{g^{-1}})_* : T_g G \rightarrow T_e G \cong \mathfrak{g}$  define a  $\mathfrak{g}$ -valued one form  $\theta$  called the **left invariant Maurer-Cartan one-form**. If  $\xi$  is a LIVF,  $\theta(\xi) = \xi_e$ .

By definition,  $\theta$  is left invariant.

**Theorem 3.7.** *The MC one form satisfies the **structure equation***

$$d\theta = -\frac{1}{2} [\theta, \theta]$$

i.e. for  $\xi, \eta \in \mathfrak{X}(G)$ ,  $d\theta(\xi, \eta) = -[\theta(\xi), \theta(\eta)]$

*Proof.* We will need the following result:

Claim: For  $\theta \in \Omega^1(M)$ ,  $X, Y \in \mathfrak{X}(M)$

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])$$

To show this take coordinates such that  $\theta = \theta_a dx^a$ ,  $X = X^a \partial_a$ ,  $Y = Y^a \partial_a$ . Then

$$\begin{aligned} d\theta(X, Y) &= (\partial_b \theta_a X^c Y^d)(dx^b \wedge dx^a)(\partial_c, \partial_d) \\ &= \partial_b \theta_a (X^b Y^a - X^a Y^b) \\ &= X^b \partial_b (\theta_a Y^a) - Y^b \partial_b (\theta_a X^a) - \theta_a (X^b \partial_b Y^a - Y^b \partial_b X^a) \\ &= X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) \end{aligned}$$

Now if  $X, Y$  are LIVFs,  $\theta(X), \theta(Y)$  are constant, so on these

$$d\theta(X, Y) + \theta([X, Y]) = 0$$

Moreover for LIVFs  $\theta([X, Y]) = [\theta(X), \theta(Y)]$ . Now LIVFs span the space of vector fields, and all the operations are linear, so we are done.  $\square$

**Proposition 3.8.** *If  $G$  is a matrix Lie group,  $\theta_g = g^{-1} dg$ .*

*Proof.* In a matrix group, we have the correspondence  $X \in \mathfrak{g} \Leftrightarrow \exp(tX) \in G$ . Take a basis  $\{T_a\}$  of  $T_e G$  and give  $g \in G$  coordinates  $x^a$  if  $g = \exp(\sum_a x^a T_a)$ . Then let  $g$  be constant and take a curve through  $g$ ,  $\gamma : \mathbb{R} \rightarrow G$ ,  $\gamma(t) = \exp[\sum_a (x^a + t\xi^a) T_a]$  with tangent vector  $g(\sum_a \xi^a T_a) \in T_g G$ . Under  $L_{g^{-1}}$ , this is a curve through  $e$  with tangent vector  $(\sum_a \xi^a T_a) \in T_e G$ . Hence if we write  $\xi = \sum_a \xi^a \frac{\partial}{\partial x^a}$  for the the vector generating  $\gamma$  we get

$$\theta_g = \sum_a T_a dx^a = g^{-1} dg$$

$\square$

Every  $g \in G$  defines a diffeomorphism  $L_g R_{g^{-1}} : G \rightarrow G$ ,  $h \mapsto ghg^{-1}$ . Since  $e = geg^{-1}$  its derivative belongs to  $GL(T_e G) = GL(\mathfrak{g})$ .

**Definition 3.9.** *The **adjoint representation** of  $G$  on  $\mathfrak{g}$  is given by  $\text{Ad}_g = (L_g)_*(R_{g^{-1}})_*$*

**Lemma 3.10.**  $R_g^* \theta = \text{Ad}_{g^{-1}} \theta$

*Proof.*

$$\begin{aligned} R_g^* \theta_{hg} &= \theta_{hg}(R_g)_* \\ &= (L_{(hg)^{-1}})_*(R_g)_* \\ &= (L_{g^{-1}})_*(L_{h^{-1}})_*(R_g)_* \\ &= (L_{g^{-1}})_*(R_g)_*(L_{h^{-1}})_* \\ &= \text{Ad}_{g^{-1}} \theta_h \end{aligned}$$

$\square$

**Definition 3.11.** The **left action** of a Lie group  $G$  on a manifold  $M$  is a smooth map  $G \times M \rightarrow M$ ,  $(g, a) \mapsto ga$  satisfying the axioms  $\forall g, h \in G, \forall a \in M$

- $g(ha) = (gh)a$
- $ea = a$

Right action is defined equivalently.

Left and right actions are equivalent if we take  $ga = ag^{-1}$ .

**Definition 3.12.** An action is **transitive** if the  $G$ -orbit of any point is  $M$ , equivalently  $\forall a, b \in M, \exists g \in G, b = ga$

**Definition 3.13.** An action is **free** if the only element which fixes any point is the identity.

**Definition 3.14.** A  **$G$ -torsor** (or principally homogeneous  $G$ -space) is a manifold  $M$  on which  $G$  acts freely and transitively

Given a  $G$ -torsor  $M$ , any point in  $M$  defines a diffeomorphism  $g \cong M$ , and as such  $G$ -torsors are said to be like a Lie group where we have 'forgotten' the identity.

**Definition 3.15.** A **principal  $G$ -bundle** is a fibre bundle  $P \xrightarrow{\pi} M$  together with a smooth right  $G$ -action  $(p, g) \mapsto r_g(p)$  which preserves fibres ( $\pi \circ r_g = \pi$ ) and acts freely and transitively.

It follows that fibres are  $G$ -orbits and hence  $M = P/G$ . The condition of local triviality now says that the local trivialisation  $\pi^{-1}(U) \xrightarrow{\varphi} U \times G$  are  $G$ -equivariant, i.e. where  $\varphi(p) = (\pi(p), \gamma(p))$ ,  $\gamma : \pi^{-1}(U) \rightarrow G$  a  $G$ -equivariant ( $\gamma \circ r_g = R_g \circ \gamma$ ) fibrewise diffeomorphism

**Definition 3.16.** A principal  $G$ -bundle is **trivial** if  $\exists$  a  $G$ -equivariant diffeomorphism  $P \xrightarrow{\psi} M \times G$ .

**Proposition 3.17.** A principal  $G$ -bundle  $P \xrightarrow{\pi} M$  admits a section iff it is trivial

*Proof.* If  $P \xrightarrow{\pi} M$  is trivial,  $\psi : P \rightarrow M \times G$  defines a section  $s : M \rightarrow P$  by  $s(a) = \psi^{-1}(a, e)$ . Conversely, if  $s$  is a section, define  $\psi$  by  $\psi(p) = (\pi(p), \chi(p))$  where  $\chi(p)$  is uniquely defined by  $p = s(\pi(p))\chi(p)$ . Notice that since  $pg = s(\pi(p))\chi(p)g = s(\pi(pg))\chi(p)g$  so  $\chi(pg) = \chi(p)g$ .  $\square$

**Example 3.18.** Let  $G$  be a Lie group and  $H \leq G$  a closed subgroup. Then  $G \xrightarrow{\pi} G/H$  is a principal  $H$ -bundle. Therefore homogeneous spaces are examples of principal bundles.

Since principal fibre bundles are locally trivial, they admit local sections. Let  $\{(U_\alpha, \varphi_\alpha)\}$  be a trivialising atlas for  $G \rightarrow P \xrightarrow{\pi} M$ . The canonical local sections  $s_\alpha : U_\alpha \rightarrow \pi^{-1}(U_\alpha)$  are given by  $s_\alpha(a) = \varphi_\alpha^{-1}(a, e)$ . On  $U_{\alpha\beta}$  we have sections  $s_\alpha, s_\beta$ . Writing  $\varphi_\alpha(p) = (\pi(p), g_\alpha(p))$  for  $g_\alpha : U_\alpha \rightarrow G$  equivariant we have that for  $p \in \pi^{-1}(U_{\alpha\beta})$ .

$$\begin{aligned} (\pi(p), g_\alpha(p)) &= \varphi_\alpha(p) = (\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta)(p) = (\varphi_\alpha \circ \varphi_\beta^{-1})(\pi(p), g_\beta(p)) \\ &\Rightarrow (\pi(p), \underbrace{g_\alpha(p)g_\beta^{-1}(p)}_{\equiv \hat{g}_{\alpha\beta}(p)}) = (\varphi_\alpha \circ \varphi_\beta^{-1})(\pi(p), g_\beta(p)) \end{aligned}$$

Note that  $\hat{g}_{\alpha\beta}(pg) = g_\alpha(pg)g_\beta^{-1}(pg) = g_\alpha(p)gg_\beta^{-1}(p) = \hat{g}_{\alpha\beta}(p)$  and so is constant along the fibres. Hence  $\exists g_{\alpha\beta} : U_{\alpha\beta} \rightarrow G$  s.t.  $\hat{g}_{\alpha\beta} = \pi^* g_{\alpha\beta}$  and  $(\varphi_\alpha \circ \varphi_\beta^{-1})(a, g) = (a, g_{\alpha\beta}(a)g)$ . It follows that the  $g_{\alpha\beta}$  obey the cocycle

conditions.

Now note  $g_\alpha \circ s_\alpha : U_\alpha \rightarrow G$  is a constant map taking value  $e$ , and so letting  $p = s_\beta(a)$

$$\begin{aligned} g_\alpha(p) &= \hat{g}_{\alpha\beta}(p)g_\beta(p) \Rightarrow g_\alpha(s_\beta(a)) = g_{\alpha\beta}(a)(g_\beta \circ s_\beta)(a) \\ &= (g_\alpha \circ s_\alpha)(a)g_{\alpha\beta}(a) \\ &= g_\alpha(s_\alpha(a)g_{\alpha\beta}(a)) \\ &\Rightarrow s_\beta(a) = s_\alpha(a)g_{\alpha\beta}(a) \quad \text{as } g_\alpha \text{ a diffeomorphism} \end{aligned}$$

## 4 Ehresmann Connections

Let  $P \xrightarrow{\pi} M$  be a principal  $G$ -bundle. Taking  $p \in P$ , the derivative  $(\pi_*)_p : T_p P \rightarrow T_{\pi(p)} M$  is a surjective map.

**Definition 4.1.** The kernel  $V_p$  is called the **vertical subspace**. A vector field  $\xi \in \mathfrak{X}(P)$  is called **vertical** if  $\forall p \in P, \xi_p \in V_p$ .

**Lemma 4.2.** The Lie bracket of two vertical vector fields is vertical

**Lemma 4.3.** The vertical subspaces span a  $G$ -invariant integrable distribution

*Proof.* Note  $\pi \circ r_g = \pi \Rightarrow \pi_*(r_g)_* = \pi_* \Rightarrow (r_h)_* V_p = V_{pg}$  so  $G$ -invariant. Integrable by the previous lemma.  $\square$

**Definition 4.4.** An **Ehresmann connection** on  $P$  is a smooth choice of horizontal subspaces  $H_p \subset T_p P$  s.t.  $T_p P = V_p \oplus H_p$  and  $(r_g)_* H_p = H_{pg}$ . Equivalently an Ehresmann connection is a  $G$ -invariant distribution  $H \subset TP$  complementary to  $V$ .

**Example 4.5.** A  $G$ -invariant Riemannian metric on  $P$  defines an Ehresmann connection by  $H_p = V_p^\perp$ .

The  $G$  action on  $P$  defines a smooth map  $\mathfrak{g} \rightarrow \mathfrak{X}(P)$  assigning to every  $X \in \mathfrak{g}$  the **fundamental vector field**  $\xi_X$  defined at  $p \in P$  by

$$(\xi_X)_p = \left. \frac{d}{dt} (pe^{tX}) \right|_{t=0}$$

**Lemma 4.6.**  $\xi_X$  is vertical

*Proof.*

$$\pi_* \xi_X|_p = \left. \frac{d}{dt} \pi(pe^{tX}) \right|_{t=0} = \left. \frac{d}{dt} \pi(p) \right|_{t=0} = 0$$

$\square$

As the  $G$  action is free,  $\forall p \in P$  the map  $X \mapsto (\xi_X)_p$  is an isomorphism  $\mathfrak{g} \xrightarrow{\cong} V_p$ .

**Lemma 4.7.**  $(r_g)_* \xi_X = \xi_{\text{Ad}_{g^{-1}}(X)}$

*Proof.*

$$(r_g)_*(\xi_X)_p = \left. \frac{d}{dt} r_g(p e^{tX}) \right|_{t=0} = \left. \frac{d}{dt} (p e^{tX} g) \right|_{t=0} = \left. \frac{d}{dt} (p g g^{-1} e^{tX} g) \right|_{t=0} = \left( \xi_{\text{Ad}_{g^{-1}}(X)} \right)_{pg}$$

□

**Definition 4.8.** The *connection one form* of a connection  $H \subset TP$  is the  $\mathfrak{g}$ -valued one form  $\omega \in \Omega^1(P; \mathfrak{g})$  defined by

$$\omega(\xi) = \begin{cases} X & \xi = \xi_X \\ 0 & \xi \in H \end{cases}$$

**Proposition 4.9.** The connection one form obeys  $r_g^* \omega = \text{Ad}_{g^{-1}} \circ \omega$

*Proof.* Let  $\xi$  be horizontal. Then  $(r_g)_* \xi$  is also horizontal as  $H$   $G$ -invariant. Then  $(r_g^* \omega)(\xi) = \omega((r_g)_* \xi) = 0$ . Note in this case  $(\text{Ad}_{g^{-1}} \circ \omega)(\xi) = 0$  too. Now if  $\xi = \xi_X$ ,  $(\text{Ad}_{g^{-1}} \circ \omega)(\xi) = \text{Ad}_{g^{-1}}(X) = \omega(\xi_{\text{Ad}_{g^{-1}}(X)}) = \omega((r_g)_* \xi_X) = (r_g^* \omega)(\xi)$  □

It turn out we also have a converse:

**Proposition 4.10.** If  $\omega \in \Omega^1(P; \mathfrak{g})$  satisfies  $r_g^* \omega = \text{Ad}_{g^{-1}} \circ \omega$  and  $\omega(\xi_X) = X$ , then  $H \equiv \ker \omega$  is a connection on  $P$ .

Now define the pullback of  $\omega$  along local sections to be  $A_\alpha \equiv s_\alpha^* \omega \in \Omega^1(U_\alpha; \mathfrak{g})$ .

**Proposition 4.11.** Let  $\omega_\alpha \equiv \text{Ad}_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^* \theta$  where  $\theta$  is the LI Maurer-Cartan one form on  $G$ . Then  $\omega_\alpha = \omega|_{\pi^{-1}U_\alpha}$

*Proof.* The proof will have two steps:

Claim:  $\omega$  and  $\omega_\alpha$  agree on the image of  $s_\alpha$

Since  $\pi \circ s_\alpha = \text{id}|_{U_\alpha}$ ,  $T_p P = \text{Im}(s_\alpha \circ \pi)_* \oplus V_p$  for  $p = s_\alpha(a)$ . Hence  $\forall \xi \in T_p P$ ,  $\exists! \xi^v \in V_p$  s.t.  $\xi = (s_\alpha)_* \pi_* \xi + \xi^v$ . Then using  $g_\alpha(p) = (g_\alpha \circ s_\alpha)(a) = e$

$$\begin{aligned} \omega_\alpha(\xi) &= (\pi^* s_\alpha^* \omega)(\xi) + (g_\alpha^* \theta_e)(\xi) \text{ (at } p, \text{Ad}_{g_\alpha^{-1}} = \text{id}) \\ &= \omega((s_\alpha)_* \pi_* \xi) + \theta_e((g_\alpha)_* \xi) \\ &= \omega((s_\alpha)_* \pi_* \xi) + \theta_e((g_\alpha)_* \xi^v) \text{ as } (g_\alpha)_*(s_\alpha)_* = (g_\alpha \circ s_\alpha)_* = 0 \\ &= \omega((s_\alpha)_* \pi_* \xi) + \omega(\xi^v) \\ &= \omega(\xi) \end{aligned}$$

Claim:  $\omega$  and  $\omega_\alpha$  transform in the same way under the right  $G$  action.

$$\begin{aligned} r_g^*(\omega_\alpha)_{pg} &= \text{Ad}_{g_\alpha(pg)^{-1}} \circ r_g^* \pi^* s_\alpha^* \omega + r_g^* g_\alpha^* \theta \\ &= \text{Ad}_{(g_\alpha(p)g)^{-1}} \circ r_g^* \pi^* s_\alpha^* \omega + g_\alpha^* R_g^* \theta \\ &= \text{Ad}_{g^{-1}g_\alpha(p)^{-1}} \circ \pi^* s_\alpha^* \omega + g_\alpha^* (\text{Ad}_{g^{-1}} \circ \theta) \\ &= \text{Ad}_{g^{-1}} (\text{Ad}_{g_\alpha(p)^{-1}} \circ \pi^* s_\alpha^* \omega + g_\alpha^* \theta) \\ &= \text{Ad}_{g^{-1}} \circ (\omega_\alpha)_p \end{aligned}$$

Hence we are done. □

Now as  $\omega$  is a global one form,  $\omega_\alpha$  and  $\omega_\beta$  must agree on  $U_{\alpha\beta}$ , allowing us to relate  $A_\alpha$  and  $A_\beta$ , namely on  $U_{\alpha\beta}$

$$\begin{aligned} A_\alpha &= s_\alpha^* \omega_\alpha = s_\alpha^* \omega_\beta = s_\alpha^* (\text{Ad}_{g_\beta(s_\alpha)^{-1}} \circ \pi^* A_\beta + g_\beta^* \theta) \\ &= \text{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^* \theta \end{aligned}$$

**Example 4.12.** For matrix Lie groups,  $g_{\beta\alpha}^* \theta = g_{\beta\alpha}^{-1} dg_{\alpha\beta} = -dg_{\alpha\beta} g_{\alpha\beta}^{-1}$ , so

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta} g_{\alpha\beta}^{-1}$$

Similarly, one can ask how  $\{A_\alpha\}$  depends on the choice of local section.

**Fact 4.13.** If  $s'_\alpha$  is another local section for  $U_\alpha$ ,  $\exists h_\alpha : U_\alpha \rightarrow G$  s.t.  $s'_\alpha(a) = s_\alpha(a)h_\alpha(a)$  and then

$$A'_\alpha = \text{Ad}_{h_\alpha^{-1}} \circ A_\alpha + h_\alpha^* \theta$$

**Idea.** We now have three different ways to understand connections on a principal  $G$ -bundle  $P \xrightarrow{\pi} M$ , namely;

1. a  $G$ -invariant horizontal distribution  $H \subset TP$
2. a one form  $\omega \in \Omega^1(P; \mathfrak{g})$  satisfying  $\omega(\xi_X) = X$  and  $r_g^* \omega = \text{Ad}_{g^{-1}} \circ \omega$
3. a family of one forms  $\{A_\alpha \in \Omega^1(U_\alpha; \mathfrak{g})\}$  satisfying  $A_\alpha = \text{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^* \theta$  on  $U_{\alpha\beta} \neq \emptyset$

If  $P \xrightarrow{\pi} M$  is a principal  $G$ -bundle,  $G$ -equivariant bundle diffeomorphisms are called **gauge transformations** and one can ask how an Ehresmann connection transforms. Let  $H \subset TP$  be a  $G$ -invariant horizontal distribution. Then let  $H^\Phi \equiv \Phi_* H$  be the gauge-transformed distribution.

**Lemma 4.14.**  $H^\Phi \subset TP$  is an Ehresmann connection

*Proof.*

$$(r_g)_* H_{\Phi(p)}^\Phi = (r_g)_* \Phi_* H_p = \Phi_* (r_g)_* H_p = \Phi_* H_{pg} = H_{\Phi(pg)}^\Phi = H_{\Phi(p)g}^\Phi$$

and  $H^\Phi$  is complementary to  $V$  because  $\Phi_* T_p P \xrightarrow{\cong} T_{\Phi(p)} P$  and  $\Phi_*$  preserves  $V = \ker \pi_*$  because  $\pi \circ \Phi = \pi$  □

**Exercise 4.15.** Let  $\Phi$  be a gauge transformation in a principal  $G$ -bundle  $P \xrightarrow{\pi} M$ . Let  $\xi_X$  denote a fundamental vector fields for the  $G$ -action on  $P$ . Show that  $\xi_X$  is gauge invariant, i.e.  $\Phi_* \xi_X = \xi_X$ . Further, show that if  $\omega$  is the connection one form for an Ehresmann connection  $H$  then  $(\Phi^{-1})^* \omega$  is the connection one form for  $H^\Phi$ .

Let  $\{A_\alpha\}, \{A_\alpha^\Phi\}$  be the gauge fields corresponding to the Ehresmann connections  $H, H^\Phi$ . Since  $\Phi$  preserves fibres it makes sense to restrict to  $\pi^{-1}U_\alpha$ . Applying the trivialisation  $\varphi_\alpha(\Phi(p)) = (\pi(p), g_\alpha(\Phi(p)))$  which defines  $\bar{\phi}_\alpha : \pi^{-1}U_\alpha \rightarrow G$  by  $\bar{\phi}_\alpha(p) = g_\alpha(\Phi(p))g_\alpha(p)^{-1}$ .

**Lemma 4.16.**  $\bar{\phi}_\alpha$  is constant on the fibres

*Proof.*

$$\begin{aligned} \bar{\phi}_\alpha(pg) &= g_\alpha(\Phi(pg))g_\alpha(pg)^{-1} \\ &= g_\alpha(\Phi(p)g)g_\alpha(pg)^{-1} \\ &= g_\alpha(\Phi(p))g(g_\alpha(p)g)^{-1} \\ &= g_\alpha(\Phi(p))g_\alpha(p)^{-1} \\ &= \bar{\phi}_\alpha(p) \end{aligned}$$

□



Hence  $\bar{\phi}_\alpha$  defines a smooth map  $\phi_\alpha : U_\alpha \rightarrow G$ . On overlaps  $U_{\alpha\beta} \neq \emptyset$  we have that  $\forall a \in U_{\alpha\beta}, p \in \pi^{-1}(a)$ , hence

$$\begin{aligned}\phi_\alpha(a) &= g_\alpha(\Phi(p))g_\alpha(p)^{-1} \\ &= g_\alpha(\Phi(p)) \cdot \underbrace{g_\beta(\Phi(p))^{-1}g_\beta(\Phi(p))}_e \underbrace{g_\beta(p)^{-1}g_\beta(p)}_e g_\alpha(p)^{-1} \\ &= g_{\alpha\beta}(a)\phi_\beta(a)g_{\alpha\beta}(a)^{-1} \text{ since } \pi(p) = \pi(\Phi(p)) = a\end{aligned}$$

**Remark.** We will see later that  $\{\phi_\alpha\}$  defines a section of a fibre bundle  $\text{Ad} P$  on  $M$  associated to the principal bundle  $P$ .

**Exercise 4.17.** Show that on  $U_\alpha$ ,  $A_\alpha^\Phi = \text{Ad}_{\phi_\alpha} \circ (A_\alpha - \phi_\alpha^* \theta) = \phi_\alpha A_\alpha \phi_\alpha^{-1} - d\phi_\alpha \phi_\alpha^{-1}$ , which is a gauge transform

## 5 Kozul Connections

**Definition 5.1.** A real, rank  $k$ , **vector bundle**  $E \xrightarrow{\pi} M$  is a fibre bundle whose fibres are  $k$ -dimensional real vector spaces and whose local trivialisations  $\psi : \pi^{-1}U \rightarrow U \times \mathbb{R}^k$  restrict fibrewise to isomorphisms  $\psi : E_a \rightarrow \{a\} \times \mathbb{R}^k$  of real vector spaces.

Let  $P \xrightarrow{\pi} M$  be a principal  $G$ -bundle and let  $\rho : G \rightarrow GL(V)$  be a Lie group homomorphism (i.e. a representation of  $G$ ), where  $V$  is a f.d. vector space. Since  $G$  acts freely on  $P$ , it also acts freely on  $P \times V$  via the right action

$$(p, v)g = (pg, \rho(g^{-1})v)$$

We let  $E \equiv P \times_G V$  denote the quotient  $(P \times V)/_G$  via the above action. It is the total space of a vector bundle  $E \xrightarrow{\pi} M$  where

$$\begin{aligned}\varpi : P \times_G V &\rightarrow M \\ [(p, v)] &\mapsto \pi(p)\end{aligned}$$

**Definition 5.2.**  $E \xrightarrow{\pi} M$  is called an **associated vector bundle** to the PFB  $P \rightarrow M$ , associated via the representation  $\rho$ .

Let  $\{(U_\alpha, \varphi_\alpha)\}$  be a trivialising atlas for  $P$  with transition function  $\{g_{\alpha\beta} : U_{\alpha\beta} \rightarrow G\}$  obeying the cocycle conditions. We may then trivialise  $P \times_G V$  on each  $U_\alpha$ , and the transition functions are  $\{\rho \circ g_{\alpha\beta} : U_{\alpha\beta} \rightarrow GL(V)\}$ .

More concretely we define  $P \times_G V \equiv \sqcup_\alpha U_\alpha \times V / \sim$  where  $(a, v) \sim (a, \rho(g_{\alpha\beta}(a))v)$

Let  $P \xrightarrow{\pi} M$  be a  $G$ -PFB and  $E \equiv P \times_G V \xrightarrow{\pi} M$  an associated VB with  $\rho : G \rightarrow GL(V)$ . Let  $\Gamma(E) = \{s : M \rightarrow E \mid \varpi \circ s = \text{id}_M\}$  denote the  $C^\infty(M)$ -module of sections of  $E$ , and  $C_G^\infty(P, V) = \{\zeta : P \rightarrow V \mid \forall g \in G, r_g^* \zeta = \rho(g)^{-1} \zeta\}$  the  $G$ -equivariant functions  $P \rightarrow V$ . We can give  $C_G^\infty(P, V)$  the structure of a  $C^\infty(M)$ -module by declaring that for  $f \in C^\infty(M)$ ,  $f\zeta = \pi^* f \zeta$

**Proposition 5.3.** There is a  $C^\infty(M)$ -module isomorphism

$$\Gamma(E) \cong C_G^\infty(P, V)$$

*Proof.* Let  $\sigma \in \Gamma(E)$ . Let  $\psi_\alpha : \varpi^{-1}U_\alpha \rightarrow U_\alpha \times V$  be a local trivialisation and define  $\sigma_\alpha : U_\alpha \rightarrow V$ ,  $(\psi_\alpha \circ \sigma)(a) = (a, \sigma_\alpha(a))$ . On overlaps the local functions  $\sigma_\alpha, \sigma_\beta$ , are related by  $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$ , where  $g_{\alpha\beta}$  are the transition functions of  $P \rightarrow M$ . We now define  $\zeta_\alpha : \pi^{-1}U_\alpha \rightarrow V$  by  $\zeta_\alpha((\pi^* s_\alpha)(p)) = \sigma_\alpha(\pi(p))$  and extend by  $\zeta_\alpha((\pi^* s_\alpha)(p)g) = \rho(g)^{-1} \sigma_\alpha(\pi(p))$ .

Let  $\pi(p) = a \in U_{\alpha\beta}$ . Then

$$\begin{aligned}\zeta_\beta(p) &= \zeta(s_\alpha(a)g_\alpha(p)) = \zeta(s_\beta(a)g_{\beta\alpha}(a)g_\alpha(p)) \\ &= \rho(g_{\beta\alpha}(a)g_\alpha(p))^{-1} \circ \sigma_\beta(a) \\ &= \rho(g_\alpha(p))^{-1} \circ \rho(g_{\alpha\beta}(a)) \circ \sigma_\beta(a) \\ &= \rho(g_\alpha(p))^{-1} \circ \sigma_\alpha(a) \\ &= \rho(g_\alpha(p))^{-1} \zeta_\alpha(s_\alpha(a)) \\ &= \zeta_\alpha(s_\alpha(a)g_\alpha(p)) = \zeta_\alpha(p)\end{aligned}$$

The  $\{\zeta_\alpha\}$  are constructed to define a function  $\zeta : P \rightarrow V$  such that  $r_g^* \zeta = \rho(g)^{-1} \circ \zeta$ . If  $f \in C^\infty(M)$ , then  $f\sigma \in \Gamma(E)$  and  $(f\sigma)_\alpha = f\sigma_\alpha$  since  $\psi_\alpha$  is fibrewise linear. Then by definition

$$\begin{aligned}\rho(g_\alpha(p))^{-1} \circ \pi^*(f\sigma_\alpha) &= \rho(g_\alpha(p))^{-1} \circ (\pi^*f)(\pi^*\sigma_\alpha) \\ &= (\pi^*f)\rho(g_\alpha(p))^{-1} \circ (\pi^*\sigma_\alpha) \\ &= (\pi^*f)\zeta_\alpha(p)\end{aligned}$$

so the map  $\Gamma(E) \rightarrow C_G^\infty(P, V)$ , thus defined, is  $C^\infty(M)$ -linear.

Conversely, given a  $G$ -equivariant  $\zeta : P \rightarrow V$ , we define  $\sigma \in \Gamma(E)$  as follows: let  $s_\alpha : U_\alpha \rightarrow P$  be the canonical local sections. Then let  $\sigma_\alpha = s_\alpha^* \zeta$ . For  $a \in U_{\alpha\beta}$ ,

$$\sigma_\beta(a) = \zeta(s_\beta(a)) = \zeta(s_\alpha(a)g_{\alpha\beta}(a)) = \rho(g_{\alpha\beta}(a))^{-1} \zeta(s_\alpha(a)) = \rho(g_{\beta\alpha}(a)) \sigma_\alpha(a)$$

□

**Example 5.4.** Let  $\omega, \omega'$  be connection one forms for Ehresmann connections  $\mathcal{H}, \mathcal{H}'$  on  $P \rightarrow M$ . Then  $r_g^* \omega = \text{Ad}_{g^{-1}} \circ \omega$  and similarly for  $\omega'$ . Now if  $\xi$  is vertical,  $\omega(\xi) = \omega'(\xi)$ , and hence  $\tau \equiv \omega - \omega' \in \Omega^1(P; \mathfrak{g})$  is **horizontal** (i.e.  $\tau(\xi) = 0$  if  $\xi$  vertical).

Now let  $\tau_\alpha = s_\alpha^* \tau \in \Omega^1(U_\alpha; \mathfrak{g})$ . Then  $\tau_\alpha = s_\alpha^* \omega - s_\alpha^* \omega' = A_\alpha - A'_\alpha$ . On  $U_{\alpha\beta}$ ,  $A_\alpha = \text{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^* \theta$ , and likewise for  $A'_\alpha$ ,  $\Rightarrow \tau_\alpha = \text{Ad}_{g_{\alpha\beta}} \circ \tau_\beta$ . Hence  $\{\tau_\alpha\}$  defines  $\tau \in \Omega^1(M; \text{ad } P)$  where  $\text{ad } P \equiv P \times_G \mathfrak{g}$ .

**Example 5.5.** Take  $H \leq G$  closed and  $M = G/H$ . Then  $G \xrightarrow{\pi} M$  is a principal  $H$ -bundle. Let  $\rho : H \rightarrow GL(V)$  be a representation. Then  $E \equiv G \times_H V \rightarrow M$  is a **homogeneous vector bundle**. Then  $\Gamma(E) \cong \{f : G \rightarrow V \mid f(ph) = \rho(h)^{-1} f(p)\}$  as  $C^\infty(M)$ -modules. On  $\Gamma(E)$  we have a rep of  $G$  given by  $(g \cdot f)(g_1) = f(g^{-1}g_1)$ .

There is a sort of converse to the associated VB construction. If  $E \xrightarrow{\pi} M$  is a real rank  $k$  vector bundle, we may associate with it a principal  $GL(k, \mathbb{R})$ -bundle in one of two ways as follows:

1. Let  $\{(U_\alpha, \psi_\alpha)\}$  be a trivialising atlas for  $E$ , with  $\psi_\alpha : \pi^{-1}U_\alpha \rightarrow U_\alpha \times \mathbb{R}^k$  and transition functions  $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow GL(k, \mathbb{R})$ . We can then glue  $U_\alpha \times GL(k, \mathbb{R})$  and  $U_\beta \times GL(k, \mathbb{R})$  along  $U_{\alpha\beta}$  by

$$(a, A) \sim (a, g_{\alpha\beta}(a)A)$$

which is equivariant under right multiplication by  $GL(k, \mathbb{R})$ . The resulting principal  $GL(k, \mathbb{R})$ -bundle is denoted  $GL(E) \xrightarrow{\varpi} M$  and it follows that  $E \rightarrow M$  is the vector bundle associated to  $GL(E)$  via the identity rep

2. The PFB  $GL(E) \xrightarrow{\varpi} M$  can be understood as the **bundle of frames** of  $E \xrightarrow{\pi} M$ . Let  $GL(E)_a = \{\text{ordered bases for } E_a\}$ . Let  $u = (u_1, \dots, u_n)$  be a frame for  $E_a$ . Then  $\varpi(u) = a$  defines  $\varpi : GL(E) \rightarrow M$ . If  $A \in GL(k, \mathbb{R})$ ,  $uA$  defined by  $(uA)_i = \sum_j u_j A_{ji}$  is another frame for  $E_a$ . Given frames  $u, u'$  for  $E_a$ ,  $\exists! A \in GL(k, \mathbb{R})$  s.t.  $u' = uA$ . Let  $(U, \psi)$  be a local trivialisation for  $E$ . We define a reference frame  $\bar{u}(a)$  for each  $a \in U$  by  $\psi(\bar{u}_i(a)) = (a, e_i)$ , where  $\{e_i\}$  is the standard bases for  $\mathbb{R}^k$ . This defines a trivialisation  $\Psi : \varpi^{-1}U \rightarrow U \times GL(k, \mathbb{R})$  by  $\Psi(u) = (a, A(u))$  where  $u$  is a frame for  $E_a$  and  $A(u) \in GL(k, \mathbb{R})$  is the unique element sending  $u$  to  $\bar{u}(a)$ . Now for  $B \in GL(k, \mathbb{R})$ , we have

$$\bar{u}(a)A(uB) = uB = (\bar{u}(a)A(u))B \Rightarrow A(uB) = A(u)B$$

Hence  $\Psi$  is  $GL(k, \mathbb{R})$ -equivariant. Let  $\{(U_\alpha, \Psi_\alpha)\}$  denote the resulting trivialising atlas. Then if  $a \in U_{\alpha\beta}$  and  $u$  is a frame for  $E_a$ , then  $\Psi_\alpha(u) = (a, A_\alpha(u))$  where  $\bar{u}_\alpha(u)A_\alpha(u) = u$ . Now note

$$\begin{aligned}
\bar{u}_\beta(a)_i &= \psi^{-1}(a, e_i) \\
&= \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_\beta^{-1}(a, e_i) \\
&= \psi_\alpha^{-1}(a, g_{\alpha\beta}(a)e_i) \\
&= \psi_\alpha^{-1}(a, \sum_j e_j(g_{\alpha\beta}(a))_{ji}) \\
&= \sum_j \psi_\alpha^{-1}(a, e_j)g_{\alpha\beta}(a)_{ji} \\
&= \sum_j \bar{u}_\alpha(a)_j g_{\alpha\beta}(a)_{ji} \\
&\Rightarrow \bar{u}_\beta(a) = \bar{u}_\alpha(a)g_{\alpha\beta}(a) \\
&\Rightarrow A_\alpha(u) = g_{\alpha\beta}(a)A_\beta(u)
\end{aligned}$$

**Definition 5.6.** Let  $E \xrightarrow{\pi} M$  be a vector bundle. A **Kozul connection** on  $E$  is an  $\mathbb{R}$ -bilinear map

$$\begin{aligned}
\nabla : \mathfrak{X}(M) \times \Gamma(E) &\rightarrow \Gamma(E) \\
(X, s) &\mapsto \nabla_X s
\end{aligned}$$

satisfying that,  $\forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E)$

1.  $\nabla_{fX}s = f\nabla_X s$
2.  $\nabla_X(fs) = X(f)s + f\nabla_X s$

Suppose that  $E = P \times_G V$  for some  $G$ -PFB  $P \xrightarrow{\pi} M$ . Then an Ehresmann connection on  $P$  induces a Kozul connection on  $E$ . For this it is convenient to use the  $C^\infty(M)$ -module isomorphism  $\Gamma(E) \cong C_G^\infty(P, V)$  and we will define  $\nabla$  on  $C_G^\infty(P, V)$ :

Let  $\mathcal{H} \subset TP$  be an Ehresmann connection. We define  $h : T_p P \rightarrow T_p P$  to be the projector onto  $\mathcal{H}$  along  $\ker(\pi_*)$ . If we write  $\xi \in T_p P$  as  $\xi^h + \xi^v$  where  $\xi^h \in \mathcal{H}_p$  and  $\pi_*(\xi^v) = 0$ , then  $h(\xi) = \xi^h$ . Let  $h^* : T_p^* P \rightarrow T_p^* P$  be the dual (i.e.  $(h^*\alpha)(\xi) = \alpha(h(\xi))$ ). Let  $X \in \mathfrak{X}(M)$ . Then given  $p \in P_a$  let  $\xi \in T_p P$  be s.t.  $\pi_*\xi = X(a)$ . We define  $\nabla_X \psi|_p = (d\psi)_p(h\xi)$ , i.e.  $d^\nabla \psi = h^*d\psi$ . This is well defined because if  $\pi_*\xi = \pi_*\xi', h\xi = h\xi'$ . Further,  $\nabla_X \psi \in C_G^\infty(P, V)$  because the split  $TP = \mathcal{V} \oplus \mathcal{H}$  is  $G$ -invariant, and hence  $r_g^* h^* = h^* r_g^*$ . Hence

$$\begin{aligned}
r_g^* d^\nabla \psi &= r_g^* h^* d\psi \\
&= h^* r_g^* d\psi \\
&= h^* d(\rho(g)^{-1} \circ \psi) \\
&= \rho(g)^{-1} \circ h^* d\psi = \rho(g)^{-1} d^\nabla \psi
\end{aligned}$$

**Proposition 5.7.**  $\nabla$  defines a Kozul connection on  $E$

*Proof.*

$$\begin{aligned}
\nabla_{fX}\psi &= d\psi(h(f\xi)) \\
&= d\psi(h[(\pi^*f)\xi]) \\
&= \pi^*f d\psi(h\xi) \\
&= f\nabla_X\psi \\
\nabla_X(f\psi) &= \nabla_X[(\pi^*f)\psi] \\
&= d[(\pi^*f)\psi](h\xi) \\
&= (\pi^*df)(h\xi) + (\pi^*f)\nabla_X\psi \\
&= \pi^*(df(\pi_*h\xi))\psi + f\nabla_X\psi \\
&= \pi^*(df(\pi_*\xi))\psi + f\nabla_X\psi \\
&= \pi^*(Xf)\psi + f\nabla_X\psi \\
&= X(f)\psi + f\nabla_X\psi
\end{aligned}$$

□

We will now define a more calculationally useful formula for the Kozul connection of  $P \times_G V$  induced by the Ehresmann connection on  $P$ . Let  $\psi \in C_G^\infty(P, V)$  and let  $\xi \in \mathfrak{X}(P)$ . We decompose  $\xi = h\xi + \xi^v$  where  $\pi_*\xi^v = 0$ . Then

$$d\psi(h\xi) = d\psi(\xi - \xi^v) = d\psi(\xi) - d\psi(\xi^v)$$

The derivative  $\xi^v\psi$  only depends on the value of  $\xi^v$  at a point, so we can take  $\xi^v$  to be the fundamental vector field  $\xi_{\omega(\xi^v)} = \xi_{\omega(\xi)}$  corresponding to the  $G$ -action. Therefore

$$\begin{aligned}
\xi^v\psi &= \xi_{\omega(\xi)}\psi = \left. \frac{d}{dt}\psi \circ r_{\exp(t\omega(\xi))} \right|_{t=0} \\
&= \left. \frac{d}{dt}\rho(\exp(-t\omega(\xi))) \circ \psi \right|_{t=0} \\
&= -\rho(\omega(\xi)) \circ \psi
\end{aligned}$$

Therefore  $d\psi(h\xi) = d\psi(\xi) + \rho(\omega(\xi)) \circ \psi$ , or abstracting  $\xi$ ,

$$d^\nabla\psi = d\psi + \rho(\omega) \cdot \psi$$

Finally, we give a formula for  $\nabla_X\sigma$ , where  $\sigma \in \Gamma(P \times_G V)$ , now viewed as a family  $\{\sigma_\alpha : U_\alpha \rightarrow V\}$  of functions transforming in overlaps as  $\sigma_\alpha(A) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$ ;

$$\begin{aligned}
d^\nabla\sigma_\alpha &= d^\nabla s_\alpha^*\psi = d^\nabla(\psi \circ s_\alpha) = d(\psi \circ s_\alpha) \circ h \\
&= d(s_\alpha^*\psi) \circ h = s_\alpha^*(d\psi) \circ h \\
&= s_\alpha^*d^\nabla\psi = s_\alpha^*(d\psi + \rho(\omega) \circ \psi) \\
&= ds_\alpha^*\psi + \rho(s_\alpha^*\omega) \circ s_\alpha^*\psi \\
&= d\sigma_\alpha + \rho(A_\alpha) \circ \sigma_\alpha
\end{aligned}$$

Hence, if  $X \in \mathfrak{X}(M)$ ,

$$\nabla_X\sigma_\alpha \equiv X(\sigma_\alpha) + \rho(A_\alpha(X)) \cdot \sigma_\alpha$$

**Exercise 5.8.** Show that  $\nabla_X\sigma_\alpha$  transforms like  $\sigma_\alpha$  on overlaps, that is

$$\nabla_X\sigma_\alpha = \rho(g_{\alpha\beta}) \circ \nabla_X\sigma_\beta$$

Note this justifies the name **covariant derivative**.

In summary, given a  $G$ -PFB,  $P \rightarrow M$ , and a f.d. rep  $\rho : G \rightarrow GL(V)$ , we construct a VB  $P \times_G V \rightarrow M$ . Every VB is obtained in this way from its frame bundle. We then introduced the notion of a Kozul connection on a VB and showed that an Ehresmann connection on  $P$  induces a Kozul connection on  $P \times_G V$ . The converse is also true: a Kozul connection on  $E$  induces an Ehresmann connection on  $GL(E)$ .

## 6 Curvature

Let  $P \xrightarrow{\pi} M$  be a principal  $G$ -bundle and  $\rho : G \rightarrow GL(V)$  a Lie group homomorphism. Let  $E \equiv P \times_G V \xrightarrow{\varpi} M$  be the associated VB. We saw in the last lecture that we have a  $C^\infty(M)$ -module isomorphism

$$\{s : M \rightarrow E \mid \varpi \circ s = \text{id}_M\} = \Gamma(E) \cong C_G^\infty(P, V) = \{\zeta : P \rightarrow V \mid r_g^* \zeta = \rho(g^{-1}) \circ \zeta\}$$

with module actions  $f \cdot \zeta = (\pi^* f)\zeta$ .

We wish to generalise this from functions to forms. We define  $\Omega^k(P, V)$  to be the  $k$ -forms on  $P$  with values in  $V$ . If  $p \in P$ ,  $\omega \in \Omega^k(P, V)$ , then  $\omega_p : \Lambda^k T_p P \rightarrow V$  is linear. Let  $\Omega_G^k(P, V) \subset \Omega^k(P, V)$  denote those  $V$ -valued  $k$ -forms  $\omega$  which are both

- **horizontal:**  $\forall \xi$  vertical,  $i_\xi \omega = 0$
- **invariant:**  $\forall g \in G$ ,  $r_g^* \omega = \rho(g^{-1}) \circ \omega$ .

Forms  $\omega \in \Omega^k(P, V)$  are said to be basic since they come from bundle valued forms on the base. Indeed, we have

**Proposition 6.1.** *There is an isomorphism of  $C^\infty(M)$ -modules*

$$\Omega_G^k(P, V) \cong \Omega^k(M, P \times_G V)$$

where for  $\omega \in \Omega_G^k(P, V)$ ,  $f \cdot \omega = (\pi^* f)\omega$

*Proof.* Similar to  $k = 0$  case. Define  $\sigma \in \Omega^k(M, P \times_G V)$  locally by  $\{\sigma_\alpha \in \Omega^k(U_\alpha, V)\}$  obeying  $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$ . Then  $\zeta_\alpha(p) = \rho(g_\alpha(p))^{-1} \circ \pi^* \sigma_\alpha$  is clearly horizontal. It can be shown to be invariant and that  $\forall p \in \pi^{-1}U_{\alpha\beta}$ ,  $\zeta_\alpha(p) = \zeta_\beta(p)$ . Conversely, if  $\zeta \in \Omega_G^k(P, V)$ , we define  $\sigma_\alpha = s_\alpha^* \zeta$  and one can show that  $\forall a \in U_{\alpha\beta}$ ,  $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$   $\square$

If  $\sigma \in \Gamma(P \times_G V)$ ,  $d^\nabla \sigma_\alpha = \rho(g_{\alpha\beta})d^\nabla \sigma_\beta$ , and hence  $d^\nabla \sigma \in \Omega^1(M, P \times_G V)$ .

**Lemma 6.2.** *Let  $\alpha \in \Omega_G^k(P, V)$ . Then  $h^* d\alpha \in \Omega_G^{k+1}(P, V)$ .*

*Proof.*  $h^* d\alpha$  is horizontal by construction, so we check invariance;

$$r_g^* h^* d\alpha = h^* r_g^* d\alpha = h^* d(r_g^* \alpha) = h^* d(\rho(g)^{-1} \circ \alpha) = \rho(g)^{-1} \circ h^* d\alpha$$

$\square$

**Definition 6.3.** *Let  $\omega \in \Omega^1(P, \mathfrak{g})$  be the connection one form of an Ehresmann connection  $\mathcal{H} \subset TP$ . Its **curvature** is  $\Omega \equiv h^* d\omega$ .*

**Lemma 6.4.**  $\Omega \in \Omega_G^2(P, V)$ .

*Proof.* Horizontal by construction, and by the same calculation as the lemma above it is invariant because  $\omega$  is.  $\square$

**Proposition 6.5.**  $\Omega = 0$  iff  $\mathcal{H} \subset TP$  is (Frobenius) integrable.

*Proof.* we see

$$\begin{aligned} \Omega(\xi, \eta) &= d\omega(h\xi, h\eta) = h\xi \underbrace{\omega(h\eta)}_{=0} - h\eta \underbrace{\omega(h\xi)}_{=0} - \omega([h\xi, h\eta]) \\ &= \omega([h\xi, h\eta]) \end{aligned}$$

Hence

$$\begin{aligned}\Omega = 0 &\Leftrightarrow \forall \xi, \eta \text{ } [h\xi, h\eta] \text{ is horizontal} \\ &\Leftrightarrow [\mathcal{H}, \mathcal{H}] \subset \mathcal{H} \\ &\Leftrightarrow \mathcal{H} \subset TP \text{ is integrable.}\end{aligned}$$

□

**Proposition 6.6** (Structure equation).  $\Omega = d\omega + \frac{1}{2} [\omega, \omega]$

*Proof.* We need to show  $\Omega(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$ .

Let  $\xi, \eta$  be horizontal. Then  $h\xi = \xi$  and  $h\eta = \eta$ , . hence  $\Omega(\xi, \eta) = d\omega(\xi, \eta)$  and  $\omega(\xi) = 0 = \omega(\eta)$ .

Let  $\eta$  be horizontal and  $\xi = \xi_X$  be vertical. Then  $h\xi = 0$ ,  $h\eta = \eta$ , and  $\omega(\eta)$ . Hence we need

$$0 = d\omega(\xi_X, \eta) = -\eta\omega(\xi_X) - \omega([\xi_X, \eta]) = -\underbrace{\eta X}_{=0} - \omega([\xi_X, \eta])$$

i.e that  $[\xi_X, \mathcal{H}] \subset \mathcal{H}$ . This is the case as  $\mathcal{H}$  is invariant.

Let  $\xi = \xi_X, \eta = \xi_Y$  vertical. Then  $h\xi_X = 0 = h\xi_Y$  and  $\omega(\xi_X), \omega(\xi_Y) = Y$ . So we must show that

$$\begin{aligned}0 &= d\omega(\xi_X, \xi_Y) + [\omega(\xi_X), \omega(\xi_Y)] \\ &= \xi_X Y - \xi_Y X - \omega([\xi_X, \xi_Y]) + [X, Y] \\ &= -\omega(\xi_{[X, Y]}) + [X, Y]\end{aligned}$$

so done.

□

**Corollary 6.7** (Bianchi Identity).  $h^* d\Omega = 0$

*Proof.*

$$h^* d\Omega = h^* d(d\omega + \frac{1}{2} [\omega, \omega]) = h^* [d\omega, \omega] = [h^* d\omega, h^* \omega] = 0$$

since  $h^* \omega = 0$

□

Let's define  $d^\nabla : \Omega_G^k(P, V) \rightarrow \Omega_G^{k+1}(P, V)$  by  $d^\nabla = h^* d$ . Then, unlike  $d$ ,  $d^\nabla$  need not be a differential, and the obstruction is the curvature:

**Proposition 6.8.**  $\forall \alpha \in \Omega_G^k(P, V), d^\nabla(d^\nabla \alpha) = \rho(\Omega) \wedge \alpha$

*Proof.*

$$\begin{aligned}d^\nabla \alpha &= d\alpha + \rho(\omega) \wedge \alpha \\ \Rightarrow d^\nabla(d^\nabla \alpha) &= d(d\alpha + \rho(\omega) \wedge \alpha) + \rho(\omega) \wedge (d\alpha + \rho(\omega) \wedge \alpha) \\ &= \rho(d\omega) \wedge \alpha - \rho(\omega) \wedge d\alpha + \rho(\omega) \wedge d\alpha + \rho(\omega) \wedge \rho(\omega) \wedge \alpha \\ &= \rho(d\omega) \wedge \alpha + \frac{1}{2} [\rho(\omega), \rho(\omega)] \wedge \alpha \\ &= \rho(d\omega + \frac{1}{2} [\omega, \omega]) \wedge \alpha \\ &= \rho(\Omega) \wedge \alpha\end{aligned}$$

□

**Exercise 6.9.** Write  $F_\alpha = s_\alpha^* \Omega$ . Express  $F_\alpha$  in terms of  $A_\alpha = s_\alpha^* \omega$  and relate  $F_\alpha, F_\beta$  on  $U_{\alpha\beta} \neq \emptyset$

## 7 Homogeneous spaces and Invariant Connections I

Let  $G$  be a Lie group acting transitively on a manifold  $M$ . Pick  $a \in M$  and let  $H \subset G$  be the stabiliser subgroup. It is a closed subgroup, and then  $M \cong G/H$ , where the diffeomorphism is  $G$ -equivariant and  $G \curvearrowright G/H$  is induced by left multiplication in  $G$ . If  $g \in G$ , we let  $\phi_g : M \rightarrow M$  denote the corresponding diffeomorphism. If  $X \in \mathfrak{g}$ , we define a vector field  $\xi_X \in \mathfrak{X}(M)$  by

$$(\xi_X f)(m) = \left. \frac{d}{dt} f(\phi_{\exp(-tX)}(m)) \right|_{t=0}$$

Then  $[\xi_X, \xi_Y] = \xi_{[X, Y]}$ .

Since  $H$  stabilises  $a \in M$ ,  $\forall h \in H, (\phi_h)_* : T_a M \rightarrow T_a M$ , and we get a Lie group homomorphism  $\lambda : H \rightarrow GL(T_a M)$  called the **linear isotropy representation**. We will use the same notation for the induced Lie algebra rep  $\lambda : \mathfrak{h} \rightarrow \mathfrak{gl}(T_a M)$ . Evaluating at  $a \in M$ , we get a surjective linear map  $\mathfrak{g} \rightarrow T_a M$ ,  $X \mapsto \xi_X|_a$  whose kernel is  $\mathfrak{h}$ .

**Definition 7.1.** We say that  $G/H$  is **reductive** if the short exact sequence

$$0 \rightarrow \mathfrak{h} \rightarrow \mathfrak{g} \rightarrow T_a M \rightarrow 0$$

splits as  $H$ -modules. In other words if  $\exists \mathfrak{m} \subset \mathfrak{g}$  such that  $\mathfrak{g} \oplus \mathfrak{m}$  and  $\forall h \in H, \text{Ad}_h : \mathfrak{m} \rightarrow \mathfrak{m}$ . In that case  $T_a M \cong \mathfrak{m}$  as  $H$ -modules.

If  $g \in G$  and  $\phi_g \in \text{Diff}(M)$ , we define  $\phi_g \cdot f = f \circ \phi_{g^{-1}}$  and  $\phi_g \cdot \xi = (\phi_g)_* \xi$  where

$$((\phi_g)_* \xi)_a = ((\phi_g)_*)_{\phi_g^{-1}(a)} \xi_{\phi_g^{-1}(a)}$$

It follows that

$$\begin{aligned} \phi_g \cdot (Xf) &= (\phi_g \cdot X)(\phi_g \cdot f) \\ \phi_g \cdot (fX) &= (\phi_g \cdot f)(\phi_g \cdot X) \end{aligned}$$

Now let  $\nabla$  be an affine connection, (i.e.  $\nabla_{fX} Y = f \nabla_X Y$ ,  $\nabla_X (fY) = X(f)Y + f \nabla_X Y$ ). Let  $\phi \in \text{Diff}(M)$ . Define  $\nabla^\phi$  by

$$\nabla_X^\phi Y = \phi \cdot \nabla_{\phi^{-1} \cdot X} (\phi^{-1} \cdot Y)$$

**Lemma 7.2.**  $\nabla^\phi$  is an affine connection

*Proof.*

$$\begin{aligned} \nabla_{fX}^\phi Y &= \phi \cdot \nabla_{\phi^{-1} \cdot (fX)} (\phi^{-1} Y) \\ &= \phi \cdot \nabla_{(\phi^{-1} \cdot f)(\phi^{-1} \cdot X)} (\phi^{-1} Y) \\ &= \phi \cdot (\phi^{-1} \cdot f \nabla_{\phi^{-1} \cdot X} (\phi^{-1} \cdot Y)) \\ &= (\phi \cdot \phi^{-1} f)(\phi \cdot \nabla_{\phi^{-1} X} (\phi^{-1} \cdot Y)) \\ &= f \nabla_X^\phi Y \\ \nabla + X^\phi(fY) &= \phi \cdot (\nabla_{\phi^{-1} \cdot X} \phi^{-1}(fY)) \\ &= \phi \cdot (\nabla_{\phi^{-1} \cdot X} (\phi^{-1} f)(\phi^{-1} Y)) \\ &= \phi \cdot ((\phi^{-1} X)(\phi^{-1} f)(\phi^{-1} Y) + (\phi^{-1} f) \nabla_{\phi^{-1} X} (\phi^{-1} Y)) \\ &= (\phi \cdot \phi^{-1} \cdot X(f))(\phi \cdot \phi^{-1} \cdot Y) + (\phi \cdot \phi^{-1} \cdot f) \nabla_{\phi^{-1} X} (\phi^{-1} Y) \\ &= X(f)Y + f \nabla_X^\phi Y \end{aligned}$$

□

**Definition 7.3.** An affine connection  $\nabla$  on a reductive homogeneous  $M = G/H$  is said to be ***G*-invariant** if  $\forall g \in G, \nabla^{\phi_g} = \nabla$ . i.e

$$\phi_g \cdot \nabla_X Y = \nabla_{\phi_g X} (\phi_g Y)$$

If  $H = \{e\}$ ,  $M = G$ , then  $\nabla$  is **left invariant** if

$$L_g \cdot \nabla_X Y = \nabla_{L_g \cdot X} (L_g \cdot Y)$$

Suppose that  $X, Y$  are left invariant, so that  $L_g \cdot X = X, L_g \cdot Y = Y$ . In that case, the left invariance of  $\nabla$  implies that  $\nabla_X Y$  is also left invariant. Now, on a Lie group we may trivialise the tangent bundle via left translations. That means that we have a global frame  $(X_1, \dots, X_n)$  consisting of left invariant vector fields. The connection is therefore uniquely determined by  $n^3$  numbers  $\Gamma_{ij}^k$  defined by

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$$

These are the components relative to the basis  $\{X_i\}$  of a linear map  $\Lambda : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ . The torsion and curvature tensors are also left-invariant and are given in terms of  $\Lambda$  by

$$\begin{aligned} T(X, Y) &= \Lambda_X Y - \Lambda_Y X - [X, Y] \\ R(X, Y)Z &= [\Lambda_X, \Lambda_Y] Z - \Lambda_{[X, Y]} Z \end{aligned}$$

for LI  $X, Y, Z \in \mathfrak{X}(G)$ . We see that curvature measures the failure of  $\Lambda$  to be a Lie algebra homomorphism. In particular, taking  $\Lambda = 0$ , we see that there exists a flat connection with torsion given by  $T(X, Y) = -[X, Y]$  relative to which LI vf on  $G$  are **parallel** (i.e.  $\nabla X = 0$ ). Of course, there exists another flat connection annihilating the right-invariant vector fields.

## 8 Invariant Connections

What did we do last time? We were looking at Homogeneous spaces  $M \equiv G/H$ ,  $H \leq G$  a closed subgroup. We had fibre

$$H \rightarrow G \xrightarrow{\pi} M$$

and for  $g \in G$  we have  $\phi_g : M \rightarrow M$  acting by multiplication, i.e.  $\phi_g(a) = g \cdot a$ . As a result of the quotient have  $eH = o \in M$  s.t

$$\forall h \in H \phi_h(o) = o$$

Then

$$(\phi_h)_* : T_o M \rightarrow T_o M$$

Hence we may make the following def:

**Definition 8.1.** The **linear isotropy representation**

$$\lambda : H \rightarrow GL(T_o M)$$

is given by  $\lambda_h = (\phi_h)_*$ .

We also have the map

$$\begin{aligned} \xi : \mathfrak{g} &\rightarrow \mathfrak{X}(M) \\ X &\mapsto \xi_X \end{aligned}$$

s.t.  $[\xi_X, \xi_Y] = \xi_{[X, Y]}$ . Composing with evaluation yields

$$ev_o \circ \xi : \mathfrak{g} \rightarrow T_o M$$



where  $\ker(ev_o \circ \xi) = \mathfrak{h} \subset \mathfrak{g}$ . This is bijective so in fact

$$T_o M \cong \mathfrak{g}/\mathfrak{h}$$

and we get commuting diagram

$$\begin{array}{ccc} T_o M & \xrightarrow{\lambda_h} & T_o M \\ \cong \downarrow & & \downarrow \cong \\ \mathfrak{g}/\mathfrak{h} & \xrightarrow{\text{Ad}(h)} & \mathfrak{g}/\mathfrak{h} \end{array}$$

**Definition 8.2.** An affine connection  $\nabla$  on  $TM$  is ***G*-invariant** if

$$\begin{aligned} \forall g \in G, \nabla^{\phi_g} &= \nabla \\ \phi_g \nabla_\xi \eta &= \nabla_{\phi_g \xi}(\phi_g \eta) \end{aligned}$$

If  $H = \{e\}$ ,  $M = G$ , then  $\nabla$  is left invariant if for all left invariant vector fields  $\xi_X, \xi_Y$   $\nabla_{\xi_X} \xi_Y$  is also LI. This is then uniquely determined by its value at  $e$ . Hence  $\nabla$  defines a bilinear map

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\xrightarrow{\alpha} \mathfrak{g} \\ (X, Y) &\mapsto \nabla_{\xi_X} \xi_Y|_e \end{aligned}$$

We can then **curry** a map as given  $\alpha : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  we can get

$$\begin{aligned} \Lambda : \mathfrak{g} &\rightarrow \text{End}(\mathfrak{g}) \\ X &\mapsto \Lambda_X \end{aligned}$$

where  $\Lambda_X(Y) = \alpha(X, Y)$ .

**Exercise 8.3.** Show that the torsion  $T$  and curvature  $R$  of  $\Lambda$  are left invariant and given by

$$\begin{aligned} T(X, Y) &= \Lambda_X Y - \Lambda_Y X - [X, Y] \\ R(X, Y)Z &= [\Lambda_X, \Lambda_Y]Z - \Lambda_{[X, Y]}Z \end{aligned}$$

Note  $R$  is the obstruction to  $\Lambda$  being a Lie algebra homomorphism.

Claim:  $\exists$  a LI connection  $\nabla$  corresponding to  $\Lambda = 0$ .

Such  $\Lambda$  is flat, but has torsion  $T(X, Y) = -[X, Y]$ . As such  $\nabla$  is characterised by  $\forall$  LI  $\xi$ ,  $\nabla \xi = 0$ . Now let  $H \neq \{e\}$  be closed and reductive:  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  where  $\text{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$ . Note  $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$ , so in the previous case  $\mathfrak{m} \cong T_o M$

**Aside.** There is a "holonomy principle" that

$$\left\{ G\text{-invariant tensor fields on } G/H \right\} \xleftrightarrow{ev_o} \left\{ \text{Ad}(H)\text{-invariant tensors on } \mathfrak{m} \right\}$$

This comes about, as if we take a tensor  $T$  at  $o$ , we can define a tensor field on  $G/H$  by

$$\mathcal{T}(a) = \phi_g T$$

for and  $g \in G$  s.t.  $\phi_g o = a$ . Then if we have another representative  $g'$  then

$$g^{-1}g' \in H \Leftrightarrow \phi_{g^{-1}g'} o = o$$

so

$$T = \phi_{g^{-1}g'} \mathcal{T}(a)$$

Claim: An invariant connection  $\nabla$  is determined by a bilinear map

$$\alpha : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$$

which is  $H$  invariant, the **Nomizu map**.

We can take natural coordinates for  $M$  in the neighbourhood  $V \subset \mathfrak{m}$  of  $o$  by exponentiating  $\mathfrak{m}$ . The projection  $\pi$  is a local diffeo on  $U = \exp(V)$ .

In a basis for  $\mathfrak{m}$ ,  $\{e_i\}$ ,

$$\begin{aligned} V &\rightarrow U \\ \sum x^i e_i &\mapsto \exp\left(\sum x^i e_i\right) \end{aligned}$$

Now  $\forall g \in U$ ,  $\pi(g) = \phi_g \cdot o$ , Let  $\bar{V} = \{\phi_g \cdot o | g \in U\}$ . For  $X \in \mathfrak{m}$  define  $\xi_X \in \mathfrak{X}(\bar{V})$  by

$$\begin{aligned} (\xi_X)_{\phi_g o} &\equiv ((\phi_g)_*)(\pi_*)_e X \\ &= ((\phi_g \circ \pi)_*)_e X \\ &= ((\pi \circ L_g)_*)_e X = (\pi_*)_g X_g^L \end{aligned}$$

where  $X^L$  is the LIVF defined by  $X^L|_e = X$ . Hence  $\xi_X$  is  $\pi$ -related to  $X^L$ . Then  $[\xi_X, \xi_Y]$  is  $\pi$ -related to  $[X^L, Y^L] = [X, Y]^L$ .

Now let  $W \subset V$  s.t.  $\forall h \in H$ ,  $\text{Ad}_h W \subset V$ , and def  $\bar{W}$  accordingly. Then for  $h \in H$ ,  $\phi_h : \bar{W} \rightarrow \bar{V}$ . As such

$$\begin{aligned} \phi_h \phi_g \cdot o &= \phi_h \phi_g \phi_{h^{-1}} \phi_h \cdot o \\ &= \phi_{hgh^{-1}} \cdot o \in \bar{V}. \end{aligned}$$

We will now need the following lemma

**Lemma 8.4.**  $\forall g \in \exp(W)$ ,  $h \in H$ ,

$$(\phi_h)_* \xi_X = \xi_{\text{Ad}_h X}$$

at  $\phi_g o$ , i.e. at all point in  $\bar{V}$ .

*Proof.*

$$\begin{aligned} [(\phi_h)_* \xi_X]_{\phi_h \phi_g o} &= (\phi_h)_* (\xi_X)_{\phi_g o} \\ &= (\phi_h)_* (\phi_g)_* \pi_* X \\ &= (\phi_{hg})_* \pi_* X \\ &= (\phi_{hgh^{-1}})_* (\phi_h)_* \pi_* X \\ &= (\xi_{\text{Ad}(h)X})_{\phi_{hgh^{-1}} o} = (\xi_{\text{Ad}_h X})_{\phi_h \phi_g o} \end{aligned}$$

recalling the commuting diagram

$$\begin{array}{ccc} T_o M & \xrightarrow{(\phi_h)_*} & T_o M \\ \pi_* \uparrow & & \uparrow \pi_* \\ \mathfrak{m} & \xrightarrow{\text{Ad}(h)} & \mathfrak{m} \end{array}$$

□

**Lemma 8.5.** Let  $X, Y \in \mathfrak{m}$ , and  $\xi_X, \xi_Y \in \mathfrak{X}(\bar{V})$ . Then  $[\xi_X, \xi_Y]|_o = \pi_* [X, Y]_{\mathfrak{m}}$

*Proof.* We saw above that  $[\xi_X, \xi_Y]$  is  $\pi$ -related to  $[X^L, Y^L] = [X, Y]^L$ . Hence  $[\xi_X, \xi_Y] = \xi_{[X, Y]}$  and evaluating at  $o \in M$  gives

$$[\xi_X, \xi_Y]|_o = \xi_{[X, Y]}|_o = \xi_{[X, Y]_{\mathfrak{m}}}|_o = \pi_* [X, Y]_{\mathfrak{m}}$$

□

**Theorem 8.6** (Nomizu). *There is a bijective correspondence*

$$\{G\text{-invariant affine connections on } M\} \leftrightarrow \{\text{Ad}(h)\text{-invariant bilinear maps } \alpha : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}\}$$

given by  $\alpha(X, Y) = \nabla_{\xi_X} \xi_Y|_o$

Note  $\exists!$   $G$ -invariant connection  $\nabla$  with  $\alpha = 0$ , and this is called the **canonical connection**. If you curvy this map again you can show

$$\begin{aligned} T(X, Y) &= \alpha(X, Y) = \alpha(Y, X) - [X, Y]_{\mathfrak{m}} \\ R(X, Y)Z &= \alpha(X, \alpha(Y, Z)) - \alpha(Y, \alpha(X, Z)) - \alpha([X, Y]_{\mathfrak{m}}, Z) - [X, Y]_{\mathfrak{h}} Z \end{aligned}$$

If  $\alpha = 0$  we get

$$\begin{aligned} T(X, Y) &= -[X, Y]_{\mathfrak{m}} \\ R(X, Y) &= -[X, Y]_{\mathfrak{h}} \end{aligned}$$

If  $T = 0$ ,  $M$  is said to be **symmetric**.

## 9 Cartan Connections

Again consider homogeneous reductive spaces

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \downarrow \pi \\ & & M = G/H \end{array}$$

With a local section  $\sigma : U \rightarrow G$  we can pull back the LI MC 1-form  $\vartheta_G \in \Omega^1(G; \mathfrak{g})$

$$\sigma^* \vartheta_G \in \Omega^1(U; \mathfrak{g})$$

Recall the MC 1-form satisfies structure equation s

$$d\vartheta_G + \frac{1}{2} [\vartheta_G, \vartheta_G] = 0$$

Then given two such sections  $\sigma_i$  we have

$$\forall a \in U, \sigma_2(a) = \sigma_1(a)h(a)$$

for some  $h : U \rightarrow H$ , a uniquely defined function.

**Lemma 9.1.**

$$\sigma_2^* \vartheta_G = \text{Ad}(h^{-1}) \cdot \sigma_1^* \vartheta_G + h^* \vartheta_H$$

*Proof.* We will notationally use the idea of matrix groups but in general the proof works. Then

$$\sigma^* \vartheta_g = \sigma^{-1} d\sigma.$$

Then

$$\begin{aligned} \sigma_2^* \vartheta_G &= \sigma_2^{-1} d\sigma_2 \\ &= (\sigma_1 h)^{-1} d(\sigma_1 h) \\ &= h^{-1} \sigma_1^{-1} (d\sigma_1 h + \sigma_1 dh) \\ &= h^{-1} (\sigma_1^{-1} d\sigma_1) h + h^{-1} dh \end{aligned}$$

so done. □

As we are in the reductive case,  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  and we can decompose. Write  $\sigma_1^* \vartheta_G = \theta_1 + \omega_1$  for  $\theta_1 \in \Omega^1(U, \mathfrak{m})$ ,  $\omega_1 \in \Omega^1(U, \mathfrak{h})$ . Then

$$\theta_2 + \omega_2 = \text{Ad}(h)^{-1}(\theta_1 + \omega_1) + h^* \vartheta_H$$

so

$$\begin{aligned}\theta_2 &= \text{Ad}(h)^{-1} \theta_1 \\ \omega_2 &= \text{Ad}(h)^{-1} \omega_1 + h^* \vartheta_H\end{aligned}$$

decomposing. Hence  $\theta_2$  transforms as a tensor,  $\omega_2$  as a gauge field. Now if we let  $\sigma = \sigma_1$  and the structure equation becomes

$$\begin{aligned}d(\theta + \omega) + \frac{1}{2} [\theta + \omega, \theta + \omega] &= 0 \\ d\theta + d\omega + \frac{1}{2} [\theta, \theta] + \frac{1}{2} [\omega, \omega] + [\omega, \theta] &= 0\end{aligned}$$

As such decomposing

$$\begin{aligned}d\theta + \frac{1}{2} [\theta, \theta]_{\mathfrak{m}} + [\omega, \theta] &= 0 & \Rightarrow & \Theta \equiv d\theta + [\omega, \theta] = -\frac{1}{2} [\theta, \theta]_{\mathfrak{m}} \\ d\omega + \frac{1}{2} [\theta, \theta]_{\mathfrak{h}} + \frac{1}{2} [\omega, \omega] &= 0 & \Rightarrow & \Omega \equiv d\omega + \frac{1}{2} [\omega, \omega] = -\frac{1}{2} [\theta, \theta]_{\mathfrak{h}}\end{aligned}$$

As such

$$\begin{aligned}\Theta(\xi_X, \xi_Y) &= -[X, Y]_{\mathfrak{m}} \\ \Omega(\xi_X, \xi_Y) &= -[X, Y]_{\mathfrak{h}}\end{aligned}$$

Gauge fields for the canonical invariant connection on  $G/H$  are  $\sigma^* \vartheta_G$ .

With this motivation with us, the Cartan connections are going to be generalisation of these where in the gauge field descriptions these are local 1-forms on the base. The Cartan viewpoint is to view  $TM$  not as a linear rep of  $GL(n, \mathbb{R})$ , but as a homogeneous space of the affine group  $\mathbb{A}(n, \mathbb{R}) \cong GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$  such that  $T_a M \cong \mathbb{A}(n, \mathbb{R}) / GL(n, \mathbb{R})$ .

**Definition 9.2.** A **Cartan gauge** (def from Sharpe, Jose doesn't like) with model  $G/H$  on  $M$  is a pair  $(U, \theta)$  where  $U \subset M$  open and  $\theta \in \Omega^1(U, \mathfrak{g})$  satisfying **regularity**

$$T_a M \xrightarrow{\theta_a} \mathfrak{g} \xrightarrow{pr} \mathfrak{g}/\mathfrak{h}$$

is an isomorphism  $\forall a \in U$ .

This is the analogue of a chart

**Definition 9.3.** A **Cartan atlas** is a collection of Cartan gauges  $\{(U_\alpha, \theta_\alpha)\}$  s.t

- $\bigcup_\alpha U_\alpha = M$
- on  $U_{\alpha\beta}$

$$\theta_\beta = \text{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha + h_{\alpha\beta}^* \vartheta_H$$

for some  $h_{\alpha\beta} : U_{\alpha\beta} \rightarrow H$ .

This is very analogous to atlases.

**Definition 9.4.** Two atlases are **equivalent** if their union is an atlas.

**Definition 9.5.** A **Cartan structure** on  $M$  is an equivalence class (equivalently maximal atlas) of Cartan atlases. A **Cartan geometry** is a manifold  $M$  together with a Cartan structure.

**Definition 9.6.** The **curvature** of a Cartan gauge  $(U, \theta)$  is  $\Omega \in \Omega^2(U, \mathfrak{g})$  given by

$$\Omega = d\theta + \frac{1}{2} [\theta, \theta]$$

If I have a Cartan atlas, I can ask how respective curvatures  $\Omega_\alpha$  change on overlaps.

**Lemma 9.7.** On  $U_{\alpha\beta}$

$$\Omega_\beta = \text{Ad}(h_{\alpha\beta}^{-1})\Omega_\alpha$$

*Proof.*

$$\begin{aligned} \theta_\beta &= \text{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha + h_{\alpha\beta}^* \vartheta_H \\ \Rightarrow d\theta_\beta + \frac{1}{2} [\theta_\beta, \theta_\beta] &= d \left( \underbrace{\text{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha}_{h_{\alpha\beta}^{-1}\theta_\alpha h_{\alpha\beta}} + h_{\alpha\beta}^* \vartheta_H \right) + \frac{1}{2} [\text{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha + h_{\alpha\beta}^* \vartheta_H, \text{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha + h_{\alpha\beta}^* \vartheta_H] \\ &= \text{Ad}(h_{\alpha\beta}^{-1})d\theta_\alpha - [\text{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha, h_{\alpha\beta}^* \vartheta_H] - \frac{1}{2} h_{\alpha\beta}^* [\vartheta_H, \vartheta_H] + \frac{1}{2} \text{Ad}(h_{\alpha\beta}^{-1}) [\theta_\alpha, \theta_\alpha] \\ &\quad + \frac{1}{2} [h_{\alpha\beta}^* \vartheta_H, h_{\alpha\beta}^* \vartheta_H] + [\text{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha, h_{\alpha\beta}^* \vartheta_H] \\ &= \text{Ad}(h_{\alpha\beta}^{-1}) \left( d\theta_\alpha + \frac{1}{2} [\theta_\alpha, \theta_\alpha] \right) \end{aligned}$$

□

Hence setting  $\Omega_\alpha = 0$  is an *extrinsic* statement of an atlas.

**Definition 9.8.** A Cartan structure is **flat** if  $\forall \alpha, \Omega_\alpha = 0$

**Example 9.9.** Flat Cartan structures:

- $G \rightarrow G/H$  with  $(U_\alpha, \sigma_\alpha^* \vartheta_G)$
- an open subset  $V \subset G/H$  as above.
- $\Gamma \subset G$  acting by covering transformations, locally like  $G/H$ .

**Definition 9.10.** A Klein geometry  $G/H$  has **kernel**  $K$ : the largest subgroup of  $H$  that is normal in  $G$ . If  $K = 1$  we say that  $G/H$  is **effective**. If  $K$  is discrete we say the geometry is **locally effective**.

**Lemma 9.11.** If  $K \neq 1$  then  $(G/K)/(H/K)$  is effective.

**Proposition 9.12.** If  $G/H$  is effective, and  $\exists k : U \rightarrow H$  s.t.  $\theta = \text{Ad}(k^{-1}) \cdot \theta + k^* \vartheta_H$ , then  $k = 1$ .

This means that, given a Cartan atlas  $\{(U_\alpha, \theta_\alpha)\}$  modelled on an effective  $G/H$ , then in overlaps  $U_{\alpha\beta}$ ,  $\theta_\beta = \text{Ad}(h_{\alpha\beta}^{-1}) \circ \theta_\alpha + h_{\alpha\beta}^* \vartheta_H$  for a unique  $h_{\alpha\beta} : U_{\alpha\beta} \rightarrow H$ . Indeed if  $\theta_\beta = \text{Ad}(\tilde{h}_{\alpha\beta}^{-1}) \circ \theta_\alpha + \tilde{h}_{\alpha\beta}^* \vartheta_H$ , then letting  $k = \tilde{h}_{\alpha\beta}^{-1} h_{\alpha\beta}$  we would have

$$\begin{aligned} \theta_\alpha &= \text{Ad}(\tilde{h}_{\beta\alpha}^{-1}) \circ \theta_\beta + \tilde{h}_{\beta\alpha}^* \vartheta_H \\ \Rightarrow \theta_\beta &= \text{Ad}(h_{\alpha\beta}^{-1}) \circ \left[ \text{Ad}(\tilde{h}_{\beta\alpha}^{-1}) \circ \theta_\beta + \tilde{h}_{\beta\alpha}^* \vartheta_H \right] + h_{\alpha\beta}^* \vartheta_H \\ &= \text{Ad}(k^{-1}) \circ \theta_\beta + \underbrace{\text{Ad}(h_{\alpha\beta}^{-1}) \circ \tilde{h}_{\beta\alpha}^* \vartheta_H + h_{\alpha\beta}^* \vartheta_H}_{k^* \vartheta_H} \end{aligned}$$

It follows from uniqueness then that  $\{h_{\alpha\beta} : U_{\alpha\beta} \rightarrow H\}$  defines a (Cech) cocycle. Therefore they are the transition functions of a principle  $H$ -bundle  $P \xrightarrow{\pi} M$ , where  $P = \sqcup_\alpha (\{\alpha\} \times U_\alpha \times H) / \sim$ ,  $(\alpha, a, h) \sim (\beta, a, h_{\alpha\beta}^{-1}(a)h)$ , and  $\pi(\alpha, a, h) = a$ . The right action is given by  $r_h[(\alpha, a, \tilde{h})] = [(\alpha, a, \tilde{h}h)]$ . This is well defined since the identification uses left multiplication.

Let  $X \in \mathfrak{h}$ . Then  $X^L \in \mathfrak{X}(H)$  is the corresponding LIVF. We extend it to  $U \times H$  as  $(0, X^L) \equiv \xi_X \in \mathfrak{X}(U \times H)$ . Since  $X^L$  is LI and the identifications involve left multiplication the vector fields  $\xi_X$  glue to give a well defined vector field  $\xi_X \in \mathfrak{X}(P)$ . We then have

**Lemma 9.13.** *Let  $r_h : P \rightarrow P$  denote the right action of  $h \in H$  on  $P$ . Then  $\forall X \in \mathfrak{h}$ ,  $(r_h)_* \xi_X = \xi_{\text{Ad}(h)^{-1}X}$ .*

*Proof.* It is sufficient to check locally on  $U \times H$ . Here  $r_h = \text{id} \times R_h$  where  $R_h : H \rightarrow H$  is right multiplication by  $h$ . Let  $L_h : H \rightarrow H$  be left multiplication and then on  $U \times H$  we have

$$\begin{aligned} (r_h)_* \xi_X &= (\text{id} \times R_h)_*(0, X^L) \\ &= (0, (R_h)_* X^L) \\ &= (0, (r_h)_*(L_{h^{-1}})_* X^L) \text{ since } X^L \text{ is LI} \\ &= (0, (\text{Ad}(h^{-1}) \cdot X)^L) \\ &= \xi_{\text{Ad}(h)^{-1}X} \end{aligned}$$

□

The Cartan atlas  $(U_\alpha, \theta_\alpha)$  does not first just give  $P \xrightarrow{\pi} M$ , but also a one-form  $\omega \in \Omega^1(P; \mathfrak{g})$  defined locally by

$$\begin{aligned} \omega : T_{(a,h)}(U_\alpha \times H) &\rightarrow T_a U_\alpha \times \mathfrak{h} \rightarrow \mathfrak{g} \\ (v, y) &\mapsto (v, \vartheta_H(y)) \mapsto \text{Ad}(h^{-1})\theta_\alpha(v) + \vartheta_H(y) \equiv \omega_\alpha(v, y) \end{aligned}$$

On overlaps, we also have  $\omega_\beta(v, y) = \text{Ad}(h^{-1})\theta_\beta(v) + \vartheta_H(y)$ . The transition function is then  $U_{\alpha\beta} \times H \xrightarrow{f_{\alpha\beta}} U_{\alpha\beta} \times H$  sending  $(a, h) \mapsto (a, h_{\alpha\beta}(a)^{-1}h)$ .

We will claim that the  $\omega_\alpha$  glue together properly to give a consistent  $\omega$ . To prove this we will need a preparatory lemma:

**Lemma 9.14.** *Let  $\mu : H \times H \rightarrow H$  and  $i : H \rightarrow H$  denote multiplication and inversion as groups maps on  $H$ . Letting  $\vartheta_H \in \Omega^1(H; \mathfrak{h})$  be the LI MC one-form we have*

$$\begin{aligned} \forall v \in T_{(h_1, h_2)}(H \times H), (\mu^* \vartheta_H)(v) &= \text{Ad}(h_2^{-1})\vartheta_H((pr_1)_* v) + \vartheta_H((pr_2)_* v) \\ \forall v \in T_h H, (i^* \vartheta_H)(v) &= -\text{Ad}(h)\vartheta_H(v) \end{aligned}$$

*Proof.* It is simpler notationally for matrix groups where  $\vartheta_H|_h = h^{-1}dh$ . Hence

$$i^* \vartheta_H|_h = h dh^{-1} = -h h^{-1} dh h^{-1} = -\text{Ad}(h) \vartheta_H|_h$$

Moreover we have

$$\mu^* \vartheta_H|_{(h_1, h_2)} = (h_1 h_2)^{-1} d(h_1 h_2) = h_2^{-1} h_1^{-2} dh_1 h_2 + h_2^{-1} dh_2 = \text{Ad}(h_2^{-1}) \vartheta_H|_{h_1} + \vartheta_H|_{h_2}$$

□

Now we are ready to state what we want:

**Proposition 9.15.** *The following diagram commutes:*

$$\begin{array}{ccc}
 T_a U_{\alpha\beta} \times T_h H & \xrightarrow{(f_{\alpha\beta})_*} & T_a U_{\alpha\beta} \times T_{h_{\alpha\beta}(a)^{-1}h} H \\
 & \searrow \omega_\alpha & \swarrow \omega_\beta \\
 & \mathfrak{g} &
 \end{array}$$

*Proof.* We notice that  $f_{\alpha\beta}(a, h) = (a, h_{\alpha\beta}(a)^{-1}h) = (\text{id} \circ pr_1, \mu \circ (i \circ h_{\alpha\beta} \circ pr_1 \times pr_2))(a, h)$ , so that if  $(v, y) \in T_a U_{\alpha\beta} \times T_h H$ ,  $(f_{\alpha\beta})_*(v, y) = (v, \mu_*(i_* \circ (h_{\alpha\beta})_* v, y)) \in T_a U_{\alpha\beta} \times T_{h_{\alpha\beta}(a)^{-1}h} H$ . Hence

$$\begin{aligned}
 (\omega_\beta \circ (f_{\alpha\beta})_*)(v, y) &= \omega_\beta(v, \mu_*(i_* \circ (h_{\alpha\beta})_* v, y)) \\
 &= \text{Ad}(h_{\alpha\beta}(a)^{-1}h)^{-1} \theta_\beta(v) + \vartheta_H(\mu_*(i_* \circ (h_{\alpha\beta})_* v, y))
 \end{aligned}$$

Using the lemma we have that

$$\begin{aligned}
 \vartheta_H(\mu_*(i_* \circ (h_{\alpha\beta})_* v, y)) &= (\mu^* \vartheta_H)(i_*(h_{\alpha\beta})_* v, y) \\
 &= \text{Ad}(h^{-1}) \vartheta_H(i_*(h_{\alpha\beta})_* v) + \vartheta_H(y) \\
 \vartheta_H(i_*(h_{\alpha\beta})_* v) &= (i^* \vartheta_H)(h_{\alpha\beta}^* v) \\
 &= -\text{Ad}(h_{\alpha\beta}(a))(h_{\alpha\beta}^* \vartheta_H)(v)
 \end{aligned}$$

Hence

$$\begin{aligned}
 (\omega_\beta \circ (f_{\alpha\beta})_*)(v, y) &= \text{Ad}(h)^{-1} \text{Ad}(h_{\alpha\beta}(a)) \theta_\beta(v) - \text{Ad}(h)^{-1} \text{Ad}(h_{\alpha\beta}(a))(h_{\alpha\beta}^* \vartheta_H)(v) + \vartheta_H(y) \\
 &= \text{Ad}(h)^{-1} \text{Ad}(h_{\alpha\beta}(a)) [\theta_\beta(v) - (h_{\alpha\beta}^* \vartheta_H)(v)] + \vartheta_H(y) \\
 &= \text{Ad}(h)^{-1} \circ \theta_\alpha(v) + \vartheta_H(y) \\
 &= \omega_\alpha(v, y)
 \end{aligned}$$

□

**Definition 9.16.** *The one-form  $\omega \in \Omega^1(P; \mathfrak{g})$  is called a **Cartan connection***

**Proposition 9.17.** *The Cartan connection  $\omega \in \Omega^1(P; \mathfrak{g})$  obeys the following:*

1.  $\forall p \in P, \omega_p : T_p P \rightarrow \mathfrak{g}$  is a vector space isomorphism
2.  $\forall h \in H, r_h^* \omega = \text{Ad}(h^{-1}) \circ \omega$
3.  $\forall X \in \mathfrak{h}, \omega(\xi_X) = X$

*Proof.* We may separate the proof:

1.  $\dim P = \dim H + \dim M = \dim \mathfrak{h} + \dim \mathfrak{g}/\mathfrak{h} = \dim \mathfrak{g}$ , so it suffices to show that  $\omega_p$  is injective. Now if  $(v, y) \in T_a U \times T_h H$  is such that  $\omega(v, y) = \text{Ad}(h^{-1})\theta(v) + \vartheta_H(y) = 0$ , we have  $\text{Ad}(h^{-1})\theta(v) = -\vartheta_H(y) \in \mathfrak{h}$  and hence  $\theta(v) \in \text{Ad}(h)\mathfrak{h} = \mathfrak{h} \Rightarrow pr_{\mathfrak{g}/\mathfrak{h}}\theta(v) = 0$ . By the regularity property of  $\theta$ ,  $v = 0$ . Hence  $\vartheta_H(y) = 0$ , but as  $\vartheta_H$  is injective, we have  $y = 0$
2. It is sufficient to check in a Cartan gauge  $(U, \theta)$ . Let  $(v, y) \in T_a U \times T_h H$ . Then for  $k \in H$ :

$$(r_k^* \omega)(v, y) = \omega(v, (R_k)_* y) = \text{Ad}(hk)^{-1} \circ \theta(v) + \vartheta_H((R_k)_* y)$$

and using  $R_k^* \vartheta_H = \text{Ad}(k^{-1}) \circ \vartheta_H$

$$\begin{aligned} (r_k^* \omega)(v, y) &= \text{Ad}(k^{-1}) \text{Ad}(h)^{-1} \theta(v) + \text{Ad}(k^{-1}) \vartheta_H(y) \\ &= \text{Ad}(k^{-1}) [\text{Ad}(h)^{-1} \theta(v) + \vartheta_H(y)] \\ &= \text{Ad}(k^{-1}) \omega(v, y) \end{aligned}$$

3. In a Cartan chart  $\xi_X = (0, X^L) \in \mathfrak{X}(U \times H)$ , hence

$$\omega(\xi_X) = \text{Ad}(h)^{-1} \theta(0) + \vartheta_H(X^L) = 0 + X = X$$

□

**Remark.** Properties 2 and 3 are reminiscent of an Ehresmann connection except that  $\omega$  takes values in  $\mathfrak{g}$  not  $\mathfrak{h}$ .

Notice that if  $\{(U_\alpha, \theta_\alpha)\}$  is a Cartan atlas trivialising  $P$ , then if  $s_\alpha : U_\alpha \rightarrow P|_{U_\alpha}$  are the canonical sections,  $s_\alpha(a) = [(a, e)]$ ,  $(s_\alpha^* \omega)(v) = \omega(v, 0) = \theta_\alpha(v)$ . So  $\theta_\alpha$  are the 'gauge fields' of the Cartan connection. Let  $\Omega = d\omega + \frac{1}{2} [\omega, \omega] \in \Omega^2(P; \mathfrak{g})$  denote the **curvature** of the Cartan connection. Then  $s_\alpha^* \Omega = d\theta_\alpha + \frac{1}{2} [\theta_\alpha, \theta_\alpha]$ . Hence bundle automorphisms of  $P$  (covering the identity) are the **gauge symmetries** of the Cartan geometry.

**Remark.**  $\omega$  parallelises  $P$ , just like  $\vartheta_G$  parallelises  $G$  in the Klein model. Given  $X \in \mathfrak{g}$  we get a vector field  $\xi_X \in \mathfrak{X}(P)$  defined by  $\xi_X|_p = \omega_p^{-1}(X)$ , but unlike the case of  $(G, \vartheta_G)$  this is not a Lie algebra morphism. This is despite that for  $X \in \mathfrak{h}, Y \in \mathfrak{g}$  we do have  $[\xi_X, \xi_Y] = \xi_{[X, Y]}$ . The curvature  $\omega$  is the obstruction to  $X \mapsto \xi_X$  defining a Lie algebra morphism  $\mathfrak{g} \rightarrow \mathfrak{X}(P)$ . To see this, calculate

$$\begin{aligned} \omega(\xi_{[X, Y]} - [\xi_X, \xi_Y]) &= [X, Y] + (d\omega(\xi_X, \xi_Y) - \xi_X \omega(\xi_Y) + \xi_Y \omega(\xi_X)) \\ &= [X, Y] + (d\Omega(\xi_X, \xi_Y) - [\omega(\xi_X), \omega(\xi_Y)]) + \xi_X Y - \xi_Y X \\ &= [X, Y] + \Omega(\xi_X, \xi_Y) - [X, Y] \\ &= \Omega(\xi_X, \xi_Y) \end{aligned}$$

We can now give the standard definition of a Cartan geometry modelled on a Klein geometry:

**Definition 9.18.** A **Cartan geometry**  $(P, \omega)$  on  $M$  modelled on  $G/H$  consists of the following:

1. a principal  $H$ -bundle  $P \rightarrow M$
2.  $\omega \in \Omega^1(P; \mathfrak{g})$  satisfying
  - (a)  $\forall p \in P \omega_p : T_p P \rightarrow \mathfrak{g}$  is a vector space isomorphism
  - (b)  $\forall h \in H, r_h^* \omega = \text{Ad}(h^{-1}) \omega$
  - (c)  $\forall X \in \mathfrak{h}, \omega(\xi_X) = X$

**Definition 9.19.** Let  $\Omega = d\omega + \frac{1}{2} [\omega, \omega] \in \Omega^2(P; \mathfrak{g})$  be the curvature of  $\omega$ . Then projection  $\text{pr}_{\mathfrak{g}/\mathfrak{h}} \circ \Omega \in \Omega^2(P; \mathfrak{g}/\mathfrak{h})$  is the **torsion** of  $\omega$ . The Cartan geometry is **torsion free** if  $\Omega \in \Omega^2(P; \mathfrak{h})$

**Lemma 9.20.** Let  $(P, \omega)$  be a Cartan geometry on  $M$  modelled on  $G/H$ . Let  $\psi : P \rightarrow H$  be a smooth and  $f : P \rightarrow P$  be such that  $f(p) = r_{\psi(p)}(p)$ . Then  $f^* \omega = \text{Ad}(\psi^{-1}) \omega + \psi^* \vartheta_H$  and  $f^* \Omega = \text{Ad}(\psi) \circ \Omega$ .

*Proof.* The expression for  $f^* \Omega$  follows from that of  $f^* \omega$ . To calculate  $f^* \omega$ , we work relative to a Cartan gauge  $(U, \theta)$  on  $U \times H$ . Then  $f : U \times H \rightarrow U \times H$  by  $f(a, h) = (a, h\psi(a, h))$  can be written as  $f =$



$(\text{id} \circ pr_1, \mu \circ (pr_2 \times \psi))$ . Hence if  $(v, y) \in T_a U \times T_h H$

$$\begin{aligned}
f_*(v, y) &= (v, \mu_*(y, \psi_*(v, y))) \in T_a U \times T_{h\psi(a, h)} H \\
\Rightarrow (f^* \omega)(v, y) &= \omega(v, \mu_*(y, \psi_*(v, y))) \\
&= \text{Ad}(h\psi(a, h))^{-1} \circ \theta(v) + \vartheta_H(\mu_*(y, \psi_*(v, y))) \\
&= \text{Ad}(\psi^{-1}) \circ \text{Ad}(h^{-1}) \circ \theta(v) + (\mu^* \vartheta_H)(y, \psi_*(v, y)) \\
&= \text{Ad}(\psi^{-1}) \circ \text{Ad}(h^{-1}) \circ \theta(v) + \text{Ad}(\psi^{-1}) \circ \vartheta_H(y) + \vartheta_H(\psi_*(v, y)) \\
&= \text{Ad}(\psi^{-1}) \circ [\text{Ad}(h^{-1}) \circ \theta(v) + \vartheta_H(y)] + (\psi^* \vartheta_H)(v, y) \\
&= [\text{Ad}(\psi^{-1}) \circ \omega + \psi^* \vartheta_H](v, y)
\end{aligned}$$

□

**Corollary 9.21.**  $\Omega$  is horizontal, i.e. if either  $u, v$  are tangent to the fibre,  $\Omega(u, v) = 0$ .

*Proof.* Let  $u, v \in T_p P$  and  $v$  tangent to the fibre. Let  $\psi : P \rightarrow H$  be any smooth map sending  $p \mapsto e$  s.t.  $(\psi_*)_p v = -\omega_p(v) \in \mathfrak{h}$ . define  $f : P \rightarrow P$  by  $f(q) = q \cdot \psi(q)$ . Then from the previous lemma we have that  $p \in P$

$$\begin{aligned}
f^* \omega &= \text{Ad}(\psi^{-1}) \omega + \psi^* \vartheta_H = \omega + \psi^* \vartheta_H \\
f^* \Omega &= \Omega
\end{aligned}$$

Hence

$$\begin{aligned}
\omega_p(f_* v) &= \omega_p(v) + \vartheta_H(\psi_* v) = \omega_p(v) - \omega_p(v) = 0 \\
\Rightarrow f_* v &= 0 \\
\Rightarrow \Omega(u, v) &= \Omega(f_* u, f_* v) = \Omega(f_* u, 0) = 0
\end{aligned}$$

□

It follows that  $\Omega$  defines a 2-form on  $TP / \ker \pi_* \cong \pi^* TM$ .

Note that each fibre  $F$  of  $P$  is identified with  $H$  up to left multiplication by some element of  $H$ . Since  $\vartheta_H$  is left-invariant, it defines a "Maurer-Cartan" form  $\vartheta_F$  on the fibre. The fact that  $\forall X \in \mathfrak{h}, \vartheta_F(\xi_X) = X$  shows that  $\vartheta_F = \omega|_F$ . It then follows that  $\Omega$  vanishes when restricted to any fibre. As such we can interpret a Cartan geometry  $(P, \omega)$  as deforming  $(G, \vartheta_G)$  in a way that fibrewise we still have  $(H, \vartheta_H)$ .

The tangent bundle of  $G/H$  is a vector bundle associated to  $G \rightarrow G/H$  via the linear isotropy representation  $\text{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \rightarrow GL(\mathfrak{g}/\mathfrak{h})$  s.t.  $T(G/H) \cong G \times_H \mathfrak{g}/\mathfrak{h}$ . In a similar way, the tangent bundle of a Cartan geometry  $(P, \omega)$  modelled on  $G/H$  is isomorphic to an associated vector bundle  $P \times_H \mathfrak{g}/\mathfrak{h}$ .

**Proposition 9.22.** Let  $(P, \omega)$  be a Cartan geometry on  $M$  modelled on  $G/H$ . There is a canonical bundle isomorphism  $\varphi : TM \xrightarrow{\cong} P \times_H \mathfrak{g}/\mathfrak{h}$  such that  $\forall p \in \pi^{-1}(x), \exists \varphi_p : T_x M \rightarrow \mathfrak{g}/\mathfrak{h}$  a  $H$ -equivariant vector space isomorphism s.t.  $\forall h \in H, \varphi_{p \cdot h} = \text{Ad}(h^{-1}) \circ \varphi_p$

*Proof.* Consider the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_p(F_x) & \longrightarrow & T_p P & \xrightarrow{(\pi_*)_p} & T_x M \longrightarrow 0 \\
& & \cong \downarrow \vartheta_H & & \cong \downarrow \omega & & \cong \downarrow \varphi_p \\
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{g}/\mathfrak{h} \longrightarrow 0
\end{array}$$

If  $v \in T_x M$ , we may write  $v = (\pi_*)_p(u) = (\pi_*)_{ph}((r_h)_*u)$  for some  $u \in T_p P$ . Thus

$$\begin{aligned}\varphi_{ph}(v) &= \varphi_{ph}((\pi_*)_{ph}((r_h)_*u)) \\ &= \rho(\omega_{ph}((r_h)_*u)) \\ &= \rho(\text{Ad}(h)^{-1} \circ \omega_p(u)) \\ &= \text{Ad}(h)^{-1}(\varphi_p((\pi_*)_p u)) \\ &= \text{Ad}(h)^{-1}\varphi_p(v)\end{aligned}$$

This allows us to define a bundle map

$$\begin{aligned}q : P \times \mathfrak{g} &\rightarrow TM \\ (p, X) &\mapsto (\pi(p), \varphi_p^{-1}(\rho(X)))\end{aligned}$$

Then

$$\begin{aligned}q(ph, \text{Ad}(h)^{-1}X) &= (\pi(ph), \varphi_{ph}^{-1}(\rho(\text{Ad}(h)^{-1}X))) \\ &= (\pi(p), (\text{Ad}(H)\varphi_{ph})^{-1}\rho(X)) \\ &= (\pi(p), \varphi_p^{-1}(\rho(X))) \\ &= q(p, X)\end{aligned}$$

Hence  $q$  induces  $\bar{q} : P \times_H \mathfrak{g}/\mathfrak{h} \rightarrow TM$ , which covers the identity and is a linear iso on the fibres.  $\square$

**Corollary 9.23.** *Let  $(P, \omega)$  be a Cartan geometry on  $M$  modelled on  $G/H$ . Then vector fields  $\xi \in \mathfrak{X}(M)$  are in bijective correspondence with functions  $\bar{\xi} : P \rightarrow \mathfrak{g}/\mathfrak{h}$  such that  $\forall p \in P, h \in H, \bar{\xi}(ph) = \text{Ad}(h^{-1}) \circ \bar{\xi}(p)$  by*

$$\xi \mapsto \bar{\xi} = \left\{ p \in P : \varphi_p(\xi_{\pi(p)}) \in \mathfrak{g}/\mathfrak{h} \right\}$$

**Definition 9.24.** *The **curvature function**  $K : P \rightarrow \text{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{g})$  of a Cartan connection  $\omega$  is defined by*

$$\forall p \in P, \forall X, Y \in \mathfrak{g}, K(p)(X, Y) \equiv \Omega_p(\omega_p^{-1}(X), \omega_p^{-1}(Y))$$

**Lemma 9.25.** *The curvature function is well defined and is  $H$ -equivariant, i.e.*

$$\forall h \in H, K(ph)(X, Y) = \text{Ad}(h^{-1})K(p)(\text{Ad}(h)X, \text{Ad}(h)Y)$$

*Proof.* Fix  $p \in P$  and let  $\tilde{X} = X + W, \tilde{Y} = Y + Z$  for some  $W, Z \in \mathfrak{h}$ . Then  $\Omega_p(\omega_p^{-1}(\tilde{X}), \omega_p^{-1}(\tilde{Y})) = \Omega_p(\omega_p^{-1}(X), \omega_p^{-1}(Y))$  since  $\omega_p^{-1}(Z), \omega_p^{-1}(W)$  are tangent to the fibres and  $\Omega$  is horizontal. Therefore  $K(p) \in \text{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{g})$ . The equivariance follows from the equivariance of  $\omega, \Omega$ .  $\square$

It follows that the curvature of a Cartan connection defines a **curvature section** of the bundle  $P \times_H \text{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{g})$ .

**Proposition 9.26.** *A Cartan connection is torsion free iff the curvature function takes values in  $\text{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{h}) \subset \text{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{g})$ .*

**Exercise 9.27.** *Show that  $K(p)(X, Y) = [X, Y] - \omega_p([\omega_p^{-1}X, \omega_p^{-1}Y])$*

**Lemma 9.28** (Bianchi identity).  $d\Omega = [\Omega, \omega]$

*Proof.* Follows *Mutatis Mutandis* as for Ehresmann connections.  $\square$

Let  $V$  be a vector space and  $f : P \rightarrow V$  a function. A Cartan connection  $\omega \in \Omega^1(P; \mathfrak{g})$  defines a universal covariant derivative as follows: if  $X \in \mathfrak{g}$  and if  $\xi_X = \omega^{-1}(X)$ , then  $\tilde{D}_X f \equiv \xi_X f$ . Since this is linear in  $X \in \mathfrak{g}$ , we get

$$\begin{aligned} \tilde{D} : \Omega^0(P; V) &\rightarrow \Omega^0(P; V \otimes \mathfrak{g}^*) \\ f &\mapsto \tilde{D}f \end{aligned}$$

where we define  $\tilde{D}f$  by  $(i_X)_* \tilde{D}f = \tilde{D}_X f$  for

$$\begin{aligned} i_X : V \otimes \mathfrak{g}^* &\rightarrow V \\ v \otimes \eta &\mapsto \eta(X)v \end{aligned}$$

**Definition 9.29.** Let  $\rho : H \rightarrow GL(V)$  be a representation. We define

$$\Omega^k(P; \rho) \equiv \{ \alpha \in \Omega^k(P; V) \mid \forall h \in H, r_h^* \alpha = \rho(h^{-1}) \circ \alpha \}$$

the  $k$ -forms on  $P$  transforming according to  $\rho$ .

**Proposition 9.30.**  $\tilde{D} : \Omega^0(P; \rho) \rightarrow \Omega^1(P; \rho) \cong \Omega^0(P; \rho \otimes \text{Ad}^*)$

*Proof.* Let  $p \in P$ ,  $X \in \mathfrak{g}$ ,  $f \in \Omega^0(P; \mathfrak{g})$ . Then

$$\begin{aligned} (i_X)_*(r_h^*(\tilde{D}f))(p) &= (i_X)_*(\tilde{D}f(ph)) \\ &= (\tilde{D}_X f)(ph) \\ &= \omega_{ph}^{-1}(X)f \end{aligned}$$

Now  $r_h^* \omega = \text{Ad}(h^{-1}) \circ \omega \Rightarrow \omega_{ph} \circ (r_h)_* = \text{Ad}(h^{-1}) \circ \omega_p \Rightarrow (r_{h^{-1}})_* \circ \omega_p^{-1} = \omega_{ph}^{-1} \circ \text{Ad}(h)$  so

$$(i_X)_*(r_h^*(\tilde{D}f))(p) = [(r_h)_* \omega_p^{-1}(\text{Ad}(h)X)] f$$

If  $Y \in \mathfrak{X}(P)$  we have

$$((r_h)_* Y)f = Y(r_h^* f) = Y(\rho(h^{-1}) \cdot f) = \rho(h^{-1})Yf$$

so taking  $Y = \omega_p^{-1}(\text{Ad}(h)X)$  yields

$$[(r_h)_* \omega_p^{-1}(\text{Ad}(h)X)] f = \rho(h^{-1})\omega_p^{-1}(\text{Ad}(h)X)f = \rho(h^{-1})\tilde{D}_{\text{Ad}(h)X} f$$

and so

$$(i_X)_*(r_h^*(\tilde{D}f))(p) = \rho(h^{-1})\tilde{D}_{\text{Ad}(h)X} f$$

$\square$

Even if  $(V, \rho)$  is irreducible,  $(V \otimes \mathfrak{g}^*, \rho \otimes \text{Ad}^*)$  need not be. Decomposing  $V \otimes \mathfrak{g}^*$  into irreducibles decomposes  $\tilde{D}$  and in this way we get 'famous' differential operators such as  $\partial, \bar{\partial}, \nabla \cdot, \nabla \times$ .

**Lemma 9.31.** Let  $X \in \mathfrak{h}$  and  $f \in \Omega^0(P; \mathfrak{g})$ . Then  $(i_X)_* \tilde{D}f = -\rho_*(X)f$  where  $\rho_* : \mathfrak{h} \rightarrow \text{End}(V)$  is the LA hom induced by  $\rho : H \rightarrow GL(V)$

*Proof.*

$$\begin{aligned}
(i_X)_*(\tilde{D}f)(p) &= \omega_p^{-1}(X)f \\
&= \left. \frac{d}{dt} f(pe^{tX}) \right|_{t=0} \\
&= \left. \frac{d}{dt} \rho(e^{-tX})f(p) \right|_{t=0} \\
&= -\rho_*(X)f(p)
\end{aligned}$$

□

## 9.1 Reductive Cartan geometries

Now assume that  $(P, \omega)$  is reductive, s.t.  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $\text{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$ . Then the Cartan connection decomposes as  $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{m}}$ , so does the Cartan gauge  $\theta = \theta_{\mathfrak{h}} + \theta_{\mathfrak{m}}$ , and so does  $\tilde{D} = \tilde{D}_{\mathfrak{h}} + \tilde{D}_{\mathfrak{m}}$ . If for  $X \in \mathfrak{h}$ ,  $\tilde{D}_X f = -\rho(X)f$ , then  $\tilde{D}_{\mathfrak{h}} = -\rho$ . As we will see below,  $\tilde{D}_{\mathfrak{m}}$  defines a Kozul connection on any associated vector bundle  $P \times_H V$ . It follows from the defining properties of a Cartan connection that  $\omega_{\mathfrak{h}} \in \Omega^1(P; \mathfrak{h})$  is the connection one-form for an Ehresmann connection on the principal  $H$ -bundle  $P \rightarrow M$ . In contrast, the component  $\omega_{\mathfrak{m}} \in \Omega^1(P; \mathfrak{m})$  satisfies

1. It is horizontal, i.e.  $\forall X \in \mathfrak{h}, \omega_{\mathfrak{m}}(\xi_X) = 0$
2.  $r_h^* \omega_{\mathfrak{m}} = \text{Ad}(h^{-1}) \circ \omega_{\mathfrak{m}}$

The above two mean that  $\omega_{\mathfrak{m}}$  induces a one-form on  $M$  with values in the associated vector bundle  $P \times_H \mathfrak{m}$ , which is isomorphic to  $TM$ . Thus  $\omega_{\mathfrak{m}}$  is a **soldering form** on  $P$ .

As  $\omega$  splits, so does  $\Omega = \Omega_{\mathfrak{h}} + \Omega_{\mathfrak{m}}$  where the structure equation  $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$  gives

$$\begin{aligned}
\Omega_{\mathfrak{h}} &= d\omega_{\mathfrak{h}} + \frac{1}{2}[\omega_{\mathfrak{h}}, \omega_{\mathfrak{h}}] + \frac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]_{\mathfrak{h}} \\
\Omega_{\mathfrak{m}} &= d\omega_{\mathfrak{m}} + [\omega_{\mathfrak{h}}, \omega_{\mathfrak{m}}] + \frac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]_{\mathfrak{m}}
\end{aligned}$$

Therefore, the  $\mathfrak{h}$ -component of the curvature of the Cartan connection is not necessarily the curvature of the Ehresmann connection, but receives a correction from the soldering form:

$$\Omega_{\mathfrak{h}}^{\text{Cartan}} = \Omega^{\text{Ehresmann}} + \frac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]_{\mathfrak{h}}$$

whereas the torsion of the Cartan connection is not necessarily the torsion of the affine connection defined by  $\omega_{\mathfrak{h}}$ :

$$\Theta^{\text{Cartan}} = \Omega_{\mathfrak{m}}^{\text{Cartan}} = \Theta + \frac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]_{\mathfrak{m}}$$

Let's now consider the universal covariant derivative  $\tilde{D} = \tilde{D}_{\mathfrak{h}} + \tilde{D}_{\mathfrak{m}}$ . The  $\mathfrak{m}$ -component defines a Kozul connection on any associated vector bundle  $E \equiv P \times_H V$  for  $(V, \rho)$  a representation of  $H$ . Indeed, let  $\psi : \Gamma(E) \rightarrow \Omega^0(P; \rho)$  be the  $C^\infty(M)$ -modules isomorphism. We define  $\nabla_{\zeta} : \Gamma(E) \rightarrow \Gamma(E)$  by the commutativity of the following square:

$$\begin{array}{ccc}
\Gamma(E) & \xrightarrow{\nabla_{\zeta}} & \Gamma(E) \\
\psi \downarrow \cong & & \cong \downarrow \psi \\
\Omega^0(P; \rho) & \xrightarrow{\tilde{\zeta}} & \Omega^0(P; \rho)
\end{array}$$

i.e.  $\psi(\nabla_{\zeta}s) = \tilde{\zeta}\psi(s)$ , where  $\tilde{\zeta}$  is the **horizontal lift** of  $\zeta$ , i.e the unique<sup>1</sup> vector field on  $P$  s.t.  $(\pi_*)_p \tilde{\zeta} = \zeta_{\pi(p)}$  and  $\omega_{\mathfrak{h}}(\tilde{\zeta}) = 0$

**Proposition 9.32.**  $\nabla$  defines a Kozul connection on  $E$

<sup>1</sup>Is it clear why this vector field is unique

*Proof.*  $\nabla_\zeta$  is  $\mathbb{R}$ -linear and if  $f \in C^\infty(M)$ ,  $(\pi^*f)\tilde{\zeta}$  is the horizontal lift of  $f\zeta$ , so we have  $\nabla_{f\zeta}s = f\nabla_\zeta s$ . Finally, to get that  $\nabla$  is a derivation, see

$$\begin{aligned}\psi(\nabla_\zeta(fs)) &= \tilde{\zeta}\psi(fs) = \tilde{\zeta}(\pi^*f\psi(s)) = \tilde{\zeta}(\pi^*f)\psi(s) + (\pi^*f)\tilde{\zeta}\psi(s) \\ &= \pi^*(\zeta f)\psi(s) + (\pi^*f)\psi(\nabla_\zeta s) = \psi((\zeta f)s) + \psi(f\nabla_\zeta s) \\ &= \psi((\zeta f)s + f\nabla_\zeta s)\end{aligned}$$

□

**Proposition 9.33.** *Let  $(U, \theta)$  be a gauge for a reductive Cartan geometry,  $\sigma : U \rightarrow P|_U$  the section such that  $\theta = \sigma^*\omega$ ,  $\zeta \in \mathfrak{X}(U)$ , and  $\phi = \sigma^*\Phi$  where  $\Phi \in \Omega^0(P; \rho)$ . Then*

$$\nabla_\zeta \phi \equiv \zeta(\phi) - \rho_*(\theta_\mathfrak{h}(\zeta))\phi$$

*is the expression of the covariant derivative of  $\Phi$  in the gauge  $(U, \theta)$ .*

## 9.2 Special geometries

We may define 'special geometries' via curvature constraints

**Lemma 9.34.** *Let  $V \subset \mathfrak{g}$  be the vector subspace spanned by the values of the curvature form  $\Omega$ . Then  $V$  is a  $H$ -submodule*

*Proof.* Let  $v = \Omega_p(\xi_p, \eta_p)$ . Then

$$\begin{aligned}\text{Ad}(h^{-1})v &= \text{Ad}(h^{-1})(\Omega_p(\xi_p, \eta_p)) \\ &= (r_h^*\Omega_p)(\xi_p, \eta_p) \\ &= \Omega_{ph}((r_h)_*\xi_p, (r_h)_*\eta_p)\end{aligned}$$

which is a value of  $\Omega$

□

In particular if  $V \subset \mathfrak{h}$  is s.t. the Cartan geometry is torsion-free, then  $V$  is an ideal. If the geometry is torsion-free and the action of  $H$  on  $\mathfrak{h}$  is irreducible, there are no special geometries arising from  $\mathfrak{g}$ -curvature conditions. However, the  $H$ -modules  $\text{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h})$  need not be irreducible and we can define special geometries by demanding that the curvature function  $K : P \rightarrow \text{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h})$  takes values in a  $H$ -submodule. If  $H$  is compact, then  $\text{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h})$  is fully reducible

**Example 9.35.**  $\mathfrak{g} = \mathfrak{so}_n \ltimes \mathbb{R}^n$  and  $\mathfrak{h} = \mathfrak{so}_n$ . Have  $\text{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}) = \text{Hom}(\Lambda^2\mathbb{R}^n, \mathfrak{so}_n)$ .

The subspace corresponding to those curvature functions obeying the (algebraic) Bianchi identity breaks up into three submodules: scalar, trace-free Ricci, and Weyl.

Cartan connections are special types of Ehresmann connections. Let  $P \rightarrow M$ ,  $G \rightarrow G/H$ , be principal  $H$ -bundles. There is an associated fibre bundle  $Q = P \times_H G$  where  $H$  acts on  $G$  by left multiplication. This is a (right) principal  $G$ -bundle and  $M$ , and we have a natural inclusion  $P \subset Q$  sending  $p \mapsto (p, e)$ . An Ehresmann connection on  $Q$  is a  $\mathfrak{g}$ -valued one-form and its restriction to  $P$  gives a candidate for a Cartan connection on  $P$ .

**Theorem 9.36.** *Let  $G/H$  be a Klein geometry and let  $P, Q$  be principal  $H, G$ -bundles respectively over a manifold  $M$ . Assume that  $\dim P = \dim G$  and  $\varphi : P \rightarrow Q$  is a  $H$ -bundle map. Then there is a bijection of sets*

$$\{\text{Ehresmann connections on } Q, \text{ kernels not } \varphi_*(TP)\} \xrightarrow{\varphi^*} \{\text{Cartan connections on } P\}$$

*Proof.* Let  $\varpi \in \Omega^1(Q; \mathfrak{g})$  be an Ehresmann connection s.t.  $\varpi_*(TP) \cap \ker \varpi = 0$ . It follows that  $\omega = \varphi^*\varpi \in \Omega^1(P; \mathfrak{g})$  with zero kernel. Since  $\dim P = \dim \mathfrak{g}$ ,  $\omega_p : T_p P \rightarrow \mathfrak{g}$  is injective and so an isomorphism.

Since  $\varphi : P \rightarrow Q$  is a  $H$ -bundle map,  $\forall X \in \mathfrak{h}$  the vector fields  $\xi_X$  on  $P$  and  $\zeta_X$  on  $Q$  are  $\varphi$ -related : i.e

$$\forall p \in P, (\varphi_*)_p \xi_X(p) = \zeta_X(\varphi(p))$$

Also,

$$r_h^* \omega = r_h^* \varphi^* \varpi = \varphi^* r_h^* \varpi = \varphi^* (\text{Ad}(h^{-1}) \circ \varpi) = \text{Ad}(h^{-1}) \circ \varphi^* \varpi = \text{Ad}(h^{-1}) \circ \omega$$

so  $\omega$  is a Cartan connection. Next we define a correspondence

$$\{\text{Cartan connections on } P\} \xrightarrow{j} \{\text{Ehresmann connections on } Q, \text{ kernels not } \varphi_*(TP)\}$$

Given a Cartan connection  $\omega$  on  $P$  we extend it to a form  $\varpi = j(\omega)$  on  $P \times G$  by

$$\varpi_{(p,g)} = \text{Ad}(g^{-1}) \circ \pi_P^* \omega_p + \pi_G^* \vartheta_G|_g$$

where  $\pi_{P/G} : P \times G \rightarrow P/G$  are the canonical projections. We notice that  $\forall X \in \mathfrak{g}$ ,  $\varpi(0, X^L) = X$ . Also, if  $i : P \rightarrow P \times G$  is the injection  $p \mapsto (p, e)$  then  $i^* \varpi = \omega$ . In particular,  $\varpi$  does not vanish on  $T(P \times \{e\})$ . Let  $\gamma \in G$  and consider  $\text{id} \times R_\gamma : P \times G \rightarrow P \times G$ :

$$\begin{aligned} (\text{id} \times R_\gamma)^* \varpi_{(p,g\gamma)} &= \varpi_{(p,g\gamma)} \circ (\text{id} \times R_\gamma)_* \\ &= (\text{Ad}(g\gamma)^{-1} \circ \pi_P^* \omega_p + \pi_G^* \vartheta_G) \circ (\text{id} \times R_\gamma)_* \\ &= \text{Ad}(g\gamma)^{-1} \circ \omega \circ (\pi_P)_* \circ (\text{id} \times R_\gamma)_* + \vartheta_G \circ (\pi_G)_* \circ (\text{id} \times R_\gamma)_* \\ &= \text{Ad}(g\gamma)^{-1} \circ \omega \circ (\pi_P)_* + \vartheta_G \circ (R_\gamma)_* \circ (\pi_G)_* \\ &= \text{Ad}(\gamma)^{-1} (\text{Ad}(g)^{-1} \circ \pi_P^* \omega + \pi_G^* \vartheta_G) \\ &= \text{Ad}(\gamma)^{-1} \circ \varpi_{(p,g)} \end{aligned}$$

We now check that  $\varpi$  is basic for  $P \times G \rightarrow P \times_H G$  which means that it is both horizontal and 'invariant'. The latter condition requires that for  $\alpha_h : P \times G \rightarrow P \times G$ ,  $(p, g) \mapsto (phh^{-1}g)$ , we have  $\alpha_h^* \varpi = \varpi$ . We calculate

$$\begin{aligned} (\alpha_h^* \varpi)_{(p,g)} &= \varpi_{(ph, h^{-1}g)} \circ (\alpha_h)_* \\ &= \text{Ad}(h^{-1}g)^{-1} \pi_P^* \omega \circ (\alpha_h)_* + \pi_G^* \vartheta_G \circ (\alpha_h)_* \\ &= \text{Ad}(h^{-1}g)^{-1} \omega \circ (\pi_P)_* \circ (\alpha_h)_* + \vartheta_G \circ (\pi_G)_* \circ (\alpha_h)_* \\ &= \text{Ad}(g^{-1}) \circ \text{Ad}(h) \circ \omega \circ (R_h)_* \circ (\pi_P)_* + \vartheta_G \circ (L_{h^{-1}})_* \circ (\pi_G)_* \\ &= \text{Ad}(g^{-1}) \circ \pi_P^* \omega + \pi_G^* \vartheta_G \quad (\text{as } R_h^* \omega = \text{Ad}(h)^{-1} \omega \text{ and } \vartheta_G \text{ is LI}) \\ &= \varpi_{(p,g)} \end{aligned}$$

To show  $\varpi$  is horizontal, let  $X \in \mathfrak{h}$  and  $\xi_X \in \mathfrak{X}(P \times G)$  corresponding to the right  $H$ -action on  $P \times G$ :

$$\begin{aligned} P \times G \times H &\rightarrow P \times G \\ (p, g, h) &\mapsto (ph, h^{-1}g) = ((\mu_P \times \mu_G) \circ (\text{id} \times \text{id} \times 1 \times \text{id}) \circ (\text{id} \times \Delta \times \text{id}) \circ \varrho)(p, g, h) \end{aligned}$$

where we have

$$\begin{aligned}\varrho : P \times G \times H &\rightarrow P \times H \times G \\ (p, g, h) &\mapsto (p, h, g)\end{aligned}$$

$$\begin{aligned}\text{id} \times \Delta \times \text{id} : P \times G \times H &\rightarrow P \times H \times H \times G \\ (p, h, g) &\mapsto (p, h, h, g)\end{aligned}$$

$$\begin{aligned}\text{id} \times \text{id} \times 1 \times \text{id} : P \times H \times H \times G &\rightarrow P \times H \times H \times G \\ (p, h, h, g) &\mapsto (p, h, h^{-1}, g)\end{aligned}$$

$$\begin{aligned}\mu_P \times \mu_G : P \times H \times H \times G &\rightarrow P \times G \\ (p, h, h^{-1}, g) &\mapsto (ph, h^{-1}g)\end{aligned}$$

Then

$$\begin{aligned}(\xi_X)_{(p,g)} &= ((\mu_P \times \mu_G) \circ (\text{id} \times \text{id} \times 1 \times \text{id}) \circ (\text{id} \times \Delta \times \text{id}) \circ \varrho)_{*,(p,g,e)}(0, 0, X) \\ &= (\mu_P \times \mu_G)_* \circ (\text{id} \times \text{id} \times 1 \times \text{id})_* \circ (\text{id} \times \Delta \times \text{id})_{*,(p,e,g)}(0, X, 0) \\ &= (\mu_P \times \mu_G)_* \circ (\text{id} \times \text{id} \times 1 \times \text{id})_{*,(p,e,e,g)}(0, X, X, 0) \\ &= (\mu_P \times \mu_G)_{*,(p,e,e,g)}(0, X, -X, 0) \\ &= (\mu_P)_{*,(p,e)}(0, X), (\mu_G)_{*,(e,g)}(-X, 0) \\ &= (\omega_p^{-1}(X), -(\vartheta_G)_g^{-1}(\text{Ad}(g^{-1})X)) \\ \Rightarrow \varpi_{(p,g)}(\xi_X) &= \varpi_{(p,g)}(\omega_p^{-1}(X), -(\vartheta_G)_g^{-1}(\text{Ad}(g^{-1})X)) \\ &= (\text{Ad}(g^{-1}) \cdot (\pi_P^* \circ \omega) + \pi_G^* \vartheta_G)(\omega_p^{-1}(X), -(\vartheta_G)_g^{-1}(\text{Ad}(g^{-1})X)) \\ &= \text{Ad}(g^{-1})X = \text{Ad}(g^{-1})X = 0\end{aligned}$$

Therefore  $\varpi$  descends to  $\varpi \in \Omega^1(P \times_H G, \mathfrak{g})$  and satisfies the properties of an Ehresmann connection which in addition obeys  $\ker \varpi \cap \varphi_*(TP) = 0$ .

Finally, we need to show that  $\varphi^*$  and  $j$  are mutual inverses:

$$\begin{aligned}\varphi^*(j(\omega_p)) &= \varphi^* \varpi_{(p,e)} = \text{Ad}(e)^{-1} \circ \varphi^* \pi_P^* \omega_p + \varphi^* \pi_G^* \vartheta_{Ge} \\ &= (\pi_P \circ \varphi)^* \omega_p + 0 \quad (\text{since } \pi_G \circ \varphi \text{ is constant}) = \omega_p\end{aligned}$$

shows that  $\varphi^* \circ j = \text{id}$ . To do the other direction, it suffices to show  $\varphi^*$  is injective. Now if  $\varphi^* \varpi = \varphi^* \varpi_2$  then  $\varpi_1, \varpi_2$  agree on the image  $\varphi_*(TP)$  and hence on all the right translations. But  $\varpi_1, \varpi_2$  agree on  $\xi_X$  and these two kinds of vectors span  $TQ$   $\square$

## References