

# Topics in Rings and Representation Theory - Kac Moody Algebras

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# 1 Introduction

A set of lecture notes on a masters course on Kac moody algebras

## 2 Groups, Algebras, and their Representations

### 2.1 Algebras

Throughout this course we will take  $k$  to be a field.

**Definition 2.1.** An *algebra* is a triple  $(A, m, i)$  of

- a  $k$ -vector space  $A$
- a linear map  $m : A \otimes A \rightarrow A$
- an element  $i : k \rightarrow A$

satisfying associativity and unitality.

**Notation.** For  $a, b \in A$  we will denote  $m(a, b) = a \cdot b$ .

**Remark.** Linearity of  $m$  gives distributivity of the multiplication over  $k$ .

**Proposition 2.2.** If a unit exists for  $(A, \cdot)$ , it is unique

*Proof.* Let  $1, 1' \in A$  be the units. Then

$$1 = 1 \cdot 1' = 1'$$

□

**Example 2.3.** Some examples of algebras are

- The base field  $k$
- polynomials over  $k$ ,  $k[X]$ .
- $\text{End}(V)$  where  $V$  is a vector space, with multiplication given by composition

**Example 2.4.** The **free algebra**  $k\langle x_1, \dots, x_n \rangle$  is the vector space consisting formally of all possible combinations of the  $x_i$  in order to make it a vector space, namely

$$k\langle x_1, \dots, x_n \rangle = \bigoplus_{m=0}^{\infty} k \cdot \prod_{1 \leq j_i \leq n} x_{j_1} \cdots x_{j_m}$$

**Example 2.5.** Given a group  $G$  we have the **group algebra**  $A \equiv kG$  with

- basis  $\{x_g \mid g \in G\}$
- multiplication  $x_g \cdot x_h = x_{gh}$
- unit  $x_{e_G}$

**Definition 2.6.**  $(A, \cdot)$  is **commutative** if  $\forall a, b \in A, a \cdot b = b \cdot a$ .

**Example 2.7.**  $kG$  is abelian iff  $G$  is abelian.

**Definition 2.8.** A homomorphism of algebras  $f : A \rightarrow B$  is a linear map of vector spaces compatible with  $\cdot$  s.t.

- $\forall a, b \in A, f(a \cdot b) = f(a) \cdot f(b)$
- $(f(1_A) = 1_B)$

## 2.2 Representations

**Definition 2.9.** A **representation** of  $(A, \cdot)$  is a vector space  $V$  with  $\rho : A \rightarrow \text{End}(V)$  a homomorphism of algebras.

**Notation.** We will often, for simplicity, abuse notation and write for  $a \in A, v \in V$

$$\rho(a)(v) = a \cdot v$$

**Remark.** A representation is also call a **left A-module**. A right  $A$ -module has  $\sigma : A \rightarrow \text{End}(V)$  an antihomomorphism. We define an algebra  $(A^{op}, m^{op})$  s.t.  $A^{op} = A, \forall a, b \in A, m^{op}(a, b) = m(b, a)$ . We can then say that a right  $A$ -module is a representation of  $A^{op}$ .

**Example 2.10.** We have a few standard examples of reps:

- $V = 0$
- $V = A$  and for  $a \in A, \rho : a \mapsto \rho(a)$  s.t.  $\forall b \in A, \rho(a)(b) = m(a, b)$ . This is called the **regular rep**.
- $A = k$ , then any rep is just a vector space over  $k$
- If  $A = k\langle x_1, \dots, x_n \rangle$  then a rep is a vector space with  $\rho(x_i) \in \text{End}(V)$  specified.

**Definition 2.11.** Given two representations  $V_1, V_2$ , the **direct sum** representation  $V_1 \oplus V_2$  is given with

$$\rho_{V_1 \oplus V_2}(a)(v_1 + v_2) = \rho_{V_1}(a)(v_1) + \rho_{V_2}(a)(v_2)$$

for  $a \in A, v_i \in V_i$ .

**Definition 2.12.** A **subrepresentation** is subspace  $W \subset V$  s.t.  $\forall a \in A, \rho(a)(W) \subset W$ .

**Definition 2.13.** Given  $v \in V$  the **minimal subrep** containing  $v$  is

$$A \cdot v = \{w \in V \mid \exists a \in A, w = a \cdot v\}$$

**Definition 2.14.** A rep is **irreducible** if the only subreps are  $W = 0, V$

**Definition 2.15.** Let  $V_1, V_2$  be reps of  $A$ . Then a homomorphism of reps (an **intertwiner**) is linear map  $\phi : V_1 \rightarrow V_2$  s.t.  $\forall v \in V_1, a \in A$ , with  $\rho_i : A \rightarrow \text{End}(V_i)$  we have

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_2 \\ \rho_1(a) \downarrow & & \downarrow \rho_2(a) \\ V_1 & \xrightarrow{\phi} & V_2 \end{array}$$

commutes, i.e.  $\phi(a \cdot v) = a \cdot \phi(v)$

**Proposition 2.16.** Let  $f : V \rightarrow W$  be an intertwiner. Then

- $\ker f \subset V$  is a subrep
- $\text{Im } f \subset W$  is a subrep

**Lemma 2.17** (Schur). Let  $V_1, V_2$  be two  $A$ -reps and let  $f : V_1 \rightarrow V_2$  be a non-zero intertwiner. Then

- $V_1$  irreducible  $\Rightarrow f$  is injective
- $V_2$  irreducible  $\Rightarrow f$  is surjective

**Definition 2.18.** A representation is **indecomposable** is when  $V = V_1 \oplus V_2$ , either  $V_1 = 0$  or  $V_2 = 0$

**Proposition 2.19.** Any irreducible rep is indecomposable

*Proof.*  $V_1, V_2$  are subreps of  $V_1 \oplus V_2$ . □

**Remark.** The converse to the above is not true,

**Aside.** Coming from the workshop, we have some points that we want to have made clear in our mind:

1. Rep theory is linear algebra. If  $f : V \rightarrow W$  is an intertwiner, the condition that for  $a \in A, v \in V$   $f(a \cdot v) = a \cdot f(v)$ , is essentially saying that  $f$  is  $A$ -linear. We now have a correspondence

$A$ -linear	$k$ -linear
Irreducible reps	eigenspaces
indecomposable reps	generalised eigenspaces

### 3 Lie algebras

#### 3.1 preliminaries

**Definition 3.1.** A *Lie algebra* is a vector space  $\mathfrak{g}$  endowed with a bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying

- *antisymmetry*:  $\forall x \in \mathfrak{g}, [x, x] = 0$
- *Jacobi identity*:  $\forall x, y, z \in \mathfrak{g}, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

**Example 3.2.** Given an associative algebra  $A$ , we can make  $A$  a Lie algebra using

$$[a, b] = ab - ba$$

for  $a, b \in A$ .

**Example 3.3.** Let  $\mathfrak{sl}_n(k) = \{X \in M_n(k) \mid \text{Tr}(X) = 0\}$ . Then as  $\text{Tr}(XY) = \text{Tr}(YX)$  we have a bracket given by the commutator

$$[X, Y] = XY - YX$$

This gives a Lie algebra structure to  $\mathfrak{sl}_n(k)$  which is not inherited from matrix multiplication, as  $\mathfrak{sl}_n(k)$  is not closed under multiplication, so is not an associative algebra. To motivate looking at such a vector space, note that

$$\text{Tr}(X) = 0 \Rightarrow \det \exp(X) = e^{\text{Tr}(X)} = 1$$

and  $SL_n(k) = \{Y \in M_n(k) \mid \det(Y) = 1\}$  has a natural operation of matrix multiplication, as  $\det(XY) = \det(X)\det(Y)$ .

#### 3.2 Universal Enveloping Algebras

**Definition 3.4.** Let  $\{x_i\}$  be a basis of  $\mathfrak{g}$  a Lie algebra and suppose the bracket is specified by

$$[x_i, x_j] = \sum_k c_{ij}^k x_k$$

for some **structure constants**  $c_{ij}^k$ . Then the **universal enveloping algebra**  $\mathcal{U}_{\mathfrak{g}}$  is the associative algebra generated by the  $x_i$  with the relations

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k$$

i.e.

$$\mathcal{U}_{\mathfrak{g}} = k \langle x_1, \dots, x_n \rangle / \left\langle x_i x_j - x_j x_i - \sum_k c_{ij}^k x_k \right\rangle$$

We get the map

$$\begin{aligned} \iota : \mathfrak{g} &\rightarrow \mathcal{U}_{\mathfrak{g}} \\ x_i &\mapsto x_i \end{aligned}$$

The above definition involves a choice of basis of  $\mathfrak{g}$ . In general we want to remove this to give a universal property of  $\mathcal{U}_{\mathfrak{g}}$ .

**Proposition 3.5.** *For any associative algebra  $A$ , and Lie algebra map  $\mathfrak{g} \xrightarrow{f} A^{Lie}$ , there exists a unique map of associative algebras  $\mathcal{U}_{\mathfrak{g}} \xrightarrow{\mathcal{U}(f)} A$  s.t.*

$$\begin{array}{ccc} & \mathcal{U}_{\mathfrak{g}} & \\ \nearrow \iota & \downarrow \mathcal{U}(f) & \\ \mathfrak{g} & \xrightarrow{f} & A \end{array}$$

commutes

**Exercise 3.6.** *Prove that  $\mathcal{U}_{\mathfrak{g}}$  is uniquely defined (up to isomorphism) by this universal property.*

*Proof.* Consider

$$\mathcal{U}_{\mathfrak{g}} = \left[ \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} \right] / \langle x \otimes y - y \otimes x - [x, y] \rangle$$

□

**Proposition 3.7.** *If  $\mathcal{U}_{\mathfrak{g}}$  satisfies the universal property then there is a bijection*

$$\{\text{rep of Lie algebra } \mathfrak{g}\} \leftrightarrow \{\text{reps of associative algebra } \mathcal{U}_{\mathfrak{g}}\}$$

### 3.2.1 Construction

**Definition 3.8.** *Recall that for a vector space  $V$  over  $k$ , the **tensor algebra** is*

$$T(V) \equiv \bigoplus_{n \geq 0} V^{\otimes n}$$

where  $V^{\otimes 0} = k$ . It comes with the map

$$\begin{aligned} V^{\otimes n} \times V^{\otimes m} &\rightarrow V^{\otimes(n+m)} \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

**Definition 3.9.** *The **symmetric algebra** is*

$$S(V) \equiv T(V) / \langle v \otimes w - w \otimes v \mid v, w \in V \rangle$$

**Definition 3.10.** *Given a Lie algebra  $\mathfrak{g}$ , the **universal enveloping algebra** is*

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$$

**Theorem 3.11** (Poincare-Birkhoff). *We have the following two properties:*

1.  $\mathcal{U}(\mathfrak{g})$  satisfies the universal property

2. As a vector space,  $\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$ .

**Remark.** If  $\{x_i \mid i = 1, \dots, n\} \subset \mathfrak{g}$  is a basis, then the set of ordered monomials  $x_1^{i_1} \cdots x_n^{i_n} = x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$  is a basis of  $\mathcal{U}(\mathfrak{g})$ .

**Remark.** Note that in  $\mathcal{U}(\mathfrak{g})$ , by our quotient it must be that

$$x_i x_j - x_j x_i = [x_i, x_j]$$

which is what we wanted to see.

**Example 3.12.** If  $\mathfrak{g}$  is abelian, then  $\forall x, y \in \mathfrak{g}$ ,  $[x, y] = 0$  and we see by definition  $\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$ .

### 3.3 Representations

#### 3.3.1 Simple Lie algebra

**Definition 3.13.** An *ideal* of a Lie algebra  $\mathfrak{g}$  is a subspace  $\mathfrak{g}'$  s.t.  $[\mathfrak{g}', \mathfrak{g}] \subset \mathfrak{g}'$

**Definition 3.14.**  $\mathfrak{g}$  is *simple* if the only ideals of  $\mathfrak{g}$  are  $0, \mathfrak{g}$ .  $\mathfrak{g}$  is *semi-simple* if it is a direct sum of simple Lie algebras.

**Remark.**  $\mathfrak{g} \odot \mathfrak{g}$  by  $x \cdot y = [x, y]$  This is the **adjoint rep**. Then we have ideals of  $\mathfrak{g}$  correspond to subreps of the adjoint rep.

**Example 3.15.** Consider  $\mathfrak{gl}_n(k) = M_n(k)^{\text{Lie}}$ . Recall  $\mathfrak{sl}_n(k) \subset \mathfrak{gl}_n(k)$  is the set of traceless matrices. This is an ideal so  $\mathfrak{gl}_n(k)$  is non-simple.

**Exercise 3.16.** Prove  $\mathfrak{sl}_n(k)$  is simple.

**Theorem 3.17** (Weyl complete reducibility). Let  $\mathfrak{g}$  be a finite dimensional semi-simple Lie algebra and  $V \in \text{Rep}(\mathfrak{g})$ . If  $W \subseteq V$  is a subrepresentation, then  $\exists W' \subseteq V$  s.t  $V \cong W \oplus W'$  as representations.

#### 3.3.2 Classification of complex f.d simple Lie algebras

- $(A_n, n \geq 1)$ :  $\mathfrak{sl}_{n+1}(\mathbb{C}) = \{\text{Tr}(X) = 0\} \subset \mathfrak{gl}_n$
- $(B_n, n \geq 2)$ :  $\mathfrak{so}_{2n+1}(\mathbb{C}) = \{\text{Tr}(X) = 0, X^T + X = 0\} \subset \mathfrak{gl}_{2n+1}$
- $(C_n, n \geq 3)$ :  $\mathfrak{sp}_n(\mathbb{C}) = \{J_n X = X^T J_n\} \subset \mathfrak{gl}_{2n}$
- $(D_n, n \geq 4)$ :  $\mathfrak{so}_{2n}(\mathbb{C}) = \{\text{Tr}(X) = 0, X^T + X = 0\} \subset \mathfrak{gl}_{2n}$
- Exceptionals,  $E_6, E_7, E_8, F_4, G_2$ , dimensions 52, 133, 248, 52, 14.

**Remark.** Suppose  $\mathfrak{g}$  is simple. Take  $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  a morphism of Lie algebras. If  $\pi \neq 0$ , then as  $\ker \pi \subsetneq \mathfrak{g}$  is an ideal it must be the case that  $\ker \pi = 0$

Now if we define  $\mathfrak{gl}_n = \text{End}_i(k^n)^{\text{Lie}}$ , this has a basis  $\{E_{ab} = (\delta_{ia} \delta_{jb})_{i,j=1}^n\}$  called the **elementary matrices**. These obey

$$\begin{aligned} E_{ij} E_{kl} &= \delta_{jk} E_{il} \\ \Rightarrow [E_{ij}, E_{kl}] &= \delta_{jk} E_{il} - \delta_{il} E_{jk} \end{aligned}$$

### 3.4 $\mathfrak{sl}(n)$

**Example 3.18.** Consider  $\mathfrak{sl}(2)$ . Taking  $n = 2$ , we have the basis

$$\begin{aligned} e &= E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f &= E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &= E_{11} - E_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

These get the commutation relations

$$\begin{aligned} [h, e] &= 2e \\ [h, f] &= -2f \\ [e, f] &= h \end{aligned}$$

More generally, we see  $\dim_{\mathbb{C}}(\mathfrak{sl}(n)) = n^2 - 1$  by considering the trace condition, but has only  $3(n - 1)$  generators

$$\begin{aligned} e_i &= E_{i,i+1} \\ f_i &= E_{i+1,i} \\ h_i &= E_{ii} - E_{i+1,i+1} \end{aligned}$$

for  $i = 1, \dots, n - 1$ .

**Exercise 3.19.** Moreover we have

$$\begin{aligned} [e_i, e_{i+2}] &= 0 \\ [e_i, [e_i, e_{i+1}]] &= 0 \end{aligned}$$

These generate as

$$[e_i, e_{i+1}] = E_{i,i+2}$$

and this can be iterated to get all upper triangular matrices, likewise for lower triangular with  $f$  and diagonal with all. Explicitly

$$\begin{aligned} [h_i, e_j] &= a_{ji} e_j \\ [h_i, f_j] &= -a_{ji} f_j \\ [e_i, f_j] &= \delta_{ij} h_i \end{aligned}$$

where

$$a_{ij} = \begin{cases} 2 & |i - j| = 0 \\ -1 & |i - j| = 1 \\ 0 & |i - j| > 1 \end{cases}$$

We call  $A = (a_{ij})$  the **Cartan matrix**.



**Theorem 3.20** (Serre). *If  $\mathfrak{g}$  is a finite dimensional simple Lie algebra over  $\mathbb{C}$  then  $\mathfrak{g}$  has a similar presentation.*

**Proposition 3.21.**  *$A$  has the following properties:*

- $A \in M_n(\mathbb{Z})$
- $\forall i \neq j, a_{ii} = 2 \text{ and } a_{ij} \leq 0$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$
- $A$  is indecomposable
- $\det(A) \neq 0$  and  $A$  is positive definite.

## 4 Kac-Moody Algebras

**Idea.** *We can try to reverse engineer the Cartan matrix, to generalise it and then assign a Lie algebra  $\mathfrak{g}(A)$  to the resulting matrix  $A$ .*

**Definition 4.1.** *A **realisation** of  $A$  is a triple  $(\mathfrak{h}, \Pi^\vee, \Pi)$  where*

- $\mathfrak{h}$  is a vector space
- $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subseteq \mathfrak{h}$
- $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$

*s.t.*

- $\Pi^\vee$  is a linearly independent set
- $\Pi$  is a linearly independent set
- $\alpha_i(\alpha_j^\vee) = a_{ji}$

*We call  $\Pi$  the **simple roots**, and  $\Pi^\vee$  the **simple coroots***

**Exercise 4.2.** *Show the following results:*

- *If  $(\mathfrak{h}, \Pi^\vee, \Pi)$  is a realisation,  $\dim \mathfrak{h} \geq 2n - \text{rank}(A)$*
- *A **minimal realisation** (i.e  $\dim \mathfrak{h} = 2n - \text{rank}(A)$ ) always exists*

**Definition 4.3.** *A morphism  $(\mathfrak{h}, \Pi^\vee, \Pi) \rightarrow (\mathfrak{h}', (\Pi')^\vee, \Pi')$  is*

- $\phi : \mathfrak{h} \rightarrow \mathfrak{h}'$
- $\phi(\Pi^\vee) = (\Pi')^\vee$
- $\phi(\Pi) = \Pi'$

**Proposition 4.4.** *For any  $A$ ,  $\exists!$  minimal realisation  $\Sigma$  up to isomorphism.*

**Definition 4.5.** Let  $(\mathfrak{h}, \Pi^\vee, \Pi)$  be a realisation of  $A$ . Then  $\tilde{\mathfrak{g}}(A)$  is the Lie algebra with generators  $e_i, f_i$  for  $i = 1, \dots, n$  containing  $\mathfrak{h}$  s.t.

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} \alpha_i^\vee \in \mathfrak{h} \\ \forall h \in \mathfrak{h}, [h, e_i] &= \alpha_i(h) e_i \\ \forall h \in \mathfrak{h}, [h, f_i] &= -\alpha_i(h) f_i \\ \forall h, h' \in \mathfrak{h}, [h, h'] &= 0 \end{aligned}$$

**Example 4.6.** If  $A = [2]$ , we can follow the procedure and find  $\tilde{\mathfrak{g}}(A) = \mathfrak{sl}(2)$ . To see we see  $\dim \mathfrak{h} = 1$  for a minimal realisation, so we only need  $\alpha^\vee \in \mathfrak{h}, \alpha \in \mathfrak{h}^*$  s.t.  $\alpha(\alpha^\vee) = 2$ . With this  $\mathfrak{h} = \text{Span}\{\alpha^\vee\}$  We then need  $e, f$  to satisfy

$$\begin{aligned} [e, f] &= \alpha^\vee \\ [\alpha^\vee, e] &= \alpha(\alpha^\vee) e = 2e \\ [\alpha^\vee, f] &= -\alpha(\alpha^\vee) f = -2f \end{aligned}$$

This is just  $\mathfrak{sl}_2$  if we relabel  $h = \alpha^\vee$ .

**Idea.**  $\tilde{\mathfrak{g}}(A)$  is still currently too large, for example as the  $e_i$  are not yet related, and we want to try make it look like  $\mathfrak{sl}_n$ , i.e maybe simple. Hence we want to consider all ideals of the form

$$\text{trivial ideals} = \{r \subset \tilde{\mathfrak{g}}(A) \text{ ideals} \mid r \cap \mathfrak{h} = 0\}$$

and let

$$r_{\max} = \sum_{r \in \text{trivial}} r$$

and quotient by this to go

$$A \rightarrow (\mathfrak{h}, \Pi^\vee, \Pi) \rightarrow \tilde{\mathfrak{g}}(A) \rightarrow \mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/r_{\max}$$

**Definition 4.7.** We define

- $Q = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i$  the **root lattice**
- $Q \supset Q_+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$  the **positive root lattice**
- $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i^\vee$  the **coroot lattice**
- $Q^\vee \supset Q_+^\vee = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i^\vee$  the **positive coroot lattice**

Note that the relations required for  $\tilde{\mathfrak{g}}(A)$  give us all the commutators we need, e.g

$$\begin{aligned} [h, [e_1, e_2]] &= -[e_2, [h, e_1]] - [e_1, [e_2, h]] \\ &= \alpha_1(h) [e_2, e_1] + \alpha_2(h) [e_1, e_2] \\ &= (\alpha_1 + \alpha_2)(h) [e_1, e_2] \end{aligned}$$

**Definition 4.8.** Now for  $\alpha \in Q$  we can define

$$\tilde{\mathfrak{g}}_\alpha \equiv \{x \in \tilde{\mathfrak{g}}(\Sigma) \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$$

**Example 4.9.** We can consider examples of this:

- $\mathfrak{h} \supset \tilde{\mathfrak{g}}_0 = \mathfrak{h}$ .
- $\tilde{\mathfrak{g}}_{\alpha_i} = \mathbb{C}e_i$
- $\tilde{\mathfrak{g}}_{-\alpha_i} = \mathbb{C}f_i$
- $\tilde{\mathfrak{g}}_{\alpha_1 - \alpha_2} = 0$

We can then also state the following

**Theorem 4.10.** We have the following

1. As a vector space,  $\tilde{\mathfrak{g}}(\Sigma) = \tilde{n}_+ \oplus \mathfrak{h} \oplus \tilde{n}_-$  where  $\tilde{n}_+$  is the free Lie algebra generated by the  $e_i$ , and  $\tilde{n}_-$  by the  $f_i$ .
2. We have

$$\begin{aligned}\tilde{n}_+ &= \bigoplus_{\alpha \in Q_+ \setminus 0} \tilde{\mathfrak{g}}_\alpha \\ \tilde{n}_- &= \bigoplus_{\alpha \in Q_+ \setminus 0} \tilde{\mathfrak{g}}_{-\alpha}\end{aligned}$$

3.  $[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] \subset \tilde{\mathfrak{g}}_{\alpha+\beta} \Rightarrow \tilde{\mathfrak{g}}(\Sigma)$  is  $Q$ -graded.

Now

**Lemma 4.11.** 1.  $I \subset \tilde{\mathfrak{g}}(\Sigma)$  is an ideal then  $I = \bigoplus_{\alpha \in Q} (I \cap \tilde{\mathfrak{g}}_\alpha)$

2.  $\exists!$  maximal ideal  $r \subseteq \tilde{\mathfrak{g}}(\Sigma)$  s.t.  $r \cap \mathfrak{h} = 0$

3.  $r = r_+ \oplus r_-$  with  $r_\pm = r \cap \tilde{n}_\pm$

**Definition 4.12.** The **Kac-Moody** Lie algebra associated to  $\Sigma$  is  $\mathfrak{g}(\Sigma) \equiv \tilde{\mathfrak{g}}(\Sigma)/_r$

**Remark.** As the minimal realisation is unique (up to iso) we have no problem notationally using either  $\mathfrak{g}(A)$  or  $\mathfrak{g}(\Sigma)$ .

**Example 4.13.**  $\mathfrak{sl}(2)$  is simple and so when  $A = [2]$ ,  $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A) = \mathfrak{sl}(2)$ .

**Exercise 4.14.** Show that if  $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$  we get  $\mathfrak{g}(A) = \mathfrak{sl}_3$ .

*Proof.*  $\text{rank}(A) = 2$ , so the dimension of the minimal realisation is  $2 \times 2 - 2 = 2$ . Hence  $\mathfrak{h} = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$ . The corresponding  $\tilde{\mathfrak{g}}(A)$  gives us all the generators we want letting  $h_i = \alpha_i^\vee$ .  $\square$

**Definition 4.15.** Let  $\tilde{w} : \tilde{\mathfrak{g}}(A) \rightarrow \tilde{\mathfrak{g}}(A)$  given by

$$\begin{aligned}\tilde{w}(e_i) &= f_i \\ \tilde{w}(f_i) &= e_i \\ \tilde{w}(h) &= -h\end{aligned}$$

This descends to a map  $w : \mathfrak{g}(A) \rightarrow \mathfrak{g}(A)$  called the **Chevalley involution**.

The next result is useful for making calculations in  $\mathfrak{g}(\Sigma)$ :

**Lemma 4.16.** *We have the following two classifications*

1. For  $x \in n_+$ ,  $x = 0 \Leftrightarrow \forall i, [f_i, x] = 0$
2. For  $x \in n_-$ ,  $x = 0 \Leftrightarrow \forall i, [e_i, x] = 0$

*Proof.* Set  $\mathfrak{g}_1 = \bigoplus_{i=1}^n \mathbb{C}e_i \subset n_+$ . Then define the vector space

$$J_x \equiv \sum_{k \geq 0} \text{ad}(\mathfrak{g}_1)^k(x) \ni x$$

Then we can note

- $[n_+, J_x] \subset J_x$
- $[\mathfrak{h}, J_x] \subset J_x$

and further

Claim:  $[f_i, \text{ad}(\mathfrak{g}_1)^k(x)] \subset J_x$ . We can show this by induction. Certainly if we have assumed  $[f_i, x] = 0$ , then  $[f_i, x] \in J_x$ . Now for  $a \in \mathfrak{g}_1, b \in \text{ad}(\mathfrak{g}_1)^{k-1}(x)$  we have

$$[f_i, [a, b]] = \left[ \underbrace{[f_i, a]}_{\in \mathfrak{h}}, b \right] + \left[ a, \underbrace{[f_i, b]}_{\in J_x} \right]$$

So we have that  $J_x$  is an ideal and that  $J_x \cap \mathfrak{h} = 0$ , so  $J_x = 0$ . □

**Definition 4.17.** Define  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(\Sigma) \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$  in analogy to  $\tilde{\mathfrak{g}}_\alpha$ .

**Remark.** We have the  $\forall \alpha \in Q_+ \setminus 0$ ,

- $\tilde{\mathfrak{g}}_{\pm\alpha} \neq 0$
- $\dim \tilde{\mathfrak{g}}_{\pm\alpha} < \infty$
- If we define  $ht(\alpha) = \sum_i k_i$  for  $\alpha = \sum_i k_i \alpha_i$  then  $\dim \mathfrak{g}_{\pm\alpha} \leq n^{|ht(\alpha)|}$

**Definition 4.18.** We call  $R = \{\alpha \in Q \setminus 0 \mid \mathfrak{g}_\alpha \neq 0\} \subset Q$  the **set of roots**

**Proposition 4.19.** We have

1.  $\mathfrak{g}(\Sigma) = \bigoplus_{\alpha \in Q_+ \setminus 0} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in Q_+ \setminus 0} \mathfrak{g}_{-\alpha}$ .
2.  $R = R_+ \cup R_-$  where  $R_\pm = R \cap (\pm Q_\pm)$ .

**Exercise 4.20.** We can show

$$\begin{aligned} A = (2) & \Rightarrow R = \{\alpha_1\} \cup \{-\alpha_1\} \\ A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \Rightarrow R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, R_- = -R_+ \\ A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} & \Rightarrow |R| = \infty \end{aligned}$$

## 4.1 Bilinear forms on $\mathfrak{g}(A)$

Sometimes  $\mathfrak{g}(\Sigma)$  has a non-degenerate, symmetric, invariant bilinear form

$$(\cdot, \cdot) : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$$

i.e.

- $\ker(\cdot, \cdot) = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g}, (x, y) = 0\} = 0$
- $\forall x, y \in \mathfrak{g}, (x, y) = (y, x)$
- $\forall x, y, z \in \mathfrak{g}, ([x, y], z) = (x, [y, z])$

This will turn out to be the analogue of the Killing form.

**Theorem 4.21.** *If  $A$  is symmetrisable, i.e.  $\exists D = \text{diag}(d_1, \dots, d_n)$  s.t.  $\det D \neq 0$  and  $B = DA$  is symmetric, then  $\exists$  such a form on  $\mathfrak{g}(\Sigma)$ .*

Note that in the above theorem, we have fixed a choice by asking for  $D$ . It is then natural to ask how many choices we have.

**Example 4.22.**  $\mathfrak{g}(A) \cong \mathfrak{g}(DA)$ , as this simply scales generators  $e_i \mapsto d_i e_i$ .

We can define a non-degenerate symmetric bilinear form on  $\mathfrak{h}$  by

- $\forall h \in \mathfrak{h}, (\alpha_i^\vee, h) = d_i \alpha_i(h)$
- $\forall h_1, h_2 \in \mathfrak{h}'', (h_1, h_2) = 0$

where we have defined  $\mathfrak{h}' \equiv \langle \Pi^\vee \rangle = \bigoplus_{i=1}^N \mathbb{C} \alpha_i^\vee$  and then required  $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ . It is the coupling to  $D$  that gives the symmetry e.g

$$\begin{aligned} (\alpha_i^\vee, \alpha_j^\vee) &= d_i \alpha_i(\alpha_j^\vee) \\ &= d_i a_{ji} \\ &= d_j a_{ij} \\ &= (\alpha_j^\vee, \alpha_i^\vee) \end{aligned}$$

**Theorem 4.23.**  $(\cdot, \cdot)$  extends to a non-degenerate symmetric bilinear invariant form on  $\mathfrak{g}(A)$  by setting

- $(e_i, f_j) = \delta_{ij}$
- $(e_i, e_j) = 0 = (f_i, f_j)$
- $(e_i, h) = 0 = (f_i, h)$

**Remark.** Note that these conditions are imposed on us in order to have invariance, e.g

$$([e_1, e_2], f_1) = (e_1, [e_2, f_1]) = 0$$

or

$$([e_1, e_2], f_1) = -(e_2, \underbrace{[e_1, f_1]}_{\in \mathfrak{h}}) = 0$$

**Corollary 4.24.** *Let  $\alpha \in Q$ , and recall  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$ . Then*

$$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) \neq 0 \Leftrightarrow \alpha + \beta = 0$$

*Then identifying  $\mathfrak{g} \cong \mathfrak{g}^*$  by  $x \mapsto (x, \cdot)$ , we have*

$$\mathfrak{g}_\alpha \cong \mathfrak{g}_{-\alpha}^*$$

Now let us make the prop:

**Proposition 4.25.** *Let  $\nu : \mathfrak{h} \xrightarrow{\cong} \mathfrak{h}^*$  be the function*

$$\begin{aligned} \nu(h) &= (h, \cdot) \\ \nu(\alpha_i^\vee) &= d_i \alpha_i \end{aligned}$$

*Then for  $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ :*

$$[x, y] = (x, y) \cdot \nu^{-1}(\alpha)$$

**Theorem 4.26** (Serre). *Suppose we have  $A \in M_n(\mathbb{Z})$  satisfying  $a_{ii} = 2, a_{ij} \leq 0$ . Then in  $\mathfrak{g}(A)$*

$$\begin{aligned} \text{ad}(e_i)^{1-a_{ij}}(e_j) &= 0 \\ \text{ad}(f_i)^{1-a_{ij}}(f_j) &= 0 \end{aligned}$$

*These are called the **Serre relations**.*

*Proof.*

□

**Theorem 4.27** (Gabber - Kac). *If we have  $A \in M_n(\mathbb{Z})$  satisfying  $a_{ii} = 2, a_{ij} \leq 0$  **and**  $A$  is symmetrisable, then the only relations on  $\mathfrak{g}(A)$  are the Serre relations.*

**Remark.** *If we have the conditions of the above theorem, then we know  $\mathfrak{g}(A)$  is generated by  $e_i, f_i, \mathfrak{h}$  s.t.*

- $[h, h'] = 0$
- $[h, e_i] = \alpha_i(h)e_i$
- $[h, f_i] = -\alpha_i(h)f_i$
- $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$
- $\text{ad}(e_i)^{1-a_{ij}}(e_j) = 0$
- $\text{ad}(f_i)^{1-a_{ij}}(f_j) = 0$

*and this entirely determines  $\mathfrak{g}(A)$ .*

**Example 4.28.** *Consider  $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ , which has  $\mathfrak{g}(A) = \mathfrak{sl}_3$ . Then we found*

$$\text{ad}(e)^{1-(-1)}(e) = [e_1, [e_1, e_2]] = 0$$

Now we have a copy of  $\mathfrak{sl}_2$ , called  $\mathfrak{sl}_2^{(i)} = \langle e_i, f_i, \alpha_i^\vee \rangle$  sitting in  $\mathfrak{g}(A)$ . We prove a lemma about it's adjoint action:

**Lemma 4.29.** *Let  $V \in \text{Rep}(\mathfrak{sl}_2)$  and  $v \in V$  s.t.  $h \cdot v = \lambda v$ ,  $e \cdot v = 0$ . Defining  $v_m = \frac{f^m}{m!} \cdot v$  we have*

$$\begin{aligned} h \cdot v_m &= (\lambda - 2m)v_m \\ e \cdot v_m &= (\lambda - m + 1)v_{m-1} \end{aligned}$$

*Proof.* Note

$$\begin{aligned} hf^m &= ([h, f] + fh)f^{m-1} \\ &= -2f^m + fhf^{m-1}. \end{aligned}$$

and so so repeating this procedure we get

$$hf^m = f^m(h - 2m) \Rightarrow hf^m \cdot v = (\lambda - 2m) \cdot v$$

We do the same with  $e$  seeing

$$\begin{aligned} ef^m &= (h + fe)f^{m-1} = hf^{m-1} + fhf^{m-2} + \dots + f^{m-1}h + f^me \\ &= f^{m-1}[(h - 2m + 2) + (h - 2m + 4) + \dots + (h)] + f^me \\ &= f^{m-1}[m(h - m + 1)] + f^me \end{aligned}$$

yielding the result □

**Definition 4.30.** *Given  $V \in \text{Rep}(\mathfrak{g})$ ,  $x \in \mathfrak{g}$  we say  $x$  acts **locally nilpotently** if  $\forall v \in V, \exists N$  s.t.  $x^N \cdot v = 0$ .*

**Definition 4.31.** *We say  $V \in \text{Rep}(\mathfrak{g}(A))$  is **integrable** if*

1.  $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$  where  $V_\lambda = \{v \in V \mid \forall h \in \mathfrak{h}, h \cdot v = \lambda(h)v\}$
2.  $e_i, f_i$  acts locally nilpotently on  $V$ .

**Definition 4.32.** *If  $V_\lambda \neq 0$ , we call it a **weight space of weight  $\lambda$***

**Example 4.33.**  $\mathfrak{g}(A)$  is integrable over itself as the adjoint rep because of the Serre relations. Here the weight spaces are  $\mathfrak{g}_\alpha$ . Note if  $h \in \mathfrak{h}$  and  $x \in \mathfrak{g}_\alpha$  we will have

$$\begin{aligned} h \cdot x &= \alpha(h)x \\ \Rightarrow \exp(h) \cdot x &= e^{\alpha(h)}x \end{aligned}$$

**Example 4.34.** *Ever finite dimensional representation is integrable. This is as automatically  $V = \bigoplus V_\lambda$ , and then as eigenspaces cannot mix, there is no way to keep acting.*

**Proposition 4.35.** *Let  $V \in \text{Rep}(\mathfrak{g}(A))^{Int}$  and let  $\mathfrak{g}_{(i)} = \langle e_i, f_i, \alpha_i^\vee \rangle \subset \mathfrak{g}(A)$ . Then*

1. As  $\mathfrak{g}_{(i)}$  modules,

$$V = \bigoplus_{d \geq 0} V_d^{\oplus m_d}$$

where  $V_d$  is a irreducible rep of  $\mathfrak{sl}_2$ ,  $\dim V_d = d + 1$ ,  $m_d \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ .

2. Take  $\lambda \in \text{wt}(V) \equiv \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$  and fix an  $\alpha_i$ -string through  $\lambda$ ,  $M = \{t \in \mathbb{Z} \mid \lambda + t\alpha_i \in \text{wt}(V)\}$ . Then

- (a)  $\exists p, q \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$  s.t.  $M = [-p, q]$
- (b)  $\text{mult}_V(\lambda) = \dim V_\lambda < \infty \Rightarrow p, q < \infty$ .
- (c)  $p, q < \infty \Rightarrow p - q = \lambda(\alpha_i^\vee)$
- (d)  $t \mapsto m(t) \equiv \dim V_{\lambda+t\alpha_i}$  is symmetric at  $t = -\frac{1}{2}\lambda(\alpha_i^\vee)$

*Proof.* Take  $v_\lambda \in V_\lambda$ , and define

$$U_{v_\lambda} = \sum_{k,l \geq 0} \mathbb{C} \cdot f_i^k \cdot e_i^l \cdot v_\lambda$$

Now we must have  $\dim U_{v_\lambda} \leq 0$  from nilpotency, and we have an action  $\mathfrak{g}_{(i)} \curvearrowright U_{v_\lambda}$ . By Weyl reducibility

$$U_{v_\lambda} = \bigoplus_{d \geq 0} V_d^{\oplus m_d}$$

Do this for all  $v \in V$ .

Now let  $U = \sum_{t \in M} V_{\lambda+t\alpha_i} \curvearrowright \mathfrak{g}_{(i)}$ , and define  $p = -\inf M$ ,  $q = \sup M$ . As  $0 \in M$  it must be the case  $p, q \geq 0$ . Now we can calculate

$$(\lambda + t\alpha_i)(\alpha_i^\vee) = \lambda(\alpha_i^\vee) + 2t$$

and so we have

$$(\lambda + t\alpha_i)(\alpha_i^\vee) = 0 \Leftrightarrow t = -\frac{1}{2}\lambda(\alpha_i^\vee)$$

□

**Corollary 4.36.**  $\lambda \in \text{wt}(V) \Rightarrow \lambda - \lambda(\alpha_i^\vee)\alpha_i \in \text{wt}(V)$

**Example 4.37.** Take  $\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$ ,  $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ . Then  $\text{wt}(\mathfrak{g}(A)) = \{\text{roots}\} \cup \{0\}$

## 4.2 Weyl group

**Definition 4.38** (Fundamental reflections). We define the **fundamental reflections**  $r_i \in GL(\mathfrak{h}^*)$  by  $r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i$

**Proposition 4.39.**  $r_i$  are reflections with fixed points  $\ker(\alpha_i^\vee) = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) = 0\}$ . Moreover  $r_i(\alpha_i) = -\alpha_i$ .

**Definition 4.40** (Weyl group). We define the **Weyl group** to be

$$W = \langle r_i \rangle \subseteq GL(\mathfrak{h}^*)$$

**Proposition 4.41.** Take  $V \in \text{Rep}(\mathfrak{g}(A))^{Int}$ ,  $\lambda \in \mathfrak{h}^*$ ,  $w \in W$ , then

1.  $\text{mult}_V(\lambda) = \text{mult}_V(w(\lambda))$



$$2. W \circ R \equiv \{\alpha \in Q \setminus 0 \mid \mathfrak{g}_\alpha \neq 0\}$$

$$3. \dim \mathfrak{g}_\alpha = \text{mult}(\alpha) = \text{mult}(w(\alpha)) = \dim \mathfrak{g}_{w(\alpha)}$$

**Remark.**  $W \circ Q$

**Exercise 4.42.**  $W \cong \langle r_i^\vee \rangle \subseteq GL(\mathfrak{h})$  where  $r_i^\vee(h) = h - \alpha_i(h)\alpha_i^\vee$

Now assume we have  $x, y : V \rightarrow V$  locally nilpotent s.t  $\text{ad}(x)^N(y) = 0$  for some  $N \gg 0$ . Then

$$\exp(x) \cdot y \cdot \exp(-x) = \text{Ad}(\exp(x))(y) = \exp(\text{ad}(x))(y)$$

**Theorem 4.43.** Take  $V \in \text{Rep}(\mathfrak{g}(A))^{Int}$  with  $\pi : \mathfrak{g}(A) \rightarrow \mathfrak{gl}(V)$  the rep. We define  $r_i^\pi : \exp(f_i) \exp(-e_i) \exp(f_i)$ . Then

$$1. r_i^\pi(V_\lambda) = V_{r_i(\lambda)}$$

$$2. r_i^{\text{ad}} : \mathfrak{g}(A) \rightarrow \mathfrak{g}(A), r_i^{\text{ad}}|_{\mathfrak{h}} = r_i^\vee.$$

**Example 4.44.** Take  $\mathfrak{g}(A) = \mathfrak{sl}_2$ , then  $W = \langle r_1 \mid r_1^2 = 1 \rangle \cong C_2$

**Example 4.45.**  $\mathfrak{g}(A) = \mathfrak{sl}_3$ . Then we have

	$\alpha_1$	$\alpha_2$
$r_1$	$-\alpha_1$	$\alpha_1 + \alpha_2$
$r_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$
$r_1 r_2$	$\alpha_2$	$-(\alpha_1 + \alpha_2)$
$r_2 r_1$	$-(\alpha_1 + \alpha_2)$	$\alpha_1$
$r_1 r_2 r_1$	$-\alpha_2$	$-\alpha_1$
$r_2 r_1 r_2$	$-\alpha_2$	$-\alpha_1$

Hence we have

$$W = \langle r_1, r_2 \mid r_1^2 = 1 = r_2^2, r_1 r_2 r_1 = r_2 r_1 r_2 \rangle \cong S_3$$

We get this rep by taking  $r_1 \mapsto (12), r_2 \mapsto (23)$

**Remark.** We can see that the above would satisfy what we want by using the braid relations.

Now we can decompose

$$\mathfrak{sl}_3 = \underbrace{\mathfrak{g}_{\alpha_1}}_{e_1} \oplus \underbrace{\mathfrak{g}_{\alpha_2}}_{e_2} \oplus \underbrace{\mathfrak{g}_{\alpha_1 + \alpha_2}}_{[e_1, e_2]} \oplus \dots$$

and then

$$R = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \cup \{-\alpha_1, -\alpha_2, -(\alpha_1 + \alpha_2)\}$$

**Proposition 4.46.**  $W$  is generated by the  $r_i$  with the relations

$$\bullet r_i^2 = 1$$

- $(r_i r_j)^{m_{ij}} = 1$  or equivalently  $\underbrace{r_j r_i r_j \dots}_{m_{ij}} = \underbrace{r_i r_j r_i \dots}_{m_{ij}}$

where

$$\frac{m_{ij}}{a_{ij}a_{ji}} \parallel \begin{array}{c|c|c|c|c} 2 & 3 & 4 & 6 & \infty \\ \hline 0 & 1 & 2 & 3 & \geq 4 \end{array}$$

**Remark.** A group of this form is called a **Coxeter group**.

**Example 4.47.** We have correspondences

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} &\mapsto \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} &\mapsto \mathfrak{sl}_3 \\ \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} &\mapsto \mathfrak{so}_4 \\ \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} &\mapsto G_2 \\ \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} &\mapsto \widehat{\mathfrak{sl}_2} = a * b / (a^2 = 1 = b^2) \quad (\text{affine } \mathfrak{sl}_2) \end{aligned}$$

Now we could consider working with realisations, and we would get the following results:

- $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$
- $Q^{\vee} = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^{\vee} \subset \mathfrak{h}_{\mathbb{R}} \hookrightarrow W$

**Definition 4.48** (Fundamental chamber). The **fundamental chamber** is  $\mathcal{C} \equiv \{h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(h) \geq 0\}$ . Correspondingly the **Tits cone** is  $X = \bigcup_{w \in W} w(\mathcal{C})$

**Proposition 4.49.** TFAE:

- $|W| < \infty$
- $|R| < \infty$
- $X = \mathfrak{h}_{\mathbb{R}}$ .

### 4.3 Finite vs Affine

We aim now to construct completely  $\mathfrak{g}(\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}) \equiv \hat{g}$ . We start by finding a realisation. We need

- $\hat{\mathfrak{h}}$  s.t.  $\dim_{\mathbb{C}} \hat{\mathfrak{h}} = 3$
- $\{\alpha_0, \alpha_2\} \subset \hat{\mathfrak{h}}^*$ ,  $\{\alpha_0^{\vee}, \alpha_1^{\vee}\} \subset \hat{\mathfrak{h}}$  linearly indep s.t.  $\alpha_i(\alpha_j^{\vee}) = \pm 2$  if  $i = j$  or  $i \neq j$ .

We can take

$$\begin{aligned}\hat{\mathfrak{h}} &= \mathbb{C}\alpha_0^\vee \oplus \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}d \\ \hat{\mathfrak{h}}^* &= \mathbb{C}\alpha_0 \oplus \mathbb{C}\alpha_1 \oplus \mathbb{C}\Lambda\end{aligned}$$

with

$$\begin{aligned}\alpha_0(d) &= 1 \\ \alpha_1(d) &= 0 \\ \Lambda(\alpha_0^\vee) &= 1 \\ \Lambda(\alpha_1^\vee) &= 0 \\ \Lambda(d) &= 0\end{aligned}$$

Now we know we will also have a bilinear form given by

$$\begin{aligned}(\alpha_i^\vee, \alpha_j^\vee) &= a_{ij} \\ (\alpha_0^\vee, d) &= 1 \\ (\alpha_1^\vee, d) &= 0 \\ (d, d) &= 0 \\ \text{also} \\ (\alpha_i, \alpha_j) &= a_{ij} \\ (\alpha_0, \Lambda) &= 1 \\ (\alpha_1, \Lambda) &= 0 \\ (\Lambda, \Lambda) &= 0\end{aligned}$$

Now we recognise this gives a special element  $c \equiv \alpha_0^\vee + \alpha_1^\vee$ ,  $\delta = \alpha_0 + \alpha_1$ . This gives a map

$$\begin{aligned}\nu : \hat{\mathfrak{h}} &\rightarrow \hat{\mathfrak{h}}^* \\ \alpha_i^\vee &\mapsto \alpha_i \\ d &\mapsto \Lambda \\ c &\mapsto \delta\end{aligned}$$

These now have inner product

$$\begin{aligned}(\delta, \alpha_0) &= 0 = (\delta, \alpha_1) \\ \Rightarrow (\delta, \delta) &= 0 \quad \text{but} \quad (\delta, \Lambda) = 1 \\ (c, \alpha_0^\vee) &= 0 = (c, \alpha_1^\vee) \\ \Rightarrow (c, c) &= 0 \quad \text{but} \quad (c, d) = 1\end{aligned}$$

Now we have orthogonal decompositions

$$\begin{aligned}\hat{\mathfrak{h}} &= \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}c \oplus \mathbb{C}d \\ \hat{\mathfrak{h}}^* &= \mathbb{C}\alpha_1 \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda\end{aligned}$$

We want to relate  $\hat{\mathfrak{g}}$  to  $\mathfrak{g} = \mathfrak{sl}_2$ . In  $\hat{\mathfrak{g}}$

$$\begin{aligned} [d, e_1] &= 0 = [d, f_1] \\ [d, e_0] &= e_0 \\ [d, f_0] &= -f_0 \\ [c, e_i] &= 0 = [c, f_i] \end{aligned}$$

Hence  $c \in Z(\hat{\mathfrak{g}})$ .

#### 4.3.1 Central extension of the loop algebra

**Definition 4.50.** Define  $\mathcal{L} = \mathbb{C}[t, t^{-1}] = \{\text{Laurent polynomials}\}$  and  $\text{res} : \mathcal{L} \rightarrow \mathbb{C}$ ,  $\text{res}(\sum_k c_k t^k) = c_{-1}$ .

**Lemma 4.51.**  $\text{res} : \mathcal{L} \rightarrow \mathbb{C}$  is actually a linear functional with

- $\text{res}(t^{-1}) = 1$
- $\text{res} \frac{dP}{dt} = 0$

**Corollary 4.52.** We can construct  $\varphi : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{C}$  by

$$\varphi(P, Q) = \text{res} \left( Q \frac{dP}{dt} \right)$$

which has the properties

- $\varphi(P, Q) = -\varphi(Q, P)$
- $\varphi(PQ, R) + \varphi(QR, P) + \varphi(RP, Q) = 0$

**Definition 4.53.** The *loop algebra* of  $\mathfrak{g}$  is

$$\mathcal{L}\mathfrak{g} \equiv \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{L} = \text{Maps}(\mathbb{C}^\times, \mathfrak{g})$$

We give it the Lie bracket

$$[x \otimes P, y \otimes Q]_0 = [x, y] \otimes PQ$$

**Definition 4.54.** A *central extension* of the loop algebra is  $\mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$  with the Lie bracket where  $\forall a \in \mathcal{L}\mathfrak{g}$ ;

$$\begin{aligned} [c, a] &= 0 \\ [a, b] &= [a, b]_0 + \psi(a, b)c \end{aligned}$$

for some antisymmetric bilinear map  $\psi$  which makes that Jacobi identity hold.

Restricting back to our example of  $\mathfrak{g} = \mathfrak{sl}_2$  we have a bilinear form, so we can set

$$\psi : \mathcal{L}\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{L}\mathfrak{g} \rightarrow \mathbb{C}$$

with

$$\psi(x \otimes P, y \otimes Q) = (x, y) \cdot \varphi(P, Q)$$

This  $\psi$  satisfies

- $\forall a, b, \psi(a, b) = -\psi(b, a)$
- $\psi([a, b]_0, c) + \psi([b, c]_0, a) + \psi([c, a]_0, b) = 0$

Now set  $\tilde{\mathcal{L}}\mathfrak{g} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$  with the bracket as above to get a central extension of the loop algebra of  $\mathfrak{sl}_2$ .

So in  $\tilde{\mathcal{L}}\mathfrak{g}$ ,

- $Z(\tilde{\mathcal{L}}\mathfrak{g}) = \mathbb{C}c$
- $\alpha_1^\vee = \alpha_1^\vee \otimes 1 \in \tilde{\mathcal{L}}\mathfrak{g}$  satisfies

$$\begin{aligned} [\alpha_1^\vee, y \otimes Q]_0 &= [\alpha_1^\vee, y] \otimes Q = \text{wt}(y)(\alpha_1^\vee)(y \otimes Q) \\ \psi(\alpha_1^\vee, y \otimes Q) &= (\alpha_1^\vee, y) \underbrace{\varphi(1, Q)}_{\text{res}(0)} = 0 \end{aligned}$$

This gives us a well defined subalgebra  $\mathbb{C}\alpha_1^\vee \oplus \mathbb{C}c \subset \tilde{\mathcal{L}}\mathfrak{g}$ .  
Finally we can define

$$\hat{g} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

s.t.

- $[a, b] = [a, b]_0 + \psi(a, b)c$
- $[d, c] = 0$
- $[d, x \otimes P] = x \otimes t \frac{dP}{dt}$

**Theorem 4.55.** *With the above definition we have*

$$\begin{aligned} \mathfrak{g}\left(\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\right) &\rightarrow \hat{\mathfrak{g}} \\ e_1, \alpha_1^\vee, f_1 &\mapsto e_1, \alpha_1^\vee, f_1 \\ c &\mapsto c \\ d &\mapsto t \frac{d}{dt} \\ e_0 &\mapsto f_1 \otimes t \\ f_0 &\mapsto e_1 \otimes t^{-1} \end{aligned}$$

## 5 Witt and Virasoro Algebras

### 5.1 The Witt algebra

Let  $A = \mathbb{C}[z, z^{-1}]$  be Laurent polynomials, and then define

$$\text{Der}(A) = \{\phi : A \rightarrow A \mid \phi \text{ } \mathbb{C}\text{-linear, } \phi(fg) = \phi(f)g + f\phi(g)\}$$

**Proposition 5.1.** *Der(A) is a Lie algebra with bracket*

$$[\phi, \psi](f) = \phi(\psi(f)) - \psi(\phi(f))$$

**Proposition 5.2.** *The operators  $\{L_n = -z^{n+1} \frac{d}{dz} \mid n \in \mathbb{Z}\}$  are a basis for  $\text{Der}(A)$*

*Proof.* They are obviously independent. Write  $\phi(z) = -\sum_n a_n z^{n+1}$  (with all but finitely many  $a_n \neq 0$ ). Now the Leibniz rule gives

$$\phi(z^k) = k z^{k-1} \phi(z)$$

and so

$$\begin{aligned} \phi(f)(z) \frac{df}{dz} \phi(z) &= -\sum_n a_n z^{n+1} \frac{d}{dz} f(z) \\ &= \sum_n a_n L_n(f)(z) \end{aligned}$$

□

**Definition 5.3.** *The **Witt algebra**  $\text{Witt}$  is the Lie algebra with basis  $\{L_n\}$  and bracket as above.*

## 5.2 Central extension

**Definition 5.4.** *Let  $\mathfrak{a}$  be a Lie algebra. A **central extension** of  $\mathfrak{a}$  is a pair  $(\tilde{\mathfrak{a}}, \pi)$  s.t.*

- $\tilde{\mathfrak{a}}$  is a Lie algebra
- $\pi : \tilde{\mathfrak{a}} \rightarrow \mathfrak{a}$  is a surjective LA hom
- $\dim_{\mathbb{C}} \ker \pi = 1$
- $\forall x \in \tilde{\mathfrak{a}}, y \in \ker \pi, [x, y] = 0$

**Definition 5.5.** *Two central extensions  $(\tilde{\mathfrak{a}}, \pi), (\tilde{\mathfrak{a}}', \pi')$  are **equivalent** if  $\exists \phi : \tilde{\mathfrak{a}} \rightarrow \tilde{\mathfrak{a}}'$  a LA iso s.t.  $\pi' \circ \phi = \pi$ .*

**Example 5.6.** *The trivial extension is  $\tilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathbb{C}K$  where  $K$  is the centre of  $\mathfrak{a}$ , and we take the same bracket for  $\tilde{\mathfrak{a}}$*

**Proposition 5.7.** *Up to equivalence,  $\exists!$  non trivial central extension of the Witt algebra, the **Virasoro algebra**  $\text{Vir}$ , written as*

$$0 \rightarrow \underbrace{\mathbb{C}c}_{\ker(\pi_{\text{Vir}})} \rightarrow \text{Vir} \xrightarrow{\pi_{\text{Vir}}} \text{Witt} \rightarrow 0$$

*Explicitly we may say that  $\text{Vir}$  has basis  $\{L_n, c\}$  with bracket*

$$\begin{aligned} [c, \text{Vir}] &= 0 \\ [L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12} \delta_{m,-n} c \end{aligned}$$

*The map to Witt is  $L_n \mapsto L_n, c \mapsto 0$*

*Proof.* We check it exists, and this is done simply by observing that the relations given above give a central extension. We can check it is not trivial, as in the trivial extension we would have  $[L_2, L_{-2}] = 2[L_1, L_{-1}]$ .

For uniqueness, let  $(\mathfrak{b}, \pi)$  be another central extension. Choose a splitting  $i : \text{Witt} \rightarrow \mathfrak{b}$  with  $\pi \circ i = \text{id}$ . We then have  $\mathfrak{b} = \mathbb{C}k \oplus i(\text{Witt})$ . The bracket is given by

$$\begin{aligned} [i(\text{Witt}), k] &= 0 \\ [i(L_m), i(L_n)] &= (m-n)i(L_{m+n}) + a(m, n)k \end{aligned}$$

for some antisymmetric  $a : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$ . Define a new splitting  $i'$  by

$$i'(L_n) = \begin{cases} i(L_0) & n = 0 \\ i(L_n) - \frac{a(0, n)}{n}k & n \neq 0 \end{cases}$$

Then  $[i'(L_0), i'(L_n)] = -ni'(L_n)$ . so wlog we may assume  $a(0, n) = 0$ . Applying the Jacobi identity we get

$$\begin{aligned} 0 &= [[i(L_0), i(L_m)], i(L_n)] + [[i(L_n), i(L_0)], i(L_m)] + [[i(L_m), i(L_n)], i(L_0)] \\ &= (m+n)a(m, n)k \end{aligned}$$

Hence  $a(m, n) = a(m)\delta_{m, -n}$  for some odd function  $a : \mathbb{Z} \rightarrow \mathbb{C}$ . Applying Jacobi again for the triple  $i(L_0), i(L_n), i(L_{-n-1})$  gives

$$(n-1)a(n+1) = (n+2)a(n) - (2n+1)a(1)$$

This is a linear recurrence completely determined by  $a(1), a(2)$ , so the space of solutions is 2-dimensional. It can be found that  $a(n) = n, a(n) = n^3$  are both solutions, so the general solution is  $a(n) = \alpha n + \beta n^3$  for  $\alpha, \beta \in \mathbb{C}$ .

If  $\beta = 0$ , we have a map

$$\begin{aligned} \text{Witt} \oplus \mathbb{C}k &\rightarrow \mathfrak{b} \\ L_n &\mapsto i(L_n) + \frac{1}{2}\alpha\delta_{0n}k \\ k &\mapsto k \end{aligned}$$

which is a LA iso, and so  $(\mathfrak{b}, \pi)$  is trivial.

If  $\beta \neq 0$ , we have the LA iso

$$\begin{aligned} \text{Vir} &\rightarrow \mathfrak{b} \\ L_n &\mapsto i(L_n) + (\alpha + \beta)\delta_{0n}k \\ c &\mapsto 12\beta k \end{aligned}$$

□

### 5.3 Heisenberg algebra

**Definition 5.8.** The *Heisenberg algebra*, *Heis*, has basis  $\{\hbar, a_n\}$  and bracket

$$\begin{aligned} [a_m, a_n] &= m\delta_{m, -n}\hbar \\ [\hbar, a_n] &= 0 \end{aligned}$$

**Example 5.9** (Natural reps of Heis). *Fixing  $\mu, h \in \mathbb{C}$ , define*

$$B(\mu, h) = \mathbb{C}[x_1, x_2, \dots]$$

*with rep  $\rho : \text{Heis} \rightarrow \mathfrak{gl}(B(\mu, h))$  given by*

$$\begin{aligned} \rho(\hbar) &= h \\ \rho(a_n) &= \begin{cases} \frac{\partial}{\partial x_n} & n > 0 \\ \mu & n = 0 \\ -hnx_{-n} & n < 0 \end{cases} \end{aligned}$$

*This is called the **Bosonic Fock space** or **oscillator reps**.*

**Remark.** *The reps  $V = B(\mu, h)$  satisfy*

$$\forall v \in V, \exists N \text{ s.t. } \forall n > N, a_nv = 0$$

Let  $V$  be any rep satisfying the above condition. For any  $n \in \mathbb{Z}$ , define

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_{-k} a_{n+k} :$$

where  $: \cdot :$  is the normal ordered product, i.e.

$$: a_i a_j := \begin{cases} a_i a_j & i < j \\ a_j a_i & i > j \end{cases}$$

By our requirement on  $V$ , we have ensured that  $\forall v \in V$ ,  $L_nv$  is well defined as the sum has only finitely many non-zero terms.

We will work towards proving a big theorem now, so we will need some results:

**Lemma 5.10.**  $\forall k, n \in \mathbb{Z}, [a_k, L_n] = ka_{k+n}$

*Proof.* Define  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\psi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

Choose  $\epsilon > 0$  and then set

$$L_n(\epsilon) = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{j+n} : \psi(\epsilon j)$$

Note that  $\forall v \in V, \exists \delta > 0$  s.t.  $\forall \epsilon < \delta, L_n(\epsilon)v = L_nv$ . Now  $L_n(\epsilon) = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j} a_{j+n} \psi(\epsilon j)$  acts by a  $\mathbb{C}$ -scalar as

$$\begin{aligned} [a_k, L_n(\epsilon)] &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [a_k, a_{-j} a_{j+n}] \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [[a_k, a_{-j}] a_{j+n} \psi(\epsilon j) + a_{-j} [a_k, a_{j+n}] \psi(\epsilon j)] \\ &= \frac{1}{2} [ka_{k+n} \psi(\epsilon k) + ka_{k+n} \psi(\epsilon(-n-k))] \\ &= ka_{k+n} \quad (\text{for } \epsilon \text{ small}) \end{aligned}$$

□



With the above we can state and prove the following:

**Theorem 5.11.** *Let  $V$  be as above and assume  $\forall v \in V, \hbar v = v$ . (e.g.  $V = B(\mu, 1)$ ). Then*

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12}$$

*Proof.* Using notation from before calculate

$$\begin{aligned} [L_m(\epsilon), L_n] &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [a_{-j} a_{j+m}, L_n] \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [[a_{-j}, L_n] a_{j+m} + a_{-j} [a_{j+m}, L_n]] \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [(-j)a_{n+j}a_{j+m} + (j+m)a_{-j}a_{j+m+n}] \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [(-j) : a_{n+j}a_{j+m} : + (j+m) : a_{-j}a_{j+m+n} :] \psi(\epsilon j) \\ &\quad - \frac{1}{2} \delta_{m,-n} \sum_{j < -m} (-m-j)j \psi(\epsilon j) + \frac{1}{2} \delta_{m,-n} \sum_{j < 0} (-j)(j+m) \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [(-j) : a_{n+j}a_{j+m} : + (j+m) : a_{-j}a_{j+m+n} :] \psi(\epsilon j) \\ &\quad + \frac{1}{2} \delta_{m,-n} \sum_{j=-1}^{-m} j(j+m) \psi(\epsilon j) \end{aligned}$$

The first sum telescopes (reindexing) giving a finite sum, so we can then take the limit  $\epsilon \rightarrow 0$ , and we get

$$[L_m, L_n] = \frac{1}{2} \sum_{j \in \mathbb{Z}} (m-n) : a_{-j}a_{j+m+n} : + \frac{1}{2} \delta_{m,-n} \sum_{j=-1}^{-m} j(j+m)$$

and answer follows. □

**Definition 5.12.** *A Vir rep  $V$  has **central charge**  $c \in \mathbb{C}$  if*

$$\forall v \in V, c_{\text{Vir}} \cdot v = cv$$

*where  $c_{\text{Vir}}$  is the central charge of Vir.*

**Remark.** *The theorem says that the Heis rep  $V$  has central charge 1. If the central charge was 0, these reps would be reps of Witt, and moreover this is a bijection.*

## 5.4 Connection to affine Lie algebras

Recall that if  $\mathfrak{g}$  is a finite-dimensional Lie algebra then  $\mathcal{L}\mathfrak{g} = \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$  with bracket  $[xf, yg] = [x, y]fg$ , then the affine Lie algebra  $\hat{\mathfrak{g}}$  is a natural central extension with SES

$$0 \rightarrow \mathbb{C}k \rightarrow \tilde{\mathcal{L}}\mathfrak{g} \rightarrow \mathcal{L}\mathfrak{g} \rightarrow 0$$

If  $\mathfrak{g}$  is simple, then  $\hat{\mathfrak{g}} = \tilde{\mathcal{L}}\mathfrak{g} \oplus \mathbb{C}d$  is Kac-Moody.

**Example 5.13.** If  $\mathfrak{g} = \mathfrak{a} = \mathbb{C}\mathfrak{a}$  is a 1-dimensional abelian Lie algebra then

$$\begin{aligned}\mathcal{L}\mathfrak{a} &\rightarrow \text{Heis} \\ at^n &\mapsto a_n \\ k &\mapsto \hbar\end{aligned}$$

is LA hom.

## 6 Highest weight representations

**Definition 6.1.** Let  $V$  be a rep space of  $\text{Vir}$ . We say  $v \in V$  is **singular** of weight  $(h, c) \in \mathbb{C}^2$  if

- $L_0 v = hv$
- $c_{\text{Vir}} v = cv$
- $\forall n > 0, L_n v = 0$

**Definition 6.2.** Let  $v \in V$  be singular. We say it is a **highest weight vector** if

$$V = \text{Span} \{L_{-n_1} \dots L_{-n_k} v \mid k, n_1, \dots, n_k > 0\}$$

**Remark.** There is a similar definition for Kac-Moody algebras.

**Example 6.3.**  $v = 1 \in B(\mu, 1)$  is a singular vector of weight  $(\frac{1}{2}\mu^2, 1)$ .

**Proposition 6.4.** Let  $V$  be a highest weight rep of  $\text{Vir}$  with highest weight  $(h, c)$ . Then

- The module  $V$  has central charge  $c$ , i.e.  $\forall w \in V, c_{\text{Vir}} w = cw$
- We have

$$V = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} V_{h+k}$$

where  $V_\lambda = \{w \in V \mid L_0 w = \lambda w\}$ .

- Each  $V_{h+k}$  is finite dimensional
- $\dim V_h = 1$ .

**Proposition 6.5.** Let  $V$  be a highest weight module. Then there is a unique maximal proper submodule  $V'' \subseteq V$ . Hence  $V' = V/V''$  is an irreducible highest weight rep with the same highest weight as  $V$ .

*Proof.* Let  $V''$  be the sum of all proper submodules of  $V$ . It remains to be shown that  $V'' \neq V$ . Assume  $U \subsetneq V$  is a submodule, and then we know  $U \cap V_h = \{0\}$  as the intersection is either 0 (as  $V_h$  is 1 dimensional) or contains the highest weight vector, and in the latter case we would have  $U = V$ . So

$$U = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (U \cap V_{h+k}) \subseteq \bigoplus_{k \in \mathbb{Z}_{> 0}} V_{h+k}$$

and then

$$V'' = \sum U = \bigoplus_{k \in \mathbb{Z}_{>0}} V_{h+k}$$

so  $V'' \neq V$ . To get the second part, note that  $W \subsetneq V'$  has preimage under the quotient which must lie in  $V'' \Rightarrow W = 0$ .  $\square$

## 6.1 Verma modules

**Proposition 6.6.** *Let  $(h, c) \in \mathbb{C}^2$ . Then  $\exists$  a highest weight Vir-module  $M(h, c)$  with highest weight  $(h, c)$  and highest weight vector  $v_M$  s.t.*

- $\forall V$  another rep of highest weight  $(h, c)$  with h.w.v  $v \in V$   $\exists!$  Vir-module hom

$$\begin{aligned} M(h, c) &\rightarrow V \\ v_M &\mapsto v \end{aligned}$$

- $V$  is isomorphic to a quotient of  $M(h, c)$

*Proof.* As a vector space

$$\text{Vir} = \text{Vir}_+ \oplus \mathfrak{h} \oplus \text{Vir}_-$$

where  $\text{Vir}_{\pm} = \text{Span}\{L_n \mid n \gtrless 0\}$  and  $\mathfrak{h} = \text{Span}\{L_0, c_{\text{Vir}}\}$ . Let  $\text{Vir}_{\geq 0} = \text{Vir}_+ \oplus \mathfrak{h}$ . Then we have

$$\begin{aligned} \rho : \text{Vir}_{\geq 0} &\rightarrow \mathfrak{gl}_1 = \mathfrak{gl}(\mathbb{C}_{(h,c)}) \\ \forall n > 0, L_n &\mapsto 0 \\ L_0 &\mapsto h \\ c_{\text{Vir}} &\mapsto c \end{aligned}$$

Hence

$$\rho : U(\text{Vir}_{\geq 0}) \rightarrow \mathfrak{gl}_1$$

is an extension to an associative algebra hom from the UEA. We can then construct  $M(h, c)$  by

$$\begin{aligned} M(h, c) &= U(\text{Vir}) \otimes_{U(\text{Vir}_{\geq 0})} \mathbb{C}_{(h,c)} \\ &\cong U(\text{Vir}) / U(\text{Vir})(x, -\rho(x), x \in \text{Vir}_+) \end{aligned}$$

**Exercise 6.7.** *Show that this constructed  $M(h, c)$  has the properties required, with h.w.v 1.*  $\square$

**Exercise 6.8.** *Show that if  $M, M'$  are two modules satisfying the above, then  $M \cong M'$*

**Definition 6.9.**  $M(h, c)$  is called the **Verma module** of highest weight  $(h, c)$

**Corollary 6.10.**  $\forall (h, c) \in \mathbb{C}^2$ , there is a unique irreducible Vir-module  $V(h, c)$  of highest weight  $(h, c)$

*Proof.* Let  $V(h, c) = M(h, c)/J(h, c)$  be the unique irreducible quotient of  $M(h, c)$ . Then by def any other such  $V$  irreducible of highest weight is isomorphic to a quotient so  $V \cong V(h, c)$   $\square$

**Proposition 6.11.** *The Verma module  $M(h, c)$  has basis*

$$\{L_{-n_k} \dots L_{-n_1} v_M \mid k \geq 0, 0 < n_1 \leq \dots \leq n_k\}$$

*Proof.* By the Poincare-Birkhoff-Witt theorem we know

$$\{L_{-n_k} \dots L_{-n_1} c^i h^j L_{m_1} \dots L_{m_l} \mid i, j > 0, k, l \geq 0, 0 < n_1 \leq \dots \leq n_k, 0 < m_1 \leq \dots \leq m_l\}$$

Then as the latter part is a basis for  $U(\text{Vir}_{\geq 0})$ , it gets cancelled in the quotient.  $\square$

## 6.2 Unitary reps

Recall we knew that 1 is a singular vector of weight  $(\frac{1}{2}\mu^2, 1)$  for the rep  $B(\mu, 1)$ . If we define

$$B'(\mu, 1) = \text{Span} \{L_{-n_k} \dots L_{-n_1} 1 \mid k \geq 0, n_i > 0\} \subseteq B(\mu, 1)$$

then  $B'(\mu, 1)$  is now a highest weight rep with highest weight  $(\frac{1}{2}\mu^2, 1)$ .

**Definition 6.12.** *Let  $\mathfrak{a}$  be a complex Lie algebra. An **anti-involution** on  $\mathfrak{a}$  is a function  $\omega : \mathfrak{a} \rightarrow \mathfrak{a}$  s.t.*

- $\omega^2 = \text{id}$
- $\omega(ax + by) = \bar{a}x + \bar{b}y$
- $\omega([x, y]) = -[\omega(x), \omega(y)]$

**Definition 6.13.** *If  $V$  is an  $\mathfrak{a}$ -rep, then a Hermitian form on  $V$  is **contravariant** if*

$$\forall u, v \in V, x \in \mathfrak{a}, \langle x \cdot u, v \rangle = \langle u, \omega(x) \cdot v \rangle$$

**Definition 6.14.** *A rep  $V$  is **unitary** if it admits a contravariant inner product.*

**Example 6.15.** *Anti-involutions on Heis and Vir are given by*

1.  $\omega_{\text{Heis}}(a_n) = a_{-n}, \omega_{\text{Heis}}(\hbar) = \hbar$
2.  $\omega_{\text{Vir}}(L_n) = L_{-n}, \omega_{\text{Vir}}(c) = c$

*Note*

$$\omega_{\text{Heis}}(L_n) = \frac{1}{2} \sum_{j \in \mathbb{Z}} \omega_{\text{Heis}}(: a_{-j} a_{j+n} :) = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j-n} a_j := L_{-n} = \omega_{\text{Vir}}(L_n)$$

**Proposition 6.16.** *Assume  $\mu \in \mathbb{R}$ , then the Heis rep  $B(\mu, 1)$  has a unique contravariant inner product s.t.  $\langle 1, 1 \rangle = 1$ . Explicitly*

$$\langle P, Q \rangle = \langle \omega(P)Q \rangle$$

where  $\langle \cdot \rangle = \text{take constant term and}$

$$\omega : \mathbb{C}[x_1, \dots, ] \rightarrow \text{Heis}$$

is the complex anti-linear ring hom given by  $\omega(x_n) = \frac{1}{n} a_n$

*Proof.*

**Exercise 6.17.** *Do this*

□

**Corollary 6.18.**  $B(\mu, 1)$  is a unitary Vir rep.

**Lemma 6.19.** *Let  $V$  be a unitary Vir-module such that  $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$  is a direct sum of  $L_0$ -eigenspaces and  $\dim V_\lambda < \infty$ .*

*If  $U \subseteq V$  is a submodule, then  $\exists U^\perp \subseteq V$  another submodule s.t.  $V = U \oplus U^\perp$ .*

*Proof.* Let

$$U^\perp = \{v \in V \mid \forall u \in U, \langle u, v \rangle = 0\}$$

It is simple to check  $U^\perp \subseteq V$  is a submodule, and  $U \cap U^\perp = 0$ . To show  $V = U + U^\perp$ , note we can decompose  $v \in V$  into eigenvectors of  $L_0$ , so it is sufficient to show  $V_\lambda \subseteq U + U^\perp$ . But

$$V_\lambda = (V_\lambda \cap U) \oplus (V_\lambda \cap U^\perp)$$

as  $\dim V_\lambda < \infty$

□

**Lemma 6.20.** *Let  $V$  be a unitary highest weight rep. Then  $V$  is irreducible.*

*Proof.* Let  $V'' \subseteq V$  be the unique maximal proper submodule. Then  $(V'')^\perp \subseteq V$  is a submodule and  $V'' \cap (V'')^\perp = 0$ . Then either

1.  $(V'')^\perp = V \Rightarrow$  done
2.  $(V'')^\perp = 0 \Rightarrow V = V''$  contradiction.

□

**Proposition 6.21.** *Assume  $\mu \in \mathbb{R}$ . Then the highest weight module  $B'(\mu, 1)$  is irreducible.*

*Proof.* Use that  $B'$  is unitary. then done by lemma.

□

**Proposition 6.22.** *Assume  $h, c \in \mathbb{R}$ . Then*

1. *If  $M(h, c)$  is unitary, then  $h, c > 0$*
2. *If  $h \geq 0, c \geq 1$ , then the irreducible representation  $V(h, c) = M(h, c)/J(h, c)$  is unitary*
3. *If  $h > 0, c > 1$  then  $M(h, c) = V(h, c)$ .*

**Proposition 6.23.** *Assume  $h, c \in \mathbb{R}$ . Let  $v \in M(h, c)$  be the highest weight vector. Then*

1.  $\exists!$  *contravariant Hermitian form on  $M$  s.t.  $\langle v, v \rangle = 1$*
2. *The eigenspaces of  $L_0$  are pairwise orthogonal*
3.  $\ker \langle \cdot, \cdot \rangle = J(h, c)$  *is a maximal proper submodule*

*Hence  $V(h, c)$  carries a non-degenerate Hermitian form s.t.  $\langle v, v \rangle = 1$*

*Proof.* See notes

□

### 6.3 Kac Determinant formula

Recall we have a basis for  $M(h, c)$ . Kac found a formula for the determinant of  $\langle \cdot, \cdot \rangle|_{M(h, c)_{h+n}}$

## 7 Lie algebra of infinite matrices

**Definition 7.1.** *Define*

$$\mathfrak{gl}_\infty = \{(a_{ij}) \mid a_{ij} \in \mathbb{C}, \text{ almost all entries } 0\}$$

*It has basis  $\{E_{ij}\}$ , the natural extension of that for finite  $\mathfrak{gl}_n$ .*

**Proposition 7.2.**  $\mathfrak{gl}_\infty$  is a Lie algebra with bracket given by matrix commutation.

Recall the definition of a grading:

**Definition 7.3.** A **graded Lie algebra** is a Lie algebra  $\mathfrak{g}$  with decomposition  $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$  s.t.

$$[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l}$$

We write  $\forall X \in \mathfrak{g}_k, \deg X = k$

**Proposition 7.4.** We can make  $\mathfrak{gl}_\infty$  into a graded Lie algebra with grading

$$(\mathfrak{gl}_\infty)_k = \text{Span}\{E_{ij} \mid i - j = k\}$$

**Definition 7.5.** We define the associated group to be

$$GL_\infty = \{(A_{ij}) \mid A_{ij} \in \mathbb{C}, \text{ invertible, almost all } A_{ij} = \delta_{ij} \ 0\}$$

*with operation given by matrix multiplication*

**Proposition 7.6.**  $GL_\infty$  is a Lie group with Lie algebra  $\mathfrak{gl}_\infty$

It turns out we need a bigger Lie algebra

**Definition 7.7.** Let

$$\mathfrak{gl}_\infty^\Delta = \{(a_{ij}) \mid \forall |i - j| \gg 0, a_{ij} = 0\}$$

**Proposition 7.8.**  $\mathfrak{gl}_\infty \subset \mathfrak{gl}_\infty^\Delta$

### 7.1 Central extension

**Definition 7.9.** Consider the central extension  $\hat{\mathfrak{gl}}_\infty^\Delta$  defined by

$$0 \rightarrow \mathbb{C}c \rightarrow \hat{\mathfrak{gl}}_\infty^\Delta \rightarrow \mathfrak{gl}_\infty^\Delta \rightarrow 0$$

*with bracket given by, for  $a, b \in \mathfrak{gl}_\infty^\Delta$ ,*

$$[a, b] = ab - ba + \gamma(a, b)c$$

$\gamma$  is called the **cocycle** and satisfies

$$\gamma(E_{ij}, E_{ji}) = 1 = -\gamma(E_{j-i}, E_{ij}) \quad \text{for } i \leq 0, j \geq 1$$

*and is 0 otherwise.*

We have representations of Heis and Vir inside  $\hat{\mathfrak{gl}}_\infty^\Delta$  given by

$$V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j$$

There is then a natural action on  $V$  by multiplication.

## 7.2 Shift operators

**Definition 7.10.** Define the **shift operator**  $\Delta_k : V \rightarrow V$ ,  $v_j \mapsto v_{j-k}$ . We can write explicitly

$$\Lambda_k = \sum_{i \in \mathbb{Z}} E_{i, i+k}$$

**Proposition 7.11.**  $[\Delta_k, \Delta_j] = 0$

Let  $\eta = \bigoplus_k \mathbb{C} \Lambda_k$  be the subalgebra of  $\mathfrak{gl}_\infty^\Delta$ . We can then let  $\hat{\eta}$  be the central extension given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \hat{\mathfrak{gl}}_\infty^\Delta & \longrightarrow & \mathfrak{gl}_\infty^\Delta \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \hat{\eta} & \longrightarrow & \eta \longrightarrow 0 \end{array}$$

**Proposition 7.12.** We have

- $\gamma(\Lambda_n, \Lambda_k) = n\delta_{n, -k}$
- $\hat{\eta} = \text{Heis}$

*Proof.* The first point is a calculation, Secondly, there is an explicit isomorphism, and the relations are the same in each.  $\square$

**Proposition 7.13.** For the Witt algebra we can say

- $\exists$  a family of embeddings depending on  $\alpha, \beta \in \mathbb{C}$  given by

$$\begin{aligned} i_{\alpha, \beta} : \text{Witt} &\hookrightarrow \hat{\mathfrak{gl}}_\infty^\Delta \\ L_n &\mapsto \sum_{k \in \mathbb{Z}} [k - \alpha - \beta(n+1)] E_{k+n, k} \end{aligned}$$

- Let  $\hat{\text{Witt}} \subset \hat{\mathfrak{gl}}_\infty^\Delta$  be the central extension. Then

$$\gamma(L_i, L_j) = \delta_{i, -j} \left( \frac{i^3 - i}{12} c_\beta + 2ih_0 \right)$$

where  $c_\beta = -12\beta^2 + 12\beta - 2$  and  $h_0 = \frac{1}{2}\alpha(\alpha + 2\beta - 1)$

- Let  $\hat{L}_n = L_n + \delta_{n,0}h_0c$ . Then

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \delta_{n, -m} \left( \frac{n^3 - n}{12} \right) c_\beta c$$

## 8 Fermionic Fock space

We can then consider  $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ , let  $I = \langle x \otimes x \rangle$ , and then get

$$\Lambda(V) = T(V)/I$$

This comes equipped with a projection map  $p : T(V) \rightarrow \Lambda(V)$ . Letting  $p(T_k(V)) = \Lambda^k(V)$ , we get the decomposition

$$\Lambda(V) = \bigoplus_{k \geq 0} \Lambda^k(V)$$

There are also linear maps

$$\begin{aligned} \phi_{s,k}^{(m)} : \Lambda^k(V) &\rightarrow \Lambda^s(V) \\ u &\mapsto u \wedge (v_{-k+m} \wedge \cdots \wedge v_{-s+k+m}) \end{aligned}$$

for fixed  $m \in \mathbb{Z}$ ,  $k \leq s$ . These maps obey

$$\begin{aligned} \phi_{r,s}^{(m)} \circ \phi_{s,k}^{(m)} &= \phi_{r,k}^{(m)} \\ \phi_{k,k}^{(m)} &= \text{id} \end{aligned}$$

Hence  $(\Lambda^k(V), \phi_{r,k}^{(m)})$  for a **direct system** for each  $m \in \mathbb{Z}$ . I can then take the direct limit to get

**Definition 8.1.** *The **Fermionic Fock space of charge  $m$**  is*

$$F^{(m)} = \Lambda_{(m)}^\infty(V) = \lim_{\rightarrow} \Lambda^k(V)$$

*The construction works as*

$$\lim_{\rightarrow} \Lambda^k(V) = \bigsqcup_k \Lambda^k(V) / \sim$$

*where the equivalence relation is given by*

$$x_i \in \Lambda^i(V) \sim x_j \in \Lambda^j(V) \Leftrightarrow \exists h, i, j \leq h, \phi_{h,i}^{(m)}(x_i) = \phi_{h,j}^{(m)}(x_j)$$

We have a basis given by

$$\psi = v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$$

called the **semi-infinite monomials**, requiring the conditions

- $i_0 > i_{-1} > \cdots$
- $i_k = k + m$  for  $k \ll 0$ .



This generalises naturally to get

$$\psi_m = v_m \wedge v_{m-1} \wedge \dots$$

which is the **vacuum vector of charge m**.

The FFS comes with a grading given by

$$\deg \psi = \sum_{s \geq 0} i_{-s} + s - m$$

which is forced to be finite by our condition on  $i_k$  for  $k \ll 0$ . We then have

$$F_k^{(m)} = \text{Span} \{ \psi \mid \deg \psi = k \}$$

Now let  $\lambda = (\lambda_0, \dots, \lambda_{n-1}) + k$  be a partition if  $h$ , that is

- $\lambda_0 + \dots + \lambda_{n-1} + k = h$
- $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1}$

We can then get semi-infinite monomials  $\psi_\lambda$  from partitions by saying  $j_{-i} = \lambda_i - i + m$  for  $i = 0, \dots, n-1$  and then  $j_{-n-i} = -n + m - i$

**Example 8.2.** Take  $\lambda = (5, 3, 3, 1) + 12$  is a partition of 24. Take  $m = 0$ . We then find

$$\begin{aligned} j_0 &= 5j_{-1} & &= 2 \\ j_{-2} &= 1 \\ j_{-3} &= -2 \end{aligned}$$

and so we get

$$\psi_\lambda = (v_5 \wedge v_2 \wedge v_1 \wedge v_{-2}) \wedge v_{-4} \wedge v_{-5} \wedge \dots$$

**Proposition 8.3.** We have

- $F^{(m)} = \bigoplus_{k \geq 0} F_k^{(m)}$ ,  $F_0^{(m)} = \mathbb{C}\psi_m$
- $\dim F_k^{(m)} = p(k) = \text{number of partitions of } k$
- $\dim_q F^{(m)} \equiv \sum_{k \geq 0} (\dim F_k^{(m)}) q^k = \prod_{l \geq 1} (1 - q^l)^{-1}$

*Proof.* • Clear

- $\{ \psi_\lambda \mid \lambda \text{ partition of } h \}$  is a basis of  $F_h^{(m)}$
- The first part is the definition. Then we have

$$\dim_q F^{(m)} = \sum_{k \geq 0} p(k) q^k$$

□

## 9 Representations of $GL_\infty$ , $\mathfrak{gl}_\infty$ , on $F^{(m)}$

### 9.1 Actions on tensor products

Let  $\mathfrak{g}$  be a Lie algebra and  $M, N$   $\mathfrak{g}$ -reps. Then  $M \otimes_{\mathbb{C}} N$  is a representation space of  $\mathfrak{g}$  given by

$$x \in \mathfrak{g}, m, n \in M, N, x \cdot (m \otimes n) = (x \cdot m) \otimes n + m \otimes (x \cdot n)$$

If  $G$  is a group, then we get a rep on  $M \otimes_{\mathbb{C}} N$  by

$$g \cdot (m \otimes n) = (gm) \otimes (gn)$$

**Remark.**