# Linearising Flows and a Cohomological Interpretation of Lax Equations - Unpacking the Paper

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# 1 Introduction

The purpose of this document is to facilitate the understanding of [1] by discussing the terms and how they fit into the wider picture of algebraic geometry.

# 2 The Preliminaries

## 2.1 Divisors

**Definition 2.1.** *A* ***divisor*** *on $C$ is a formal finite sum of points, i.e. $D = \sum_i n_i p_i$ for $n_i \in \mathbb{Z}$, $p_i \in C$. The group of divisors under addition is denoted* $\mathrm{Div}(C)$.

**Definition 2.2.** *The* ***degree*** *of a divisor $D = \sum_i n_i p_i$* $\deg D = \sum_i n_i$

**Definition 2.3.** *Given a meromorphic function $f : C \to \mathbb{C}$ we define $(f) \in \mathrm{Div}(C)$ by*

$$(f) = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p$$

*For $D \in \mathrm{Div}(C)$, if $\exists f$ s.t. $D = (f)$ we say $D$ is a* ***principal divisor***.

**Lemma 2.4.** $(fg) = (f) + (g)$

**Corollary 2.5.** *Principal divisors form a subgroup* $\mathrm{Lin}(C) \leq \mathrm{Div}(C)$.

**Lemma 2.6.** *If $X$ is a compact Riemann surface and $f : X \to \mathbb{C}$ meromorphic then* $\deg(f) = 0$.

**Proposition 2.7.** *Let $C$ be compact. Then* $\mathrm{Lin}(C) = \{D \in \mathrm{Div}(C) \,|\, \deg(D) = 0\}$.

**Definition 2.8.** *The **divisor class group** of $C$ is* $\mathrm{Cl}(C) = {}^{\mathrm{Div}(C)}\!\big/_{\mathrm{Lin}(C)}$.

**Remark.** $\deg : \mathrm{Div}(C) \to \mathbb{Z}$ *is a group homomorphism and as the kernel is* $\mathrm{Lin}(C)$ *we see* $\mathrm{Cl}(C) \cong \mathrm{Im}\deg$

**Corollary 2.9.** $\mathrm{Cl}(\mathbb{CP}^n) \cong \mathbb{Z}$.

**Definition 2.10.** *Two divisors $D, E$ are **linearly equivalent** if $D - E$ is a principal.*

**Lemma 2.11.** *Linear equivalence of divisors is an equivalence relation.*

**Lemma 2.12.** *$f : X \to Y$ induces a group morphism $f : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ by*

$$f\left(\sum_i n_i p_i\right) = \sum_i n_i f(p_i)$$

**Proposition 2.13.** *If $f : X \to Y$ is a map of Riemann surfaces and $D \in \mathrm{Div}(X)$, then* $\deg(f(D)) = \deg f \cdot \deg D$.

**Definition 2.14.** *A divisor $D = \sum_i n_i p_i$ is **effective** if each $n_i \geq 0$.*

**Proposition 2.15.** *We have a partial ordering on $\mathrm{Div}(C)$ by saying $D \geq D'$ id $D - D'$ is effective.*

## 2.2 Abel-Jacobi

Suppose $C$ has genus $g$, then we know that $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ where the generators are the loops $\{\gamma_i\}_{i=1}^{2g}$. There is an alternative way to say this condition:

**Definition 2.16.** *The **canonical bundle** on a space $X$ with $\dim X = n$ is the line bundle of exterior $n$-forms on $X$.*

**Remark.** *Note we know the canonical bundle is a line bundle as there is only 1 basis element of $n$-forms on an $n$-dimensional space.*

**Proposition 2.17.** *If $X = C$ is a Riemann surface of genus $g$ then $H^0(C, K) \cong \mathbb{C}^g$.*

*Proof.* Find this ☐

**Corollary 2.18.** *We can take a basis $\{\omega_i\}_{i=1}^g$ of 1-forms on $C$.*

**Definition 2.19.** *The **Jacobian** of $C$ is defined to be*

$$J(C) = {}^{\mathbb{C}^g}\!\big/_{\Lambda}$$

*where $\Lambda$ is the lattice generated over $\mathbb{R}$ by the vectors*

$$\Omega_j = \left(\int_{\gamma_j} \omega_1, \ldots, \int_{\gamma_j} \omega_g\right), \quad 1 \leq j \leq 2g$$

**Definition 2.20.** *The **Abel-Jacobi map** for $p_0 \in C$ is*

$$\begin{aligned} u &: C \to J(C) \\ p &\mapsto \left( \int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g \right) \mod \Lambda \end{aligned}$$

*This is independent of the path of integration as we have quotiented by $\Lambda$.*

**Theorem 2.21** (Abel's Theorem)**.** *Let $u$ be the Abel-Jacobi map and $D, E$ effective divisors. Then $u(D) = u(E) \Leftrightarrow D \sim E$.*

**Theorem 2.22** (Jacobi's Theorem)**.** *The map Abel-Jacobi map is surjective.*

**Corollary 2.23.** *There is an isomorphism from the space of degree-0 divisors to the Jacobian.*

## 2.3 Bundles and Sheaves

We recall a few necessary bundle definitions and results:

**Definition 2.24.** *The tensor product of vector bundles $E, F \to M$ is $E \otimes F \to M$ s.t. $(E \otimes F)_m = E_m \otimes F_m$ for $m \in M$.*

**Lemma 2.25.** *If $O$ is the trivial line bundle then $E \otimes O = E$.*

**Definition 2.26.** *The **dual bundle** of a vector bundle $E \to M$ is $E^* \to M$ where the fibres of $E^*$ are the dual spaces of the fibres of $E$, with the transition functions $g_{ij}^* = \left(g_{ij}^T\right)^{-1}$.*

**Remark.** *We can check the cocycle condition here as*

$$g_{kj}^* g_{ji}^* = \left(g_{kj}^T\right)^{-1} \left(g_{ji}^T\right)^{-1} = \left(g_{ji}^T g_{kj}^T\right)^{-1} = \left([g_{kj} g_{ji}]^T\right)^{-1} = \left(g_{ki}^T\right)^{-1} = g_{ki}^*$$

**Example 2.27.** *The dual bundle to the tangent bundle is the cotangent bundle, i.e. $(TM)^* = T^*M$*

**Lemma 2.28.** $E \otimes E^* \cong \mathrm{End}(E)$.

**Lemma 2.29.** *Line bundles have tensor inverses, i.e given $L$, $\exists L^{-1}$ s.t. $L \otimes L^{-1} \cong O$ the trivial bundle.*

*Proof.* We will show this by showing $L^{-1} = L^*$. To trivialise $\mathrm{End}(L)$ we note here the transition maps are $g_{ij} \otimes g_{ij}^{-1} = 1 \otimes 1$ as $g_{ij}, g_{ij}^* \in \mathbb{F}$. Hence any section is globally defined. □

**Remark.** *Why is the identity section not global on any other vector bundle.*

These results motivate the definition of the **Picard group** which we will cover now:

**Definition 2.30.** *A **ringed space** is a pair $(X, O_X)$ where $X$ is a topological space and $O_X$ is a sheaf of rings on $X$. $O_X$ is called the **structure sheaf**.*

**Example 2.31.** *Given a topological space $X$, if we take $O_X$ to be $\mathbb{R}$-valued continuous functions on open subsets of $X$ then $(X, O_X)$ is a ringed space.*

**Definition 2.32.** *The **Picard group** of a locally ringed space $X$ is $\mathrm{Pic}(X)$ the group of isomorphism classes of line bundles on $X$ with the group operation being $\otimes$.*

**Remark.** *In place of line bundles we can actually say **invertible sheaves***

**Theorem 2.33.** $\mathrm{Cl}(C) \cong \mathrm{Pic}(C)$ *naturally.*

**Corollary 2.34.** *We get a group homomorphism* $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ *giving the degree of the corresponding divisor in* $\mathrm{Cl}(C)$.

**Corollary 2.35.** $\mathrm{Pic}(\mathbb{CP}^1) \cong \mathbb{Z}$.

**Notation.** *We denote the isomorphism class of line bundles degree $d$ as* $\mathrm{Pic}^d(C)$

**Remark.** *With this new notation we may rephrase the corollary of the Abel-Jacobi theorem to say* $J(C) \cong \mathrm{Pic}^0(C)$.

**Proposition 2.36.** *There is a canonical isomorphism* $\mathrm{Pic}(X) \cong H^1(X, O_X^\times)$.

**Corollary 2.37.** $T_L(\mathrm{Pic}^d(X)) \cong H^1(X, O_X)$

*Proof.* You need to use the exponential sheaf sequence. □

Let us now consider a specific class of bundles:

**Definition 2.38.** *The **hyperplane bundle on** $\mathbb{CP}^n$ is the bundle $\mathbb{C}^{n+1} \to \mathbb{CP}^n$ given by the standard projection $(z_0, \dots, z_n) \to [z_0 : \cdots : z_n]$. It is often denoted $\mathcal{O}(1)$. We denote $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$.*

**Definition 2.39.** *The **tautological line bundle** on projective space is $\mathcal{O}(-1) = \mathcal{O}(1)^*$. We denote $\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n}$.*

**Proposition 2.40.** *The canonical bundle on the projective space is* $K = \mathcal{O}(-n-1)$.

**Proposition 2.41.** $\mathrm{Pic}(\mathbb{CP}^n)$ *is generated by* $\mathcal{O}(\pm 1)$.

We make a few more useful definitions.

## 2.4 Lax Pairs and Spectral Curves

**Definition 2.42.** *A Lax pair is a pair of $\mathfrak{g} \subset \mathfrak{gl}_n$-valued matrices $A, B$, functions of a spectral parameter $\xi$ and time $t$ satisfying $\dot{A} = [A, B]$.*

**Proposition 2.43.** *The Lax equation is invariant under the substitution*

$$B \mapsto B + P(A, \xi)$$

*for polynomial* $P(x, \xi) \in \mathbb{C}[x, \xi]$.

**Definition 2.44.** *The **spectral curve** is $C$ given by the solution in $\mathbb{P}^1$ of*

$$\det\left[\eta I - A(\xi, t)\right] = 0$$

**Proposition 2.45.** *The flow $t \mapsto A(\xi, t)$ is isospectral.*

It will be the understanding of this isospectral flow that we want to gain. We formulate this flow as the family of holomorphic map gained by the eigenvectors

$$f_t : C \to \mathbb{CP}^{n-1}$$

Suppose that $C$ has degree $d$, then we know we can define

$$L_t = f_t^* \left(\mathcal{O}(1)\right) \in \mathrm{Pic}^d(C)$$

Lets choose a reference bundle $L_0 \in \mathrm{Pic}^d(X)$

**Lemma 2.46.** *The map*

$$\begin{aligned} \mathrm{Pic}^d(C) &\to J(C) \\ L &\mapsto L \otimes L_0^{-1} \end{aligned}$$

*is an isomorphism.*

Now knowing our result about the tangent space to the Picard group we can say $\frac{dL_t}{dt} \in H^1(C, O_C)$.

# References


[1] Phillip A. Griffiths. Linearizing flows and a cohomological interpretation of lax equations. *American Journal of Mathematics*, 107(6):pp. 1445–1484, 1985. ISSN 00029327, 10806377. doi: 10.2307/2374412.