Affine Toda

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1 Introduction

These will be a set of notes dedicated to a project looking at the affine toda lattice, but in situ we will cover some theory from Lie algebras and representations. See my notes on Kac-Moody algebras and Symmetries, Fields, and Particles for additional background which I will omit here as it is covered there.

2 Lie Algebra Conventions

Let $\mathfrak{g}$ be a simple Lie algebra of rank r and $\mathfrak{h} \subset \mathfrak{g}$ a fixed Cartan subalgebra with a inner product $(\ ,\ ) := (\ ,\ )_{\mathfrak{h}^*}$. Let Φ denote the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$ and W the associated Weyl group. By averaging we may always take $(\ ,\ )$ to be Weyl-invariant. We begin with

(i) the linearly independent set $\Delta := \{\alpha_1, \dots, \alpha_r\} \subset \Phi \subset \mathfrak{h}^*$, the simple roots. To each $\alpha \in \Phi$ set
$$\epsilon_\alpha := \frac{2}{(\alpha, \alpha)}, \quad \alpha^\vee := \epsilon_\alpha \alpha := \frac{2\alpha}{(\alpha, \alpha)}.$$

Here $\alpha^\vee \in \mathfrak{h}^*$ are the **coroots** (or **dual** roots) and $\Phi^\vee := \{\alpha^\vee \,|\, \alpha \in \Phi\}$[1]. We write $\epsilon_i := 2/(\alpha_i, \alpha_i)$ for $\alpha_i \in \Delta$.

(ii) The Cartan matrix is $A := (a_{ij})$ with $a_{ij} := (\alpha_i^\vee, \alpha_j)$. Then $A = DB$ where $D = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$ and $B := (b_{ij})$, $b_{ij} = (\alpha_i, \alpha_j)$ is symmetric; A is symmetrizable. Then

$$(\alpha_i^\vee, \alpha_j^\vee) = \epsilon_i(\alpha_i, \alpha_j)\epsilon_j = \epsilon_i\, \alpha_i(\alpha_j^\vee).$$

The choice of ϵ_α is so as to make the Cartan matrix have two's along the diagonal,

(iii) Let $\{H_a\}$ $(a = 1, \ldots, r)$ be a basis of $\mathfrak{h}$. The Cartan-Weyl basis $\{H_a\}$ and $\{E_\alpha\}$, $\alpha \in \Phi$ satisfies

$$[H_a, H_b] = 0, \quad [H_a, E_\alpha] = \alpha_a\, E_\alpha, \quad \alpha_a := \alpha(H_a).$$

The Jacobi identity then yields for $\alpha, \beta \in \Phi$ that

$$[H_a, [E_\alpha, E_\beta]] = (\alpha + \beta)_a\, [E_\alpha, E_\beta]$$

and so

$$[E_\alpha, E_\beta] = \begin{cases} c_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \Phi. \end{cases}$$

Finally, using the fact that the $\mathrm{centraliser}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ we see that $[E_\alpha, E_{-\alpha}] \in \mathfrak{h}$.

(iv) Denote the Killing form by

$$\kappa(x, y) := \mathrm{Tr}\, \mathrm{ad}_x \circ \mathrm{ad}_y, \qquad x, y \in \mathfrak{g}. \tag{2.0.1}$$

Then

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

The non-degeneracy of the Killing form means we get an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ such that $\kappa(h_1, h_2)_{\mathfrak{h}} = \nu(h_1)(h_2)$. For each $\alpha \in \Phi$ define $t_\alpha \in \mathfrak{h}$ by $\nu(t_\alpha) = \alpha$. Thus $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha)$. Then for all $h \in \mathfrak{h}$

$$\begin{aligned} \kappa(h, [E_\alpha, E_{-\alpha}]) &= \kappa([h, E_\alpha], E_{-\alpha}]) = \alpha(h)\kappa(E_\alpha, E_{-\alpha}) = \kappa(t_\alpha, h)\kappa(E_\alpha, E_{-\alpha}) \\ &= \kappa(\kappa(E_\alpha, E_{-\alpha})\, t_\alpha, h). \end{aligned}$$

and the non-degeneracy of the Killing form now yields that

$$[E_\alpha, E_{-\alpha}] = \kappa(E_\alpha, E_{-\alpha})\, t_\alpha.$$

(v) Upon noting that

$$\begin{aligned} \mathrm{ad}_{H_a} \circ \mathrm{ad}_{H_b}(h) &= 0 \\ \mathrm{ad}_{H_a} \circ \mathrm{ad}_{H_b}(E_\alpha) &= \alpha_a \alpha_b\, E_\alpha \end{aligned}$$

we find

$$\kappa(H_a, H_b) = \sum_{\alpha \in \Phi} \alpha_a \alpha_b.$$

[1]Caution: Kac's notation has $\alpha^\vee \in \mathfrak{h}$

(vi) The Weyl group acts irreducibly on the vector space $\mathfrak{h}^*$. If we write the W-invariant metric as $(\alpha, \beta) = \alpha_a g^{ab} \beta_b$ then

$$\sum_{w \in W} (w\alpha)_a (w\alpha)_b = \frac{(\alpha, \alpha)}{r} \, |\mathcal{O}(\alpha)| \, g_{ab}.$$

Now a root system Φ consists of at most root vectors of two lengths two (long L and short S), and those vectors of the same length form a single orbit. Then

$$\sum_{\alpha \in \Phi} \alpha_a \alpha_b = \left((\alpha_L, \alpha_L) \, |\mathcal{O}(\alpha_L)| + (\alpha_S, \alpha_S) \, |\mathcal{O}(\alpha_S)| \right) \, g_{ab} = 2g \, \frac{(\alpha_L, \alpha_L)}{2} \, g_{ab}.$$

Here g is the **dual Coxeter** number. Therefore

$$\kappa(H_a, H_b) = 2g \, \frac{(\alpha_L, \alpha_L)}{2} \, g_{ab}.$$

(vii) Let us set $c := 2g \, (\alpha_L, \alpha_L)/2$ so that $\kappa_{ab} := \kappa(H_a, H_b) = c \, g_{ab}$. We wish to express t_α in terms of the basis $\{H_a\}$. Now

$$\kappa(t_\alpha, H_a) = \nu(t_\alpha)(H_a) = \alpha(H_a) = \alpha_a.$$

If $t_\alpha = x^b H_b$ then $x^b \kappa_{ba} = \alpha_a$ and so $x^b = \alpha_a g^{ab}/c = \alpha^b/c$ and

$$t_\alpha = \frac{1}{c} \alpha^a H_a = \frac{1}{c} \alpha \cdot H.$$

Note that

$$\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) = \frac{\alpha^a}{c} \, \kappa(H_a, H_b) \, \frac{\alpha^b}{c} = \frac{\alpha^a}{c} \, c \, g_{ab} \, \frac{\alpha^b}{c} = \frac{(\alpha, \alpha)}{c}.$$

(viii) Set

$$H_\alpha := \frac{2 \, t_\alpha}{\kappa(t_\alpha, t_\alpha)} = \frac{2 \, \alpha \cdot H}{(\alpha, \alpha)} = \alpha^\vee \cdot H.$$

Upon noting that $[t_\alpha, E_\alpha] = \alpha(t_\alpha) E_\alpha = (\alpha, \alpha) E_\alpha / c$ then for all $\alpha \in \Phi$,

$$[H_\alpha, E_\alpha] = 2 \, E_\alpha.$$

Now

$$[E_\alpha, E_{-\alpha}] = \kappa(E_\alpha, E_{-\alpha}) \, t_\alpha = \left[\frac{1}{2} \kappa(E_\alpha, E_{-\alpha}) \kappa(t_\alpha, t_\alpha) \right] H_\alpha.$$

Setting

$$E_\alpha^{Ch} := E_\alpha \Big/ \sqrt{\frac{1}{2} \kappa(E_\alpha, E_{-\alpha}) \kappa(t_\alpha, t_\alpha)}$$

we then have for all $\alpha \in \Phi$ the standard sl_2 relations

$$[H_\alpha, E_\alpha^{Ch}] = 2 \, E_\alpha^{Ch}, \quad [E_\alpha^{Ch}, E_{-\alpha}^{Ch}] = H_\alpha.$$

Further

$$[H_\alpha, E_\beta^{Ch}] = \epsilon_\alpha \alpha^a \beta(H_a) E_\beta^{Ch} = (\alpha^\vee, \beta) \, E_\beta^{Ch}$$

and

$$\kappa(H_\alpha, H_\beta) = c \, (\alpha^\vee, \beta^\vee), \quad \kappa(E_\alpha^{Ch}, E_{-\alpha}^{Ch}) = c \, \epsilon_\alpha.$$

(ix) The Chevalley basis consists of $\{H_\alpha\}$ for $\alpha \in \Delta$ and $\{E^{Ch}_\beta\}_{\beta\in\Phi}$, where

$$[H_\alpha, E^{Ch}_\beta] = (\alpha^\vee, \beta)\, E^{Ch}_\beta,$$
$$[E^{Ch}_\alpha, E^{Ch}_\beta] = \begin{cases} H_\alpha & \text{if } \alpha+\beta = 0, \\ N_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi, \\ 0 & \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Phi. \end{cases}$$

with

$$\kappa(H_\alpha, H_\beta) = c\,(\alpha^\vee, \beta^\vee), \quad \kappa(E^{Ch}_\alpha, E^{Ch}_{-\alpha}) = c\,\epsilon_\alpha, \quad c = 2g\,\frac{(\alpha_L, \alpha_L)}{2}.$$

(x) There is a unique maximal root, which we denote as $\Theta = \sum_{\alpha\in\Delta} n_\alpha\,\alpha$ be the highest root. Set $\overline{\Delta} = \Delta \cup \{-\Theta\}$

3 Background Theory

3.1 Lie Algebras and Representation Theory

We start with a recap of Chapters II and III of [3]. Denote the base of simple roots as Δ.

Proposition 3.1. *There exists a unique root of highest weight $\theta = \sum_{\alpha\in\Delta} n_\alpha \alpha \in \mathfrak{h}^*$.*

Proposition 3.2. *Let A be the cartan matrix corresponding to $\mathfrak{g}$ of finite type, rank n, and let $h_\theta = \sum_i n_i h_i \in \mathfrak{h}$ be the element corresponding to θ under the natural iso $\mathfrak{h} \cong \mathfrak{h}^*$. Define $\hat{A}$ by*

$$\begin{aligned} \hat{A}_{ij} &= A_{ij},\ 1 \leq i, j \leq n \\ \hat{A}_{00} &= 2 \\ \hat{A}_{i0} &= -\sum_j m_j A_{ij} \\ \hat{A}_{0j} &= -\sum_i n_i A_{ij} \end{aligned}$$

*Then $\hat{A}$ is an affine generalised Cartan matrix corresponding to an **untwisted affine Dynkin diagram**.*

Proposition 3.3. *The Lie algebra corresponding to $\hat{A}$ is isomorphic to the affine Kac-Moody Lie algebra $\mathcal{L}\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$*

3.2 Monopoles

Nahm established an equivalence between non-singular charge-n $SU(2)$-monopoles and $\{T_i(s)\,|\,T_i \in M_n(\mathbb{C}),\ s \in [0,2]\}$ subject to

1. Nahm's equation:

$$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}\,[T_j, T_k]$$

2. $T_i(s)$ is regular for $s \in (0,2)$ and has simple poles at $s = 0, 2$ with residues that form an irreducible, n-dimensional rep of $\mathfrak{su}(2)$

3. $T_i(2) = -T_i^\dagger(s)$, $T_i(s) = T_i^T(2-s)$.

Remark. *We could have here included another matrix $T_4 = -T_4^\dagger$ by modifying Nahm's equations to*

$$\dot{T}_i = [T_4, T_i] + \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j(z), T_k(z)]$$

but this can always be gauged away by the transform

$$\begin{aligned} T_i &\mapsto uT_iu^{-1} \\ T_4 &\mapsto uT_0u^{-1} - \frac{du}{ds}u^{-1} \end{aligned}$$

for $u : (0,2) \to U(n)$ satisfying $u(2-s) = \left(u^T(s)\right)^{-1}$.

Proposition 3.4. *Nahm's equation is equivalent to the Lax equation*

$$\left[\frac{d}{ds} + M, L\right] = 0$$

where

$$\begin{aligned} L(\zeta) &= (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2 \\ M(\zeta) &= -iT_3 + (T_1 - iT_2)\zeta \end{aligned}$$

Proof. It will be convenient to intoduce the notation

$$\begin{aligned} \alpha &= iT_3 \\ \beta &= T_1 + iT_2 \end{aligned}$$

s.t. this Lax pair can be rewritten as

$$\begin{aligned} L(\zeta) &= \beta - (\alpha + \alpha^\dagger)\zeta - \beta^\dagger\zeta^2 \\ M(\zeta) &= -\alpha - \beta^\dagger\zeta \end{aligned}$$

Then the Lax equation says, splitting by order in ζ

$$\frac{d\beta}{ds} - [\alpha, \beta] = 0 \qquad -\frac{d}{ds}(\alpha + \alpha^\dagger) + \left[\alpha, \alpha^\dagger\right] + \left[\beta, \beta^\dagger\right] = 0 \qquad -\frac{d\beta^\dagger}{ds} + \left[\beta^\dagger, \alpha^\dagger\right] = 0$$

and the last equation is redundant leaving just

$$\begin{aligned} \frac{d\beta}{ds} - [\alpha, \beta] &= 0 \quad \text{(the \textbf{complex equation})} \\ \frac{d}{ds}(\alpha + \alpha^\dagger) - \left(\left[\alpha, \alpha^\dagger\right] + \left[\beta, \beta^\dagger\right]\right) &= 0 \quad \text{(the \textbf{real equation})} \end{aligned}$$

To show these are equivalent to Nahm's equations, it now just a matter of explicitly writing these out. ∎

Lemma 3.5. *The complex equation is invariant under the complex gauge group $GL_k(\mathbb{C})$ acting as*

$$\begin{aligned}\alpha &\mapsto g\alpha g^{-1} + \frac{dg}{ds}g^{-1} \\ \beta &\mapsto g\beta g^{-1}\end{aligned}$$

Proposition 3.6. *The general local solution to the complex equation is*

$$\begin{aligned}\alpha &= -g^{-1}\frac{dg}{ds} \\ \beta &= g^{-1}\beta' g\end{aligned}$$

for some constant matrix β'.

4 Affine Toda

We start by introducing affine Toda from a field theory perspective, following [2]:

4.1 Independent Definition

Definition 4.1. *Let $\mathfrak{g}$ be a rank-r Lie algebra with simple roots α_i, taking a particular realisation of these as vectors in $\mathbb{R}^r$. The* ***Toda field theory*** *is that with $\mathbb{R}^r$-valued field $\mathbf{\Phi} = (\phi^a)$ and Lagrangian*

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{\lambda}{\beta^2}\sum_{i=1}^{r} e^{\beta\alpha_i\cdot\mathbf{\Phi}}$$

for parameters λ, β.

Proposition 4.2. *The corresponding classical equations of motion are*

$$\partial^2\phi_j = -\frac{\lambda}{\beta}\sum_{i=1}^{r} C_{ji}e^{\beta\phi_i}$$

where $\phi_j = \alpha_j \cdot \mathbf{\Phi}$ and

$$C_{ij} = \alpha_i \cdot \alpha_j$$

Proof. The e.o.m are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi^a} &= \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a} \\ \Rightarrow -\frac{\lambda}{\beta}\sum_{i=1}^{r}(\alpha_i)^a e^{\beta\phi_i} &= \partial^2\phi^a\end{aligned}$$

and the result follow from contracting with α_j. □

Remark. *If we shift* $\phi_i \mapsto \phi_i + \frac{1}{\beta}\log\left(\frac{2}{\alpha_i^2}\right)$ *the matrix* C *is replaced with*

$$A_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}$$

which we recognise to be the Cartan matrix.

This field theory does not have a unique minimum for us to consider as the classical vacuum, so we will want to deform it in some way. The following result motivates how do this deformation:

Proposition 4.3. $1+1$*-dimensional Toda field theory has a zero-curvature representation*

Proof. We follow [4]. Define light-cone coordinates

$$u = \frac{1}{2}(x+t)$$
$$v = \frac{1}{2}(x-t)$$

s.t.

$$\partial_u \partial_v = -\partial_t^2 + \partial_x^2 = -\partial_\mu \partial^\mu$$

and a gauge potential with

$$A_u = \sum_{i=1}^{r} \left(\frac{1}{2}\right)$$

finish this off... □

This is thus integrable, and so when we want to generalise this system, we look for integrable deformations.

Definition 4.4. *The field theory obtained by perturbing Toda by*

$$\delta V(\mathbf{\Phi}) = \frac{\epsilon\lambda}{\beta^2} e^{\beta\alpha_0 \cdot \mathbf{\Phi}}$$

s.t. $\{\alpha_0, \alpha_j\}$ *are roots of an affine Lie algebra is called* ***affine Toda field theory***.

Affine Toda has a minimum $\mathbf{\Phi}^{(0)}$ satisfying

$$\sum_i \alpha_i e^{\beta\alpha_i \cdot \mathbf{\Phi}^{(0)}} = -\epsilon\alpha_0 e^{\beta\alpha_0 \cdot \mathbf{\Phi}^{(0)}}$$

If we centre around this by letting $\mathbf{\Phi} = \mathbf{\Phi}^{(0)} + \boldsymbol{\phi}$ we have

$$\begin{aligned} V(\boldsymbol{\phi}) &= \frac{\epsilon\lambda}{\beta^2} e^{\beta\alpha_0 \cdot \mathbf{\Phi}^{(0)}} \left[e^{\beta\alpha_0 \cdot \boldsymbol{\phi}} - \sum_{i,j} e^{\beta\alpha_i \cdot \boldsymbol{\phi}} \left(C^{-1}\right)_{ij} \alpha_j \cdot \alpha_0 \right] \\ &= \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta\alpha_i \cdot \boldsymbol{\phi}} \end{aligned}$$

where we have let $m^2 = \epsilon\lambda e^{\beta\alpha_0 \cdot \mathbf{\Phi}^{(0)}}$ and written $\alpha_0 = \sum_{i=1}^{r} n_i \alpha_i$, $n_0 = 1$. The simplest deformation is to take $\alpha_0 = -\Theta$, the maximal root.

Example 4.5. *Consider the Lie algebra A_n. We may find a realisation of the n simple where $\alpha_i = e_i - e_{i+1}$, where e_i are the canonical basis vectors of $\mathbb{R}^n$ and we take $e_{n+1} = e_1$. Moreover, the maximal root has all $n_i = 1$, $1 \leq i \leq n$.*

4.2 Braden Approach

We will now obtain the affine Toda Field theory through a different lens.
Let $\mathfrak{g}$ be a compact semisimple Lie algebra of rank r with a fixed Cartan subalgebra $\mathfrak{h}$. Let $\{X_\mu\} = \{H_i, E_\alpha\}$ be a Cartan-Weyl basis where $\{H_i\}$ is a basis of $\mathfrak{h}$ and $\{E_\alpha\}$ the set of step operators (labelled by the root system Φ of $\mathfrak{g}$) and

$$[H_i, E_\alpha] = \alpha_i\, E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha \cdot H, \quad [E_\alpha, E_\beta] = N_{\alpha,\beta}\, E_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \Phi.$$

Recalling we have the maximal root $\Theta = \sum_{\alpha\in\Delta} n_\alpha \alpha$, and extend $\overline{\Delta} = \Delta \cup \{-\Theta\}$, $n_{-\Theta} = 1$ consider

$$E = \sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, E_\alpha, \quad E^\dagger = \sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, E_{-\alpha}.$$

Lemma 4.6. $\left[E, E^\dagger\right] = 0.$

Proof.

$$[E, E^\dagger] = \sum_{\alpha\in\Delta} n_\alpha\, [E_\alpha, E_{-\alpha}] + [E_{-\Theta}, E_\Theta] = 0$$

□

Consider a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\kappa\left(\partial_\mu\phi, \partial^\mu\phi\right) - \kappa\left(e^{b\phi} E e^{-b\phi}, E^\dagger\right)$$

for $\phi = \phi^i H_i$.

Proposition 4.7. *The corresponding field equations are*

$$\partial_\mu\partial^\mu\phi + b\,[e^{b\phi} E e^{-b\phi}, E^\dagger] = 0.$$

These are the affine Toda field equations if we can make the normalisation $\kappa(E_\alpha, E_{-\alpha}) = \frac{m^2}{\lambda^2}$.

Proof. We first expand

$$\kappa(\partial_\mu\phi, \partial^\mu\phi) = \partial_\mu\phi^i \partial^\mu\phi^j\, \kappa(H^i, H^j)$$

As $\mathfrak{g}$ is compact and semi-simple we can have chosen the basis of the Cartan subalgebra s.t. $\kappa(H^i, H^j) = -\kappa\delta^{ij}$, and so we get the $\partial^\mu\partial_\mu\phi$ from the kinetic part of the equations of motion. Then

$$e^{b\phi} E e^{-b\phi} = \exp(\mathrm{ad}_{b\phi}) E = \sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha} e^{b\alpha(\phi)} E_\alpha$$

giving

$$\begin{aligned}\kappa\left(e^{b\phi}Ee^{-b\phi}, E^\dagger\right) &= \sum_{\alpha,\beta\in\overline{\Delta}} \sqrt{n_\alpha n_\beta}\, e^{b\alpha(\phi)}\kappa(E_\alpha, E_{-\beta}) \\ &= \sum_{\alpha\in\overline{\Delta}} n_\alpha e^{b\alpha(\phi)}\kappa(E_\alpha, E_{-\alpha}) \quad \text{as } \alpha-\beta\notin\Phi \text{ for } \alpha,\beta\in\overline{\Delta}\end{aligned}$$

Taking the differential wrt to ϕ we find that the EL equations are

$$\begin{aligned}-\partial_\mu\partial^\mu\phi &= -b\sum_{\alpha\in\overline{\Delta}} n_\alpha(\alpha\cdot H)e^{b\alpha(\phi)}\kappa(E_\alpha, E_{-\alpha}) \\ &= b\sum_{\alpha\in\overline{\Delta}} n_\alpha t_\alpha e^{b\alpha(\phi)}\kappa(E_\alpha, E_{-\alpha}) \\ &= b\sum_{\alpha\in\overline{\Delta}} n_\alpha e^{b\alpha(\phi)}\left[E_\alpha, E_{-\alpha}\right] = b\left[e^{b\phi}Ee^{-b\phi}, E^\dagger\right]\end{aligned}$$

□

Remark. *We can rewrite this Lagrangian as*

$$\mathcal{L} = \frac{1}{2}\operatorname{Tr}\left[\operatorname{ad}(\partial_\mu\phi)\operatorname{ad}(\partial^\mu\phi)\right] - \operatorname{Tr}\left[\operatorname{ad}(e^{b\phi}Ee^{-b\phi})\operatorname{ad}(E^\dagger)\right]$$

by definition of the Killing form, and so we often make the adjoint rep implicit in this equation to just write

$$\mathcal{L} = \operatorname{Tr}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - e^{b\phi}Ee^{-b\phi}E^\dagger\right]$$

Given the following remark, to make this notation more sensible we show the following results:

Lemma 4.8. *If A, B are matrices s.t. $A^\dagger = A$ and $[A,B]\lambda B$, then $\left[A, B^\dagger\right] = -\overline{\lambda}B^\dagger$.*

Lemma 4.9. *Wrt to the Cartan-Weyl basis,* $\operatorname{ad}(H)^\dagger = \operatorname{ad}(H)$, $\operatorname{ad}(E_\alpha)^\dagger = \operatorname{ad}(E_{-\alpha})$.

Proof. We first assume we have chosen our Lie algebra to a vector space over $\mathbb{R}$. Then, we know that the elements of the Cartan subalgebra are diagonalised wrt the CW basis, and their diagonal must be real, so we instantly have that $\operatorname{ad}(H) = \operatorname{ad}(H)^T = \operatorname{ad}(H)^\dagger$.
Now we apply the previous lemma as

$$\left[\operatorname{ad}(H), \operatorname{ad}(E_\alpha)\right] = \operatorname{ad}\left([H, E_\alpha]\right) = \operatorname{ad}\left(\alpha(H)E_\alpha\right) = \alpha(H)\operatorname{ad}(E_\alpha)$$

to see that $\operatorname{ad}(E_\alpha)^\dagger$ is in the same eigenspace of $\operatorname{ad}(\operatorname{ad}(H))$ as $\operatorname{ad}(E_{-\alpha})$. Whether it is then possible to make them equal depends on the scaling freedom we have of the E_α, and the freedom of the choice of basis of $\mathfrak{h}$. We now draw

$$\mathrm{ad}(E_\alpha) = \left(\begin{array}{ccc|c|c|ccc}
0 & \cdots & 0 & 0 & c\left(\kappa_{i1}^{-1}\alpha^i\right) & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & c\left(\kappa_{ir}^{-1}\alpha^i\right) & 0 & \cdots & 0 \\
\hline
-\alpha^1 & \cdots & -\alpha^r & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & 0 & 0 & & & \\
\vdots & \ddots & \vdots & \vdots & \vdots & & M_{\beta\gamma} & \\
0 & \cdots & 0 & 0 & 0 & & &
\end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c}
0_{ij} & 0_j & c\left(\kappa_{ij}^{-1}\alpha^i\right) & 0_{\beta j} \\
\hline
-\alpha^i & 0 & 0 & 0_\beta \\
\hline
0_i & 0 & 0 & 0_\beta \\
\hline
0_{i\gamma} & 0_\gamma & 0_\gamma & M_{\beta\gamma}
\end{array}\right)$$

where

- M has only non-zero entries $M_{\beta,\beta+\alpha} = N_{\alpha,\beta}$ when $\alpha+\beta$ is a root
- we have ordered our CW basis as $(H_i)_{1\leq i\leq r}, E_\alpha, E_{-\alpha}, (E_\beta)_{\beta\in\Phi\setminus\{\pm 1\}}$
- $c = \kappa\left(E_\alpha, E_{-\alpha}\right)$.

We therefore see that by choosing a basis of $\mathfrak{h}$ that diagonalises κ and then removing any residual scaling through the effect on c of scaling $E_{\pm\alpha}$, we can make the upper left hand s.t. $\mathrm{ad}(E_\alpha)^\dagger = \mathrm{ad}(E_{-\alpha})$. To complete we need to know what happens to M. What we need is that $N_{\alpha,\gamma} = N_{-\alpha,\gamma}$. Simple manipulation using the Jacobi rule gives us that

$$N_{\alpha,\gamma}N_{-\alpha,\alpha+\gamma} + N_{-\alpha,\gamma}N_{\alpha,\gamma-\alpha} = c(\gamma,\alpha)$$

so we see that if we normalise the basis vectors corresponding to the simple roots first, we may inductively proceed the basis vectors corresponding to higher roots to get the desired result. $\square$

Proposition 4.10. *$1+1$-dimsensional Toda field theory has a zero curvature representation.*

Proof. With $ds^2 = dt^2 - dx^2 = -dx^+dx^-$, $x^\pm = x \pm t$, $\partial_x = \partial_+ + \partial_-$, $\partial_t = \partial_+ - \partial_-$ the field equations become

$$-\partial_{+-}\phi + \frac{b}{4}\left[e^{b\phi}Ee^{-b\phi}, E^\dagger\right] = 0$$

which are the consistency of

$$0 = [\partial_+ + A_+, \partial_- + A_-], \quad A_+ = \frac{b}{2}e^{b\phi/2}\,E\,e^{-b\phi/2} + \frac{b}{2}\partial_+\phi, \quad A_- = \frac{b}{2}e^{-b\phi/2}\,E^\dagger\,e^{b\phi/2} - \frac{b}{2}\partial_-\phi.$$

To see this recall that $e^\phi E_\alpha e^{-\phi} = e^{\mathrm{ad}_\phi}E_\alpha = e^{\alpha(\phi)}E_\alpha$, giving

$$A_+ = \frac{b}{2}\sum_{\alpha\in\overline{\Delta}}\sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}E_\alpha + \frac{b}{2}\partial_+\phi, \quad A_- = \frac{b}{2}\sum_{\alpha\in\overline{\Delta}}\sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}E_{-\alpha} - \frac{b}{2}\partial_-\phi.$$

Then

$$A_1 = A_+ + A_- = \frac{b}{2}\partial_0\phi + b\sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, e^{b\alpha(\phi)/2} X_\alpha^+$$
$$A_0 = A_+ - A_- = \frac{b}{2}\partial_1\phi + b\sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, e^{b\alpha(\phi)/2} X_\alpha^-$$

where $X_\alpha^\pm = (E_\alpha \pm E_{-\alpha})/2$. Thus the zero curvature condition is

$$0 = [\partial_0 + A_0, \partial_1 + A_1] = \frac{b}{2}(\partial_0^2 - \partial_1^2)\phi + \frac{b^2}{2}\sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, e^{b\alpha(\phi)/2}\left[\alpha\,(\partial_0\phi)\; X_\alpha^+ - \alpha\,(\partial_1\phi)\; X_\alpha^-\right]$$
$$+ \frac{b^2}{2}\sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\, e^{b\alpha(\phi)/2}\left([\partial_1\phi, X_\alpha^+] - [\partial_0\phi, X_\alpha^-]\right) + \frac{b^2}{2}\sum_{\alpha\in\overline{\Delta}} n_\alpha\, e^{b\alpha(\phi)}[E_\alpha, E_{-\alpha}]$$

so giving, using $[\partial_\mu\phi, X_\alpha^\pm] = \alpha\,(\partial_\mu\phi)\, X_\alpha^\mp$,

$$0 = \partial_\mu\partial^\mu\phi + b\sum_{\alpha\in\overline{\Delta}} n_\alpha\, e^{b\alpha(\phi)}[E_\alpha, E_{-\alpha}] = \partial_\mu\partial^\mu\phi + b\,[e^{b\phi}Ee^{-b\phi}, E^\dagger].$$

Finally we note

$$[\partial_0 + A_0, \partial_1 + A_1] = 2\,[\partial_+ + A_+, \partial_- - A_-]$$

□

Remark. *Recall we have the field strength tensor defined as*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]$$

so the fact that the commutators in light cone coordinates are effectively those in standard Minkowski is not surprising.

Remark. *In the zero curvature equation there so far has been no appearance of a spectral parameter. We see that taking*

$$X_\alpha^\pm = \frac{1}{2}\left(\zeta^{r_\alpha}E_\alpha \pm \zeta^{-r_\alpha}E_{-\alpha}\right)$$

will result in the same equations of motion. Two common choices in the literature are

1. $r_\alpha = 1$ *for all* $\alpha \in \overline{\Delta}$,
2. $r_{-\Theta} = 1$ *and* $r_\alpha = 0$ *for all* $\alpha \in \Delta$.

4.3 The Spectral Curve

Having now found the zero curvature representation of affine Toda in 1 + 1-dimensions we make the following observation:

Proposition 4.11. *Assuming that $\partial_0\phi = 0$, the zero curvature equation becomes a Lax equation with $L = A_0$, $M = A_1$.*

Proof. Then the independence from the 0-coordinate gives $0 = [\partial_1 + A_1, \partial_0 + A_0] = \partial_1 A_0 + [A_1, A_0]$ □

With this we can start to calculate the spectral curve, which for notational purposes we will fix to be

$$\mathcal{C} = \{\det\left(L(\zeta) - \eta I\right) = 0\}$$

I have coded in Sage a worksheet that takes the fundamental rep corresponding to a Dynkin diagram and finds the corresponding curve as a function of $q_i = \phi^i$, $p_i = -\dot{\phi}^i$.

Example 4.12. *The simplest example we can consider is the A_1 Lie algebra, which has roots $\Phi = \{\pm\alpha\}$, highest root $\Theta = \alpha$. A bit of explicit calculation then finds our Lax matrix to be*

$$\frac{2}{b}L(\zeta) = -pR(H) + \underbrace{\left(e^{bq}\zeta^{r_1} - e^{-bq}\zeta^{-r_0}\right)}_{e(\zeta)} R(E) + \underbrace{\left(e^{-bq}\zeta^{r_0} - e^{bq}\zeta^{-r_1}\right)}_{f(\zeta)} R(F)$$

where we have now made the representation R explicit and called $E = E_\alpha, F = E_{-\alpha}$. We also remark the symmetry $e(\zeta) = -f(\zeta^{-1})$. We will drop the factor of $\frac{2}{b}$ from now on as it can be absorbed wlog into η. We now recall that the reps of A_1 are simple when considered as highest weight reps $R = R_\Lambda$, s.t. L is tridiagonal with (using Wikipedia's notation)

$$\begin{aligned} a_n &= -p\left[\Lambda - 2(n-1)\right] \\ b_n &= es_n, \quad (s_n = (\Lambda - n + 1)n \textit{ a coefficient from rep theory}) \\ c_n &= f \end{aligned}$$

with $1 \le n \le \Lambda + 1$. Hence it is simple to find that the curve can be written as

$$N \textit{ odd } \Rightarrow \mathcal{C} = w \prod_{k \textit{ even}}^{N} (w^2 - k^2 y)$$

$$N \textit{ even } \Rightarrow \mathcal{C} = \prod_{k \textit{ odd}}^{N} (w^2 - k^2 y)$$

where $y = ef + p^2$ is a function of z.

This reducibility into factors is quite general as the following result shows:

Proposition 4.13. *$\forall\lambda$ a dominant weight, $\exists p_\lambda : \mathfrak{g} \times \mathbb{C} \to \mathbb{C}$ a polynomial s.t. for $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation with weight multiplicities m_λ*

$$\det\left[\rho(x) - z\right] = \prod_\lambda \left[p_\lambda(x, z)\right]^{m_\lambda}$$

where λ runs through dominant weights.

Proof. For $x \in \mathfrak{g}$ write $x = x_s + x_n$, and let $\mathfrak{h}$ be the CSA wrt the weights are defined. x_s must belong to a Cartan subalgebra, and as all CSAs are conjugate $\exists g \in G$ s.t. $\mathrm{Ad}_g\, x_s \in \mathfrak{h}$. Define

$$p_\lambda(x,z) = \prod_{\gamma \in W\cdot\lambda} \left[\gamma\left(\mathrm{Ad}_g\, x_s\right) - z\right]$$

where W is the Weyl group acting on dominant weights, and then viewing $\gamma \in \mathfrak{h}^*$. The choice of g does not affect p_λ as the ambiguity is up to conjugation by an element of the Weyl group. As x and x_s have the same eigenvalues with the same multiplicities we have

$$\det\left[\rho(x) - z\right] = \det\left[\rho(x_s) - z\right] = \prod_\lambda \left[p_\lambda(x,z)\right]^{m_\lambda}$$

$\square$

Questions:

1. What is the effect on the spectral curve of the different scalings r_α? Are the curves birational?
2. We have an action of the Weyl group on the roots, so does this preserve the spectral curve, or what is its effect on the curve?

4.4 Perturbative Theory

To make contact with perturbative affine Toda theory we note the expansion

$$\begin{aligned}
\mathrm{Tr}\, e^{b\phi} E e^{-b\phi} E^\dagger &= \mathrm{Tr}(1 + b\phi + \frac{b^2}{2}\phi^2 + \frac{b^3}{6}\phi^3 + \ldots)E(1 - b\phi + \frac{b^2}{2}\phi^2 - \frac{b^3}{6}\phi^3 + \ldots)E^\dagger \\
&= \mathrm{Tr}\left(EE^\dagger + b\phi[E,E^\dagger] + \frac{b^2}{2}\phi[E,[E^\dagger,\phi]] + \frac{b^3}{6}\phi[[\phi,E^\dagger],[\phi,E]] + \ldots\right) \\
&= \mathrm{Tr}\, EE^\dagger + \frac{b^2}{2}\,\mathrm{Tr}\,\phi[E,[E^\dagger,\phi]] + \frac{b^3}{6}\,\mathrm{Tr}\,\phi[[\phi,E^\dagger],[\phi,E]] + \ldots
\end{aligned}$$

which is further simplified upon specifying the normalisations $\mathrm{Tr}\, E_\alpha E_{-\alpha}$. This form of the affine Toda equation has been chosen so that $\phi = 0$ is a classical solution. If we work with

$$\mathrm{Tr}\, E_\alpha E_{-\alpha} = \epsilon_\alpha^{-1}$$

then

$$\mathrm{Tr}\, EE^\dagger = \sum_{\alpha \in \overline{\Delta}} n_\alpha^\vee = g, \qquad n_\alpha^\vee := n_\alpha/\epsilon_\alpha,$$

where g is the dual Coxeter number. If we work with the (unshifted) Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \sum_{\alpha \in \overline{\Delta}} \epsilon_\alpha e^{(\alpha,\psi)}$$

and expand $\psi = \psi^i \epsilon_i \lambda_i$ with $(\alpha_i^\vee, \lambda_j) = \delta_{ij}$ for the simple roots, then we obtain equations of motion

$$\epsilon_i(\lambda_i,\lambda_j)\epsilon_j\partial_\mu\partial^\mu\psi^j = -\sum_{\alpha \in \overline{\Delta}} \epsilon_\alpha(\alpha, \epsilon_i\lambda_i)\, e^{(\alpha,\psi)} = -\epsilon_i\, e^{\psi^i} + n_i\epsilon_{-\Theta}\, e^{-(\Theta,\psi)}.$$

Then with $K_{ij} = (\alpha_i^\vee, \alpha_j) = \epsilon_i(\alpha_i, \alpha_j) := \epsilon_i b_{ij}$ and $(\lambda_i, \lambda_j) = G_{ij} = \epsilon_i^{-1} b_{ij}^{-1} \epsilon_j^{-1} = \epsilon_i^{-1} K_{ij}^{-1}$ we obtain

$$-\partial_\mu \partial^\mu \psi^j = b_{ji}\epsilon_i \, e^{\psi^i} - b_{ji} n_i \epsilon_{-\Theta} \, e^{-(\Theta,\psi)} = \overline{K}_{ji}^T \, e^{\psi^i} + \overline{K}_{ji}^T \, e^{-(\Theta,\psi)} = \overline{K}_{ja}^T \, e^{\psi^a}$$

and $\psi^0 := -(\Theta, \psi)$.

5 Monopoles and Toda

From the starting point now of an affine Toda field theory in $1+0$ dimensions (now with time coordinate s) as discussed above, real s.t. $\phi = \phi^\dagger$, now define T_i by

$$\beta = T_1 + iT_2 = e^{\phi/2} E e^{-\phi/2}, \quad \beta^\dagger = -T_1 + iT_2 = e^{-\phi/2} E^\dagger e^{\phi/2}, \quad \alpha + \alpha^\dagger = 2iT_3 = \dot{\phi}.$$

Proposition 5.1. *The T_i satisfy $T_i^\dagger = -T_i$ and Nahm's equations are the Toda equations, i.e.*

$$\ddot{\phi} = [e^\phi E e^{-\phi}, E^\dagger] \iff \begin{cases} \frac{d}{ds}\beta - [\alpha, \beta] = 0 \\ \frac{d}{ds}(\alpha + \alpha)^\dagger - \left(\left[\alpha, \alpha^\dagger\right] + \left[\beta, \beta^\dagger\right]\right) = 0 \end{cases}$$

Proof. We first note that if we define $g = e^{-\phi/2}$ then

$$\alpha = -g^{-1}\dot{g}, \quad \beta = g^{-1} E g$$

and this is the form of the general solution to the complex equation. Next

$$[\beta, \beta^\dagger] = e^{-\phi/2}[e^\phi E e^{-\phi}, E^\dagger] e^{\phi/2}, \quad \left[\alpha, \alpha^\dagger\right] = 0$$

so we get the equivalence of the equations. □

Remark. *In the above, we are hiding an important point: elements of the algebra are not matrices, and so we need to be actually considering a representation. $e^{\phi/2} E e^{-\phi/2}$ is actually an algebra element in and of itself recall, as $\mathrm{Ad}_{e^\phi} = e^{\mathrm{ad}_\phi}$, but we still need a rep to make it into a matrix. A question I have is what reps ρ ensure that $\rho(E)^\dagger = \rho(E^\dagger)$? Note moreover that we have chosen an algebra $\mathfrak{g}$ s.t. there are* rank $\mathfrak{g}$ *fields ϕ. This ensures that we can have all ϕ commuting.*

Remark. *This coincides with the notation of* Cyclic Monopoles, Affine Toda and Spectral Curves

[1]

$$T_1 + iT_2 = \begin{pmatrix} 0 & e^{(q_1-q_2)/2} & 0 & \dots & 0 \\ 0 & 0 & e^{(q_2-q_3)/2} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{(q_{n-1}-q_n)/2} \\ e^{(q_n-q_1)/2} & 0 & 0 & \dots & 0 \end{pmatrix} \tag{5.0.1}$$

$$T_1 - iT_2 = -\begin{pmatrix} 0 & 0 & \dots & 0 & e^{(q_n-q_1)/2} \\ e^{(q_1-q_2)/2} & 0 & \dots & 0 & 0 \\ 0 & e^{(q_2-q_3)/2} & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & e^{(q_{n-1}-q_n)/2} & 0 \end{pmatrix} \tag{5.0.2}$$

$$T_3 = -\frac{i}{2}\begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & p_n \end{pmatrix} \tag{5.0.3}$$

where p_i, q_i are real in the case of the Lie algebra A_n.

Upon using $0 = \operatorname{Tr} E^2 = \operatorname{Tr} \dot\phi(\beta - \beta^\dagger)$

$$\frac{1}{2}\operatorname{Tr} L^2 = \frac{1}{2}\operatorname{Tr}\left[\beta - (\alpha+\alpha^\dagger)\zeta - \beta^\dagger\zeta^2\right]^2 = \zeta^2 \operatorname{Tr}\left(\frac{1}{2}\dot\phi^2 - e^{\phi}Ee^{-\phi}E^\dagger\right) := \zeta^2 H$$

and this Hamiltonian is not bounded below[2]. This is necessary as the monopole boundary conditions require $T_a \sim \rho_a/s$ as $s \sim 0$ (and similarly at $s \sim 2$), where ρ_a is an irreducible n-dimensional representation of $su(2)$, thus the momenta are unbounded for $s \sim 0$ and so the potential must also be unbounded below.

Observe that the Lax matrix for the monopoles may be written

$$\begin{aligned} L/\zeta &= -\dot\phi + e^{\phi/2}\,Ee^{-\phi/2}/\zeta - e^{-\phi/2}\,E^\dagger e^{\phi/2}\zeta = -\dot\phi + \sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}\left(\zeta^{-1}E_\alpha - \zeta E_{-\alpha}\right) \\ &= -2A_0^\dagger \\ M &= -\frac{1}{2}\dot\phi - e^{-\phi/2}\,E^\dagger e^{\phi/2}\zeta = \frac{1}{2}\frac{L}{\zeta} - \frac{1}{2}e^{\phi/2}\,\frac{E}{\zeta}e^{-\phi/2} - \frac{1}{2}e^{-\phi/2}\,E^\dagger e^{\phi/2}\zeta \\ &= \frac{1}{2}\frac{L}{\zeta} - \frac{1}{2}\sum_{\alpha\in\overline{\Delta}} \sqrt{n_\alpha}\,e^{b\alpha(\phi)/2}\left(\zeta^{-1}E_\alpha + \zeta E_{-\alpha}\right) \\ &= -A_0^\dagger - A_1^\dagger \end{aligned}$$

and where $\partial_0\phi = 0$ and $\partial_1\phi = \dot\phi$ in the previous section. Then the independence from the 0-coordinate gives $0 = [\partial_1 + A_1, \partial_0 + A_0] = \partial_1 A_0 + [A_1, A_0]$ and $0 = \partial_1 A_0^\dagger - [A_1^\dagger, A_0^\dagger] = [\partial_1 - A_1^\dagger, A_0^\dagger]$ and hence the Lax equation $0 = [\partial_1 + M, L]$.

Questions:

[2]Here the Lagrangian is $\mathfrak{L} := \operatorname{Tr}\left(\frac{1}{2}\dot\phi^2 + e^{\phi}Ee^{-\phi}E^\dagger\right)$ corresponding to a potential of the wrong sign (see the expansion below).

1. What is the effect on the spectral curve of the different scalings r_α? Are the curves birational?

2. What is the analogue of the characteristic polynomial and determinant for the matrices

$$a \cdot H + \sum_{\alpha \in \overline{\Delta}} (b_\alpha E_\alpha + c_\alpha E_{-\alpha})?$$

   (We may view these as generalizations of tridiagonal matrices.)

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