GR + Effective Field Theory

Linden Disney-Hogg

Contents

1 Introduction

Let's note with some good sources to start with

- Donaghue : 1702.00319, 9512024
- Manohar : 1804.0563, 9606222

In GR we will take signature $(-,+,+,+)$.

2 GR+EFTs - Lecture 1

2.1 Effective Field Theories

Definition 2.1 (EFT). *The steps to define an* ***effective field theory (EFT)*** *are*

1. *Write down the most general Lagrangian L compatible with the symmetry of your problem. (Note if you are doing this in GR this will include a graviton)*
2. *Keep all the terms with a fixed number of derivatives.*
3. *Fix dimensions of coefficients with dimensional analysis*
4. *Do QFT with this action (loops)*
5. *Identify regime of validity.*

Recall that really there is a step 0 here (for example read Carroll). In GR we have a massless spin 2 field with dynamical degrees of freedom $g_{\alpha\beta}$ (a metric) with 10 degrees of freedom (using symmetry). We then lose 4 degrees of freedom due to gauge transformation, i.e. spacetime diffeomorphisms. We again lose 4 degrees of freedom which are "non-dynamical". This gives us the 2 we would expect for a spin 2 field. These latter 4 turn out to come from 4 of Einstein's equations which are degree 1 in time, and are the equivalent of Gauss' law in electrodynamics.

Remark. *The above was in $d = 4$ dimensions. In general we would get*

$$\frac{1}{2}d(d+1) - d - d = \frac{1}{2}d(d-3)$$

degrees of freedom for this field. This would mean that in $d = 3$, gravity is non-dynamical (correct "depending on what aspects of the theory you are interested in"). GR is only invariant under local diffeomorphisms, not global (or large diffeos), in fact if we are in a spacetime with a boundary we can get boundary gravitons (gravitons whose wavefunction has support on the boundary).

Recall that gravitons in the quantum theory correspond to gravitational waves in the classical theory.

Example 2.2. *1) Let's take actions*

$$S = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\left(-2\Lambda + R + c + 1R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \ldots\right) .$$

2) Recall $R_{\mu\nu\rho\sigma} \sim \partial\partial g$. g is our metric, Λ is our cosmological constant. Λ is not fixed in the theory, and has dimensions, but one in practice uses the experimental value so we can make predictions and move on. We could have also included matter fields.
3) We can calculate the dimension of these constants to be

$$c_i \sim M_S^{-3}\,, \quad \frac{1}{G_N} \sim M_P^2\,,$$

where M_P is our planck mass and M_S is ... *If we took $c_2 = 0 = c_3$ and $c_1 \neq 0$, then we get field equations that are approximately*

$$\Box h + \frac{1}{M_S^2}\Box\Box h = 8\pi G_N T\,. \tag{2.1.1}$$

4) We are taking the perturbative expansion $g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P}h_{\mu\nu}$, so our action looks like

$$S \sim \int \underbrace{(\partial h)^2 + \frac{1}{M_P}h(\partial h)^2}_{R} + \underbrace{\frac{1}{M_S^2}\left[(\partial^2 h)^2 + \frac{1}{M_P}h(\partial^2 h)^2\right]}_{h.o.t} + \ldots$$

This means our Feynmann rules associate a $\sqrt{G_N}$ to each vertex, so the contribution from the 4pt function is

$$\left(\frac{E^2}{M_P^2}\right)^{4+loops} .$$

5) Ultimately this makes this theory non-renormalisable as the coupling constant is not dimensionless. However if $E \ll M_P$ we are ok to do QFT, so we can work at lower-than-plank-scale energies we can make predictions.

The propagator for 2.1.1 looks like

$$\frac{1}{q^2 + \frac{q^4}{M_S^2}} = \frac{1}{q^2} - \frac{1}{q^2 + M_S^2}$$

and because of the sign of the latter term, we call this a ghost, as it corresponds to a massive particle of imaginary mass. This means that extra degrees of freedom are necessary. From a phase space perspective, this is because the initial value problem of gravity represented by 2.1.1 needs more initial values because it is a 4th order equation (as opposed to classical gravity, which is second order). We could also think of this semi-classically, as imagining the higher order terms in the actions as modifying the standard Newtonian potential to be

$$V(r) = -\frac{Gm_1m_2}{r}\left[1 - e^{-rM_S}\right] ,$$

and so for distances $r \gg M_S^{-1}$, these corrections are suppressed. Ultimately this is saying GR is valid in "low" energies, which is good.
So far we have been focusing on the effective part of EFT.

2.2 Gauge Field Theories - Symmetries and Hamiltonians

2.2.1 The Classical Version

Recall classical mechanics, where we have

$$I = \int dt\, L$$

and we compute a Hamiltonian via Legendre transform

$$H = p\dot{q} - L\big|_{p=\frac{\partial L}{\partial \dot{q}}}$$

which generates t-translations. Here t is an independent variable, and we can parameterise by it. GR is different, because t is one of our dependent variables, and so to construct a Hamiltonian we need a clock. In classical mechanics t is this physical clock, but if we want to write our theory in terms of some non-physical (arbitrary parameter) clock τ, one thing we can do is the following:
We might expect introducing our new arbitrary parameter to correspond to introducing a gauge freedom. We can use gauge symmetry to "eat" degrees of freedom, so to reformulate our theory we know we will want to increase our degrees of freedom. Hence lets add to $q(t), p(t)$ the fields $T(\tau)$ (clock) and $\pi(\tau)$ (conjugate momentum to T). We now write down a new action (which we will justify in the posterior)

$$\tilde{I} = \int d\tau\, \left[pq' + \pi T' - N\left(\pi + H\right)\right]$$

where N is a Lagrange multiplier field, and $\,' = \frac{d}{d\tau}$. Not my conjugate momenta are (for example) $p = \frac{\partial L}{\partial q'}$. The variation wrt N imposes

$$\pi = -H(q,p)$$

Then going on-shell wrt the field N gives

$$\tilde{I}\Big|_{N=0} = \int d\tau\, \frac{dT}{d\tau}\left[p\dot{q} - H(p,q)\right]$$

where $\dot{} = \frac{d}{dT}$. This action $\tilde{I}$ is invariant under reparameterisation of τ. The new Hamiltonian is

$$\tilde{H} = N(\pi + H)\,.$$

This vanishes on-shell, which is exactly what we want. Recall we said that $\tilde{H}$ generates τ-translations, but our theory is invariant under such reparameterisations (in the Hilbert space of physical states $|\psi\rangle$).
The reason for seeing this is to assure us that, despite this being a toy model, when we go through the same process to make GR a Hamiltonian theory, we should be ok.

2.2.2 ADM Hamilton

The paper by Arnowitt, Deser, Misner (ADM) is [1]. The idea is to formulate GR as an initial value problem, which can be related to the identification of the degrees of freedom as we mentioned earlier. We will follow their paper.

Lets take manifold (M, g_{ab}), assumed to be globally hyperbolic, which means that we can foliate our manifold by Cauchy surfaces Σ_t, defining a time function t telling you which surface a point is on. Further define a unit normal to Σ_t, n^a, defined globally s.t.

$$n^a n_a = g_{ab} n^a n^b = -1 \, .$$

As such we can write the induced metric on Σ_t using standard differential geometry as

$$h_{ab} = g_{ab} + n_a n_b \, .$$

We will define a vector associated to t, confusingly named t^a, s.t.

$$t^a \nabla_a t = 1 \, .$$

We think of this as giving us a flow of t. Given this t^a we can project it into our normal and parallel directions at a given t to define

$$\begin{aligned} -t^a n_a &= N, \quad \text{(normal component, lapse function)} \, , \\ h_{ab} t^b &= N_a, \quad \text{(tangential component, shift vector)} \, . \end{aligned}$$

We can calculate

$$\begin{aligned} h^a{}_b &= g^{ac} h_{cb} = \delta^a{}_b + n^a n_b \\ \Rightarrow h^a{}_b t^b &= t^a - N n^a \\ \Rightarrow n^a &= \frac{1}{N}(t^a - N^a) \, . \end{aligned}$$

This means that we can write our full metric as

$$g^{ab} = h^{ab} - \frac{1}{N^2}(t^a - N^a)(t^b - N^b) \, ,$$

which suggest that the degrees of freedom that appear in our Hamiltonian formulation of GR correspond to the freedom in the choice of foliation, because our metric is written in terms of the surface metrics and our choice of t.

We will now want to pick a particular coordinate chart where $t^a = (\partial_t)^a$. In this gauge the metric is

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \, .$$

Remark. *Recall we spoke about a connection to the non-dynamical parts of Einstein's equations. Taking the usual Einstein tensor G_{ab} we project to calculate*

$$G_{ab} n^a n^b = h^b{}_d G_{bc} n^c = 0 \, .$$

These equations have no second order derivatives in time, and there are four of them, so they represent non-dynamical constraints in the initial value problem we mentioned earlier. If we were to quantise this theory we would want these operators to act trivially on physical states of the Hilbert space. We may hope to replicate this trick from our classical analogue.
The action we will write is

$$I = \frac{1}{16\pi G_N} \int_M d^4x \, \sqrt{-g} R - \frac{1}{8\pi G_N} \int_{\partial M} d^3x \, \sqrt{-\gamma}(K - K_0) \, .$$

This latter term is the Gibbons-Hawking-York term, where K is the extrinsic curvature, γ a metric on the boundary, and K_0 correspond to the curvature of the boundary in the vacuum. It turns out that $\delta I = 0$ gives me GR. We can compute the Hamiltonian, the details of which are in Wald's book. We first rewrite in ADM variables as

$$I = \int_M d^4x \, \left[\pi^{ij} h - ij + N\mathcal{H} - N^i \mathcal{H}_i \right] + \textit{boundary term} \,,$$

where the $\mathcal{H}, \mathcal{H}_i$ are just some functions of our fields, and the we can write the Hamiltonian (which is a functional of t^a) as

$$H[t^a] = \int_{\Sigma_t} d^3x \, \left[N\mathcal{H} + N^i \mathcal{H}_i \right] + \textit{boundary term} \,.$$

This integral contains only Lagrange multipliers, so we get on-shell constrains of $\mathcal{H} = 0 = \mathcal{H}_i$ (of which there are 4), and this means that if we didn't have any boundary terms then our on-shell Hamiltonian acts as 0 on our Hilbert space. However, generically our boundary term will not be zero, and so we get a non-zero energy value. The boundary term looks like

$$\int_{\partial\Sigma_t} d^2x \, \sqrt{\sigma} n^\mu T_{\mu\nu} t^\nu$$

so we get

$$\delta I_{on\ shell} = \frac{1}{2} \int_{\partial M} d^3x \, \sqrt{-\gamma} T^{ij} \delta g_{ij} \,.$$

This calculation is in Brown-York gr-qc 9209012. This boundary term reproduces Komar's integral

3 Asymptotic Symmetry Groups (ASGs) - Lecture 2

Recall we saw that the ADM Hamiltonian had boundary terms that did not vanish on shell. Indeed if one evaluates the energy on the vector $t^\mu = (1,0,0,0) = (\partial_t)^\mu$ and take that $N \to 1$, $N^i \to 0$ on $\partial\Sigma$, then we explicitly get

$$E = \int_{\partial\Sigma} d^2x \, \sqrt{\sigma} n^\mu T_{\mu t} \,.$$

We will now aim to put this in quite a general framework

3.1 Local vs Large Diffeos

If you compute

$$\delta_\xi \left(\sqrt{-g} L \right) = \mathcal{L}_\xi \left(\sqrt{-g} L \right) = \nabla_\mu (L \xi^\mu)$$

for some Lagrangian density L, we note that if ξ acts non-trivially at the boundary the by Stokes' theorem we may have a boundary contribution to our action. Hence the assertion of gauge-invariance in a boundary-dependent statement. This gives us reason to introduce the following definition,

Definition 3.1. *The ASG of a theory is*

$$ASG = \frac{symmetries}{trivial\ symmetries}\,,$$

where these are symmetries at the boundary, so the trivial symmetries are equivalently the local symmetries.

Example 3.2. *At null infinity in $d = 4$ GR, we have the BMS group.*

Example 3.3. *In AdS_3 we have $Vir \oplus Vir$.*

3.2 Quantum Gravity (QG)

Treating now GR as an EFT for QG, we should have some UV completion, and this is the open problem. There are a few things we think we know.

- QG has no local observables (gauge invariant quantities). If we were in a gauge QFT in flat space, then the local observables are
  - correlators $\langle O(x_1) \dots O(x_n) \rangle$
  - S-matrix elements

  These observables however depend on spacetime points, so if we have a quantum theory of gravity where points fluctuate, these objects are no longer gauge invariant. If one were to do a semi-classical approximation and expand around a curved background, these may appear gauge-invariant, but in the full scale these arguments break down. One can overcome this fact by using what are called "dressed operators", where we connect the operators to the boundary, thus fixing gauge invariance, but now this operator is non-local.

- Gravitons are not a composite object. Note this is a valid question to ask, as we have particle theories where particles like mesons are the fundamental objects at low energies, but when we go to the UV completion we see that these are composite. This result is rigorous enough in the sense that we have the following theorem:

  Theorem 3.4. *A theory containing a Poincaré covariant conserved stress tensor $T^{\mu\nu}$ forbids massless particles of spin $j > 1$.*

  Proof. We sketch the idea. Labelling massless states with 4-momenta p and spin j_3, consider the terms

  $$\langle p', \pm j | T^{\mu\nu} | p, \pm j \rangle$$

  in the limit $p' \to p$. Up to normalisations we can write

  $$p^\mu = \int d^3x\, T^{0\mu}$$

  and

  $$\langle p', \pm j | p^\mu | p, \pm j \rangle = p^\mu \delta(p - p')$$

but rewriting the LHS of the above using the integral gives

$$(2\pi)^3\delta(p-p')\lim_{p'\to p}\langle p',\pm j|T^{\mu\nu}|p,\pm j\rangle = p^\mu\delta(p-p')$$

and hence

$$\lim_{p'\to p}\langle p',\pm j|T^{\mu\nu}|p,\pm j\rangle = \frac{p^\mu p^\nu}{E(2\pi)^3}\neq 0\,.$$

Given 2 massless particles we have

$$(p+p')^2 = 2p\cdots p' = 2\left|\boldsymbol{p}\right|\left|\boldsymbol{p}'\right|(\cos\phi - 1)\leq 0$$

and we will assume $aphi \neq 0$. Hence $\exists$ a frame s.t. $p=(\left|\boldsymbol{p}\right|,\boldsymbol{p})$, $p'=(\left|\boldsymbol{p}\right|,-\boldsymbol{p})$. Consider now a rotation by angle θ about $\boldsymbol{p}$ and take $\theta'=-\theta$. The bra and ket transform as

$$\begin{aligned}|p,\pm j\rangle &\mapsto e^{\pm i\theta j}\,|p,\pm j\rangle\\ |p',\pm j\rangle &\mapsto e^{\mp i\theta j}\,|p',\pm j\rangle\ ,\end{aligned}$$

so we have

$$e^{\pm 2i\theta j}\lim_{p'\to p}\langle p',\pm j|T^{\mu\nu}|p,\pm j\rangle = \lim_{p'\to p}\Lambda(\theta)^\mu{}_\rho\Lambda(\theta)^\rho{}_\sigma\,\langle p',\pm j|T^{\rho\sigma}|p,\pm j\rangle\ .$$

The Λ are rotation matrices, so their eigenvalues can only be $\sim e^{\pm i\theta}$. This transform on the RHS does not depend on the quantum number, so we get a bound on j because either matrix elements vanish (not possible) or $2j\in\{0,1,2\}$. □

It is important to note about this theory:

- it does not remove gravitons in the UV, because GR does not have a conserved stress tensor that includes gravitons, we only have $\nabla_\mu T^{\mu\nu}_{\text{matter}}=0$, but what is important is that there in no local gauge-invariant energy momentum tensor in GR, which is connected to the fact that you shouldn't be able to measure energy locally (think about choosing a coordinate frame so locally Schwarzschild looks like Minkowski, but we know Schwarzschild should have a mass)
- if you thought that the graviton was composite as graviton $\sim$ 2gluon, then because we have a conserved stress tensor for the gluons we could construct a stress tensor for the graviton, and this would be obstructed via the theorem. One might wonder how this intersects with the work of the double-copy community.

3.2.1 Summary

The sum of the above is that

- gravitons appear in the UV together with some other particles
- it is possible that the graviton is an emergent degree of freedom when the UV theory is not an ordinary 4d QFT. Here emergent is in the sense that we have a QFT of sort on another manifold with associated tensor $\hat{T}_{\alpha\beta}$ from which the graviton $g_{\alpha\beta}$ arises on our original manifold under some correspondence.

3.3 BH Thermodynamics

Let us list some main features of black holes.

- $\exists$ an event horizon ($r = r_+$ when spherically symmetric) (this is a global requirement, looking at the penrose diagram we need to have global knowledge of spacetime to draw this),
- picking coordinates (e.g. (t, r) for Schwarzschild), they flip at the horizon, so the interior geometry changes with "real time which is r",
- $\exists$ gravitational redshift, $\delta\tau = \sqrt{-g_{tt}}\delta t$, $E_r = \frac{E_t}{\sqrt{-g_{tt}}}$.
- $\exists$ BH uniqueness theorems
- we can phrase BHs locally in terms of trapped surfaces
- In Hawking 1971, it was proven that the area of the event hoizon A always grows (i.e. $\delta A \geq 0$) under any classical process if
  1. the null energy condition holds (i.e. $T_{\mu\nu}k^\mu k^\nu \geq 0$ for any null k^μ),
  2. cosmic censorship holds.

Let us take a warm-up example.

Example 3.5. *Let's write 4d RN metric, Einstein coupled to Maxwell,*

$$
\begin{aligned}
ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \\
A &= -\frac{Q}{r}dt
\end{aligned}
$$

where $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, M the BH mass and Q the BH electric charge, A a Maxwell field (not to be confused with area). Note this is a vacuum solution so $T^{\mu\nu}_{matter} = 0$.

- *We can find $r_\pm = M \pm \sqrt{M^2 - Q^2}$ s.t $f(r)\frac{(r-r_-)(r-r_+)}{r^2}$, and then the event horizon corresponds to $r = r_+$.*
- *we can compute $A(r_+) = 4\pi r_+^2$. If we consider two solutions (M, Q) and $(M + \delta M, Q + \delta Q)$ then*

  $$\delta A = 8\pi r_+ \delta r_+ = 8\pi r_+ \left[\left(1 + \frac{M}{\sqrt{M^2 - Q^2}} \right) \delta M - \frac{Q}{\sqrt{M^2 - Q^2}} \delta Q \right] .$$

  Using the result that $r_+^2 f'(r_+) = r_+ - r_-$ one rewrites

  $$\frac{f'(r_+)}{16\pi} \delta A = \delta M - \frac{Q}{r_+} \delta Q \, .$$

Comparing this last equation with the 1st law of thermodynamics $TdS = dM - \Phi dQ$ (Φ a gauge potential, dM a matter content) and the second $\delta S \geq 0$ we would be inclined to write $S = \eta A$ and then

$$\Phi = - \left. A_t \right|_{r=r_+} \quad (\textit{Maxwell})\,,$$
$$T_{BH} = \frac{f'(r_+)}{16\pi\eta}\,.$$

Hence the "temperature" of the BH is constant over the entire horizon, which looks like the 0th law of thermodynamics.

The rest of the analogy is completed in

- Israel, 1986, (3rd law)
- Bardeen, Carter, Hawking, 1973, (law of classical BH vs thermodynamics)

What remains is to calculate the appropriate η. Hawking did it first with a very complicated calculation, so next time we will see a simpler trick.

4 Trick to Compute T_{BH} - Lecture 3

Remark. *Note that in QFT at vanishing temperature, we use $\mathbb{R}^{1,3}$, but at finite temperature (β is inverse temperature) use $S^1_\beta \times \mathbb{R}^3$. We use this to motivate going to Euclidean BHs to do this calculation.*

Now take $t = i\tau$ to get line element

$$ds_E^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2\,.$$

We wonder what happens at r_+, as $f(r_+) = 0$, but note now this is not a horizon because we have changed signature.
Let $r - r_+ = \rho^2$, $\rho^2 \ll r_+$. Then as $\rho \to 0$ our line element has the limimt

$$ds_E^2 \to \frac{\beta}{4\pi\eta}\left(\rho^2 d\overline{\tau}^2 + d\rho^2\right) + r_+^2 d\Omega_2^2$$

where $\overline{\tau} = (8\pi\eta T_{BH})\,\tau$ and $\beta = T_{BH}^{-1}$. This first term gives a smooth metric "at the horizon" iff $\overline{\tau}$ is a polar coordinate, so has period 2π (otherwise we could have a conical singularity). Then

$$\overline{\tau} \sim \overline{\tau} + 2\pi \Rightarrow \tau \sim \tau + \frac{\beta}{4\eta}\,.$$

If we take $r \to \infty$ in this limit metric, our asymptotic manifold is $S^1_\beta \times \mathbb{R}^3$. Hence here we would want $\tau \sim \tau + \beta$, and this forces $\eta = \frac{1}{4}$.

Remark. *One might wonder why we expect this transformed BH to not have a singularity, and this is valid (think about what gravitational instantons we get in Euclidean metric). We will talk about this more later.*

4.1 Covariant Phase Space

We will now want to re-derive the thermodynamics of the black hole in covariant GR. As good references we have

- Iyer, Wald, 9403028
- Wald, Zoupas, 991095
- Wald, 9307038

Example 4.1. *Starting with $L = \mathcal{L}(q, \dot{q})dt$ and the Euler-Lagrange eqns give*

$$\delta_1 L = \left(\frac{\partial \mathcal{L}}{\partial q^a} - \frac{d}{dt} \frac{del \mathcal{L}}{\partial \dot{q}^a} \right) dt \delta_1 q^a + d\Theta$$

with $\Theta = \frac{\partial \mathcal{L}}{\partial \dot{q}^a} \delta_1 q^a = p_a \delta_1 q^a$. This gives a symplectic form

$$\Omega = \delta_2 \Theta(\delta_1 q^a, q^a, \dot{q}^a) - \delta_1 \Theta(\delta_2 q^a, q^a, \dot{q}^a) = \delta_2 p_a \delta_1 q^a - \delta_1 p_a \delta_2 q^a$$

We can define a Noether current associated with a vector field ξ

$$J_\xi = \Theta(q^a, \dot{q}^a, \mathcal{L}_\xi q^a) - \underbrace{i_\xi L}_{\xi \cdot L} \,.$$

For example if $\xi = \partial_t$, then $\mathcal{L}_\xi q^a = \dot{q}^a$ and $i_\xi L = \mathcal{L}$ so

$$J_\xi = p_a \dot{q}^a - \mathcal{L} = H$$

Think of this example as giving a blueprint for how to find the Noether current for a generic Lagrangian, as we will next attempt to do in GR.
Define:

- our manifold M, globally hyperbolic,
- $\phi = \phi(g_{ab}, \psi, \dots)$ our fields, ψ matter fields,
- $\mathcal{F}$ the space of kinematically allowed configurations, with requirements such as smoothness and asymptotic boundary conditions.
- A bulk Lagrangian

  $$L_{\text{bulk}}(g_{ab}, R_{abcd}, \nabla_a R_{abcd}, \dots, \psi, \nabla_a \psi, \dots)$$

  where we can can have arbitrary, but finite, number of derivatives in our terms.

Our variation is defined as

$$\delta L_{\text{bulk}} = E[\phi]\delta\phi + d\Theta \,.$$

Remark. *Note our asymptotic boudnary conditions will not affect our EL equations, but will affect Θ. Hence there are potential ambiguities, discussed further in (Harlow, 1906.08616) (1912.06025).*

In pure GR, we will take

$$(L_{\text{bulk}})_{abcd} = \frac{1}{16\pi G_N}\epsilon_{abcd}R \Rightarrow \Theta_{abc} = \frac{1}{16\pi G_N}\epsilon_{abcd}g^{de}g^{fh}\left(\nabla_f \delta g_{eh} - \nabla_e \delta g_{fh}\right) .$$

Following our algorithm our symplectic current is

$$\omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_2\Theta(\delta_1\phi, \phi, \dots) - \delta_1\Theta(\delta_2\phi, \phi, \dots) ,$$

(an $(n-1)$-form in general). The symplectic form is

$$\Omega_\Sigma = \int_\Sigma \omega(\phi, \delta_1\phi, \delta_2\phi) ,$$

depending on some hypersurface Σ. We now want to calculate the Noether current associated to a vector field ξ, with action on $\phi \in \mathcal{F}$ given by $\mathcal{L}_\xi\mathcal{F}$. We will define a tangent vector in $\mathcal{F}$ if it maps $\mathcal{F} \to \mathcal{F}$, thus taking $\mathcal{L}_\xi\phi = \delta_\xi\phi$. Let us also define $\overline{\mathcal{F}}$ to be the submanifold of on-shell configurations, and a projection onto physical phase space $\mathcal{F} \to \Gamma$ (with associated $\overline{\mathcal{F}} \to \overline{\Gamma}$).

Definition 4.2. *Suppose we have $\mathcal{F}$, ξ, and Σ defining $\int_\Sigma \omega(\phi, \delta\phi, \delta_\xi\phi)$ that converges $\forall\phi \in \overline{\mathcal{F}}$, $\forall\delta\phi$ tangent to $\mathcal{F}$. Then a function $H_\xi : \mathcal{F} \to \mathbb{R}$ is called the **Hamiltonian conjugate to ξ on** Σ if*

$$\forall\phi \in \overline{\mathcal{F}},\ \forall\delta\phi \text{ tangent to } \mathcal{F},\ \delta H_\xi = \Omega_\Sigma(\phi, \delta\phi, \delta_\xi\phi) = \int_\Sigma \omega(\phi, \delta\phi, \delta_\xi\phi) .$$

If such a H_ξ exists, it's value gives a conserved charge associated with transform by ξ on Σ. We now want to wonder when H_ξ exists. Let us write our Noether current, an $(n-1)$-form $J_\xi = \Theta(\phi, \delta_\xi\phi) - i_\xi L_{\text{bulk}}$. Then

$$\begin{aligned} dJ_\xi &= d\Theta - d(i_\xi L_{\text{bulk}}) \\ &= [\delta_\xi L_{\text{bulk}} - E[\phi]\delta\phi] - [(\mathcal{L}_\xi - i_\xi d)L_{\text{bulk}}] \\ &= -E[\phi]\delta\phi , \end{aligned}$$

where we knew $dL = 0$ as L is a top form. This vanishes on-shell, so we can write

$$J = dQ + \xi^a c_a$$

where c_a vanishes on-shell, and Q is an $(n-2)$-form called the **Noether charge**. In pure GR we can calculate

$$J_{abc} = \frac{1}{8\pi G_N}\epsilon_{dabc}\nabla_e\nabla^{[e}\xi^{d]}$$

and so

$$Q_{ab} = -\frac{1}{16\pi G_N}\epsilon_{abcd}\nabla^c\xi^d .$$

Returning to generality, we can write the variation of our current as

$$\begin{aligned} \delta J_\xi &= \omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) + di_\xi\Theta - i_\xi\left(E[\phi]\delta\phi\right) \\ \Rightarrow \delta J &= d(\delta Q) + \xi^a\delta c_a \quad \text{(as } \xi \text{ fixed)} \\ \Rightarrow \omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) &= \xi^a\delta x_a + d(\delta Q) - di_\xi\Theta \\ \Rightarrow \delta H_\xi &= \int_\Sigma \xi^a\delta c_a + \int_{\partial\Sigma}[\delta Q - i_\xi\Theta] \\ \Rightarrow\ \delta H_\xi\big|_{\text{on-shell}} &= \int_{\partial\Sigma}[\delta Q - i_\xi\Theta] . \end{aligned}$$

Wald and Zoupas derive an integrability condition for the existence of H_ξ that

$$0 = (\delta_1\delta_2 - \delta_2\delta_1)H_\xi =_{\text{on-shell}} -\int_{\partial\Sigma} i_\xi \omega(\phi, \delta_1\phi, \delta_2\phi)$$

It is worth remarking that the above was derived under very general conditions.

4.1.1 Connection to ADM

Recall on shell we have

$$\delta H_\xi = \delta \int_{\partial\Sigma} Q - \int_{\partial\Sigma} i_\xi \Theta$$

The latter term is the tricky on for the existence of H_ξ. Pick $\partial\Sigma$ to be at spacelike infinity, and assume $\exists B$ satisfying

$$\delta \int_{\partial\Sigma} i_\xi B = \int_{\partial\Sigma} i_\xi \Theta$$

This would give us a solution given as

$$H_\xi = \int_{\partial\Sigma} (Q - i_\xi B)\,.$$

This is a boundary quantity, depending on ξ, and in pure GR we can compute this B. If we took $\xi = \partial_t$ or $\xi = \partial_\varphi$, we would reproduce ADM with $H =$ energy or angular momentum.
Let us now go to a BH situation. Here $\partial\Sigma$ has two contributions, at infinity and at the horizon. Let us take the vector

$$\xi^a = t^a + \Omega_i \varphi_i^a\,,$$

a quantity vanishing at the horizon, meaning that the difficult term $\int_{\partial\Sigma_{\text{Hor}}} i_\xi \Theta = 0$. Further, because $\omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) = 0$ because ξ is Killing and so $\mathcal{L}_\xi\phi$, $\delta H_\xi = \int_\Sigma \omega = 0$. This combines to mean

$$\delta \int_{\partial\Sigma_{\text{Hor}}} Q = \delta M - \Omega_i \delta J_i\,,$$

where M, J_i are the ADM charges of energy and angular momentum. Assuming that the Lagrangian in the bulk is pure GR, we can use $Q_{ab} = -\frac{1}{16\pi G_N}\epsilon_{abcd}\nabla^c\xi^d$. We will also want that

- the volume form on Σ_{Hor} is

  $$\epsilon_{ab} = \epsilon_{abcd}N^c\xi^d\,,$$

  where N is our future directed null normal to Σ_{Hor} s.t $N^c\xi_c = -1$.

- We contract ϵ^{ab} with the integrand for Q to get

  $$\epsilon^{ab}\epsilon_{abcd}\nabla^c\xi^d = 4N_c\xi_d\nabla^d\xi^c\,.$$

- As ξ is Killing, $\xi^a\xi_a|_{\Sigma_{\text{Hor}}} = 0 \Rightarrow \nabla_a\xi^b\xi_b = -2\kappa\xi^a$ on the horizon (for some κ).Using that ξ is Killing gives $\nabla_{(a}\xi_{b)} = 0$ and hence

  $$\xi^b\nabla_a\xi_b = -\xi^b\nabla_b 1_a = -\kappa\xi_a$$

  This is called the **surface gravity**.

Hence
$$\epsilon^{ab}\epsilon_{abcd}\nabla^c\xi^d = -4\kappa$$
By covariance, we must have
$$\epsilon^{ab}\epsilon_{abcd}\nabla^c\xi^d = \gamma\epsilon_{ab}\epsilon^{ab}\,,$$
which gives in total that
$$\delta\int_{\Sigma_{\rm Hor}} Q = \delta\left(\int_{\Sigma_{\rm Hor}} \frac{\kappa}{8\pi}\epsilon_{ab}\right) = \delta M - \Omega_i\delta J_i$$

Remark. *we may a few remarks.*

- *The Euclidean $T_{BH} = \frac{\kappa}{2\pi}$ for RN.*
- *One can show the 0th law of BH mechanics:*
$$\xi_{[d}\nabla_{c]}\kappa = -\xi_{[d}R^f_{c]}\xi_f$$
using $\nabla_a\nabla_b\xi_c = -R^d{}_{bca}\xi_d$. Now for RN we can write
$$\delta\int_{\Sigma_{Hor}} \frac{\kappa}{8\pi}\epsilon^{ab} = \frac{\kappa}{2\pi}\delta\left(\frac{A}{4}\right)\,,$$
and if the dominant energy condition holds, we can write $\xi_{[d}\nabla_{c]}\kappa = 0$ this is true more generally, so
$$\delta M - \Omega_i\delta J_i = T_{E,RN}\frac{\delta A}{4}\,.$$
- *The sum effect of this is to show that entropy is a Noether charge, and is related to the surface gravity, a quantity interesting in differential geometry in its own right.*

5 Thermal States - Lecture 4

Let's start with the question "what is a thermal state?" Let's take two perspectives

- From statistical mechanics, we have density $\rho_\beta = \frac{e^{-\beta H}}{Z(\beta)}$, $\beta = \frac{1}{T}$, $\langle X\rangle_\beta = \mathrm{Tr}(\rho_\beta X)$.
- Open subsystems, thermal states appear as a reduced density matrix of a closed Hilbert space. i.e. we have a Hilbert space $H = H_1\otimes I_2 + I_1\otimes H_2$, with state
$$|\psi\rangle_{TFD} = \frac{1}{\sqrt{Z}}\sum_i e^{-\beta E_i/2}\,|i\rangle_1\otimes|i\rangle_2\,,$$
and then we calculate
$$\rho = |\psi\rangle\langle\psi| \Rightarrow \rho_\beta := \rho_1 = \mathrm{Tr}_{H_2}\,\rho\,.$$
This means
$$\langle\psi\,|\,X_1\otimes I_2\,|\,\psi\rangle = \mathrm{Tr}_{H_1}(\rho_1 X_1)\,.$$
Note 'TFD' stands for thermal field double.

Remark. *Consider 2 harmonic oscillators with Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, corresponding creation an annihilation operators $a_i^\dagger, a_i$, with ground states $|0\rangle_i$ s.t.*

$$a_i |0\rangle_i = 0\,.$$

We thus have a natural vacuum state of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ given by $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$. The analogue of the TFD state here is

$$|\psi\rangle \propto \exp\left[e^{-w\beta/2} a_1^\dagger a_2^\dagger\right] |0\rangle = \sum_n e^{-n\beta w/2} |nw\rangle_1 \otimes |nw\rangle_2\ .$$

We observe that the operators

$$b_1 = \cosh(\theta) a_1 - \sinh(\theta) a_2^\dagger\,,$$
$$b_2 = \cosh(\theta) a_2 - \sinh(\theta) a_1^\dagger\,,$$

where $\cosh(\theta) = \frac{1}{\sqrt{1-e^{-\beta w}}}$*. This is a unitary transform (an example of a bogoluibov transform), and we have that*

$$b_1 |\psi\rangle = 0 = b_2 |\psi\rangle\ .$$

Hence some unitarily related observes will see $|\psi\rangle$ as a vaccuum, where their annihilation and creation operators are $b_i^\dagger, b_i$. Later we will show this to be related to the fact that, implicitly we have picked a time to do quantum mechanics here, and by changing time we will change frequencies, hence energies. The Hamiltonian corresponding to this new set of energies will be the boosted Hamiltonian. This will somehow apply to the thermodynamics of BHs later, where we expect the time to change.

Remark. *Consider a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with entangled state $|\psi\rangle \neq |\chi\rangle_A \otimes |\tilde{\chi}\rangle_B$. For example we could take the qubit Bell state*

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)\ .$$

If we let $\rho = |\psi\rangle\langle\psi|$, but we can can only see system A, then we have $\rho_A = \frac{1}{2} I_A$. This has maximum ignorance, in the sense that it maximises Shannon entropy, but also that as an operator it doesn't even know the basis wrt which it is defined.

5.1 How do we measure entanglement

We will want two properties of the entanglement we measure

1. quantum entanglement $\Rightarrow$ quantum correlation
2. if the system is in a pure state of $\mathcal{H}_A \otimes \mathcal{H}_B$, then $S(\rho_A) = -\operatorname{Tr}_{\mathcal{H}_A} \rho_A \log \rho_A$

If $\mathcal{H}$ has finite dimensions, then

$$\forall O_A, O_B, \quad I(A,B) \geq \frac{(\langle O_A O_B\rangle - \langle O_A\rangle \langle O_B\rangle)}{2\,|O_A|^2\,|O_B|^2}$$

where I is the mutual information

$$I(A,B) := S(A) - S(A|B)\,,$$

which, despite how its definition looks, is symemtric.
In QFT, for a free massive scalar with vacuum $|\Omega\rangle$, we recall

$$\langle\Omega\,|\,\phi(0,x)\phi(0,y)\,|\,\Omega\rangle = \frac{1}{4\pi^2}\frac{m}{|x-y|}k_A(m\,|x-y|)\,.$$

We hence get behaviour

1. $|x-y| \ll m^{-1} \Rightarrow$ decays as $|x-y|^{-2}$,
2. $|x-y| \ll m^{-2} \Rightarrow$ decays as $e^{-m|x-y|}$.

If $m=0$, the decay is $|x-y|^{-2}$ all the way.
Comparing with our dicsussion of the discrete case, we can think of $\mathcal{H}_A$ as a region of spacetime where a local observer can measure $\phi(0,x)$ by restricting $x \in A$, $y \in B$, where $A \cap B = \emptyset$ and $A \cup B$ is our Cauchy slice we are doing QFT on. This would make it look like our entanglement entropy diverges, as we could make $|x-y|$ arbitrarily small by take points close to the split of the Cauchy surface. Hence UV behaviour introduces a UB cutoff ξ_{UV}.
In gauge theories, there exist constraints (think of the Gauss law) which hold locally. Hence the split $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ must be compatible with these. One proposal of how to deal with this is just to handle the physical phase space, and not deal with a splitting, while another is to increase the Hilbert space by adding "edge modes", which are degrees of freedom living on the boundary (which can be measured in the lab via the Quantum Hall Effect).

Remark. *In the discrete (i.e. the spin) story, where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, the Von-Neumann entropy S_A is invariant under local unitary transformations in the subsytem, i.e $\rho'_A = U_A^\dagger \rho_A U_A$. In a f.d. system these transforms are square matrices, but in QFT one type is time evolution with Hamiltonian, which deforms the Cauchy surface either side of the boundary.*

5.2 Review of path integrals, states, and operators in QFT

5.2.1 Transition Amplitudes and Ground State Wave Functions in QM

In QM, let's recall the position basis and operators $\hat{X}\,|x\rangle = x\,|x\rangle$ with

$$I = \int |x\rangle\,\langle x|\; dx\,.$$

The transition amplitude between x_i as t_i, $i = 1,2$ is

$$\begin{aligned} G(x_2,t_2;x_1,t_1) = \langle x_2,t_2\,|\,x_1,t_1\rangle &= \left\langle x_2\,|\,e^{-iH(t_2-t_1)}\,|\,x_1\right\rangle\,, \\ &= \int_{\overline{x}(t_1)=x_1}^{\overline{x}(t_2)=x_2} e^{iS[\overline{x}(t)]}\,[D\overline{x}(t)]\,, \end{aligned}$$

using Feynmann's path integral formulation.
To write the vacuum wave function introduce $\sum_n |n\rangle\,\langle n|$, $|n\rangle$ the eigenstates of H, from which we can write

$$G(x,t;x_0,t_0) = \sum_n e^{-iE_n(t-t_0)}\,\langle x\,|\,n\rangle\,\langle n\;\; x_0\rangle = \sum_n e^{-iE_n(t-t_0)}\psi_n(x)\psi_n^*(x_0)\,.$$

We observe that taking Euclidean time $t_E = it$ and the limit $t_E \to \infty$, this of G is dominated by the contribution of the vacuum where $E = E_0$. Hence

$$\lim_{t_E \to \infty} G(x,t;0,0) - \left(e^{-E_0 t_E} \psi_0^*(0)\right) \psi_0(x)\,,$$

up to normalisation. This means we can compute $\psi_0(x)$, by taking the limit in Euclidean time of a quantity G that we are taught to compute with path integrals.
Moving now to QFT, introduce operator $\hat{\phi}(x)$ s.t. $\hat{\phi}(x)\,|\phi\rangle = \phi(x)\,|\phi\rangle$, where $|\phi\rangle = \bigotimes_{y\in\Sigma} |\phi(y)\rangle$, Σ our Cauchy slice. We then write the vacuum

$$|\Omega\rangle = \int [D\phi]\,|\phi\rangle\,\langle \phi\,|\,\Omega\rangle\ .$$

Transition amplitudes in Euclidean time τ between $\phi_1 = \phi(\tau_1 = 0, x)$ to $\phi_2 = \phi(\tau_2, x)$ are

$$\left\langle \phi_2\,\middle|\, e^{-\tau_2 H}\,\middle|\,\phi_1\right\rangle = \int_{\phi(\tau_1)=\phi_1}^{\phi(\tau_2)=\phi_2} [D\phi] e^{-S_E[\phi]}$$

6 Path Integrals - Lecture 5

Recall we want to be handling transition amplitudes of the form $\left\langle \phi_2\,\middle|\, e^{-\tau H}\,\middle|\,\phi_1\right\rangle$. If we split this conceptually as the inner product $\langle \phi_2\,|\,\Psi\rangle$ where

- $|\Psi\rangle = |\phi_1(\tau)\rangle := e^{-\tau H}\,|\phi_1\rangle$,
- we let $\Phi(\phi_2) = \langle \phi_2\,|\,\Psi\rangle$ be the wavefunction evaluated at field configuration ϕ_2.

We can then have a formal description of the ket $|\Psi\rangle$ as

$$\int_{\phi(\tau_1)=\phi_1}^{\phi(\tau_2)=??} [D\phi] e^{-S_E[\phi]}\,,$$

where the upper limit is free.

Remark. *We might wonder why we are using the Euclidean path integral? States defined at the 2nd cut, i.e.* $\phi_2(\tau_2) = ??$*, remain states in the Lorentzian theory, and so we are fine to Wick rotate.*

6.1 Wavefunction of the Vacuum

We now want to compute the wavefunction of the vacuum using our transition amplitudes. The difficulty here is that we have an infinitely long interval of Euclidean time.

$$\langle \phi\,|\,\Omega\rangle = \frac{1}{\langle \Omega\,|\,\chi\rangle} \lim_{T_E\to\infty} \left\langle \phi\,\middle|\, e^{-T_E H}\,\middle|\,\chi\right\rangle \propto \int_{\phi(T_E\to-\infty)=0}^{\phi(T_E=0)=\phi} [D\phi] e^{-S_E[\phi]}\,.$$

We can look at certain correlation functions.

$$\langle \Omega\,|\,\Omega\rangle = \sum_\phi \langle \Omega\,|\,\phi\rangle\,\langle \phi\,|\,\Omega\rangle$$

Inserting this $\sum_\phi |\phi\rangle\langle\phi|$ is equivalent to gluing the path integral at at the finite limit at configuration ϕ. Note that the complex conjugation occurring in going from $|\Omega\rangle$ to $\langle\Omega|$ switches the sign of Euclidean time (as $T_E = it$), so the $T_E \to -\infty$ limit becomes a $T_E \to \infty$ limit. This is a useful perspective, as it means that we can now write correlation functions like $\langle \Omega \,|\, \mathcal{O}(T_{E_1}, x_1) \ldots \mathcal{O}(T_{E_n}, x_n)\rangle$ by inserting these operators at points in our path integral over $T_E \in \mathbb{R}$.

6.2 Density Matrices

Recall our density matrix is $\rho = |\Psi\rangle\langle\Psi|$. This is an operator, so object of the form $\langle \phi_2 \,|\, \rho \,|\, \phi_1\rangle$ gives us complex numbers. Hence the path integral associated to ρ is a path integral of finite length in T_E evolving between two cuts. Contracting with states $|\phi_1\rangle$ is the process of fixing states on those cuts.

Example 6.1. *If we took the example of $\rho_\beta = Z^{-1}e^{-\beta H}$, then our partition function is $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$. This expression is basis independent, so we can express it in our local basis as*

$$Z(\beta) = \sum_{\phi_1} \left\langle \phi_1 \,|\, e^{-\beta H} \,|\, \phi_1 \right\rangle .$$

Here, this sum then becomes a sum over tori corresponding to our path integral. If we were to now compute correlation functions at finite temperature of the form

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n)\rangle_\beta = \mathrm{Tr}(\rho_\beta \mathcal{O}(x_1) \ldots \mathcal{O}(x_n))$$

we then are seeing a sum over $M_{1,n}$, the moduli space of genus 1 curves with n fixed punctures.

6.3 Unruh's Effect / QFT in Rindler Spactime

6.3.1 Special Relativity

Let's go back to special relativity for a minute (working in $\mathbb{R}^{1,1}$). Recall that the relation between the acceleration of two intertial observers is

$$a_x = \frac{a'_x}{\gamma^3\left(1 + \frac{v u'_x}{c^2}\right)} ,$$

where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$. In the particular case of instantaneous rest frame where $u'_x = 0$, $a'_x = \gamma^3 a_x$.
As $a_x = \frac{dv}{dt}$, we have

$$\gamma^3 \frac{dv}{dt} = \frac{d}{dt}(\gamma v) .$$

Assuming $a_x = \alpha = a'_x$ a constant, this is a differential equation that we can solve for $\alpha t = \gamma(v) v$. Integrating again with $v = \frac{dx}{dt}$, gives

$$x(t) = \frac{c^2}{\alpha}\sqrt{1 + \frac{\alpha^2 t^2}{c^4}} .$$

Trajectories as above are hyperbolic, and define what is called **Rindler observer**. Drawing these as trajectories in $\mathbb{R}^{1,1}$ shows why these are hyperbolic, and they have asymptotes $x \pm t = 0$. Light emitted by this observer can only live in the **right Rindler wedge**, the region $x \geq |t|$. For time $t > t_c$, t_c some critical time, light will not reach our observer. This means we have a horizon, but this horizon is observer dependent. This is called a **Rindler horizon**.

6.3.2 General Relativity

We now write down the spacetime metric percieved by a Rindler observer covering the Rindler wedge. We will use proper time τ, which must be given by

$$d\tau = \frac{dt}{\gamma} \Rightarrow t = \frac{c}{\alpha}\sinh\frac{\alpha\tau}{c} \Rightarrow x(|tau) = \frac{c^2}{\alpha}\cosh\frac{\alpha\tau}{c}\,.$$

We also introduce coordinates ρ, η ($\eta = \tau$) related by

$$\begin{aligned} ct &= \rho\sinh\frac{\alpha\eta}{c}\,, \\ x &= \rho\cosh\frac{\alpha\eta}{c}\,, \end{aligned}$$

which maker the metric

$$ds^2 = -dt^2 + dx^2 = -\rho^2\left[d\left(\frac{\alpha\eta}{c}\right)\right]^2 + d\rho^2 = -\rho^2 d\eta^2 + d\rho^2\,.$$

This metric looks like our BH metric very close to the Killing horizon, which can be seen geometrically by drawing the Penrose diagram.
Let's again Wick rotate by setting $T_E = it$ and $\theta = i\eta$ s.t.

$$ds^2 = dT_E^2 + dx^2 = \rho^2 d\theta^2 + d\rho^2\,.$$

Remark. *In the current normalisation, if θ has the periodicity condition $\theta \sim \theta + 2\pi$, this is the metric of the Euclidean plane, which is the Wick rotation of Minkowski, and so both Euclidean continuations are equivalent. Hence Euclidean observables in QFT are identical.*

We want to ask whether this interpretation holds when we continue back to Lorentzian QFT. We will aim to show the following:

- In Minkowski, Euclidean correlation functions give rise to correlation functions in $|\Omega\rangle = |0\rangle_M$.
- in Rindler's right wedge, continuation computes correlation functions in $|0\rangle_M$ for observable restrict to the wedge.

Remark. *Note what we are working towards here is the idea that the fact that this spacetime is split into a left and right part, so our Cauchy surface $\Sigma(= t = 0)$ is also split into a L/R part, will end up meaning that the Hilbert space $\mathcal{H}_M$ will factorise in to the tensor product $\mathcal{H}_L \otimes \mathcal{H}_R$. The problem we will then run up against is that $|0\rangle_M \neq |0\rangle_L \otimes |0\rangle_R$.*
To spell this out, imagine we have a QFT of scalar field (as we don't want to deal with gauge questions) with arbitrary (i.e. possibly interacting) Lagrangian. Our Minkowski Hilbert space $\mathcal{H}_M$ will contain states of the form $\Psi[\phi(x)]$, and we will have Hamiltonian H_M conjugate to time t. Write Ψ_0 for the wavefunction corresponding to the vacuum $|\Omega\rangle = |0\rangle_M$.
On the right Rindler wedge, the wavefunctions will be of the form $\Psi[\phi_R]$ where $\phi_R(x) = \phi(x >$

$0, t = 0)$. The Hamiltonian here H_R will be conjugate to the clock η. Now because the information of $\phi|_\Sigma = \phi(x, t = 0)$ is contained in $(\phi_L(x), \phi_R(x)$, we get the factorisation $\mathcal{H}_M = \mathcal{H}_L \otimes \mathcal{H}_R$. Note that it is because our Hamiltonians have been take as conjugate to different coordinates, despite the fact that the metrics are the same, that they will be different and so the vacua will not agree on the intersect of the Cauchy surfaces. Indeed if one does the algebra, the vector field ∂_η is a boost generator in x, t coordinates.

7 Vacuum States - Lecture 6

Define

$$\Psi_0(\phi) := \Psi_0[\phi_L, \phi_R]$$

Using our polar coordinates ρ, θ, we can do our path integral now, which goes from $\theta = -\pi$ to $\theta = \pi$.

Remark. *We should have*

$$\langle \phi(x_2, t_2) \,|\, \phi(x_1, t_1) \rangle = \left\langle \phi_2 \,|\, e^{-i(t_2 - t_1)H} \,|\, \phi_1 \right\rangle ,$$

which should suggest to us that

$$\Psi_0(\phi) \propto \left\langle \phi_R \,|\, e^{-i(-i\pi)H_R} \,|\, \phi_L \right\rangle = \left\langle \phi_R \,|\, e^{-\pi H_R} \,|\, \phi_L \right\rangle .$$

We can introduce a resolution of the identity $\{|n\rangle_R\}$. so

$$\Psi_0(\phi(x)) \propto \sum_n e^{-\pi n} \chi_n(\phi_R) \chi_n^*(\phi_L)$$

where $\chi_n(\phi_R) = \langle \phi_R \,|\, n \rangle_R$, and likewise for L. Rewrite this as

$$\sum_n e^{-\pi n} \chi_n(\phi_R) \tilde{\chi}_n(\phi_L)$$

where we interpret $\tilde{\chi}$ as a wave function in a Hilbert space where evolution flows in the opposite direction to the right wedge (as it does in the left wedge). Note our $\Psi_0(\phi(x))$ is really $\langle \phi_L, \phi_R \,|\, 0 \rangle_M$, so what we have is that

$$|\Omega\rangle = |0\rangle_M \propto \sum_n e^{-\pi n} \,|n\rangle_R \otimes |n\rangle_L$$

7.1 Direct Density Matrix Calculation (Second Round)

Recall $\rho = |\Omega\rangle \langle\Omega|$. Taking two states in the right Hilbert space ϕ_i^R, $i = 1, 2$, we can calculate the reduced density matrix on the right as

$$\left\langle \phi_2^R \,|\, \rho_R \,|\, \phi_1^R \right\rangle = \sum_{\tilde{\phi}} \left\langle \tilde{\phi}, \phi_2^R \,|\, \rho_R \,|\, \phi_1^R, \tilde{\phi} \right\rangle$$

We will take the foliation $ds^2 = d\rho^2 + \rho^2 d\theta^2$, which means time translation is

$$\frac{1}{\hbar} [H_R, \mathcal{O}] = \partial_\theta \mathcal{O} ,$$

and using our transition amplitude intuition we write

$$\left\langle \phi_2^R \,|\, \rho_R \,|\, \phi_1^R \right\rangle \propto \left\langle \phi_2 \,|\, e^{-2\pi H_R} \,|\, \phi_1 \right\rangle \Rightarrow \rho_R = \frac{e^{-2\pi H_R}}{Z} \,.$$

Remark.

- *Note ρ_R appears as a thermal state with $T = \frac{1}{2\pi}$ with respect to H_R.*
- *Note also this is a non-perturbative derivation independent of the Lagrangian.*
- *$H_R \sim K_x$ a boost generator after wick rotation, where*

$$K_x \propto X\partial_T - T\partial_X \overset{T\to \pm iT_E}{\to} X\partial_{T_E} + T_E\partial_X = \partial_\theta$$

- *$|\Omega\rangle = |0\rangle_M$ is annihilated by $H_R - H_L = H_R \otimes I_L - I_R \otimes H_L$. The minus sign is related to time running backward on the left Rindler wedge. That is*

$$e^{-i\eta(H_R - H_L)} \,|\Omega\rangle = |\Omega\rangle$$

- *Note that because $ds^2 = -\rho^2 d\eta^2 + d\rho^2$, we can write $d\tau_{obs} = R_{obs} d\eta = \frac{1}{a} d\eta$, so it is R_{obs} that contains units, as τ is dimensionful and η is dimensionless. These give $\eta = a\tau_{obs}$. If we were to compute temperature wrt τ_{obs}, we would get dimensionful temperature*

$$T_{Unruh} = \frac{a}{2\pi}\frac{\hbar}{ck_b}$$

7.2 Proper Analytic Derivation (Third Round)

We know

$$\langle \phi_L, \phi_R \,|\, \Omega\rangle = \frac{1}{Z}\left\langle \phi_R \,|\, e^{-\pi K_R}\Theta \,|\, \phi_L \right\rangle$$

where

- K_R is the restriction of the analytic continuation of a boost to the right wedge,
- Θ is the generator of a CPT transformation in QFT, i.e

$$\Theta^\dagger \Phi(t,x,y)\Theta = \Phi^\dagger(-t,-x,y) \,,$$

  so it is antiunitary and

$$\left\langle u \,|\, \Theta^\dagger \,|\, v \right\rangle = \langle v \,|\, \Theta \,|\, u\rangle \,.$$

  Note Θ is necessary as it maps $\mathcal{H}_{R/L} \to \mathcal{H}_{L/R}$. If our states were charged under some gauge field, Θ would have to act on those quantum numbers too.

Now using our expression

$$\langle \phi_L, \phi_R \,|\, \Omega\rangle = \frac{1}{Z}\sum_n e^{-\pi n} \langle n \,|\, \Theta\phi_L\rangle \langle \phi_R \,|\, n\rangle_R \,,$$

and introducing $|n^*\rangle_L = \Theta^\dagger \,|n\rangle_R$ we get

$$\langle \phi_L, \phi_R \,|\, \Omega\rangle = \frac{1}{Z}\sum_n e^{-\pi n} \langle \phi_R \quad n\rangle_R \langle \phi_L \,|\, n^*\rangle_L \,,$$

and so we deduce

$$|\Omega\rangle = \frac{1}{Z}\sum_n e^{-\pi n}\,|n^*\rangle_L\,|n\rangle_R\ .$$

Remark. *Consider now a free field and take the coordinate change from* $\mathbb{R}^{1,1}$

$$x = e^{\xi_R}\cosh(\eta_R) = -e^{-\xi_L}\cosh(\eta_L)$$
$$t = e^{\xi_R}\sinh(\eta_R) = e^{-\xi_L}\sinh(\eta_L)$$

The corresponding metrices in the two wedges are

$$e^{-2\xi_L}(-d\eta_L^2 + d\xi_L^2) = -dt^2 + dx^2 = e^{2\xi_R}(-d\eta_R^2 + d\xi_R^2)\,.$$

We have boost generator $K_x = \partial_{\eta_R}$ *in the right wedge (as we have seen previously) but we can also check in the left wedge this is* $-\partial_{\eta_L}$*.*

Remark. *We make some philosophical remarks on the relevance of entanglement wrt the the smoothness of the horizon.*

- *Note that if the state is* $|0\rangle_R$*, the right Rindler observer sees no particles, precisely because this is the vacuum state in* $\mathcal{H}_R$*.*
- *While to a Rindler observer there is a horizon corresponding to the wedge, any inertial observer will not see these horizons, and eventually will cross out of the region.*
- $|0\rangle_R$ *is singular at the horizon. More precisely, if an inertial observer is in* $|0\rangle_R$*, and you compute the stress tensor, the expectation* $\langle 0\,|\,T_{\mu\nu}\,|\,0\rangle_R$ *diverges at the horizon. To interpret this, go back to our open subsystem perspective where* $\rho = \rho_L \otimes \rho_R$*. Here there are no correlations between L and R, and the two subsystems do not know about the existence of the other. Hence a right wedge observer does not know about the existence of the left wedge.*

Remark. *Note we started with classical black holes, and we showed that there was a formal analogy to classical thermodynamics, giving a 'temperature'. We then went on a discussion where* $G_N = 0$*, and saw that for non-inertial observers we could get situations where they perceive the vacuum state of the whole system as excited from a subsystem perspective - this was Unruh's effect. We ask - what can we learn about the physical reality of* T_{BH} *from Unruh's effect?*

- *Unruh's effect is a purely QFT effect, so it cannot apply directly.*
- *For BHs we have two scenarios, maximal Kruskal extensions, and the formation of BHs by gravitational collapse, which are two different systems. For the first, we do recall that the Schwarzschild metric is, close to the horizon, similar to Rindler, and so our previous work will apply. For the latter, we do not have white holes, and so it is harder (and this is what Hawking discussed in his work).*

7.3 QFT in Curved Backgrounds

Let's lay the groundwork to answer these questions raised at the end of the last section. In order to turn of G_N without removing the black hole, we let $G_N \to 0$ s.t. $r_S = GM$ is constant. We do this by letting the dimensionless quantity $\frac{M}{M_P} \to \infty$. This keeps the dynamics of quantum matter, but kills graviton interactions (while still keeping free gravitons) while preserving the geometry. Recall that when we used the Euclidean path integral to compute $|0\rangle_M$, in order to have the coordinates (T_E, x) and (θ, ξ) be equivalent, we had to have $\theta \sim \theta + 2\pi$ (Hartle-Hawking precsription). If θ was non-compact we have a degree of freedom, and this changes the vacuum.

8 HH Vacuum - Lecture 6

Remark. *For some references to cover what we have seen, look at*

- *Unruh, Notes on BH evaporation, 1976*
- *Fulling, 1973*
- *Davoies, 1975*
- *Unruh, Bisogano, Wichmann, 1975, 1976*
- *Crispins, Itiguchi, Metsas, 0710.5373(?)*

Remark. *To make concrete a remark from before, recall the Schwarzschild metric*

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \,,$$

and as $r \to \infty$, $f(r) \to 1$, and $f(r) \sim \rho^2$ when $r - r_+ := \rho^2 \ll r_+$. Hence we really do see the canonical asymptotic clock at infinity is the same clock up to constant rescalings as the on in the right Rindler wedge.

If we perform canonical quantisation in the Schwarzschild BH, the corresponding vacuum $|0\rangle_R$ has t_E non-compact. In (cite, Hartle Hawking 1976), HH discovered a vacuum state $|0\rangle_{HH}$ corresponding to $t_E \sim t_E + \beta_{BH}$. Taking again $t = 0$ as the Cauchy slice, we have a Euclidean preparation for $|0\rangle_{HH}$ on the lower half plane $\times S^2$. We can thus write our state as

$$|0\rangle_{HH} \propto \sum_i e^{-\beta_{BH} E_i/2} \, |i^*\rangle_L \otimes |i\rangle_R \,,$$

and so our asymptotic observer only sees the ρ_R, a thermal state. Hence β_{BH} is physical.

8.1 Towards 1-sided

So far we have been looking at the Penrose diagram with both sides, but we will want to see if we can restrict to the 1-sided case (which corresponds to BH formed by collapse rather than always being there?)

Let's have a perturbative discussion. We have a field ϕ satisfying

$$\nabla_\mu \nabla^\mu \phi = m^2 \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right) \phi \,.$$

Taking the ansatz

$$\phi = \frac{1}{r} Y_{lm}(\Omega) f(r,t)$$

with tortoise coordinate
$$r_* = r + 2M \log\left(\frac{r}{2M} - 1\right) ,$$
this gives
$$\partial_t^2 f = \partial_{r_*}^2 f + V_{eff}(r) = 0$$
where
$$V_{eff}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right] .$$
This is a Schrödinger-like equation, which has solutions
$$f(r,t) = e^{-iwt}\psi_{wl}(r) .$$
We have the regimes

1. $r \to 2M$, $r_* \to -\infty$ where $V_{eff} \to 0$ and we get free solutions
2. $r \to \infty$, $r_* \to \infty$, in which case
$$V_{eff} \to \begin{cases} \frac{l(l+1)}{r^2} & m^2 = 0 \\ m^2\left(1 - \frac{2M}{r}\right) & m^2 \neq 0 \end{cases}$$

Importantly $\exists r_{max}$ around which $V_{eff} \sim \rho^2$ where $\rho = r - r_{max}$. We picture that we are scattering off this potential V_{eff}.
Consider first where $m^2 = 0$. We expect to get
$$\phi \sim \begin{cases} f_+(u)[\text{support } J^+] + g_+(v)[\text{BH horizon}] & t \to \infty \\ f_-(u)[\text{support WH}] + g_-(v)[J^-] & t \to -\infty \end{cases}$$
where u, v are the coordinates $t \mp r_*$ respectively.
Looking at the asymptotic past, consider a single particle Hilbert space $\mathcal{H}_{in} = \mathcal{H}_{in,\infty} \oplus \mathcal{H}_{in,WH}$. We need coordinates to reference states in this space, and we will use t on $\mathcal{H}_{in,\infty}$, but we have ambiguity on $\mathcal{H}_{in,WH}$ (for example, use t or Kruskal).
We can do the same thing for asymptotic future, and write $\mathcal{H}_{out} = \mathcal{H}_{out,\infty} \oplus \mathcal{H}_{out,BH}$.
Now for our 1-sided case, part of our Hilbert space decomposition will not be there, we will only have
$$L^2\left(\mathcal{H}_{out,BH}\right) \otimes L^2\left(\mathcal{H}_{out,BH}\right) .$$

Remark. *For references on this section, see*

- *Hawking, "Particle creation ...", 1976*
- *Wald, "On particle creation by BHs", 1975*

The problem we need to solve, is that the wave equation in Minkowski is glued with what happened earlier on, which is time dependent and could depend on details of the collapse. Our strategy will be as follows:

1. use late times, because frequencies suffer from gravitational redshift, making excitations very massive. Hence one can use geometric optics approximation to the KG equation $\phi \sim \phi_0 e^{iS}$, and so the field is determined by null geodesics γ,

2. γ goes through the J^- at $v = v_1$, close to $v = v_0$ (the last inward ray on the surface of the collapsing body, i.e. in the WH case $v = v_0$ corresponds with the WH horizon).

3. Use geodesic deviation to γ, s.t this deviation can be parameterise by ϵn^a where n^a is the tangent to an ingoing null geodesic at the horizon, hence $n^a l_a = -1$, where l^a is the null geodesic generating the horizon.

4. Let $p^a = \frac{\partial u}{\partial \lambda}\partial u$, tangent to the ingoing null geodesic. $p^a = A^2 n^a$, λ is an affine parameter $B^2 U = -B^2 e^{-\kappa u}$.

5. solve the deviation equation, which connects the behaviour at the horizon ($\lambda = 0$) with γ ($\lambda < 0$),
$$\frac{dp^a}{d\lambda} = \frac{d^2 x^a}{d\lambda^2} \Rightarrow \lambda p^a = x^a(\lambda) - x^a(0) = -\epsilon n^a \Rightarrow \epsilon = -\lambda A^2 \,.$$

6. Following γ up to J^-,
$$v_0 - v = \epsilon = -\lambda A^2 = C^2 e^{-\kappa u}$$

The solution to the scattering problem according to this strategy is
$$\phi \sim \begin{cases} e^{-iwu} & J^+ \\ e^{i(w/\kappa)\log(v_0 - v)/C^2} & v < v_0 \end{cases}$$
Doing this analysis properly using Fourier analysis, Wald shows
$$\forall f > 0, \quad \tilde{\phi}(-f) = -e^{-\pi w/\kappa}\tilde{\phi}(f)\,,$$
and so for any positive frequencies in the wavefunction, the negative frequency also shows up. Hence there will be non-trivial Bogoliubov transformations.

8.2 Summary/End-result/Outcomes

Hawking measured $\frac{dE}{dt}$, the outward energy flux at ∞. We think of this as radiation, and he showed that
$$\frac{dE_{\text{rad}}}{dt} = \frac{wdw}{2\pi}\frac{P_{\text{absorbtion}}(w,l)}{e^{\beta_{BH}w} - 1}\,.$$
This is independent of the details of the collapse. We can make an approximation to this by using that when $wr_s \ll 1$, $P_{abs}(w,l) \sim (wr_s)^2$ ($r_s = 2GM$) for low l, and as higher powers for higher l. Hence we can approximate
$$\frac{dE_{\text{tot}}}{dt} \approx \int_0^{1/r_s} fracwdw2\pi \frac{(wr_s)^2}{\beta_{BH} w} \sim \frac{c}{\beta_{BH} r_s} = \frac{\tilde{c}}{r_s^2}\,.$$
Taking $E = M$, $r_S \sim G_N M$, then
$$M^2 dM = -\frac{\tilde{D}}{G_N^2}dt\,,$$
and so $t_{\text{evaporation}} \sim G_N^2 M^3$ ($d = 4$). This is long, but finite.
In the paper by Wald, he computed the reduced density matrix to be
$$\rho \propto \bigotimes_{w,l,m}\left[\sum_n P_{abs}(w,l)e^{-\beta w n}\left|n\right\rangle_{w,l,m}\left\langle n\right|\right]\,.$$

Hawking's conclusion was that, if $G_N \neq 0$, the time evolution of a pure state photon ends up in a mixed state. This violates unitarity in quantum mechanics.

Remark. *At early times, some radiation is asymptotic, so entanglement is growing while the BH is becoming small. Hawking's process always operates, so any state is in a space $(BH) \times (Asymp\ Rad)$. These two systems must have the same entanglement entropy at all time, but a small BH must have small entropy because it has a small number of states, unless quantum gravity does something here. One possible way of solving the problems occuring is to believe that BHs give a 'coarse-graining', and so $\exists$ subtle quantum correction that may save the day. That is, suppose your state is pure, we probe it with correlators*

$$\langle \psi_i \,|\, \mathcal{O}_1(x_1) \dots \mathcal{O}_N(x_N) \quad \psi_i \rangle = \mathrm{Tr}(\rho_\beta \mathcal{O}_1(x_1) \dots \mathcal{O}_N(x_N)) + \mathcal{O}\left(e^{-S_{BH} - \tilde{N}}\right) ,$$

and if N is not large enough, then one cannot distinguish ψ_i from another close state, so if you have an operator complex enough to distinguish the BH microstates then it may back-react and change your Hilbert space.

9 The Holographic Principle - Lecture 7

write this in

10 Partition Functions in GR - Lecture 8

This was first considered by Gibbons and Hawking in 1977, the paper "Action Integrals and Partition Functions in Quantum Gravity". Our partition function will be, in Euclidean signature,

$$Z = \int [Dg][D\phi] e^{-S_E[g,\phi]} ,$$

where

$$S_E[g, \phi] = \frac{1}{16\pi G_N} \int \sqrt{g}(R + \dots) + S_{\text{matter}} .$$

We will take finite temperature in order to make this object more tractable, and so have

$$t_E \sim t_E + \beta\,, \text{ and } \lim_{r \to \infty} g_{tt} = 1\,,$$

to have asymptotic flatness. We will take what is called **semi-classical gravity**, where $\hbar = 1$ but $G_N \to 0$. This means gravity is classical, but matter is quantum.
Moreover, we assume that our Euclidean action has the form

$$S_E = \frac{1}{\kappa^2} \left[\dots \right] ,$$

and so the partition function can be approximated by saddles, $\overline{g}_i$, $\overline{\phi}_i$. Hence

$$Z \approx \sum_i e^{-S_E[\overline{g}_i, \overline{\phi}_i]} \left(1 + \text{loop}\right) .$$

10.0.1 Flat Space Thermodynamics

At finite T, we know that there is the saddle (which for us will just mean a stationary point of the action, so a solution to the Einstein equations) given by the Euclidean BH. Hence

$$\log Z \approx -S_E[\bar{g}_{BH}] = S - \beta E$$

where $S = (1-\beta\partial_\beta)\log Z$ and $E = -\partial_\beta \log Z$ (recall $Z = Z[\beta]$).

10.1 Evaluation Euclidean Action

Introduce cutoff r_0 which we let $\to \infty$. Taking $G_N = 1$ for now we get

$$s_E = -\frac{1}{16\pi}\int_M \sqrt{g}R - \frac{1}{8\pi}\int_{\partial M}\sqrt{h}K\,,$$

where the boundary corresponds to r_0, h the emtric induced on ∂M, and $K = h^{ij}K_{ij}$ and $K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij}$, N the inward pointing unit normal to ∂_M.
We will impose the boundary condition $\delta g|_{\partial M} = 0$, which means that we don't get a boundary term when integrating by parts the bulk term. The saddle is, as we have written,

$$ds^2 = \left(1 - \frac{2M}{r}\right)d\tau^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2\,,$$

where τ is periodic $\tau \sim \tau + \beta$, where for the Schwarzschild BH $\beta = 8\pi M$. On shell here $R = 0$ as it is a vacuum solution, and we know

$$ds^2\big|_{r_0} = \left(1 - \frac{2M}{r_0}\right)d\tau^2 + r_0^2 d\Omega_2^2\,,$$

so letting $f = 1 - \frac{2M}{r_0}$ our boundary has normal $n = \sqrt{f}\partial_r$. This means we can calculate

$$K_t t = \sqrt{f}\frac{M}{r_0^2}\,, \quad K_{\theta_i\theta_j} = \sqrt{f}r_0\hat{h}_{\theta_i\theta_j}\,,$$

hence $K = \frac{M}{\sqrt{f}r_0^2} + \sqrt{f}\frac{2}{r_0}$ and $\sqrt{h} = \sqrt{f}r_0^2\sqrt{\hat{h}}$. This all combines to mean

$$\int_{\partial M}\sqrt{h}K = 4\pi\beta\left[8\pi r_0 - 12\pi M\right]\,.$$

This expression diverges when $r_0 \to \infty$, as one might expect. Like in QFT therefore, we add a counter term

$$S_{ct} = \frac{1}{8\pi}\int_{\partial M}\sqrt{h}K_0\,,$$

where K_0 is the extrinsic curvature of an on-shell metric sharing the same boundary metric on the BH. This background metric will be

$$ds^2_{\text{background}} = \left(1 - \frac{2M}{r_0}\right)d\tau^2 + dr^2 + r^2 d\Omega_2^2\,,$$

which it is not too hard to see is flat and will have the same boundary metric. One can work out that for this backgroun $K_0 = \frac{2}{r_0}$, and so

$$S_{ct} = \frac{1}{8\pi}\left[8\pi\beta\sqrt{f}r_0\right] .$$

check factors here. This is designed s.t now we get no divergence of the GHY action, but it modifies the finite piece.
According to our saddle point approximation, we now have

$$\lim_{r_0\to\infty}\log Z \approx -\frac{\beta M}{2} = -4\pi M^2 = -\frac{\beta^2}{16\pi} .$$

This gives

$$Z[\beta] \approx e^{-\frac{\beta^2}{16\pi}} .$$

10.1.1 Check Thermodynamic Quantities

If our partition function calculation is to be good then we should have entropy

$$S = \log Z + \frac{\beta^2}{4\pi} = 4\pi M^2$$

which is equal to our S_{BH}, good. Moreover the energy E should be

$$E = -\partial_\beta \log Z = M .$$

again agreeing with E_{BH}.

10.1.2 Comparison with Wald

Recall the 1st law said that in Lorentzian signature there is a horzion, and it is necessarily cloaking a singularity. Our Euclidean partition function said that in Euclidean signature there does not exist as horizon that's a smooth saddle.
To compare, note in our calculation there is a boundary term. Wald's story, a covariant formalism, had ambiguity $L \to L+dB$. However, both Euclidean and Lorentzian spacetime are time translation invariant, and with the inclusion of the boundary term Wald also finds that

$$-S_E = S_{BH} - \beta(M - \Omega_i J_i)$$

where the RHS really is the $\log Z$, or indeed the free energy $-\beta F$. The problem arises because when we go to Euclidean time we compactify the time dimension.

10.1.3 Final Remarks

- Note we have other saddles, for example Euclidean Minkowski (often thought of as a thermal gas). When one is doing this calculation with multiple saddle, we have multiple free energies, and so there may be regimes where different saddles dominate at finite temperature.

- Gross, Perry, Yalle, proved in "instability of flat space at finite T" (1982) that hot flat space is unstable due to what is called Jeans' instability. Moreover, the Schwarzschild BH has a negative mode (responsible for specific heat < 0) also giving instability.

- Atick and Witten in "Hagedorn transition and number of degrees of freedom in string theory" (1988) showed that closed strings can wind around t_E, giving a new quantum number (corresponding to the winding number). Tachyon instabilities arise in the spectrum below some temperature. Hence closed strings know about the difference between Euclidean and Lorentzian time.

10.2 de Sitter

Recall de Sitter the maximally symmetric spacetime with $\Lambda > 0$ given by

$$R_{\mu\nu} = \frac{d-1}{l^2} g_{\mu\nu} \,,$$

and we take $\Lambda = \frac{(d-1)(d-2)}{2l^2}$. This can be viewed as a hyperboloid in $\mathbb{R}^{1,d}$ given by

$$-x_0^2 + \sum_i x_i^2 = l^2$$

and so we take coordinates $x_0 = l \sinh(T/l)$, $x_i = l w_i \cosh(T/l)$, where w_i are coordinates on a unit sphere. Then

$$ds^2 = -dT^2 + l^2 \cosh^2(T/l) d\Omega_{d-1}^2 \,.$$

Alternatively if we let $\tan(\eta/2) = \tanh(T/2l)$ then on the boundary where $T \to \infty$ we get

$$ds^2 = \frac{l^2}{\cos^2\eta} \left(-d\eta^2 + d\Omega_{d-1}^2\right) \,.$$

Remark. *Following from causality, interial observers cannot observe full de Sitter.*

Isolating out one particular θ on the S^{d-1} we can talk about its north and south pole, and then we call the causally connected patch to the south pole the **causal diamond**. This has coordinates

$$\begin{aligned} x_0 &= \sqrt{l^2 - r^2} \sinh(t/l) \,, \\ x_1 &= \sqrt{l^2 - r^2} \cosh(t/l) \,, \\ x_i &= r z_i \,, \quad (2 \le i \le d) \end{aligned}$$

and this gives metric

$$ds^2 = -\left(1 - \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{l^2}} + r^2 d\Omega_{d-2}^2 \,.$$

$r = l$ is horizon seen just by the south pole observer.

10.2.1 de Sitter thermodynamics

Now take $r = l(1-\rho^2)$ for $\rho^2 \ll 1$. This gives

$$ds^2 = 4l^2(d\rho^2 + \frac{\rho^2}{l^2}dt^2)$$

We identigy $t \sim t + 2\pi i l$ and so the Euclidean time has $t_E \sim t_E + 2\pi l$ giving a temperature $T_{dS} = \frac{1}{2\pi l}$, and a horizon area
In $d = 4$, taking the saddle point approximation we have

$$S_{dS} = \pi l^2$$

Euclidean de Sitter is a sphere, so has no boundary. and so we get

$$S_E = -\frac{1}{16\pi}\int \sqrt{g}(R - 2\Lambda) = -pil^2$$

Recall we would want to see $-S_E = \log Z = -\beta F = S_{dS} - \beta E_{dS}$, and the lack of boundary means $E_{dS} = 0$, so this story agrees with thermodynamics.

10.3 Remarks

Let's collate what we know about QG so far.

- Recall the holographic principle slightly rephrased: the number of d.o.f of QG scale like the number of d.o.f in QFT of one dimension less.
- Weinberg-Witten theory, which prevents gravitons in a relativistic QFT with a covariantly conserved $T_{\mu\nu}$.

Some possible conclusions are

- gravitons may live in one extra dimension,
- the QM description capturing the physics of graviton should be strongly coupled.

We can ask some questions.

- How can we test these ideas? We might look to large N limits of gauge theories, which look like string theories. It will turn out that pairing this with the holographic principle with CFTs will force you onto AdS.
- How could we show that $A = B$? We could go to a low energy limit (sometimes called the decoupling limit), where A (resp B) becomes A_1, A_2 (resp B_1, B_2), and in this limit show $A_i = B_i$.

11 Large N 't Hooft limit - Lecture 9

The references for the following section will be

- 't Hooft, 1974, "a planar diagram theory for strong interactions"
- review 9802419

Consider the QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{g_{YM}^2}\left[-\frac{1}{4}\operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}) - i\overline{\psi}(\not{D} - m)\psi\right] .$$

This is strongly coupled at low energies. The fields are $SU(3)$ valued, we hope that if we change that to $SU(N)$, we get $1/N$ perturbative parameter.

11.1 Toy Model

Consider first

$$\mathcal{L} = -\frac{1}{g^2}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\Phi + \frac{1}{4}\Phi^4\right]$$

where $\Phi\Phi^a{}_b$ and $N \times N$ Hermitian matrix. We have global $U(N)$ symmetry given by $\Phi \mapsto U\Phi U^\dagger$. The Feynmann rules are that we have propagator from ab to cd

$$\langle \Phi^a{}_b(x)\Phi^c{}_d(y)\rangle = g^2\delta^a_d\delta^c_b G(x-0)$$

and 4-point function of interactions ab, cd, ef, gh given by

$$\frac{1}{g^2}\delta^a_h\delta^c_b\delta^e_d\delta^g_f .$$

To simplify the combinatorics of these, 't Hooft introduced **double line** notation, where instead of 1 line for a propagator we draw 2, where the upper index of one goes to the lower of the other, and they are given arrows in the opposite directions.

11.1.1 Vacuum Diagrams

We can now try to consider vacuum diagrams with a single vertex. One can work out that we get two diagrams, giving contributions

1. g^2N^3 (a vertex with external legs on the same side joined - this can be drawn in the plane)
2. g^2N (a vertex with external legs on the opposite side joined - this cannot be drawn in the plane) (note this looks like the Möbius band)

The diagrams we can draw without intersections are called **planar**.
One can then consider 2-vertex diagrams, and now we get contributions

1. g^4N^4 (planar)
2. g^4N^2 (non planar)

One might wonder whether we can construct general rules to understand the behvaiour of these diagrams at any order.

Remark. *Both non-planar diagrams can be drawn on a torus without crossing lines (the diagram has been 'straightened out'). The power of N is the number of faces in each diagram after being straightened out.*

In order to generalise this, we use the following facts.

- Any orientable 2d surface is topologically classified by an integer h (the genus)
- $\forall$ non-planar diagram, $\exists h$ s.t. the diagram can be straightened out on a surface of genus h ($h \in \mathbb{Z}_{\geq 0}$), but not on a surface of smaller genus.
- For any diagram, the power of N is given by a number of faces F.
- We have Euler's formula $V - E + F = 2 - 2h := \chi$. One can derive this by thinking of the diagram as a triangulation of the surface.

Using these we can say that for any diagram

$$\mathcal{A} \sim N^F (g^2)^{E-V}$$

where V is the number of vertices and E is the number of edges.

11.1.2 't Hooft Limit

To make this tractable, we might have Euler's formula in mind, so we choose to take $N \to \infty$, $g \to 0$ s.t. $\lambda = g^2 N$ is fixed. This is called the **'t Hooft coupling / 't Hooft limit**. Then we get

$$\mathcal{A} \sim \lambda^{E-V} N^{V-E+F} .$$

Notes $E - V = L - 1$, where L is the number of loops in the diagram, i.e. the number of undetermined momenta (this is as edges correspond to momenta, vertices impose a delta conservation, and there is an overall conservation in the diagram). Hence we have

$$\mathcal{A} \sim \lambda^{L-1} N^{2-2h} .$$

What we learn is that our answer for the total vacuum amplitude is

$$N^2 \left[c_0 + c_1 \lambda + \ldots\right] + \mathcal{O}(N^{2-2h}) ,$$

and non-perturbatively, in the large N limit,

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda) .$$

In our toy model approach, we had leading order term in our partition function $\log Z \sim N^2$. Recall that if we did a path integral to calculate Z and took a saddle approximation, we would expect to get leading contribution to $\log Z$ of S_E at the saddle. Because we had $\mathcal{L} = -\frac{N}{\lambda} \operatorname{Tr}[\ldots]$, we should then expect to get $\operatorname{Tr}[\ldots] \sim \mathcal{O}(N)$. This is a heuristic argument, but it will hold more generally that if we have $\mathcal{L} = -\frac{N}{\lambda} \operatorname{Tr}[\ldots]$, and we have matrix degrees of freedom, then we should expect a partition function of the form

$$\log Z \sim \sum_h N^{2-2h} f_h .$$

The expansion becomes an expansion in the topology of Feynmann diagrams.

11.2 General Observables

In QCD, we have gauge transform (in some convention)

$$A_\mu \mapsto UAU^\dagger - i(\partial_\mu U)U^\dagger \,.$$

Φ^2 was a good observable in our toy model, but it will not be so in QCD, and in that sense gauge theories are more restrictive than our toy model. If we have a Lagrangian $\mathcal{L}[A,\psi] = -\frac{N}{\lambda_{YM}}\,\mathrm{Tr}[\ldots]$. We will restrict to local operatrors, i.e involving trace.

- single trace $\mathcal{O}_i$ ($i \in I$ a label),
- multi-trace operators $\mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)$.

If we want to compute observables $\langle\mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)\rangle$, we want to know if there is an N expansion, similar to how there was for our vacuum correlator, but now we may have external legs.
Recall the way we calculate these objects with a generator

$$Z[J_1,\ldots,J_n] = \int [DA_\mu][D\psi] e^{iS_0 + i\int J_i(x)\mathcal{O}_i(x)} \,,$$

which gives the correlator as

$$\langle\mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)\rangle = \left.\frac{\delta^n \log Z}{i^n \delta J_1 \ldots \delta J_n}\right|_{J_i=0} \,.$$

For us $S_0 \sim \frac{N}{\lambda}\,\mathrm{Tr}[\ldots]$, and if we scale the J_i by N we get $iS_{eff} = N\,\mathrm{Tr}[\ldots]$, which is the right N scaling. Hence

$$\langle\mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)\rangle = \left.\frac{\delta^n \log Z}{i^n N^n \delta J_1 \ldots \delta J_n}\right|_{J_i=0} = \sum_{h=0}^{\infty} N^{2-n-2h} F_n^h(\ldots) \,.$$

Remark. *We there are no operators, the dominant scaling of the vacuum is $\langle I\rangle \sim \mathcal{O}(N^2)$, and likewise the dominant contribution with k operators is $\mathcal{O}(N^{2-k})$.*

11.2.1 Physical Consequences

Remark. *Single and multi-trace operators at large N behave like n-particle states, in the sense that*

$$\mathcal{O}_i(x)\,|0\rangle \sim \textit{ single particle}$$

12 Large N as a String Theory - Lecture 10

12.1 Finishing Off from Last Time - Fluctions in Large N

Suppose we have a state $\langle\mathcal{O}\rangle \neq 0$. The variance is $\langle O^2\rangle - \langle O\rangle^2 = \langle O^2\rangle_C$, the connected part of the expectation. We have that

$$\frac{\sqrt{\langle\mathcal{O}^2\rangle_C}}{\langle\mathcal{O}\rangle} \sim \frac{1}{N} \to 0 \,.$$

This idea can be extended to higher product operators.
We can reinterpret this remark as a classical theory at leading order, doing for example

$$\underbrace{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_C}_{\mathcal{O}(N^{-1})} = \sum_i \underbrace{\langle\mathcal{O}_1\mathcal{O}_i\rangle}_{\mathcal{O}(1)}\underbrace{\langle\mathcal{O}_i\mathcal{O}_2\prime_3\rangle}_{\mathcal{O}(N^{-1})} + \sum_{i,j}\underbrace{\langle\mathcal{O}_1 : \mathcal{O}_i\mathcal{O}_j :\rangle}_{\mathcal{O}(N^{-1})}\underbrace{\langle : \mathcal{O}_i\mathcal{O}_j : \mathcal{O}_2\mathcal{O}_3\rangle}_{\mathcal{O}(N^{-2})} + \dots .$$

We see that only the first term in the expansion contributes at leading order, so there are no loops at leading order, which is exactly what we consider a classical theory. We interpret this as saying that gauge theory in which $N \to \infty$ with $\hbar$ finite is equivalent to a 'glueball theory' or 'generalised free field theory' with coupling $\tilde{h}$. There is a $\frac{1}{N}$ expansion that is equivalent in a semiclassical expansion in $\tilde{h}$.

12.2 String Theory

Recall first the moral of how we get the quantum theory of a particle, we have $S_{\text{particle}} = m\int_\gamma dp$, and then to quantise we do a path integral $\int[Dx^m]e^{iS_{\text{particle}}}$. We might wonder whether it is possible to reproduce QFT amplitudes including interactions in this picture. It turns out that this is possible, but it must be done 'by hand', and so historically this picture was abandoned.
Now if we do string theory, we have action $S_{\text{string}} = T\int_\Sigma dA$, ($T = 1/2\pi\alpha'$), and then if we quantise we do the path integral $\int[Dx^\mu(\tau,\sigma)]e^{iS_{\text{string}}}$. Again, we wonder how to include interactions. The analogy between these two pictures will be worth study.
Let's work in Euclidean signature.

12.2.1 Vacuum Strings

Our vacuum energy partition function is a sum over all 2d surfaces with no external string states (i.e. $\partial\Sigma = \emptyset$), so they are defined only by their genus and so

$$Z_{\text{string}} = \sum_{h=0}^{\infty}\sum_{\Sigma_h} e^{-S_{\text{string}}} .$$

There is one (known) wy of including interactions consistently with QM and the symmetries of the string action, which is to have

$$Z_{\text{string}} = \sum_{h=0}^{\infty} e^{-\lambda\chi}\sum_{\Sigma_h} e^{-S_{\text{string}}} .$$

From a perspective of statistical mechanics, λ looks like a chemical potential for the topology of Σ_h (it comes from the string coupling as $e^\lambda = g_S$).

Example 12.1. *The sphere gets a contributions $\sim g_S^{-2}$, and the torus gets a contributions $\sim g_S^0$.*

Our conclusion is that summing over topologies encodes the interactions of strings.

Remark. *We can view the sphere the contribution as the nucleation of a closed string from the vacuum, and decaying afterwards.*
Likewise, we can view the as nucleation of a closed string, that splits, recombines later, and then

decays.
If one were to play this game with higher genus surfaces, one can take different viewpoints on this (based on how one views 'time' across the surface), but they will be equivalent under the symmetry of the theory.

Each split/recombination gives a factor g_S, and as such we can think of them as vertices, and then we know every hole comes with a split and a recombination.

12.2.2 External Strings

If we allow for the analogues of external legs to the diagram, this is equivalent to allowing boundaries (or punctures) to the surface Σ, and now the Euler formula is modified to be

$$\chi = 2 - 2h - n\,,$$

where n is the number of boundaries. The amplitude for the string is now, at fixed n,

$$\mathcal{A}_{\text{string}} = \sum_{h=0} g_S^{n-2+2h} G_h^(n)(\dots)\,.$$

When we look back at the expectation values for the product of n observables, we see we have the same expression, with the dictionary that $g_S \leftrightarrow N^{-1}$, external strings $\leftrightarrow$ single trace operators, and sum over genus-h $\Sigma \leftrightarrow$ sum over genus-h Feynmann diagrams. The hope is to now the show that this isn't a coincidence, but something deeper.

Remark. *In our operator expectation expression, $F_n^h = \sum_M G$ was the sum over Feynmann diagrams of genus h G, whereas $G_n^h = -\int_{\Sigma_h}[Dx^m]e^{-S_{string}}$. These might initially look different, but recall we thought of diagrams as triangulations of a surface of genus h, so $\sum_M G$ is the sum over all triangulations of the surface, which one might hope is a discrete version of $\int_{\Sigma_h}$.*
If we believe this is a correct interpretation, and we can identify the contribution G with $e^{-S_{string}}$, then we have exactly that a large N gauge theory is a string theory, s.t. the classical theory of single trace operators can be identified with the tree level classical string theory.
This is not at all obvious, as G and e^{-S} look very different.

Remark. *On the string side, we have many choices, such as the choice of spacetime, degrees of freedom (e.g. bosonic, superstring), and choice of the actions S_{string}.*

Remark. *In order for large genus diagrams, which are in some way the diagrams that provide the better approximation to the continuum limit, we need the strong string coupling limit (large g_S).*

This problem is hard to make progress with, but there was

- Klebanov 9108019, 'String theory in Two Dimensions'.
- Simpler models did realise the correspondence quantitatively for
  - matrix Hermitian models,
  - matrix quantum mechanics.

  These are non-critical string theories.

We might still wonder about QCD. One might hope that we get a string theory in $\mathbb{R}^{1,3}$, but these string theories only have propagators in very specific spacetimes and dimensions ($D = 26$ for bosons, $D = 10$ for fermions). One could then consider $\mathbb{R}^{1,3} \times \Sigma_6$, where Σ_6 is a compact manifold (this is the story of Calabi-Yaus), but this cannot work as closed string theories always contain gravitons, which QCD doesn't (recall our no-go theorem).

Remark. *Historically, Polyakov wrote a paper a few months before Maldacena published AdS/CFT correspondence, proposing that QCD came from a 5d string theory.*

Since strings always contain gravity, we wonder is there any property in gravity theories that gives a way to search for a string theory description of QCD. The holographic principle (as we saw earlier) then motivates that we try strings in 5d non-compact space. We start by writing the most general metric that is invariant under QCD symmetries (namely Lorentz symmetry)

$$ds^2 = Q(z)^2 \left[dz^2 + d\boldsymbol{x}^2 - dt^2\right] .$$

Remark. *If we replace QCD with a conformal gauge theory, then we also have a scaling symmetry* $x^m u \mapsto \lambda x^\mu$ *(*$(x^\mu) = (\boldsymbol{x}, t)$*) that the metric must respect. We can choose to respect this symmetry by having* $z \mapsto \lambda z$*, and then we get the condition on* Q

$$Q(\lambda z) = \frac{Q(z)}{\lambda} \Rightarrow Q(z) = \frac{R}{z}$$

for some constant R*. The metric*

$$ds^2 = \frac{R^2}{z^2}\left[dz^2 + d\boldsymbol{x}^2 - dt^2\right]$$

is that of AdS_{d+1}*.*
The gauge-gravity correspondence that we are looking for now suggests that a large N *CFT on* $\mathbb{R}^{1,d-1}$ *can be viewed as a string theory on* AdS_{d+1}*, and vice versa.*

13 Motivating AdS/CFT - Lecture 11

Consider we have a system X, which decouples at low energy into descriptions A and B, i.e. their partition functions are equal. We will introduce the notation $\mathcal{N}$ for the degrees of freedom of N Relevant sources will be

- Maldacena, 9711200
- Witten, 9802150
- Gubser et al, 9802109
- Aharaony et al, 9905111

13.0.1 Description A

We first describe X as a theory in flat space. We have action

$$S_{total} = S_{grav} + S_{matter} + S_{int} .$$

At low energy $G_N E^{d-2} \to 0$, where d is the dimension of the theory, so we see

- free gravitons propagating in flat space,
- matter dynamics (eventually large N gauge/CFTs)

13.0.2 Description B

This description is GR-like. $\mathcal{N}$ is large enough and energy s sufficiently localised s.t. X will be source BH-like solutions

$$ds^2 = -f(r)dt^2 + \dots$$

where r is the coordinate distance to the source. Clocks and energy depend on time in the sense that

$$\text{prop time } d\tau = \sqrt{f}dt \Rightarrow E_\tau = \frac{E_\infty}{\sqrt{f}} .$$

Assuming a fundamental scale α', then the low energy limit is

$$E_\infty \alpha' \to 0 \Leftrightarrow f E_\tau^2 \alpha' \sim \kappa\rho\alpha' E_\tau^2 \to 0 ,$$

where ρ is the radial coordinate close to the horizon (i.e. $\rho = 0$ corresponds to the horizon). We hence see

- free gravitons as before
- any excitation is in the low energy limit close to the horizon.

One could explore the $\kappa \to 0$ limit here, which corresponds to an extremal BH. Here $f(r) \sim \frac{\rho^2}{R^2}$ close to the horizon, where R is approximately the curvature, assumed constant.
We now want to go from our system (A, d) to $(B, d+1)$. We write down such a metric

$$\begin{aligned} ds^2 &= f(r)\left[-dt^2 + d\boldsymbol{x}^2\right] + \frac{dr^2}{f(r)} + g(r)ds_Y^2 \\ &\to \frac{\rho^2}{R^2}\left[-dt^2 + d\boldsymbol{x}^2\right] + \frac{R^2}{\rho^2}d\rho^2 + g(r_M)ds_Y^2 , \end{aligned}$$

This is $AdS_{d+1} \times Y$ in the near-horizon limit.

13.0.3 Matchup of Parameters

If, in A, the matter system allots the 't Hooft limit, we have a constant g_{YM} in the scaling, s.t. $g_{YM}^2 \sim g_S$. Constrastingly for B, through dimensional analysis, we get the scaling requirement $R^2 \sim \alpha' N^\gamma h(g_S)$ (where $h(g_S)$ is often $\sqrt{g_S}$). Similarly in A, our large N theory had an expansion in N^{-1}. In B we get expansions in $G_{d+1}^{-1} \sim \mathrm{Vol}(Y) G_N^{-1} \sim R^{9-d} g_S^{-2} (\alpha')^{-4}$, and so using algebra we have

$$\frac{G_{d+1}}{R^{d-1}} \sim \frac{g_S^2}{(\sqrt{g_S})^4} \frac{1}{\mathcal{N}^{-4\gamma}} \,.$$

Recall also the 't Hooft limit had the dimensionless parameter $\lambda = g_{YM}^2 N$. We might want that to be scaling as the dimensionless parameter on the other side $\frac{R^2}{\alpha'} \sim \mathcal{N}^\gamma h(g_S)$.

13.0.4 Further General Comments

In some limits, certain CFTs have gravity duals. However, we know that there does not exists a non-perturbative description for QG (right now) and so we will be interested in exploring a duality when gravity is weakly coupled. We will then ask ourselves which features must hold in CFT to have a gravity dual.

13.1 Tackling These Questions

13.1.1 Weakly Coupled Gravity

Recall

$$I_{gravity} \sim \frac{1}{G_N} \int d^d x \, (R - 2\Lambda)$$

where $G_N \sim l_P^{d-2}$. We get effective coupling constant $g_{eff} = \left(\frac{l_P}{L}\right)^{d-2}$ where L is the length scale. Hence the EFT description breaks down when $L \sim l_P$.

Example 13.1. *Take metric*

$$ds^2 = \frac{R_{AdS}^2}{z^2} \left(-dt^2 + d\boldsymbol{x}^2 + dz^2\right)$$

This is scale invariant, and has $L \sim R_{AdS}$. This is clearly AdS, which is with $\Lambda < 0$.
One can also take dS when $\Lambda > 0$ which has metric

$$ds^2 = \frac{l^2}{\eta^2} \left(-d\eta^2 + d\boldsymbol{x}^2\right) = -dt^2 + e^{2t/l} d\boldsymbol{x}^2 \,.$$

This is also scale invariant (under the map $(\eta, \boldsymbol{x}) \mapsto \lambda(\eta, \boldsymbol{x})$), and we have scale $L \sim l \sim H^{-1} \sim \Lambda^{-1/2}$.

On the CFT side, we have universal operator

$$\langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle \sim \mathcal{N} g(x, y) \,.$$

In a $1+1$ CFT, $\mathcal{N}$ is exactly the central charge. Since $T_{\mu\nu}^{CFT}$ measures the response of the theory to a change in the background metric, but in the description B this operator is

$$\frac{\delta}{\delta g_{\alpha\beta}} \frac{\delta}{\delta g_{\mu\nu}} e^{-I_{gravity}} \,,$$

we suspect that the scaling parameter $\mathcal{N}$ matches to

$$\frac{R_{AdS}^{d-2}}{G_N} \sim \frac{1}{g_{eff}^2} \,.$$

This means that the weak coupling limit for gravity would be $\mathcal{N} \gg 1$.

13.1.2 CFTs with Gravity Duals

A CFT is characterised by its spectrum of conformal dimensions, and the corresponding algebra (e.g. $\{\Delta_i, C_{ijk}\}$). This spectrum already appears when we consider OPE operators in higher n-point functions. This data appears in the operator equations

$$\mathcal{O}_i^{\Delta_i}\mathcal{O}_j^{\Delta_j} \sim \sum_k C_{ijk}\mathcal{O}_k^{\Delta_k}\,.$$

We wonder what data can come from a gravity theory. General OPEs generate excitations that a priori have arbitrary spin, so certainly contain gravitons ($s = 2$), as well as other particles. In gravity, the number of particles/excitations that can occur in loops is much smaller than could be generated by arbitrary non-zero C_{ijk} (though one can consider 'higher spin theories' of gravity which include more excitations). One possible out of this is that, to reproduce Einstein-like theories, we need strong interactions as strongly coupled CFTs generically get large anomalous dimension contributions to the Δ_i, so they do not appear in the spectrum (as the Δ_i behave like mass, so if they are large they are not observed in low energy limits).

13.1.3 A Particular Example

A particular example will be with X a $D3$-brane, $\mathcal{N} = N$.
The description A is $N = 4$ sYM, with $U(N)$ corresponding to open strings.
As the branes carry energy, they back-react, and this gives a gravity metric that is description B.
The metric is explicitly given by

$$ds^2 = H^{-1/2}(r)\underbrace{\left[-dt^2 + d\boldsymbol{x}^2\right]}_{\mathbb{R}^4} + H^{1/2}(r)\underbrace{\left[dr^2 + r^2 d\Omega_5^2\right]}_{\mathbb{R}^6}\,,$$

where $H(r) = 1 + \frac{R^4}{r^4}$. The constant R is fixed by the condition

$$R^4 = \frac{4}{\pi^2}G_N T_{D3} N = 4\pi g_S N(\alpha')^2\,.$$

Looking at our previous discussion, we indeed see $h(g_S) = \sqrt{g_S}$, and $\gamma = \frac{1}{2}$.

Remark. *We have a few comments on the previous example.*

- *Looking back at the Schwarzschild solution, we can see that the quantity $G_N T_{D3} N$ plays the role of mass per volume of the source.*
- *For $r \gg R$, $H \sim 1 + \mathcal{O}((R/r)^r)$, and so the $D3$-brane is a point-like object from the perspective of $\mathbb{R}^6$.*
- *When $r \sim R$, the geometry is not flat, though the metric is asymptotically so, and hence the length scale is $L \sim R$.*
- *When $r \ll R$, $H \sim (R/r)^4$, and the metric has limit*

  $$ds^2 \to \frac{r^2}{R^2}\left[-dt^2 + d\tau^2\right] + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2\,,$$

  which is AdS.

- *This metric is a solution to Einstein's equations everywhere except for $r = 0$, where we get a delta contribution.*

To analyse the singularity at $r = 0$, introduce l by

$$\frac{dr^2}{r^2} = dl^2 \Rightarrow l = \log r + l_0 \,.$$

As $r \to 0$, $l \to -\infty$, and this may support why decoupling works. Explicitly description A describes N-many $D3$ branes in $\mathbb{R}^{1,9}$, while description B describes the spacetime as seen by closed strings. We can view a tree-level closed string diagram as a 1-loop open string diagram. At low energy we have $E^2\alpha' \to 0$. For description A (open strings) the two excitations that survive are

1. massless excitation of free gravitons (closed strings), and
2. $N = 4$ sYM with group $U(N)$.

In description B, the energy E_∞ associated with $r \to \infty$ is related by redshift to $E_\tau = H^{1/4}E_\infty$, and hence with $r \gg R$ we get free gravitons, whereas with $r \ll R$ the low energy condition is

$$\frac{r^2 E_\tau^2}{\sqrt{4\pi g_S^N}} \to 0$$

which doesn't give constraints by letting r be as small as desired. If we expect these two descriptions to match up, as the free graviton sectors would match up, we claim that the $N = 4$ sYM with group $U(N)$ is equal to (type IIB) string theory on $AdS_5 \times S^5$.

14 Formulating AdS/CFT - Lecture 12

14.1 Geometry and Isometries of AdS

In the decoupling limit we have metric

$$ds^2 = \frac{r^2}{R^2}\left[-dt^2 + d\boldsymbol{x}^2\right] + \frac{R^2}{r^2}dr^2$$

This has horizon at $r = 0$. We call the portion this metric describes the **Poincaré patch**. Surfaces of constant r are isomorphic to $\mathbb{R}^{1,d}$ in AdS_{d+1}. We note some points

- Time is warped
- curvature $\mathcal{R} = -\frac{d(d+1)}{R^2}$
- solves Einstein's equations with cosmological constant $\Lambda = -\frac{d(d-1)}{2R^2}$.
- It has timelike boundary at $r \to \infty$.

If we introduce the coordinate $z = \frac{R^2}{r^2}$, then

$$ds^2 = \frac{R^2}{z^2}\left[-dt^2 + d\boldsymbol{x}^2 + dz^2\right]$$

Definition 14.1. *AdS_{d+1} is the maximally symmetric spacetime given by the hyperboloid*

$$X_{-1}^2 + X_0^2 + \sum_i X_i^2 = R^2$$

in $\mathbb{R}^{2,d}$. It has isometry group $SO(2,d)$. We get the original coordinates back by $r = X_{-1} + X_d > 0$ and $x^\mu = (t, \boldsymbol{x}) = \frac{R}{r} X^\mu$.

14.1.1 Global AdS

We can describe the entire hyperboloid with coordinates ρ, τ given by

$$X_{-1} = R\sqrt{1+\rho^2} cos\tau, \quad X_0 = R\sqrt{1+\rho^2}\sin tau, \quad \sum_i X_i^2 = \rho^2 R^2\,,$$

wrt which the metric is

$$ds^2 = R^2 \left[-(1+\rho^2)d\tau^2 + \frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_{d-1}^2 \right]$$

Remark. *In this metric τ is periodic. AdS is defined as the universal cover where $\tau \in \mathbb{R}$.*

Performing $\rho = \tan s$ for $s \in [0, \pi/2]$ we get metric

$$ds^2 = \frac{R^2}{\cot^2 s}\left[-d\tau^2 + ds^2 + \sin^2 s d\Omega_{d-1}^2 \right] .$$

Fixed s hypersurfaces are isomorphic to $\mathbb{R} \times S^{d-1}$ (think a cylinder), whereas AdS is the "filled cylinder".

Remark. *There is a conformal boundary at $s \to \pi/2$ which is timelike.*

Remark. *There are null radial geodesics $z = 0 = s$. It takes finite proper time $\tau = \pi/2$ to reach the boundary from the centre. Hence from a kinematics perspective we need reflective boundary conditions at the boundary to prevent leakage of energy.*
Contrastingly, one can check that massive particles never reach the boundary as a warp factor generates a potential preventing escape. Hence they get periodic motion around $\rho = 0$. It is for this reason that AdS is sometimes called a defining box.

One can draw the cyclinder, and see that the Poincaré patch is indeed a patch in this cylinder. We can now try to realise the $SO(2,d)$ symmetry we know we have. This will turn out to have generators on the Poincaré patch

- translations $P^\mu : x^\mu \mapsto x^\mu + \alpha^\mu$ (d-many)
- Lorentz transforms $M^{\mu\nu} : x^\mu \mapsto \Lambda^\mu_\nu x^\nu$ ($d(d-1)/2$-many)
- scaling transform $D : x^\mu \mapsto \lambda x^\mu$, $z \mapsto \lambda z$ (one)

- special conformal transforms K_μ, which are

$$z \mapsto \frac{z}{1 + 2b \cdot x + Ab^2} \,,$$
$$x^\mu \mapsto \frac{x^\mu + Ab^\mu}{1 + 2b \cdot x + Ab^2} \,,$$

  where $A = z^2 + x^2$, and contractions are done with $\eta_{\mu\nu}$. There are d-many.

Counting we find a total of $\frac{1}{2}(d+1)(d+2)$ transforms. Note we could also consider the group $O(2,d)$ and then have generators of inversion $z \mapsto z/A$, $x^\mu \mapsto x^\mu/A$, translation, and then the extra inversion.
Now recall we want to talk about a "string theory on $AdS_{d+1} \times Y$". There is no non-perturbative definition of this theory, but perturbatively we can begin to tackle it. It will have 2 dimensionless parameters g_S and $\frac{\alpha'}{R^2}$. There are two regimes to handle this

1. classical string theory where we let $g_S \to 0$, have $\frac{\alpha'}{R^2}$ fixed, which keeps string excitations,
2. classical 'gravity' theory where $g_S, \frac{\alpha'}{R^2} \to 0$.

We restrict to the case of compact Y^{9-d} (where recall we assume we have a total of 10 dimensions). Introducing coordinates $\{y_j\}$ on Y, general theory will tell us that have an orthonormal basis of harmonics on Y which we will denote as $\{Y_l\}$. This will allow us to write fields $\Phi = \Phi(x^\mu, z, y_j)$ as $\sum_l \phi_l(x^\mu, z) Y_l(y_j)$. We can also then integrate out the Y part of our actions to get

$$\frac{1}{16\pi G_N} \int d^{d+1}x \, d^{9-d}y \sqrt{-g} \, [\mathcal{R}_{10} + \ldots] = \frac{\mathrm{Vol}(Y)}{16\pi G_N} \int d^{d+1}x \sqrt{-g_{d+1}} \, [\mathcal{R}_{d+1} + \ldots] \,.$$

This gives us the relation $G_{d+1} \sim \frac{G_N}{R^{9-d}}$.

Example 14.2. *In the particular case of D3-branes, one finds*

- $16\pi G_N = (2\pi)^7 g_S^2 (\alpha')^4$,
- $R^4 = 4\pi g_S N (\alpha')^2$,
- $G_5 = \frac{G_N}{\pi^3 R^5} \Rightarrow \frac{G_5}{R^3} = \frac{\pi}{2N^2}$.

This will be somewhat generic, that $G_{d+1} \sim \frac{1}{N^2}$.

Remark. *An important diversion to note is that, if our 10d theory came from an 11d reduction, our g_S might correspond to an expectation of a dilatation of the extra dimension, and so when computing perturbtaively one needs to be careful to check that in our expansions we are not pushed into a regime where $g_S \sim 1$.*

14.2 Comments on Matching of Parameters

We describe our dictionary of parameters as

$N = 4\, sYM$	$AdS_5 \times S^5$
g_{YM}^2	$4\pi g_S$
$\mathrm{rk}\, U(N)$	N
$\lambda = g_{YM}^2 N$	$\frac{R^Y}{(\alpha')^2} = 4\pi g_S N$
$\frac{1}{N^2}$ exp	$\frac{G_5}{R^8} = \frac{\pi}{2N^2}$ exp

15 CFT - Lecture 13

Let's start with come remarks about CFTs. We have

- Local operators with definite scaling dimension
$$\hat{\mathcal{O}}(x) \mapsto \lambda^{-\Delta} \hat{\mathcal{O}}(x)\,.$$
- Observables are correlation functions
$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \delta_{\Delta_1\Delta_2}\frac{c}{|x_1-x_2|^{2\Delta_1}}\,,$$
$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \frac{c_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}\,|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}\,|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}\,.$$
- There is an operator state correspondence (isomorphism).
- In a 2d CFT with have the complex coordinate $w = \sigma + i\tau$, and we let $z = e^{-iw} = e^{-i\sigma}e^{\tau}$, i.e. the infinite past is mapped to the origin.

15.0.1 Symmetries

In $N = 4$ sYM, $d = 4$, we have $SO(2,4)$ symmetries as well as an $SO(6)$ R-symmetry. Under our correspondence these are mapped to the isometries of AdS_5 and S^5 respectively.

15.0.2 Corrections

Since we work in units with $\hbar = 1$, we take semiclassical gravity with $G_N \to 0$, $\frac{\alpha'}{\sqrt{R}} \to 0$. Since $\frac{G_5}{R^8} \sim \frac{1}{N^2}$, the limit $G_N \to 0$ corresponds to $N \to \infty$, and $\frac{\alpha'}{\sqrt{R}} \to 0$ corresponds to $g_{YM}^2 N := \lambda \to \infty$. Hence the slogan is that the strongly coupled limit of a large N gauge theory is approximately semiclassical gravity in the bulk.
There are two types of correction to this this, either letting $G_N \to \frac{1}{N^2}$ or having expansion in $\frac{1}{\sqrt{\lambda}}$ (the massive string correction).

15.1 UV/IR connection

We will look at the work of Susskind, Witten, 9805119.
We ask a question, what is the meaning of the extra dimension in the bulk relative to the boundary? Recall the low energy limit meant $r \to 0$, so we were approaching the source of the D3-brane. As such we can think of r as giving us the length scale of an open string, equivalently the energy scale in the CFT. Now the coordinates $x^\mu = (t, x^i)$ in YM on ∂AdS give metric $f(r)(-dt^2 + d\boldsymbol{x}^2) + g(r)dr$ ($f(r) = \frac{R^2}{z^2}$). Local proper time is given by $d\tau = \frac{R}{z}dt$, and length is given by $dl = \frac{R}{z}dx$, so we get warp factors $E_{YM} = \frac{R}{z}E_{\text{local}}$ and analogously distance$_{YM} := D_{YM} = \frac{z}{R}dl$.
If we consider the same bulk process at different z, it corresponds to different YM scales, with $E \sim \frac{1}{z}$, $D \sim z$. Hence as $z \to 0$, $E_{YM} \to \infty$, $D_{YM} \to \infty$, which corresponds to the UV regime of YM. Conversely, the IR regime of YM corresponds to $z \to \infty$.
Taking the AdS perspective on the above, $z \to 0$ corresponds to the boundary of AdS, i.e. spacelike infinity, so the length scale $\to \infty$, and we associate this with the IR regime. Again, conversely the $z \to \infty$ limit corresponds to stretched open strings that carry lots of energy, hence the UV limit.

15.1.1 Matching Boundary/Bulk Cutoffs

Consider a CFT observable

$$\langle \mathcal{O}(X_1) mcO(X_2)\rangle \sim \frac{\mu^{-P}}{|X_1 - X_2|^P}.$$

where μ is the CFT cutoff. Let's assume the points X_i (nearly) on the boundary are connected by an open string, then as $z \to 0$ the length of this string tends to infinity. At infinite distance in the bulk, we can replace this cutoff with δ the AdS cutoff, and then if we assume we have massive fields corresponding to this correlator in the bulk we can use the WKB approximation to take a geodesic approximation for the length of this string (see next week). Hence

$$l_{\text{geodesic}} = \frac{\log|X_1 - X_2|}{\delta} \Rightarrow \langle \Phi_1 \Phi_2 \rangle \sim e^{-m_\Phi l} \sim \frac{\delta^{m_\Phi}}{|X_1 - X_2|^{m_\Phi}}.$$

15.1.2 Lattice cutoff in YM

Each independent quantum field in $N = 4$ sYM gives a degree of freedom for a lattice, where each d.o.f carries a bit of information. Energy density of fields then goes as $\sim \frac{1}{\delta^4}$, hence the maximal temperature goes as $\sim \delta^{-4}$.
Suppose we take sYM on $\mathbb{R} \times S^3$. The number of cells on S^3 goes as $\sim \delta^{-3}$, and we have N^2 matrix elements as this is a $U(N)$ theory, so $N_{dof} \sim \frac{N^2}{\delta^3}$. We can calculate the area of the cutoff surface to be $A_\delta \sim \frac{R^3}{\delta^3}$, and moreover recall on AdS we have $G_N \sim g_S^2 (\alpha')^4$, so one can alternatively think of $N_{dof} \sim \frac{A_\delta}{G_5}$.
At finite T_{YM}, taking dimensionless T defined by $T_{YM} = TR$, we have

$$S_{YM} \sim N^2 T_{YM}^3 \Rightarrow (TR)^3 \sim \frac{S_{YM}}{N^2} \sim S_{YM} \frac{g_S^2 (\alpha')^4}{R^8} \sim \frac{A}{R^3}.$$

Using the holographic principle we get that $S_{bulk} \sim \frac{A}{G}$, and so $A_{max} \sim \frac{R^3}{\delta^3}$.

15.2 Operators vs Bulk Field Correspondence

Given the duality, we expect the spectrum to be equal on both side. In CFT, we have a representation of $SO(2, d)$, so there are local primary operators, and we wonder what do they correspond to in the bulk. The intuition is that they should correspond to bulk fields which transform under $SO(2, d)$ in the same way.
To approach this, let's deform the CFT by adding $\int dx\, \phi_0(x) \hat{\mathcal{O}}(x)$ to $S_{CFT} = \int ds \mathcal{L}_{CFT}$. Note if ϕ_0 was constant, then this would modify/generate the coupling of $\hat{O}$ in the Lagrangian. What does this deformation mean in the bulk? To gain some intuition on this question, recall we have $g_{YM}^2 \leftrightarrow g_S \sim e^{\langle \phi \rangle}$, and that g_{YM} appears explicitly as the Lagrangian density is $\sim \frac{1}{g_{YM}^2} \operatorname{Tr}[\ldots]$. $\langle \phi \rangle$ is the expectation of the dilaton, and typically the constant piece of $\langle \phi \rangle$ is the value fo the dilaton at the boundary. Hence let's write $\langle \phi \rangle = \phi|_{\partial AdS}$. If we expect this value to correspond to ϕ_0, we see that the dilaton would correspond to the Lagrangian density as a field.

Remark. *The takeaway from this argument is that turning on a boundary deformation $\int dx\, \phi_0(x) \hat{\mathcal{O}}(x)$ is equivalent to associating a bulk field $\phi(x)$ to the boundary operator $\hat{O}$ s.t the boundary value $\phi|_{\partial AdS}$ corresponds to ϕ_0.*

Example 15.1. *Take a $U(1)$ current CFT, $J^\mu(x) \to \int dx\, a_\mu(x) J^\mu(x)$, o $a_\mu(x) := A_\mu(x, z)|_{z=0}$.*

References

[1] Richard Arnowitt, Stanley Deser, Charles W. Misner. Republication of: The dynamics of general relativity. *General Relativity and Gravitation*, 40(9):p. 1997–2027, 2008. ISSN 1572-9532. doi: 10.1007/s10714-008-0661-1.