Linearising Flows and a Cohomological Interpretation of Lax Equations - Unpacking the Paper

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Contents

1 Introduction

The purpose of this document is to facilitate the understanding of [1] by discussing the terms and how they fit into the wider picture of algebraic geometry.

2 The Preliminaries

2.1 Divisors

Definition 2.1. *A **(Weil) divisor** on C is a formal finite sum of points, i.e. $D = \sum_i n_i p_i$ for $n_i \in \mathbb{Z}$, $p_i \in C$. The group of divisors under addition is denoted* $\mathrm{Div}(C)$*.*

Definition 2.2. *The **degree** of a divisor $D = \sum_i n_i p_i$ $\deg D = \sum_i n_i$*

Definition 2.3. *Given a meromorphic function $f : C \to \mathbb{C}$ we define $(f) \in \mathrm{Div}(C)$ by*

$$(f) = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p$$

*For $D \in \mathrm{Div}(C)$, if $\exists f$ s.t. $D = (f)$ we say D is a **principal divisor**.*

Lemma 2.4. $(fg) = (f) + (g)$

Corollary 2.5. *Principal divisors form a subgroup* $\mathrm{Lin}(C) \leq \mathrm{Div}(C)$*.*

Lemma 2.6. *If X is a compact Riemann surface and $f : X \to \mathbb{C}$ meromorphic then* $\deg(f) = 0$.

Proposition 2.7. *Let C be compact. Then* $\mathrm{Lin}(C) \subset \{D \in \mathrm{Div}(C) \mid \deg(D) = 0\}$.

Definition 2.8. *The **divisor class group** of C is* $\mathrm{Cl}(C) = {}^{\mathrm{Div}(C)}\!/\!_{\mathrm{Lin}(C)}$. *The equivalence class corresponding to D is often denoted as $|D|$ and is called the **complete linear system** associated with D.*

Remark. $\deg : \mathrm{Div}(C) \to \mathbb{Z}$ *is a group homomorphism and as the kernel is* $\mathrm{Lin}(C)$ *we see* $\mathrm{Cl}(C) \cong \mathrm{Im}\deg$

Corollary 2.9. $\mathrm{Cl}(\mathbb{CP}^n) \cong \mathbb{Z}$.

Definition 2.10. *Two divisors D, E are **linearly equivalent** if $D - E$ is a principal.*

Lemma 2.11. *Linear equivalence of divisors is an equivalence relation.*

Lemma 2.12. *$f : X \to Y$ induces a group morphism $f : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ by*

$$f\left(\sum_i n_i p_i\right) = \sum_i n_i f(p_i)$$

Proposition 2.13. *If $f : X \to Y$ is a map of Riemann surfaces and $D \in \mathrm{Div}(X)$, then* $\deg(f(D)) = \deg f \cdot \deg D$.

Definition 2.14. *A divisor $D = \sum_i n_i p_i$ is **effective** if each $n_i \geq 0$.*

Proposition 2.15. *We have a partial ordering on $\mathrm{Div}(C)$ by saying $D \geq D'$ if $D - D'$ is effective.*

Definition 2.16. *A Weil divisor on C defines a coherent sheaf $O_C(D)$ as meromorphic functions f s.t $(f) + D \geq 0$.*

2.2 Abel-Jacobi

Suppose C has genus g, then we know that $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ where the generators are the loops $\{\gamma_i\}_{i=1}^{2g}$. There is an alternative way to say this condition:

Definition 2.17. *The **canonical bundle** on a space X with $\dim X = n$ is the line bundle of exterior n-forms on X. It is often denoted K, not to be confused with the canonical divisor.*

Remark. *Note we know the canonical bundle is a line bundle as there is only 1 basis element of n-forms on an n-dimensional space.*

Proposition 2.18. *If $X = C$ is a Riemann surface of genus g then $H^0(C, K) \cong \mathbb{C}^g$.*

Proof. See Farkas & Kra, III.2.7. $\square$

Corollary 2.19. *We can take a basis $\{\omega_i\}_{i=1}^g$ of 1-forms on C.*

Definition 2.20. *The* ***Jacobian*** *of C is defined to be*

$$J(C) = {}^{\mathbb{C}^g}\!\big/_{\Lambda}$$

where Λ is the lattice generated over $\mathbb{R}$ by the vectors

$$\Omega_j = \left(\int_{\gamma_j} \omega_1, \dots, \int_{\gamma_j} \omega_g \right), \quad 1 \le j \le 2g$$

Definition 2.21. *The* ***Abel-Jacobi map*** *for $p_0 \in C$ is*

$$\begin{aligned} u : C &\to J(C) \\ p &\mapsto \left(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_g \right) \mod \Lambda \end{aligned}$$

This is independent of the path of integration as we have quotiented by Λ.

Theorem 2.22 (Abel's Theorem)**.** *Let u be the Abel-Jacobi map and D, E effective divisors. Then $u(D) = u(E) \Leftrightarrow D \sim E$.*

Theorem 2.23 (Jacobi's Theorem)**.** *The map Abel-Jacobi map is surjective.*

Corollary 2.24. *There is an isomorphism from the space of degree-0 divisors to the Jacobian.*

2.3 Bundles and Sheaves

We recall a few necessary bundle definitions and results:

Definition 2.25. *The tensor product of vector bundles $E, F \to M$ is $E \otimes F \to M$ s.t. $(E \otimes F)_m = E_m \otimes F_m$ for $m \in M$.*

Lemma 2.26. *If O is the trivial line bundle then $E \otimes O = E$.*

Definition 2.27. *The* ***dual bundle*** *of a vector bundle $E \to M$ is $E^* \to M$ where the fibres of E^* are the dual spaces of the fibres of E, with the transition functions $g_{ij}^* = \left(g_{ij}^T\right)^{-1}$.*

Remark. *We can check the cocycle condition here as*

$$g_{kj}^* g_{ji}^* = \left(g_{kj}^T\right)^{-1} \left(g_{ji}^T\right)^{-1} = \left(g_{ji}^T g_{kj}^T\right)^{-1} = \left([g_{kj} g_{ji}]^T\right)^{-1} = \left(g_{ki}^T\right)^{-1} = g_{ki}^*$$

Example 2.28. *The dual bundle to the tangent bundle is the cotangent bundle, i.e. $(TM)^* = T^*M$*

Lemma 2.29. *$E \otimes E^* \cong \mathrm{End}(E)$.*

Lemma 2.30. *Line bundles have tensor inverses, i.e given L, $\exists L^{-1}$ s.t. $L \otimes L^{-1} \cong O$ the trivial bundle.*

Proof. We will show this by showing $L^{-1} = L^*$. To trivialise $\mathrm{End}(L)$ we note here the transition maps are $g_{ij} \otimes g_{ij}^{-1} = 1 \otimes 1$ as $g_{ij}, g_{ij}^* \in \mathbb{F}$. Hence any section is globally defined. $\square$

Remark. *Why is the identity section not global on any other vector bundle.*

These results motivate the definition of the **Picard group** which we will cover now:

Definition 2.31. *A* ***ringed space*** *is a pair* (X, O_X) *where* X *is a topological space and* O_X *is a sheaf of rings on* X*.* O_X *is called the* ***structure sheaf****.*

Example 2.32. *Given a topological space* X*, if we take* O_X *to be* $\mathbb{R}$*-valued continuous functions on open subsets of* X *then* (X, O_X) *is a ringed space.*

Example 2.33. *An example that will be relevant for later discussions is that an affine variety* X *with sheaf* O_X *given by* $O_X(U)$ *being the regular functions on* U*, regular functions being those given locally by polynomials.*

Definition 2.34. *The* ***Picard group*** *of a locally ringed space* X *is* $\mathrm{Pic}(X)$ *the group of isomorphism classes of line bundles on* X *with the group operation being* $\otimes$*.*

Remark. *In place of line bundles we can actually say* ***invertible sheaves***

Theorem 2.35. $\mathrm{Cl}(C) \cong \mathrm{Pic}(C)$ *naturally.*

Proof. See Vakil's notes. Alternatively this is covered in more generality in "The Rising Sea" (§14.2). □

Corollary 2.36. *We get a group homomorphism* $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ *giving the degree of the corresponding divisor in* $\mathrm{Cl}(C)$*.*

Corollary 2.37. $\mathrm{Pic}(\mathbb{CP}^1) \cong \mathbb{Z}$*.*

Notation. *We denote the isomorphism class of line bundles degree* d *as* $\mathrm{Pic}^d(C)$

Remark. *With this new notation we may rephrase the corollary of the Abel-Jacobi theorem to say* $J(C) \cong \mathrm{Pic}^0(C)$*.*

Proposition 2.38. *There is a canonical isomorphism* $\mathrm{Pic}(X) \cong H^1(X, O_X^\times)$*.*

Corollary 2.39. $T_L(\mathrm{Pic}^d(X)) \cong H^1(X, O_X)$

Proof. You need to use the exponential sheaf sequence. □

Let us now consider a specific class of bundles:

Definition 2.40. *The* ***hyperplane bundle on*** $\mathbb{CP}^n$ *is the bundle* $\mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$ *given by the standard projection* $(z_0, \dots, z_n) \to [z_0 : \cdots : z_n]$*. It is often denoted* $\mathcal{O}(1)$*. We denote* $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$*.*

Definition 2.41. *The* ***tautological line bundle*** *on projective space is* $\mathcal{O}(-1) = \mathcal{O}(1)^*$*. We denote* $\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n}$*.*

Proposition 2.42. *The canonical bundle on the projective space is* $K = \mathcal{O}(-n-1)$*.*

Proposition 2.43. $\mathrm{Pic}(\mathbb{CP}^n)$ *is generated by* $\mathcal{O}(\pm 1)$*.*

We make a few more useful definitions.

Definition 2.44. *Let* X *be an algebraic surface and* $\pi : L \to X$ *a line bundle. Then the* ***tautological section*** *of* $\pi^* L$ *as a bundle over* L *is given by* $\sigma(l) = (l, l)$*.*

Remark. *Not that the tautological section is indeed valid as we have*

$$\pi^* L = \{(l, l') \in L \times L \mid \pi(l) = \pi(l')\}$$

so certainly $(l, l) \in \pi^* L$*.*

2.4 Lax Pairs and Spectral Curves

Notation. *We start by laying out some notation that will be necessary for the following section. Let:*

- $P = \mathbb{CP}^1$ *with coordinates* $[\xi_0 : \xi_1]$*. We take* $\xi = \frac{\xi_1}{\xi_0}$*.*
- O_P *be the natural structure sheaf on the variety* P
- V *be a* m*-dimensional vector space,* $\mathcal{V} = V \otimes O_P$*,* $\mathcal{V}(k) = V \otimes \mathcal{O}_P(k)$ *where we view* V *as either the constant sheaf or trivial bundle over* P*.*
- $A(t, \xi) = \sum_{k=0}^{n} A_k(t)\xi^k \in H^0(P, \mathrm{Hom}(\mathcal{V}, \mathcal{V}(n)))$ *for some* n*, where we see* $A_i(t) \in \mathrm{End}(V)$ *as a time dependent* $m \times m$ *matrix and* $\xi^k \in H^0(P, \mathcal{O}(n))$ *as*

  $$[\xi_0 : \xi_1]^k = \underbrace{\xi_0 \otimes \cdots \otimes \xi_0}_{\times(n-k)} \otimes \underbrace{\xi_1 \otimes \cdots \otimes \xi_1}_{\times k}$$

  This is homogeneous of degree n*, so we allow* A *to not have a scale?*
- $B(\xi, t) \in H^0(P, \mathrm{Hom}(\mathcal{V}, \mathcal{V}(N)))$ *for some* N *likewise .*
- $Q(\xi, \eta) = \det[\eta I - A(\xi, t)]$ *be the characteristic polynomial of* A*.*
- σ *be the tautological section of* $\mathcal{O}_P(n)$*.*

Lemma 2.45. $Q(\xi, \sigma) \in H^0(\mathcal{O}_P(n), \pi^*\mathcal{O}_P(mn))$

Definition 2.46. *The pair* A, B *is a Lax pair if* $\dot{A} = [A, B]$*.*

Proposition 2.47. *The Lax equation is invariant under the substitution*

$$B \mapsto B + p(A, \xi)$$

for polynomial $p(x, \xi) \in \mathbb{C}[x, \xi]$*.*

Definition 2.48. *The* ***spectral curve*** *is* C *given by the solution in* P *of*

$$Q(\xi, \eta) = 0$$

Proposition 2.49. *The flow* $t \mapsto A(\xi, t)$ *is isospectral.*

It will be the understanding of this isospectral flow that we want to gain. We formulate this flow as the family of holomorphic map gained by the eigenvectors

$$f_t : C \to \mathbb{CP}^{m-1}$$

Suppose that C has degree d, then we know we can define

$$L_t = f_t^*(\mathcal{O}(1)) \in \mathrm{Pic}^d(C)$$

Lets choose a reference bundle $L_0 \in \mathrm{Pic}^d(X)$

Lemma 2.50. *The map*

$$\begin{aligned} \mathrm{Pic}^d(C) &\to J(C) \\ L &\mapsto L \otimes L_0^{-1} \end{aligned}$$

is an isomorphism.

Now knowing our result about the tangent space to the Picard group we can say $\frac{dL_t}{dt} \in H^1(C, O_C)$.

References

[1] Phillip A. Griffiths. Linearizing flows and a cohomological interpretation of lax equations. *American Journal of Mathematics*, 107(6):pp. 1445–1484, 1985. ISSN 00029327, 10806377. doi: 10.2307/2374412.