Monopoles

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Contents

1 Introduction

1.1 Preamble

I already have notes on Gauge Theory, Algebraic Geometry, Solitons, and Algebraic Topology, but I have yet to actually make any notes on Monopoles. The purpose of these notes is to be a comprehensive cover of the knowledge required to understand [6]. This will include previous works by Atiyah, Donaldson, Hitchin, Nahm, and more.

2 Preliminaries

As with all my projects, the preliminaries will undoubtedly end up being too long, but I will try keep this minimal this time:

Definition 2.1. *The* ***annihilator*** *of $U \leq V$ is*

$$U^0 = \{f \in V^* \,|\, \forall u \in U,\ f(u) = 0\} \leq V^*$$

If V has bilinear $\langle \cdot, \cdot \rangle$ we can use the isomorphism of $V^* \cong V$ to understand

$$U^0 = \{v \in V \,|\, \forall u \in U,\ \langle u, v\rangle = 0\} \leq V$$

Lemma 2.2. *THe annihilator is a subspace,* $\dim U^0 = \dim V - \dim U$.

Definition 2.3. *A subspace U is called* ***isotropic*** *if $U \subset U^0$.*

2.1 The Dirac Monopole

The standard maxwell equations prohibit monopoles, by which we mean point magnetic field sources, as $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{0}$. Dirac showed in [7] that it is possible to escape this conclusion by giving non-trivial topology to the space by allowing $\boldsymbol{B} = \frac{g}{4\pi r^2}\hat{\boldsymbol{x}}$ to have a singularity at $\boldsymbol{x} = \boldsymbol{0}$. We can calculate $\boldsymbol{\nabla} \cdot \boldsymbol{B} = g\delta(\boldsymbol{x})$. Removing this circle gives $\mathbb{R}^3 \setminus 0$, homotopic to S^2, and the corresponding magnetic two form on this sphere is

$$f = \frac{g}{4\pi} \sin\theta d\theta \wedge d\phi$$

and so the flux through a 2-sphere enclosing the origin is $\int_{S^2_R} f = g$. For $g \neq 0$, we know $f \neq da$ for a global $a \in \Omega^1(S^2)$ by Stokes' theorem, but if we take a cover of the sphere $U_{N/S}$ (north/south) and define gauge potentials

$$a_N = \frac{g}{4\pi}(1 - \cos\theta)d\phi \in \Omega^1(U_N)$$
$$a_S = \frac{g}{4\pi}(-1 - \cos\theta)d\phi \in \Omega^1(U_S)$$

On the intersect $U_N \cap U_S$ we have $da_N = f = da_S$ and $a_N = a_S + \frac{g}{2\pi}d\phi$.
Now taking $A = ia$, $F = if$, we have that $g_{NS}(\theta, \phi) = e^{-i\frac{g\phi}{2\pi}}$. Requiring that this is a well-defined

transition function gives $g \in \mathbb{Z}$. This is equivalent to the integrality of the Chern number.
We will not want to consider this as this solution is not solitonic (it has infinite mass), but for further discussion see [13].

2.2 Pauli Matrices

Definition 2.4 (Pauli Matrices)**.** *The* ***Pauli matrices*** *are*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note they are all Hermitian and traceless.

Fact 2.5. $\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k \Rightarrow \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$

2.3 $SU(2)$

We can write

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

This can be expressed as, for $A \in SU(2)$

$$A = a_0 I + i\boldsymbol{a} \cdot \boldsymbol{\sigma}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, and $a_0^2 + |\boldsymbol{a}|^2 = 1$. Hence $SU(2) \cong S^3$. In addition, by parametrising $SU(2)$ by the a_i, it can be seen that $\{ i\sigma_i \}$ forms a basis of $\mathfrak{su}(2)$. It is typical to normalise this basis to $\{ T^a = -\frac{1}{2} i\sigma_a \}$.

Lemma 2.6. *The structure constants in this basis* $\{ T^a \}$ *are* $f^{ab}_c = \epsilon_{abc}$.

Corollary 2.7. *The Killing form is given by* $\kappa\left(T^a, T^b\right) = \kappa^{ab} = -2\delta^{ab} = 4\,\mathrm{Tr}(T^a T^b)$. *Hence* $\kappa(X, Y) = 4\,\mathrm{Tr}(XY)$

2.4 Degree of a Map

We will want to consider continuous maps $f : X \to Y$ between connected oriented n-dimensional manifold. We state the following lemma:

Lemma 2.8. *X orientable iff $H_n(X) = \mathbb{Z}$.*

Proof. See [11]. □

Denote $[X], [Y]$ to be the generators of $H_n(X), H_n(Y)$ respectively. Recalling that we get an induced homomorphism $f_* : H_n(X) \to H_n(Y)$ we make the following definition:

Definition 2.9. *The* ***degree*** *of f is defined s.t.*

$$f_*([X]) = \deg(f)[Y]$$

Now recall we have a pairing between k-forms and k-chains on a manifold given by

$$\langle c, \omega \rangle = \int_c \omega$$

and this descends to a pairing between cohomology and homology. With this we have

$$\langle f_*[c], \omega \rangle = \langle [c], f^*[\omega] \rangle$$

Meaning that we can also express the degree via

$$\deg(f) \int_Y \omega = \int_X f^*\omega$$

for some top form on Y. This reformulation gives a convenient way to interpret the degree:

Proposition 2.10. *Take $y \in Y$ s.t. $f^{-1}(y) \subset X$ is a set of isolated points, say $\{x_i\}_{i=1}^m$. Then*

$$\deg f = \sum_{i=1}^{m} \operatorname{sgn}(J(x_i))$$

Proof. See [13], the idea is to choose a volume form on Y localised around y, which pulls back to a volume form on X localised around the x_i, with the ± 1 coming from the change of orientation. □

2.5 Spinors

2.5.1 Spinor Bundles

Definition 2.11. *The **complexification** of a vector space V is the vector space $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$.*

Proposition 2.12. *$V_{\mathbb{C}}$ comes with a well defined map of conjugation $v \otimes z \to v \otimes \overline{z}$. This is a $\mathbb{C}$-linear isomorphism onto $\overline{V_{\mathbb{C}}}$.*

Lemma 2.13. $(V^*)_{\mathbb{C}} = (V_{\mathbb{C}})^*$.

We will now state a few results about $SL_2(\mathbb{C})$ bundles:

Lemma 2.14. *A rank-2 complex vector bundle $S \to M$ has a reduction of structure group to a $SL_2(\mathbb{C})$-bundle iff $\exists \varepsilon \in \Gamma(M, \wedge^2 S)$ a non-degenerate form.*

Remark. *Such an ϵ will be a symplectic form on S.*

Corollary 2.15. *The cotangent bundle on a 2-dimensional manifold M reduces to a $SL_2(\mathbb{C})$ bundle iff the canonical bundle is trivial.*

Definition 2.16. *Suppose M is a 4-dimensional manifold s.t $\exists S \to M$ a $SL_2(\mathbb{C})$-bundle s.t.*

$$TM_{\mathbb{C}} \cong S \otimes \overline{S}$$

*Then we call S a **spinor bundle** on M. A section $s \in \Gamma(S)$ is called a **spinor**.*

Remark. *This is a quite specific definition, so to see this done in more generality check out [14].*

Notation. *Given a spinor bundle $S \to M$ we will use the notation*

$$S^A = S^- = S \qquad S^{A'} = S^+ = \overline{S} \qquad S_A = S_- = S^* \qquad S_{A'} = S_+ = \overline{S}^*$$

Each of the above comes with an associated non-degenerate two form $\epsilon^\pm \in \Gamma(M; \wedge^2 S^\pm)$, $\epsilon_\pm \in \Gamma(M; \wedge^2 S_\pm)$, and these provide isomorphisms to the dual as follows:

$$\epsilon^{A'B'} = \epsilon^+ : S_+ \to S^+ \quad \epsilon^{AB} = \epsilon^- : S_- \to S^- \quad \epsilon_{A'B'} = \epsilon_+ : S^+ \to S_+ \quad \epsilon_{AB} = \epsilon_- : S_- \to S_-$$

If we take an open $U \subset M$ and trivialise s.t $S|_U \cong \mathbb{C}^2$, we have the standard action of $SL_2(\mathbb{C})$ given by $v \mapsto gv$ for $g \in SL_2(\mathbb{C})$. We think of this as saying that a spinor in S transforms in the fundamental representation of $SL_2(\mathbb{C})$. We can then build the following table:

Bundle	Representation	Actions
S^-	Fundamental	$v \mapsto gv$
S_-	Dual	$v \mapsto v^T g^T$
S^+	Congujate	$v \mapsto \overline{g}v$
S_+	Conjugate Dual	$v \mapsto v^T \overline{g}^T$

Definition 2.17. *A* ***spinor frame*** *is a choice of frame of S^- wrt which*

$$\epsilon^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

2.5.2 Spinors on Minkowski Space

One important example of Spinors will be those on Minkowski space, which we will now denote as M^4, so we shall cover it in slightly more detail.

Remark. *Here, as in the twistor section, we shall follow Ward & Wells [17]. They remark that we should really consider Minkwoski space as an affine space in order to not distinguish the origin, and cite [16] for a further discussion. The result of this is that if we choose an origin $0 \in M^4$ we have $M^4 \cong T_0 M^4$.*
Note also, [17] uses the mostly-minus Minkowski metric, so initially this section should be assumed to be using this convention unless stated otherwise.

We first introduce a useful coordinate system via the following map to $H_2 = \{2 \times 2 \text{ Hermitian matrices}\}$:

$$\begin{aligned} M^4 &\xrightarrow{\cong} H_2 \\ x = (x^\mu) \mapsto \tilde{x} &= \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \end{aligned}$$

Note the correspondence can be written as $\tilde{x} = x^\mu \sigma_\mu / \sqrt{2}$ where $\sigma_\mu = (I, \boldsymbol{\sigma})$ corresponds to the Pauli matrices.

Remark. *Braden will use the definition that $x = x_0 - i\boldsymbol{x} \cdot \boldsymbol{\sigma}$. This may seem somewhat strange here, but will end up being a sensible definition when we want to let the $x_i = T_i$ be matrices satisfying Nahm's equations in the charge-1 case.*

Proposition 2.18. $\det \tilde{x} = \frac{1}{2} |x|^2$.

Proof. It is simple to calculate

$$\det \tilde{x} = \frac{1}{2}\left[(x^0 + x^3)(x^0 - x^3) - (x^1 - ix^2)(x^1 + ix^2)\right] = \frac{1}{2}\left[(x^0)^2 - (x^3)^2 - (x^1)^2 - (x^2)^2\right]$$

□

Note that under the same isomorphism, when we complexify, we get that $M^4_{\mathbb{C}} \cong M_2(\mathbb{C})$. This is as every complex can be decomposed into a Hermitian and anti-Hermitian part.
Now let $S = \mathbb{C}^2$, where we consider an element $s \in S$ as a column vector. Then $S^* = \mathbb{C}^2$ with elements $\tilde{s}$ considered as row vectors.

Proposition 2.19. *S is a spinor bundle on M^4.*

Proof.

$$T_0M^4 \otimes \mathbb{C} \cong M^4 \otimes \mathbb{C} \cong M_2(\mathbb{C}) \cong S \otimes S^*$$

We are then done as $\overline{S} = S^*$ here. □

Under this correspondence, we get the following result

Lemma 2.20. *$v \in T_0M^4_{\mathbb{C}} \otimes \mathbb{C}^4$ is null iff $\exists s \in S, \tilde{s} \in S^*$ s.t. $v = s\tilde{s}$. Moreover v corresponds to a real vector iff $\tilde{s} = rs$ for $r \in \mathbb{R}$.*

Remark. *By the above a vector $v \in T_0M^4_{\mathbb{C}}$ is real iff $\exists s \in S$ s.t. $v = ss^\dagger$. In this sense we can think of spinors as the square root of real null vectors.*

2.5.3 Spinor Fields

We want to extend the above work looking at spinors on Minkowski space to write explicitly the isomorphism claimed in Definition 2.16

Definition 2.21. *We define the mixed spinor tensor ${\sigma_a}^{AA'}$ by the conditions*

- ${\overline{\sigma}_a}^{AA'} = {\sigma_a}^{AA'}$
- ${\sigma_a}^{AA'}{\sigma^b}_{AA'} = \delta^b_a$
- ${\sigma_a}^{AA'}{\sigma^a}_{BB'} = {\epsilon_B}^A{\epsilon_{B'}}^{A'}$.
- ${\sigma_{[a}}^{AA'}{\sigma_{b]A}}^{B'} = -\frac{1}{2}i\epsilon_{abcd}\sigma^{cAA'}{\sigma^d}_A{}^{B'}$

Given a manifold M with spin bundle S this gives the isomorphism $TM = S \otimes \overline{S}$ as

$$v^{AA'} = {\sigma_a}^{AA'}v^a, \quad v^a = {\sigma^a}_{AA'}v^{AA'}$$

Lemma 2.22. *Given a metric η_{ab} on M the spinor equivalent is*

$$\eta_{ab}{\sigma^a}_{AA'}{\sigma^b}_{BB'} = \epsilon_{AB}\epsilon_{A'B'}$$

Remark. *We will often omit the σ to simply write $v^a = v^{AA'}$.*

Lemma 2.23. *If ξ_{AB} is a skew-spinor then $\xi_{AB} = \epsilon_{AB}\xi_C{}^C$.*

Example 2.24. *If F_{ab} is a Maxwell tensor (i.e. will eventually be a curvature tensor in a gauge theory) then*

$$F_{AA'BB'} = \phi_{AB}\epsilon_{A'B'} + \phi_{A'B'}\epsilon_{AB}$$

where

$$\phi_{AB} = \frac{1}{2}F_{AC'B}{}^{C'}, \quad \phi_{A'B'} = \frac{1}{2}F_{CB'}{}^{C}{}_{A'}$$

are symmetric. We also have the following result:

Lemma 2.25. *$\phi_{A'B'}\epsilon_{AB}$, $\phi_{AB}\epsilon_{A'B'}$ are the self dual and anti self dual parts of F respectively.*

Now recall a vector $v \in TM$ is null if $v_a v^a = 0$. We can translate this result into a comment on spinors:

Proposition 2.26. *$v^{AA'}$ is null iff $v^{AA'} = \lambda^A \xi^{A'}$ for spinors λ, ξ.*

2.6 Twistors

2.6.1 Grassmannians and Flag Manifolds

Definition 2.27. *Let V be an n-dimensional vector space over $\mathbb{F}$. Define the* ***Grassmannian manifold of k-dimensional subspaces*** *as $G_k(V)$.*

Lemma 2.28. *If $\mathbb{F} = \mathbb{R}$, $G_k(V) \cong {}^{O(n)}\!\diagup_{[O(k) \times O(n-k)]}$*

Proof. We can view the Grassmannian as the quotient of $GL(V)$ by the stabiliser of a k-dim subspace. □

Remark. *Likewise if $\mathbb{F} = \mathbb{C}$, $G_k(V) \cong {}^{U(n)}\!\diagup_{[U(k) \times U(n-k)]}$,*

Corollary 2.29. $\dim G_k(V) = k(n-k)$.

Example 2.30. $G_1(\mathbb{C}) \cong \mathbb{CP}^1$.

Notation. *If we pick a basis of complex vector space V to make $V \cong \mathbb{C}^n$, we identify $G_k(V) \cong G_k(\mathbb{C}^n)$ which we will notation $G_{k,n}$.*

If we denote $M^*_{n\times k}(\mathbb{C})$ to be the full rank $m \times k$ matrices we can construct G via the projection

$$\begin{aligned} M^*_{n\times k}(\mathbb{C}) &\to G_{k,n} \\ M &\mapsto \text{Span}\,\{\text{columns of } M\} \end{aligned}$$

$M^*_{n\times k}(\mathbb{C})$ carries a transitive left action via multiplication that doesn't affect span, and so we get a bundle structure. We can then consider coordinates on $G_{k,n}$ as $Z \in M_{(n-k)\times k}(\mathbb{C}) \cong \mathbb{C}^{k(n-k)}$ via the map

$$\begin{aligned} \phi : M_{(n-k)\times k}(\mathbb{C}) &\to G_{k,n} \\ Z &\mapsto \left[\begin{pmatrix} Z \\ I_k \end{pmatrix}\right] \end{aligned}$$

It will now be necessary to generalise this concept slightly:

Definition 2.31. *Given a fixed complex n-dimensional vector space V and $\{d_i\}_{i=1}^m$ integers satisfying $1 \le d_1 < d_2 < \cdots < d_m < n$ we define a* ***flag manifold of type*** $(d_1, \ldots, d_m)$ *as*

$$F_{d_1 \ldots d_m}(V) = \{(S_1, \ldots, S_m) \,|\, S_i \text{ a } d_i\text{-dimensional subspace of } V,\ S_1 \subset S_2 \subset \cdots \subset S_m\}$$

Lemma 2.32. *The dimension of a flaf manifold of type $(d_1, \ldots, d_m)$ is*

$$d_1(n - d_1) + (d_2 - d_1)(n - d_2) + \cdots + (d_m = d_{m-1})(n - d_m)$$

Definition 2.33. *If V is a 4-dimensional vector space we define the* ***Plücker embedding*** *to be the map*

$$\begin{gathered} pl : G_2(V) \to \mathbb{P}(\wedge^2 V) \\ \mathrm{Span}(Z, W) \mapsto [\mathrm{Span}(Z \wedge W)] \end{gathered}$$

Theorem 2.34. *pl is an embedding, with image $Q_4 = pl(G_{2,4})$ given by*

$$Q_4 = \left\{ z^{ij} \,|\, z^{12}z^{34} - z^{13}z^{24} + z^{14}z^{23} = 0 \right\} = \{[z] \,|\, z \wedge z = 0\} \subset \mathbb{P}^5$$

where $[z^{ij}]$ are homogeneous coordinates on $\mathbb{P}(\wedge^2 V)$. Q_4 is called the ***Klein quadric****.*

Generically a line L in $\mathbb{P}^5$ will intersect Q_4 in isolated points, but when this doesn't happen we can give such lines a special name:

Definition 2.35. *A line $L \subset \mathbb{P}^5$ is* ***null*** *if $L \subset Q_4$. A plane $N \subset \mathbb{P}^5$ is null if every line $L \subset N$ is null.*

2.6.2 Twistor Space

Now given $\mathbb{T}$ a 4d $\mathbb{C}$ vector space we can get a **double fibration**

$$F_1(\mathbb{T}) \overset{\pi_1}{\twoheadleftarrow} F_{12}(\mathbb{T}) \overset{\pi_2}{\twoheadrightarrow} F_2(\mathbb{T})$$

from the projections of (S_1, S_2)

Proposition 2.36. *The π_i are holomorphic.*

Definition 2.37. *Given seta A, B a* ***correspondence*** *is a map $f : A \to \mathcal{P}(B)$ sending each $a \in A$ to $f(a) \subset B$.*

The double fibration induces natural correspondences $c = \pi_2 \circ \pi_1^{-1}$ and $c^{-1} = \pi_1 \circ \pi_2^{-1}$.

Notation. *We shall call $\mathbb{T}$ the* ***twistor space****, and denote*

- $\mathbb{PT} = F_1(\mathbb{T}) \cong \mathbb{CP}^3$ ***projective twistor space***
- $M_{\mathbb{C}} = F_2(\mathbb{T}) \cong G_{2,4}(\mathbb{C})$ ***compactified complexified Minkowski space***
- $F = F_{12}(\mathbb{T})$ *the* ***correspondence*** *between $\mathbb{PT}$ and $M_{\mathbb{C}}$.*

For $A \subset \mathbb{PT}$ we shall let $c(A) = \tilde{A}$, and for $B \subset M_{\mathbb{C}}$ let $\hat{B} = c^{-1}(A)$.

Idea. *The point of twistor geometry is understanding how to use the correspondence*

$$\mathbb{PT} \leftarrow F \rightarrow M_{\mathbb{C}}$$

to transfer information about (subsets of) $\mathbb{PT}$ to (subsets of) $M_{\mathbb{C}}$.

Remark. *F is sometimes denoted as $\mathbb{PS}$ and called the* ***projective spinor bundle****. We will hopefully return to this later.*

Proposition 2.38. *For $p \in \mathbb{PT}$, $q \in M_{\mathbb{C}}$, $\tilde{p} \cong \mathbb{P}^2$ and $\hat{q} \cong \mathbb{P}^1$*

We now wish to understand the correspondence in coordinates. We consider the map ϕ as described before, though now we send

$$z \mapsto \left[\begin{pmatrix} iz \\ I_2 \end{pmatrix}\right]$$

for convenience, let $M^I_{\mathbb{C}} = \phi(M_2(\mathbb{C}))$, $\mathbb{PT}^I = c^{-1}(M^I_{\mathbb{C}})$ and $F^I = \pi_2^{-1}(M^I)$.

Remark. *I will be very slack on noting where we use this I notation. Be careful.*

Proposition 2.39. *$F^I \cong M^I \times \mathbb{P}^1$.*

Proof. The isomorphism is given as

$$\begin{aligned} M^I_{\mathbb{C}} \times \mathbb{P}^1 &\rightarrow F^I \\ (z, [v]) &\mapsto \left(\left[\begin{pmatrix} izv \\ v \end{pmatrix}\right], \left[\begin{pmatrix} iz \\ I_2 \end{pmatrix}\right] \right) \end{aligned}$$

□

Corollary 2.40. *In coordinates we can see the double fibration as*

$$\begin{aligned} \mathbb{PT}^I \leftarrow \quad & F^I \quad \rightarrow M^I_{\mathbb{C}} \\ (izv, v)^T \leftarrow & (z, [v]) \rightarrow z \end{aligned}$$

We can further construct the following double fibrations of flag manifolds over $\mathbb{T}$

$$F_3 \twoheadleftarrow F_{23} \twoheadrightarrow F_2 \qquad\qquad F_{13} \twoheadleftarrow F_{123} \twoheadrightarrow F_2$$

As $\dim \mathbb{T} = 4$ we have that canonically $F_3 \cong \mathbb{PT}^*$, $F_{23} \cong F^*$.

Definition 2.41. *We call F_3 the* ***dual projective twistor space****, $F_{13} = \mathbb{A}$ the* ***ambitwistor space****. We will denote $F_{123} = \mathbb{G}$.*

With this notation the fibrations look as

$$\mathbb{PT}^* \twoheadleftarrow F^* \twoheadrightarrow M_{\mathbb{C}} \qquad\qquad \mathbb{A} \twoheadleftarrow \mathbb{G} \twoheadrightarrow M_{\mathbb{C}}$$

We now can extend the result of prop 2.38

Proposition 2.42. *If $p \in \mathbb{PT}$, $q \in \mathbb{PT}^*$, $r \in \mathbb{A}$ then*

$$\tilde{p} \cong \mathbb{P}^2 \qquad \tilde{q} \cong \mathbb{P}^2 \qquad \tilde{r} \cong \mathbb{P}^1$$

and for $s \in M_{\mathbb{C}}$

$$\hat{s}_{\mathbb{PT}} \cong \mathbb{P}^1 \qquad \hat{s}_{\mathbb{PT}^*} \cong \mathbb{P}^1 \qquad \hat{s}_{\mathbb{A}} \cong \mathbb{P}^1 \times \mathbb{P}^1$$

Definition 2.43. *Given $p \in \mathbb{PT}$, $\tilde{p} \subset M_{\mathbb{C}}$ is called an α-**plane**, and given $q \in \mathbb{PT}^*$, $\tilde{q} \subset M_{\mathbb{C}}$ is called an β-**plane**.*

By the Plücker embedding, we have that $M_{\mathbb{C}} \cong Q_4 \subset \mathbb{P}^5$, and so we can describe α, β-planes in terms of null planes via the following result:

Proposition 2.44. *We have that*

- *All α, β-planes in $M_{\mathbb{C}}$ are null in $\mathbb{P}^5$*
- *Any null planes in $\mathbb{P}^5$ is either an α-plane or β-plane.*
- *Null lines in $M_{\mathbb{C}}$ are the 5-dimensional family of lines of the form $\tilde{r}$ for $r \in \mathbb{A}$.*
- *Any null line in $M_{\mathbb{C}}$ is the intersection of an α-plane and β-plane.*

2.6.3 Actions on Twistor Space

We will often want to equip $\mathbb{T}$ with a Hermitian form Φ. We state a quick lemma:

Lemma 2.45. *Φ determines a volume form $\Omega \in \wedge^4\mathbb{T}^*$ on $\mathbb{T}$ by*

$$\Omega = \operatorname{Im}\Phi \wedge \operatorname{Im}\Phi$$

Definition 2.46. *We define $SU(\mathbb{T}, \Phi) \subset GL(\mathbb{T})$ to be the subset preserving Φ and Ω.*

Example 2.47. *If Φ has signature $(2,2)$ then $SU(\mathbb{T}, \Phi) \cong SU(2,2)$.*

Given a $(2,2)$-signature Hermitian form on $\mathbb{T}$ $\exists (Z^\alpha)$ coordinates wrt which we have

$$\Phi = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \Leftrightarrow \Phi(Z, Z) = Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1}$$

Notation. *We denote the vector space $\mathbb{T}$ with the above coordinates adapted to Φ as $\mathbb{T}^\alpha$ and its dual as $\mathbb{T}_\alpha$. We call the corresponding dual coordinates W_α.*

$SL_2(\mathbb{C}) \leq SU(2,2)$ will act reducibly on $\mathbb{T}^\alpha$ meaning we have $\mathbb{T}^\alpha = S_1 \oplus S_2$ where the S_i are spinor bundles in the fundamental and conjugate dual representations respectively. This means we can write our coordinates as $Z^\alpha = (\omega^A, \pi_{A'})$.

Remark. *Recalling the double fibration, we can see that in order for a point $Z = (\omega, \pi) \in \mathbb{PT}$ to correspond to a $z \in M_{\mathbb{C}}$ we must have*

$$iz\pi = \omega$$

*This is often called the **incidence relation**. From this point of view, we see that z must get the indices $z^{AA'}$ s.t. in coordinates the incidence relation is*

$$\omega^A = iz^{AA'}\pi_{A'}$$

Definition 2.48. *Define the antiholomorphic involution $\sigma : \mathbb{PT} \to \mathbb{PT}$ by*

$$(\sigma(Z)^0, \sigma(Z)^1, \sigma(Z)^2, \sigma(Z)^3) = (\overline{Z^1}, -\overline{Z^0}, \overline{Z^3}, -\overline{Z^2})$$

Lemma 2.49. *σ preserves $\Phi(Z, Z)$.*

Definition 2.50. *A lines $\hat{p} \subset \mathbb{PT}$ is called **real** if it is invariant under σ.*

Lemma 2.51. *If $\hat{z}$ is a real line, $iz \in SU(2)$.*

Proof. Let $\omega = iz\pi$ so $\omega^0 = i(z^{00'}\pi_{0'} + z^{01'}\pi_{1'})$, $\omega^1 = i(z^{10'}\pi_{0'} + z^{11'}\pi_{1'})$. Then

$$\begin{aligned}\sigma(\omega) &= (-i(\overline{z}^{10'}\overline{\pi}_{0'} + \overline{z}^{11'}\overline{\pi}_{1'}), i(\overline{z}^{00'}\overline{\pi}_{0'} + \overline{z}^{01'}\overline{\pi}_{1'})) \\ \sigma(\pi) &= (\overline{\pi}_{1'}, -\overline{\pi}_{0'})\end{aligned}$$

Hence if $\hat{z}$ is a real line we get $\overline{z}^{11'} = -z^{00'}, \overline{z}^{01'} = z^{10'}$, and z has unit det. □

Lemma 2.52. *$\forall Z \in \mathbb{PT}$, the line joining Z and $\sigma(Z)$ is real.*

2.6.4 Complexified Minkowski Space

It is at this point useful to discuss slightly more concretely what is meant by complexified Minkowski space. The resource to use here is [1].

Definition 2.53. *Let (M, g) be a smooth real manifold, and take a coordinate system x^a s.t. $g = g_{ab}dx^a dx^b$. The **complexification** is defined by allowing the x^a to take complex values and extending $g_{ab}(x)$ holomorphically.*

2.6.5 Single Fibration Picture

Recall the antiholomorphic fibration σ on $\mathbb{PT}$.

Proposition 2.54. *All real lines on $\mathbb{PT}$ are lines joining $Z \to \sigma(Z)$, and no two such lines intersect*

Proposition 2.55. *$\mathbb{PT}$ is fibered by its real lines and the quotient space is S^4.*

3 The Monopole Equations

3.1 Yang-Mills-Higgs equations

Definition 3.1. *Take a principal G-bundle $P \to M$, ω_{vol} an orientation on M, and $\langle \cdot, \cdot \rangle$ to be an* ad*-invariant inner product on $\mathfrak{g}$. Then the **Yang-Mills-Higgs actions** on M is*

$$S_{YMH}[A, \phi] = \int_M \left[-|F|^2 - |D\phi|^2 - V(\phi) \right] \omega_{vol}$$

where $F = dA + A \wedge A$ is the curvature associated to a section $A \in \Gamma(T^*M \otimes \mathrm{ad}(P))$, $D = d + A$ is the associated covariant derivative, and $\phi \in \Gamma(\mathrm{ad}(P))$.

To connect with physical theory we want our Lagrangian to be of the form kinetic-potential. This will manifest itself in our choice of signs by requiring that

$$-|D\phi|^2 = (\partial_0\phi)^2 + \dots$$
$$-|F|^2 = E_i^2 + \dots$$

This is the reason for the somewhat strange looking sign choice at this point. Obviously in the end it will be equivalent to take

$$S_{YMH} = \int |F|^2 + |D\phi|^2 + V$$

when it comes to the variational equations, but this will not be true when we consider the energy functional.

Remark. *A common choice of potential function V is $V(\phi) = \lambda\left(1 - |\phi|^2\right)^2$, the ϕ^4-**potential**. By our choice of signs, we want $\lambda > 0$. We can check $V'(\phi) = -4\lambda(1 - |\phi|^2)\,|\phi|$.*

Definition 3.2. *A **monopoles** will be a soliton-like solution to the Yang-Mills-Higgs equations when $G = SU(2)$, $M = \mathbb{R}^4$ with the Minkowski metric, the principal bundle is $P = M \times G$, and the potential is ϕ^4.*

Proposition 3.3. *The variational equations corresponding to S_{YMH} in Minkowski $\mathbb{R}^4$ are the **Yang-Mills-Higgs equations***

$$DF = 0 \quad \textit{(Bianchi)}$$
$$\star D \star F = -[\phi, D\phi]$$
$$\star D \star D\phi = -\frac{1}{2|\phi|}V'(\phi)\phi$$

Proof. We first consider the equation that comes from the variation of A. Let $A_t = A + t\beta$, then $F_t = F + t\,(d\beta + \beta \wedge A + A \wedge \beta) + \mathcal{O}(t^2)$ and $D_t\phi = D\phi + t\,[\beta, \phi]$. Hence

$$S_t = S + 2t \int_M \left[-\langle F, D\beta\rangle - \langle D\phi, [\beta, \phi]\rangle\right]\omega_{vol} + \mathcal{O}(t^2)$$

Hence to be at a stationary point of the action variation we want

$$\int_M \left[-\langle F, D\beta\rangle - \langle D\phi, [\beta, \phi]\rangle\right]\omega_{vol} = 0$$

Using the fact that inner product is ad-invariant and letting D^* be the formal adjoint of D wrt to inner product $\langle\langle \eta, \omega\rangle\rangle = \int_M \langle \eta, \omega\rangle\,\omega_{vol}$ we can rewrite this as

$$\int_M \langle -D^*F + [D\phi, \phi]\,, \beta\rangle\,\omega_{vol} = 0$$

Using results on the dual of the covariant derivative we can say that

$$D^*F = (-1)^{4(2-1)+1}(-1) \star D \star F = \star D \star F$$

Hence as β was a generic variation we must have $\star D \star F - [D\phi, \phi] = 0$.
We now consider a ϕ variation so $\phi_t = \phi + t\psi$. Note

$$|\phi_t| = \sqrt{\langle \phi_t, \phi_t \rangle} = \sqrt{|\phi|^2 + 2t\langle \phi, \psi \rangle + \mathcal{O}(t^2)} = |\phi| \sqrt{1 + \frac{2t\langle \phi, \psi \rangle}{|\phi|^2} + \mathcal{O}(t^2)} = |\phi| + t\frac{\langle \phi, \psi \rangle}{|\phi|} + \mathcal{O}(t^2)$$

so if we consider V as $V(\phi) = V(|\phi|)$ (i.e. as a function of a real variable) then

$$V(\phi_t) = V(\phi) + t\,|\phi|^{-1}\, V'(\phi)\,\langle \phi, \psi \rangle + \mathcal{O}(t^2)$$

Then a variational argument as before means that we need to set

$$\begin{aligned}
& \int_M \left[-2\langle D\phi, D\psi \rangle - |\phi|^{-1}\, V'(\phi)\,\langle \phi, \psi \rangle \right] \omega_{vol} = 0 \\
\Rightarrow \quad & -2D^* D\phi - |\phi|^{-1}\, V'(\phi)\phi = 0 \\
\Rightarrow \quad & (-1)^{4(1-1)+1}(-1)(-1) \star D \star D\phi - \frac{1}{2\,|\phi|} V'(\phi)\phi = 0 \\
\Rightarrow \quad & \star D \star D\phi + \frac{1}{2\,|\phi|} V'(\phi)\phi = 0
\end{aligned}$$

□

Remark. *The Dirac monopole is a solution in the case of $G = U(1)$.*

We may make these equations explicit in coordinates. The first approach is to try and substitute in coordinate expressions into the YMH equations. Taking coordinates x^μ on $\mathbb{R}^4$ and writing $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$

$$\begin{aligned}
\star F &= \frac{1}{4} F_{\mu\nu}\epsilon^{\mu\nu}{}_{\rho\sigma} dx^\rho \wedge dx^\sigma \\
\Rightarrow D \star F &= \frac{1}{4} D_\tau F_{\mu\nu}\epsilon^{\mu\nu}{}_{\rho\sigma} dx^\tau \wedge dx^\rho \wedge dx^\sigma \\
\Rightarrow \star D \star F &= \frac{1}{4} D_\tau F_{\mu\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}\epsilon^{\tau\rho\sigma}{}_\lambda dx^\lambda \\
&= \frac{1}{4} D_\tau F_{\mu\nu}\epsilon_{\rho\sigma}{}^{\mu\nu}\epsilon^{\rho\sigma\tau}{}_\lambda dx^\lambda \\
&= -D_\tau F^\mu{}_\nu\, \delta^\tau_{[\mu}\delta^\nu_{\lambda]} dx^\lambda \\
&= -D_\tau F^\tau{}_\lambda\, dx^\lambda
\end{aligned}$$

Hence the first monopole equation reads

$$D_\mu F^{\mu\nu} + [D^\nu \phi, \phi] = 0$$

Next we have

$$\begin{aligned}
\star D\phi &= \frac{1}{6}(D_\mu\phi)\epsilon^\mu{}_{\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \\
\Rightarrow D \star D\phi &= \frac{1}{6}(D_\tau D_\mu\phi)\epsilon^\mu{}_{\nu\rho\sigma} dx^\tau \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\
\Rightarrow \star D \star D\phi &= \frac{1}{6}(D_\tau D_\mu\phi)\epsilon^\mu{}_{\nu\rho\sigma}\epsilon^{\tau\nu\rho\sigma} \\
&= -(D_\tau D^\mu\phi)\delta^\tau_\mu = -D_\mu D^\mu\phi
\end{aligned}$$

which yields (taking the ϕ^4 potential)

$$D_\mu D^\mu \phi + 2\lambda(1 - |\phi|^2)\phi = 0$$

Collecting this with the Bianchi identity (taking it as $\star DF = 0$) we have

$$\begin{aligned} \epsilon^{\rho\mu\nu\tau} D_\rho F_{\mu\nu} &= 0 \\ D_\mu F^{\mu\nu} + [D^\nu \phi, \phi] &= 0 \\ D_\mu D^\mu \phi + 2\lambda(1 - |\phi|^2)\phi &= 0 \end{aligned}$$

An alternative approach to deriving these equations is to first write the Lagrangian in coordinate form and then derive the variational equations. We take the inner product on $\mathfrak{g} = \mathfrak{su}(2)$ to be $\langle X, Y\rangle = -\frac{1}{2}\kappa(X, Y) = -2\,\mathrm{Tr}(XY)$ for concreteness, which gives $\langle t^a, t^b\rangle = \delta^{ab}$, and as such we need the mostly-positive Minkowski metric.

Remark. *It is possible to change around the signs here in order to assure that we use the mostly-minus metric, as Manton & Sutcliffe do. We will want to avoid this as the mostly positive makes more sense when reducing from Minkowski $\mathbb{R}^{n+1}$ to Euclidean $\mathbb{R}^n$.*

We have

$$\begin{aligned} \langle F, F\rangle &= \frac{1}{4}\langle F_{\mu\nu}, F_{\rho\sigma}\rangle \langle dx^\mu \wedge dx^\nu, dx^\rho \wedge dx^\sigma\rangle \\ &= \frac{1}{4}\langle F_{\mu\nu}, F_{\rho\sigma}\rangle (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ &= \frac{1}{2}\langle F_{\mu\nu}, F^{\mu\nu}\rangle \\ &= -\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \\ \langle D\phi, D\phi\rangle &= \langle D_\mu\phi, D_\nu\phi\rangle \langle dx^\mu, dx^\nu\rangle \\ &= \langle D_\mu\phi, D_\nu\phi\rangle \eta^{\mu\nu} \\ &= -2\,\mathrm{Tr}(D_\mu\phi D^\mu\phi) \end{aligned}$$

Hence the corresponding Lagrangian density is

$$\mathcal{L} = \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + 2\,\mathrm{Tr}(D_\mu\phi D^\mu\phi) - V(\phi) = \sum_a \left[-\frac{1}{2}F^{(a)}_{\mu\nu}F^{(a)\mu\nu} - (D_\mu\phi)^{(a)}(D^\mu\phi)^{(a)}\right] - \lambda\left[1 - \sum_a (\phi^{(a)})^2\right]^2$$

To check that we have the correct signs in this Lagrangian we verify that it takes the form kinetic-potenial with

$$\begin{aligned} \text{kinetic} &= -2\,\mathrm{Tr}(E_iE_i) - 2\,\mathrm{Tr}(D_0\phi D_0\phi) \\ \text{potential} &= -\mathrm{Tr}(F_{ij}F_{ij}) - 2\,\mathrm{Tr}(D_i\phi D_i\phi) + V(\phi) \end{aligned}$$

where $E_i = F_{0i}$.

Remark. *Something we can immediately recognise is that in order to get finite energy solutions when $\lambda \neq 0$, we need $|\phi| \to 1$.*

We now recall the Euler-Lagrange equations for Lagrangian with field ψ

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0$$

The fields here are really the coefficients in $\mathfrak{su}(2)$ of A_μ, ϕ, that is they are $A_\mu^{(a)}, \phi^{(a)}$, so we expand

$$\begin{aligned} F_{\mu\nu}^{(a)} &= \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + A_\mu^{(b)} A_\nu^{(c)} \epsilon_{abc} \\ (D_\mu \phi)^{(a)} &= \partial_\mu \phi^{(a)} + A_\mu^{(b)} \phi^{(c)} \epsilon_{abc} \end{aligned}$$

giving

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^{(a)})} &= -2F^{(a)\mu\nu} \\ \frac{\partial \mathcal{L}}{\partial A_\mu^{(a)}} &= -2\epsilon_{bac} A_\nu^{(c)} F^{(b)\mu\nu} - 2\phi^{(c)} \epsilon_{bac} (D^\mu \phi)^{(b)} \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(a)})} &= -2(D^\mu \phi)^{(a)} \\ \frac{\partial \mathcal{L}}{\partial \phi^{(a)}} &= -2\epsilon_{cba} A_\mu^{(b)} (D^\mu \phi)^{(c)} + 4\lambda \phi^{(a)} \left[1 - \sum_b (\phi^{(b)})^2\right] \end{aligned}$$

We can now write the Euler-Lagrange equations

$$\begin{aligned} 0 &= -\partial_\nu F^{(a)\nu\mu} - \left[-\epsilon_{bac} A_\nu^{(c)} F^{(b)\mu\nu} - \phi^{(c)} \epsilon_{bac} (D^\mu \phi)^{(b)}\right] \\ &= -\left[\partial_\nu F^{(a)\nu\mu} + A_\nu^{(c)} F^{(b)\nu\mu} \epsilon_{cba}\right] - (D^\mu \phi)^{(b)} \phi^{(c)} \epsilon_{bca} \\ \Rightarrow 0 &= D_\nu F^{\nu\mu} + [D^\mu \phi, \phi] \end{aligned}$$

and

$$\begin{aligned} 0 &= -\partial_\mu (D^\mu \phi)^{(a)} - \left[-\epsilon_{cba} A_\mu^{(b)} (D^\mu \phi)^{(c)} + 2\lambda \phi^{(a)} \left[1 - \sum_b (\phi^{(b)})^2\right]\right] \\ &= -\left[\partial_\mu (D^\mu \phi)^{(a)} + A_\mu^{(b)} (D^\mu \phi)^{(c)} \epsilon_{bca}\right] - \lambda \phi^{(a)} \left[1 - \sum_b (\phi^{(b)})^2\right] \\ \Rightarrow 0 &= D_\mu D^\mu \phi + 2\lambda(1 - |\phi|^2)\phi \end{aligned}$$

We happily see that these two approaches agree, and we should see that these are indeed the sort of equations we want (e.g the ϕ equation looks like Klein-Gordon if we linearise around $|\phi| = 1$.)

3.2 BPS limit

The monopole equations we have found so far are second order, but we want to apply the classic strategy when working with topological solitons: write the energy functional of a static configuration as the integral of a square term plus a topological term, and then we locally must have a minimising solution by setting the squared term to 0. This will be possible if we set $\lambda = 0$ but retain that $|\phi| = 1$ at infinity. More specifically we take the conditions

$$\begin{aligned} |\phi| &= 1 - \frac{m}{2r} + \mathcal{O}\left(\frac{1}{r^2}\right) \\ \frac{\partial\,|\phi|}{\partial\Omega} &= \mathcal{O}\left(\frac{1}{r^2}\right) \\ |D\phi| &= \mathcal{O}\left(\frac{1}{r^2}\right) \end{aligned}$$

With $\lambda = 0$ we can rewrite the energy functional of static configurations at any time as

$$E[A,\phi] = \int_M \left[-\operatorname{Tr}(F_{ij}F_{ij}) - 2\operatorname{Tr}(D_i\phi D_i\phi)\right] d^3x$$

If we keep the same inner product on $\mathfrak{su}(2)$, but now take Euclidean $\mathbb{R}^3$ we can express

$$E[A,\phi] = \int_{\mathbb{R}^3} \left[|F|^2 + |D\phi|^2\right] d^3x$$

As we are in Euclidean $\mathbb{R}^3$ we can write

$$|\star D\phi|^2\,\omega_{vol} = \left\langle \star D\phi \wedge \star^2 D\phi\right\rangle = (-1)^{1(3-1)}\left\langle \star D\phi \wedge D\phi\right\rangle = (-1)^{2\times 1}\left\langle D\phi \wedge \star D\phi\right\rangle = |D\phi|^2\,\omega_{vol}$$

and, using the Bianchi identity,

$$\langle F, \star D\phi\rangle\,\omega_{vol} = \left\langle F \wedge \star^2 D\phi\right\rangle = (-1)^{1(3-1)}\langle F \wedge D\phi\rangle = (-1)^2\left[d\langle F\wedge\phi\rangle - \langle DF \wedge \phi\rangle\right] = d\langle F \wedge \phi\rangle$$

so

$$\begin{aligned} E[A,\phi] &= \int_{\mathbb{R}^3} \left[\langle F \mp \star D\phi, F \mp \star D\phi\rangle \pm 2\langle F, \star D\phi\rangle\right] d^3x \\ &= \int_{\mathbb{R}^3} |F \mp \star D\phi|^2\, d^3x \pm 2\lim_{R\to\infty}\int_{S^2_R} \langle F \wedge \phi\rangle \end{aligned}$$

Remark. *In the above discussion, make sure to check that as this the 3d Hodge star, our inner products are well defined in the sense that the are the inner product of a k-form with a k-form.*

This boundary term turns out to be a topological contribution, and we can see this in two ways. We first need to prove the following lemma

Lemma 3.4. *In the $\mathbb{R}^3$ bulk*

$$\langle \phi \wedge (D\phi \wedge D\phi)\rangle = \langle \phi \wedge (d\phi \wedge d\phi)\rangle + |\phi|^2\left[\langle F\wedge\phi\rangle - d\langle A \wedge \phi\rangle\right] - \frac{1}{2}\langle A\wedge\phi\rangle \wedge d|\phi|^2$$

Proof. Exercise. □

Corollary 3.5. *Given the decay conditions, on S^2_∞ we have*

$$\langle F \wedge \phi \rangle - d \langle A \wedge \phi \rangle = - \langle \phi \wedge (d\phi \wedge d\phi) \rangle$$

and hence the topological term is $\mp 2 \int_{S^2_\infty} \langle \phi \wedge (d\phi \wedge d\phi) \rangle$.

The two interpretations are then as follows:

1. Degree of a map of spheres - We will want to recall some standard vector calculus results stated in the language of forms. Firstly recall that if we have a manifold M with volume form ω_M and orientable submanifold Σ with normal N we get a volume form on Σ given by

$$\omega_\Sigma = i_N \omega_M$$

   If we parametrise Σ on an open patch U, that is find $\psi : \mathbb{R}^d \to U \subset \Sigma$, and do an area integral over $U \subset \Sigma$ in these coordinates, that is equivalent to pulling back ω_Σ by ψ.
   By noting that, on S^2_∞, $\phi : S^2_\infty \to S^2_1 \subset \mathfrak{su}(2)$, where on this S^2 ϕ is also the normal, and that we have chosen the metric on $\mathfrak{su}(2)$ to be Euclidean, we have that

$$\int_{S^2_\infty} \langle \phi \wedge (d\phi \wedge d\phi) \rangle = \int_{S^2_\infty} \boldsymbol{\phi} \cdot (\partial_u \boldsymbol{\phi} \times \partial_v \boldsymbol{\phi})\, dudv = \int_{S^2_\infty} \phi^* \omega_{S^2} = 4\pi \deg \phi$$

   where we have the used the notation $\phi = \boldsymbol{\phi} \cdot \boldsymbol{t}$, and the degree here refers to of ϕ as a map of spheres. This degree is a topological invariant.

2. Chern class of a line bundle - Consider the matrix $\boldsymbol{\phi} \cdot \boldsymbol{\sigma}$. We give the following lemma

   Lemma 3.6. *Let $P = \frac{1}{2}(I + \boldsymbol{\phi} \cdot \boldsymbol{\sigma})$. Then P is a projection operator, i.e. $P^2 = P$.*

   Proof. Do the multiplication ☐

   Corollary 3.7. *$\boldsymbol{\phi} \cdot \boldsymbol{\sigma}$ has eigenvalues ± 1.*

   Proof. Projection operators have eigenvalues $0, 1$. Hence we get the result, and as we can never have $\boldsymbol{\phi} \cdot \boldsymbol{\sigma} = \pm I$, both eigenvalues must occur. ☐

   Now we can consider the eigenvector bundle

$$L = \left\{ (\boldsymbol{x}, \psi) \in S^2_\infty \times \mathbb{C}^2 \mid [\boldsymbol{\phi}(\boldsymbol{x}) \cdot \boldsymbol{\sigma}]\, \psi = \psi \right\}$$

   Remark. *This will be a complex line bundle, so the only possible Chern class we can relate to it will be c_1.*

   We can get a connection on the bundle by viewing it as a subbundle of $S^2_\infty \times \mathbb{C}^2$, and getting covariant derivative on vector fields by projection, that is $DX = PdX$.

   Remark. *Note that here the variations in ψ that preserve the the eigenvector condition are exactly those that scale ψ by some element of $\mathbb{C}^\times$. This really makes the bundle look like the tangent bundle to the sphere S^2_∞ in the sense that there is a plane attached at every point of S^2.*

Now at every point $\boldsymbol{x} \in S^2_\infty$ if $\hat{\psi}(\boldsymbol{x})$ is a normalised eigenvector we have that a local section $\sigma : U \subset S^2_\infty \to E$ is given by $\sigma = h\hat{\psi}$ where $h : U \to \mathbb{C}$ is some scale. Then as we are dealing with 2×2 non-trivial projection matrices we can write $P = \hat{\psi}\hat{\psi}^\dagger$ giving

$$\begin{aligned} D(h\hat{\psi}) &= \hat{\psi}\hat{\psi}^\dagger \left[(dh)\hat{\psi} + h(d\hat{\psi})\right] \\ &= (dh)\hat{\psi} + h\hat{\psi}\hat{\psi}^\dagger(d\hat{\psi}) \end{aligned}$$

What we have done by choosing a local section $\hat{\psi}$ is given a local trivialisation of the bundle. To find the connection locally we need to see how the covariant derivaitve acts locally on $\mathbb{C}$, that is how it acts on h. By writing

$$D(h\hat{\psi}) = \hat{\psi}\left[dh + h\hat{\psi}^\dagger d\hat{\psi}\right]$$

We can read off that $A = \hat{\psi}^\dagger d\hat{\psi}$ locally, giving $F_L = d\hat{\psi}^\dagger \wedge d\hat{\psi}$.
We now want to relate this to our map ϕ. As we are dealing with 2×2 non-trivial projection matrices we can write

$$P = \hat{\psi}\hat{\psi}^\dagger \Rightarrow \phi = -i\hat{\psi}\hat{\psi}^\dagger + \frac{i}{2}I \Rightarrow d\phi = -i\left[(d\hat{\psi})\hat{\psi}^\dagger + \hat{\psi}(d\hat{\psi}^\dagger)\right]$$

We can see that, as the inner product corresponds to the trace, $\langle d\phi \wedge d\phi\rangle = \langle(\partial_\mu\phi)(\partial_\nu\phi)\rangle\, dx^\mu \wedge dx^\nu = \frac{1}{2}\langle[\partial_\mu\phi, \partial_\nu\phi]\rangle\, dx^\mu \wedge dx^\nu = 0$ which means

$$\begin{aligned} \langle\phi \wedge (d\phi \wedge d\phi)\rangle &= -i\left\langle\left(\hat{\psi}\hat{\psi}^\dagger - \frac{1}{2}I\right)(d\phi \wedge d\phi)\right\rangle \\ &= -i\left\langle\hat{\psi}\hat{\psi}^\dagger\left[(d\hat{\psi})\hat{\psi}^\dagger + \hat{\psi}(d\hat{\psi}^\dagger)\right]^{\wedge 2}\right\rangle \end{aligned}$$

We expand out the latter, using the cyclicity of the trace and Leibniz' rule with $\hat{\psi}^\dagger\hat{\psi} = 1$, for example

$$\begin{aligned} \left\langle\hat{\psi}\hat{\psi}^\dagger(d\hat{\psi})\hat{\psi}^\dagger \wedge (d\hat{\psi})\hat{\psi}^\dagger\right\rangle &= \left\langle\hat{\psi}^\dagger\hat{\psi}\hat{\psi}^\dagger(d\hat{\psi})\hat{\psi}^\dagger \wedge (d\hat{\psi})\right\rangle \\ &= \left\langle\hat{\psi}^\dagger(d\hat{\psi})\hat{\psi}^\dagger \wedge (d\hat{\psi})\right\rangle \\ &= \left\langle\left[d(\hat{\psi}^\dagger\hat{\psi}) - (d\hat{\psi}^\dagger)\hat{\psi}\right]\hat{\psi}^\dagger \wedge (d\hat{\psi})\right\rangle \\ &= -\left\langle(d\hat{\psi}^\dagger)\hat{\psi} \wedge \hat{\psi}^\dagger(d\hat{\psi})\right\rangle \\ &= -\left\langle\hat{\psi}\hat{\psi}^\dagger\hat{\psi}(d\hat{\psi}^\dagger) \wedge \hat{\psi}(d\hat{\psi}^\dagger)\right\rangle \end{aligned}$$

$$\begin{aligned} \left\langle\hat{\psi}\hat{\psi}^\dagger(d\hat{\psi}) \wedge (d\hat{\psi}^\dagger)\right\rangle &= -\left\langle\hat{\psi}(d\hat{\psi}^\dagger)\hat{\psi} \wedge (d\hat{\psi}^\dagger)\right\rangle \\ &= \left\langle\hat{\psi}(d\hat{\psi}^\dagger) \wedge (d\hat{\psi})\hat{\psi}^\dagger\right\rangle \\ &= \left\langle d\hat{\psi}^\dagger \wedge d\hat{\psi}\right\rangle \\ &= \left\langle\hat{\psi}\hat{\psi}^\dagger\hat{\psi}(d\hat{\psi}^\dagger) \wedge (d\hat{\psi})\hat{\psi}^\dagger\right\rangle \end{aligned}$$

and as a result we get

$$\langle\phi\wedge(d\phi\wedge d\phi)\rangle = -2i\left\langle d\hat{\psi}^{\dagger}\wedge d\hat{\psi}\right\rangle = -4\pi\left(\frac{i}{2\pi}\langle F_L\rangle\right)$$

We can then see that

$$\int_{S^2_\infty}\langle\phi\wedge(d\phi\wedge d\phi)\rangle = -4\pi c_1(L)$$

Remark. *A point to be made about the above is that, a priori, the connection is not related to the Higgs field on S^2_∞. It is the decay condition on $D\phi$ which enforces that on S^2_∞ we have $\partial_\mu\phi = [\phi, A_\mu]$.*

Through either discussion we have $E \geq \pm 8\pi k$ for some $k \in \mathbb{Z}$ with equality iff $F = \mp \star D\phi$ where we choose the sign to make the bound positive. This is the **BPS equation**.

Remark. *We might wonder what is the connection between these two viewpoints, where we have that $\deg\phi = -c_1(L)$. To make this connection we recall a result stated in [5]:*

Proposition 3.8. *If L is a complex line bundle on a Riemann surface and f is a meromorphic section then*

$$\deg(D(f)) = c_1(L)$$

Proof. See the Riemann Surfaces notes by Joel Robbin, University of Wisconsin, which in turn reference [9]. □

We can think of L as a complex line bundle over $\mathbb{P}^1$ viewed as the Riemann sphere. If we take a section $\sigma = h\hat{\psi}$ where we choose $h = \hat{\psi}_2$, this section has poles exactly where the eigenvector is $(1,0) \Rightarrow \boldsymbol{\phi} = (1,0,0)$, and has no zeros. Hence the Chern class is counting the number of preimages of $(1,0,0)$, that is

$$-c_1(L) = |\{\text{preimages of } (1,0,0)\}|$$

This gives $c_1(L) = \deg\phi$, provided we have an argument to tell us that these all should be counted with sign 1.

Example 3.9. *We can make an ansätze of spherical symmetry for a static monopole in $\mathbb{R}^3$ to assume our solution has the form*

$$\phi = ih(r)\frac{x^i\sigma_i}{r} \qquad\qquad A_i = -\frac{i}{2}\left[1-k(r)\right]\frac{\epsilon_{ijk}x^j\sigma_k}{r^2}$$

The BPS equations here become

$$\frac{dh}{dr} = \frac{1}{2r^2}(1-k^2), \qquad\qquad \frac{dk}{dr} = -2hk$$

and these give the solutions

$$h(r) = \coth(2r) - \frac{1}{2r}, \qquad\qquad k(r) = \frac{2r}{\sinh(2r)}$$

be careful of scaling and conventions here

3.3 Self-Dual Reduction

Suppose now we consider pure Yang-Mills on $\mathbb{R}^{n+1}$ with contant diagonal metric g (i.e we are considering it to be either Minkowski or Euclidean). Take coordinates x^μ and ask that the connection $A = A_\mu dx^\mu$ is x^0-independent. Then writing $A = \phi dx^0 + A_i dx^i$ we have

$$\begin{aligned} F &= \frac{1}{2} F_{ij} dx^i \wedge dx^j + (D_i\phi) dx^i \wedge dx^0 \\ &= {}^3F + D\phi \wedge dx^0 \end{aligned}$$

as $F_{i0} = \partial_i \phi + [A_i, \phi] = D_i \phi$ and denoting ${}^3F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$. We calculate

$$\begin{aligned} \left\langle {}^3F, D\phi \wedge dx^0 \right\rangle &= \frac{1}{2} \left\langle F_{ij}, D_k\phi \right\rangle \underbrace{\left\langle dx^i \wedge dx^j, dx^k \wedge dx^0 \right\rangle}_{=0} \\ \left\langle D\phi \wedge dx^0, D\phi \wedge dx^0 \right\rangle &= \left\langle D_i\phi, D_j\phi \right\rangle \underbrace{\left\langle dx^i \wedge dx^0, dx^j \wedge dx^0 \right\rangle}_{=g^{ij} g^{00}} \end{aligned}$$

and so on Euclidean $\mathbb{R}^{n+1}$, $|F|^2 = \left|{}^3F\right|^2 + |D\phi|^2$. This means we recover the action for Yang-Mills-Higgs with 0 potential from this reduction.

Remark. *It is not absurd to consider Euclidean $\mathbb{R}^4$, as we should view this as performing a Wick rotation from Minkowski space, which is natural when quantising the theory as it means that the path integral is now well defined (see [17] for more of a discussion on this).*

The Yang-Mills equation for F reads

$$D \star F = 0 \Rightarrow g^{\mu\nu} D_\mu F_{\nu\rho} = 0$$

This splits to give

$$\begin{aligned} g^{\mu\nu} D_\mu F_{\nu 0} = 0 &\Rightarrow g^{ij} D_i D_j \phi = D_i D^i \phi = 0 \\ g^{\mu\nu} D_\mu F_{\nu k} = 0 &\Rightarrow -g^{00} D_0 D_k \phi + g^{ij} D_i F_{jk} = 0 \\ & \quad\; - g^{00} \left[\phi, D_k\phi\right] + D_i F^i{}_k = 0 \end{aligned}$$

Moreover we have the stronger result:

Proposition 3.10. *F is (anti-)self-dual iff $({}^3F, \phi)$ satisfy the Bogomolny equations.*

Proof. The (anti-)self-duality equations for F say $\star_4 F = (-)F$, where we are now making explicit the dimension wrt which $\star$ is acting. We can calculate

$$\begin{aligned} \star_4 F &= \frac{1}{4} F_{ij} \epsilon^{ij}{}_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{2} (D_i\phi) \epsilon^{i0}{}_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= -\frac{1}{2} (D_k\phi) \epsilon^k{}_{ij} dx^i \wedge dx^j - \frac{1}{2} F_{ij} \epsilon^{ij}{}_k dx^k \wedge dx^0 \\ &= - \star_3 D\phi - \star_3 F \wedge dx^0 \end{aligned}$$

which we see to mean, using $\star_3^2 = 1$.

$$\star_4 F = \pm F \Leftrightarrow {}^3F \pm \star_3 D\phi = 0$$

$\square$

4 Self-Dual Gauge Fields

4.1 Gauge Fields from Holomorphic Vector Bundles

We now want to study self-dual gauge fields more, using Ward & Wells section 8 as our reference. For this section we will denote the gauge potential as Φ as A will be a spinor index eventually, so this should not be confused with a Higgs field. The gauge fields we will be studying we now extend to be those over $M_{\mathbb{C}}$. As such we will be using the machinery of the twistor correspondence.

Definition 4.1. *An open set $U \subset M_{\mathbb{C}}$ is* ***elementary*** *if $\forall \tilde{Z}$ a self-dual plane (i.e. an α-plane) s.t. $\tilde{Z} \cap U \neq \emptyset$, $\tilde{Z} \cap U$ is connected and simply connected.*

This lets us give a novel relation to self-dual curvature:

Proposition 4.2. *The curvature F on a given elementary U is anti-self-dual iff $\forall$ self dual $\tilde{Z}$ that intersects U, the restriction of the covariant derivative D to $\tilde{Z} \cap U$ is integrable.*

Proof. Because $\tilde{Z} \cap U$ is connected and simply connected, the covariant derivative being integrable is equivalent to the curvature being flat, that is

$$\forall v, w \in T\tilde{Z},\ v^a w^b F_{ab} = 0$$

Suppose $\tilde{Z}$ corresponds to a projective twistor $Z = (\omega^A, \pi_{A'})$. Then as $\tilde{Z}$ is null, a tangent vector to $\tilde{Z}$ will be null hence of the form $v^a = \lambda^A \pi^{A'}$ for some spinor λ. This means we get the condition

$$(\phi_{A'B'}\epsilon_{AB} + \phi_{AB}\epsilon_{A'B'})\lambda_1^A \lambda_2^B \pi^{A'}\pi^{B'} = 0$$

The term $\epsilon_{A'B'}\pi^{A'}\pi^{B'}$ vanishes, and then for this to vanish $\forall \lambda_1, \lambda_2, \tilde{Z}$, we have that $\phi_{A'B'} = 0$, i.e. the self dual part of F is 0. □

This in turn leads to a useful equivalence coming from the twistor geometry.

Theorem 4.3. *Let $\hat{U} \subset \mathbb{PT}$ be the open subset s.t $\forall Z \in \hat{U}, \tilde{Z} \cap U \neq \emptyset$, then there is a bijection (up to equivalence) between*

1. *anti-self-dual $GL_n(\mathbb{C})$-bundles on U*
2. *holomorphic rank-n vector bundles $E \to \hat{U}$ s.t. E is trivial over $\hat{x}$ for each $x \in U$.*

Proof. Ward and Wells [17] gives a very thorough proof of this, we will only sketch it.
We define the map $1 \mapsto 2$ by making the fibres of E to be

$$E_Z = \left\{ \psi \,|\, D\psi = 0 \text{ on } \tilde{Z} \right\}$$

As the bundle is ASD, the covariant derivative is integrable and so a value of ψ is defined by its value at a point $x \in U$. Hence $E_z \cong \mathbb{C}^n$. Moreover, any Z' s.t $\tilde{Z}'$ intersects x will also have its value of ψ fixed by the value of ψ on x. Hence $E|_{\hat{x}}$ is trivial.
To map $2 \mapsto 1$ note as the vector bundle E is trivial over $\hat{x}$ we can get the rank n vector bundle $\tilde{E} \to U$ by taking the copy of $\mathbb{C}^n$ over $\hat{x}$ in $E|_{\hat{x}}$. To define a connection we need a way to parallel transport. Given $x, y \in \tilde{Z}$ we know they have corresponding lines $\hat{x}, \hat{y} \subset U$ s.t. $\hat{x} \cap \hat{y} = Z$. As E is trivial over $\hat{x}, \hat{y}$, we may transport along these lines under the connection induced from the base M (sometimes called the **spacetime connection**, denoted ∇) via Z to transport from x to y. This connection will turn out to satisfy the conditions we need. □

The theorem offers a potential way to construct solutions to the ASD Yang Mills equations via holomorphic vector bundles. These are vector bundles on $\hat{U} \subset \mathbb{PT}$ which can be covered with two charts $W, \underline{W}$ (analogous to the north/south charts on the Riemann sphere), and so they are defined by a single holomorphic $n \times n$ matrix transition function $F(Z) = F(\omega^A, \pi_{A'})$ on $W \cap \underline{W}$. This acts as

$$\underline{\xi} = F(Z)\xi$$

where $\xi, \underline{\xi}$ are coordinate vectors on $W, \underline{W}$ respectively.

Remark. *We say that $W, \underline{W}$ are analogous to the N/S charts on the Riemann sphere in the sense that on any line $W \cap \underline{W} \cap \hat{x} \cong \mathbb{P}^1$ we have that $W = \{\pi_{1'} \neq 0\}$, $\underline{W} = \{\pi_{0'} \neq 0\}$, where we are viewing π as the projective coordinates on the line. Hence we know the intersection contains the circle $\{|\pi_{0'}/\pi_{1'}| = 1\}$.*

To do the transport along $\hat{x}$ we need to restrict to the line, that is take

$$G(x, \pi) = F(ix^{AA'}\pi_{A'}, \pi_{A'})$$

and then find $H, \underline{H}$ non-singular $n \times n$ matrices holomorphic on $W \cap \hat{x}$, $\underline{W} \cap \hat{x}$ respectively s.t. on $W \cap \underline{W} \cap \hat{x}$

$$G = \underline{H}H^{-1} \tag{4.1.1}$$

This is called the **spliting formula**. Such $H, \underline{H}$ must exists as the triviality of $E|_{\hat{x}}$ means that we know

$$\xi = H\psi, \quad \underline{\xi} = \underline{H}\psi$$

for some constant vector ψ.

Proposition 4.4. *With the above notation we find the connection on the bundle to be given by*

$$\pi^{A'}\Phi_{AA'} = H^{-1}\pi^{A'}\nabla_{AA'}H$$

Example 4.5. *In the case $n = 1$ the splitting reduces to finding a Taylor-Laurent splitting, that is can be done by writing $F = e^f \Rightarrow G = e^g$ then $g = \underline{h} - h$.*

There are many versions of this theorem that consider either bundles over S^4 or bundles with different gauge groups. To see all of these check out Ward & Wells §8.1. Here we will restate the final most useful one, after a few definitions:

Definition 4.6. *A* ***real form*** *on $E \to \hat{U}$ a holomorphic vector bundle is an antilinear isomorphism $\tau : E \to E^*$ s.t.*

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E^* \\ \downarrow & & \downarrow \\ \hat{U} & \xrightarrow{\sigma} & \hat{U} \end{array}$$

commutes and

$$\forall \xi \in E_Z,\, \eta \in E_{\sigma(Z)},\ \langle \xi, \tau\eta \rangle = \overline{\langle \eta, \tau\xi \rangle}$$

Lemma 4.7. *If $\forall x \in U$, $E|_{\hat{x}}$ is trivial then τ induces a non-degenerate Hermitian form τ_0 on the space of holomorphic section of $E|_{\hat{x}}$.*

Definition 4.8. *We say τ is **positive** if the induced form τ_0 is positive definite.*

Theorem 4.9. *Let $U \subset S^4$ be an elementary open set and $\hat{U} \subset \mathbb{PT}$ the corresponding open subset in twistor space. Then there is a bijection (up to equivalence) between*

1. *ASD $SU(n)$ gauge fields on U and*
2. *holomorphic rank-n vector bundles $E \to \hat{U}$ s.t.*
   (a) *$\forall x \in U$, $E|_{\hat{x}}$ is trivial*
   (b) *$\det E$ is trivial*
   (c) *E admits a positive real form.*

Remark. *If, given a holomorphic matrix valued function $K(Z)$ we define*

$$K^{\dagger}(Z) = K(\sigma(Z))^*$$

then the additional conditions in the above theorem may be restated as conditions on the transition function F that $\det F = 1$ and $F^{\dagger} = F$ (up to a similarity condition, that is $F = \underline{K}^{-1}\tilde{F}K$ for some unit-determinant holomorphic matrices, and $\tilde{F}$ satisfies these conditions. Wlog we may take $\underline{K} = 1$). It will be useful to know $\dagger : \zeta \mapsto -\zeta^{-1}$.

4.2 Extension Ansätze

Definition 4.10. *A rank-2 vector bundle $E \to M$ is an **extension** of a line bundle L_2 if $\exists L_1$ a line bundle with SES*

$$0 \to L_1 \to E \to L_2 \to 0$$

We want to be considering rank-2 vector bundles $E \to \hat{U}$ as in the previous subsection. If E is an extension then we know the transition function F on $W \cap \underline{W}$ is given by

$$F = \begin{pmatrix} \Xi_1 & \Gamma \\ 0 & \Xi_2 \end{pmatrix}$$

where Ξ_i is the transition function of L_i and Γ defines and element of $H^1(\hat{U}, O(L_1 \otimes L_2^{-1}))$. We will assume we have done a reduction of structure group to get E to be a $SL_2(\mathbb{C})$-bundle, so $\Xi_1 = \Xi_2^{-1}$.

Remark. *There is some homological algebra going into this result on the transition function. Firstly note that by tensoring with L_2^{-1} we only need to understand classifying extensions of the trivial line bundle O. We now need two results:*

Lemma 4.11 (See [18])**.** *Given R-modules A, B, equivalence classes of extensions of A by B are in 1-to-1 correspondence with $\mathrm{Ext}^1_R(A, B)$, where the identity element of Ext^1 corresponds to the split extension $A \oplus B$.*

Lemma 4.12. *Let F be a sheaf of R-modules on a ringed space (X, R). Then*

$$\forall i \geq 0,\ H^i(X, F) = \mathrm{Ext}^i_R(R, F)$$

Proof. The LHS and RHS are the right-derived functor of the same functor $\Gamma(X, \cdot) = \mathrm{Hom}_R(R, \cdot)$, the global section functor (Recall the sheaf cohomology is **not** the Čech cohomology, but they agree by a theorem). □

Now to any vector bundle $E \to X$ we can associated a sheaf of O_X-modules where O_X is the sheaf of holomorphic functions, and an extension of vector bundles corresponds to an extension of sheaves [2]. The trivial bundle then corresponds to the sheaf O_X and we have that extensions

$$0 \to L_1 \to E \to L_2 \to 0$$

are in 1-1 correspondence with elements $\Gamma \in H^1(\hat{U}, O(L_1 \otimes L_2^{-1}))$, and moreover Γ should be zero when $E = L_1 \oplus L_2$. For more background see [3] and [10]
Another more direct way to see this is that we take two sections on L_2, $s_W, s_{\underline{W}}$, and call the projection $\pi : E \to L_1$, then $\forall l \in L_2$ we must have that on $W \cap \underline{W}$

$$\pi(s_W(l) - s_{\underline{W}}(l)) = l - l = 0$$

as $\pi \circ s = \mathrm{id}$ for any section. By exactness this must mean that $s_W(l) - s_{\underline{W}}(l) \in L_1$, giving a function $\Gamma : L_2 \to L_1$. This map must satisfy the cocycle condition and so we equivalently get

$$\Gamma \in H^1(\hat{U}, \mathrm{End}(L_2, L_1)) \cong H^1(\hat{U}, L_1 \otimes L_2^{-1})$$

Taking the ansẗze that a holomorphic rank-2 vector bundle $E \to \hat{U}$ has the form of an extension of a line bundle will allow us to calculate the corresponding ASD $SL_2(\mathbb{C})$-gauge field explicitly, as given by the following theorem:

Theorem 4.13. *Let B_a be a potential for an ASD $GL_1(\mathbb{C})$-gauge field and let (for $k \geq 1$) $\{\Delta_r\}_{r=1-k}^{k-1}$ be a set of fields satisfying*

$$\begin{aligned} &\text{if } k > 1: \quad (\nabla_{A0'} + 2B_{A0'})\Delta_r = (\nabla_{A1'} + 2B_{A1'})\Delta_{r+1} \quad (1-k \leq r \leq k-2) \\ &\text{if } k = 1: \quad (\nabla_a + 2B_a)(\nabla^a + 2B^a)\Delta_0 = 0 \end{aligned} \tag{4.2.1}$$

Suppose that B_a, Δ_r are holomorphic on the region $U \subset M_{\mathbb{C}}$. Define the matrix $k \times k$ matrix M by $M_{rs} = \Delta_{r+s-k-1}$ (assumed invertible), let

$$E = (M^{-1})_{11},\ F = (M^{-1})_{1k} = (M^{-1})_{k1},\ G = (M^{-1})_{kk}$$

and define a gauge potential Φ by

$$\Phi_{A0'} = \frac{1}{2F}\begin{pmatrix} \hat{\partial}_{A0'}F & 0 \\ -2\hat{\partial}_{A1'}G & -\hat{\partial}_{A0'}F \end{pmatrix}, \qquad \Phi_{A1'} = \frac{1}{2F}\begin{pmatrix} -\hat{\partial}_{A1'}F & -2\hat{\partial}_{A0'}E \\ 0 & \hat{\partial}_{A1'}F \end{pmatrix}$$

where $\hat{\partial} = \nabla - 2B$. Then Φ is the potential for an ASD $SL_2(\mathbb{C})$-gauge field.

Proof. The idea will be to assume that we have a $SL_2(\mathbb{C})$-bundle coming from a holomorphic bundle E that is an extension, and then we will see that the conditions of the theorem are natural from the extension condition. First let $k = -c_1(L_1)$ and recalling that the line bundle on the sphere with transition function $z \mapsto z^n$ has Chern class $-n$ we know that

$$\Xi_1(Z) = \zeta^k \exp(f(Z))$$

where $\zeta = \pi_{0'}/\pi_{1'}$ is the homogeneous coordinate on the $\mathbb{P}^1$ at each line $\hat{x} \in \hat{U}$, and f is holomorphic in $W \cap \underline{W}$. If we denote g, Ω the restrictions of f, Γ to the line $\hat{x}$ then the splitting formula is rephrased as

$$\underbrace{\begin{pmatrix} \zeta^k e^g & \Omega \\ 0 & \zeta^{-k} e^{-g} \end{pmatrix}}_{G} = \underbrace{\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix}}_{\underline{H}} \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{H^{-1}}$$

s.t.

$$ad - bc = 1 = \underline{ad} - \underline{bc}$$

Remark. *Note here a, b, c, d ($\underline{a}, \underline{b}, \underline{c}, \underline{d}$) are functions of x, π holomorphic where ($|\zeta| \leq 1$ $|\zeta| \geq 1$).*

We can deduce from these first that

$$c = \underline{c} e^g \zeta^k, \qquad d = \underline{d} e^g \zeta^k$$

We can, as per example 4.5, split $g = \underline{h} - h$ and so

$$c e^h = \underline{c} e^{\underline{h}} \zeta^k, \qquad d e^h = \underline{d} e^{\underline{h}} \zeta^k$$

Through a holomorphicity argument we must have $k \geq 0$, and then we have

$$c = e^{-h} \sum_{j=0}^{k} c_j \zeta^j \qquad \underline{c} = e^{-\underline{h}} \sum_{j=0}^{k} c_j \zeta^{j-k}$$

$$d = e^{-h} \sum_{j=0}^{k} d_j \zeta^j \qquad \underline{d} = e^{-\underline{h}} \sum_{j=0}^{k} d_j \zeta^{j-k}$$

where he c_j, d_j are arbitrary functions of x. We can now get an equation for $a, \underline{a}$

$$e^{-\underline{h}} \underline{a} - \zeta^k e^{-h} a = e^{-\underline{h} - h} \Omega \sum_{j=0}^{k} c_j \zeta^j$$

Expanding as a Laurent series $e^{-\underline{h}-h}\Omega = \sum_{r \in \mathbb{Z}} \Delta_r \zeta^{-r}$, a solution for $a, \underline{a}$ is given by

$$a = -e^h \zeta^{-k} \sum_{r=1}^{\infty} \theta_r \zeta^r, \qquad \underline{a} = e^{\underline{h}} \sum_{r=-\infty}^{0} \theta_r \zeta^r$$

where $\theta_r = \sum_{j=0}^k c_j \Delta_{j-r}$. Moreover, this is unique when $k > 0$, and so we will further assume that for now. Similarly we find

$$b = -e^h \zeta^{-k} \sum_{r=1}^{\infty} \phi_r \zeta^r, \qquad \underline{b} = e^{\underline{h}} \sum_{r=-\infty}^{0} \phi_r \zeta^r$$

where $\phi_r = \sum_{j=0}^k d_j \Delta_{j-r}$. In order to have that a, b are holomorphic we get the condition

$$\forall 1 \leq r < k, \ \theta_r = 0 = \phi_r \tag{4.2.2}$$

($\underline{a}, \underline{b}$ don't give additional conditions) and having $ad - bc = 1$ requires

$$c_0 \phi_k - d_0 \theta_k = 1 \tag{4.2.3}$$

(doesn't it imply more?). Given $\{\theta_r, \phi_r\}$ satisfying 4.2.2, any choice of solutions c_j, d_j to 4.2.3 gives a splitting, and the freedom corresponds (exactly) to the gauge freedom.
To look for solutions to 4.2.3, let

$$N = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \Delta_{-k} & \Delta_{-k+1} & \dots & \Delta_0 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{-1} & \Delta_0 & \dots & \Delta_{k-1} \end{pmatrix}$$

Writing $\boldsymbol{c} = (c_0, \dots, c_k)$, $\boldsymbol{d} = (d_0, \dots, d_k)$ we have

$$N\boldsymbol{c} = \begin{pmatrix} c_0 \\ \theta_k \\ \vdots \\ \theta_1 \end{pmatrix}, \quad N\boldsymbol{d} = \begin{pmatrix} d_0 \\ \phi_k \\ \vdots \\ \phi_1 \end{pmatrix},$$

Hence, assuming N is invertible we can get

$$\boldsymbol{c} = N^{-1} \begin{pmatrix} \alpha \\ \theta_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad N^{-1} \begin{pmatrix} \beta \\ \phi_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

from any solutions α, β to $\alpha\phi_k - \beta\theta_k = 1$, easily found and corresponding to $N\boldsymbol{c}, N\boldsymbol{d}$ being linearly independent. Conversely, if N is not invertible, the first row must be a linear combination of the other k, so we must have

$$c_0 = \sum_{i=1}^k v_i \theta_i = v_k \theta_k, \quad d_0 = \sum_{i=1}^k v_i \phi_i = v_k \phi_k$$

for some v_i. Hence there is no linearly indep combination. Hence it is a necessary and sufficient condition for the splitting under the ansẗz that N be invertible, which is equivalent to M being

invertible as defined in the theorem.
To finish off the connection with the theorem, first note that if we have a splitting of g the corresponding gauge field on the $GL_1(\mathbb{C})$ bundle is given by

$$\pi^{A'}B_{AA'} = \left[e^h\right]^{-1}\pi^{A'}\nabla_{AA'}\left[e^h\right] = \pi^{A'}\nabla_{AA'}h = \pi^{A'}\nabla_{AA'}\underline{h}$$

We then take a particular gauge (the **R-gauge**) introduced in [19] where $c_k = 0 = d_0$ and $c_0 = d_k$. In the R-gauge the solution for $\boldsymbol{c}, \boldsymbol{d}$ is

$$\forall 1 \le r \le k, \quad d_r = F^{-1/2}(M^{-1})_{1r}, \quad c_{r-1} = F^{-1/2}(M^{-1})_{kr}$$

We then find

$$H_0 \equiv H(\zeta = 0) = \begin{pmatrix} -e^h\theta_k & -e^h\phi_k \\ c_0e^{-h} & 0 \end{pmatrix}$$

We can calculate

$$\phi_k = \sum_{j=1}^{k} d_j\Delta_{j-k} = \sum_{j=1}^{k} F^{-1/2}(M^{-1})_{1j}M_{j1} = F^{-1/2}$$

(no such simple form for θ_k exists in this gauge), and the unit-det conditions forces $c_0 = \phi_k^{-1} = F^{1/2}$, so

$$\begin{aligned}
\Phi_{A0'} &= H_0^{-1}\nabla_{A0'}H_0 \\
&= \begin{pmatrix} 0 & e^hF^{-1/2} \\ -c_0e^{-h} & -e^h\theta_k \end{pmatrix}\begin{pmatrix} -[B_{A0'}\theta_k + \nabla_{A0'}\theta_k]e^h & -[B_{A0'}F^{-1/2} + \nabla_{A0'}F^{-1/2}]e^h \\ [-B_{A0'}c_0 + \nabla_{A0'}c_0]e^{-h} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}F^{-1}\hat{\partial}_{A0'}F & 0 \\ P & -\frac{1}{2}F^{-1}\hat{\partial}_{A0'}F \end{pmatrix}
\end{aligned}$$

where $P = 2c_0\theta_kB_{A0'} + c_0\nabla_{A0'}\theta_k - \theta_k\nabla_{A0'}c_0$, and we have used $\nabla_{A0'}F^{1/2} = \frac{1}{2}F^{-1}\nabla_{A0'}F$. Likewise using $\underline{H}_\infty = \underline{H}(\zeta = \infty)$ we can show

$$\Phi_{A1'} = \begin{pmatrix} -\frac{1}{2}F^{-1}\hat{\partial}_{A1'}F & Q \\ 0 & \frac{1}{2}F^{-1}\hat{\partial}_{A1'}F \end{pmatrix}$$

If remains to show that if 4.2.1 hold, then

$$P = -F^{-1}\hat{\partial}_{A1'}G, \qquad\qquad Q = -F^{-1}\hat{\partial}_{A0'}E$$

□

Remark. *When $k = 0$ a splitting can also be found, but it will not lead to an $SU(2)$ gauge potential, and so we will disregard it. In the remaining cases of the previous theorem, it will be possible to construct corresponding $SU(2)$ fields.*

Remark. *The Δ_r should be viewed as massless fields of helicity $k-1$ couples to the Maxwell field* $2B$

Remark. *Under the extension ansätze, as F is upper triangular, it is not possible to have $F^\dagger = F$ unless F is diagonal. We hence need to have a non-identity K s.t. $\tilde{F} = FK$ satisfies $\tilde{F}^\dagger = \tilde{F}$.*

Example 4.14. *Let $k > 0$, suppose f is a real holomorphic function, and*

$$\Gamma = Q^{-1}\left[e^f + (-1)^k e^{-f}\right]$$

where $Q = (\pi_{0'}\pi_{1'})^{-k}P$ and P is a homogeneous polynomial in Z of degree $2k$ satisfying $P^\dagger = (-1)^k P$. Then we get a real gauge field under the isomorphism coming from the extension taking

$$K = \begin{pmatrix} 0 & -1 \\ 1 & \zeta^k Q \end{pmatrix} \Rightarrow \tilde{F} = \begin{pmatrix} \Gamma & (-\zeta)^k e^{-f} \\ \zeta^{-k} e^{-f} & Q e^{-f} \end{pmatrix}$$

4.3 Monopoles from ASD gauge fields

We may now use our previous results to recall that under a reduction, a time invariant Yang-Mills gauge field gives us a solution to the monopole equations. We will consider only specialisations of example 4.14, and in order to get a gauge field that is x^0-independent we require that f, Q depends on ω only through the combination

$$\gamma = \frac{i\omega^1}{\pi_{1'}} - \frac{i\omega^0}{\pi_{0'}}$$

Example 4.15 (Hedgehog solution)**.** *Take $k = 1$ and have $f = \gamma/\sqrt{2} = Q$, here giving*

$$\Gamma = 2f^{-1}\sinh f$$

On the line $\hat{x}$ we have

$$\begin{aligned} g &= \frac{-1}{\sqrt{2}}\left[\frac{x^{10'}\pi_{0'} + x^{11'}\pi_{1'}}{\pi_{1'}} - \frac{x^{00'}\pi_{0'} + x^{01'}\pi_{1'}}{\pi_{0'}}\right] \\ &= \frac{1}{\sqrt{2}}\left[-x^{10'}\zeta + \left(x^{00'} - x^{11'}\right) + x^{01'}\zeta^{-1}\right] \\ &= -\frac{1}{2}(x^1 + ix^2)\zeta + x^3 + \frac{1}{2}(x^1 - ix^2)\zeta^{-1} \end{aligned}$$

To get $h, \underline{h}$ here we will just spot the splitting

$$h = \frac{1}{2}\left[(x^1 + ix^2)\zeta - x^3\right], \qquad \underline{h} = \frac{1}{2}\left[(x^1 - ix^2)\zeta^{-1} + x^3\right]$$

This means we can write

$$e^{-h-\underline{h}}\Omega = \frac{e^{-2h} - e^{-2\underline{h}}}{g} = \frac{2\left\{e^{-\left[(x^1+ix^2)\zeta - x^3\right]} - e^{-\left[(x^1-ix^2)\zeta^{-1}+x^3\right]}\right\}}{-(x^1 + ix^2)\zeta + 2x^3 + (x^1 - ix^2)\zeta^{-1}}$$

To extract Δ_0 we will apply the operator $\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{d\zeta}{\zeta}$ to the part that is holomorphic on $|\zeta| \le 1$, likewise to the part holomorphic on $|\zeta| \ge 1$, and sum their contributions. We define $\zeta_\pm$ by

$$-(x^1 + ix^2)\zeta^2 + 2x^3\zeta + (x^1 - ix^2) = -(x^1 + ix^2)(\zeta - \zeta_+)(\zeta - \zeta_-)$$

that is

$$\zeta_\pm = \frac{1}{2}\left\{\frac{2x^3}{x^1+ix^2} \pm \sqrt{\frac{(2x^3)^2}{(x^1+ix^2)^2} + \frac{4(x^1-ix^2)}{(x^1+ix^2)}}\right\} = \frac{1}{x^1+ix^2}(x^3 \pm r)$$

where $r = \sqrt{\sum_{i=1}^3 (x^i)^2}$. *We can check*

$$|\zeta_\pm|^2 = 1 + \frac{2x^3(x^3 \pm r)}{(x^1)^2+(x^2)^2}$$

and so the two roots are separated by the line $|\zeta| = 1$ *(except on the plane* $x^3 = 0$*). Wlog we will take* $x^3 > 0$ *so* $|\zeta_-| < 1$. *It will be necessary to note* $\zeta_\pm^{-1} = -\frac{x^3 \mp r}{x^1 - ix^2}$. *Then*

$$\frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{2\left\{e^{-\left[(x^1+ix^2)\zeta - x^3\right]}\right\}}{-(x^1+ix^2)\zeta + 2x^3 + (x^1-ix^2)\zeta^{-1}}\frac{d\zeta}{\zeta} = \frac{2e^{-\left[(x^1+ix^2)\zeta_- - x^3\right]}}{-(x^1+ix^2)(\zeta_- - \zeta_+)} = \frac{e^r}{r}$$

$$\frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{2\left\{e^{-\left[(x^1-ix^2)\zeta + x^3\right]}\right\}}{-(x^1+ix^2)\zeta^{-1} + 2x^3 + (x^1-ix^2)\zeta}\frac{d\zeta}{\zeta} = \frac{2e^{-\left[(x^1-ix^2)\zeta_+^{-1} + x^3\right]}}{(x^1-ix^2)(\zeta_+^{-1} - \zeta_-^{-1})} = \frac{e^{-r}}{r}$$

so

$$\Delta_0 = 2r^{-1}\sinh r$$

From this we can (do this) reconstruct the hedgehog solution found by Prasad & Sommerfeld of

$$\Phi = ir^{-2}(r\coth r - 1)(x^1\sigma_1 + x^2\sigma_2 - x^3\sigma_3)$$

5 Constructions

5.1 Nahm's Construction

Relevant reading for this section is Manton & Sutcliffe [13] and Hitchin [12]. The original paper is [15].

Nahm established an equivalence between non-singular charge-n $SU(2)$-monopoles and $\{T_i(s)\,|\,T_i \in M_n(\mathbb{C}),\ s \in [0,2]\}$ subject to

1. Nahm's equation:

$$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}\,[T_j, T_k]$$

2. $T_i(s)$ is regular for $s \in (0,2)$ and has simple poles at $s = 0, 2$ with residues that form an irreducible, n-dimensional rep of $\mathfrak{su}(2)$

3. $T_i(s) = -T_i^\dagger(s)$, $T_i(s) = T_i^T(2-s)$.

Remark. *We could have here included another matrix $T_4 = -T_4^\dagger$ by modifying Nahm's equations to*

$$\dot{T}_i = [T_4, T_i] + \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j(z), T_k(z)]$$

but this can always be gauged away by the transform

$$\begin{aligned} T_i &\mapsto uT_iu^{-1} \\ T_4 &\mapsto uT_0u^{-1} - \frac{du}{ds}u^{-1} \end{aligned}$$

for $u : (0,2) \to U(n)$ satisfying $u(2-s) = \left(u^T(s)\right)^{-1}$.

5.1.1 Donaldson Rational Maps

Here we comment on [8]. We start with a result linking to the integrability of Nahm's equations.

Proposition 5.1. *Nahm's equation is equivalent to the Lax equation*

$$\left[\frac{d}{ds} + M, L\right] = 0$$

where

$$\begin{aligned} L(\zeta) &= (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2 \\ M(\zeta) &= -iT_3 + (T_1 - iT_2)\zeta \end{aligned}$$

Proof. It will be convenient to intoduce the notation

$$\begin{aligned} \alpha &= iT_3 \\ \beta &= T_1 + iT_2 \end{aligned}$$

s.t. this Lax pair can be rewritten as

$$\begin{aligned} L(\zeta) &= \beta - (\alpha + \alpha^\dagger)\zeta - \beta^\dagger\zeta^2 \\ M(\zeta) &= -\alpha - \beta^\dagger\zeta \end{aligned}$$

Then the Lax equation says, splitting by order in ζ

$$\frac{d\beta}{ds} - [\alpha, \beta] = 0 \qquad -\frac{d}{ds}(\alpha + \alpha^\dagger) + \left[\alpha, \alpha^\dagger\right] + \left[\beta, \beta^\dagger\right] = 0 \qquad -\frac{d\beta^\dagger}{ds} + \left[\beta^\dagger, \alpha^\dagger\right] = 0$$

and the last equation is redundant leaving just

$$\begin{aligned} \frac{d\beta}{ds} - [\alpha, \beta] &= 0 \quad \text{(the \textbf{complex equation})} \\ F(\alpha, \beta) \equiv \frac{d}{ds}(\alpha + \alpha^\dagger) - \left(\left[\alpha, \alpha^\dagger\right] + \left[\beta, \beta^\dagger\right]\right) &= 0 \quad \text{(the \textbf{real equation})} \end{aligned}$$

To show these are equivalent to Nahm's equations, it now just a matter of explicitly writing these out. ☐

Remark. *These definitions are slightly different to Donaldson where he uses* $\alpha = \frac{i}{2}T_1$, $\beta = \frac{1}{2}(T_2 + iT_3)$. *Hence the real and complex equation are slightly different. To go from those in [8] send* $\alpha \mapsto -\frac{1}{2}\alpha$, $\beta \mapsto \frac{1}{2}\beta$.

We will now want to understand the real and complex equation a little better. First start with the complex equation: we introduce the following operators on complex vector functions f (i.e. $f(s) \in \mathbb{C}^n$) and complex matrices γ (i.e. $\gamma(s) \in M_n(\mathbb{C})$)

$$\begin{aligned} \overline{d}_\alpha f &= \frac{df}{ds} - \alpha f, & \overline{d}_\beta f &= \beta f \\ \overline{d}_\alpha \gamma &= \frac{d\gamma}{ds} - [\alpha, \gamma], & \overline{d}_\beta \gamma &= [\beta, \gamma] \end{aligned}$$

We will define their conjugates analogously with $d_\alpha = \frac{d}{ds} + \alpha^\dagger$, $d_\beta = -\beta^\dagger$.

Remark. *Note how these sort of look like covariant derivatives associated to a connection*

Lemma 5.2. *The complex equation can be written as* $\left[\overline{d}_\alpha, \overline{d}_\beta\right] = 0$, *and the real equation as* $F(\alpha, \beta) = -(\left[d_\alpha, \overline{d}_\alpha\right] + \left[d_\beta, \overline{d}_\beta\right])$.

Proof. Pick an f then

$$\left[\overline{d}_\alpha, \overline{d}_\beta\right] f = \left(\frac{d(\beta f)}{ds} - \alpha(\beta f)\right) - \left(\beta \frac{df}{ds} - \beta(\alpha f)\right) = \left(\frac{d\beta}{ds} - [\alpha, \beta]\right) f$$

giving the complex equation. To further get the real equation note

$$\begin{aligned} -\left[d_\alpha, \overline{d}_\alpha\right] f &= \left[\frac{d}{ds}\left(\frac{df}{ds} + \alpha^\dagger f\right) - \alpha\left(\frac{df}{ds} + \alpha^\dagger f\right)\right] - \left[\frac{d}{ds}\left(\frac{df}{ds} - \alpha f\right) + \alpha^\dagger\left(\frac{df}{ds} - \alpha f\right)\right] \\ &= \left(\frac{d}{ds}(\alpha + \alpha^\dagger) - \left[\alpha, \alpha^\dagger\right]\right) f \\ -\left[d_\beta, \overline{d}_\beta\right] f &= \beta(-\beta^\dagger f) + \beta^\dagger(\beta f) = -\left[\beta, \beta^\dagger\right] f \end{aligned}$$

□

Lemma 5.3. *The complex equation is invariant under the complex gauge group* $GL_n(\mathbb{C})$ *as a fibre over* $(0, 2)$ *acting as*

$$\begin{aligned} \alpha &\mapsto \alpha' = g(\alpha) = g\alpha g^{-1} + \frac{dg}{ds} g^{-1} \\ \beta &\mapsto \beta' = g(\beta) = g\beta g^{-1} \end{aligned}$$

Proof. We have

$$\begin{aligned} \frac{d\beta'}{ds} &= \frac{dg}{ds}\beta g^{-1} + g\frac{d\beta}{ds} g^{-1} - g\beta g^{-1}\frac{dg}{ds} g^{-1} \\ [\alpha', \beta'] &= \left(g\alpha\beta g^{-1} + \frac{dg}{ds}\beta g^{-1}\right) - \left(g\beta\alpha g^{-1} + g\beta g^{-1}\frac{dg}{ds} g^{-1}\right) \\ \Rightarrow \frac{d\beta'}{ds} - [\alpha', \beta'] &= g\left(\frac{d\beta}{ds} - [\alpha, \beta]\right) g^{-1} \end{aligned}$$

Alternatively, and somewhat more simply, note $\overline{d}_{g(\alpha)} = g \circ \overline{d}_\alpha \circ g^{-1}$, $\overline{d}_{g(\beta)} = g \circ \overline{d}_\beta \circ g^{-1}$ and so gauge invariance is clear from the previous lemma. □

Notation. *We will often write* $(\alpha', \beta') = g(\alpha, \beta)$.

Proposition 5.4. *The general local solution to the complex equation is*

$$\alpha = -g^{-1}\frac{dg}{ds}$$
$$\beta = g^{-1}\beta' g$$

for some constant matrix β'.

To go from a local soluion to a global solution we will need to handle the poles at $s = 0, 2$. We now want to deal with the real equation. As Nahm's equations have a $U(n)$-gauge invariance as seem previously, it will be more natural to consider a our transforms in the homogeneous space $H = GL_n(\mathbb{C})/U(n)$. To extract the corresponding coset for a $g : (0, 2) \to GL_n(\mathbb{C})$ we take

$$h = h(g) = g^\dagger g : (0, 2) \to H$$

Lemma 5.5. *Under the gauge transformation* g *we have*

$$g^{-1} \circ d_{g(\alpha)} \circ g = d_\alpha + (h^{-1}d_\alpha h) \qquad\qquad g^{-1} \circ d_{g(\beta)} \circ g = d_\beta + (h^{-1}d_\beta h)$$

Proof. As before take f so

$$\begin{aligned}
(g^{-1} \circ d_{g(\alpha)} \circ g)f &= g^{-1}\left[g\frac{df}{ds} + \frac{dg}{ds}f + \left(g\alpha g^{-1} + \frac{dg}{ds}g^{-1}\right)^\dagger gf\right] \\
&= \frac{df}{ds} + h^{-1}\alpha^\dagger hf + g^{-1}\frac{dg}{ds}f + h^{-1}\frac{dg^\dagger}{ds}gf \\
&= \frac{df}{ds} + h^{-1}\left(\left[\alpha^\dagger, h\right] + h\alpha^\dagger + \frac{dh}{ds}\right) f \\
&= \left[d_\alpha + (h^{-1}d_\alpha h)\right] f \\
\text{and} \quad (g^{-1} \circ d_{g(\beta)} \circ g)f &= -g^{-1}(g\beta g^{-1})^\dagger gf \\
&= -h^{-1}\beta^\dagger hf \\
&= -h^{-1}\left(\left[\beta^\dagger, h\right] + h\beta^\dagger\right) f \\
&= \left[d_\beta + (h^{-1}d_\beta h)\right] f
\end{aligned}$$

□

Corollary 5.6. $g^{-1}F(\alpha', \beta')g = F(\alpha, \beta) + \left[\overline{d}_\alpha(h^{-1}d_\alpha h) + \overline{d}_\beta(h^{-1}d_\beta h)\right]$.

Proof. Using how we know the operators $d_\alpha, \ldots$ transform we can say

$$\begin{aligned}
g^{-1}F(\alpha', \beta')g &= -(\left[g^{-1}d_{\alpha'}g, g^{-1}\overline{d}_{\alpha'}g\right] + \left[g^{-1}d_{\beta'}g, g^{-1}\overline{d}_{\beta'}g\right]) \\
&= -(\left[d_\alpha + (h^{-1}d_\alpha h), \overline{d}_\alpha\right] + \left[d_\beta + (h^{-1}d_\beta h), \overline{d}_\beta\right]) \\
&= F(\alpha, \beta) - \left(\left[h^{-1}d_\alpha h, \overline{d}_\alpha\right] + \left[h^{-1}d_\beta h, \overline{d}_\beta\right]\right)
\end{aligned}$$

Now $\overline{d}_\alpha$, $\overline{d}_\beta$ satisfy the Leibniz rule □

Now to study the real equation further we can to give it a variational description:

Proposition 5.7. *Fix α, β and for each $0 < \epsilon \ll 1$ define the functional*

$$L_\epsilon(g) = \frac{1}{2} \int_\epsilon^{2-\epsilon} \left| g(\alpha) + g(\alpha)^\dagger \right|^2 + 2 \left| g(\beta) \right|^2 \, ds$$

where the norm on $GL_n(\mathbb{C})$ is coming from the trace inner product $\langle X, Y \rangle = \mathrm{Tr}(XY^\dagger)$. The Euler-Lagrange equations of these functionals are $F(\alpha', \beta') = 0$.

Proof. Through an appropriate choice of α, β, we need only consider variations about $g = I$. Moreover note that $U(n)$ variations leave the integrand invariant, so wlog we can consider self-adjoint variations $g \mapsto g + \delta g, (\delta g)^\dagger = \delta g$. We can see

$$\delta\alpha = [\delta g, \alpha] + \frac{d(\delta g)}{ds}, \quad \delta\beta = [\delta g, \beta]$$

and that for this inner product ad-invariance is expressed as

$$\langle [X, Y], Z \rangle - \left\langle Y, \left[X^\dagger, Z\right] \right\rangle = 0$$

so

$$\begin{aligned}
\delta L &= \frac{1}{2} \int \left[\left\langle \alpha + \alpha^\dagger, \delta(\alpha + \alpha^\dagger) \right\rangle + \left\langle \delta(\alpha + \alpha^\dagger), \alpha + \alpha^\dagger \right\rangle + 2 \left(\langle \beta, \delta\beta \rangle + \langle \delta\beta, \beta \rangle \right) \right] ds \\
&= \mathrm{Re} \int \left[\left\langle \alpha + \alpha^\dagger, \delta(\alpha + \alpha^\dagger) \right\rangle + 2 \langle \beta, \delta\beta \rangle \right] ds \\
&= \mathrm{Re} \int \left[\left\langle \alpha + \alpha^\dagger, \left[\delta g, \alpha - \alpha^\dagger\right] + 2 \frac{d(\delta g)}{ds} \right\rangle + 2 \langle \beta, [\delta g, \beta] \rangle \right] ds \\
&= \mathrm{Re} \int \left\langle -2 \frac{d}{ds}(\alpha + \alpha^\dagger) - \left[\alpha^\dagger - \alpha, \alpha + \alpha^\dagger\right] - 2 \left[\beta^\dagger, \beta\right], \delta g \right\rangle ds \\
&= \int -2 \langle F(\alpha, \beta), \delta g \rangle \, ds
\end{aligned}$$

□

Now we recall on $(\epsilon, 2 - \epsilon)$ we can trivialise the complex equation to write $(\alpha', \beta') = g(0, \beta)$ for some constant β. We want to know what conditions the real equation imposes on this solution, and this is given through the following proposition:

Proposition 5.8. *Let $(\alpha', \beta') = g(0, \beta)$, β constant, be a solution to the real equation. Then*

$$\frac{d}{ds} \left(h^{-1} \frac{dh}{ds} \right) - \left[\beta, h^{-1} \beta^\dagger h\right] = 0$$

Proof. We have from 5.6 that

$$\begin{aligned}
g^{-1} F(\alpha', \beta') g &= F(0, \beta) + \left[\overline{d}_0(h^{-1} d_0 h) + \overline{d}_\beta(h^{-1} d_\beta h)\right] \\
&= -\left[\beta, \beta^\dagger\right] + \frac{d}{ds}\left(h^{-1} \frac{dh}{ds}\right) + \left[\beta, h^{-1}\left[-\beta^\dagger, h\right]\right] \\
&= \frac{d}{ds}\left(h^{-1} \frac{dh}{ds}\right) - \left[\beta, h^{-1} \beta^\dagger h\right]
\end{aligned}$$

and we know the LHS is 0 as (α', β') is a solution of the real equation.
An alternative way to see this result is to identify that when we set $\alpha = 0$, β constant the integrand of L_ϵ is

$$\begin{aligned}
\mathcal{L}_\epsilon &= \left| \frac{dg}{ds} g^{-1} + (g^\dagger)^{-1} \frac{dg^\dagger}{ds} \right|^2 + 2\left| g\beta g^{-1} \right|^2 \\
&= \left| g \left(h^{-1} \frac{dh}{ds} \right) g^{-1} \right|^2 + 2\left| g\beta g^{-1} \right|^2 \\
&= \mathrm{Tr}\left[\left(h^{-1} \frac{dh}{ds} \right)^2 \right] + 2\,\mathrm{Tr}(\beta h^{-1} \beta^\dagger h)
\end{aligned}$$

We now make a variation $h \mapsto h + \delta h$. This gives $\delta(h^{-1}) = -h^{-1}(\delta h)h^{-1}$ and so

$$\begin{aligned}
\delta\left(h^{-1} \frac{dh}{ds} \right) &= h^{-1} \frac{d(\delta h)}{ds} - h^{-1}(\delta h) h^{-1} \frac{dh}{ds} \\
\delta(\beta h^{-1} \beta^\dagger h) &= -\beta h^{-1} (\delta h) h^{-1} \beta^\dagger h + \beta h^{-1} \beta^\dagger (\delta h)
\end{aligned}$$

Hence

$$\begin{aligned}
\delta\mathcal{L}_\epsilon = 2&\left\{ \mathrm{Tr}\left[\left(h^{-1}\frac{dh}{ds} \right) \left(h^{-1} \frac{d(\delta h)}{ds} - h^{-1}(\delta h) h^{-1} \frac{dh}{ds} \right) \right] + \mathrm{Tr}\left[-\beta h^{-1}(\delta h) h^{-1} \beta^\dagger h + \beta h^{-1} \beta^\dagger (\delta h) \right] \right\} \\
&= 2\,\mathrm{Tr}\left\{ \left[-\frac{d}{ds}\left(h^{-1} \frac{dh}{ds} h^{-1} \right) - \left(h^{-1} \frac{dh}{ds} \right)^2 h^{-1} - h^{-1} \beta^\dagger h \beta h^{-1} + \beta h^{-1} \beta^\dagger \right] (\delta h) \right\} \\
&= 2\,\mathrm{Tr}\left\{ \left[-\frac{d}{ds}\left(h^{-1} \frac{dh}{ds} \right) - \left(h^{-1} \frac{dh}{ds} \right) \frac{d(h^{-1})}{ds} h - \left(h^{-1} \frac{dh}{ds} \right)^2 - h^{-1} \beta^\dagger h \beta + \beta h^{-1} \beta^\dagger h \right] h^{-1}(\delta h) \right\} \\
&= 2\,\mathrm{Tr}\left\{ \left[-\frac{d}{ds}\left(h^{-1} \frac{dh}{ds} \right) + \left[\beta, h^{-1} \beta^\dagger h \right] \right] h^{-1}(\delta h) \right\}
\end{aligned}$$

The result follows. $\square$

5.2 The ADHM construction

This section follows the work first laid out in [4]. Suppose we have the following information:

- W a k-dimensional vector space
- V a $2k+2$-dimensional vector space with skew, non-degenerate bilinear form $(\cdot,\cdot) : \wedge^2 V \to \mathbb{C}$.
- $z = (z_i) \in \mathbb{C}^4$
- $A(z) = \sum_i A_i z_i \in \mathrm{End}(W, V)$ s.t.

$$\forall z \neq 0,\ U_z \equiv A(z)W \subset V \text{ is isotropic and } k\text{-dimensional}$$

We now state some important properties:

Lemma 5.9. *Let* $E_z = {}^{U_z^0}\!/_{U_z}$*, then*

- $\dim E_z = 2$
- E_z *inherits a non-degenerate skew bilinear*
- $\forall \lambda \in \mathbb{C}^\times$, $E_z = E_{\lambda z}$.

Proof. We go point by point:

- $\dim E_z = \dim U_z^0 - \dim U_z = (\dim V - \dim U_z) - \dim U_z = 2k + 2 - 2k = 2.$
- The bilinear on W is only degenerate in U_z^0 on U_z, so by quotienting by this it descends directly to E_z.
- $A(\lambda z) = \lambda A(z)$, so $A(\lambda z)(\lambda^{-1}\boldsymbol{w}) = A(z)(\boldsymbol{w})$. Hence we can see $U_{\lambda z} = U_z$ and so result.

□

Corollary 5.10. *We get a vector bundle* $E \to \mathbb{CP}^3$ *with group* $SL(2, \mathbb{C})$.

A break to introduce Bernd's notes

5.2.1 Connection from Projection

Given an rank-n vector bundle $E \to M$ a subbundle of $\mathbb{R}^{n+k} \times M$. We can decompose $E_x + E_x^\perp = \mathbb{R}^{n+k}$ and define projectors P, Q onto $E, E^\perp$ respectively. Then

$$E = \left\{(x, v) \in M \times \mathbb{R}^{n+k} \mid P_x(v) = v\right\}$$

and sections are maps $x \mapsto (x, f(x))$ s,t $P_x f(x) = f(x)$. We define a connection on E via

$$Df = Pdf$$

Now we want to pick a gauge via $u_x : \mathbb{R}^n \to \mathbb{R}^{n+k}$, i.e $u^\dagger u = I$, and so we can write $P = uu^\dagger$. Then if we write our section as $f = ug$ for $g : U \subset M \to \mathbb{R}^n$. Then

$$Df = uu^\dagger \left[(du)g + u(dg)\right] = u\left[dg + u^\dagger(du)g\right]$$

and we can read off that the connection must be $A = u^\dagger du$. and so

$$D = du^\dagger \wedge du + u^\dagger du \wedge u^\dagger du$$

Now in terms of the other projection, we have the following results:

Lemma 5.11. $dQ = Q(dQ) + (dQ)Q$ *and* $(dQ)f = -Q(df)$.

If we now define $B = QdQ$ we get for $f : M \to \mathbb{R}^{n+k}$

$$D_B f = df + Bf = df + (dQ)f - (dQ)Qf$$

On sections of E we have $Qf = 0$ and so $D_B f = Df$. As we can calculate $F_b = dQ \wedge dQ$ we must get

$$F = P(dQ) \wedge (dQ)P$$

Choose a gauge fo $E^\perp$ $v : \mathbb{R}^k \to \mathbb{R}^{n+k}$ with $u^\dagger v = 0$. Letting $\rho = v^\dagger v$ we get

$$Q = v\rho^{-2}v^\dagger$$

giving

$$F = \cdots = P(dv)\rho^{-2} \wedge (dv^\dagger)P$$

5.2.2 Quaternionic Bundles

We now take $M = S^4 = \mathbb{HP}^1$ with matrix multiplication form the left but scalar $\mathbb{H}$ action on the right. Our plan will be to try find good v s.t. we can get F (anti)-self dual, and $\overline{F} = -F$ to ensure F is $\mathfrak{su}(2)$-valued.
The ADHM idea is

1. take $v(x, y) = Cx + Dy$ for $x, y \in \mathbb{H}$, $C, D \in \mathbb{H}^{(k+n)k}$.
2. Assume maximal rank for $(x, y) \neq 0$. Then image of v is a k-dimensional subspace of $\mathbb{H}^{n+k}$ depending on xy^{-1}.
3. Project to $\mathbb{R}^4 \subset S^4$ where $y \neq 0$ and take affine coordinate $(x, 1)$
4. assume $\rho^2 = (\overline{x}C^\dagger + D^\dagger)(Cx + d)$ is real for $x \in \mathbb{H}$.

We now write L for the tautological quaternionic line bundle over $\mathbb{HP}^1$ with $c_2 = -1$. Take

$$E^\perp = \underbrace{L \oplus \cdots \oplus L}_{\times k}$$

Then since $E \oplus E^\perp$ is trivial and Chern number adds we have $c_2(E) = k$. Thus taking $n = 1$,

$$F = PC(dx)\rho^{-2} \wedge (d\overline{x})C^\dagger P$$

is an $SU(2)$-instanton with charge k.

Remark. *We needed $n = 1$ as we actually get an $Sp(n)$ bundle*

Now we can pick a gauge where

$$v(x) = \begin{pmatrix} \Lambda \\ B - xI_k \end{pmatrix}$$

with $n \times k$ quaternionic matrix Λ and $k \times k$ constant quaternionic matrix B. We have to solve the non-linear condition

$$\overline{\Lambda^\dagger\Lambda + B^\dagger B} = \Lambda^\dagger\Lambda + B^\dagger B$$

for symmetric B. We then solve the linear, x-dependent matrix equation $v^\dagger u = 0$ for u to find an explicit formula for A, F.

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