Symmetries, Fields, and Particles Revision Notes

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1 Introduction

A brief overview of some key ideas, concepts, and facts that I find useful in revising SFP.

2 Lie Groups

2.1 Basics

Definition 2.1 (Lie Group)**.** *A **Lie group** is a group that has a smooth manifold structure such that the group operations are smooth functions on the manifold.*

Idea. *Lie groups are introduced here to give a foundation to the idea of a continuous symmetry group, which will act in some way. It is a consequence of the Peter-Weyl theorem that every compact Lie groups is isomorphic to a subgroup of $GL(n, \mathbb{C})$ for some n so often one can restrict their thought process to matrix Lie groups.*

Idea. *When showing that something is a structure is a Lie group, typically the hardest step is showing that it is a manifold. One approach is to find an explicit parametrisation of the manifold, often a good approach if the group has very obvious parameters upon which it depends (e.g. the Heisenberg group). Alternatively, one can use the preimage theorem, useful if the group is defined by some constraints (e.g. the orthogonal group).*

Theorem 2.2 (Closed Subgroup Theorem)**.** *If G is a Lie group and $H \leq G$ is closed, then H is an embedded Lie group with the subspace topology.*

3 Algebras

Throughout we will take k to be a field.

Definition 3.1. *An **algebra** is a triple (A, m, i) of*

- *a k-vector space A*
- *a linear map $m : A \otimes A \to A$*
- *an element $i : k \to A$*

satisfying associativity and unitality.

Notation. *For $a, b \in A$ we will denote $m(a, b) = a \cdot b$.*

Remark. *Linearity of m gives distributivity of the multiplication over k.*

Proposition 3.2. *If a unit exists for $(A, \cdot)$, it is unique*

Proof. Let $1, 1' \in A$ be the units. Then

$$1 = 1 \cdot 1' = 1'$$

$\square$

Example 3.3. *Some examples of algebras are*

- *The base field* k
- *polynomials over* k*,* $k[X]$*.*
- $\mathrm{End}(V)$ *where* V *is a vector space, with multiplication given by composition*

Example 3.4. *The* ***free algebra*** $k\langle x_1, \ldots, x_n \rangle$ *is the vector space consisting formally of all possible combinations of the* x_i *in order to make it a vector space, namely*

$$k\langle x_1, \ldots, x_n \rangle = \bigoplus_{m=0}^{\infty} k \cdot \prod_{1 \leq j_i \leq n} x_{j_1} \cdots x_{j_m}$$

Example 3.5. *Given a group* G *we have the* ***group algebra*** $A \equiv kG$ *with*

- *basis* $\{x_g \,|\, g \in G\}$
- *multiplication* $x_g \cdot x_h = x_{gh}$
- *unit* x_{e_G}

Definition 3.6. $(A, \cdot)$ *is* ***commutative*** *if* $\forall a, b \in A$*,* $a \cdot b = b \cdot a$*.*

Example 3.7. kG *is abelian iff* G *is abelian.*

Definition 3.8. *A homomorphism of algebras* $f : A \to B$ *is a linear map of vector spaces compatible with* $\cdot$ *s.t.*

- $\forall a, b \in A,\ f(a \cdot b) = f(a) \cdot f(b)$
- *(*$f(1_A) = 1_B$*)*

Definition 3.9. *A* ***derivation*** *is a map* $\partial : A \to A$ *satisfying* $\forall a, b, \in A,\ \partial(ab) = a\partial(b) + \partial(a)b$

4 Lie Algebras

4.1 Basics

Definition 4.1 (Lie Algebra)**.** *A* ***Lie algebra*** *is a vector space, equipped with a bracket that:*

- *is bilinear*
- *is antisymmetric*
- *obeys the Jacobi identity*

Idea. *Given a matrix Lie group G, it has a natural associated Lie algebra $\mathfrak{g} = T_I G$ with bracket given by the matrix commutator. This will be the prototypical Lie algebra to think of. The Lie algebra corresponding to a Lie group is useful to work with as we can take a basis, which will commute under addition, rather than work with a difficult presentation of the group. Using the exponential map, we can map back into the original Lie group. This is not generally injective or surjective, but map injectively onto the connected component containing the identity.*

Fact 4.2. *The dimension of the tangent space to a manifold is equal to the dimension of the manifold. Hence, a Lie group and its associated Lie algebra have the same dimension*

Definition 4.3 (Structure Constants)**.** *Given a basis $\{ T^a \}$ of a Lie algebra, the **structure constants** f^{ab}_c are defined such that*

$$\left[T^a, T^b\right] = f^{ab}_c T^c$$

Definition 4.4 (Complexification and Real Form)**.** *Given a real Lie algebra $\mathfrak{g}_{\mathbb{R}} \equiv \mathrm{Span}_{\mathbb{R}} \{ T^a : a = 1, \ldots, D \}$ the **complexification** is $\mathfrak{g}_{\mathbb{C}} \equiv \mathrm{Span}_{\mathbb{C}} \{ T^a : a = 1, \ldots, D \}$. Conversely, given such a $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{R}}$ is called a **real form**.*

4.2 Ideals and Simplicity

Definition 4.5 (ideal)**.** *An **ideal** is a subalgebra $\mathfrak{h} \leq \mathfrak{g}$ s.t.*

$$\forall X \in \mathfrak{g},\ \forall Y \in \mathfrak{h} \quad [X, Y] \in \mathfrak{h}$$

Definition 4.6 (Derived Algebra)**.** *The **derived algebra** of a Lie algebra $\mathfrak{g}$ is*

$$i(\mathfrak{g}) = \mathrm{Span}\{ [X, Y] : X, Y \in \mathfrak{g} \}$$

Note this will sometimes be notated as $[\mathfrak{g}, \mathfrak{g}]$.

Definition 4.7 (Centre)**.** *The **centre** of a Lie algebra $\mathfrak{g}$ is*

$$\zeta(\mathfrak{g}) = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g}\ [X, Y] = 0 \}$$

Definition 4.8 (Simple Lie Algebra)**.** *A Lie Algebra is **simple** if it non-abelian and contains no non-trivial ideals.*

Idea. *An ideal is to a Lie algebra what a normal subgroup is to a Lie group. Hence simple Lie algebras are the analogy of simple Lie groups.*

Definition 4.9 (Semi-Simple Lie Algebra)**.** *A Lie Algebra is **semi-simple** if it contains no non-trivial abelian ideals.*

4.3 Adjoint Map

Definition 4.10 (Adjoint Map)**.** *The **adjoint map** of an element $X \in \mathfrak{g}$ is $ad_X : \mathfrak{g} \to \mathfrak{g}$ defined such that for $Y \in \mathfrak{g}$*

$$ad_X(Y) = [X, Y]$$

Lemma 4.11.

$$[ad_X, ad_Y] = ad_{[X,Y]}$$

Proof. Act on an element $Z \in \mathfrak{g}$ and apply the Jacobi identity. $\square$

Definition 4.12 (ad-diagonalisable)**.** *An element $X \in \mathfrak{g}$ is* ***ad-diagonalisable*** *if the linear map ad_X is diagonalisable.*

Definition 4.13 (Killing Form)**.** *The* ***Killing form*** *is a map $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined such that for $X, Y \in \mathfrak{g}$*

$$\kappa (X, Y) = \mathrm{Tr} \left(ad_X \circ ad_Y \right)$$

In components

$$\begin{aligned} \kappa(X, Y) &= \kappa^{ab} X_a Y_b \\ \kappa^{ab} &= f^{ac}_d f^{bd}_c \end{aligned}$$

It is symmetric and bilinear

Theorem 4.14. *Let $\mathfrak{g}$ be a Lie algebra and κ the Killing form. Then*

$$\forall X, Y, Z \in \mathfrak{g} \quad \kappa \left([Z, X] , Y \right) + \kappa \left(X, [Z, Y] \right) = 0$$

Proof. Expand out using $ad_{[X,Y]} = [ad_X, ad_Y]$. $\square$

Theorem 4.15 (Cartan)**.** *Let $\mathfrak{g}$ be a Lie algebra with Killing form κ. Then*

$$\kappa \text{ non-degenerate} \Leftrightarrow \mathfrak{g} \text{ semi-simple}$$

Proof. The direction non-degenerate $\Rightarrow$ semi-simple is as follows. Assume $\mathfrak{g}$ not semi-simple. Let $\left\{T^a\right\}$ be a basis of a abelian ideal of $\mathfrak{g}$, and extend with $\left\{T^i\right\}$ to a basis of the full algebra. Then

$$\begin{aligned} \left[T^a, T^b\right] = 0 &\Rightarrow f^{ab}_\alpha = 0 \\ \left[T^a, T^\alpha\right] \in \mathrm{Span}\left\{T^b\right\} &\Rightarrow f^{a\alpha}_i = 0 \end{aligned}$$

so

$$\begin{aligned} \kappa^{ab} &= f^{a\alpha}_\beta f^{b\beta}_\alpha \\ &= f^{ai}_\beta f^{b\beta}_i \\ &= f^{ai}_j f^{bj}_i = 0 \end{aligned}$$

so κ degenerate. $\square$

4.4 Cartan Subalgebras

Definition 4.16 (Cartan Subalgebra)**.** *A* ***Cartan subalgebra*** *$\mathfrak{h} \leq \mathfrak{g}$ is a maximal abelian subalgebra containing only ad-diagonalisable elements. It is typically written as $\mathfrak{h} = \mathrm{Span}\left\{H^i\right\}$. Write* $\dim \mathfrak{h} = \mathrm{rank}\, \mathfrak{g} = r$

Fact 4.17. *All Cartan subalgebras of a given Lie algebra have the same dimension. Though Cartan subgalgebras are not unique, they are all conjugate.*

Idea. *It is a fact that Cartan subalgebras correspond to maximal toral subgroups of the corresponding Lie group. This gives a route to visualisation of what the subalgebra might be. A Cartan subalgebra is defined such that representations of the basis* $\{ H^i \}$ *are simultaneously diagonalisable.*

Definition 4.18 (κ^{ij})**.** *Given a Cartan subalgebra* $\mathfrak{h} = \mathrm{Span}\{ H^i \}$ *and Killing form* κ *the notation will be used* κ^{ij} *for the matrix elements of* $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$*,*

$$\kappa^{ij} = \kappa(H^i, H^j)$$

Definition 4.19 (Compact Type)**.** *A real Lie algebra* $\mathfrak{g}_{\mathbb{R}}$ *is of* ***compact type*** *if* $\exists$ *a basis in which* $\kappa^{ij} = -K\delta^{ij}$ *for some* $K \in \mathbb{R}^+$*.*

Fact 4.20. *Every complex semi-simple Lie algebra of finite dimension has a real form of compact type.*

5 Representations

5.1 Basics

Definition 5.1 (Representations)**.** *A* ***representation*** *of a Lie group* G *is a smooth group homomorphism*

$$D : G \to GL(V)$$

where V *is a* n*-dimensional vector space called the* ***representation space****. In analogy a representation of a Lie algebra is a Lie algebra homomorphism*

$$d : \mathfrak{g} \to \mathfrak{gl}(V)$$

Definition 5.2 (Dimension)**.** *The* ***dimension*** *of a representation is the dimension of the representation space.*

Definition 5.3 (Trivial Representation)**.** *The* ***trivial representation*** *is the unique one dimensional representation that sends every element to the identity.*

Definition 5.4 (Fundamental Representation)**.** *For a matrix Lie group* $G \leq GL(V)$ *the* ***fundamental representation*** *is the identity map* $R = id_{GL(V)}$*. Note* $dim(R) = dim(V)$*.*

Definition 5.5 (Adjoint Representation)**.** *Given a Lie algebra* $\mathfrak{g}$ *the* ***adjoint representation*** *is* $d_{adj} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$*,* $d_{adj}(X) = ad_X$*. Note* $dim(d_{adj}) = dim(\mathfrak{g})$

Definition 5.6 (Faithful Representation)**.** *A representation is* ***faithful*** *if it is injective.*

Definition 5.7 (Isomorphic Representations)**.** *Two representations* R_1*,* R_2 *of a Lie algebra* $\mathfrak{g}$ *are* ***isomorphic****, written* $R_1 \cong R_2$ *if* $\exists S$ *an invertible matrix such that*

$$\forall X \in \mathfrak{g} \quad R_2(X) = SR_1(X)S^{-1}$$

5.2 Constructing Representations

Theorem 5.8. *A representation D of a Lie group G induces a representation on the the corresponding Lie algebra $\mathfrak{g}$ as such: For $X \in \mathfrak{g}$ let $g : (-\epsilon, \epsilon) \to G$ be a curve in G defined such that $g(0) = e$ the identity of G and $g'(0) = X$. This curve necessarily exists for some $\epsilon > 0$. Then define d by*

$$d(X) \equiv \frac{d}{dt} D\left(g(t)\right)|_{t=0}$$

Conversely, a representation of a Lie algebra $\mathfrak{g}$ induces a representation in some neighbourhood of e in G given by

$$D(\exp X) \equiv \exp d(X)$$

Definition 5.9 (Direct Sum Representation)**.** *Given representations R_1, R_2 with representation spaces V_1, V_2 the* ***direct sum representation*** *is $R_1 \oplus R_2$ with representation space $V_1 \oplus V_2$ acting by*

$$(R_1 \oplus R_2)(X)(v_1 \oplus v_2) = (R_1(X)v_1) \oplus (R_2(X)v_2)$$

Definition 5.10 (Tensor Product Representation)**.** *Given representations R_1, R_2 with representation spaces V_1, V_2 the* ***tensor product representation*** *is $R_1 \otimes R_2$ with representation space $V_1 \otimes V_2$ with*

$$(R_1 \otimes R_2)(X) = R_1(X) \otimes I_{V_2} + I_{V_1} \otimes R_2(X)$$

Idea. *Considering a representation as a visualisation of the action of the algebra induced by a group on a representation space, then $R_1 \oplus R_2$ can be viewed as two non-interacting systems, whereas $R_1 \otimes R_2$ can be viewed as two entangled systems.*

5.3 Reducibility

Definition 5.11 (Invariant Subspace)**.** *Given a representation R of Lie algebra $\mathfrak{g}$ with representation space V, an* ***invariant subspace*** *is $U \leq V$ such that*

$$\forall X \in \mathfrak{g},\ \forall u \in U \quad R\left(X\right) u \in U$$

Definition 5.12 (Irreducible Representation)**.** *A representation is* ***irreducible*** *if it has no invariant subspaces.*

Definition 5.13 (Full Reducible)**.** *A representation is* ***fully reducible*** *if it can be expressed as the direct sum of irreducible representations.*

Fact 5.14. *If R_i are finite dimensional irreducible representations of a simple Lie algebra for $i = 1, \ldots, m$ then*

$$R_1 \otimes R_2 \otimes \cdots \otimes R_m$$

is fully reducible.

5.4 Weights

Definition 5.15 (Weights)**.** *Given a representation R of a Lie algebra with Cartan subalgebra $\mathfrak{h} = \operatorname{Span}\{ H^i \}$ the **weights** of the representation are the eigenvalues $S_R = \{ \lambda : \exists v \in V \; R(H^i)v = \lambda^i v \}$*

Fact 5.16. *Weights can be viewed as maps in $\mathfrak{h}^*$ given by*

$$\lambda : \mathfrak{h} \to \mathbb{C}$$
$$\lambda : e_i H^i \mapsto e_i \lambda^i$$

Thus

$$R(H)v = \lambda(H)v$$

Idea. *This idea is made more explicit in the concept of roots, which are the weights of the adjoint representation.*

Fact 5.17. *Given weights set S_R*

$$V = \bigoplus_{\lambda \in S_R} V_\lambda$$

where $V_\lambda = \{ v \in V : R(H^i)v = \lambda^i v \}$

6 Useful Facts

6.1 Miscellaneous

Theorem 6.1. *If X is a complex square matrix then*

$$\det e^X = e^{\operatorname{Tr} X}$$

Theorem 6.2 (Baker-Campbell-Hausdorff Formula)**.** *Let $\mathfrak{g}$ be a Lie algebra, and $X, Y \in \mathfrak{g}$. Then under sufficient existence conditions $e^X e^Y = e^Z$ where*

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \dots$$

Corollary 6.3 (Lie Product Formula)**.** *For $X, Y \in \mathfrak{g}$, a Lie algebra*

$$e^{X+Y} = \lim_{N \to \infty} \left(e^{\frac{X}{N}} e^{\frac{Y}{N}} \right)^N$$

Fact 6.4. *Rotation matrices corresponding to a rotation θ degrees about an axis n have the form*

$$R_{ij} = \cos\theta \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta \epsilon_{ijk} n_k$$

6.2 Pauli Matrices

Definition 6.5 (Pauli Matrices)**.** *The **Pauli matrices** are*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note they are all Hermitian and traceless.

Fact 6.6.

$$\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k \tag{6.2.1}$$

7 SU(2)

7.1 Definition and Basic Properties

Definition 7.1 (U(n))**.** *The **unitary group** of dimension n is*

$$U(n) = \{ U \in GL(n, \mathbb{C}) : U^\dagger U = I \}$$

It is a path connected, compact, Lie group.

Definition 7.2 (SU(n))**.** *The **special unitary group** of dimension n is*

$$SU(n) = \{ U \in U(n) : \det U = 1 \}$$

Through the det homomorphism, it can be seen $SU(n)$ is a closed subgroup of $U(n)$, hence also a Lie group. The corresponding Lie algebra is

$$\mathfrak{su}(n) = \{ Z \in GL(n, \mathbb{C}) : Z^\dagger + Z = 0, \operatorname{Tr} Z = 0 \}$$

The real dimension is

$$\underbrace{2 \times \frac{1}{2}(n-1)n}_{\text{off diagonal elements}} + \underbrace{n}_{\text{diagonal elements}} - \underbrace{1}_{\text{trace constraint}} = n^2 - 1$$

In the case $n = 2$ we have

$$SU(2) = \{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \}$$

This can be expressed as, for $A \in SU(2)$

$$A = a_0 I + i\boldsymbol{a} \cdot \boldsymbol{\sigma}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, and $a_0^2 + |\boldsymbol{a}|^2 = 1$. Hence $SU(2) \cong S^3$. In addition, by parametrising $SU(2)$ by the a_i, it can be seen that $\{ i\sigma_i \}$ forms a basis of $\mathfrak{su}(2)$. It is typical to normalise this basis to $\{ T^a = -\frac{1}{2} i\sigma_a \}$.

Fact 7.3. *The structure constants in this basis* $\{ T^a \}$ *are*

$$f_c^{ab} = \epsilon_{abc}$$

Fact 7.4. *The function*

$$\begin{aligned} R &: SU(2) \to SO(3) \\ R(A)_{ij} &= \frac{1}{2} \operatorname{Tr} \left(\sigma_i A \sigma_j A^\dagger \right) \end{aligned}$$

Is a double cover of $SO(3)$*, with* $R(A) = R(-A)$*, and inverse given by*

$$A = \pm \frac{I + \sigma_i R(A)_{ij} \sigma_j}{2\sqrt{1 + \operatorname{Tr} R(A)}}$$

Hence $SO(3) \cong {}^{SU(2)}\!/_{\mathbb{Z}_2}$

7.2 Representations

Definition 7.5 (Cartan-Weyl basis of $\mathfrak{su}_{\mathbb{C}}(2)$)**.** $\{ H, E_\pm \}$*, where* $H = \sigma_3$ *and* $E_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$*, is the* ***Cartan-Weyl basis*** *for* $\mathfrak{su}_{\mathbb{C}}(2)$*. The elements satisfy the commutation relations*

$$\begin{aligned} [H, E_\pm] &= \pm 2 E_\pm \\ [E_+, E_-] &= H \end{aligned}$$

Fact 7.6. H *spans a Cartan subalgebra of* $\mathfrak{su}_{\mathbb{C}}(2)$*, ad-diagonalisable w.r.t the Cartan Weyl basis, with eigenvalues* $0, \pm 2$ *respectively.*

Theorem 7.7. *Finite dimensional irreducible representations of* $\mathfrak{su}_{\mathbb{C}}(2)$ *with* $R(H)$ *diagonalisable are determined by the highest weight* $\Lambda \in \mathbb{N}_0$*. Such a representation is called* R_Λ*. It has weight set*

$$S_\Lambda = \{ \Lambda - 2n : n = 0, \ldots, \Lambda \}$$

The eigenvectors of $R_\Lambda(H)$ *are* $v_{\Lambda - 2n}$ *defined such that*

$$\begin{aligned} R_\Lambda(H) v_\lambda &= \lambda v_\lambda \\ R_\Lambda(E_+) v_\Lambda &= 0 \\ \left(R_\Lambda(E_-)\right)^n v_\Lambda &= v_{\Lambda - 2n} \\ R_\Lambda(E_+) v_{\Lambda - 2n} &= r_n v_{\Lambda - 2n + 2} \end{aligned}$$

with

$$r_n = (\Lambda + 1 - n) n$$

Note that $\dim(R_\Lambda) = \Lambda + 1$, so

- R_0 is the trivial representation d_0.
- R_1 is the fundamental representation.
- R_2 is the adjoint representation.

8 Cartan Classification

Élie Cartan classified finite dimensional, simple, complex Lie algebras in 1894. This sections will therefore restrict to treatment of such Lie algebras. Note that $\mathfrak{su}_{\mathbb{C}}(2)$ will be a prototypical such Lie algebra.

8.1 Cartan-Weyl Basis

Definition 8.1 (Cartan-Weyl Basis)**.** *Given a Lie algebra $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h} = \mathrm{Span}\{ H^i : i = 1, \ldots, r \}$, let $\{ E^\alpha : \alpha \in \Phi \}$ be the simultaneous eigenvectors of ad_{H^i} s.t.*

$$ad_{H^i}(E^\alpha) = \alpha^i E^\alpha$$

Then

$$B = \{ H^i : i = 1, \ldots, r \} \cup \{ E^\alpha : \alpha \in \Phi \}$$

*is the **Cartan-Weyl basis** for $\mathfrak{g}$.*

Definition 8.2 (Roots)**.** *The **roots** of a Lie Algebra are $\alpha \in \Phi$. These are the weights corresponding to the adjoint representation. The roots can be seen as elements of the dual space to the Cartan subalgebra $\mathfrak{h}^*$ by*

$$\alpha : \mathfrak{h} \to \mathbb{R}$$
$$\alpha(H^i e_i) = \alpha^i e_i$$

Fact 8.3. *The roots are all non-degenerate i.e. the eigenspace is 1 dimensional.*

Theorem 8.4. *Let κ be the Killing form on $\mathfrak{g}$. Then,*

- $\forall H \in \mathfrak{h}, \forall \alpha \in \Phi,\ \kappa(H, E^\alpha) = 0$
- $\forall \alpha, \beta \in \Phi, \alpha + \beta \neq 0,\ \kappa\left(E^\alpha, E^\beta\right) = 0$
- $\forall H \in \mathfrak{h}, \exists H' \in \mathfrak{h},\ \kappa(H, H') \neq 0$
- $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$ *and* $\kappa(E^\alpha, E^{-\alpha}) \neq 0$

Hence $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, so invertible.

Fact 8.5. *Given $\alpha \in \Phi$, $\{ \lambda : \lambda\alpha \in \Phi \} = \{ \pm 1 \}$. Note one direction of this inclusion follows from the previous result.*

Definition 8.6 (Dual Space Inner Product)**.** *The Killing form gives the dual space $\mathfrak{h}^*$ a natural inner product $(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ defined by*

$$(\alpha, \beta) = \left(\kappa^{-1}\right)_{ij} \alpha^i \beta^j = k^{ij} \alpha_i \beta_j$$

Defining $\alpha_i = \left(\kappa^{-1}\right)_{ij} \alpha^j$. Note that it is immediate that the bracket is symmetric and bilinear. Positive definiteness follows from the fact

$$(\alpha, \beta) = \sum_{\delta \in \Phi} (\alpha, \delta)(\beta, \delta)$$

(proven by noting the trace of an operator is the sum over its eigenvalues, so $k^{ij} = \sum_{\delta\in\Phi} \delta^i \delta^j$)

$$(\alpha, \alpha) = \sum_{\delta\in\Phi} (\alpha, \delta)^2$$

Note that the above definition shows that we actually have an inner product on the span of the roots. The following lemma closes that hole

Lemma 8.7.

$$\mathfrak{h}^* = \operatorname{Span}_{\mathbb{C}} \Phi$$

Hence r roots may be chosen to form a basis $\{\alpha_{(i)} : i = 1, \ldots, r\}$

Proof. Suppose not. Then $\exists \lambda \in \mathfrak{h}^*$ s.t.

$$\forall \alpha \in \Phi \quad (\lambda, \alpha) = 0$$

now let $H_\lambda = \lambda_i H^i$, where $\lambda^i = \lambda(H^i)$. Then

$$\begin{aligned} \forall H \in \mathfrak{h} &\quad [H_\lambda, H] = 0 \\ \forall \alpha \in \Phi &\quad [H_\lambda, E^\alpha] = (\lambda, \alpha) E^\alpha \\ \Rightarrow \forall X \in \mathfrak{g} &\quad [H_\lambda, X] = 0 \end{aligned}$$

Hence $\mathfrak{g}$ has the non-trivial ideal $\operatorname{Span}_{\mathbb{C}}\{H_\lambda\}$ and so $\mathfrak{g}$ is not simple. Contradiction. □

Definition 8.8 (Coroots)**.** *Given a root $\alpha \in \Phi$ the **coroot** is*

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)} \alpha$$

Note $(\alpha^\vee)^\vee = \alpha$

Definition 8.9 (H^α)**.** *Define*

$$H^\alpha = \frac{[E^\alpha, E^{-\alpha}]}{\kappa(E^\alpha, E^{-\alpha})}$$

In components

$$H^\alpha = \left(\kappa^{-1}\right)_{ij} \alpha^j H^i$$

Theorem 8.10 (Cartan-Weyl Basis Algebra)**.** *The algebra of the Cartan-Weyl basis is*

$$\begin{aligned} \left[H^i, H^j\right] &= 0 \\ \left[H^i, E^\alpha\right] &= 0 \\ \left[E^\alpha, E^\beta\right] &= \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta} & \alpha + \beta \in \Phi \\ \kappa(E^\alpha, E^{-\alpha}) H^\alpha & \alpha + \beta = 0 \\ 0 & \textit{otherwise} \end{cases} \end{aligned}$$

Fact 8.11. $\forall H \in \mathfrak{h}$, $\alpha, \beta \in \Phi$

- $\kappa\left(H^{\alpha}, H\right)=\alpha(H)$
- $\kappa\left(H^{\alpha}, E^{\beta}\right)=0$
- $\left[H^{\alpha}, H\right]=0$
- $\left[H^{\alpha}, E^{\beta}\right]=(\alpha, \beta) E^{\beta}$

Theorem 8.12. *Letting*

$$
\begin{aligned}
e^{\alpha} &= \sqrt{\frac{2}{(\alpha, \alpha) \kappa\left(E^{\alpha}, E^{-\alpha}\right)}} E^{\alpha} \\
h^{\alpha} &= \frac{2}{(\alpha, \alpha)} H^{\alpha}
\end{aligned}
$$

yields the algebra

$$
\begin{aligned}
\left[h^{\alpha}, h^{\beta}\right] &= 0 \\
\left[h^{\alpha}, e^{\beta}\right] &= \frac{2(\alpha, \beta)}{(\alpha, \alpha)} e^{\beta} \\
\left[e^{\alpha}, e^{\beta}\right] &= \begin{cases} n_{\alpha, \beta} e^{\alpha+\beta} & \alpha+\beta \in \Phi \\ h^{\alpha} & \alpha+\beta=0 \\ 0 & \textit{otherwise} \end{cases}
\end{aligned}
$$

Definition 8.13 ($sl(2)_{\alpha}$)**.** *For $\alpha \in \Phi$ there is a $\mathfrak{su}_{\mathbb{C}}(2)$ subalgebra*

$$
sl(2)_{\alpha}=\operatorname{Span}\left\{h^{\alpha}, e^{\pm \alpha}\right\}
$$

8.2 Root Geometry

Definition 8.14 (α-string through β)**.** *The α-**string through** β is*

$$
S_{\alpha, \beta}=\{\beta+\rho \alpha \in \Phi: \rho \in \mathbb{Z}\}
$$

The corresponding subspace of $\mathfrak{g}$ is

$$
V_{\alpha, \beta}=\left\{e^{\beta+\rho \alpha}: \beta+\rho \alpha \in S_{\alpha, \beta}\right\}
$$

*The **length** of the string is*

$$
l_{\alpha, \beta}=\left|S_{\alpha, \beta}\right|
$$

Fact 8.15. *$V_{\alpha, \beta}$ is a representation space of the adjoint representation $R=d_{adj}$ of $sl(2)_{\alpha}$. The non-degeneracy of the roots ensures that the weight set*

$$
\begin{aligned}
S_{R} &= \left\{\frac{2(\alpha, \beta)}{(\alpha, \alpha)}+2 \rho: \beta+\rho \alpha \in S_{\alpha, \beta}\right\} \\
&= \{-\Lambda,-\Lambda+2, \ldots, \Lambda\}=S_{\Lambda}
\end{aligned}
$$

Hence $R_{\alpha, \beta}=\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$

Fact 8.16. *Writing*

$$S_{\alpha,\beta} = \{ \beta + \rho\alpha \in \Phi : n_- \leq \rho \leq n_+ \}$$

with $n_+ \geq 0$ *and* $n_- \leq 0$ *yields*

- $l_{\alpha,\beta} = n_+ - n_- + 1$
- $R_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -(n_+ + n_-) \in \mathbb{Z}$

Definition 8.17 (Weyl Reflections and Group)**.** *For* $\alpha \in \Phi$*, the* ***Weyl reflection*** *in the hyperplane orthogonal to* α *is*

$$\begin{aligned} w_\alpha &: \mathfrak{h}^* \to \mathfrak{h}^* \\ w_\alpha(x) &= x - \frac{2(\alpha, x)}{(\alpha,\alpha)}\alpha \end{aligned}$$

Noting

- $(w_\alpha)^2 = id$
- $w_\alpha(\alpha) = -\alpha$
- $(\alpha, x) = 0 \Rightarrow w_\alpha(x) = x$
- $(w_\alpha(x), w_\alpha(y)) = (x, y)$

It can be seen w_α *is indeed a reflection in the hyperplane orthogonal to* α*, and hence is an isometry. The* ***Weyl group*** *is*

$$W = \{ w_\alpha : \alpha \in \Phi \}$$

with composition as the group operation. Using the previous fact it can be seen w_α *restricts to an isometry on* Φ*. Hence* W *acts via permutation on* Φ*, and so* $W \leq S_\Phi$

Lemma 8.18.

$$\forall \alpha, \beta \in \Phi \ (\alpha, \beta) \in \mathbb{R}$$

Proof. Write

$$\begin{aligned} \frac{(\alpha,\alpha)}{2} R_{\alpha,\beta} &= \sum_{\delta\in\Phi} \left[\frac{(\alpha,\alpha)}{2} R_{\alpha,\delta}\right] \left[\frac{(\beta,\beta)}{2} R_{\beta,\delta}\right] \\ \Rightarrow \frac{2}{(\beta,\beta)} R_{\alpha,\beta} &= \sum_{\delta\in\Phi} R_{\alpha,\delta} R_{\beta,\delta} \in \mathbb{R} \\ &\Rightarrow (\beta,\beta) \in \mathbb{R} \\ &\Rightarrow (\alpha,\beta) = \frac{(\alpha,\alpha)}{2} R_{\alpha,\beta} \in \mathbb{R} \end{aligned}$$

□

Definition 8.19 ($\mathfrak{h}^*_{\mathbb{R}}$)**.** *Given a basis of roots* $\{ \alpha_{(i)} : i = 1, \ldots, r \}$*,* $\mathfrak{h}^*_{\mathbb{R}}$ *is defined as*

$$\mathfrak{h}^*_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}} \{ \alpha_{(i)} : i = 1, \ldots, r \}$$

Fact 8.20. $\Phi \subset \mathfrak{h}_{\mathbb{R}}^*$

Theorem 8.21. $(\cdot,\cdot) : \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ *is a Euclidean inner product.*

Definition 8.22 (Length and angle between roots)**.** *Given $\alpha \in \Phi$ the **length** of α is defined as*

$$|\alpha| = (\alpha, \alpha)^{\frac{1}{2}}$$

*Given also $\beta \in \Phi$, the **angle** between α and β is ϕ defined such that*

$$(\alpha, \beta) = |\alpha||\beta| \cos \phi$$

Theorem 8.23.

$$\cos \phi = \pm \frac{\sqrt{n}}{2}$$

for $n \in \{ 0, 1, 2, 3, 4 \}$

Definition 8.24 (Positive roots)**.** *Given a hyperplane $H \leq \mathfrak{h}^*$ s.t. $H \cap \Phi = \emptyset$ and $\mathfrak{h}^*$ separated into $\mathfrak{h}_{\pm}^*$ let*

$$\Phi_{\pm} = \Phi \cap \mathfrak{h}_{\pm}^*$$

*Define the **positive roots** to be $\alpha \in \Phi_+$*

Fact 8.25. *Let $\Phi_{\pm}$ be as defined.*

- $\alpha \in \Phi_{\pm} \Rightarrow -\alpha \in \Phi_{\mp}$
- $\alpha, \beta \in \Phi_{\pm},\ \alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Phi_{\pm}$

Definition 8.26 (Simple Roots)**.** *A roots δ is **simple** if*

- $\delta \in \Phi_+$
- $\nexists \alpha, \beta \in \Phi_+ \ s.t.\ \delta = \alpha + \beta$

The set of simple roots is Φ_S

Theorem 8.27 (Properties of simple roots)**.** *Let $\alpha, \beta \in \Phi_S$, $\alpha \neq \beta$. Then*

- $\alpha - \beta \notin \Phi$
- $l_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$
- $(\alpha, \beta) \leq 0$
- $\Phi_+ \subset \mathrm{Span}_{\mathbb{N}_0} \Phi_S$
- *The simple roots are linearly independent.*
- $|\Phi_S| = r$

Hence $\Phi_S = \{ \alpha_{(i)} : i = 1, \ldots, r \}$ is a basis for $\mathfrak{h}_{\mathbb{R}}^*$

8.3 Cartan Matrix and Dynkin Diagrams

Definition 8.28 (Cartan Matrix)**.** *The **Cartan matrix** A is an $r \times r$ matrix with elements*

$$A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})}$$

Definition 8.29 (Chevalley Basis)**.** *For each $\alpha_{(i)} \in \Phi_S$ the basis of $sl(2)_{\alpha_{(i)}}$*

$$\left\{h^i = h^{\alpha_{(i)}}, e^i_\pm = e^{\pm\alpha_{(i)}}\right\}$$

*is the **Chevalley basis***

Theorem 8.30. *In the Chevalley basis*

- $\left[h^i, h^j\right] = 0$
- $\left[h^i, e^j_\pm\right] = \pm A^{ji} e^j_\pm$
- $\left[e^i_+, e^j_-\right] = \delta_{ij} h^i$

Theorem 8.31 (Serre Relation)**.** *The **Serre relation** is*

$$\left(ad_{e^i_\pm}\right)^{1-A^{ji}} e^j_\pm = 0$$

Idea. *The previous two results show that the Cartan Matrix completely determines a finite dimensional, simple, complex Lie algebra.*

Theorem 8.32 (Properties of the Cartan Matrix)**.** *The Cartan matrix satisfies*

- $\forall i\ A^{ii} = 2$
- $A^{ij} = 0 \Rightarrow A^{ji} = 0$
- *for $i \neq j$ $A^{ij} \in \mathbb{Z}_{\leq 0}$*
- $|A^{ij}| \leq 4$
- $\det A > 0$
- *A irreducible (for simple Lie algebras)*

Definition 8.33 (Dynkin Diagrams)**.** *The **Dynkin diagrams** are constructed from the Cartan matrix as follows:*

- *Draw a node for each simple root.*
- *Connect the i^{th} and j^{th} root with $\max\{\ |A^{ij}|, |A^{ji}|\}$ lines.*
- *If the roots have different length, draw an arrow from the longer root to the shorter.*

8.4 Lattices

Definition 8.34 (Root Lattice)**.** *Given a Lie algebra* $\mathfrak{g}$ *with simple roots* $\Phi_S = \{\, \alpha_{(i)} : i = 1, \ldots, r \,\}$ *the* ***root lattice*** *is*

$$\mathcal{L}[\mathfrak{g}] = \left\{ \sum_{i=1}^{r} m_i \alpha_{(i)} : m_i \in \mathbb{Z} \right\}$$

Definition 8.35 (Coroot Lattice)**.** *The* ***coroot lattice*** *is given by*

$$\mathcal{L}^{\vee}[\mathfrak{g}] = \left\{ \sum_{i=1}^{r} m_i \alpha^{\vee}_{(i)} : m_i \in \mathbb{Z} \right\}$$

Definition 8.36 (Quantisation Condition)**.** *Given a representation* R *of the* $sl(2)_\alpha$ *subalgbera, and* $v \in V_\lambda$ *for* λ *a weight of the representation,*

$$R(h^\alpha)v = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} v$$

Hence as the weights of $sl(2)_\alpha$ *are integers*

$$\frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z}$$

Definition 8.37 (Weight Lattice)**.** *The* ***weight lattice****,* $\mathcal{L}_W[\mathfrak{g}]$*, is the dual of the coroot lattice, defined as*

$$\mathcal{L}_W[\mathfrak{g}] = \left(\mathcal{L}^{\vee}[\mathfrak{g}] \right)^* = \left\{ \lambda \in \mathfrak{h}^*_{\mathbb{R}} : \forall \mu \in \mathcal{L}^{\vee}[\mathfrak{g}] \, (\lambda, \mu) \in \mathbb{Z} \right\}$$

Theorem 8.38. *Note*

$$(\lambda, \alpha^{\vee}_{(i)}) = \frac{2(\alpha_{(i)}, \lambda)}{(\alpha_{(i)}, \alpha_{(i)})} \in \mathbb{Z}$$

Hence all weights Lie in the weight lattice.

Definition 8.39 (Fundamental Weights)**.** *The dual to the coroot basis, i.e. the weights* $\omega_{(i)}$ *such that*

$$(\alpha^{\vee}_{(i)}, \omega_{(j)}) = \delta_{ij}$$

are the ***fundamental weights***

Fact 8.40. *Writing* $\omega_{(i)} = \sum_{j=1}^{r} B^{ij} \alpha_{(j)}$ *yields* $\delta_{ij} = A^{ki} B_{jk}$*. Hence*

$$\alpha_{(i)} = \sum_{j=1}^{r} A^{ij} \omega_{(j)}$$

where A^{ij} *is the Cartan matrix.*

Definition 8.41 (Highest Weight)**.** *Given a finite dimensional representation, the* ***highest weight*** *is* Λ *such that*

$$\forall v_\Lambda \in V_\Lambda, \forall \alpha \in \Phi_+ \quad R(E^\alpha)v_\Lambda = 0$$

Note that a highest weight must exist if the rep is f.d. If the representation is irreducible, all other weight are generated as product of $R(E^{-\alpha})$ *for* $\alpha \in \Phi_+$*.*

Theorem 8.42 (Theorem of the Highest Weight)**.** *Every finite dimension, irreducible, representation has a highest weight. Moreover, if two such representation have the same highest weight, they are isomorphic. Call the representation R_Λ.*

Fact 8.43. *Given a representation R, for $v \in V_\lambda$,*

$$\lambda + \alpha \in S_R \Rightarrow R(E^\alpha)v \in V_{\lambda+\alpha}$$

Hence if $R = R_\Lambda$ all weights are of the form

$$\lambda = \Lambda - \mu$$

for $\mu = \sum_{i=1}^r \mu_i \alpha_{(i)}$, $\mu_i \in \mathbb{N}_0$. Hence writing

$$\lambda = \sum_{i=1}^r \lambda^i \omega_{(i)}$$

gives that

$$\forall m_{(i)} \in \mathbb{Z},\ 0 \leq m_{(i)} \leq \lambda^i, \quad \lambda - m_{(i)}\alpha_{(i)} \in S_R$$

Lemma 8.44. *If $R_\Lambda, R_{\Lambda'}$ are two representations with weights λ, λ' respectively, then $\lambda + \lambda'$ is a weight of $R_\Lambda \otimes R_{\Lambda'}$*

9 Symmetries in QM

Consider a Hamiltonian $\hat{H}$, and the associated Hilbert space corresponding to the sum of the energy eigenspaces

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$
$$\mathcal{H}_n = \{|\psi\rangle : \hat{H}|\psi\rangle = E_n|\psi\rangle\}$$

Definition 9.1 (Symmetry Transformation)**.** *A* ***symmetry transformation*** *is a map*

$$|\psi\rangle \mapsto |\psi'\rangle = U|\psi\rangle$$

where U is a unitary transform s.t.

$$U\hat{H}U^\dagger = \hat{H}$$

Proposition 9.2. *A symmetry transformation*

- *preserves inner products : $\langle\psi_1'|\psi_2'\rangle = \langle\psi_1|\psi_2\rangle$*
- *preserves the energy eigenspaces : $|\psi\rangle \in \mathcal{H}_n \Leftrightarrow |\psi'\rangle \in \mathcal{H}_n$*

For a symmetry transform $U = e^{i\partial}$, where ∂ is a Hermitian operator,

$$\left[\partial, \hat{H}\right] = 0$$

and so ∂ corresponds to a conserved quantity.

Proof. The first two properties are immediate from the definition of a symmetry transformation. Now we seek to evaluate

$$f(s) = e^{is\partial} \hat{H} e^{-is\partial}$$

Note

$$f'(s) = e^{is\partial} i \left[\partial, \hat{H} \right] e^{-is\partial}$$
$$f(0) = \hat{H}$$

So in order to have $f(s) = \hat{H}$ we need $\left[\partial, \hat{H} \right] = 0$. $\square$

Given a Hamiltonian system it is natural to consider

$$\{ \partial^a : \left[\partial^a, \hat{H} \right] = 0 \}$$

the *maximal* set of conserved quantities, which leads to the lie algebra

$$\mathfrak{g}_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}} \{ i\partial^a \}$$

Fact 9.3. *The group $G = \exp \mathfrak{g}_{\mathbb{R}}$ will be unitary, so will have unitary representations. Hence the induced representations on $\mathfrak{g}_{\mathbb{R}}$ will be antihermitian.*

9.1 Gauge Theories

Example 9.4. *Consider the Lagrangian for scalar field ϕ*

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - W(\phi^* \phi) \, .$$

This exhibits the ***global symmetry***

$$\phi \to g\phi \, ,$$
$$\phi^* \to g^{-1} \phi^* \, ,$$

where $g \in U(1)$[1]*. This symmetry is global as it acts in the same way at all spacetime points.*

Definition 9.5 (Gauge Symmetry)**.** *A* ***gauge symmetry*** *is a symmetry arising from a non-phsyical redundancy whose action varies in spacetime. i.e. let G be a Lie group with representation D and representation space V*

$$g : \mathbb{R}^{1,3} \to G$$
$$g(x) = \exp\left(\epsilon X(x)\right)$$
$$X : \mathbb{R}^{1,3} \to L(G)$$

which acts on field

$$\phi : \mathbb{R}^{1,3} \to V$$

[1] We need $U(1)$ here so that $g^* = g^{-1}$

by

$$\phi(x) \to D(g(x))\phi(x)$$

Recall D induces a representation R on $L(G)$ by

$$D(g) = \exp(R(X))$$

We will take D to have dimension N, so $V \cong \mathbb{C}^N$, and take the standard inner product on V[2]

$$\forall u, v \in V, \ \ (u, v) = \boldsymbol{u}^\dagger \boldsymbol{v}$$

. We will also take D to be a unitary representation.

Idea. *By taking D to be a unitary representation we tie back in with the idea of symmetries in quantum mechanics, which is what we will be aiming for here. Explicitly, by requiring D to be a unitary representation gives that, under a gauge transform*

$$(\phi, \phi) \to (D\phi, D\phi) = (\phi, \phi)$$

so the norm of the state is conserved.

Example 9.6 ($U(1)$ gauge symmetry)**.** *A $U(1)$ gauge symmetry is a gauge symmetry that takes values in $U(1)$, i.e.*

$$\begin{aligned} g &: \mathbb{R}^{1,3} \to U(1) \\ g(x) &= \exp\left(\epsilon X(x)\right) \\ X &: \mathbb{R}^{1,3} \to L(U(1)) \end{aligned}$$

Under this transform, to linear order, a scalar field ϕ transforms as

$$\begin{aligned} \phi &\to \phi + \delta_X \phi \\ \delta_X \phi &= \epsilon X \phi \end{aligned}$$

and

$$\begin{aligned} \partial_\mu \phi &\to \partial_\mu \phi + \delta_X(\partial_\mu \phi) \\ \delta_X(\partial_\mu \phi) &= \partial_\mu(\delta_X \phi) = \epsilon(\partial_\mu X)\phi + \epsilon X(\partial_\mu \phi) \end{aligned}$$

Hence the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - W(\phi^* \phi)$$

does not have $U(1)$ gauge symmetry if $\partial_\mu X \neq 0$

Definition 9.7 (Covariant Derivative)**.** *A **covariant derivative** is*

$$D_\mu = \partial_\mu + R(A_\mu)$$

where

$$A_\mu : \mathbb{R}^{1,3} \to L(G)$$

*is a **gauge field** with defined variation*

$$\delta_X A_\mu = -\epsilon \partial_\mu X + \epsilon \left[X, A_\mu\right]$$

[2]Here $u, v \in V$ are vectors in our vector space and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^N$ are their images under some isomorphism.

Lemma 9.8.

$$\delta_X(D_\mu\phi) = \epsilon R(X) D_\mu \phi$$

Proof.

$$\begin{aligned}
\delta_X(D_\mu\phi) &= \delta_X(\partial_\mu\phi + R(A_\mu)\phi) \\
&= \partial_\mu(\delta_X\phi) + R(A_\mu)(\delta_X\phi) + R(\delta_X A_\mu)\phi \\
&= \partial_\mu(\epsilon R(X)\phi) + \epsilon R(A_\mu)R(X)\phi - \epsilon R(\partial_\mu X)\phi + \epsilon R([X, A_\mu])\phi \\
&= \epsilon R(\partial_\mu X)\phi + \epsilon R(X)\partial_\mu\phi + \epsilon R(X)R(A_\mu)\phi + \epsilon\left[R(A_\mu), R(X)\right]\phi - \epsilon R(\partial_\mu X)\phi + \epsilon\left[R(X), R(A_\mu)\right]\phi \\
&= \epsilon R(X)(\partial_\mu\phi + R(A_\mu)\phi)
\end{aligned}$$

□

Corollary 9.9. *The term*

$$(D_\mu\phi)^\dagger(D^\mu\phi)$$

is gauge invariant.

Idea. *The idea of the covariant derivative is to introduce a gauge field to cancel out the effect of the changing gauge, in order to make terms*

$$(D_\mu\phi, D^\mu\phi) = (D_\mu\phi)^\dagger(D^\mu\phi)$$

gauge invariant.

Definition 9.10 (Field Strength Tensor)**.** *Define the* ***field strength tensor***

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Proposition 9.11.

$$\delta_X(F_{\mu\nu}) = \epsilon\left[X, F_{\mu\nu}\right]$$

Proof.

$$\begin{aligned}
\delta_X(F_{\mu\nu}) =& \partial_\mu(\delta_X A_\nu) - \partial_\nu(\delta_X A_\mu) + [\delta_X A_\mu, A_\nu] + [A_\mu, \delta_X A_\nu] \\
=& \partial_\mu(-\epsilon\partial_\nu X + \epsilon\left[X, A_\nu\right]) - \partial_\nu(-\epsilon\partial_\mu X + \epsilon\left[X, A_\mu\right]) + [-\epsilon\partial_\mu X + \epsilon[X, A_\mu], A_\nu] \\
&+ [A_\mu, -\epsilon\partial_\nu X + \epsilon\left[X, A_\nu\right]] \\
=& \epsilon\left[\partial_\mu X, A_\nu\right] + \epsilon\left[X, \partial_\mu A_\nu\right] - \epsilon\left[\partial_\nu X, A_\mu\right] - \epsilon\left[X, \partial_\nu A_\mu\right] - \epsilon\left[\partial_\mu X, A_\nu\right] \\
&+ \epsilon[\left[X, A_\mu\right], A_\nu] - \epsilon\left[A_\mu, \partial_\nu X\right] + \epsilon\left[A_\mu, [X, A_\nu]\right] \\
&= \epsilon\left[X, \partial_\mu A_\nu - \partial_\nu A_\mu\right] - \epsilon\left[A_\nu, [X, A_\mu]\right] - \epsilon\left[A_\mu, [A_\nu, X]\right] \\
&= \epsilon\left[X, \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right]
\end{aligned}$$

□

Corollary 9.12. *Letting κ be the Killing form,*

$$\kappa(F_{\mu\nu}, F^{\mu\nu})$$

is gauge invariant.

Idea. *We introduced the gauge field to give us gauge invariant derivative terms. We now see that the field strength tensor allows us to construct gauge invariant terms from this gauge field, which can then be included in any Lagrangian for a theory. Note that for a simple, compact Lie group, $\kappa^{ab} = -\kappa\delta^{ab}$ and*

$$\kappa(F_{\mu\nu}, F^{\mu\nu}) = -\kappa \sum F_{\mu\nu}^{(a)} F^{\mu\nu(a)}$$

A general gauge invariant Lagrangian thus looks like

$$\mathcal{L} = \frac{1}{g^2}\kappa(F_{\mu\nu}, F^{\mu\nu}) + (D_\mu\phi, D^\mu\phi) - W((\phi,\phi))$$