Linearising Flows and a Cohomological Interpretation of Lax Equations - Unpacking the Paper

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1 Introduction

The purpose of this document is to facilitate the understanding of [1] by discussing the terms and how they fit into the wider picture of algebraic geometry.

2 The Paper

2.1 Laying out the Ingredients

Notation. *We start by laying out some notation that will be necessary for the following section. Let:*

- $P = \mathbb{CP}^1$ *with coordinates* $[\xi_0 : \xi_1]$*. We take* $\xi = \frac{\xi_1}{\xi_0}$.
- O_P *be the natural structure sheaf on the variety* P
- V *be a* m*-dimensional vector space,* $\mathcal{V} = V \otimes O_P$, $\mathcal{V}(k) = V \otimes \mathcal{O}_P(k)$ *where we view* V *as either the constant sheaf or trivial bundle over* P.
- $A(t,\xi) = \sum_{k=0}^n A_k(t)\xi^k \in H^0(P, \mathrm{Hom}(\mathcal{V}, \mathcal{V}(n)))$ *for some* n*, where we see* $A_i(t) \in \mathrm{End}(V)$ *as a time dependent* $m \times m$ *matrix and* $\xi^k \in H^0(P, \mathcal{O}(n))$ *as*

$$[\xi_0 : \xi_1]^k = \underbrace{\xi_0 \otimes \cdots \otimes \xi_0}_{\times(n-k)} \otimes \underbrace{\xi_1 \otimes \cdots \otimes \xi_1}_{\times k}$$

This is homogeneous of degree n*, so we allow* A *to not have a scale?*

- $B(\xi, t) \in H^0(P, \mathrm{Hom}(\mathcal{V}, \mathcal{V}(N)))$ *for some* N *likewise.*
- $Q(\xi, \eta) = \det[\eta I - A(\xi, t)]$ *be the characteristic polynomial of* A.
- σ *be the tautological section of* $\mathcal{O}_P(n)$.

Lemma 2.1. $Q(\xi, \sigma) \in H^0(\mathcal{O}_P(n), \pi^*\mathcal{O}_P(mn))$

Definition 2.2. *The pair* A, B *is a Lax pair if* $\dot{A} = [A, B]$.

Proposition 2.3. *The Lax equation is invariant under the substitution*

$$B \mapsto B + p(A, \xi)$$

for polynomial $p(x, \xi) \in \mathbb{C}[x, \xi]$.

Definition 2.4. *The* ***spectral curve*** *is* C *given by the solution in* P *of*

$$Q(\xi, \eta) = 0$$

Proposition 2.5. *The flow* $t \mapsto A(\xi, t)$ *is isospectral.*

It will be the understanding of this isospectral flow that we want to gain. We formulate this flow as the family of holomorphic map gained by the eigenvectors

$$f_t : C \to \mathbb{P}V \cong \mathbb{P}^{m-1}$$

Suppose that C has degree d, then we know we can define

$$L_t = f_t^* \mathcal{O}_{\mathbb{P}V}(1) \in \mathrm{Pic}^d(C)$$

Lets choose a reference bundle $L_0 = L \in \mathrm{Pic}^d(X)$

Lemma 2.6. *The map* $\mathrm{Pic}^d(C) \overset{\otimes L^{-1}}{\to} J(C)$ *is an isomorphism.*

Now knowing our result about the tangent space to the Picard group we can say $\dot{L} = \frac{dL_t}{dt} \in H^1(C, O_C)$.

2.2 The Eigenvector Mapping as a Deformation

Recall we have a 1-parameter family of maps

$$f_t : C \to \mathbb{P}V\,.$$

We want to interpret this a as a deformation of the map

$$f_0 : C \to \mathbb{P}V$$

and characterise it as such, so we need to develop a little theory.
Given a map

$$f : X \to Y$$

we think of a deformation as a 1-parameter family of maps

$$f_t : X_t \to Y$$

with $X_0 = X$, $f_0 = f$. This gives a point in the tangent space to the moduli space of such arrangements, which by deformation theory is $H^0(X, \mathcal{N}_{X/Y})$ where $\mathcal{N}$ is the normal bundle given by the SES

$$0 \to \mathcal{T}_X \to f^*\mathcal{T}_Y \to \mathcal{N} \to 0\,.$$

This short exact sequence gives rise to the segment of an LES

$$H^0(X, \mathcal{T}_X) \to H^0(X, f^*\mathcal{T}_Y) \to H^0(X, \mathcal{N}) \xrightarrow{\tilde{\delta}} H^1(X, \mathcal{T}_X)\,.$$

Again, from deformation theory, we know that $H^1(X, \mathcal{T}_X)$ is the tangent space to the moduli space of X, so if we wanted to look at deformations that kept X fixed, we would need the kernel of $\tilde{\delta}$. Hence the eigenvector mapping gives a cohomology class

$$\dot{f} \in {}^{H^0(C, f^*\mathcal{T}_{\mathbb{P}V})}\!\big/_{H^0(C, \mathcal{T}_C)} \subset H^0(C, \mathcal{N})\,.$$

2.3 Combining with the Euler Sequence

Recall we also have the sequence

$$0 \to O_{\mathbb{P}V} \to \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}V}(1) \to \mathcal{T}_{\mathbb{P}V} \to 0$$

which pulls back under f to give

$$0 \to O_C \xrightarrow{\nu} \mathcal{V} \otimes L \to f^*\mathcal{T}_{\mathbb{P}V} \to 0\,,$$

where

$$\nu : O_C \to \mathcal{V} \otimes L\,,$$
$$\phi \mapsto \phi\nu\,,$$

and ν is the vector defined s.t. for $z \in C$, $f_t(z) = \mathbb{C}\nu(z,t)$.

Definition 2.7. *We define $\dot{\nu}$ by*

$$\dot{\nu}(z) = \left.\frac{\partial \nu(z,t)}{\partial t}\right|_{t=0} \mod \nu$$

Lemma 2.8. *$\dot{\nu}$ is well defined.*

Proof. Suppose we had chosen a different representative $\tilde{\nu}$. Writing $\tilde{\nu} = \rho\nu$ we see

$$\dot{\tilde{\nu}} = \rho\dot{\nu} + \dot{\rho}\nu = \rho\dot{\nu} \mod \nu\,.$$

∎

Combined with the normal sheaf sequence this gives the cohomology diagram

$$\begin{tikzcd}
 & H^0(\mathcal{V}\otimes L) \arrow[d, "\tau"] & & \\
H^0(C,\mathcal{T}_C) \arrow[r] & H^0(C,f^*\mathcal{T}_{\mathbb{P}V}) \arrow[r, "j"] \arrow[d, "\delta"] & H^0(C,\mathcal{N}) \arrow[r, "\tilde{\delta}"] & H^1(C,\mathcal{T}_C) \\
 & H^1(C,O_C) & &
\end{tikzcd}$$

and we can interpret $\dot{\nu}$ as a cohomology class

$$\dot{\nu} \in H^0(C, {}^{\mathcal{V}\otimes L}\!/\!{}_{O_C}) = H^0(C, f^*\mathcal{T}_{\mathbb{P}V})$$

Proposition 2.9. *We have*

- $j(\dot{\nu}) = \dot{f}$,
- $\delta(\dot{\nu}) = \dot{L}$.

Corollary 2.10. $\dot{L} = 0 \Leftrightarrow \exists w \in H^0(C, \mathcal{V}\otimes L)$, $\dot{\nu} = \tau(w)$.

References

[1] Phillip A. Griffiths. Linearizing flows and a cohomological interpretation of lax equations. *American Journal of Mathematics*, 107(6):pp. 1445–1484, 1985. ISSN 00029327, 10806377. doi: 10.2307/2374412.