The Eisenhart Lift

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1 The Eisenhart Lift

1.1 The metric and equations of motion

Consider the $(d+2)$-dimensional line element,

$$ds^2 = \hat{g}_{\mu\nu}\, dx^\mu\, dx^\nu = h_{ij}\, dx^i\, dx^j + 2dt\left(dv - \Phi dt + N_i dx^i\right), \tag{1.1.1}$$

where $i, j = 1, \ldots, d$, $x^{d+1} = t$, $x^{d+2} = v$ and Φ, N_i and h_{ij} are independent of the coordinate v. Then $\xi = \partial_v$ is a Killing vector. We have

$$\hat{g} = \begin{pmatrix} h_{ij} & N_i & 0 \\ N_j & -2\Phi & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{g}^{-1} = \begin{pmatrix} h^{ij} & 0 & -h^{ik}N_k \\ 0 & 0 & 1 \\ -h^{jk}N_k & 1 & 2\Phi + N_i h^{ij} N_j \end{pmatrix},$$

where h^{ij} is the inverse of h_{ij}. The geodesic Lagrangian is

$$\mathcal{L} = \frac{1}{2}\hat{g}_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu = \frac{1}{2}h_{ij}\, \dot{x}^i\, \dot{x}^j + \dot{t}\dot{v} - \Phi\dot{t}^2 + N_i\, \dot{x}^i\dot{t} := \tilde{L} + \dot{t}\dot{v},$$

where $\dot{x}^\mu = dx^\mu/d\lambda$ for an affine geodesic parameter λ ($\tilde{L}$ is defined below). Calculating the equations of motion from $\mathcal{L}$ enables a simple determination of (appropriate combinations of) the Christoffel symbols for $\hat{g}$. Recall

$$\hat{\Gamma}^\mu_{\nu\rho} = \frac{1}{2}\hat{g}^{\mu\delta}\left(\hat{g}_{\delta\nu,\rho} + \hat{g}_{\delta\rho,\nu} - \hat{g}_{\nu\rho,\delta}\right) := \hat{g}^{\mu\delta}[\nu\rho, \delta]_{\hat{g}}.$$

and the equations of motion are

$$0 = \ddot{x}^\mu + \hat{\Gamma}^\mu_{\nu\rho}\dot{x}^\mu\dot{x}^\rho.$$

Setting

$$A := A_\mu dx^\mu = N_i dx^i - \Phi dt, \qquad F = dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu = \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

we get

$$\begin{aligned} F_{ij} &= \partial_i N_j - \partial_j N_i = -F_{ji} \\ F_{it} &= -(\partial_t N_i + \partial_i \Phi) = -F_{ti} \end{aligned}$$

and using that the equations of motion for v, x^i and t (from $\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \frac{\partial\mathcal{L}}{\partial x^\mu}$) yield

$$0 = \frac{d}{d\lambda}\dot{t} = \ddot{t}$$

(for v) then

$$\begin{aligned}\frac{1}{2}(\partial_i h_{jk})\dot{x}^j\dot{x}^k - (\partial_i\Phi)\dot{t}^2 + (\partial_i N_j)\dot{x}^j\dot{t} &= \frac{d}{d\lambda}\left(h_{ij}\dot{x}^j + N_i\dot{t}\right)\\ &= h_{ij}\ddot{x}^j + (\partial_k h_{ij})\dot{x}^j\dot{x}^k + (\partial_t h_{ij})\dot{x}^j\dot{t} + (\partial_j N_i)\dot{x}^j\dot{t} + (\partial_t N_i)\dot{t}^2\end{aligned}$$

(for $\dot{x}^i$) and

$$\begin{aligned}\frac{1}{2}(\partial_t h_{ij})\dot{x}^i\dot{x}^j - (\partial_t\Phi)\dot{t}^2 + (\partial_t N_i)\dot{x}^i\dot{t} &= \frac{d}{d\lambda}\left(\dot{v} - 2\Phi\dot{t} + N_i\dot{x}^i\right)\\ &= \ddot{v} - 2(\partial_i\Phi)\dot{t}\dot{x}^i - 2(\partial_t\Phi)\dot{t}^2 - 2\Phi\ddot{t} + (\partial_j N_i)\dot{x}^i\dot{x}^j + (\partial_t N_i)\dot{x}^i\dot{t} + N_i\ddot{x}^i\end{aligned}$$

we get (collating them together)

$$\begin{aligned}0 &= \ddot{t},\\ 0 &= h_{ij}\,\ddot{x}^j + [jk,i]_h\,\dot{x}^j\dot{x}^k + (\partial_t h_{ij} + \partial_j N_i - \partial_i N_j)\,\dot{t}\dot{x}^j + (\partial_i\Phi + \partial_t N_i)\,\dot{t}^2,\\ &= h_{ij}\,\ddot{x}^j + [jk,i]_h\,\dot{x}^j\dot{x}^k + (\partial_t h_{ij} - F_{ij})\,\dot{t}\dot{x}^j + F_{ti}\dot{t}^2,\\ 0 &= \ddot{v} + N_i\ddot{x}^i + \left[\frac{1}{2}\left(\partial_j N_i + \partial_i N_j\right) - \frac{1}{2}\partial_t h_{ij}\right]\,\dot{x}^i\dot{x}^j - 2\partial_i\Phi\,\dot{t}\dot{x}^i - \partial_t\Phi\,\dot{t}^2,\\ &= \ddot{v} + \left[\frac{1}{2}\left(\partial_j N_i + \partial_i N_j\right) - N_k\Gamma^k_{ij} - \frac{1}{2}\partial_t h_{ij}\right]\,\dot{x}^i\dot{x}^j + \left[-N^k(\partial_t h_{ki} - F_{ki}) - 2\partial_i\Phi\right]\,\dot{t}\dot{x}^i + \left(-\partial_t\Phi + N^i F_{it}\right)\,\dot{t}^2\end{aligned}$$

where we have substituted F in the latter, and usde the notation

$$[jk,i]_h = \frac{1}{2}\left(\partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk}\right)\,.$$

Recall

$$\Gamma^i_{jk} = h^{il}[jk,l]_h\,.$$

Note that to raise the index of N has required we recognise that

$$N^i = \hat{g}^{ij}N_j = h^{ij}N_j$$

1.2 Equivalence of equations of motion

The canonical momenta are given by $p_\mu = \partial\mathcal{L}/\partial\dot{x}^\mu = \hat{g}_{\mu\nu}\dot{x}^\nu$ giving

$$p_v = \dot{t}, \qquad p_i = h_{ij}\dot{x}^j + N_i\dot{t}, \qquad p_t = \dot{v} - 2\Phi\dot{t} + N_i\dot{x}^i,$$

and so

$$\dot{t} = p_v, \qquad \dot{x}^i = h^{ij}(p_j - N_j p_v), \qquad \dot{v} = p_t - N^i p_i + [2\Phi + N^2]\,p_v.$$

Likewise, the geodesic Hamiltonian is

$$\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2}\hat{g}^{\mu\nu}\, p_\mu\, p_\nu = \frac{1}{2}h^{ij}\,(p_i - N_i p_v)(p_j - N_j p_v) + p_t p_v + \Phi\, p_v^2.$$

The equations of motion are

$$\begin{aligned}
&\frac{dt}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_t} = p_v, && \frac{dv}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_v}, && \frac{dx^i}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_i} = h^{ij}\,(p_j - N_j p_v),\\
&\frac{dp_t}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial t}, && \frac{dp_v}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial v} = 0, && \frac{dp_i}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^i}.
\end{aligned}$$

Because v is a cyclic coordinate its conjugate momentum p_v is conserved along geodesics: thus $p_v = m$ is a constant and we may write

$$\mathcal{H} := H + m\, p_t, \qquad H := \frac{1}{2}h^{ij}\,(p_i - mN_i)(p_j - mN_j) + m^2\,\Phi.$$

We observe that we have the geodesics have the conserved quantities,

$$\begin{aligned}
\frac{1}{2}\hat{g}^{\mu\nu}\, p_\mu p_\nu &= m\left[\frac{p^i p_i}{2m} - N^i p_i + mN^i N_i + p_t + m\Phi\right] := -mE_0,\\
\hat{g}^{\mu\nu}\, p_\mu \xi_\nu &= p_v = m.
\end{aligned}$$

Following the identifications of [3] we view $p_v = m$ as the mass, $-p_t = E$ as the energy, E_0 as the internal energy, and $m\Phi = V$ as the potential energy. Taking the internal energy to vanish in the nonrelativistic limit the null geodesics of $\hat{g}$ may be identified with the motion in the d-dimensional space with potential energy V. We note that two conformally related metrics have the same null geodesics, and so the d-dimensional world lines will be the same. For $m \neq 0$ the equations of motion for t then give $dt/d\lambda = m$, whence $dt = m\, d\lambda$ and we may eliminate the affine geodesic parameter λ for t. The equations of motion are then precisely those coming from the standard mechanical system

$$\tilde{L} = \frac{1}{2}h_{ij}\,\dot{x}^i\,\dot{x}^j + N_i\,\dot{x}^i - \Phi$$

where $\dot{x}^i$ is now the standard dx^i/dt (and $\dot{t} = 1$). Now

(a) in the case of a non-null geodesic, if we parameterised the curve by arc length, $\lambda = s$ and $t = ms$, then from (1.1.1) we have

$$\frac{dv}{dt} = \frac{1}{2m^2} - \tilde{L}.$$

The equations of motion for v follow from this and

$$v = \frac{t}{2m^2} - \int \tilde{L}\, dt + b.$$

(b) in the case of a null geodesics we have

$$\frac{dv}{dt} = -\tilde{L}, \qquad v = -\int \tilde{L}\, dt + b.$$

Thus we have for each $m \neq 0$ and b a bijection between the geodesics of $\hat{g}$ and the equations of motion of $\tilde{L}$.

1.3 Connection and Curvature

From the equations of motion we read that the nonvanishing Christoffel symbols for $\hat{g}$ are

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk}, \qquad \hat{\Gamma}^i_{jt} = -\frac{1}{2}F^i{}_j + \frac{1}{2}h^{ik}\partial_t h_{kj}, \qquad \hat{\Gamma}^i_{tt} = h^{ik}\left(\partial_t N_k + \partial_k \Phi\right) = -F^i{}_t,$$

$$\hat{\Gamma}^v_{tt} = -\partial_t\Phi + N^k F_{kt}, \quad \hat{\Gamma}^v_{ij} = \frac{1}{2}\left[\nabla_i N_j + \nabla_j N_i - \partial_t h_{ij}\right], \quad \hat{\Gamma}^v_{ti} = -\frac{1}{2}N^k(\partial_t h_{ki} - F_{ki}) - \partial_i\Phi.$$

Here we have used ∇_i for the Levi-Civita connection from the metric h. Recall now the equation for the Riemann tensor

$$\hat{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho\hat{\Gamma}^\mu_{\nu\sigma} - \partial_\sigma\hat{\Gamma}^\mu_{\nu\rho} + \hat{\Gamma}^\mu_{\rho\lambda}\hat{\Gamma}^\lambda_{\nu\sigma} - \hat{\Gamma}^\mu_{\sigma\lambda}\hat{\Gamma}^\lambda_{\nu\rho}$$

We immediately notice

$$\hat{R}^i{}_{jkl} = R^i{}_{jkl} + \hat{\Gamma}^i_{kt}\hat{\Gamma}^t_{jl} - \hat{\Gamma}^i_{lt}\hat{\Gamma}^t_{jk} + \hat{\Gamma}^i_{kv}\hat{\Gamma}^v_{jl} - \hat{\Gamma}^i_{lv}\hat{\Gamma}^v_{jk} = R^i{}_{jkl}$$

as there are non-vanishing Christoffel symbols with v as lower index, or t as an upper index. Further, as all Christoffel symbols are independent of v (as the metric is) we can then say that $\hat{R}^\mu{}_{\nu v\sigma} = 0$. As such $\hat{R}^\mu{}_{\nu\rho\sigma} = 0$ if any of $\nu, \rho, \sigma = v$. We can also see that $\hat{R}^t{}_{\nu\rho\sigma} = 0$ by the formula. so we now need only determine

1. $\hat{R}^i{}_{jtl}$
2. $\hat{R}^i{}_{tkl}$
3. $\hat{R}^i{}_{ttl}$
4. $\hat{R}^v{}_{jkl}$
5. $\hat{R}^v{}_{jtl}$
6. $\hat{R}^v{}_{tkl}$
7. $\hat{R}^v{}_{ttl}$

Making the observation

$$\hat{R}^v{}_{\nu\rho\sigma} = -h^{ik}N_k R_{i\nu\rho\sigma} + R_{t\nu\rho\sigma}$$

and seeing that

$$\begin{aligned}\hat{R}_{i\nu\rho\sigma} &= \hat{g}_{i\mu}\hat{R}^\mu{}_{\nu\rho\sigma} \\ &= h_{ij}\hat{R}^j{}_{\nu\rho\sigma}\end{aligned}$$

we can simplify

$$\hat{R}^v{}_{j\rho\sigma} = -N_i\hat{R}^i{}_{j\rho\sigma} - h_{ji}\hat{R}^i{}_{t\rho\sigma}$$

and

$$\hat{R}^v{}_{t\rho\sigma} = -N_i\hat{R}^i{}_{t\rho\sigma}$$

This lets us get the seoncd column of terms immediately after we have the first.
The Bianchi identity also tells us that

$$\hat{R}^\mu{}_{\nu\rho\sigma} = -\hat{R}^\mu{}_{\rho\sigma\nu} - \hat{R}^\mu{}_{\sigma\nu\rho}$$

This means

$$\begin{aligned}\hat{R}^i{}_{tkl} &= -\hat{R}^t{}_{klt} - \hat{R}^i{}_{lkt} \\ &= 2\hat{R}^i{}_{[k|t|l]}\end{aligned}$$

and so we just need to work out 1 and 3.
Let us begin the slog:

$$\begin{aligned}\hat{R}^i{}_{jtl} &= \partial_t \hat{\Gamma}^i_{jl} - \partial_l \hat{\Gamma}^i_{jt} + \hat{\Gamma}^i_{t\mu}\hat{\Gamma}^\mu_{jl} - \hat{\Gamma}^i_{l\mu}\hat{\Gamma}^\mu_{jt} \\ &= \Gamma^i_{jl,t} - \nabla_l \hat{\Gamma}^i_{jt} \\ &= \Gamma^i_{jl,t} + \frac{1}{2}\nabla_l \left[h^{ik}\left(F_{kj} - h_{kj,t}\right)\right] \\ &= \Gamma^i_{jl,t} - \frac{1}{2}h^{ik}\nabla_l h_{kj,t} + \frac{1}{2}\nabla_l F^i{}_j\end{aligned}$$

(This form will be useful to give coherence with [3]). Now note

$$\begin{aligned}h^{ik}\nabla_l(\partial_t h_{kj}) &= h^{ik}\left[h_{kj,tl} - \Gamma^m_{lk}h_{mj,t} - \Gamma^m_{lj}h_{km.t}\right] \\ &= h^{ik}\left[h_{kj,lt} - \partial_t(h_{mj}\Gamma^m_{lk}) + h_{mj}\Gamma^m_{lk,t} - \partial_t(h_{km}\Gamma^m_{lj}) + h_{mk}\Gamma^m_{lj,t}\right] \\ &= h^{ik}\left[\partial_t(h_{kj,l} - [lk,j]_h - [lj,k]_h) + h_{mj}\Gamma^m_{lk,t}\right] + \Gamma^i_{lj,t}\end{aligned}$$

Calculating

$$h_{kj,l} - [lk,j]_h - [lj,k]_h = h_{kj,l} - \frac{1}{2}(h_{lj,k} + h_{kj,l} - h_{lk,j}) - \frac{1}{2}(h_{lk,j} + h_{jk,l} - h_{jl,k}) = 0$$

we have

$$\hat{R}^i{}_{jtl} = \frac{1}{2}\left[\nabla_l F^i{}_j + \Gamma^i_{jl,t} - h^{ik}h_{jm}\Gamma^m_{lk,t}\right]$$

Hence

$$\begin{aligned}\hat{R}^i{}_{tkl} = 2\hat{R}^i{}_{[k|t|l]} &= \nabla_{[l}F^i{}_{k]} - h^{ij}h_{m[k}\Gamma^m_{l]j,t} \\ \text{or } &= \nabla_{[l}F^i{}_{k]} - \nabla_{[l}h^{ij}h_{k]j,t}\end{aligned}$$

Further

$$\begin{aligned}\hat{R}^i{}_{ttl} &= \partial_t \hat{\Gamma}^i_{tl} - \partial_l \hat{\Gamma}^i_{tt} + \hat{\Gamma}^i_{t\mu}\hat{\Gamma}^\mu_{tl} - \hat{\Gamma}^i_{l\mu}\hat{\Gamma}^\mu_{tt} \\ &= -\frac{1}{2}\partial_t\left(F^i{}_l - h^{ij}h_{jl,t}\right) + \frac{1}{4}\left(F^i{}_j - h^{ik}h_{kj,t}\right)\left(F^j{}_l - h^{jm}h_{ml,t}\right) + \nabla_l F^i{}_t \\ &= -\frac{1}{2}\partial_t\left(F^i{}_l - h^{ij}h_{jl,t}\right) + \frac{1}{4}\left(F^i{}_j + h^{ik}{}_{,t}h_{kj}\right)\left(F^j{}_l - h^{jm}h_{ml,t}\right) + \nabla_l F^i{}_t \\ &= -\frac{1}{2}\left[\partial_t\left(F^i{}_l - h^{ij}h_{jl,t}\right) - \frac{1}{2}\left(F^{ij} + h^{ij}{}_{,t}\right)\left(F_{jl} - h_{jl,t}\right) - 2\nabla_l F^i{}_t\right] \quad \text{(useful to give coherence with [3])} \\ &= -\frac{1}{2}\left(F^i{}_{l,t} - h^{ij}h_{jl,tt}\right) + \frac{1}{4}\left(F^i{}_j F^j{}_l - F^{ij}h_{jl,t} + F_{jl}h^{ij}{}_{,t} + h^{ij}{}_{,t}h_{jl,t}\right) + \nabla_l F^i{}_t\end{aligned}$$

With these three we can read off

$$\begin{aligned}
\hat{R}^{v}{}_{jkl} &= -N_i R^i{}_{jkl} - h_{ji}\left[h^{ia} h_{m[l}\Gamma^m_{k]a,t} - \nabla_{[k}F^i{}_{l]}\right] \\
&= -N_i R^i{}_{jkl} - h_{m[l}\Gamma^m_{k]j,t} + \nabla_{[k}F_{|j|l]} \\
&= -N_i R^i{}_{jkl} - h_{m[l}\Gamma^m_{k]j,t} + \frac{1}{2}\nabla_j F_{kl}
\end{aligned}$$

using

$$\begin{aligned}
\nabla_{[k}F_{|j|l]} &= F_{j[l,k]} - F_{jm}\Gamma^m_{[kl]} - F_{m[l}\Gamma^m_{k]j} \\
&= N_{[l,|j|k]} - N_{j,[lk]} - F_{m[l}\Gamma^m_{k]j} \\
&= \frac{1}{2}\left(F_{kl,j} - \Gamma^m_{kj}F_{ml} + \Gamma^m_{jl}F_{mk}\right) \\
&= \frac{1}{2}\nabla_j F_{kl}
\end{aligned}$$

$$\begin{aligned}
\hat{R}^{v}{}_{jtl} &= -\frac{1}{2}N_i\left[\nabla_l F^i{}_j + \Gamma^i_{jl,t} - h^{ik}h_{jm}\Gamma^m_{lk,t}\right] - h_{ji}\left\{-\frac{1}{2}\left(F^i{}_{l,t} - h^{ik}h_{kl,tt}\right)\right. \\
&\quad \left. +\frac{1}{4}\left(F^i{}_k F^k{}_l - F^{ik}h_{kl,t} + F_{kl}h^{ik}{}_{,t} + h^{ik}{}_{,t}h_{kl,t}\right) + \nabla_l F^i{}_t\right\} \\
&= -\frac{1}{2}N_i\left[\nabla_l F^i{}_j + \Gamma^i_{jl,t} - h^{ik}h_{jm}\Gamma^m_{lk,t}\right] + \frac{1}{2}\left(h_{ji}F^i{}_{l,t} + h_{jl,tt}\right) \\
&\quad + \frac{1}{4}\left(F_{jk}F^k{}_l - F_j{}^k h_{kl,t} + F_{kl}h_{ji}h^{ik}{}_{,t} + h_{ji}h^{ik}{}_{,t}h_{kl,t}\right) + \nabla_l F_{jt} \\
&= -\frac{1}{2}N_i\left[\nabla_l F^i{}_j + \Gamma^i_{jl,t} - h^{ik}h_{jm}\Gamma^m_{lk,t}\right] + \frac{1}{2}\left(h_{ji}F^i{}_{l,t} + h_{jl,tt}\right) \\
&\quad + \frac{1}{4}\left(F_{jk}F^k{}_l - F_j{}^k h_{kl,t} - F^i{}_l h_{ji,t} + h_{ji}h^{ik}{}_{,t}h_{kl,t}\right) + \nabla_l F_{jt}
\end{aligned}$$

Now we have constructed Riemann curvature tensors, we can go on to calculate the Ricci tensor given by

$$\hat{R}_{\nu\sigma} = \hat{R}^{\mu}{}_{\nu\mu\sigma}$$

As before, we said that we can never have t as the first upper index, or a v as the lower indices, so we know that

$$\hat{R}_{v\mu} = 0$$

and that this formula reduces to

$$\hat{R}_{\nu\sigma} = \hat{R}^{i}{}_{\nu i\sigma}$$

We straight away recognise that the spatial part is the same as for the spatial manifold, i.e.

$$\hat{R}_{ij} = R_{ij}$$

All that's left to calculate is $\hat{R}_{ti} = \hat{R}_{it}$, $\hat{R}_{tt}$, which are given by

$$\begin{aligned}
\hat{R}_{tl} &= \hat{R}^i{}_{til} \\
&= \nabla_{[l} F^i{}_{k]} - \nabla_{[l} h^{ij} h_{i]j,t} \\
&= \frac{1}{2}\left[\nabla_l F^i_i - \nabla_i F^i_l - \partial_l\left(h^{ij} h_{ij,t}\right) + \nabla_i h^{ij} h_{jl,t}\right] \\
&= -\frac{1}{2}\left[\nabla^i\left(F_{il} - h_{il,t}\right) + \partial_l\left(h^{ij} h_{ij,t}\right)\right] \quad \text{(compare to [3])}
\end{aligned}$$

$$\begin{aligned}
\hat{R}_{tt} &= \hat{R}^i{}_{tit} \\
&= \frac{1}{2}\left[\partial_t\left(F^i_i - h^{ij} h_{ji,t}\right) - \frac{1}{2}\left(F^{ij} + h^{ij}{}_{,t}\right)\left(F_{ji} - h_{ji,t}\right) - 2\nabla_i F^i_t\right] \\
&= -\frac{1}{2}\left[\partial_t\left(h^{ij} h_{ij,t}\right) - \frac{1}{2}\left(F^{ji} - h^{ji}{}_{,t}\right)\left(F_{ji} - h_{ji,t}\right) + 2\nabla_i F^i_t\right] \quad \text{(compare to [3])} \\
&= -\frac{1}{2}\left[\partial_t\left(h^{ij} h_{ij,t}\right) - \frac{1}{2}\left(F^{ij} F_{ij} + F^{ij} h_{ij,t} + h^{ij}{}_{,t} F_{ij} + h^{ij}{}_{,t} h_{ij,t}\right) + 2\nabla^i F_{it}\right] \\
&= -\frac{1}{2}\left[\partial_t\left(h^{ij} h_{ij,t}\right) - \frac{1}{2}\left(F^{ij} F_{ij} - 2F_{ij,t} h^{ij} + h^{ij}{}_{,t} h_{ij,t}\right) + 2\nabla^i F_{it}\right]
\end{aligned}$$

I have slightly different answers to [1, 3]. I should check this.

1.4 The Frame

An alternative way to get this result is using the frame field formulation.
Given the metric (1.1.1) we define the frame $\{\hat{e}^A\}$,

$$ds^2 = \hat{g}_{\mu\nu}\, dx^\mu\, dx^\nu = h_{ij}\, dx^i\, dx^j + 2dt\left(dv - \Phi dt + N_i dx^i\right) = \hat{\eta}_{AB}\, \hat{e}^A \hat{e}^B = \eta_{ab}\, e^a e^b + \hat{e}^+ \hat{e}^- + \hat{e}^- \hat{e}^+.$$

Here $A \in \{+, -, a, b, \ldots\}$, $\hat{\eta}_{+-} = \hat{\eta}_{-+} = 1$, and we take

$$\hat{e}^+ := dt, \qquad \hat{e}^- := dv - \Phi\, dt + N_i\, dx^i, \qquad \hat{e}^a := \hat{e}^a_\mu dx^\mu = e^a_i dx^i = e^a,$$

and

$$e^a_i \eta_{ab} e^b_j = h_{ij}.$$

The coframe $\{\hat{E}_A\}$ with $\hat{e}^A(\hat{E}_B) = \delta^A_B$ is given by

$$\hat{E}_+ := \partial_t + \Phi\, \partial_v, \qquad \hat{E}_- := \partial_v, \qquad \hat{E}_a := E_a - N_a\, \partial_v,$$

where $N_a = N_i E^i_a$ and similarly

$$e^a(E_b) = \delta^a_b, \qquad E_b = E^i_b\, \partial_i.$$

We emphasise that N, ϕ and e^a may depend on x^i *and* t.

Denoting the structure constants $[\hat{E}_B, \hat{E}_C] = c^A{}_{BC} \hat{E}_A$ we have from

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

for a one-form α, then for the torsion free connection

$$d\hat{e}^A = -\hat{\omega}^A{}_B \wedge e^B = \hat{\omega}^A{}_{BC} e^B \wedge e^C$$

we have

$$d\hat{e}^A(\hat{E}_B, \hat{E}_C) = \hat{\omega}^A{}_{BC} - \hat{\omega}^A{}_{CB} = -\hat{e}^A([\hat{E}_B, \hat{E}_C]) = -c^A{}_{BC},$$

from which

$$\hat{\omega}^A{}_{BC} = \frac{1}{2}\hat{\eta}^{AF}(c_{CFB} + c_{BFC} - c_{FBC}).$$

The v-independence of the metric means that

$$[\hat{E}_-, \hat{E}_B] = 0, \qquad c^A{}_{-B} = 0$$

while

$$\begin{aligned}
[\hat{E}_+, \hat{E}_a] &= \partial_t E_a - (\partial_t N_a)\partial_v - (E_a\Phi)\partial_v \\
&= (\partial_t E_a^j)[\partial_j - N_j\partial_v] - E_a^j)[\partial_t N_j + \partial_j\Phi]\partial_v \\
&= (\partial_t E_a^j\, e_j^b)\, \hat{E}_b + F_{at}\, \hat{E}_-
\end{aligned}$$

and

$$\begin{aligned}
[\hat{E}_a, \hat{E}_b] &= [E_a, E_b] - (E_a N_b - E_b N_a)\,\partial_v \\
&= c^f{}_{ab}\, \hat{E}_f - F_{ab}\, \hat{E}_-
\end{aligned}$$

giving the (possibly) non-vanishing structure constants as

$$c^f{}_{ab}, \quad c^-{}_{ab} = -F_{ab}, \quad c^b{}_{+a} = (\partial_t E_a^j\, e_j^b), \quad c^-{}_{+a} = F_{at}.$$

Now

$$\begin{aligned}
d\hat{e}^+ &= 0, \\
d\hat{e}^- &= \frac{1}{2}F_{ab}\, e^a \wedge e^b + F_{it}\, dx^i \wedge dt = \frac{1}{2}F_{ab}\, \hat{e}^a \wedge \hat{e}^b + F_{at}\, \hat{e}^a \wedge \hat{e}^+, \\
d\hat{e}^a &= d\left(\hat{e}^a_\mu dx^\mu\right) = e^a_i dx^i = e^a = (\partial_j e_i^a)\, dx^j \wedge dx^i + (\partial_t e_i^a)\, dt \wedge dx^i \\
&= \omega^a{}_{bc}\, e^b \wedge e^c - (E_b^i\, \partial_t e_i^a)\, dt \wedge e^b = \omega^a{}_{bc}\, e^b \wedge e^c + (\partial_t E_b^i\, e_i^a)\, dt \wedge e^b,
\end{aligned}$$

from which we see

$$\begin{aligned}
\hat{\omega}^a{}_{BC}\, \hat{e}^B \wedge \hat{e}^C &= \omega^a{}_{bc}\, e^b \wedge e^c + (\partial_t E_b^i\, e_i^a)\, \hat{e}^+ \wedge \hat{e}^b, \\
\hat{\omega}^-{}_{BC}\, \hat{e}^B \wedge \hat{e}^C &= \frac{1}{2}F_{bc}\, \hat{e}^b \wedge \hat{e}^c + F_{at}\, \hat{e}^a \wedge \hat{e}^+.
\end{aligned}$$

Set

$$\alpha^a{}_b := e_i^a\, \partial_t E_b^i = c^a{}_{+b}, \quad \alpha_{ab} = -\alpha_{ba},$$

Using the antisymmetry of the connection then $0 = \hat{\omega}_{++A} = \hat{\omega}^{-}{}_{+A}$ and so

$$\hat{\omega}^{-}{}_{a+} = F_{at}, \quad \hat{\omega}^{-}{}_{ab} = \frac{1}{2}F_{ab}, \quad \hat{\omega}^{a}{}_{bc} = \omega^{a}{}_{bc}, \quad \hat{\omega}_{ab+} = -\frac{1}{2}F_{ab} - \frac{1}{2}\left[\partial_t E^i_a\, E_{ib} - \partial_t E^i_b\, E_{ia}\right] = -\frac{1}{2}F_{ab} + \alpha_{ab}.$$

From the structure of the connection we see that $\hat{R}^A{}_B \in \{\hat{R}^{-}{}_a, \hat{R}^a{}_+, \hat{R}^a{}_b\}$ and so $\widehat{\mathrm{Ric}}_{AB} = \hat{R}^F{}_{AFB} = \hat{R}^f{}_{AfB}$. Thus the (possible) non-vanishing components of the Ricci tensor are

$$\widehat{\mathrm{Ric}}_{++} = \hat{R}^f{}_{+f+}, \quad \widehat{\mathrm{Ric}}_{a+} = \hat{R}^f{}_{af+}, \quad \widehat{\mathrm{Ric}}_{ab} = \hat{R}^f{}_{afb}.$$

1.5 Einstein metrics

Now because $\hat{g}_{tv} = 1$ and $\hat{R}_{tv} = 0$, then $(B, \hat{g})$ is Einstein if and only if it is Ricci flat, i.e. that $\hat{R}_{\mu\nu} = 0$. This immediately forces $R_{ij} = 0$. Assuming this, we can follow the paper [1] to try and derive conditions on the functions N_i, Φ.

Insert lead in - left out to get calculation of after (43) done quicker

Define

$$\begin{aligned} H_{ij} &= h_{ij,t} \\ H &= h^{ij}H_{ij} \\ h &= \left|\det\left(h_{ij}\right)\right| . \end{aligned}$$

With this notation we have

$$\begin{aligned} \hat{R}_{tl} &= -\frac{1}{2}\left[\nabla^i(F_{il} - H_{il}) + H_{,l}\right] \\ \hat{R}_{tt} &= -\frac{1}{2}\left[H_{,t} - \frac{1}{2}\left(F^{ji} - H^{ji}\right)\left(F_{ji} - H_{ji}\right) + 2\nabla_i N^i_{,t} + \Delta\Phi\right] , \end{aligned}$$

such that the conditions for the metric to be Einstein become

$$\begin{aligned} \nabla^i(F_{il} - H_{il}) + H_{,l} &= 0 \\ H_{,t} - \frac{1}{2}\left(F^{ji} - H^{ji}\right)\left(F_{ji} - H_{ji}\right) + 2\nabla_i N^i_{,t} + \Delta\Phi &= 0 . \end{aligned}$$

The first condition contains no N_i terms directly (only derivatives) and so we can write it as

$$\begin{aligned} h^{ij}\nabla_j(N_{k,i} - N_{i,k}) &= f_k \\ \Rightarrow \frac{1}{\sqrt{h}}\partial_j\left[\sqrt{h}h^{ij}h^{kl}\left(N_{k,i} - N_{i,k}\right)\right] &= f^l , \end{aligned}$$

where

$$\begin{aligned} f_k &= \nabla_i H^i_k - \nabla_k H \\ \Rightarrow f^i &= \nabla_j H^{ij} - h^{ij}\nabla_j H \\ &= (h^{ik}h^{jl} - h^{ij}h^{kl})\nabla_j H_{kl} . \end{aligned}$$

By contraction of symmetric and antisymmetric tensors we see it is necessary that

$$\partial_l\left(\sqrt{h}f^l\right)=0\,.$$

Moreover, [1] says (and I do not understand why - he says "the only differential relations between the left hand side of (42) lead to (43)". This is effectively arguing that therefore the condition forces involutivity of the algebra of differential operators?) that this is in fact sufficient for the integrability of the first Einstein condition to find N_i.

Lemma 1.1. $\hat{R}_{ij}=0\Rightarrow\partial_l\left(\sqrt{h}f^l\right)=0$

Proof. We have shown previously

$$\begin{aligned}\nabla_j H_{kl}&=h_{mk}\Gamma^m_{lj,t}+h_{ml}\Gamma^m_{kj,t}\\ \Rightarrow f^i&=h^{jl}\Gamma^i_{lj,t}+h^{ik}\Gamma^j_{kj,t}-2h^{ij}\Gamma^l_{jl,t}\\ &=h^{jk}\Gamma^i_{jk,t}-h^{ij}\Gamma^k_{jk,t}\,.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}f^i\right)=\nabla_i f^i&=h^{jk}\nabla_i\Gamma^i_{jk,t}-h^{ij}\nabla_i\Gamma^k_{jk,t}\\ &=h^{ij}\left[\nabla_k\Gamma^k_{ij,t}-\nabla_i\Gamma^k_{jk,t}\right]\\ &=h^{ij}\left[\Gamma^k_{ij,tk}+\Gamma^k_{kl}\Gamma^l_{ij,t}-\Gamma^l_{ki}\Gamma^k_{lj,t}-\Gamma^l_{kj}\Gamma^k_{il,t}-\Gamma^k_{jk,ti}+\Gamma^l_{ij}\Gamma^k_{lk,t}\right]\\ &=h^{ij}\left[\partial_t\left(\Gamma^k_{ij,k}-\Gamma^k_{jk,i}+\Gamma^k_{kl}\Gamma^l_{ij}-\Gamma^k_{il}\Gamma^l_{jk}\right)+\Gamma^k_{il}\Gamma^l_{jk,t}-\Gamma^l_{ki}\Gamma^k_{lj,t}\right]\\ &=h^{ij}R^k{}_{jki,t}=h^{ij}R_{ij,t}\end{aligned}$$

□

Provided we then solve for N_i, we can substitute into the second Einstein condition. Along with the information of the time derivative of the metric H_{ij}, this gives an equation for Φ, namely Poisson's equation.

1.6 Bargman Structures

A Bargmann structure $(B,\hat{g},\xi)$ is a principal bundle $\pi:B\to M$, where $\dim B=\dim M+1$, equipped with a Lorentzian metric $\hat{g}$ and nowhere vanishing null vector field ξ such that with respect to the usual Levi-Civita connection $\hat{\nabla}\xi=0$. Then $M:=B/\mathbb{R}\xi$ is equipped with a Newton-Cartan geometry (M,K,θ,∇) where

$$K=\pi_*\hat{g}^{-1},\qquad\hat{g}(\xi)=\pi^*\theta,$$

K is degenerate and $\pi^*\theta$ generates $\ker K$.

In our setting we have a metric of Brinkmann form

$$\hat{g}=h+dt\otimes\omega+\omega\otimes dt,\quad\omega=dv-\Phi(x,t)\,dt+N_i(x,t)\,dx^i,\quad h=h_{ij}(x,t)dx^i\otimes dx^j.$$

Then $\xi=\partial_v$, $\theta=dt$.

2 Introduction

Let us start with a bit of back story, so we can develop and go further. This will be built off of [2].

2.1 Galilei and Newton Structures

We start with some more classical work.

Definition 2.1 (Galilei group)**.** *The* ***Galilei group*** *is the matrix group*

$$G = \left\{ \begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \mid R \in SO(d),\ \ , b, c \in \mathbb{R}^n,\ e \in \mathbb{R} \right\} \leq GL_{d+2}(\mathbb{R})$$

We think of G as acting on $(\boldsymbol{x}, t, 1)$ s.t.

$$\begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Rx + tb + c \\ t + e \\ 1 \end{pmatrix}$$

with this action we see:

1. R are rotations in space
2. b are boosts
3. c, e are translations in space and time respectively

With this interpretation we have

Definition 2.2. *The* ***Homogeneous Galilei group/Euclidean group*** *H is the group of Galilean transformations that preserve the spatio-temporal origin $(\mathbf{0}, 0, 1)$.*

Proposition 2.3. *H consists of matrices of the form*

$$\begin{pmatrix} R & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover $H \cong SO(d) \ltimes \mathbb{R}^d$ as a Lie group (not a as a Lie transformation group [4]) is faithfully represented by matrices of the form

$$\begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \in GL_{d+1}.$$

Proof. See my CQIS notes for a more built up discussion of this fact. □

We now recall the following def:

Definition 2.4. *The* ***frame bundle*** *of a k-dimensional smooth manifold M is $GL(M)$, the GL_k-principal fibre bundle with fibres at $x \in M$ given by the space of ordered bases of T_xM.*

Definition 2.5. *A* ***proper Galilei structure*** *$H(M)$ is a reduction of structure group of the frame bundle of a $(d+1)$-dimensional M via $H \hookrightarrow GL_{d+1}$.*

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