Separability of the HJE

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1 Introduction

1.1 Bookkeeping

These are notes I am writing to set into concrete my thoughts on separability and the works of Eisenhart, Kalnins, and Benenti. This work will also hopefully tie into the work of Stäckel.
Indices will be flying all over the place, and I will try to uniformise this with my KYT notes afterwards to keep their ranges standard, but be prepared to be on your toes. Moreover, I will switch back and forth between using summation notation and not in a blasé manner. To work out if I am or not, assume not, and then sum over any indices which would otherwise not be explained.

1.2 Motivation

The purpose of this subsection will be to motivate the Hamilton Jacobi equation, and then to give a reason as to why we would want to study its separability.
To start, consider two manifolds Q_1, Q_2, and a symplectomorphism $\phi : T^*Q_1 \to T^*Q_2$ where the cotangent spaces are given the natural symplectic structure from the tautological one form.[1] That is, letting $\theta_i \in \Omega^1(T^*Q_i)$ be the respective tautological one form,

$$\begin{aligned} \phi^*(-d\theta_2) &= -d\theta_1 \\ \Rightarrow d(\theta_1 - \phi^*\theta_2) &= 0 \end{aligned}$$

using that exterior derivative and pullback commute. Now suppose we can write ϕ locally as

$$\begin{aligned} p &= p(q,x) \\ y &= y(q,x) \end{aligned}$$

where $(q,p),(x,y)$ are locally trivialising coordinates for Q_1, Q_2 respectively[2]. Note that this can be done locally if $\frac{\partial x}{\partial p} \neq 0$ as we certainly have a well defined map

$$\begin{aligned} x &= x(q,p) \\ y &= y(q,p) \end{aligned}$$

and then if $\frac{\partial x}{\partial p} \neq 0$ we can invert this. In this case we have a (locally well defined) function

$$\Gamma : Q_1 \times Q_2 \to T^*Q_1 \times T^*Q_2$$

whose image is the graph of ϕ. Now using Γ the symplectomorphism condition can be written as

$$\begin{aligned} d\left[\Gamma^*(\theta_1 - \theta_2)\right] &= 0 \\ \Rightarrow \Gamma^*(\theta_1 - \theta_2) &= dS \end{aligned}$$

for some $S \in C^\infty(Q_1 \times Q_2)$ (at least locally). In coordinates this says

$$pdq - ydx = \partial_q S\, dq + \partial_x S\, dx \Rightarrow \begin{cases} p = \partial_q S \\ y = -\partial_x S \end{cases}$$

S is called a **generating function** of the transform.

Example 1.1. *Consider the function $S(q,x) = qx$. Then*

$$\begin{aligned} p &= \partial_q S = x \\ y &= -\partial_x S = q \end{aligned}$$

This demonstrates that we can treat (up to a minus sign) position and momenta equivalently in a Hamiltonian formalism.

[1] Note that ϕ being a symplectomorphism means that $\dim Q_1 = \dim Q_2$ necessarily.
[2] These could be indexed to show that there are multiple coordinates, but that will only make notation clunky so I will not do it.

Suppose now at a point $\frac{\partial x}{\partial p} = 0$. Then here $\frac{\partial y}{\partial p} \neq 0$ as ϕ is invertible. In this case we follow through the above work, but now writing the transform in the form

$$\begin{aligned} p &= p(q, y) \\ x &= x(q, y) \end{aligned}$$

This is now less natural, as instead of having a function $Q_1 \times Q_2 \to T^*Q_1 \times T^*Q_1$, we get a map from $U \times V$, where $U \subset Q_1$, and V is an open neighbourhood with coordinates y, corresponding to a region where $\frac{\partial y}{\partial p} \neq 0$. In this case we get a generating function $S = S(q, y)$ with conditions

$$\begin{aligned} x &= \partial_y S \\ p &= \partial_q S \end{aligned}$$

Note that this is in some sense a more natural generating function, as the identity transform (and any transform "close" to it) have a generating function of this type (e.g. $S(q, y) = qy$ for the identity). This is a useful class of generating functions because of its relation to time evolution: Recall that if we have a time flow on the phase space T^*Q we get a smooth 1-parameter family of diffeomorphisms $\phi_t : T^*Q \to T^*Q$ s.t.

- $\phi_0 = \mathrm{id}$
- $\phi_t \circ \phi_s = \phi_{t+s}$

As such we should believe we can find a generating function of this type for time evolution. Consider an evolution ϕ_ϵ, where $|\epsilon| \ll 1$, so to first order

$$\begin{aligned} x &= q + \epsilon \dot{q} \\ y &= p + \epsilon \dot{p} \end{aligned}$$

If the evolution is Hamiltonian flow on the manifold then

$$\begin{aligned} \dot{q} &= \partial_p H \\ \dot{p} &= -\partial_q H \end{aligned}$$

We now want to find a generating function as $S(q, y) = qy + \epsilon T(q, y)$, for this transform. Plugging in we get

$$\begin{aligned} x = q + \epsilon \partial_y T &\Rightarrow \partial_y T = \partial_p H \\ p = y + \epsilon \partial_q T &\Rightarrow \partial_q T = \partial_q H \end{aligned}$$

This is satisfied if $T(q, y) = H(q, p(q, y))$. Now note

$$\dot{T} = \dot{q}\partial_q T + \dot{y}\partial_y T = (\partial_p H)(\partial_q H) + (-\partial_q H + \epsilon \ddot{p})(\partial_p H) = \mathcal{O}(\epsilon)$$

so $T(q, y)$ is a constant. Hence to find the generating function is equivalent to finding a solution S to

$$E = H(q, \partial_q S)$$

Remark. *Note in all the above we have considered only autonomous systems as they are all I have ever cared about, but in principle it can be extended.*

can the second part of this discussion be made more geometric? I hope so. I know that the HJE can be derived as a 0th order WKB approximation to the Schrödinger equation. It'd be nice to see this sometime.

2 General Separability

2.1 Completely integrable foliations

For completeness we now reproduce here the results of [2]. Let us consider a trivial fibration $\pi : M = Q \times Z \to Q$ where Z is a N-dimensional vector space with coordinates $z = (z_A)$, and Q is an n-dimensional manifold with local coordinates $q = (q^i)$. Let C be a regular distribution giving a connection, and let

$$D_i = \frac{\partial}{\partial q^i} + C_{iA}\frac{\partial}{\partial z_A}$$

be n horizontal vector fields that span C locally. Then D_i are called the **generators** of C.

Theorem 2.1. *A first order differential system of the form*

$$\partial_i f_A(q, c) = C_{iA}(q, f(q, c))$$

is completely integrable, i.e. admits a local complete solution satifying the ***completeness condition***

$$\det \frac{\partial f}{\partial c} \neq 0$$

iff the generators D_i commute.

Proof. We may calculate

$$\begin{aligned}[D_i, D_j] &= \left(\frac{\partial}{\partial q^i} + C_{iA}\frac{\partial}{\partial z_A}\right)\left(\frac{\partial}{\partial q^j} + C_{jB}\frac{\partial}{\partial z_B}\right) - \left(\frac{\partial}{\partial q^j} + C_{jB}\frac{\partial}{\partial z_B}\right)\left(\frac{\partial}{\partial q^i} + C_{iA}\frac{\partial}{\partial z_A}\right)\\ &= \left(\frac{\partial C_{jB}}{\partial q^i} + C_{iA}\frac{\partial C_{jB}}{\partial z_A}\right)\frac{\partial}{\partial z_B} - \left(\frac{\partial C_{iA}}{\partial q^j} + C_{jB}\frac{\partial C_{iA}}{\partial z_B}\right)\frac{\partial}{\partial z_A}\end{aligned}$$

This vector field must be vertical, and so by the Frobenius theorem C is integrable iff $[D_i, D_j] = 0$. Complete integrability gives a folitation of integral manifold transversal to the fibres, and locallly represented by equations

$$z_A = f_A(q, c)$$

where $c = (c_A)$ are constant parameters uniquely determined by assigning values z at a fixed $q_0 \in Q$. This unique determination means that we must have the completeness condition Moreover, since the integral manifolds are tangent to generators, it must we the case that

$$\begin{aligned}D_i(z_a - f_A(q, c)) &= 0\\ \Rightarrow \partial_i f_A &= C_{iA}\end{aligned}$$

□

2.2 Separability of PDEs

To understand separability we now want to consider the partial differential equation

$$\mathcal{H}(q, u, u_i, u_{ij}, \dots, u_{ij\cdots h}) = E$$

for the function $u = u(q)$, where $u_{ij\cdot h} = \partial_h \dots \partial_j \partial_i u$, E is a constant, and $\mathcal{H}$ is a smooth function of the up to the lth partials of u.

Definition 2.2. *A* ***separable solution*** *to the pde is a solution of the form*

$$u(q) = \sum_{i=1}^{n} S_i(q^i, E)$$

For a separable solution the mixed partials vanish and so the pde reduces to

$$\mathcal{H}_s(q, u, u^{(1)}, u^{(2)}, \dots, u^{(l)}) = E$$

where $u^{(1)} = (u_i), u^{(2)} = (u_{ii})$, etc.

Notation. *Note that technically $\mathcal{H}_s$ is a different function from $\mathcal{H}$ (being a function of fewer variables) but we will suppress this for simplicity of notation.*

Taking the total derivative we get

$$\frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial u} u_i + \dots + \frac{\partial \mathcal{H}}{\partial u_i^{(l)}} u_i^{(l+1)} = 0$$

and so under the assumption $\frac{\partial \mathcal{H}}{\partial u_i^{(l)}} \neq 0$ we may define

$$R_i = -\left(\frac{\partial \mathcal{H}}{\partial u_i^{(l)}} \right)^{-1} \left(\frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial u} u_i + \dots + \frac{\partial \mathcal{H}}{\partial u_i^{(l-1)}} u_i^{(l)} \right)$$

so

$$u_i^{(l+1)} = R_i$$

and we get

$$\begin{aligned} \partial_i u &= u_i^{(1)} \\ \partial_i u_i^{(1)} &= u_i^{(2)} \\ \dots &= \dots \\ \partial_i u_i^{(l)} &= R_i \\ \partial_j u_i^{(k)} &= 0, \ i \neq j \end{aligned}$$

To connect this with our previous theorem, we consider the $N = nl + 1$ functions $u, u^{(1)}, \dots, u^{(l)}$ as coordinates on a space Z, i.e

$$z = (u, u^{(1)}, \dots, u^{(l)})$$

and then we have the equations for tangent integral manifolds for generators

$$D_i = \partial_i + u_i^{(1)} \frac{\partial}{\partial u} + u_i^{(2)} \frac{\partial}{\partial u_i^{(1)}} + \cdots + R_i \frac{\partial}{\partial u_i^{(l)}}$$

This gives us a characterisation of complete separability:

Theorem 2.3. *The PDE*

$$\mathcal{H}(q, u, u_i, u_{ij}, \ldots, u_{ij\cdots h}) = E$$

is separable (i.e. admits a completely separable solution) in the coordinates q iff the separability conditions

$$[D_i, D_j] = 0$$

are identically satisfied for D_i defined as above.

Remark. *In examples where $\frac{\partial \mathcal{H}}{\partial u} = 0$ we can exclude the u coordinate in z while excluding the $\frac{\partial}{\partial u}$ term in D and all the above carries through.*

3 Separation of the Hamilton Jacobi Equation

3.1 Levi-Civita separability

We now wish to apply the above theory to the Hamilton Jacobi Equation (HJE) associated with a Hamiltonian $H : M = T^*Q \to \mathbb{R}$. Note that the cotangent fibration $T^*Q \to Q$ is not in general trivial, but it is certainly locally so, and so are previous considerations can apply locally.
Now in standard notation the HJE is

$$H(q, \partial_i S) = E$$

where we have identified $\mathcal{H} = H$, $u = S, u_i^{(1)} = \partial_i S, z = (u_i) = (p_i)$. The differential system is then

$$\begin{aligned} \partial_i p_i &= R_i = -\frac{\partial_i H}{\partial^i H} \\ \partial_j p_i &= 0 \end{aligned}$$

where we have used $\partial^i = \frac{\partial}{\partial p_i}$.

Theorem 3.1. *The HJE is completely separable in coordinates q iff the* ***Levi-Civita separability conditions (LCSCs)***

$$\forall i \neq j,\ (\partial^i \partial^j H)(\partial_i H)(\partial_j H) + (\partial_i \partial_j H)(\partial^i H)(\partial^j H) - (\partial^i \partial_j H)(\partial_i H)(\partial^j H) - (\partial_i \partial^j H)(\partial^i H)(\partial_j H) = 0$$

are satisfied identically.

Proof. From the previously developed theory, the separability of the HJE is equivalent to

$$\begin{aligned}
0 &= \left[\partial_i - \frac{\partial_i H}{\partial^i H}\partial^i, \partial_j - \frac{\partial_j H}{\partial^j H}\partial^j\right] \\
&= -\left[-\frac{\partial_j\partial_i H}{\partial^i H} + \frac{(\partial_i H)(\partial_j\partial^i H)}{(\partial^i H)^2}\right]\partial^i + \frac{\partial_j H}{\partial^j H}\left[-\frac{\partial^j\partial_i H}{\partial^i H} + \frac{(\partial_i H)(\partial^j\partial^i H)}{(\partial^i H)^2}\right]\partial^i \\
&\quad - \frac{\partial_i H}{\partial^i H}\left[-\frac{\partial^i\partial_j H}{\partial^j H} + \frac{(\partial_j H)(\partial^i\partial^j H)}{(\partial^j H)^2}\right]\partial^j + \left[-\frac{\partial_i\partial_j H}{\partial^j H} + \frac{(\partial_j H)(\partial_i\partial^j H)}{(\partial^j H)^2}\right]\partial^j
\end{aligned}$$

so we must have

$$\frac{\partial_j\partial_i H}{\partial^i H} - \frac{(\partial_i H)(\partial_j\partial^i H)}{(\partial^i H)^2} + \frac{\partial_j H}{\partial^j H}\left[-\frac{\partial^j\partial_i H}{\partial^i H} + \frac{(\partial_i H)(\partial^j\partial^i H)}{(\partial^i H)^2}\right] = 0$$

any multiplying up gives the □

Definition 3.2. *A **natural Hamiltonian** is one of the form* $H(q,p) = \frac{1}{2}g^{ij}(q)p_i p_j + V(q)$

Definition 3.3. *A **standard coordinate system** is a coordinate system* $q = (q^i) = (q^a, q^\alpha)$ *with* $a = 1, \ldots, m$, $\alpha = m+1, \ldots, n$ *such that*

- *the metric tensor assumes the semi-diagonal standard form*

$$\boldsymbol{g} = g^{aa}\partial_a \otimes \partial_a + g^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$$

- *the coordinates* (q^α) *are **ignorable**, i.e.*

$$\begin{aligned}
\partial_\alpha g^{ij} &= 0 \\
\partial_\alpha V &= 0
\end{aligned}$$

The coordinates (q^a) *are called **essential**.*

Making the definition[3] that for $f \in C^\infty(M)$

$$S_{ij}(f) = \partial_i\partial_j f - (\partial_i \log g^{jj})(\partial_j f) - (\partial_j \log g^{ii})(\partial_i f)$$

we have the following result

Proposition 3.4. *In a standard coordinate system for a natural Hamiltonian the LCSCs are*

$$\begin{aligned}
S_{ab}(g^{cc}) &= 0 \\
S_{ab}(g^{\alpha\beta}) &= 0 \\
S_{ab}(V) &= 0
\end{aligned}$$

In the case of an orthogonal coordinate system we just have

$$\begin{aligned}
S_{ij}(g^{kk}) &= 0 \\
S_{ij}(V) &= 0
\end{aligned}$$

[3]Note here that $\partial_i \log g^{jj} = \frac{\partial_i g^{jj}}{g^{jj}}$ for $g^{jj} < 0$. Here we know $g^{jj} \neq 0$ as throughout we assume a non-degenerate metric.

Proof. For completeness we reproduce the beginning of the proof here. Clearly the 3 contributions from $g^{aa}, g^{\alpha\beta}, V$ will separate due to the order of the momenta in these terms. Hence we will show this initially just for the g^{aa} term. As the q^α are ignorable we do not need to think about ∂_α terms. Hence we get

$$\begin{aligned}
\partial_a H &= \frac{1}{2}(\partial_a g^{cc})p_c^2 \\
\partial^a H &= g^{ac}p_c = g^{aa}p_a \\
\partial_a\partial_b H &= \frac{1}{2}(\partial_a\partial_b g^{cc})p_c^2 \\
\partial_a\partial^b H &= (\partial_a g^{bc})p_c = (\partial_a g^{bb})p_b \\
\partial^a\partial^b H &= g^{ab} = 0
\end{aligned}$$

so the LCSCs are

$$\begin{aligned}
\frac{1}{2}(\partial_a\partial_b g^{cc})g^{aa}g^{bb}p_ap_bp_c^2 - \frac{1}{2}(\partial_b g^{aa})(\partial_a g^{cc})g^{bb}p_ap_bp_c^2 - \frac{1}{2}(\partial_a g^{bb})(\partial_b g^{cc})g^{aa}p_ap_bp_c^2 = 0 \\
\Rightarrow \frac{1}{2}g^{aa}g^{bb}p_ap_bp_c^2 S_{ab}(g^{cc}) = 0
\end{aligned}$$

$\square$

Remark. *We can consider adding a term p_iA^i to a natural Hamiltonian. This modifies the LCSCs to give*

$$\begin{aligned}
0 &= \left(g^{ir}g^{js}\partial_{ij}g^{kl} - g^{ir}\partial_i g^{js}\partial_j g^{kl} - g^{js}\partial_j g^{ir}\partial_i g^{kl} + \frac{1}{2}g^{ij}\partial_i g^{kl}\partial_j g^{rs}\right)p_kp_lp_rp_s \\
0 &= \big(2g^{ir}g^{js}\partial_{ij}A^k + A^ig^{jk}\partial_{ij}g^{rs} + A^jg^{ik}\partial_{ij}g^{rs} + g^{ij}\partial_i g^{rs}\partial_j A^k + g^{ij}\partial_j g^{rs}\partial_i A^k \\
&\quad - g^{ik}\partial_i A^j\partial_j g^{rs} - g^{jk}\partial_j A^i\partial_i g^{rs} - A^i\partial_i g^{jk}\partial_j g^{rs} \\
&\qquad - A^j\partial_j g^{ik}\partial_i g^{rs} - 2g^{ir}\partial_i g^{js}\partial_j A^k - 2g^{jr}\partial_j g^{is}\partial_i A^k\big)p_kp_rp_s \\
0 &= \Big(g^{ir}\partial_i g^{js}\partial_i V + g^{jr}\partial_j g^{js}\partial_j V - g^{ir}g^{js}\partial_{ij}V - \frac{1}{2}g^{ij}\partial_i g^{rs}\partial_j V - \frac{1}{2}g^{ij}\partial_j g^{rs}\partial_i V \\
&\quad + g^{ir}\partial_i A^j\partial_j A^s + g^{jr}\partial_j A^i\partial_i A^s - g^{ir}A^j\partial_{ij}A^s - g^{jr}A^i\partial_{ij}A^s - \frac{1}{2}A^iA^j\partial_{ij}g^{rs} \\
&\qquad + A^i\partial_i g^{jr}\partial_j A^s + A^j\partial_j g^{ir}\partial_i A^s + A^i\partial_i A^j\partial_j g^{rs} + A^j\partial_j A^i\partial_i g^{rs} - g^{ij}\partial_i A^r\partial_j A^s\Big)p_rp_s \\
0 &= g^{ij}\partial_i A^k\partial_j V + g^{ij}\partial_j A^k\partial_i V + A^ig^{jk}\partial_{ij}V + A^jg^{ik}\partial_{ij}V + A^iA^j\partial_{ij}A^k \\
&\quad - g^{ik}\partial_i A^j\partial_j V - g^{jk}\partial_j A^i\partial_i V - A^i\partial_i g^{jk}\partial_j V - A^j\partial_j g^{ik}\partial_i V - A^i\partial_i A^j\partial_j A^k - A^j\partial_j A^i\partial_i A^k \\
0 &= A^i\partial_i A^j\partial_j V + A^j\partial_j A^i\partial_i V - A^iA^j\partial_{ij}V - g^{ij}\partial_i V^j\partial_j V
\end{aligned}$$

To see this now note (using compact notation for the derivative)

$$\begin{aligned}
\partial_i H &= \frac{1}{2} g^{kl}_{,i} p_k p_l + p_k A^k_{,i} + V_{,i} \\
\partial^i H &= g^{ik} p_k + A^i \\
\partial_i \partial_j H &= \frac{1}{2} g^{kl}_{,ij} p_k p_l + p_k A^k_{,ij} + V_{,ij} \\
\partial_i \partial^j H &= g^{jk}_{,i} p_k + A^j_{,i} \\
\partial^i \partial^j H &= g^{ij}
\end{aligned}$$

So the LCSCs in their full gory detail are

$$\begin{aligned}
0 = g^{ij} &\left(\frac{1}{2} g^{kl}_{,i} p_k p_l + p_k A^k_{,i} + V_{,i}\right)\left(\frac{1}{2} g^{rs}_{,j} p_r p_s + p_r A^r_{,j} + V_{,j}\right) + \left(\frac{1}{2} g^{kl}_{,ij} p_k p_l + p_k A^k_{,ij} + V_{,ij}\right)\left(g^{ir} p_r + A^i\right)\left(g^{js} p_s + A^j\right) \\
&- \left(g^{js}_{,i} p_s + A^j_{,i}\right)\left(g^{ir} p_r + A^i\right)\left(\frac{1}{2} g^{kl}_{,j} p_k p_l + p_k A^k_{,j} + V_{,j}\right) - \left(g^{ir}_{,j} p_r + A^i_{,j}\right)\left(g^{js} p_s + A^j\right)\left(\frac{1}{2} g^{kl}_{,i} p_k p_l + p_k A^k_{,i} + V_{,i}\right)
\end{aligned}$$

To split these up we do as before, separating them by order of p in the terms. We will then use that the values of p are arbitrary to set the coefficient to be 0. As we will have many free indices to be summed over we need to symmetrise over these indices. Do not forget that the indices i, j are not summed over. We can also use the symmetry in $i \leftrightarrow j$ in the two negative terms to compactify the formula more. This process gives:

$$\begin{aligned}
0 &= \frac{1}{2} g^{ij} g^{(kl}_{,i} g^{rs)}_{,j} + g^{i(r} g^{kl}_{,ij} g^{s)j} - \left[g^{i(r} g^{kl}_{,j} g^{s)j}_{,i} + g^{i(r}_{,j} g^{kl}_{,i} g^{s)j}\right] \\
0 &= g^{ij} g^{(kl}_{(,i} A^{r)}_{,j)} + A^{(j} g^{i)(r} g^{kl)}_{,ij} + g^{i(r} A^k_{,ij} g^{l)j} - \frac{1}{2}\left[g^{i(r} g^{kl)}_{,j} A^j_{,i} + A^i_{,j} g^{(kl}_{,i} g^{r)j}\right] \\
&\quad - \frac{1}{2}\left[A^i g^{(kl}_{,j} g^{r)j}_{,i} + g^{i(r}_{,j} g^{kl)}_{,i} A^j\right] - \left[g^{i(r} A^k_{,j} g^{l)j}_{,i} + g^{i(r}_{,j} A^k_{,i} g^{l)j}\right] \\
0 &= g^{ij}\left[g^{kl}_{(,i} V_{,j)} + A^k_{(,i} A^l_{,j)}\right] + \frac{1}{2} g^{kl}_{,ij} A^i A^j + 2 A^{(j} g^{i)(l} A^{k)}_{,ij} + V_{,ij} g^{k(i} g^{j)l} - \frac{1}{2}\left[A^i g^{kl}_{,j} A^j_{,i} + A^i_{,j} g^{kl}_{,i} A^j\right] \\
&\quad - \left[A^i A^{(k}_{,j} g^{l)j}_{,i} + g^{i(l} A^{k)}_{,j} A^j_{,i} + g^{i(l}_{,j} A^{k)}_{,i} A^j + A^i_{,j} A^{(k}_{,i} g^{l)j}\right] - \left[V_{,j} g^{i(k} g^{l)j}_{,i} + V_{,i} g^{i(k}_{,j} g^{l)j}\right] \\
0 &= 2 g^{ij} A^k_{(,i} V_{,j)} + A^k_{,ij} A^i A^j + 2 A^{(j} g^{i)k} V_{,ij} - \left[A^i A^k_{,j} A^j_{,i} + A^i_{,j} A^k_{,i} A^j\right] \\
&\quad - \left[V_{,j}\left(g^{ik} A^j_{,i} + A^i g^{jk}_{,i}\right) + V_{,i}\left(g^{ik}_{,j} A^j + A^i_{,j} g^{jk}\right)\right] \\
0 &= g^{ij} V_{,i} V_{,j} + V_{,ij} A^i A^j - \left[A^i V_{,j} A^j_{,i} + A^i_{,j} V_{,i} A^j\right]
\end{aligned}$$

Proposition 3.5. *The general solution of the LCSCs in a standard coordinate system is*

$$\begin{aligned}
g^{aa} &= \varphi^a_{(m)} \\
g^{\alpha\beta} &= \phi^{\alpha\beta}_a g^{aa} \\
V &= \phi_a g^{aa}
\end{aligned}$$

where $[\varphi^{(b)}_a]$ is a $m \times m$ Stäckel matrix, the a^{th} row depending on q^a only, and $\phi^{\alpha\beta}_a, \phi_a$ are functions *depending on q^a only. In this case the HJE splits to*

$$p_\alpha = c_\alpha$$
$$\frac{1}{2}p_a^2 = \varphi_a^{(b)} c_b - \phi_a^{\alpha\beta} c_\alpha c_\beta - \phi_a$$

where $c = (c^i) = (c^a, c^\alpha)$ are arbitrary constants.

Proof. A standard proof of Stäckel's theorem shows that we must have the Stäckel matrix $\varphi_a^{(b)}$ and the ϕ_a. To get the $\phi_a^{\alpha\beta}$, knowing the p_α are ignorable, we know that in the separated HJE the equation for them will be $p_\alpha = c_\alpha$. Hence we can consider $g^{\alpha\beta} p_\alpha p_\beta$ as an additional potential term, and finding

$$g^{\alpha\beta}(q^a) p_\alpha p_\beta = \Phi_a(q^a, p_\alpha) g^{aa} = \phi_a^{\alpha\beta}(q^a) p_\alpha p_\beta$$

□

Remark. *According to [1] the integral curves of the Hamiltonian with metric g and potential V at fixed energy E are equivalent to those of the Hamiltonian with no potential and metric $(V - E)g$, which is a consequence of the Maupertuis' principle. As such we can choose to ignore the potential and absorb its effect into the metric. This requires that V be bounded below, and we take $E < \inf V$ in order to keep the new metric non-degenerate. If instead V is bounded abouve we can take $E > \sup V$ and then consider $(E - V)g$. If V is not bounded, we need to restrict to a compact subset of Q.*

3.2 Equivalent separable systems

To start to extend this theory to ask what more general coordinate systems are allowed, we make definitions which allows us to make precise questions:

Definition 3.6. *Let (q^i) be a system of separable coordinates. We say a coordinate q^i is **first class** if R_i is linear in the momenta (p_j). Otherwise we say the coordinate is **second class**.*

Remark. *Note that ignorable coordinates are first class. As such, to correspond with previous section, we will indicate first class coordinates with Greek letters $(\alpha, \beta, \dots)$ and second class coordinates with Latin indices $(a, b, \dots)$.*

Definition 3.7. *Two separable coordinate systems are **equivalent** if in everywhere they are simultaneously defined the give rise to the same foliation of the cotangent bundle M.*

Definition 3.8. *A **separated transformation** is one whose Jacobian is diagonal.*

Proposition 3.9. *Separated transformations preserve the separability property.*

Remark. *This characterisation of separability can be viewed in different ways. This way emphasises geometric aspects. A more algebraic definition would be to say that the two coordinate systems give the same S, the solution of the HJE. By "the same" I mean in the sense of the following diagram:*

$$\begin{tikzcd}
\mathbb{R}^n \arrow[drr, "S^x"'] & \mathbb{R}^{2n} \arrow[l, "\pi_x"'] & M \arrow[l, "{(x,y)}"'] \arrow[r, "{(q,p)}"] \arrow[d, dashed, "S"] & \mathbb{R}^{2n} \arrow[r, "\pi_q"] & \mathbb{R}^n \arrow[dll, "S^q"] \\
 & & \mathbb{R} & &
\end{tikzcd}$$

that is, if two local trivialisations of M, $(x,y),(q,p)$ have solutions to the HJE in these coordinates S^x, S^q respectively, then the corresponding pulled-back map on M, is the same for both.

With the above definitions we can start by making an immediate claim

Proposition 3.10. *Two equivalent separable systems have the same number of first class coordinates*

Proof. Let $(q^i), (\overline{q}^{\overline{\imath}})$ be equivalent systems of separable coordinates. Let the change of coordinate matrices be given by

$$A^i_{\overline{\jmath}} = \frac{\partial q^i}{\partial \overline{q}^{\overline{\jmath}}}$$
$$\overline{A}^{\overline{\jmath}}_i = \frac{\partial \overline{q}^{\overline{\jmath}}}{\partial q^i}$$

As the separable systems are equivalent, we must have $p_i = \partial_i S$ and $\overline{p}_{\overline{\jmath}} S$, hence we can write

$$p_i = \overline{A}^{\overline{\jmath}}_i \overline{p}_{\overline{\jmath}}$$
$$\overline{p}_{\overline{\jmath}} = A^i_{\overline{\jmath}} p_i$$

Applying $\overline{\partial}_{\overline{\jmath}} = A^i_{\overline{\jmath}} \partial_i$ to the first eq gives

$$\begin{aligned} A^j_{\overline{\jmath}} \partial_j p_i &= A^j_{\overline{\jmath}} \partial_j (\overline{A}^{\overline{\imath}}_i \overline{p}_{\overline{\imath}}) \\ &= A^j_{\overline{\jmath}} (\partial_j \overline{A}^{\overline{\imath}}_i) \overline{p}_{\overline{\imath}} + \overline{A}^{\overline{\imath}}_i \overline{\partial}_{\overline{\jmath}} \overline{p}_{\overline{\imath}} \\ \Rightarrow A^i_{\overline{\jmath}} R_i &= A^j_{\overline{\jmath}} (\partial_j \overline{A}^{\overline{\imath}}_i) \overline{p}_{\overline{\imath}} + \overline{A}^{\overline{\jmath}}_i \overline{R}_{\overline{\jmath}} \quad \text{(no sum over } i, \overline{\jmath} \text{)} \end{aligned}$$

If we choose $i = \alpha$ to be the index of a first class coordinate, and $\overline{\jmath} = \overline{a}$ to be second class, then we have that R_α is linear, in momenta, but $R_{\overline{a}}$ is not. This is a contradiction unless $\overline{A}^{\overline{a}}_\alpha = 0$. By symmetry of the argument it must also be true that $A^\alpha_{\overline{a}} = 0$. As such we know that the second class coordinates in one system must completely determine those in the other, and so there are equally many. □

We may couple this results with a useful one about first class coordinates:

Proposition 3.11. *Every separable system is equivalent to a separable system in which all first class coordinates are ignorable.*

Proof. For the first class coordinates write

$$\partial_\alpha p_\alpha = B^i_\alpha p_i$$

The Frobenius integrability condition gives that

$$\begin{aligned} \left[\partial_i + R_i \partial^i, \partial_j + R_j \partial^j\right] &= 0 \\ \Rightarrow \partial_i R_j + (\partial^j R_j) R_i &= 0 \end{aligned}$$

and so we find

$$\partial_a(B^i_\alpha p_i) + B^a_\alpha R_a \quad \text{(no sum over } a\text{)}$$

Hence R_a is linear in momenta () unless $B^a_\alpha = 0$, which forces $\partial_a B^i_\alpha = 0$. This gives an autonomous subsystem in first class coordinates given by

$$\partial_\alpha p_\alpha = B^\beta_\alpha p_\beta$$

This system is integrable and linear, and so $\exists \phi^{(\beta)}$ a system of independent solutions s.t. the general solution for the momenta is given by

$$p_\alpha = c_\beta \phi^{(\beta)}_\alpha$$

Define a new coordinate system (x^i) by

$$dx^a = dq^a$$
$$dx^\alpha = \phi^{(\alpha)}_\beta dq^\beta$$

We can calculate

$$p_\alpha dq^\alpha = dS = \frac{\partial S}{\partial x^\beta} dx^\beta = y_\beta \phi^{(\beta)}_\alpha dq^\alpha$$

where we have let $y_\beta = \frac{\partial S}{\partial x^\beta}$ be the conjugate momenta to the coordinate x^β. This shows that $y_\beta = c_\beta$, As such the x^α are ignorable. Since the second class coordinates are unchanged, the system remains separable, and so the two systems are equivalent. ☐

Proposition 3.12. *In two equivalent separable systems the second class coordinates are related by a separated transformation.*

Proof. See [1]. The proof is not replicated here as it currently does not provide any additional useful information. ☐

We finish this subsection with a result that will be uesful when talking about what separable systems correspond to Einstein metrics:

Proposition 3.13. *Let $(q^i) = (q^a, q^\alpha)$ be a separable system s.t. the q^α are ignorable. Then every equivalent separable system $(x^i) = (x^a, x^\alpha)$ is related (up to a separable transform) by transform of the kind*

$$dq^a = dx^a$$
$$dq^\alpha = A^\alpha_i dx^i$$

where $A^\alpha_i = A^\alpha_i(x^i)$ is a function only of the coordinate corresponding to the lower index.

Proof. See [1]. The proof is not replicated here as it currently does not provide any additional useful information. ☐

3.3 Generality of standard coordinate systems

In this section, we will build upon the results of equivalent coordinate systems to find the most general separable HJE for a natural Hamiltonian, and then show it is equivalent to the a standard coordinate system. A question we will eventually want to ask is what separable systems correspond to Einstein manifolds, and it is then necessary to know whether if a system is Einstein whether equivalent systems are also Einstein[4].

Proposition 3.14. *In a separable system $(q^i) = (q^a, q^\alpha)$, the second class coordinates are orthogonal*

Proof. Let $H_a = \partial_a H, H^a = \partial^a H$, etc. and then the Levi-Civita separability conditions for second class coordinates gives

$$H^a(H^b H_{ab} - H_b H^b_a) = H_a(H^b H^a_b - H_b H^{ab}) \quad \text{(no sum over } a, b)$$

Note that $H_a = \frac{1}{2}(\partial_a g^{ij})p_i p_j$, and $H^a = g^{aj}p_j$, so as $R_a = \frac{H_a}{H^a}$ is not linear in momenta, it must be that H^a is not a factor of H_a. Then the equation above gives that

$$\begin{aligned} H^a \,|\, H^b H^a_b - H_b g^{ab} \\ \Rightarrow H^b H^a_b - H_b g^{ab} = L_{ba} H^a \end{aligned}$$

for some $L_{ba} = L^i_{ba} p_i$ a linear polynomial in momenta. Substituting back into our Levi-Civita condition gives

$$H^b H_{ab} - H_b H^b_a = H_a L_{ba}$$

Note that if $L_{ba} = 0$ then $R_b = \frac{H^a_b}{g^{ab}} = \frac{(\partial_b g^{aj})p_j}{g^{ab}}$ is linear in momenta .
Hence we must have $L_{ba} \neq 0$. □

The above result allows us to distinguish second class coordinates into two cases

1. $q^{aa} = 0$, said to be **isotropic**
2. $q^{aa} \neq 0$

Taking, as has been done previously, there to be m second class coordinates, we split these into m_1 non-isotropic coordinates, and $m_1 = m - m_1$ isotropic coordinates. This allows us to make the following claim:

Proposition 3.15. *In each equivalence of separable coordinates, $\exists (q^i) = (q^{\tilde{a}}, q^{\overline{a}}, q^\alpha)$ a coordinate system s.t.*

- *The first class coordinates are q^α, which are ignorable*
- *The $q^{\tilde{a}}$ are non-isotropic second class coordinates with $g^{\tilde{a}\alpha} = 0$*
- *The $q^{\overline{a}}$ are isotropic second class coordinates*

[4] Are we inclined to think this is the case? Consider conformal equivalences at least in the next section

i.e. the metric looks like

$$(g^{ij}) = \begin{pmatrix} g^{\tilde{a}\tilde{a}} & 0 & 0 \\ 0 & 0 & g^{\overline{a}\beta} \\ 0 & g^{\alpha\overline{a}} & g^{\alpha\beta} \end{pmatrix}$$

Such coordinates are called ***normal separable coordinates***

Proof. Building upon previous results, it remains to be shown that we can find a separability-preserving transform making $g^{\tilde{a}\alpha} = 0$.
The content of the proof in [1] comes in showing that $\theta^\alpha_a = \frac{g^{a\alpha}}{g^{aa}}$ is a function of q^a only if $g^{aa} \neq 0$. Then the coordinate transform

$$dx^a = dq^a$$
$$dx^\alpha = dq^\alpha - \theta^\alpha_{\tilde{a}} dq^{\tilde{a}}$$

preserves the separability property by previously shown results, keeps the first class coordinates ignorable, and moreover gives $g^{\tilde{a}\alpha} = 0$ in the new coordinates. □

Remark. *Note that we must have at least one $g^{\overline{a}\alpha} \neq 0$ in order to have a non-degenerate metric.*

We can now remark the result that I mentioned near the beginning of this section

Theorem 3.16. *All separable coordinate systems on a manifold with positive definite metric are equivalent to a standard coordinate system where $\boldsymbol{g} = g^{aa}\partial_a \otimes \partial_a + g^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$.*

4 Einstein Manifolds

We start by recalling what an Einstein metric is:

Definition 4.1. *A metric is* ***Einstein*** *if $\exists \lambda \in \mathbb{R}$, $\mathrm{Ric} = \lambda g$.*

It is perhaps an interesting question to ask what kinds of Einstein metrics can be separated. We should possibly ask for some motivation that these two could be related to start with.

4.1 Motivation - Kerr-NUT-(A)dS metrics

We are now going to cover the essential results about a motivatin example of the Kerr-NUT-(A)dS metrics. See [3] and references therein for the origin of this.

Definition 4.2. *The canonical metric describinig the Kerr-NUT-(A)dS geometry in $D = 2n + \epsilon$ ($\epsilon = 0, 1$) dimensions is*

$$g = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu} dx_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_{j=0}^{n-1} A_\mu^{(j)} d\psi_j \right)^2 \right] + \epsilon \frac{c}{A^{(n)}} \left(\sum_{k=0}^{n} A^{(k)} d\psi_k \right)^2$$

where

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k}}^{n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k=1\\ \nu_1<\dots<\nu_k\\ \nu_i\neq\mu}}^{n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 \qquad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2)$$

and $X_\mu = X_\mu(x_\mu)$*.* c *is a free scalar parameter.*

It is not too hard to see this metric is in Stäckel form. The corresponing linear and quadratic constants of motion are given by

$$L_j = p_{\psi_j}$$
$$K_j = \sum_\mu A_\mu^{(j)} \left[\frac{X_\mu}{U_\mu} p_{x_\mu}^2 + \frac{U_\mu}{X_\mu} \left(\sum_{k=0}^{n-1+\epsilon} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} p_{\psi_k} \right)^2 \right] + \epsilon \frac{A^{(j)}}{cA^{(n)}} p_{\psi_n}^2$$

Proposition 4.3. *The metric satisfies the Einstein equations* $R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0$ *iff*

$$X_\mu = \begin{cases} -2b_\mu x_\mu + \sum_{k=0}^{n} c_k x_\mu^{2k} & \epsilon = 0 \\ -\frac{c}{x_\mu^2} - 2b_\mu + \sum_{k=1}^{n} c_k x_\mu^{2k} & \epsilon = 1 \end{cases}$$

where $\boldsymbol{b}, \boldsymbol{c}$ *are free vector parameters with* c_n *fixed by*

$$\Lambda = \frac{1}{2}(-1)^n (D-1)(D-2)c_n$$

This shows that we can generate large classes of metric which are separable that involve both first and second class coordinates. The metrics need not always satisfy the Einstein equations, but we can find conditions on the metric functions such that this holds. The structure that achieves this in the case of the Kerr-NUT-(A)dS metric is the **principal tensor**. Such a principal tensor immedately gives a separability structure of an algebra of Killing tensors and vectors which is sufficient for separability. The principal tensor has the added benefit that it gives a canonical frame basis in which the Ricci tensor and metric are immediately diagonal. This gives the Einstein conditions as DEs for the metric components.

4.2 The general question

Example 4.4. *We note that in [4] it is shown that the Riemannian manifolds of constant curvature all admit metrics which can be separated in (often multiple) coordinates system. It is a known fact that all of these manifolds are Einstein, so we are perhaps given initial hope that this isn't a foolish question.*

We can now consider the simplest example of a diagonal metric $\boldsymbol{g} = g^{aa}\partial_a \otimes \partial_a$. We will assume $g^{aa} > 0$. We have from [5] that the off-diagonal and diagonal components of the Ricci tensor are separately given by

$$4R_{ab} = \sum_{c\neq a,b} [(\partial_a \log g_{bb} - \partial_a)\partial_b \log g_{cc} + (b \leftrightarrow a) - (\partial_a \log g_{bb})(\partial_b \log g_{aa})]$$
$$4R_{aa} = (\partial_a \log g_{aa} - 2\partial_a)\, \partial_a \log \frac{g}{g_{aa}} - \sum_{c\neq a} \left[(\partial_a \log g_{cc})^2 + \left(\partial_c \log \frac{g}{g_{aa}^2} + 2\partial_c \right) g^{cc} \partial_c g_{aa} \right]$$

Immediately, we know that a general diagonal metric is non-Einstein, as the Ricci tensor is not diagonal.
Recall we know $S_{ab}(g^{cc}) = 0$, can this be used to simplify R_{ab}?

As previously mentioned, and immediately useful question to ask is whether transforms that preserve the separability property preserve the Einstein property. By our discussion above, we see that we need only consider two types of transform

1. Separable transformations, i.e. $\frac{\partial \bar{q}^{\bar{\imath}}}{\partial q^i} = \delta_i^{\bar{\imath}} \xi^i(q^i)$
2. Transformation of first class coordinates of the form $dx^\alpha = A_i^\alpha(q^i)dq^i$

Proposition 4.5. *Separable transformations preserve the Einstein property*

Proof. The Einstein equation is tensorial, so is diffeomorphism invariant. □

A separate question we might like to ask is what conformal transforms preserve both the separability property and the Einstein property. Suppose we make the transform $H \to \tilde{H} = e^{2f}H$ for $f \in C^\infty(Q)$, the LCSCs become (using the notation $\partial_j f = f_j,\ ,\partial_i\partial_j f = f_{ij}$, $\partial^i H = H^i, \dots$)

$$\begin{aligned}
0 =& e^{6f}\left\{H^{ij}(H_i + Hf_i)(H_j + Hf_j) + (Hf_if_j + Hf_{ij} + H_if_j + H_jf_i + H_{ij})H^iH^j\right.\\
&\left.-(H^i_j + H^if_j)(H_i + Hf_i)H^j - (H^j_i + H^jf_i)(H_j + Hf_j)H^i\right\}\\
\Rightarrow 0 =& H^{ij}(HH_if_j + HH_jf_i + H^2f_if_j) + H^iH^j(Hf_if_j + Hf_{ij} + H_if_j + H_jf_i)\\
&- H^j(HH^i_jf_i + H^iH_if_j + H^iHf_if_j) - H^i(HH^j_if_j + H^jH_jf_i + H^jHf_if_j)\\
=& H\left[H^{ij}(H_if_j + H_jf_i + Hf_if_j) + H^iH^j(f_{ij} - f_if_j) - (H^jH^i_j + H^iH^j_i)\right]
\end{aligned}$$

If we have a diagonal separable metric g^{aa} s.t. $H = \frac{1}{2}g^{aa}p_a^2$ we get

$$0 = \left\{f_{ab} - f_af_b - (\partial_b \log g^{aa} + \partial_a \log g^{bb})\right\} g^{aa}g^{bb}p_ap_b$$

5 Geometric Characterisation

5.1 Stäckel Webs

We will now develop the geometric theory to give conditions on the existence of separability. To have all the properties of Killing tensors under your belt, look at the KYT notes.
We start with some definitions:

Definition 5.1. *An **orthogonal web** on $(Q, \boldsymbol{g})$ is a family $(\mathcal{S}_i) = (\mathcal{S}_1, \dots, \mathcal{S}_n)$ of n orthogonal foliations of hypersurfaces defined on $M \setminus \Omega$ where Ω is some closed singular set. A **parametrisation** of the orthogonal foliation is a set of coordinates $\{q_i\}$ on $Q \setminus \Omega$ s.t. $\mathcal{S}_i = \{q_i = const\}$.*

Definition 5.2. *An orthogonal web whose coordinates give a separable HJE for the geodesic Hamiltonian is called a **Stäckel web**.*

Definition 5.3. *A **Killing-Stäckel algebra (KSA)** is a n dimensional subspace of the space of Killing tensors of order 2 on Q, $\mathcal{K}$, s.t on $Q \setminus \Omega$*

- *$\mathcal{K}$ has a basis of pointwise independent elements*
- *the elements of $\mathcal{K}$ have common eigendirections*
- *the eigendirections are normal (i.e orthogonally integrable).*

Proposition 5.4. *An orthogonal web is a Stäckel web iff its leaves are integral manifolds of a KSA.*

Proof. Let $\{q_i\}$ be a parametrisation of the orthogonal web and $\partial_i = \frac{\partial}{\partial q_i}$ We have that the condition of $\mathcal{S}_i$ being foliations gives that their tangent spaces given by Span $\{\partial_1, \ldots, \partial_n\}$ are involutive. Let ρ_i be eigenvalues of a Killing tensor that is diagonalised wrt the parametrisation, then

$$\begin{aligned}
0 =&\partial_i(\partial_j \rho_k) - \partial_j(\partial_i \rho_k) \\
=&\partial_i \left[(\rho_j - \rho_k)\partial_j \log g^{kk}\right] - \partial_j \left[(\rho_i - \rho_k)\partial_i \log g^{kk}\right] \\
=&\partial_j \log g^{kk} \left[(\rho_i - \rho_j)\partial_i \log g^{jj} - (\rho_i - \rho_k)\partial_i \log g^{kk}\right] \\
&- \partial_i \log g^{kk} \left[(\rho_j - \rho_i)\partial_j \log g^{ii} - (\rho_j - \rho_k)\partial_j \log g^{kk}\right] - (\rho_i - \rho_j)\partial_i\partial_j \log g^{kk} \\
=&(\rho_i - \rho_j) \left[(\partial_j \log g^{kk})(\partial_i \log g^{jj}) + (\partial_i \log g^{kk})(\partial_j \log g^{ii}) - (\partial_i \log g^{kk})(\partial_j \log g^{kk}) - \partial_i\partial_j \log g^{kk}\right] \\
=&(\rho_i - \rho_j) \left[\frac{\partial_j g^{kk}}{g^{kk}}\partial_i \log g^{jj} + \frac{\partial_i g^{kk}}{g^{kk}}\partial_j \log g^{ii} - \frac{\partial_i g^{kk}}{g^{kk}}\frac{\partial_j g^{kk}}{g^{kk}} - \left(\frac{\partial_i\partial_j g^{kk}}{g^{kk}} - \frac{\partial_i g^{kk}}{g^{kk}}\frac{\partial_j g^{kk}}{g^{kk}}\right)\right]
\end{aligned}$$

Hence we have

$$(\rho_i - \rho_j)(\partial_i\partial_j g^{kk} - \partial_i \log g^{jj}\partial_j g^{kk} - \partial_j \log g^{ii}\partial_i g^{kk}) = (\rho_i - \rho_j)S_{ij}(g^{kk}) = 0$$

$(\Rightarrow)$: Assume that the orthogonal coordinates are separable, then the above equations are immediately satisfied by the previous lemma, and so we have n independent solutions $(\rho_a)_i = \rho_i^{(a)}$. As such we have a Killing tensor basis of a KSA $\mathcal{K}$ given by

$$\left\{\boldsymbol{T}^{(a)} = \sum_i \rho_i^{(a)} dq^i \otimes \partial_i\right\}$$

The leaves of the foliation are then given by $\left\{q^i = \text{const}\right\}$, which are exactly the integral manifold of $\mathcal{K}$, defined on the same range $M \setminus \Omega$.
$(\Leftarrow)$: Conversely, if such a Stäckel system exists, we may take a basis corresponding to the directions ∂_i and then the functions $\rho_i^{(a)}$ form s complete solution set to integrability equation condition. As we cannot have $\rho_i^{(a)} - \rho_j^{(a)} = 0$ for all a (as then the solutions would not be independent), it must be that case that the separability condition holds. □

We can go even farther with the following theorem:

Theorem 5.5. *A KSA is uniquely determined by a Killing tensor with normal eigenvectors and pointwise simple eigenvalues.*

Proposition 5.6. *A potential $V : Q \to \mathbb{R}$ is compatible with respect to a Stäckel web generated by a KSA $\mathcal{K}$ iff for $\boldsymbol{T}$ generating $\mathcal{K}$*

$$d(\boldsymbol{T} \cdot dV) = 0$$

Proof. Let $\boldsymbol{T} = \sum_i \rho_i dq^i \otimes \partial_i$. Then $\boldsymbol{T} \cdot dV = \sum_i (\rho_i \partial_i V) dq^i$ and so the condition in the statement is equivalent to

$$\partial_j(\rho_i \partial_i V) - \partial_i(\rho_j \partial_j V) = 0$$

Which using the results from Killing tensors gives

$$\begin{aligned} 0 &= (\rho_i - \rho_j)\partial_i\partial_j V + \partial_i V \left[(\rho_j - \rho_i)\partial_i \log g^{jj}\right] - \partial_j V \left[(\rho_i - \rho_j)\partial_j \log g^{ii}\right] \\ &= (\rho_i - \rho_j)\left[\partial_i\partial_j V - (\partial_i \log g^{jj})(\partial_j V) - (\partial_j \log g^{ii})(\partial_i V)\right] \\ &= (\rho_i - \rho_j) S_{ij}(V) \end{aligned}$$

Hence we are done. □

Proposition 5.7. *Let $\boldsymbol{T}^{(a)}$ be a basis of a Stäckel system, and V a separable potential. Define $V^{(a)} : M \to \mathbb{R}$ locally by*

$$dV^{(a)} = \boldsymbol{T}^{(a)} \cdot dV$$

and the functions $c_a : M \to \mathbb{R}$ by

$$c_a = J_{T^{(a)}} + \pi^* V^{(a)}$$

where $\pi : M \to Q$ is the cotangent fibration. Then the c_a are independent functions in involution

Proof.

$$\begin{aligned} \{c_a, c_b\} &= \left\{J_{T^{(a)}} + \pi^* V^{(a)}, J_{T^{(b)}} + \pi^* V^{(b)}\right\} \\ &= J_{\left[\boldsymbol{T}^{(a)}, \boldsymbol{T}^{(b)}\right]} + J_{\boldsymbol{T}^{(a)} \cdot dV^{(b)}} - J_{\boldsymbol{T}^{(b)} \cdot dV^{(a)}} \\ &= J_{(\boldsymbol{T}^{(a)} \cdot \boldsymbol{T}^{(b)} - \boldsymbol{T}^{(b)} \cdot \boldsymbol{T}^{(a)}) \cdot dV} = 0 \end{aligned}$$

□

5.2 Properties of KSAs

A few useful characterisations of the above are as follows:

Proposition 5.8. *A tensor $\boldsymbol{T}$ has distinct eigenvalues iff*

$$D \equiv \begin{vmatrix} n & S_1 & \cdots & S_{n-1} \\ S_1 & S_2 & \cdots & S_n \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1} & S_n & \cdots & S_{2n-2} \end{vmatrix} \neq 0$$

where $S_p \equiv \mathrm{Tr}(\boldsymbol{T}^p)$

Proof. Follows from Sylvester's theorem on the discriminant D of an algebraic equation, applied to the characteristic equation of $\boldsymbol{T}$ □

Proposition 5.9. *A symmetric tensor* $\boldsymbol{T}$ *with simple eigenvalues has normal eigenvectors iff*

$$H^c_{ab}T^a_d T^b_e + 2H^b_{a[d}T^a_{e]}T^c_b + H^a_{de}T^c_b T^b_a = 0$$

where

$$H^c_{ab}(\boldsymbol{T}) \equiv 2T^d_{[a}\partial_{|d|}T^c_{b]} - 2T^c_{[d}\partial_{|a|}T^d_{b]}$$

is the ***Nijenhuis Torsion****.*

Proposition 5.10. *An* n*-dimensional space* $\mathcal{K}$ *of Killing tensors is a KSA if and only if its elements*

1. *commute as matrices:* $\forall \boldsymbol{T}, \boldsymbol{S} \in \mathcal{K}$, $\boldsymbol{TS} - \boldsymbol{ST} = 0$
2. *are in involution:* $\forall \boldsymbol{T}, \boldsymbol{S} \in \mathcal{K}$, $\{J_T, J_S\} = 0$

Proof. Note that commutation as matrices is equivalent to being simultaneously diagonalisable. Involutivity then gives normality of eigenvectors. □

5.3 Generalisation

We can now relax orthogonality as a condition and think more generally about a standard coordinate system:

Definition 5.11. *A Killing tensor* $\boldsymbol{T}$ *is said to be in* ***standard form*** *wrt to a standard coordinate system if*

$$\boldsymbol{T} = \rho_a g^{aa}\partial_a \otimes \partial_a + T^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$$

and $\partial_\alpha \rho_a = 0 = \partial_\alpha T^{\beta\gamma}$.

Definition 5.12. *A* ***separable Killing algebra*** *is a pair* $(D, \mathcal{K})$ *where*

- D *is a* r *dimensional linear space of commuting Killing vectors*
- $\mathcal{K}$ *is a* D*-invariant,* $(n-r)$*-dimensional vector space of Killing two tensors with* $m = n - r$ *normal simultaneous eigenvectors orthogonal to* D*.*

All the previous characterisations now generalise, in the sense that separability of the HJE is now equivalent to the existence of a separable Killing algebra. We remark that, moreover, we can quotient Q by the orbits of D to get $\tilde{Q}$, which is now an m-dimensional manifold with metric $\tilde{\boldsymbol{g}} = (g^{ab})$. As a consequence of the D invariance of $\mathcal{K}$ in the separable Killing algebra, it projects onto a KSA $\tilde{\mathcal{K}}$.

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