The Eisenhart Lift

Linden Disney-Hogg & Harry Braden

March 2020

1 The Eisenhart Lift

Consider the $(d+2)$-dimensional line element,

$$ds^2 = \hat{g}_{\mu\nu}\, dx^\mu\, dx^\nu = h_{ij}\, dx^i\, dx^j + 2dt\left(dv - \Phi dt + N_i dx^i\right), \tag{1.0.1}$$

where $i, j = 1, \ldots, d$, $x^{d+1} = t$, $x^{d+2} = v$ and Φ, N_i and h_{ij} are independent of the coordinate v. Then $\xi = \partial_v$ is a Killing vector. We have

$$\hat{g} = \begin{pmatrix} h_{ij} & N_i & 0 \\ N_j & -2\Phi & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{g}^{-1} = \begin{pmatrix} h^{ij} & 0 & -h^{ik}N_k \\ 0 & 0 & 1 \\ -h^{jk}N_k & 1 & 2\Phi + N_i h^{ij} N_j \end{pmatrix},$$

where h^{ij} is the inverse of h_{ij}. The geodesic Lagrangian is

$$\mathcal{L} = \frac{1}{2}\hat{g}_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu = \frac{1}{2}h_{ij}\, \dot{x}^i\, \dot{x}^j + \dot{t}\dot{v} - \Phi\dot{t}^2 + N_i\, \dot{x}^i\dot{t} := \tilde{L} + \dot{t}\dot{v},$$

where $\dot{x}^\mu = dx^\mu/d\lambda$ for an affine geodesic parameter λ ($\tilde{L}$ is defined below). Calculating the equations of motion from $\mathcal{L}$ enables a simple determination of (appropriate combinations of) the Christoffel symbols for $\hat{g}$. Recall

$$\hat{\Gamma}^\mu_{\nu\rho} = \frac{1}{2}\hat{g}^{\mu\delta}\left(\hat{g}_{\delta\nu,\rho} + \hat{g}_{\delta\rho,\nu} - \hat{g}_{\nu\rho,\delta}\right) := \hat{g}^{\mu\delta}[\nu\rho, \delta]_{\hat{g}}.$$

and the equations of motion are

$$0 = \ddot{x}^\mu + \hat{\Gamma}^\mu_{\nu\rho}\dot{x}^\mu\dot{x}^\rho.$$

Setting

$$A := A_\mu dx^\mu = N_i dx^i - \Phi dt, \qquad F = dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu = \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

the equations of motion for v, x^i and t yield

$$\begin{aligned}
0 &= \ddot{t}, \\
0 &= h_{ij}\, \ddot{x}^j + [jk, i]_h\, \dot{x}^j\dot{x}^k + (\partial_t h_{ij} + \partial_j N_i - \partial_i N_j)\, \dot{t}\dot{x}^j + (\partial_i\Phi + \partial_t N_i)\, \dot{t}^2, \\
&= h_{ij}\, \ddot{x}^j + [jk, i]_h\, \dot{x}^j\dot{x}^k + (\partial_t h_{ij} - F_{ij})\, \dot{t}\dot{x}^j + F_{ti}\dot{t}^2, \\
0 &= \ddot{v} + N_i\ddot{x}^i + \left[\frac{1}{2}(\partial_j N_i + \partial_i N_j) - \frac{1}{2}\partial_t h_{ij}\right]\dot{x}^i\dot{x}^j - 2\partial_i\Phi\, \dot{t}\dot{x}^i - \partial_t\Phi\, \dot{t}^2, \\
&= \ddot{v} + \left[\frac{1}{2}(\partial_j N_i + \partial_i N_j) - N_k\Gamma^k_{ij} - \frac{1}{2}\partial_t h_{ij}\right]\dot{x}^i\dot{x}^j + \left[-N^k(\partial_t h_{ki} - F_{ki}) - 2\partial_i\Phi\right]\dot{t}\dot{x}^i + \left(-\partial_t\Phi + N^i F_{it}\right)\dot{t}^2
\end{aligned}$$

where we have substituted the earlier equations in the latter. Note that where the indec From these we read that the nonvanishing Christoffel symbols for $\hat{g}$ are

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk}, \qquad \hat{\Gamma}^i_{jt} = -\frac{1}{2}F^i{}_j + \frac{1}{2}h^{ik}\partial_t h_{kj}, \qquad \hat{\Gamma}^i_{tt} = h^{ik}\left(\partial_t N_k + \partial_k \Phi\right) = -F^i{}_t,$$
$$\hat{\Gamma}^v_{tt} = -\partial_t\Phi + N^k F_{ku}, \quad \hat{\Gamma}^v_{ij} = \frac{1}{4}\left[\nabla^{(h)}_i N_j + \nabla^{(h)}_j N_i - \partial_u h_{ij}\right], \quad \hat{\Gamma}^v_{ti} = -\frac{1}{2}N^k(\partial_t h_{ki} - F_{ki}) - \partial_i\Phi.$$

Note that to raisethe index of N has required we recognise that

$$N^i = \hat{g}^{ij}N_j = h^{ij}N_j$$

In particular this means ∂_s is parallel with respect to the Levi-Civita metric.

The canonical momenta are given by $p_\mu = \partial\mathcal{L}/\partial\dot{x}^\mu = \hat{g}_{\mu\nu}\dot{x}^\nu$ giving

$$p_v = \dot{t}, \qquad p_i = h_{ij}\dot{x}^j + N_i\dot{t}, \qquad p_t = \dot{v} - 2\Phi\dot{t} + N_i\dot{x}^i,$$

and so

$$\dot{t} = p_v, \qquad \dot{x}^i = h^{ij}(p_j - N_j p_v), \qquad \dot{v} = p_t - N^i p_i + [2\Phi + N^2]\, p_v.$$

Likewise, the geodesic Hamiltonian is

$$\mathcal{H} = p_\mu\dot{x}^\mu - \mathcal{L} = \frac{1}{2}\hat{g}^{\mu\nu}\, p_\mu\, p_\nu = \frac{1}{2}h^{ij}\,(p_i - N_i p_v)(p_j - N_j p_v) + p_t p_v + \Phi\, p_v^2.$$

The equations of motion are

$$\frac{dt}{d\lambda} = \frac{\partial\mathcal{H}}{\partial p_t} = p_v, \qquad \frac{dv}{d\lambda} = \frac{\partial\mathcal{H}}{\partial p_v}, \qquad \frac{dx^i}{d\lambda} = \frac{\partial\mathcal{H}}{\partial p_i} = h^{ij}\,(p_j - N_j p_v),$$
$$\frac{dp_t}{d\lambda} = -\frac{\partial\mathcal{H}}{\partial t}, \qquad \frac{dp_v}{d\lambda} = -\frac{\partial\mathcal{H}}{\partial v} = 0, \qquad \frac{dp_i}{d\lambda} = -\frac{\partial\mathcal{H}}{\partial x^i}.$$

Because v is a cyclic coordinate its conjugate momentum p_v is conserved along geodesics: thus $p_v = m$ is a constant and we may write

$$\mathcal{H} := H + m\, p_t, \qquad H := \frac{1}{2}h^{ij}\,(p_i - mN_i)(p_j - mN_j) + m^2\,\Phi.$$

We observe that we have the geodesics have the conserved quantities,

$$\frac{1}{2}\hat{g}^{\mu\nu}\, p_\mu p_\nu = m\left[\frac{p^i p_i}{2m} - N^i p_i + mN^i N_i + p_t + m\Phi\right] := -mE_0,$$
$$\hat{g}^{\mu\nu}\, p_\mu \xi_\nu = p_v = m.$$

Following the identifications of [2] we view $p_v = m$ as the mass, $-p_t = E$ as the energy, E_0 as the internal energy, and $m\Phi = V$ as the potential energy. Taking the internal energy to vanish in the nonrelativistic limit the null geodesics of $\hat{g}$ may be identified with the motion in the d-dimensional space with potential energy V. We note that two conformally related metrics have the same null geodesics, and so the d-dimensional world lines will be the same. For $m \neq 0$ the equations of motion for t then give $dt/d\lambda = m$, whence $dt = m\, d\lambda$ and we may eliminate the affine geodesic parameter λ for t. The equations of motion are then precisely those coming from the standard mechanical system

$$\tilde{L} = \frac{1}{2}h_{ij}\,\dot{x}^i\,\dot{x}^j + N_i\,\dot{x}^i - \Phi$$

where $\dot{x}^i$ is now the standard dx^i/dt (and $\dot{t} = 1$). Now

(a) in the case of a non-null geodesic, if we parameterised the curve by arc length, $\lambda = s$ and $t = ms$, then from (1.0.1) we have

$$\frac{dv}{dt} = \frac{1}{2m^2} - \tilde{L}.$$

The equations of motion for v follow from this and

$$v = \frac{t}{2m^2} - \int \tilde{L}\, dt + b.$$

(b) in the case of a null geodesics we have

$$\frac{dv}{dt} = -\tilde{L}, \qquad v = -\int \tilde{L}\, dt + b.$$

Thus we have for each $m \neq 0$ and b a bijection between the geodesics of $\hat{g}$ and the equations of motion of $\tilde{L}$.

1.1 Bargman Structures

A Bargmann structure $(B, \hat{g}, \xi)$ is a principal bundle $\pi : B \to M$, where $\dim B = \dim M + 1$, equipped with a Lorentzian metric $\hat{g}$ and nowhere vanishing null vector field ξ such that with respect to the usual Levi-Civita connection $\hat{\nabla}\xi = 0$. Then $M := B/\mathbb{R}\xi$ is equipped with a Newton-Cartan geometry (M, K, θ, ∇) where

$$K = \pi_* \hat{g}^{-1}, \qquad \hat{g}(\xi) = \pi^* \theta,$$

K is degenerate and $\pi^*\theta$ generates $\ker K$.

In our setting we have a metric of Brinkmann form

$$\hat{g} = h + dt \otimes \omega + \omega \otimes dt, \quad \omega = dv - \Phi(x,t)\, dt + N_i(x,t)\, dx^i, \quad h = h_{ij}(x,t) dx^i \otimes dx^j.$$

Then $\xi = \partial_v$, $\theta = dt$.

2 Introduction

Let us start with a bit of back story, so we can develop and go further. This will be built off of [1].

2.1 Galilei and Newton Structures

We start with some more classical work.

Definition 2.1 (Galilei group)**.** *The* ***Galilei group*** *is the matrix group*

$$G = \left\{ \begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \mid R \in SO(d), \ , b, c \in \mathbb{R}^n, \ e \in \mathbb{R} \right\} \leq GL_{d+2}(\mathbb{R})$$

We think of G as acting on $(\boldsymbol{x}, t, 1)$ s.t.

$$\begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Rx + tb + c \\ t + e \\ 1 \end{pmatrix}$$

with this action we see:

1. R are rotations in space
2. b are boosts
3. c, e are translations in space and time respectively

With this interpretation we have

Definition 2.2. *The* ***Homogeneous Galilei group/Euclidean group*** H *is the group of Galilean transformations that preserve the spatio-temporal origin* $(\mathbf{0}, 0, 1)$.

Proposition 2.3. H *consists of matrices of the form*

$$\begin{pmatrix} R & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover $H \cong SO(d) \ltimes \mathbb{R}^d$ *as a Lie group (not a as a Lie transformation group [3]) is faithfully represented by matrices of the form*

$$\begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \in GL_{d+1}.$$

Proof. See my CQIS notes for a more built up discussion of this fact. □

We now recall the following def:

Definition 2.4. *The* ***frame bundle*** *of a* k*-dimensional smooth manifold* M *is* $GL(M)$*, the* GL_k*-principal fibre bundle with fibres at* $x \in M$ *given by the space of ordered bases of* T_xM.

Definition 2.5. *A* ***proper Galilei structure*** $H(M)$ *is a reduction of structure group of the frame bundle of a* $(d+1)$*-dimensional* M *via* $H \hookrightarrow GL_{d+1}$.

References

[1] C. Duval, G. Burdet, H. P. Künzle, M. Perrin. Bargmann structures and Newton-Cartan theory. *Physical Review D*, 31(8):pp. 1841–1853, 1985. ISSN 05562821. doi:10.1103/PhysRevD.31.1841.

[2] Christian Duval, Gary Gibbons, Péter Horvathy. Celestial mechanics, conformal structures, and gravitational waves. *Physical Review D*, 43(12):pp. 3907–3922, 1991. ISSN 05562821. doi:10.1103/PhysRevD.43.3907.

[3] H. P. Künzle. Galilei and Lorentz structures on space-time: comparison of the corresponding geometry and physics. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 17:pp. 337–362, 1972. ISSN 0246-0211.