Topics in Rings and Representation Theory - Kac Moody Algebras

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1 Introduction

A set of lecture notes on a masters course on Kac moody algebras

2 Groups, Algebras, and their Representations

2.1 Algebras

Throughout this course we will take k to be a field.

Definition 2.1. *An* ***algebra*** *is a triple* (A, m, i) *of*

- *a k-vector space A*
- *a linear map $m : A \otimes A \to A$*
- *an element $i : k \to A$*

satisfying associativity and unitality.

Notation. *For $a, b \in A$ we will denote $m(a, b) = a \cdot b$.*

Remark. *Linearity of m gives distributivity of the multiplication over k.*

Proposition 2.2. *If a unit exists for $(A, \cdot)$, it is unique*

Proof. Let $1, 1' \in A$ be the units. Then

$$1 = 1 \cdot 1' = 1'$$

□

Example 2.3. *Some examples of algebras are*

- *The base field k*
- *polynomials over k, $k[X]$.*
- $\mathrm{End}(V)$ *where V is a vector space, with multiplication given by composition*

Example 2.4. *The **free algebra** $k\langle x_1, \ldots, x_n\rangle$ is the vector space consisting formally of all possible combinations of the x_i in order to make it a vector space, namely*

$$k\langle x_1, \ldots, x_n\rangle = \bigoplus_{m=0}^{\infty} k \cdot \prod_{1\le j_i\le n} x_{j_1}\cdots x_{j_m}$$

Example 2.5. *Given a group G we have the **group algebra** $A \equiv kG$ with*

- *basis $\{x_g \mid g \in G\}$*
- *multiplication $x_g \cdot x_h = x_{gh}$*
- *unit x_{e_G}*

Definition 2.6. *$(A, \cdot)$ is **commutative** if $\forall a, b \in A,\ a \cdot b = b \cdot a$.*

Example 2.7. *kG is abelian iff G is abelian.*

Definition 2.8. *A homomorphism of algebras $f : A \to B$ is a linear map of vector spaces compatible with $\cdot$ s.t.*

- $\forall a, b \in A,\ f(a \cdot b) = f(a) \cdot f(b)$
- *($f(1_A) = 1_B$)*

2.2 Representations

Definition 2.9. *A **representation** of $(A, \cdot)$ is a vector space V with $\rho : A \to \mathrm{End}(V)$ a homomorphism of algebras.*

Notation. *We will often, for simplicity, abuse notation and write for $a \in A, v \in V$*

$$\rho(a)(v) = a \cdot v$$

Remark. *A representation is also call a **left A-module**. A right A-module has $\sigma : A \to \mathrm{End}(V)$ an antihomomorphism. We define an algebra (A^{op}, m^{op}) s.t. $A^{op} = A$, $\forall a, b \in A$, $m^{op}(a, b) = m(b, a)$. We can then say that a right A-module is a representation of A^{op}.*

Example 2.10. *We have a few standard examples of reps:*

- $V = 0$
- *$V = A$ and for $a \in A$, $\rho : a \mapsto \rho(a)$ s.t. $\forall b \in A$, $\rho(a)(b) = m(a, b)$. This is called the **regular** rep.*
- *$A = k$, then any rep is just a vector space over k*
- *If $A = k\langle x_1, \ldots, x_n\rangle$ then a rep is a vector space with $\rho(x_i) \in \mathrm{End}(V)$ specified.*

Definition 2.11. *Given two representations V_1, V_2, the **direct sum** representation $V_1 \oplus V_2$ is given with*

$$\rho_{V_1\oplus V_2}(a)(v_1 + v_2) = \rho_{V_1}(a)(v_1) + \rho_{V_2}(a)(v_2)$$

for $a \in A, v_i \in V_i$.

Definition 2.12. *A **subrepresentation** is subspace $W \subset V$ s.t. $\forall a \in A$, $\rho(a)(W) \subset W$.*

Definition 2.13. *Given $v \in V$ the **minimal subrep** containing v is*

$$A \cdot V = \{w \in V \mid \exists a \in A,\, w = a \cdot v\}$$

Definition 2.14. *A rep is **irreducible** if the only subreps are $W = 0, V$*

Definition 2.15. *Let V_1, V_2 be reps of A. Then a homomorphism of reps (an **intertwiner**) is linear map $\phi : V_1 \to V_2$ s.t. $\forall v \in V_1$, $a \in A$, with $\rho_i : A \to \mathrm{End}(V_i)$ we have*

$$\begin{tikzcd}
V_1 \arrow[r, "\phi"] \arrow[d, "\rho_1(a)"'] & V_2 \arrow[d, "\rho_2(a)"] \\
V_1 \arrow[r, "\phi"'] & V_2
\end{tikzcd}$$

commutes, i.e. $\phi(a \cdot v) = a \cdot \phi(v)$

Proposition 2.16. *Let $f : V \to W$ be an intertwiner. Then*

- $\ker f \subset V$ *is a subrep*
- $\mathrm{Im}\, f \subset W$ *is a subrep*

Lemma 2.17 (Schur)**.** *Let V_1, V_2 be two A-reps and let $f : V_1 \to V_2$ be a non-zero intertwiner. Then*

- *V_1 irreducible $\Rightarrow$ f is injective*
- *V_2 irreducible $\Rightarrow$ f is surjective*

Definition 2.18. *A representation is **indecomposable** is when $V = V_1 \oplus V_2$, either $V_1 = 0$ or $V_2 = 0$*

Proposition 2.19. *Any irreducible rep is indecomposable*

Proof. V_1, V_2 are subreps of $V_1 \oplus V_2$. □

Remark. *The converse to the above is not true,*

Aside. *Coming from the workshop, we have some points that we want to have made clear in our mind:*

1. *Rep theory is linear algebra. If $f : V \to W$ is an intertwiner, the condition that for $a \in A$, $v \in V$ $f(a \cdot v) = a \cdot f(v)$, is essentially saying that f is A-linear. We now have a correspondence*

A-linear	*k-linear*
Irreducible reps	*eigenspaces*
indecomposable reps	*generalised eigenspaces*

3 Lie algebras

3.1 preliminaries

Definition 3.1. *A **Lie algbera** is a vector space* $\mathfrak{g}$ *endowed with a bilinear map*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

- *antisymmetry:* $\forall x \in \mathfrak{g}$, $[x, x] = 0$
- *Jacobi identity:* $\forall x, y, z \in \mathfrak{g}$, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Example 3.2. *Given an associative algebra* A*, we can make* A *a Lie algebra using*

$$[a, b] = ab - ba$$

for $a, b \in A$*.*

Example 3.3. *Let* $\mathfrak{sl}_n(k) = \{X \in M_n(k) \mid \mathrm{Tr}(X) = 0\}$*. Then as* $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ *we have a bracket given by the commutator*

$$[X, Y] = XY - YX$$

This gives a Lie algebra structure to $\mathfrak{sl}_n(k)$ *which is not inherited from matrix multiplication, as* $\mathfrak{sl}_n(k)$ *is not closed under multiplication, so is not an associative algebra. To motivate looking at such a vector space, note that*

$$\mathrm{Tr}(X) = 0 \Rightarrow \det \exp(X) = e^{\mathrm{Tr}(X)} = 1$$

and $SL_n(k) = \{Y \in M_n(k) \mid \det(Y) = 1\}$ *has a natural operation of matrix multiplication, as* $\det(XY) = \det(X)\det(Y)$*.*

3.2 Universal Enveloping Algebras

Definition 3.4. *Let* $\{x_i\}$ *be a basis of* $\mathfrak{g}$ *a Lie algebra and suppose the bracket is specified by*

$$[x_i, x_j] = \sum_k c_{ij}^k x_k$$

*for some **structure constants*** c_{ij}^k*. Then the **universal enveloping algebra*** $\mathcal{U}_\mathfrak{g}$ *is the associative algebra generated by the* x_i *with the relations*

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k$$

i.e.

$$\mathcal{U}_\mathfrak{g} = {}^{k\langle x_1, \ldots, x_n \rangle}\Big/\left\langle x_i x_j - x_j x_i - \sum_k c_{ij}^k x_k \right\rangle$$

We get the map

$$\iota : \mathfrak{g} \to \mathcal{U}_\mathfrak{g}$$
$$x_i \mapsto x_i$$

The above definition involves a choice of basis of $\mathfrak{g}$. In general we want to remove this to give a universal property of $\mathcal{U}_\mathfrak{g}$.

Proposition 3.5. *For any associative algebra A, and Lie algebra map $\mathfrak{g} \xrightarrow{f} A^{Lie}$, there exists a unique map of associative algebras $\mathcal{U}_\mathfrak{g} \xrightarrow{\mathcal{U}(f)} A$ s.t.*

$$\begin{tikzcd} & \mathcal{U}_\mathfrak{g} \arrow[d, "\mathcal{U}(f)"] \\ \mathfrak{g} \arrow[ur, "\iota"] \arrow[r, "f"'] & A \end{tikzcd}$$

commutes

Exercise 3.6. *Prove that $\mathcal{U}_\mathfrak{g}$ is uniquely defined (up to isomorphism) by this universal property.*

Proof. Consider

$$\mathcal{U}_\mathfrak{g} = \left[\bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}\right] \Big/ \langle x \otimes y - y \otimes x - [x, y]\rangle$$

□

Proposition 3.7. *If $\mathcal{U}_\mathfrak{g}$ satisfies the universal property then there is a bijection*

$$\{\text{rep of Lie algebra } \mathfrak{g}\} \leftrightarrow \{\text{reps of associative algebra } \mathcal{U}_\mathfrak{g}\}$$

3.2.1 Construction

Definition 3.8. *Recall that for a vector space V over k, the **tensor algebra** is*

$$T(V) \equiv \bigoplus_{n\geq 0} V^{\otimes n}$$

where $V^{\otimes 0} = k$. It comes with the map

$$\begin{aligned} V^{\otimes n} \times V^{\otimes m} &\to V^{\otimes(n+m)} \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

Definition 3.9. *The **symmetric algebra** is*

$$S(V) \equiv T(V) \big/ \langle v \otimes w - w \otimes v \,|\, v, w \in V\rangle$$

Definition 3.10. *Given a Lie algebra $\mathfrak{g}$, the **universal enveloping algebra** is*

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) \big/ \langle x \otimes y - y \otimes x - [x, y] \,|\, x, y \in \mathfrak{g}\rangle$$

Theorem 3.11 (Poincare-Birkhoff)**.** *We have the following two properties:*

1. *$\mathcal{U}(\mathfrak{g})$ satisfies the universal property*

2. As a vector space, $\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$.

Remark. *If* $\{x_i \mid i = 1, \ldots, n\} \subset \mathfrak{g}$ *is a basis, then the set of ordered monomials* $x_1^{i_1} \cdots x_n^{i_n} = x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$ *is a basis of* $\mathcal{U}(\mathfrak{g})$.

Remark. *Note that in* $\mathcal{U}(\mathfrak{g})$, *by our quotient it must be that*

$$x_i x_j - x_j x_i = [x_i, x_j]$$

which is what we wanted to see.

Example 3.12. *If* $\mathfrak{g}$ *is abelian, then* $\forall x, y \in \mathfrak{g}$, $[x, y] = 0$ *and we see by definition* $\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$.

3.3 Representations

3.3.1 Simple Lie algebra

Definition 3.13. *An* ***ideal*** *of a Lie algebra* $\mathfrak{g}$ *is a subspace* $\mathfrak{g}'$ *s.t.* $[\mathfrak{g}', \mathfrak{g}] \subset \mathfrak{g}'$

Definition 3.14. $\mathfrak{g}$ *is* ***simple*** *if the only ideals of* $\mathfrak{g}$ *are* $0, \mathfrak{g}$. $\mathfrak{g}$ *is* ***semi-simple*** *if it is a direct sum of simple Lie algebras.*

Remark. $\mathfrak{g} \circlearrowleft \mathfrak{g}$ *by* $x \cdot y = [x, y]$ *This is the* ***adjoint rep****. Then we have ideals of* $\mathfrak{g}$ *correspond to subreps of the adjoint rep.*

Example 3.15. *Consider* $\mathfrak{gl}_n(k) = M_n(k)^{\mathrm{Lie}}$. *Recall* $\mathfrak{sl}_n(k) \subset \mathfrak{gl}_n(k)$ *is the set of traceless matrices. This is an ideal so* $\mathfrak{gl}_n(k)$ *is non-simple.*

Exercise 3.16. *Prove* $\mathfrak{sl}_n(k)$ *is simple.*

Theorem 3.17 (Weyl complete reducibility)**.** *Let* $\mathfrak{g}$ *be a finite dimensional semi-simple Lie algebra and* $V \in \mathrm{Rep}(\mathfrak{g})$. *If* $W \subseteq V$ *is a subrepresentation, then* $\exists W' \subseteq V$ *s.t* $V \cong W \oplus W'$ *as representations.*

3.3.2 Classification of complex f.d simple Lie algebras

- $(A_n, n \geq 1)$: $\mathfrak{sl}_{n+1}(\mathbb{C}) = \{\mathrm{Tr}(X) = 0\} \subset \mathfrak{gl}_n$
- $(B_n, n \geq 2)$: $\mathfrak{so}_{2n+1}(\mathbb{C}) = \left\{\mathrm{Tr}(X) = 0, X^T + X = 0\right\} \subset \mathfrak{gl}_{2n+1}$
- $(C_n, n \geq 3)$: $\mathfrak{sp}_n(\mathbb{C}) = \left\{J_n X = X^T J_n\right\} \subset \mathfrak{gl}_{2n}$
- $(D_n, n \geq 4)$: $\mathfrak{so}_{2n}(\mathbb{C}) = \left\{\mathrm{Tr}(X) = 0, X^T + X = 0\right\} \subset \mathfrak{gl}_{2n}$
- Exceptionals, E_6, E_7, E_7, F_4, G_2, dimensions $52, 133, 248, 52, 14$.

Remark. *Suppose* $\mathfrak{g}$ *is simple. Take* $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ *a morphism of Lie algebras. If* $\pi \neq 0$, *then as* $\ker \pi \subsetneq \mathfrak{g}$ *is an ideal it must be the case that* $\ker \pi = 0$
Now if we define $\mathfrak{gl}_n = \mathrm{End}_i(k^n)^{\mathrm{Lie}}$, *this has a basis* $\left\{E_{ab} = (\delta_{ia}\delta_{jb})_{i,j=1}^n\right\}$ *called the* ***elementary matrices****. These obey*

$$E_{ij} E_{kl} = \delta_{jk} E_{il}$$
$$\Rightarrow [E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{jk}$$

3.4 $\mathfrak{sl}(n)$

Example 3.18. *Consider $\mathfrak{sl}(2)$. Taking $n = 2$, we have the basis*

$$e = E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$f = E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$h = E_{11} - E_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These get the commutation relations

$$[h, e] = 2e$$
$$[h, f] = -2f$$
$$[e, f] = h$$

More generally, we see $\dim_{\mathbb{C}}(\mathfrak{sl}(n)) = n^2 - 1$ by considering the trace condition, but has only $3(n-1)$ generators

$$e_i = E_{i,i+1}$$
$$f_i = E_{i+1,i}$$
$$h_i = E_{ii} - E_{i+1,i+1}$$

for $i = 1, \ldots, n-1$.

Exercise 3.19. *Moreover we have*

$$[e_i, e_{i+2}] = 0$$
$$[e_i, [e_i, e_{i+1}]] = 0$$

These generate as

$$[e_i, e_{i+1}] = E_{i,i+2}$$

and this can be iterated to get all upper triangular matrices, likewise for lower triangular with f and diagonal with all. Explicitly

$$[h_i, e_j] = a_{ji} e_j$$
$$[h_i, f_j] = -a_{ji} f_j$$
$$[e_i, f_j] = \delta_{ij} h_i$$

where

$$a_{ij} = \begin{cases} 2 & |i-j| = 0 \\ -1 & |i-j| = 1 \\ 0 & |i-j| > 1 \end{cases}$$

We call $A = (a_{ij})$ the **Cartan matrix**.

Theorem 3.20 (Serre)**.** *If $\mathfrak{g}$ is a finite dimensional simple Lie algebra over $\mathbb{C}$ then $\mathfrak{g}$ has a similar presentation.*

Proposition 3.21. *A has the following properties:*

- $A \in M_n(\mathbb{Z})$
- $\forall i \neq j$, $a_{ii} = 2$ *and* $a_{ij} \leq 0$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$
- *A is indecomposable*
- $\det(A) \neq 0$ *and A is positive definite.*

4 Kac-Moody Algebras

Idea. *We can try to reverse engineer the Cartan matrix, to generalise it and then assign a Lie algebra $\mathfrak{g}(A)$ to the resulting matrix A.*

Definition 4.1. *A* ***realisation*** *of A is a triple $(\mathfrak{h}, \Pi^\vee, \Pi)$ where*

- $\mathfrak{h}$ *is a vector space*
- $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subseteq \mathfrak{h}$
- $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$

s.t.

- $\Pi^\vee$ *is a linearly independent set*
- Π *is a linearly independent set*
- $\alpha_i(\alpha_j^\vee) = a_{ji}$

We call Π the ***simple roots****, and $\Pi^\vee$ the* ***simple coroots***

Exercise 4.2. *Show the following results:*

- *If $(\mathfrak{h}, \Pi^\vee, \Pi)$ is a realisation,* $\dim \mathfrak{h} \geq 2n - \operatorname{rank}(A)$
- *A* ***minimal realisation*** *(i.e* $\dim \mathfrak{h} = 2n - \operatorname{rank}(A)$*) always exists*

Definition 4.3. *A morphism $(\mathfrak{h}, \Pi^\vee, \Pi) \to (\mathfrak{h}', (\Pi')^\vee, \Pi')$ is*

- $\phi : \mathfrak{h} \to \mathfrak{h}'$
- $\phi(\Pi^\vee) = (\Pi')^\vee$
- $\phi(\Pi) = \Pi'$

Proposition 4.4. *For any A, $\exists!$ minimal realisation Σ up to isomorphism.*

Definition 4.5. *Let* $(\mathfrak{h}, \Pi^\vee, \Pi)$ *be a realisation of* A. *Then* $\tilde{\mathfrak{g}}(A)$ *is the Lie algebra with generators* e_i, f_i *for* $i = 1, \ldots, n$ *containing* $\mathfrak{h}$ *s.t.*

$$\begin{aligned}
[e_i, f_j] &= \delta_{ij}\alpha_i^\vee \in \mathfrak{h} \\
\forall h \in \mathfrak{h},\ [h, e_i] &= \alpha_i(h)e_i \\
\forall h \in \mathfrak{h},\ [h, f_i] &= -\alpha_i(h)f_i \\
\forall h, h' \in \mathfrak{h},\ [h, h'] &= 0
\end{aligned}$$

Example 4.6. *If* $A = [2]$, *we can follow the procedure and find* $\tilde{\mathfrak{g}}(A) = \mathfrak{sl}(2)$. *To see we see* $\dim \mathfrak{h} = 1$ *for a minimal realisation, so we only need* $\alpha^\vee \in \mathfrak{h}, \alpha \in \mathfrak{h}^*$ *s.t.* $\alpha(\alpha^\vee) = 2$. *With this* $\mathfrak{h} = \mathrm{Span}\{\alpha^\vee\}$ *We then need* e, f *to satisfy*

$$\begin{aligned}
[e, f] &= \alpha^\vee \\
[\alpha^\vee, e] &= \alpha(\alpha^\vee)e = 2e \\
[\alpha^\vee, f] &= -\alpha(\alpha^\vee)f = -2f
\end{aligned}$$

This is just $\mathfrak{sl}_2$ *if we relabel* $h = \alpha^\vee$.

Idea. $\tilde{\mathfrak{g}}(A)$ *is still currently too large, for example as the* e_i *are not yet related, and we want to try make it look like* $\mathfrak{sl}_n$, *i.e maybe simple. Hence we want to consider all ideals of the form*

$$\textit{trivial ideals } = \{r \subset \tilde{\mathfrak{g}}(A) \textit{ ideals } | \, r \cap \mathfrak{h} = 0\}$$

and let

$$r_{max} = \sum_{r \in trivial} r$$

and quotient by this to go

$$A \to (\mathfrak{h}, \Pi^\vee, \Pi) \to \tilde{\mathfrak{g}}(A) \to \mathfrak{g}(A) = \tilde{\mathfrak{g}}(A) / r_{max}$$

Definition 4.7. *We define*

- $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ *the* ***root lattice***
- $Q \supset Q_+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$ *the* ***positive root lattice***
- $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ *the* ***coroot lattice***
- $Q^\vee \supset Q_+^\vee = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i^\vee$ *the* ***positive coroot lattice***

Note that the relations required for $\tilde{g}(A)$ give us all the commutators we need, e.g

$$\begin{aligned}
[h, [e_1, e_2]] &= -[e_2, [h, e_1]] - [e_1, [e_2, h]] \\
&= \alpha_1(h)\, [e_2, e_1] + \alpha_2(h)\, [e_1, e_2] \\
&= (\alpha_1 + \alpha_2)(h)\, [e_1, e_2]
\end{aligned}$$

Definition 4.8. *Now for* $\alpha \in Q$ *we can define*

$$\tilde{\mathfrak{g}}_\alpha \equiv \{x \in \tilde{\mathfrak{g}}(\Sigma) \,|\, \forall h \in \mathfrak{h},\ [h, x] = \alpha(h)x\}$$

Example 4.9. *We can consider examples of this:*

- $\mathfrak{h} \supset \tilde{\mathfrak{g}}_0 = \mathfrak{h}$.
- $\tilde{\mathfrak{g}}_{\alpha_i} = \mathbb{C}e_i$
- $\tilde{\mathfrak{g}}_{-\alpha_i} = \mathbb{C}f_i$
- $\tilde{\mathfrak{g}}_{\alpha_1 - \alpha_2} = 0$

We can then also state the following

Theorem 4.10. *We have the following*

1. *As a vector space, $\tilde{\mathfrak{g}}(\Sigma) = \tilde{n}_+ \oplus \mathfrak{h} \oplus \tilde{n}_-$ where $\tilde{n}_+$ is the free Lie algebra generated by the e_i, and $\tilde{n}_-$ by the f_i.*
2. *We have*

$$\tilde{n}_+ = \bigoplus_{\alpha \in Q_+ \setminus 0} \tilde{\mathfrak{g}}_\alpha$$
$$\tilde{n}_- = \bigoplus_{\alpha \in Q_+ \setminus 0} \tilde{\mathfrak{g}}_{-\alpha}$$

3. *$[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] \subset \tilde{\mathfrak{g}}_{\alpha+\beta} \Rightarrow \tilde{\mathfrak{g}}(\Sigma)$ is Q-graded.*

Now

Lemma 4.11.
1. *$I \subset \tilde{\mathfrak{g}}(\Sigma)$ is an ideal then $I = \oplus_{\alpha \in Q}(I \cap \tilde{\mathfrak{g}}_\alpha)$*
2. *$\exists!$ maximal ideal $r \subseteq \tilde{\mathfrak{g}}(\Sigma)$ s.t. $r \cap \mathfrak{h} = 0$*
3. *$r = r_+ \oplus r_-$ with $r_\pm = r \cap \tilde{n}_\pm$*

Definition 4.12. *The **Kac-Moody** Lie algebra associated to Σ is $\mathfrak{g}(\Sigma) \equiv {}^{\tilde{\mathfrak{g}}(\Sigma)}\!\big/_{r}$*

Remark. *As the minimal realisation is unique (up to iso) we have no problem notationally using either $\mathfrak{g}(A)$ or $\mathfrak{g}(\Sigma)$.*

Example 4.13. *$\mathfrak{sl}(2)$ is simple and so when $A = [2]$, $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A) = \mathfrak{sl}(2)$.*

Exercise 4.14. *Show that if $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ we get $\mathfrak{g}(A) = \mathfrak{sl}_3$.*

Proof. $\operatorname{rank}(A) = 2$, so the dimension of the minimal realisation is $2 \times 2 - 2 = 2$. Hence $\mathfrak{h} = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$. The corresponding $\tilde{\mathfrak{g}}(A)$ gives us all the generators we want letting $h_i = \alpha_i^\vee$. $\square$

Definition 4.15. *Let $\tilde{w} : \tilde{\mathfrak{g}}(A) \to \tilde{\mathfrak{g}}(A)$ given by*

$$\tilde{w}(e_i) = f_i$$
$$\tilde{w}(f_i) = e_i$$
$$\tilde{w}(h) = -h$$

*This descends to a map $w : \mathfrak{g}(A) \to \mathfrak{g}(A)$ called the **Chevalley involution**.*

The next result is useful for making calculations in $\mathfrak{g}(\Sigma)$:

Lemma 4.16. *We have the following two classifications*

1. *For* $x \in n_+$, $x = 0 \Leftrightarrow \forall i$, $[f_i, x] = 0$
2. *For* $x \in n_-$, $x = 0 \Leftrightarrow \forall i$, $[e_i, x] = 0$

Proof. Set $\mathfrak{g}_1 = \bigoplus_{i=1}^n \mathbb{C}e_i \subset n_+$. Then define the vector space

$$J_x \equiv \sum_{k\geq 0} \mathrm{ad}(\mathfrak{g}_1)^k(x) \ni x$$

Then we can note

- $[n_+, J_x] \subset J_x$
- $[\mathfrak{h}, J_x] \subset J_x$

and further
Claim: $\left[f_i, \mathrm{ad}(\mathfrak{g}_1)^k(x)\right] \subset J_x$. We can show this by induction. Certainly if we have assumed $[f_i, x] = 0$, then $[f_i, x] \in J_x$. Now for $a \in \mathfrak{g}_1, b \in \mathrm{ad}(\mathfrak{g}_1)^{k-1}(x)$ we have

$$[f_i, [a, b]] = \left[\underbrace{[f_i, a]}_{\in \mathfrak{h}}, b\right] + \left[a, \underbrace{[f_i, b]}_{\in J_x}\right]$$

So we have that J_x is an ideal and that $J_x \cap \mathfrak{h} = 0$, so $J_x = 0$. □

Definition 4.17. *Define* $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(\Sigma) \,|\, \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}$ *in analogy to* $\tilde{\mathfrak{g}}_\alpha$.

Remark. *We have the* $\forall \alpha \in Q_+ \setminus 0$,

- $\tilde{\mathfrak{g}}_{\pm\alpha} \neq 0$
- $\dim \tilde{\mathfrak{g}}_{\pm\alpha} < \infty$
- *If we define* $ht(\alpha) = \sum_i k_i$ *for* $\alpha = \sum_i k_i\alpha_i$ *then* $\dim \mathfrak{g}_{\pm\alpha} \leq n^{|ht(\alpha)|}$

Definition 4.18. *We call* $R = \{\alpha \in Q \setminus 0 \,|\, \mathfrak{g}_\alpha \neq 0\} \subset Q$ *the* ***set of roots***

Proposition 4.19. *We have*

1. $\mathfrak{g}(\Sigma) = \bigoplus_{\alpha\in Q_+\setminus 0} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in Q_+\setminus 0} \mathfrak{g}_{-\alpha}$.
2. $R = R_+ \cup R_-$ *where* $R_\pm = R \cap (\pm Q_\pm)$.

Exercise 4.20. *We can show*

$$A = (2) \quad \Rightarrow \quad R = \{\alpha_1\} \cup \{-\alpha_1\}$$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \Rightarrow \quad R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \; R_- = -R_+$$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \Rightarrow \quad |R| = \infty$$

4.1 Bilinear forms on g(A)

Sometimes $\mathfrak{g}(\Sigma)$ has a non-degenerate, symmetric, invariant bilinear form

$$(\cdot,\cdot):\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$$

i.e.

- $\ker(,)=\{x\in\mathfrak{g}\,|\,\forall y\in\mathfrak{g},\,(x,y)=0\}=0$
- $\forall x,y\in\mathfrak{g},(x,y)=(y,x)$
- $\forall x,y,z\in\mathfrak{g},\,([x,y]\,,z)=(x,[y,z])$

This will turn out to be the analogue of the Killing form.

Theorem 4.21. *If A is symmetrisable, i.e. $\exists D=\mathrm{diag}(d_1,\dots,d_n)$ s.t. $\det D\neq 0$ and $B=DA$ is symmetric, then $\exists$ such a form on $\mathfrak{g}(\Sigma)$.*

Note that in the above theorem, we have fixed a choice by asking for D. It is then natural to ask how many choices we have.

Example 4.22. $\mathfrak{g}(A)\cong\mathfrak{g}(DA)$, *as this simply scales generators $e_i\mapsto d_ie_i$.*

We can define a non-degenerate symmetric bilinear form on $\mathfrak{h}$ by

- $\forall h\in\mathfrak{h},\,(\alpha_i^\vee,h)=d_i\alpha_i(h)$
- $\forall h_1,h_2\in\mathfrak{h}'',\,(h_1,h_2)=0$

where we have defined $\mathfrak{h}'\equiv\langle\Pi^\vee\rangle=\bigoplus_{i=1}^N\mathbb{C}\alpha_i^\vee$ and then required $\mathfrak{h}=\mathfrak{h}'\oplus\mathfrak{h}''$. It is the coupling to D that gives the symmetry e.g

$$\begin{aligned}(\alpha_i^\vee,\alpha_j^\vee)&=d_i\alpha_i(\alpha_j^\vee)\\&=d_ia_{ji}\\&=d_ja_{ij}\\&=(\alpha_j^\vee,\alpha_i^\vee)\end{aligned}$$

Theorem 4.23. $(\cdot,\cdot)$ *extends to a non-degenerate symmetric bilinear* invariant *form on $\mathfrak{g}(A)$ by setting*

- $(e_i,f_j)=\delta_{ij}$
- $(e_i,e_j)=0=(f_i,f_j)$
- $(e_i,h)=0=(f_i,h)$

Remark. *Note that these conditions are imposed on us in order to have invariance, e.g*

$$([e_1,e_2]\,,f_1)=(e_1,[e_2,f_1])=0$$

or

$$([e_1,e_2]\,,f_1)=-(e_2,\underbrace{[e_1,f_1]}_{\in\mathfrak{h}})=0$$

Corollary 4.24. *Let $\alpha \in Q$, and recall $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) \,|\, \forall h \in \mathfrak{h}, [h,x] = \alpha(h)x\}$. Then*

$$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) \neq 0 \Leftrightarrow \alpha + \beta = 0$$

Then identifying $\mathfrak{g} \cong \mathfrak{g}^*$ by $x \mapsto (x, \cdot)$, we have

$$\mathfrak{g}_\alpha \cong \mathfrak{g}^*_{-\alpha}$$

Now let us make the prop:

Proposition 4.25. *Let $\nu : \mathfrak{h} \overset{\cong}{\to} \mathfrak{h}^*$ be the function*

$$\nu(h) = (h, \cdot)$$
$$\nu(\alpha_i^\vee) = d_i \alpha_i$$

Then for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$;

$$[x,y] = (x,y) \cdot \nu^{-1}(\alpha)$$

Theorem 4.26 (Serre)**.** *Suppose we have $A \in M_n(\mathbb{Z})$ satisfying $a_{ii} = 2$, $a_{ij} \leq 0$. Then in $\mathfrak{g}(A)$*

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0$$
$$\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$$

These are called the ***Serre relations****.*

Proof. □

Theorem 4.27 (Gabber - Kac)**.** *If we have $A \in M_n(\mathbb{Z})$ satisfying $a_{ii} = 2$, $a_{ij} \leq 0$* ***and*** *A is symmetrisable, then the only relations on $\mathfrak{g}(A)$ are the Serre relations.*

Remark. *If we have the conditions of the above theorem, then we know $\mathfrak{g}(A)$ is generated by $e_i, f_i, \mathfrak{h}$ s.t.*

- $[h, h'] = 0$
- $[h, e_i] = \alpha_i(h) e_i$
- $[h, f_i] = -\alpha_i(h) f_i$
- $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$
- $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0$
- $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$

and this entirely determines $\mathfrak{g}(A)$.

Example 4.28. *Consider $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, which has $\mathfrak{g}(A) = \mathfrak{sl}_3$. Then we found*

$$\mathrm{ad}(e)^{1-(-1)}(e) = [e_1, [e_1, e_2]] = 0$$

Now we have a copy of $\mathfrak{sl}_2$, called $\mathfrak{sl}_2^{(i)} = \langle e_i, f_i, \alpha_i^\vee \rangle$ sitting in $\mathfrak{g}(A)$. We prove a lemma about it's adjoint action:

Lemma 4.29. *Let $V \in \operatorname{Rep}(\mathfrak{sl}_2)$ and $v \in V$ s.t. $h \cdot v = \lambda v$, $e \cdot v = 0$. Defining $v_m = \frac{f^m}{m!} \cdot v$ we have*

$$\begin{aligned} h \cdot v_m &= (\lambda - 2m) v_m \\ e \cdot v_m &= (\lambda - m + 1) v_{m-1} \end{aligned}$$

Proof. Note

$$\begin{aligned} hf^m &= ([h, f] + fh) f^{m-1} \\ &= -2f^m + fhf^{m-1}. \end{aligned}$$

and so so repeating this procedure we get

$$hf^m = f^m(h - 2m) \Rightarrow hf^m \cdot v = (\lambda - 2m) \cdot v$$

We do the same with e seeing

$$\begin{aligned} ef^m = (h + fe) f^{m-1} &= hf^{m-1} + fhf^{m-2} + \cdots + f^{m-1}h + f^m e \\ &= f^{m-1}\left[(h - 2m + 2) + (h - 2m + 4) + \cdots + (h)\right] + f^m e \\ &= f^{m-1}\left[m(h - m + 1)\right] + f^m e \end{aligned}$$

yielding the result $\square$

Definition 4.30. *Given $V \in \operatorname{Rep}(\mathfrak{g})$, $x \in \mathfrak{g}$ we say x acts* ***locally nilpotently*** *if $\forall v \in V, \exists N$ s.t. $x^N \cdot v = 0$.*

Definition 4.31. *We say $V \in \operatorname{Rep}(\mathfrak{g}(A))$ is* ***integrable*** *if*

1. *$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where $V_\lambda = \{v \in V \mid \forall h \in \mathfrak{h},\, h \cdot v = \lambda(h) v\}$*
2. *e_i, f_i acts locally nilpotently on V.*

Definition 4.32. *If $V_\lambda \neq 0$, we call it a* ***weight space of weight*** *λ*

Example 4.33. *$\mathfrak{g}(A)$ is integrable over itself as the adjoint rep because of the Serre relations. Here the weight spaces are $\mathfrak{g}_\alpha$. Note if $h \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$ we will have*

$$\begin{aligned} h \cdot x &= \alpha(h) x \\ \Rightarrow \exp(h) \cdot x &= e^{\alpha(h)} x \end{aligned}$$

Example 4.34. *Ever finite dimensional representation is integrable. This is as automatically $V = \bigoplus V_\lambda$, and then as eigenspaces cannot mix, there is no way to keep acting.*

Proposition 4.35. *Let $V \in \operatorname{Rep}(\mathfrak{g}(A))^{Int}$ and let $\mathfrak{g}_{(i)} = \langle e_i, f_i, \alpha_i^\vee \rangle \subset \mathfrak{g}(A)$. Then*

1. *As $\mathfrak{g}_{(i)}$ modules,*

$$V = \bigoplus_{d \geq 0} V_d^{\oplus m_d}$$

where V_d is a irreducible rep of $\mathfrak{sl}_2$, $\dim V_d = d + 1$, $m_d \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

2. *Take* $\lambda \in \mathrm{wt}(V) \equiv \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$ *and fix an* α_i*-string through* λ, $M = \{t \in \mathbb{Z} \mid \lambda + t\alpha_i \in \mathrm{wt}(V)\}$. *Then*

   (a) $\exists p, q \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ *s.t.* $M = [-p, q]$

   (b) $\mathrm{mult}_V(\lambda) = \dim V_\lambda < \infty \Rightarrow p, q < \infty$.

   (c) $p, q < \infty \Rightarrow p - q = \lambda(\alpha_i^\vee)$

   (d) $t \mapsto m(t) \equiv \dim V_{\lambda + t\alpha_i}$ *is symmetric at* $t = -\frac{1}{2}\lambda(\alpha_i^\vee)$

Proof. Take $v_\lambda \in V_\lambda$, and define

$$U_{v_\lambda} = \sum_{k,l \geq 0} \mathbb{C} \cdot f_i^k \cdot e_i^l \cdot v_\lambda$$

Now we must have $\dim U_{v_\lambda} \leq 0$ from nilpotency, and we have an action $\mathfrak{g}_{(i)} \circlearrowright U_{v_\lambda}$. By Weyl reducibility

$$U_{v_\lambda} = \bigoplus_{d \geq 0} V_d^{\oplus m_d}$$

Do this forall $v \in V$.
Now let $U = \sum_{t \in M} V_{\lambda + t\alpha_i} \circlearrowleft \mathfrak{g}_{(i)}$, and define $p = -\inf M$, $q = \sup M$. As $0 \in M$ it must be the case $p, q \geq 0$. Now we can calculate

$$(\lambda + t\alpha_i)(\alpha_i^\vee) = \lambda(\alpha_i^\vee) + 2t$$

and so we have

$$(\lambda + t\alpha_i)(\alpha_i^\vee) = 0 \Leftrightarrow t = -\frac{1}{2}\lambda(\alpha_i^\vee)$$

□

Corollary 4.36. $\lambda \in \mathrm{wt}(V) \Rightarrow \lambda - \lambda(\alpha_i^\vee)\alpha_i \in \mathrm{wt}(V)$

Example 4.37. *Take* $\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$. *Then* $\mathrm{wt}(\mathfrak{g}(A)) = \{\textit{roots}\} \cup \{0\}$

4.2 Weyl group

Definition 4.38 (Fundamental reflections)**.** *We define the **fundamental refelctions*** $r_i \in GL(\mathfrak{h}^*)$ *by* $r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i$

Proposition 4.39. r_i *are reflections with fixed points* $\ker(\alpha_i^\vee) = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) = 0\}$. *Moreover* $r_i(\alpha_i) = -\alpha_i$.

Definition 4.40 (Weyl group)**.** *We define the **Weyl group** to be*

$$W = \langle r_i \rangle \subseteq GL(\mathfrak{h}^*)$$

Proposition 4.41. *Take* $V \in \mathrm{Rep}(\mathfrak{g}(A))^{Int}$, $\lambda \in \mathfrak{h}^*$, $w \in W$, *then*

1. $\mathrm{mult}_V(\lambda) = \mathrm{mult}_V(w(\lambda))$

2. $W \circlearrowright R \equiv \{\alpha \in Q \setminus 0 \,|\, \mathfrak{g}_\alpha \neq 0\}$

3. $\dim \mathfrak{g}_\alpha = \mathrm{mult}(\alpha) = \mathrm{mult}(w(\alpha)) = \dim \mathfrak{g}_{w(\alpha)}$

Remark. $W \circlearrowright Q$

Exercise 4.42. $W \cong \langle r_i^\vee \rangle \subseteq GL(\mathfrak{h})$ *where* $r_i^\vee(h) = h - \alpha_i(h)\alpha_i^\vee$

Now assume we have $x, y : V \to V$ locally nilpotent s.t $\mathrm{ad}(x)^N(y) = 0$ form some $N \gg 0$. Then

$$\exp(x) \cdot y \cdot \exp(-x) = \mathrm{Ad}(\exp(x))(y) = \exp(\mathrm{ad}(x))(y)$$

Theorem 4.43. *Take* $V \in \mathrm{Rep}(\mathfrak{g}(A))^{Int}$ *with* $\pi : \mathfrak{g}(A) \to \mathfrak{gl}(V)$ *the rep. We define* r_i^π : $\exp(f_i)\exp(-e_i)\exp(f_i)$. *Then*

1. $r_i^\pi(V_\lambda) = V_{r_i(\lambda)}$

2. $r_i^{\mathrm{ad}} : \mathfrak{g}(A) \to \mathfrak{g}(A)$, $r_i^{\mathrm{ad}}\big|_{\mathfrak{h}} = r_i^\vee$.

Example 4.44. *Take* $\mathfrak{g}(A) = \mathfrak{sl}_2$, *then* $W = \left\langle r_1 \,|\, r_1^2 = 1 \right\rangle \cong C_2$

Example 4.45. $\mathfrak{g}(A) = \mathfrak{sl}_3$. *Then we have*

	α_1	α_2
r_1	$-\alpha_1$	$\alpha_1 + \alpha_2$
r_2	$\alpha_1 + \alpha_2$	$-\alpha_2$
$r_1 r_2$	α_2	$-(\alpha_1 + \alpha_2)$
$r_2 r_1$	$-(\alpha_1 + \alpha_2)$	α_1
$r_1 r_2 r_1$	$-\alpha_2$	$-\alpha_1$
$r_2 r_1 r_2$	$-\alpha_2$	$-\alpha_1$

Hence we have

$$W = \left\langle r_1, r_2 \,|\, r_1^2 = 1 = r_2^2,\, r_1 r_2 r_1 = r_2 r_1 r_2 \right\rangle \cong S_3$$

We get this rep by taking $r_1 \mapsto (12)$, $r_2 \mapsto (23)$

Remark. *We can see that the above would satisfy what we want by using the braid relations.*

Now we can decompose

$$\mathfrak{sl}_3 = \underbrace{\mathfrak{g}_{\alpha_1}}_{e_1} \oplus \underbrace{\mathfrak{g}_{\alpha_2}}_{e_2} \oplus \underbrace{\mathfrak{g}_{\alpha_1+\alpha_2}}_{[e_1,e_2]} \oplus \dots$$

and then

$$R = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \cup \{-\alpha_1, -\alpha_2, -(\alpha_1 + \alpha_2)\}$$

Proposition 4.46. W *is generated by the* r_i *with the relations*

- $r_i^2 = 1$

- $(r_i r_j)^{m_{ij}} = 1$ *or equivalently* $\underbrace{r_j r_i r_j \ldots}_{m_{ij}} = \underbrace{r_i r_j r_i \ldots}_{m_{ij}}$

where

m_{ij}	2	3	4	6	∞
$a_{ij}a_{ji}$	0	1	2	3	≥ 4

Remark. *A group of this form is called a* ***Coxeter group****.*

Example 4.47. *We have correspondences*

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mapsto \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$$
$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mapsto \mathfrak{sl}_3$$
$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \mapsto \mathfrak{so}_4$$
$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \mapsto G_2$$
$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \mapsto \widehat{\mathfrak{sl}_2} = {a * b}\big/{(a^2 = 1 = b^2)} \quad \textit{(affine } \mathfrak{sl}_2\textit{)}$$

Now we could consider working with realisations, and we would get the following results:

- $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$
- $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}_{\mathbb{R}} \circlearrowleft W$

Definition 4.48 (Fundamental chamber)**.** *The* ***fundamental chamber*** *is* $\mathcal{C} \equiv \{h \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(h) \geq 0\}$. *Correspondingly the* ***Tits cone*** *is* $X = \bigcup_{w \in W} w(\mathcal{C})$

Proposition 4.49. *TFAE:*

- $|W| < \infty$
- $|R| < \infty$
- $X = \mathfrak{h}_{\mathbb{R}}$.

4.3 Finite vs Affine

We aim now to construct completely $\mathfrak{g}(\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)) \equiv \hat{g}$. We start by finding a realisation. We need

- $\hat{\mathfrak{h}}$ s.t. $\dim_{\mathbb{C}} \hat{\mathfrak{h}} = 3$
- $\{\alpha_0, \alpha_2\} \subset \hat{\mathfrak{h}}^*$, $\{\alpha_0^\vee, \alpha_1^\vee\} \subset \hat{\mathfrak{h}}$ linearly indep s.t. $\alpha_i(\alpha_j^\vee) = \pm 2$ if $i = j$ or $i \neq j$.

We can take

$$\hat{\mathfrak{h}} = \mathbb{C}\alpha_0^\vee \oplus \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}d$$
$$\hat{\mathfrak{h}}^* = \mathbb{C}\alpha_0 \oplus \mathbb{C}\alpha_1 \oplus \mathbb{C}\Lambda$$

with

$$\alpha_0(d) = 1$$
$$\alpha_1(d) = 0$$
$$\Lambda(\alpha_0^\vee) = 1$$
$$\Lambda(\alpha_1^\vee) = 0$$
$$\Lambda(d) = 0$$

Now we know we will also have a bilinear form given by

$$(\alpha_i^\vee, \alpha_j^\vee) = a_{ij}$$
$$(\alpha_0^\vee, d) = 1$$
$$(\alpha_1^\vee, d) = 0$$
$$(d, d) = 0$$
$$\text{also}$$
$$(\alpha_i, \alpha_j) = a_{ij}$$
$$(\alpha_0, \Lambda) = 1$$
$$(\alpha_1, \Lambda) = 0$$
$$(\Lambda, \Lambda) = 0$$

Now we recognise this gives a special element $c \equiv \alpha_0^\vee + \alpha_1^\vee$, $\delta = \alpha_0 + \alpha_1$. This gives a map

$$\nu : \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}^*$$
$$\alpha_i^\vee \mapsto \alpha_i$$
$$d \mapsto \Lambda$$
$$c \mapsto \delta$$

These now have inner product

$$(\delta, \alpha_0) = 0 = (\delta, \alpha_1)$$
$$\Rightarrow (\delta, \delta) = 0 \quad \text{but} \quad (\delta, \Lambda) = 1$$
$$(c, \alpha_0^\vee) = 0 = (c, \alpha_1^\vee)$$
$$\Rightarrow (c, c) = 0 \quad \text{but} \quad (c, d) = 1$$

Now we have orthogonal decompositions

$$\hat{\mathfrak{h}} = \mathbb{C}\alpha_1^\vee \oplus \mathbb{C}c \oplus \mathbb{C}d$$
$$\hat{\mathfrak{h}}^* = \mathbb{C}\alpha_1 \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda$$

We want to relate $\hat{\mathfrak{g}}$ to $\mathfrak{g} = \mathfrak{sl}_2$. In $\hat{\mathfrak{g}}$

$$[d, e_1] = 0 = [d, f_1]$$
$$[d, e_0] = e_0$$
$$[d, f_0] = -f_0$$
$$[c, e_i] = 0 = [c, f_i]$$

Hence $c \in Z(\hat{\mathfrak{g}})$.

4.3.1 Central extension of the loop algebra

Definition 4.50. *Define* $\mathcal{L} = \mathbb{C}[t, t^{-1}] = \{$*Laurent polynomials*$\}$ *and* $\operatorname{res} : \mathcal{L} \to \mathbb{C}$, $\operatorname{res}\left(\sum_k c_k t^k\right) = c_{-1}$.

Lemma 4.51. $\operatorname{res} : \mathcal{L} \to \mathbb{C}$ *is actually a linear functional with*

- $\operatorname{res}(t^{-1}) = 1$
- $\operatorname{res} \frac{dP}{dt} = 0$

Corollary 4.52. *We can construct* $\varphi : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ *by*

$$\varphi(P, Q) = \operatorname{res}\left(Q \frac{dP}{dt}\right)$$

which has the properties

- $\varphi(P, Q) = -\varphi(Q, P)$
- $\varphi(PQ, R) + \varphi(QR, P) + \varphi(RP, Q) = 0$

Definition 4.53. *The* ***loop algebra*** *of* $\mathfrak{g}$ *is*

$$\mathcal{L}\mathfrak{g} \equiv \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{L} = \operatorname{Maps}(\mathbb{C}^{\times}, \mathfrak{g})$$

We give it the Lie bracket

$$[x \otimes P, y \otimes Q]_0 = [x, y] \otimes PQ$$

Definition 4.54. *A* ***central extension*** *of the loop algebra is* $\mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$ *with the Lie bracket where* $\forall a \in \mathcal{L}\mathfrak{g}$*;*

$$[c, a] = 0$$
$$[a, b] = [a, b]_0 + \psi(a, b)c$$

for some antisymmetric bilinear map ψ *which makes that Jacobi identity hold.*

Restricting back to our example of $\mathfrak{g} = \mathfrak{sl}_2$ we have a bilinear form, so we can set

$$\psi : \mathcal{L}\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{L}\mathfrak{g} \to \mathbb{C}$$

with

$$\psi(x \otimes P, y \otimes Q) = (x, y) \cdot \varphi(P, Q)$$

This ψ satisfies

- $\forall a, b,\ \psi(a,b) = -\psi(b,a)$
- $\psi([a,b]_0, c) + \psi([b,c]_0, a) + \psi([c,a]_0, b) = 0$

Now set $\tilde{\mathcal{L}\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c$ with the bracket as above to get a central extension of the loop algebra of $\mathfrak{sl}_2$.
So in $\tilde{\mathcal{L}\mathfrak{g}}$,

- $Z(\tilde{\mathcal{L}\mathfrak{g}}) = \mathbb{C}c$
- $\alpha_1^\vee = \alpha_1^\vee \otimes 1 \in \tilde{\mathcal{L}\mathfrak{g}}$ satisfies

$$\begin{aligned}[\alpha_1^\vee, y \otimes Q]_0 &= [\alpha_1^\vee, y] \otimes Q = \mathrm{wt}(y)(\alpha_1^\vee)(y \otimes Q) \\ \psi(\alpha_1^\vee, y \otimes Q) &= (\alpha_1^\vee, y) \underbrace{\varphi(1, Q)}_{\mathrm{res}(0)} = 0\end{aligned}$$

This gives us a well defined subalgebra $\mathbb{C}\alpha_1^\vee \oplus \mathbb{C}c \subset \tilde{\mathcal{L}\mathfrak{g}}$.
Finally we can define

$$\hat{g} = \mathcal{L}\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

s.t.

- $[a,b] = [a,b]_0 + \psi(a,b)c$
- $[d,c] = 0$
- $[d, x \otimes P] = x \otimes t\frac{dP}{dt}$

Theorem 4.55. *With the above definition we have*

$$\begin{aligned}\mathfrak{g}(\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)) &\to \hat{\mathfrak{g}} \\ e_1, \alpha_1^\vee, f_1 &\mapsto e_1, \alpha_1^\vee, f_1 \\ c &\mapsto c \\ d &\mapsto t\frac{d}{dt} \\ e_0 &\mapsto f_1 \otimes t \\ f_0 &\mapsto e_1 \otimes t^{-1}\end{aligned}$$

5 Witt and Virasoro Algebras

5.1 The Witt algebra

Let $A = \mathbb{C}[z, z^{-1}]$ be Laurent polynomials, and then define

$$\mathrm{Der}(A) = \{\phi : A \to A \,|\, \phi\ \mathbb{C}\text{-linear},\ \phi(fg) = \phi(f)g + f\phi(g)\}$$

Proposition 5.1. $\mathrm{Der}(A)$ *is a Lie algebra with bracket*

$$[\phi, \psi](f) = \phi(\psi(f)) - \psi(\phi(f))$$

Proposition 5.2. *The operators* $\left\{L_n = -z^{n+1}\frac{d}{dz} \,|\, n \in \mathbb{Z}\right\}$ *are a basis for* $\mathrm{Der}(A)$

Proof. The are obviously independent. Write $\phi(z) = -\sum_n a_n z^{n+1}$ (with all but finitely many $a_n \neq 0$). Now the Leibniz rule gives

$$\phi(z^k) = kz^{k-1}\phi(z)$$

and so

$$\begin{aligned}\phi(f)(z)\frac{df}{dz}\phi(z) &= -\sum_n a_n z^{n+1}\frac{d}{dz}f(z) \\ &= \sum_n a_n L_n(f)(z)\end{aligned}$$

□

Definition 5.3. *The* ***Witt algebra*** Witt *is the Lie algebra with basis* $\{L_n\}$ *and bracket as above.*

5.2 Central extension

Definition 5.4. *Let* $\mathfrak{a}$ *be a Lie algebra. A* ***central extension*** *if* $\mathfrak{a}$ *is a pair* $(\tilde{\mathfrak{a}}, \pi)$ *s.t.*

- $\tilde{\mathfrak{a}}$ *is a Lie algebra*
- $\pi : \tilde{\mathfrak{a}} \to \mathfrak{a}$ *is a surjective LA hom*
- $\dim_{\mathbb{C}} \ker \pi = 1$
- $\forall x \in \tilde{\mathfrak{a}}, y \in \ker \pi,\ [x, y] = 0$

Definition 5.5. *Two central extensions* $(\tilde{\mathfrak{a}}, \pi), (\tilde{\mathfrak{a}}', \pi')$ *are* ***equivalent*** *if* $\exists \phi : \tilde{\mathfrak{a}} \to \tilde{\mathfrak{a}}'$ *a LA iso s.t.* $\pi' \circ \phi = \pi$.

Example 5.6. *The trivial extension is* $\tilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathbb{C}K$ *where* K *is the centre of* $\mathfrak{a}$*, and we take the same bracket for* $\tilde{\mathfrak{a}}$

Proposition 5.7. *Up to equivalence,* $\exists!$ *non trivial central extension of the Witt algebra, the* ***Virasoro algebra*** Vir*, written as*

$$0 \to \underbrace{\mathbb{C}c}_{\ker(\pi_{\mathrm{Vir}})} \to \mathrm{Vir} \overset{\pi_{\mathrm{Vir}}}{\to} \mathrm{Witt} \to 0$$

Explicitly we may say that Vir *has basis* $\{L_n, c\}$ *with bracket*

$$\begin{aligned}[c, \mathrm{Vir}] &= 0 \\ [L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}c\end{aligned}$$

The map to Witt *is* $L_n \mapsto L_n$, $c \mapsto 0$

Proof. We check it exists, and this is done simply by observing that the relations given above give a central extension. We can check it is not trivial, as in the trivial extension we would have $[L_2, L_{-2}] = 2\,[L_1, L_{-1}]$.
For uniqueness, let $(\mathfrak{b}, \pi)$ be another central extension. Choose a splitting i : Witt $\to \mathfrak{b}$ with $\pi \circ i = \mathrm{id}$. We then have $\mathfrak{b} = \mathbb{C}k \oplus i(\mathrm{Witt})$. The bracket is given by

$$\begin{aligned}
[i(\mathrm{Witt}), k] &= 0 \\
[i(L_m), i(L_n)] &= (m-n)i(L_{m+n}) + a(m,n)k
\end{aligned}$$

for some antisymmetric $a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$. Define a new splitting i' by

$$i'(L_n) = \begin{cases} i(L_0) & n = 0 \\ i(L_n) - \frac{a(0,n)}{n}k & n \neq 0 \end{cases}$$

Then $[i'(L_0), i'(L_n)] = -ni'(L_n)$. so wlog we may assume $a(0,n) = 0$. Applying the Jacobi identity we get

$$\begin{aligned}
0 &= [[i(L_0), i(L_m)]\,, i(L_n)] + [[i(L_n), i(L_0)]\,, i(L_m)] + [[i(L_m), i(L_n)]\,, i(L_0)] \\
&= (m+n)a_(m,n)k
\end{aligned}$$

Hence $a(m,n) = a(m)\delta_{m,-n}$ for some odd function $a : \mathbb{Z} \to \mathbb{C}$. Applying Jacobi again for the triple $i(L_0), i(L_n), i(L_{-n-1})$ gives

$$(n-1)a(n+1) = (n+2)a(n) - (2n+1)a(1)$$

This is a linear recurrence completely determined by $a(1), a(2)$, so the space of solutions is 2-dimensional. It can be found that $a(n) = n, a(n) = n^3$ are both solutions, so the general solution is $a(n) = \alpha n + \beta n^3$ for $\alpha, \beta \in \mathbb{C}$.
If $\beta = 0$, we have a map

$$\begin{aligned}
\mathrm{Witt} \oplus \mathbb{C}k &\to \mathfrak{b} \\
L_n &\mapsto i(L_n) + \frac{1}{2}\alpha\delta_{0n}k \\
k &\mapsto k
\end{aligned}$$

which is a LA iso, and so $(\mathfrak{b}, \pi)$ is trivial.
If $\beta \neq 0$, we have the LA iso

$$\begin{aligned}
\mathrm{Vir} &\to \mathfrak{b} \\
L_n &\mapsto i(L_n) + (\alpha + \beta)\delta_{0n}k \\
c &\mapsto 12\beta k
\end{aligned}$$

□

5.3 Heisenberg algebra

Definition 5.8. *The **Heisenberg algebra**,* Heis*, has basis* $\{\hbar, a_n\}$ *and bracket*

$$\begin{aligned}
[a_m, a_n] &= m\delta_{m.-n}\hbar \\
[\hbar, a_n] &= 0
\end{aligned}$$

Example 5.9 (Natural reps of Heis). *Fixing $\mu, h \in \mathbb{C}$, define*

$$B(\mu, h) = \mathbb{C}[x_1, x_2, \dots]$$

with rep $\rho : \text{Heis} \to \mathfrak{gl}(B(\mu, h))$ *given by*

$$\rho(\hbar) = h$$

$$\rho(a_n) = \begin{cases} \frac{\partial}{\partial x_n} & n > 0 \\ \mu & n = 0 \\ -hnx_{-n} & n < 0 \end{cases}$$

This is called the ***Bosonic Fock space*** *or* ***oscillator reps****.*

Remark. *The reps $V = B(\mu, h)$ satisfy*

$$\forall v \in V,\ \exists N \text{ s.t. } \forall n > N,\ a_n v = 0$$

Let V be any rep satisfying the above condition. For any $n \in \mathbb{Z}$, define

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_{-k} a_{n+k} :$$

where $: \cdot :$ is the normal ordered product, i.e.

$$: a_i a_j : := \begin{cases} a_i a_j & i < j \\ a_j a_i & i > j \end{cases}$$

By our requirement on V, we have ensured that $\forall v \in V$, $L_n v$ is well defined as the sum has only finitely many non-zero terms.

We will work towards proving a big theorem now, so we will need some results:

Lemma 5.10. $\forall k, n \in \mathbb{Z}$, $[a_k, L_n] = k a_{k+n}$

Proof. Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Choose $\epsilon > 0$ and then set

$$L_n(\epsilon) = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{j+n} : \psi(\epsilon j)$$

Note that $\forall v \in V, \exists \delta > 0$ s.t. $\forall \epsilon < \delta, L_n(\epsilon) v = L_n v$. Now $L_n(\epsilon) = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j} a_{j+n} \psi(\epsilon j)$ acts by a $\mathbb{C}$-scalar as

$$\begin{aligned} [a_k, L_n(\epsilon)] &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [a_k, a_{-j} a_{j+n}] \, \psi(\epsilon j) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [[a_k, a_{-j}] \, a_{j+n} \psi(\epsilon j) + a_{-j} \, [a_k, a_{j+n}] \, \psi(\epsilon j)] \\ &= \frac{1}{2} \left[k a_{k+n} \psi(\epsilon k) + k a_{k+n} \psi(\epsilon(-n-k)) \right] \\ &= k a_{k+n} \quad \text{(for } \epsilon \text{ small)} \end{aligned}$$

□

With the above we can state and prove the following:

Theorem 5.11. *Let V be as above and assume $\forall v \in V$, $\hbar v = v$. (e.g. $V = B(\mu, 1)$). Then*

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3 - m}{12}$$

Proof. Using notation from before calculate

$$\begin{aligned}
[L_m(\epsilon), L_n] &= \frac{1}{2}\sum_{j\in\mathbb{Z}} [a_{-j}a_{j+m}, L_n]\,\psi(\epsilon j) \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}} [[a_{-j}, L_n]\,a_{j+m} + a_{-j}\,[a_{j+m}, L_n]]\,\psi(\epsilon j) \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}} [(-j)a_{n+j}a_{j+m} + (j+m)a_{-j}a_{j+m+n}]\,\psi(\epsilon j) \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}} [(-j) : a_{n+j}a_{j+m} : +(j+m) : a_{-j}a_{j+m+n} :]\,\psi(\epsilon j) \\
&\quad - \frac{1}{2}\delta_{m,-n}\sum_{j<-m}(-m-j)j\psi(\epsilon j) + \frac{1}{2}\delta_{m,-n}\sum_{j<0}(-j)(j+m)\psi(\epsilon j) \\
&= \frac{1}{2}\sum_{j\in\mathbb{Z}} [(-j) : a_{n+j}a_{j+m} : +(j+m) : a_{-j}a_{j+m+n} :]\,\psi(\epsilon j) \\
&\quad + \frac{1}{2}\delta_{m,-n}\sum_{j=-1}^{-m} j(j+m)\psi(\epsilon j)
\end{aligned}$$

The first sum telescopes (reindexing) giving a finite sum, so we can then take the limit $\epsilon \to 0$, and we get

$$[L_m, L_n] = \frac{1}{2}\sum_{j\in\mathbb{Z}}(m-n) : a_{-j}a_{j+m+n} : + \frac{1}{2}\delta_{m,-n}\sum_{j=-1}^{-m} j(j+m)$$

and answer follows. □

Definition 5.12. *A* Vir *rep V has* ***central charge*** *$c \in \mathbb{C}$ if*

$$\forall v \in V,\ c_{\mathrm{Vir}} \cdot v = cv$$

where c_{Vir} is the central charge of Vir.

Remark. *The theorem says that the* Heis *rep V has central charge 1. If the central charge was 0, these reps would be reps of* Witt*, and moreover this is a bijection.*

5.4 Connection to affine Lie algebras

Recall that if $\mathfrak{g}$ is a finite-dimensional Lie algebra then $\mathcal{L}\mathfrak{g} = \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with bracket $[xf, yg] = [x, y]\,fg$, then the affine Lie algebra $\hat{\mathfrak{g}}$ is a natural central extension with SES

$$0 \to \mathbb{C}k \to \tilde{\mathcal{L}\mathfrak{g}} \to \mathcal{L}\mathfrak{g} \to 0$$

If $\mathfrak{g}$ is simple, then $\hat{\mathfrak{g}} = \tilde{\mathcal{L}\mathfrak{g}} \oplus \mathbb{C}d$ is Kac-Moody.

Example 5.13. *If $\mathfrak{g} = \mathfrak{a} = \mathbb{C}a$ is a 1-dimensional abelian Lie algebra then*

$$\begin{aligned} \mathcal{L}\mathfrak{a} &\to \text{Heis} \\ at^n &\mapsto a_n \\ k &\mapsto \hbar \end{aligned}$$

is LA hom.

6 Highest weight representations

Definition 6.1. *Let V be a rep space of* Vir. *We say $v \in V$ is* ***singular*** *of weight $(h, c) \in \mathbb{C}^2$ if*

- $L_0 v = hv$
- $c_{\text{Vir}} v = cv$
- $\forall n > 0$, $L_n v = 0$

Definition 6.2. *Let $v \in V$ be singular. We say it is a* ***highest weight vector*** *if*

$$V = \text{Span}\,\{L_{-n_1} \ldots L_{-n_k} v \,|\, k, n_1, \ldots, n_k > 0\}$$

Remark. *There is a similar definition for Kac-Moody algebras.*

Example 6.3. *$v = 1 \in B(\mu, 1)$ is a singular vector of weight $(\frac{1}{2}\mu^2, 1)$.*

Proposition 6.4. *Let V be a highest weight rep of* Vir *with highest weight (h, c). Then*

- *The module V has central charge c, i.e. $\forall w \in V$, $c_{\text{Vir}} w = cw$*
- *We have*

  $$V = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} V_{h+k}$$

  where $V_\lambda = \{w \in V \,|\, L_0 w = \lambda w\}$.
- *Each V_{h+k} is finite dimensional*
- $\dim V_h = 1$.

Proposition 6.5. *Let V be a highest weight module. Then there is a unique maximal proper submodule $V'' \subseteq V$. Hence $V' = {}^{V}\!/_{V''}$ is an irreducible highest weight rep with the same highest weight as V.*

Proof. Let V'' be the sum of all proper submodules of V. It remains to be shown that $V'' \neq V$. Assume $U \subsetneq V$ is a submodule, and then we know $U \cap V_h = \{0\}$ as the intersection is either 0 (as V_h is 1 dimensional) or contains the highest weight vector, and in the latter case we would have $U = V$. So

$$U = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (U \cap V_{h+k}) \subseteq \bigoplus_{k \in \mathbb{Z}_{>0}} V_{h+k}$$

and then

$$V'' = \sum U =\subsetneq \bigoplus_{k\in\mathbb{Z}_{>0}} V_{h+k}$$

so $V'' \neq V$. To get the second part, note that $W \subsetneq V'$ has preimage under the quotient which must lie in $V'' \Rightarrow W = 0$. $\square$

6.1 Verma modules

Proposition 6.6. *Let $(h,c) \in \mathbb{C}^2$. Then $\exists$ a highest weight Vir-module $M(h,c)$ with highest weight (h,c) and highest weight vector v_M s.t.*

- *$\forall V$ another rep of highest weight (h,c) with hwv v $\exists!$ Vir-module hom*

$$\begin{aligned} M(h,c) &\to V \\ v_M &\mapsto v \end{aligned}$$

- *V is isomorphic to a quotient of $M(h,c)$*

Proof. As a vector space

$$\mathrm{Vir} = \mathrm{Vir}_+ \oplus \mathfrak{h} \oplus \mathrm{Vir}_-$$

where $\mathrm{Vir}_\pm = \mathrm{Span}\,\{L_n \,|\, n \gtrless 0\}$ and $\mathfrak{h} = \mathrm{Span}\,\{L_0, c_{\mathrm{Vir}}\}$. Let $\mathrm{Vir}_{\geq 0} = \mathrm{Vir}_+ \oplus \mathfrak{h}$ Then we have

$$\begin{aligned} \rho : \mathrm{Vir}_{\geq 0} &\to \mathfrak{gl}_1 = \mathfrak{gl}(\mathbb{C}_{(h,c)}) \\ \forall n > 0,\, L_n &\mapsto 0 \\ L_0 &\mapsto h \\ c_{\mathrm{Vir}} &\mapsto c \end{aligned}$$

Hence

$$\rho : U(\mathrm{Vir}_{\geq 0}) \to \mathfrak{gl}_1$$

is an extension to an associative algebra hom from the UEA. We can then construct $M(h,c)$ by

$$\begin{aligned} M(h,c) &= U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_{\geq 0})} \mathbb{C}_{(h,c)} \\ &\cong U(\mathrm{Vir}) \big/ U(\mathrm{Vir})(x, -\rho(x), x \in \mathrm{Vir}_+) \end{aligned}$$

Exercise 6.7. *Show that this constructed $M(h,c)$ has the properties required, with hwv 1.*

$\square$

Exercise 6.8. *Show that if M, M' are two modules satisfying the above, then $M \cong M'$*

Definition 6.9. *$M(h,c)$ is called the **Verma module** of highest weight (h,c)*

Corollary 6.10. *$\forall (h,c) \in \mathbb{C}^2$, there is a unique irreducible Vir-module $V(h,c)$ of highest weight (h,c)*

Proof. Let $V(h,c) = {}^{M(h,c)}\!\!\diagup_{J(h,c)}$ be the unique irreducible quotient of $M(h,c)$. Then by def any other such V irreducible of highest weight is isomorphic to a quotient so $V \cong V(h,c)$ $\square$

Proposition 6.11. *The Verma module $M(h,c)$ has basis*

$$\{L_{-n_k}\dots L_{-n_1}v_M \,|\, k \geq 0,\, 0 < n_1 \leq \dots \leq n_k\}$$

Proof. By the Poincare-Birkhoff-Witt theorem we know

$$\left\{L_{-n_k}\dots L_{-n_1}c^i h^j L_{m_1}\dots L_{m_l} \,|\, i,j > 0,\, k,l \geq 0,\, 0 < n_1 \leq \dots \leq n_k,\, 0 < m_1 \leq \dots \leq m_l\right\}$$

Then as the latter part is a basis for $U(\mathrm{Vir}_{\geq 0})$, it gets cancelled in the quotient. $\square$

6.2 Unitary reps

Recall we knew that 1 is a singular vector of weight $(\frac{1}{2}\mu^2, 1)$ for the rep $B(\mu,1)$. If we define

$$B'(\mu,1) = \mathrm{Span}\,\{L_{-n_k}\dots L_{-n_1}1 \,|\, k \geq 0,\, n_i > 0\} \subseteq B(\mu,1)$$

then $B'(\mu,1)$ is now a highest weight rep with highest weight $(\frac{1}{2}\mu^2, 1)$.

Definition 6.12. *Let $\mathfrak{a}$ be a complex Lie algebra. An* ***anti-involution*** *on $\mathfrak{a}$ is a function $\omega : \mathfrak{a} \to \mathfrak{a}$ s.t.*

- $\omega^2 = \mathrm{id}$
- $\omega(ax+by) = \overline{a}x + \overline{b}y$
- $\omega([x,y]) = -[\omega(x), \omega(y)]$

Definition 6.13. *If V is an $\mathfrak{a}$-rep, then a Hermitian form on V is* ***contravariant*** *if*

$$\forall u, v \in V,\, x \in \mathfrak{a},\, \langle x \cdot u, v\rangle = \langle u, \omega(x)\cdot v\rangle$$

Definition 6.14. *A rep V is* ***unitary*** *if it admits a contravariant inner product.*

Example 6.15. *Anti-inovlutions on* Heis *and* Vir *are given by*

1. $\omega_{\mathrm{Heis}}(a_n) = a_{-n}$, $\omega_{\mathrm{Heis}}(\hbar) = \hbar$
2. $\omega_{\mathrm{Vir}}(L_n) = L_{-n}$, $\omega_{\mathrm{Vir}}(c) = c$

Note

$$\omega_{\mathrm{Heis}}(L_n) = \frac{1}{2}\sum_{j\in\mathbb{Z}} \omega_{\mathrm{Heis}}(:a_{-j}a_{j+n}:) = \frac{1}{2}\sum_{j\in\mathbb{Z}} :a_{-j-n}a_j: = L_{-n} = \omega_{\mathrm{Vir}}(L_n)$$

Proposition 6.16. *Assume $\mu \in \mathbb{R}$, then the* Heis *rep $B(\mu,1)$ has a unique contravariant inner product s.t. $\langle 1,1\rangle = 1$. Explicitly*

$$\langle P, Q\rangle = \langle \omega(P)Q\rangle$$

where $\langle\cdot\rangle$ = take constant term and

$$\omega : \mathbb{C}[x_1, \dots,] \to \mathrm{Heis}$$

is the complex anti-linear ring hom given by $\omega(x_n) = \frac{1}{n}a_n$

Proof.

Exercise 6.17. *Do this*

$\square$

Corollary 6.18. *$B(\mu, 1)$ is a unitary* Vir *rep.*

Lemma 6.19. *Let V be a unitary* Vir*-module such that $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ is a direct sum of L_0-eigenspaces and $\dim V_\lambda < \infty$.*
If $U \subseteq V$ is a submodule, then $\exists U^\perp \subseteq V$ another submodule s.t. $V = U \oplus U^\perp$.

Proof. Let

$$U^\perp = \{v \in V \mid \forall u \in U, \langle u, v \rangle = 0\}$$

It is simple to check $U^\perp \subseteq V$ is a submodule, and $U \cap U^\perp = 0$. To show $V = U + U^\perp$, note we can decompose $v \in V$ into eigenvectors of L_0, so it is sufficient to show $V_\lambda \subseteq U + U^\perp$. But

$$V_\lambda = (V_\lambda \cap U) \oplus (V_\lambda \cap U^\perp)$$

as $\dim V_\lambda < \infty$ $\square$

Lemma 6.20. *Let V be a unitary highest weight rep. Then V is irreducible.*

Proof. Let $V'' \subseteq V$ be the unique maximal proper submodule. Then $(V'')^\perp \subsetneqq V$ is a submodule and $V'' \cap (V'')^\perp = 0$. THen either

1. $(V'')^\perp = V \Rightarrow$ done
2. $(V'')^\perp = 0 \Rightarrow V = V''$ contradction.

$\square$

Proposition 6.21. *Assume $\mu \in \mathbb{R}$. Then the highest weight module $B'(\mu, 1)$ is irreducible.*

Proof. Use that B' is unitary. then done by lemma. $\square$

Proposition 6.22. *Assume $h, c \in \mathbb{R}$. Then*

1. *If $M(h, c)$ is unitary, then $h, c > 0$*
2. *If $h \geq 0$, $c \geq 1$, then the irreducible representation $V(h, c) = {}^{M(h,c)}\!/\!_{J(h,c)}$ is unitary*
3. *If $h > 0$, $c > 1$ then $M(h, c) = V(h, c)$.*

Proposition 6.23. *Assume $h, c \in \mathbb{R}$. Let $v \in M(h, c)$ be the highest weight vector. Then*

1. *$\exists!$ contravariant Hermitian form on M s.t. $\langle v, v \rangle = 1$*
2. *The eigenspaces of L_0 are pairwise orthogonal*
3. *$\ker \langle \cdot, \cdot \rangle = J(h, c)$ is a maximal proper submodule*

Hence $V(h, c)$ carries a non-degenerate Hermitian form s.t. $\langle v, v \rangle = 1$

Proof. See notes $\square$

6.3 Kac Determinant formula

Recall we have a basis for $M(h,c)$. Kac found a formula for the determinant of $\langle \cdot, \cdot \rangle|_{M(h,c)_{h+n}}$

7 Lie algebra of infinite matrices

Definition 7.1. *Define*

$$\mathfrak{gl}_\infty = \{(a_{ij}) \,|\, a_{ij} \in \mathbb{C}, \text{ almost all entries } 0\}$$

It has has basis $\{E_{ij}\}$, the natural extension of that for finite $\mathfrak{gl}_n$.

Proposition 7.2. *$\mathfrak{gl}_\infty$ is a Lie algebra with bracket given by matrix commutation.*

Recall the definition of a grading:

Definition 7.3. *A **graded Lie algebra** is a Lie algebra $\mathfrak{g}$ with decomposition $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ s.t.*

$$[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l}$$

We write $\forall X \in \mathfrak{g}_k$, $\deg X = k$

Proposition 7.4. *We can make $\mathfrak{gl}_\infty$ into a graded Lie algebra with grading*

$$(\mathfrak{gl}_\infty)_k = \operatorname{Span}\{E_{ij} \,|\, i - j = k\}$$

Definition 7.5. *We define the associated group to be*

$$GL_\infty = \{(A_{ij}) \,|\, A_{ij} \in \mathbb{C}, \text{ invertible, almost all } A_{ij} - \delta_{ij} \ 0\}$$

with operation given by matrix multiplication

Proposition 7.6. *GL_∞ is a Lie group with Lie algebra $\mathfrak{gl}_\infty$*

It turns out we need a bigger Lie algebra

Definition 7.7. *Let*

$$\mathfrak{gl}_\infty^\Delta = \{(a_{ij}) \,|\, \forall |i-j| \gg 0, \, a_{ij} = 0\}$$

Proposition 7.8. *$\mathfrak{gl}_\infty \subset \mathfrak{gl}_\infty^\Delta$*

7.1 Central extension

Definition 7.9. *Consider the central extension $\hat{\mathfrak{gl}}_\infty^\Delta$ defined by*

$$0 \to \mathbb{C}c \to \hat{\mathfrak{gl}}_\infty^\Delta \to \mathfrak{gl}_\infty^\Delta \to 0$$

with bracket given by, for $a, b \in \mathfrak{gl}_\infty^\Delta$,

$$[a,b] = ab - ba + \gamma(a,b)c$$

*γ is called the **cocycle** and satisfies*

$$\gamma(E_{ij}, E_{ji}) = 1 = -\gamma(E_{j-i}, E_{ij} \quad \text{for } i \leq 0, \, j \geq 1$$

and is 0 otherwise.

We have representations of Heis and Vir inside $\hat{\mathfrak{gl}}_\infty^\Delta$ given by

$$V = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}v_j$$

There is then a natural action on V by multiplication.

7.2 Shift operators

Definition 7.10. *Define the* ***shift operator*** $\Delta_k : V \to V$, $v_j \mapsto v_{j-k}$. *We can write explicitly*

$$\Lambda_k = \sum_{i\in\mathbb{Z}} E_{i,i+k}$$

Proposition 7.11. $[\Delta_k, \Delta_j] = 0$

Let $\eta = \oplus_k \mathbb{C}\Lambda_k$ be the subalgebta of $\mathfrak{gl}_\infty^\Delta$. We can then let $\hat{\eta}$ be the central extension given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \hat{\mathfrak{gl}}_\infty^\Delta & \longrightarrow & \mathfrak{gl}_\infty^\Delta & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{C}c & \longrightarrow & \hat{\eta} & \longrightarrow & \eta & \longrightarrow & 0 \end{array}$$

Proposition 7.12. *We have*

- $\gamma(\Lambda_n, \Lambda_k) = n\delta_{n,-k}$
- $\hat{\eta} = \text{Heis}$

Proof. The first point is a calculation, Secondly, there is an explicit isomorphism, and the relations are the same in each. ☐

Proposition 7.13. *For the* Witt *algebra we can say*

- $\exists$ *a family of embeddings depending on* $\alpha, \beta \in \mathbb{C}$ *given by*

$$\begin{aligned} i_{\alpha,\beta} : \text{Witt} &\hookrightarrow \mathfrak{gl}_\infty^\Delta \\ L_n &\mapsto \sum_{k\in\mathbb{Z}} [k - \alpha - \beta(n+1)]\, E_{k+n,k} \end{aligned}$$

- *Let* $\hat{\text{Witt}} \subset \hat{\mathfrak{gl}}_\infty^\Delta$ *be the central extension. Then*

$$\gamma(L_i, L_j) = \delta_{i,-j}\left(\frac{i^3 - i}{12} c_\beta + 2ih_0\right)$$

where $c_\beta = -12\beta^2 + 12\beta - 2$ *and* $h_0 = \frac{1}{2}\alpha(\alpha + 2\beta - 1)$

- *Let* $\hat{L_n} = L_n + \delta_{n,0}h_0 c$. *Then*

$$\left[\hat{L_n}, \hat{L}_m\right] = (n-m)\hat{L}_{n+m} + \delta_{n,-m}\left(\frac{n^3 - n}{12}\right) c_\beta c$$

8 Fermionic Fock space

We can then consider $T(V) = \bigoplus_{k\geq 0} V^{\otimes k}$, let $I = \langle x \otimes x \rangle$, and then get

$$\Lambda(V) = {}^{T(V)}\!\diagup_{I}$$

This comes equipped with a projection map $p : T(V) \to \Lambda(V)$. Letting $p(T_k(V) = \Lambda^k(V)$, we get the decomposition

$$\Lambda(V) = \bigoplus_{k\geq 0} \Lambda^k(V)$$

There are also linear maps

$$\begin{aligned} \phi^{(m)}_{s,k} : \Lambda^k(V) &\to \Lambda^s(V) \\ u &\mapsto u \wedge (v_{-k+m} \wedge \cdots \wedge v_{-s+k+m} \end{aligned}$$

for fixed $m \in \mathbb{Z}$, $k \leq s$. These maps obey

$$\begin{aligned} \phi^{(m)}_{r,s} \circ \phi^{(m)}_{s,k} &= \psi^{(m)}_{r,k} \\ \phi^{(m)}_{k,k} &= \mathrm{id} \end{aligned}$$

Hence $(\Lambda^k(V), \phi^{(m)}_{r,k})$ for a **direct system** for each $m \in \mathbb{Z}$. I can then take the direct limit to get

Definition 8.1. *The* ***Fermionic Fock space of charge m*** *is*

$$F^{(m)} = \Lambda^{\infty}_{(m)}(V) = \lim_{\to} \Lambda^k(V)$$

The construction works as

$$\lim_{\to} \Lambda^k(V) = {}^{\bigsqcup_k \Lambda^k(V)}\!\diagup_{\sim}$$

where the equivalence relation is given by

$$x_i \in \Lambda^i(V) \sim x_j \in \Lambda^j(V) \Leftrightarrow \exists h,\, i,j \leq h,\, \phi^{(m)}_{h,i}(x_i) = \phi^{(m)}_{k,j}(x_j)$$

We have a basis given by

$$\psi = v_{i_0} \wedge v_{i_{-1}} \wedge \ldots$$

called the **semi-infinite monomials**, requiring the conditions

- $i_0 > i_{-1} > \ldots$
- $i_k = k + m$ for $k \ll 0$.

This generalises naturally to get

$$\psi_m = v_m \wedge v_{m-1} \wedge \dots$$

which is the **vacuum vector of charge m**.
The FFS comes with a grading given by

$$\deg \psi = \sum_{s \geq 0} i_{-s} + s - m$$

which is forced to be finite by our condition on i_k for $k \ll 0$. We then have

$$F_k^{(m)} = \text{Span}\left\{\psi \mid \deg \psi = k\right\}$$

Now let $\lambda = (\lambda_0, \dots, \lambda_{n-1}) + k$ be a partition if h, that is

- $\lambda_0 + \dots + \lambda_{n-1} + k = h$
- $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1}$

We can then get semi-infinite monomials ψ_λ from partitions by saying $j_{-i} = \lambda_i - i + m$ for $i = 0, \dots, n-1$ and then $j_{-n-i} = -n + m - i$

Example 8.2. *Take $\lambda = (5, 3, 3, 1) + 12$ is a partition of 24. Take $m = 0$. We then find*

$$\begin{aligned} j_0 &= 5 j_{-1} & &= 2 \\ j_{-2} &= 1 \\ j_{-3} &= -2 \end{aligned}$$

and so we get

$$\psi_\lambda = (v_5 \wedge v_2 \wedge v_1 \wedge v_{-2}) \wedge v_{-4} \wedge v_{-5} \wedge \dots$$

Proposition 8.3. *We have*

- $F^{(m)} = \bigoplus_{k \geq 0} F_k^{(m)}$, $F_0^{(m)} = \mathbb{C}\psi_m$
- $\dim F_k^{(m)} = p(k) =$*number of partitions of* k
- $\dim_q F^{(m)} \equiv \sum_{k \geq 0} (\dim F_k^{(m)}) q^k = \prod_{l \geq 1} (1 - q^l)^{-1}$

Proof.

- Clear
- $\{\psi_\lambda \mid \lambda \text{ partition of h}\}$ is a basis of $F_h^{(m)}$
- The first part is the definition. Then we have

$$\dim_q F^{(m)} = \sum_{k \geq 0} p(k) q^k$$

□

9 Representations of GL_∞, $\mathfrak{gl}_\infty$, on $F^{(m)}$

9.1 Actions on tensor products

Let $\mathfrak{g}$ be a Lie algebra and M, N $\mathfrak{g}$-reps. Then $M \otimes_{\mathbb{C}} N$ is a representation space of $\mathfrak{g}$ given by

$$x \in \mathfrak{g},\, m, n \in M, N,\, x \cdot (m \otimes n) = (x \cdot m) \otimes n + m \otimes (x \cdot n)$$

If G is a group, then we get a rep on $M \otimes_{\mathbb{C}} N$ by

$$g \cdot (m \otimes n) = (gm) \otimes (gn)$$

Remark.