

Supersymmetry Notes

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1 Introduction - What is SUSY and Why do we care?

1.1 What is SUSY

In any quantum theory involving fermions, we can always split true Hilbert space into *bosonic* and *fermionic* parts:

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$$

Here, \mathcal{H}_B contains an even number of fermionic excitations and \mathcal{H}_F contains an odd number of fermion excitations, since a pair of fermions is a boson.

A theory is **supersymmetric** if there is a fermionic operator $Q : \mathcal{H}_B \rightarrow \mathcal{H}_F$, $Q : \mathcal{H}_F \rightarrow \mathcal{H}_B$, such that $\{Q, Q^\dagger\} = 2H$, where H is the Hamiltonian, and Q^\dagger is the adjoint of Q with respect to the inner product on \mathcal{H} . We also require $Q^2 = 0 = (Q^\dagger)^2$.

The consequences of this are

•

$$\begin{aligned} 2[H, Q] &= [\{Q, Q^\dagger\}, Q] \\ &= [QQ^\dagger + Q^\dagger Q, Q] \\ &= QQ^\dagger Q - QQ^\dagger Q \text{ since } Q^2 = 0 \\ &= 0 \end{aligned}$$

Hence Q is *conserved* and the transformations it generates will be *symmetries*. These symmetries are parametrised by fermionic parameters (since Q is a fermionic operator) which are called **supersymmetries**. Q is known as the **supercharge**.

• For any state $|\psi\rangle \in \mathcal{H}$ we have

$$\begin{aligned} 2\langle\psi|H|\psi\rangle &= \langle\psi|\{Q, Q^\dagger\}|\psi\rangle \\ &= \langle\psi|QQ^\dagger|\psi\rangle + \langle\psi|Q^\dagger Q|\psi\rangle \\ &= \|Q^\dagger|\psi\rangle\|^2 + \|Q|\psi\rangle\|^2 \\ &\geq 0 \end{aligned}$$

\Rightarrow all states have non-negative energy. There is equality iff the ground state obeys $Q|\psi\rangle = 0 = Q^\dagger|\psi\rangle$. In other words, $|0\rangle$ has zero energy iff it is *supersymmetric*.

1.2 Why do we care?

- **Phenomenological:** In the Standard Model, the energy scale (i.e. the ultraviolet cutoff) versus the coupling constants looks like (...) so the forces don't meet in the plot. Matter in the SM actually transforms in representation of $SO(10) \supset SU(5) \supset G_{SM} = SU(3) \times SU(2) \times U(1)$. Perhaps there was some unification, then the symmetries were broken? This does not happen in the Standard Model. If, however, we take a supersymmetric SM we get new particles \Rightarrow changes scale dependence of coupling constant, allowing for the possibility of grand unification.

- Another phenomenological reason is that it provides a *dark matter* candidate.
- (?) Perhaps related to the mass of the Higgs particle. The SM has a puzzle, why does the Higgs particle take the mass it does? We expect large quantum corrections to our predictions, but it turns out that we don't need them. What "protects" the mass of the Higgs? Could be SUSY... *but* ruled out by CERN.
- **Theoretical Motivations:** SUSY helps understand QFT. Usually in QM, we've started from idealised systems with lots of symmetry, *then* we perturb to more realistic cases. In QFT, the *only* idealised case met so far is free theory. SUSY allows us to do better. We can compute some quantities *exactly*. Better yet, these often reveal deep connections between QFT and geometry/topology.

2 Path Integrals in QFT

Consolidate this section into something

In QFT, we are interested in computing

$$z = \int_{\mathcal{C}} e^{-\frac{S[X]}{\hbar}} \mathcal{D}X \text{ (Euclidean signature)}$$

where X is some field and \mathcal{C} is the space of all field configurations. This integral is *not* well defined, and is formidably hard to compute. As a toy, suppose the whole universe is just a single point: $\mathcal{M} = \{pt\}$. Then a field is a map $X : \{pt\} \rightarrow \mathbb{R}$, and the integral is now just

$$z = \int_{\mathbb{R}} e^{-\frac{S(X)}{\hbar}} dX$$

Suppose $S(X) = \frac{1}{2}mX^2 + \frac{1}{6}\lambda X^4 + \frac{1}{6!}gX^6$, say. Even with this simple form of the integral, the integral is hard for the choice of S .

As $\hbar \rightarrow 0$, taking the semi classical limit, we can obtain an asymptotic series as $\hbar \rightarrow 0^+$:

$$z \sim \sqrt{2\pi\hbar} \frac{e^{-\frac{S[X_0]}{\hbar}}}{\sqrt{\frac{\partial^2 S}{\partial X^2}(X_0)}} (1 + A\hbar + B\hbar^2 + \dots)$$

where S has an isolated minimum at $X_0 \in \mathbb{R}$, where isolated $\Rightarrow \frac{\partial^2 S}{\partial X^2}(X_0) \neq 0$. We find (in AQFT) that the term

$$\frac{e^{-\frac{S[X_0]}{\hbar}}}{\sqrt{\frac{\partial^2 S}{\partial X^2}(X_0)}}$$

corresponds to *tree level diagrams*, the $\mathcal{O}(\hbar)$ corresponds to one-loop corrections, etc.

This is at *most* an asymptotic series. If it were to converge it would have to converge in a disk, but if $\hbar < 0$ the integral obviously diverges.

3 SUSY in d=0 dimensions

3.1 Super vector spaces and Grassman variables

Let us start with a slightly formal introduction to the algebra that will underpin supersymmetry.

Definition 3.1. A \mathbb{Z}_2 -**graded vector space** (or **super vector space**) is a vector space (over $\mathbb{F} = \mathbb{C}/\mathbb{R}$) $V = V_0 \oplus V_1$ with a parity operation $|\cdot|$ s.t. the **parity** of a vector is

$$|v| = \begin{cases} 0 & v \in V_0 \\ 1 & v \in V_1 \end{cases}$$

Vectors that lie purely in V_0 or V_1 are called **homogeneous**. Elements in V_0/V_1 are called **even/odd** or **bosonic/fermionic**. If $\dim V_0 = p$, $\dim V_1 = q$, we write $\dim V = p|q$.

Example 3.2. The basic example is $\mathbb{R}^{p|q}$.

Definition 3.3. Given a super vector space V we have the **parity reversion** ΠV also a super vector space with

$$\begin{aligned} (\Pi V)_0 &= V_1 \\ (\Pi V)_1 &= V_0 \end{aligned}$$

Definition 3.4. The **dual of a super vector space** V is the super vector space $V^* = \text{End}(V, \mathbb{F})$ with

$$\begin{aligned} (V^*)_0 &= (V_1)^0 \\ (V^*)_1 &= (V_0)^0 \end{aligned}$$

where W^0 is the annihilator of a subspace W .

Definition 3.5. The **direct sum of two super vector spaces** V, W is $V \oplus W$ with

$$\begin{aligned} (V \oplus W)_0 &= V_0 \oplus W_0 \\ (V \oplus W)_1 &= V_1 \oplus W_1 \end{aligned}$$

The **direct product of two super vector spaces** is $V \otimes W$ with

$$\begin{aligned} (V \otimes W)_0 &= (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) \\ (V \otimes W)_1 &= (V_0 \otimes W_1) \oplus (V_1 \otimes W_0) \end{aligned}$$

Definition 3.6. Given V a super vector space and $M \in \text{End}(V)$ decompose M as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then define the **supertrace** of M by $\text{sTr}(M) = \text{Tr}(A) - \text{Tr}(D)$.

Proposition 3.7. The supertrace satisfies

$$\bullet \text{sTr}(MN) = (-1)^{|M||N|} \text{sTr}(NM)$$

- $\text{sTr}(NMN^{-1}) = \text{sTr}(M)$, i.e. it is basis independent
- $\text{sTr}(M^T) = \text{sTr}(M)$

Definition 3.8. Given V a super vector space and $M \in GL(V)$ define the **superdeterminant** of M by

$$\begin{aligned}\delta [\log \text{sdet}(M)] &= \text{sTr}(M^{-1} \delta M) \\ \text{sdet}(I) &= 1\end{aligned}$$

where δM is an arbitrary variation of M .

We are going to work out a general formula for the superdet, but we will need a few lemmas first

Lemma 3.9. $\text{sdet}(MN) = \text{sdet}(M) \text{sdet}(N)$.

Proof. We have

$$\begin{aligned}\delta [\log \text{sdet}(MN)] &= \text{sTr}((MN)^{-1} \delta(MN)) \\ &= \text{sTr}(N^{-1} M^{-1} [(\delta M)N + M(\delta N)]) \\ &= \text{sTr}(M^{-1} \delta M) + \text{sTr}(N^{-1} \delta N) \\ &= \delta [\log \text{sdet}(M)] + \delta [\log \text{sdet}(N)] \\ &= \delta [\log(\text{sdet}(M) \text{sdet}(N))]\end{aligned}$$

□

Proposition 3.10. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where D is invertible. Then

$$\text{sdet}(M) = \frac{\det(A - BD^{-1}C)}{\det(D)}$$

Proof. We first consider the case where $B = 0 = C$. Then

$$\begin{aligned}M^{-1} &= \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \\ \delta M &= \begin{pmatrix} \delta A & 0 \\ 0 & \delta D \end{pmatrix}\end{aligned}$$

(Note that the assumption that M, D are invertible forces A invertible.) Hence We can see

$$\delta [\log \text{sdet}(M)] = \text{Tr}(A^{-1} \delta A) - \text{Tr}(D^{-1} \delta D) = \delta \log \det A - \delta \log \det D = \delta \log \frac{\det A}{\det D}$$

Hence evaluating when $A = I = D$ sets the constant to 1 and

$$\text{sdet } M = \frac{\det A}{\det D}$$

Now consider the case where $A = I = D$, $C = 0$. Then

$$M^{-1} = \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}$$

$$\delta M = \begin{pmatrix} 0 & \delta B \\ 0 & 0 \end{pmatrix}$$

and

$$\delta \log \text{sdet } M = 0$$

so evaluating when $B = 0$ gives

$$\text{sdet } M = 1$$

Finally, writing

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}$$

gives the answer. □

Definition 3.11. *Given two vector spaces A, B and an exchange operation $s : A \otimes B \rightarrow B \otimes A$ the **symmetric and antisymmetric powers** are respectively defined as*

$$A \odot B = A \otimes B + s(A \otimes B)$$

$$A \wedge B = A \otimes B - s(A \otimes B)$$

Example 3.12. *For standard vector spaces with no grading we typically take*

$$s : A \otimes B \rightarrow B \otimes A$$

$$a \otimes b \mapsto b \otimes a$$

This recovers the standard definitions of the symmetric and wedge product of vectors.

Definition 3.13. *For super vector spaces V, W the exchange operation is defined on homogeneous vectors as*

$$s(v \otimes w) = (-1)^{|v||w|}(w \otimes v)$$

Remark. *A consequence of the above definition is that 'swapping' fermions introduces a minus sign. Hence*

$$\text{Sym}^k V_1 \cong \wedge^k V_1$$

Definition 3.14. A **superalgebra** is a super vector space equipped \mathcal{A} with a bilinear multiplication map s.t. for $a, b, \in \mathcal{A}$

$$|a \cdot b| = |a| + |b|$$

Definition 3.15. A superalgebra is **supercommutative** if $\forall a, b \in \mathcal{A}, ab = (-1)^{|a||b|}ba$.

Example 3.16. We can make $\mathbb{R}^{p|q}$ into a superalgebra by calling $x^i \in \mathbb{R}^{p|0}$ the bosonic variables, $\psi^a \in \mathbb{R}^{0|q}$ the fermionic variables, and imposing

$$\begin{aligned} [x^i, x^j] &= 0 \\ [\psi^a, x^j] &= 0 \\ \{\psi^a, \psi^b\} &= 0 \end{aligned}$$

In particular note this implies $(\psi^a)^2 = 0$. We call the ψ **Grassman variables**.

Example 3.17. Note $\{\psi^\alpha(\mathbf{x}), \psi^\beta(\mathbf{y})\} = 0$ for the Dirac field in QFT. Further, the exterior product on differential forms is antisymmetric: $dx^a \wedge dx^b = -dx^b \wedge dx^a$

Definition 3.18. A **Lie superalgebra** is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying:

- the graded antisymmetry rule, i.e. $\forall X, Y \in \mathfrak{g}$

$$[X, Y] = -(-1)^{|X||Y|} [Y, X]$$

- the graded Jacobi identity, i.e. $\forall X, Y, Z \in \mathfrak{g}$,

$$(-1)^{|X||Z|} [X, [Y, Z]] + (-1)^{|Y||X|} [Y, [Z, X]] + (-1)^{|Z||Y|} [Z, [X, Y]] = 0$$

Proposition 3.19. Let $V = V_0 \oplus V_1$ be a superalgebra. The space of V -valued vector fields with Lie bracket

$$[X, Y](f) = X(Y(f)) - (-1)^{|X||Y|} Y(X(f))$$

is a Lie superalgebra.

Proof. The bracket trivially satisfies the graded antisymmetry rule, so we just need to show the graded Jacobi rule. This is an exercise in patience. Note we have $|[Y, Z]| = |Y| + |Z|$. Hence

$$\begin{aligned} [X, [Y, Z]]f &= X([Y, Z]f) - (-1)^{|X|(|Y|+|Z|)} [Y, Z](Xf) \\ &= X \left[Y(Zf) - (-1)^{|Y||Z|} Z(Yf) \right] \\ &\quad - (-1)^{|X|(|Y|+|Z|)} \left[Y(Z(Xf)) - (-1)^{|Y||Z|} Z(Y(Xf)) \right] \\ \Rightarrow (-1)^{|X||Z|} [X, [Y, Z]]f &= (-1)^{|X||Z|} X(Y(Zf)) - (-1)^{|Z|(|Y|+|X|)} X(Z(Yf)) \\ &\quad - (-1)^{|X||Y|} Y(Z(Xf)) + (-1)^{|Y|(|X|+|Z|)} Z(Y(Xf)) \end{aligned}$$

It can then be clearly seen that by cycling all terms will cancel. □

If we take a function of the Grassmann variables, it has an expansion that must eventually terminate:

$$F(\psi) = f + e_a \psi^a + \phi_{ab} \psi^a \psi^b + \dots + g \psi^1 \psi^2 \dots \psi^n$$

Once we have all n fermions, any other term will give zero contribution. (Note $\phi_{ab} = -\phi_{ba}$). If $F(\psi)$ is *bosonic* (i.e. commuting) then f, ϕ_{ab}, \dots (coefficients of even powers) must also be bosonic, whereas e_a, \dots (coefficients of odd powers) must be fermionic.

3.2 Differentiation and integration

Definition 3.20. A **derivation** of a commutative superalgebra \mathcal{A} is a linear map $D : \mathcal{A} \rightarrow \mathcal{A}$ satisfying

$$D(ab) = (Da)b + (-1)^{|a||D|} a(Db)$$

where $|D|$ is defined the **degree** of the derivation.

Let us develop calculus on $\mathbb{R}^{p|q}$, our prototypical commutative superalgebra.

Definition 3.21. We define *fermionic derivatives* by:

$$\frac{\partial}{\partial \psi^a} (\psi^b \dots) = \delta_a^b (\dots) - \psi^b \frac{\partial}{\partial \psi^a} (\dots) \text{ (Negative Leibniz rule)}$$

This gives us the total set of rules for differentiation on $\mathbb{R}^{p|q}$ as

$$\frac{\partial x^j}{\partial x^i} = \delta_j^i, \quad \frac{\partial \psi^b}{\partial x^i} = 0, \quad \frac{\partial x^j}{\partial \psi^a} = 0, \quad \frac{\partial \psi^b}{\partial \psi^a} = 0, \quad \frac{\partial}{\partial \psi^a} (\psi^b \psi^c) = \delta_a^b \psi^c - \delta_a^c \psi^b$$

We can also define integration simply by defining $\int 1 d\psi$ and $\int \psi d\psi$. We want our integral to be translation invariant, i.e

$$\begin{aligned} \int (\psi + \eta) d\psi &= \int \psi d\psi \\ \Rightarrow \int 1 d\psi &= 0 \end{aligned}$$

We then want to normalise by choosing

$$\int \psi d\psi = 1 \quad (\text{Berezin integration})$$

Note, suppose we have n fermions ψ^1, \dots, ψ^n . with

$$\int \psi^1 \dots \psi^n \underbrace{d\psi^1 \dots d\psi^n}_{d^n \psi} = 1$$

and

$$\int \psi^{a_1} \dots \psi^{a_n} d^n \psi = \epsilon^{a_1 \dots a_n}$$

Now let $\psi'^a = N_b^a \psi^b$ for $N \in GL(n)$. We have

$$\begin{aligned} \int \psi'^a \psi'^b \dots \psi'^d d^n \psi &= N_e^a N_f^b \dots N_g^d \int \psi^e \psi^f \dots \psi^g d^n \psi \\ &= N_e^a N_f^b \dots N_g^d \epsilon^{ef\dots g} = \det N \epsilon^{ab\dots d} \\ &= \det N \int \psi'^a \psi'^b \dots \psi'^d d^n \psi' \end{aligned}$$

Hence we see $d^n \psi' = \frac{1}{\det N} d^n \psi$ (opposite way round to usual).

In QFT, we will often need the Gaussian integral. Suppose ψ^1, ψ^2 are fermions and let $S(\psi^a) = \frac{1}{2} \psi^1 M \psi^2$. Then

$$\int e^{-S(\psi^a)} d\psi^1 d\psi^2 = \int \left(1 - \frac{1}{2} \psi^1 M \psi^2\right) d\psi^1 d\psi^2 = \frac{1}{2} M$$

More generally for $2m$ fermions we action $S(\psi^a) = \frac{1}{2} \psi^a M_{ab} \psi^b$ ($M_{ab} = -M_{ba}$).

$$\begin{aligned} \int e^{-S(\psi)} d^{2m} \psi &= \int \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{2^k} (\psi^a M_{ab} \psi^b)^k d^{2m} \psi \\ &= \frac{(-1)^m}{2^m m!} \int (\psi^a M_{ab} \psi^b)^{2m} \\ &= \frac{(-1)^m}{2^m m!} \epsilon^{a_1 b_1 a_2 b_2 \dots a_m b_m} M_{a_1 b_1} M_{a_2 b_2} \dots M_{a_m b_m} \\ &= \sqrt{\det M} \equiv \text{Pfaff}(M) \end{aligned}$$

(c.f. for bosons we have $\int e^{-\frac{1}{2} x^a M_{ab} x^b} d^{2m} x = \frac{(2\pi)^m}{\sqrt{\det M}}$)

3.3 Supersymmetric Integrals and Localisation

Consider a $d = 0$ theory of one bosonic variable, and 2 fermions ψ^1, ψ^2 . In this case the configuration space is $\mathcal{C} = \mathbb{R}^{1|2}$. Take $S(x, \psi) = V(x) - \psi^1 \psi^2 U(x)$ as our action. Even in $d = 0$, for generic V, U the integral $\int e^{-S(x, \psi^i)} dx d\psi^1 d\psi^2$ is difficult. We can do better using supersymmetry. Suppose we choose a polynomial $W(x)$ and take $S(x, \psi) = \frac{1}{2} (\partial W)^2 - \bar{\psi} \psi \partial^2 W$ (taking $\psi = \psi^1 + i\psi^2, \bar{\psi} = \psi^1 - i\psi^2$).

Lemma 3.22. *This $S(x, \psi, \bar{\psi})$ is invariant under*

$$\begin{aligned} \delta x &= \epsilon \psi - \bar{\epsilon} \bar{\psi} \\ \delta \psi &= \bar{\epsilon} \partial W \\ \delta \bar{\psi} &= \epsilon \partial W \end{aligned}$$

where $\epsilon, \bar{\epsilon}$ are fermionic parameters.

Proof. We check that

$$\delta_\epsilon S = [(\partial W)(\partial^2 W) - \psi \bar{\psi} \partial^3 W] (\epsilon \psi - \bar{\epsilon} \bar{\psi}) - [\bar{\psi} \partial^2 W] \bar{\epsilon} \partial W - \epsilon \partial W [\psi \partial^2 W] = 0$$

□

Remark. Note that in order to have the variation in x be bosonic, we multiply together two fermionic variables, the fermionic ϵ and the fermion fields.

We write $\delta = \epsilon Q + \bar{\epsilon} \bar{Q}$ where Q, \bar{Q} are called *supercharges*

$$\begin{aligned} Qx &= \psi & \bar{Q}x &= -\bar{\psi} \\ Q\psi &= 0 & \bar{Q}\psi &= \partial W \\ Q\bar{\psi} &= \partial W & \bar{Q}\bar{\psi} &= 0 \end{aligned}$$

Or

$$\begin{aligned} Q &= \psi \frac{\partial}{\partial x} + \partial W \frac{\partial}{\partial \bar{\psi}} \\ \bar{Q} &= -\bar{\psi} \frac{\partial}{\partial x} + \partial W \frac{\partial}{\partial \psi} \end{aligned}$$

Proposition 3.23. *These Generators obey*

$$\begin{aligned} \{Q, Q\} &= 2(\partial^2 W) \psi \frac{\partial}{\partial \bar{\psi}} \\ \{\bar{Q}, \bar{Q}\} &= -2(\partial^2 W) \bar{\psi} \frac{\partial}{\partial \psi} \\ \{Q, \bar{Q}\} &= (\partial^2 W) \left(\psi \frac{\partial}{\partial \psi} - \bar{\psi} \frac{\partial}{\partial \bar{\psi}} \right) \end{aligned}$$

Proof. Calculate

$$\begin{aligned} Q^2 x &= Q\psi = 0 \\ Q^2 \psi &= Q0 = 0 \\ Q^2 \bar{\psi} &= Q(\partial W) = (\partial^2 W) \psi \end{aligned}$$

Hence

$$Q^2 = \partial^2 W \psi \frac{\partial}{\partial \bar{\psi}} \Rightarrow \{Q, Q\} = 2(\partial^2 W) \psi \frac{\partial}{\partial \bar{\psi}}$$

The same calculation can be done for \bar{Q}^2 , $Q\bar{Q}$, and $\bar{Q}Q$ to get

$$\begin{aligned} \bar{Q}^2 x &= 0 & Q\bar{Q}x &= -\partial W & \bar{Q}Qx &= \partial W \\ \bar{Q}^2 \psi &= -\partial^2 W \bar{\psi} & Q\bar{Q}\psi &= \partial^2 W \psi & \bar{Q}Q\psi &= 0 \\ \bar{Q}^2 \bar{\psi} &= 0 & Q\bar{Q}\bar{\psi} &= 0 & \bar{Q}Q\bar{\psi} &= -\partial^2 \bar{\psi} W \end{aligned}$$

□

Remark. This is a Lie superalgebra, and is a symmetry algebra of the theory.

Remark. The 'equations of motion' that come from varying S wrt ψ (or $\bar{\psi}$) give $\partial^2 W = 0$. Hence 'on-shell' all anti-commutators are 0.

We will now use this machinery to show that the supersymmetric 'path' integral $\int e^{-S(x,\psi,\bar{\psi})} dx d\psi d\bar{\psi}$ in fact **localises**. Suppose we rescale $W \mapsto \lambda W$ for $\lambda \in \mathbb{R}_{\geq 0}$. Under this transform say $S \mapsto S_\lambda$ and $Q \mapsto Q_\lambda, \bar{Q} \mapsto \bar{Q}_\lambda$.

Proposition 3.24. *Define*

$$I(\lambda) = \frac{1}{\sqrt{2\pi}} \int e^{-S_\lambda(x,\psi,\bar{\psi})} dx d^2\psi$$

Then $\frac{dI}{d\lambda} = 0$

Proof. We have

$$S_\lambda = \frac{\lambda^2}{2}(\partial W)^2 - \lambda\bar{\psi}\psi\partial^2 W \Rightarrow \frac{\partial}{\partial\lambda}S_\lambda = \lambda(\partial W)^2 - \bar{\psi}\psi\partial^2 W$$

Further as $\bar{Q}(S) = 0, \bar{Q}_\lambda(S_\lambda) = 0$. Hence

$$\begin{aligned} \frac{dI}{d\lambda} &= \frac{1}{\sqrt{2\pi}} \int \frac{\partial}{\partial\lambda} e^{-S_\lambda} dx d^2\psi \\ &= -\frac{1}{\sqrt{2\pi}} \int [\lambda(\partial W)^2 - \bar{\psi}\psi\partial^2 W] e^{-S_\lambda} dx d^2\psi \\ &= -\frac{1}{\sqrt{2\pi}} \int \bar{Q}_\lambda(\partial W\psi) e^{-S_\lambda} dx d^2\psi \\ &= -\frac{1}{\sqrt{2\pi}} \int \bar{Q}_\lambda(\partial W\psi e^{-S_\lambda}) dx d^2\psi \end{aligned}$$

Since $\bar{Q}_\lambda = \bar{\psi}\frac{\partial}{\partial x} + (\lambda\partial W)\frac{\partial}{\partial\psi}$, this integral has two contributions, which can be seen to be either a total derivative or zero after Berexin integration. \square

Corollary 3.25. $I(1) = \lim_{\lambda \rightarrow \infty} I(\lambda)$

As $\lambda \rightarrow \infty$, $e^{-\frac{\lambda^2}{2}(\partial W)^2}$ suppresses the integral except near where $\partial W = 0$. The integral thus localises to critical points of $W(x)$.

In general, suppose we have a group G acting freely¹ on our space of fields, and suppose the action and integration measure are G -invariant. In this case decompose \mathcal{C} as $G \times \mathcal{C}/G$, and integrate over G to obtain a factor of $\text{Vol } G$.

Example 3.26. *Consider*

$$\int_{\mathbb{R}^2 \setminus \{0\}} e^{-S(x,y)} dx dy$$

with $G = SO(2)$ and S just a function of $r = \sqrt{x^2 + y^2}$. Then the integral can be written as

$$\int_{\mathbb{R}_{\geq 0}} e^{-S(r)} \cdot 2\pi dr$$

¹A group G acts on a set X freely if for $g \in G$, $(\exists x \in X, g(x) = x) \Rightarrow g = e = id_G$, i.e. all the stabilisers are trivial.

where

$$2\pi = \int_{SO(2)} d\theta = \text{Vol}(SO(2))$$

and r is a coordinate on $\mathbb{R}_{\geq 0} \cong \mathbb{R}^2 / SO(2)$.

If G is a fermionic group, then $\text{Vol } G = 0$ since $0 = \int_G 1 d^{\dim G} \theta$ by Berezin integration, and we would get no contribution. If G acts freely everywhere except at some fixed subset of the configuration space $\mathcal{C}_0 \subset \mathcal{C}$ we can decompose the integral to some neighbourhood of \mathcal{C}_0 , U , and its complement U^c . The integral over U^c is then 0 as it has a fermionic group acting on it freely, and the total integral receives only a contribution from a neighbourhood of \mathcal{C}_0 .

In our case $\delta\psi = \bar{\epsilon}\partial W$, $\delta\bar{\psi} = -\epsilon\partial W$ so fixed points of our SUSY are critical points of $W(x)$. Away from such critical points define

$$\begin{aligned} y &= x - \frac{\bar{\psi}\psi}{\partial W} \\ \chi &= \psi\sqrt{\partial W} \\ \bar{\chi} &= \bar{\psi} \end{aligned}$$

Exercise 3.27. Show $dx d^2\psi = \sqrt{\partial W(y)} dy d^2\chi$

The point is

•

$$\delta y = \epsilon\psi - \bar{\epsilon}\bar{\psi} - \frac{\epsilon\partial W\psi}{\partial W} - \frac{\bar{\psi}\bar{\epsilon}\partial W}{\partial W} = 0$$

•

$$\begin{aligned} S(y, 0, 0) &= \frac{1}{2}(\partial W(y))^2 \\ &= \frac{1}{2}(\partial W(x))^2 - \partial W \partial^2 W \frac{\bar{\psi}\psi}{\partial W} \\ &= S(x, \psi, \bar{\psi}) \end{aligned}$$

Hence

$$\int_{U^c} e^{-S(x, \psi, \bar{\psi})} dx d^2\psi = \int e^{-S(y, 0, 0)} \sqrt{\partial W(y)} dy d^2\chi = 0$$

where U is an open neighbourhood of $\{\partial W = 0\}$ and $U^c = \mathcal{C} \setminus U$, as there are no fermionic variables in the integrand and Berezin integration gives 0. Near any isolated critical point x_* of second order write

$$W(x) = W(x_*) + \frac{c_*}{2}(x - x_*)^2 + \dots$$

where $c_* = \partial^2 W(x_*) \neq 0$. The higher order terms will be irrelevant, so we only need consider contribution from

$$S^{(2)}(x, \psi, \bar{\psi}) = \frac{c_*^2}{2}(x - x_*)^2 - \bar{\psi}\psi c_*$$

Hence

$$\begin{aligned} I &= \frac{1}{\sqrt{2\pi}} \int e^{-S(x, \psi, \bar{\psi})} dx d\psi^2 \\ &= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{c_*^2}{2}(x-x_*)^2} (-1 + \bar{\psi}\psi c_*) dx d^2\psi \\ &= \frac{c_*}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{c_*^2}{2}(x-x_*)^2} dx = \frac{c_*}{|c_*|} = \pm 1 \end{aligned}$$

If W has several critical points the

$$I = \sum_{x_*: \partial W=0} \frac{c_*}{|c_*|}$$

Hence

- $I = 0$ if W is an odd degree polynomial
- $I = -1$ if W has even degree and $\lim_{|x| \rightarrow \infty} W(x) = -\infty$
- $I = 1$ if W has even degree and $\lim_{|x| \rightarrow \infty} W(x) = \infty$

3.4 d=0 Landau Ginzburg Theory

Now take $z \in \mathbb{C}$ and two fermions ψ_1, ψ_2 . Choose a holomorphic $W(z)$

$$S(z, \psi) = |\partial W|^2 + \partial^2 W \psi_2 \psi_2 - \overline{\partial^2 W \psi_1 \psi_2}$$

Proposition 3.28. *The action is invariant under*

$$\begin{aligned} \delta z &= \epsilon_1 \psi_1 + \epsilon_2 \psi_2 \\ \delta \psi_1 &= \epsilon_2 \overline{\partial W} \\ \delta \psi_2 &= -\epsilon_1 \overline{\partial W} \\ \bar{\delta} \bar{z} &= \bar{\epsilon}_1 \bar{\psi}_1 + \bar{\epsilon}_2 \bar{\psi}_2 \\ \bar{\delta} \bar{\psi}_1 &= \bar{\epsilon}_2 \partial W \\ \bar{\delta} \bar{\psi}_2 &= -\bar{\epsilon}_1 \partial W \end{aligned}$$

and

$$\begin{aligned} \bar{\delta} z &= 0 = \bar{\delta} \psi_i \\ \delta \bar{z} &= 0 = \delta \bar{\psi}_i \end{aligned}$$

or equivalently with supercharges

$$\begin{aligned} Q_1 &= \psi_1 \frac{\partial}{\partial z} + \overline{\partial} W \frac{\partial}{\partial \psi_2} & \overline{Q}_1 &= \overline{\psi}_1 \frac{\partial}{\partial \bar{z}} + \partial W \frac{\partial}{\partial \overline{\psi}_2} \\ Q_2 &= \psi_2 \frac{\partial}{\partial z} - \overline{\partial} W \frac{\partial}{\partial \psi_1} & \overline{Q}_2 &= \overline{\psi}_2 \frac{\partial}{\partial \bar{z}} - \partial W \frac{\partial}{\partial \overline{\psi}_1} \end{aligned}$$

so $\delta = \epsilon_1 Q_1 + \bar{\epsilon}_1 \overline{Q}_1 + \dots$.

Proposition 3.29. $\{Q_i, \overline{Q}_j\} = 0$ and $\{Q_i, Q_j\} = 0 = \{\overline{Q}_i, \overline{Q}_j\}$ only holds "on shell", i.e. $\partial^2 W = 0 = \overline{\partial}^2 W$.

Again by rescaling $W \rightarrow \lambda W$ for $\lambda \in \mathbb{R}_{\geq 0}$, localise $I = \int e^{-S} d^2 z d^4 \psi$ to its critical points of W where

$$W(z) = W(z_*) + \frac{\alpha_*}{2} (z - z_*)^2 + \dots$$

$$S^{(2)}(z, \psi_i) = |\alpha_*|^2 |z - z_*|^2 + \alpha_* \psi_1 \psi_2 - \overline{\alpha}_* \overline{\psi}_1 \overline{\psi}_2 + \dots$$

so

$$\begin{aligned} I &= \frac{1}{2\pi} \int e^{-S} d^2 z d^4 \psi \\ &= \sum_{z_*} \frac{1}{2\pi} \int e^{-|\alpha_*(z-z_*)|^2} |\alpha_*|^2 \psi_1 \psi_2 \overline{\psi}_1 \overline{\psi}_2 d^2 z d^4 \psi \\ &= \sum_{z_*} \frac{|\alpha_*|^2}{|\alpha_*|^2} = \sum_{z_*} 1 \end{aligned}$$

counting the critical points. More generally let $f(z)$ be any holomorphic functions, then

$$\langle f(z) \rangle = \int e^{-S} f(z) d^2 z d^4 \psi$$

is still invariant under $\bar{\delta}$ transform, so again localises to critical points of \overline{W} . Then

$$\begin{aligned} \langle f(z) \rangle &= \sum_{z_*} f(z_*) \frac{1}{2\pi} \int e^{-S^{(2)}} d^2 z d^4 \psi \\ &= \sum_{z_*} f(z_*) \end{aligned}$$

the sum of the values at critical points. The key fact was $\bar{\delta} f = 0$. Since $\overline{Q}_i^2 = 0$, one way to construct a \overline{Q}_i -invariant function is to take $f = \overline{Q}_i \Lambda(z, \bar{z}, \psi_i, \overline{\psi}_j)$ for some general Λ . However, if $F = \overline{Q} \Lambda$,

$$\begin{aligned} \langle F \rangle &= \langle \overline{Q} \Lambda \rangle \\ &= \int (\overline{Q} \Lambda) e^{-S} \frac{d^2 z d^4 \psi}{2\pi} \\ &= \int \overline{Q} (\Lambda e^{-S}) \frac{d^2 z d^4 \psi}{2\pi} = 0 \end{aligned}$$

So interesting functions (i.e.those with non-trivial but easily computable correlation functions) are in

$$H_{\overline{Q}} = \ker \overline{Q} / \text{Im } \overline{Q}$$

Definition 3.30. $H_{\overline{Q}} = \ker \overline{Q} / \text{Im } \overline{Q}$ is called the \overline{Q} -cohomology. In the context of supersymmetry it is called the **chiral ring**.

Proposition 3.31. The chiral ring is indeed a ring with addition and multiplication as for operators.

2

Proposition 3.32. Correlation functions of sums and products of \overline{Q} -invariant operators detect only the \overline{Q} -cohomology class of the operators.

Proof. Take \overline{Q} -invariant functions F_i . Then

$$\langle F_1 + \overline{Q}\Lambda + \sum_{i=2}^n F_i \rangle = \langle \sum_{i=1}^n F_i \rangle + \langle \overline{Q}\Lambda \rangle = \langle \sum_i F_i \rangle$$

and

$$\langle (F_1 + \overline{Q}\Lambda) \prod_{i=2}^n F_i \rangle = \langle \prod_{i=1}^n F_i \rangle + \langle \overline{Q}\Lambda \prod_{i=2}^n F_i \rangle = \langle \prod_i F_i \rangle + \langle \overline{Q}(\Lambda \prod_{i=2}^n F_i) \rangle = \langle \prod_i F_i \rangle$$

□

Example 3.33. the transform $\overline{\delta\psi}_i = \overline{\epsilon}_i \partial W$ shows that $\partial W \in \overline{Q}$. Hence if our operator contain δW as a factor, its correlators vanish. As such the chiral ring is

$$\mathcal{R} = \mathbb{C}[z] / \langle \partial W \rangle$$

If $W(z) = \frac{1}{n+1} z^{n+1} - az$, $\partial W = z^n - a$ Then we have non trivial \overline{Q} invariant operators that are polynomials subject to $z^n = a$. Hence the chiral ring is generated by $\{1, z, \dots, z^{n-1}\}$.

3.5 The Duistermaat-Heckmann localisation formula

Take (M, ω) to be a compact $2n$ -dimensional symplectic manifold without boundary, and take coordinates x^a locally on M .

Definition 3.34. Writing $\omega = \omega_{ab} dx^a \wedge dx^b$ the **Liouville measure** on M is

$$\omega^n = \det(\omega_{ab}) dx^1 \wedge \dots \wedge dx^{2n}$$

² $H_{\overline{Q}}$ represents the fact that $\overline{Q}\overline{Q}\Lambda = 0$, so $\text{Im } \overline{Q} \leq \ker \overline{Q}$, and moreover $\langle F + \overline{Q}\Lambda \rangle = \langle F \rangle$ so we extract the relevant information by quotienting out (under addition) this subspace.

Let us assume we have a Hamiltonian flow given by function H , i.e. vector field $X = \{\cdot, H\}$, and lets further assume the vector field generates a $U(1)$ action on M , with non-degenerate isolated fixed points $\{x_*\}$.

Theorem 3.35. *The Fourier transform of the measure localises, i.e.*

$$\int_m \frac{\omega^n}{n!} e^{i\alpha H} = \left(\frac{2\pi}{i\alpha}\right)^n \sum_{\substack{x_* \in M \\ X(x_*)=0}} \frac{e^{i\alpha H(x_*)}}{\prod_i k_i(x_*)}$$

where the k_i are weights of the $U(1)$ action.

Proof. As makes sense for these notes we will use a supersymmetry approach. Take x^a to be bosonic fields, and then $2n$ fermionic fields ψ^a which will transform as tangent vectors to M . (Technically $\mathcal{C} = \Pi TM$, the parity-reversed tangent bundle). A generic function of the fields can be written as

$$F(x, \psi) = f(x) + \rho_{a_1}(x) \psi^{a_1} + \dots + r_{a_1 \dots a_{2n}} \psi^{a_1} \dots \psi^{a_{2n}}$$

As such we can identify $C^\infty(\Pi TM) \cong \Omega^\bullet(M)$.

We now build an action: Let

$$S_0(x, \psi) = -i\alpha [H(x) + \omega_{ab}(x) \psi^a \psi^b]$$

and

$$Q = \psi^a \frac{\partial}{\partial x^a} + X^a(x) \frac{\partial}{\partial \psi^a}$$

which we view under the identification with multiforms as $d + \iota_X$. We can calculate

$$QS_0 = -i\alpha(d + \iota_X)[H + \omega] = -i\alpha(dH + \iota_X \omega) = 0$$

as X is the Hamiltonian vector field corresponding to H . Further

$$Q^2 = d\iota_X + \iota_X d = \mathcal{L}_X$$

and so, restricted to forms constant on the orbits of the flow, $Q^2 = 0$.

Pick now a positive definite metric \mathbf{g} on M that has X as a Killing vector (is this possible?), s.t. wrt the coordinates x^a it has components g_{ab} . Write $g(\psi, X) = g_{ab} \psi^a X^b$. Then as a multiform $g(\psi, X) = \iota_X \mathbf{g}$, and

$$\mathcal{L}_X \iota_X \mathbf{g} = \iota_X \mathcal{L}_X \mathbf{g} + [\mathcal{L}_X, \iota_X] \mathbf{g} = \iota_{[X, X]} \mathbf{g} = 0 \Rightarrow Q^2 g(\psi, X) = 0$$

As such if we deform from S_0 to the action

$$S_\lambda = S_0 + \lambda Q(g(\psi, X))$$

we retain the SUSY, i.e. $Q(S_\lambda) = 0$. The partition function us now

$$Z = \int_{\Pi TM} e^{-S_\lambda(x, \psi)} d^{2n}x d^{2n}\psi$$

Note the measure here is the canonical measure on ΠTM invariant under orientation-preserving diffeomorphisms, as the bosonic and fermionic measures transform oppositely. As we have done previously we find

$$\frac{\partial}{\partial \lambda} \int_{\Pi TM} e^{-S_\lambda(x, \psi)} d^{2n}x d^{2n}\psi = - \int_{\Pi TM} Q \left[g(\psi, X) e^{-S_\lambda(x, \psi)} \right] d^{2n}x d^{2n}\psi = 0$$

i.e the integral is independent of λ . Evaluating at $\lambda = 0$ we get

$$\begin{aligned} Z &= \int_{\Pi TM} e^{-S_0(x, \psi)} d^{2n}x d^{2n}\psi \\ &= \int_{\Pi TM} e^{i\alpha H(x)} \left[1 + i\alpha \omega_{a_1 b_1} \psi^{a_1} \psi^{b_1} + \dots + \frac{(i\alpha)^n}{n!} \omega_{a_1 b_1} \dots \omega_{a_n b_n} \psi^{a_1} \psi^{b_1} \dots \psi^{a_n} \psi^{b_n} \right] d^{2n}x d^{2n}\psi \\ &= \frac{(i\alpha)^n}{n!} \int_M e^{i\alpha H(x)} \epsilon^{a_1 b_1 \dots a_n b_n} \omega_{a_1 b_1} \dots \omega_{a_n b_n} d^{2n}x = \frac{(i\alpha)^n}{n!} \int_M e^{i\alpha H(x)} \omega^n \end{aligned}$$

Now we rewrite

$$\begin{aligned} Q(g(\psi, X)) &= g_{ab} X^a X^b - \psi^a \psi^b \partial_b (g_{ac} X^c) \\ &= g(X, X) - \psi^a \psi^b \partial_b X_a \end{aligned}$$

where $X_a = g_{ac} X^c$. As the metric is positive definite, $g_{ab} X^a X^b \geq 0$, with equality only where $X = 0$. Hence taking Z in the limit $\lambda \rightarrow \infty$ we get contributions only where $X = 0$. Hence taking the leading order term of the asymptotic expansion gained from the method of steepest descent.

$$Z = \frac{(2\pi)^n}{n!} \sum_{\substack{x_* \in M \\ X(x_*)=0}} e^{i\alpha H(x_*)} \frac{\epsilon^{a_1 b_1 \dots a_n b_n} (\partial_{a_1} X_{b_1}) \dots (\partial_{a_n} X_{b_n})}{\sqrt{\det(\partial_a \partial_b g(X, X))}} \Big|_{x_*}$$

We may simplify this term further: around each x_* choose Darboux coordinates s.t. we can write

$$X = \sum_{i=1}^n k_i \left(q^i \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial q^i} \right)$$

An invariant choice of metric is then $g = dq^2 + dp^2$. If we did this in the case $n = 1$ we would then find

$$\begin{aligned} \epsilon^{ab} \partial_a (g_{bc} X^c) &= 2k \\ \sqrt{\det(\partial_a \partial_b g(X, X))} &= 2k^2 \end{aligned}$$

so in the case of general n we get

$$\frac{\epsilon^{a_1 b_1 \dots a_n b_n} (\partial_{a_1} X_{b_1}) \dots (\partial_{a_n} X_{b_n})}{\sqrt{\det(\partial_a \partial_b g(X, X))}} \Big|_{x_*} = \prod_i \frac{1}{k_i(x_*)}$$

The result follows. □

4 Supersymmetric Quantum Mechanics

There are 2 perspectives of quantum mechanics: canonical frameworks and path integral framework.

4.1 Path integrals in QM

Path integrals for bosons

Consider a free particle travelling on \mathbb{R} , Time evolution operator e^{-iHt} becomes $e^{-H\tau}$ under Wick rotation $t \rightarrow -i\tau$. If our particle is located at y_0 at $\tau = 0$ the amplitude to find the particle at y_1 at $\tau = \beta$ is

$$\langle y_1 | e^{-\beta H} | y_0 \rangle = K_\beta(y_1, y_0) = \frac{1}{\sqrt{2\pi\beta}} \exp\left(-\frac{(y_1 - y_0)^2}{2\beta}\right)$$

We break this evolution into steps of length $\Delta\tau = \frac{\beta}{N}$

$$\begin{aligned} \langle y_1 | e^{-\beta H} | y_0 \rangle &= \int \langle y_1 | e^{-\Delta\tau H} | x_{N-1} \rangle \langle x_{N-1} | e^{-\Delta\tau H} | x_{N-2} \rangle \dots \langle x_1 | e^{-\Delta\tau H} | y_0 \rangle d^{N-1}x \\ &= \int K_{\Delta\tau}(y_1, x_{N-1}) \dots K_{\Delta\tau}(x_1, y_0) d^{N-1}x \\ &= \frac{1}{\sqrt{2\pi\Delta\tau}} \int \exp\left[-\sum_{i=0}^N \frac{\Delta\tau}{2} \left(\frac{x_{i+1} - x_i}{\Delta\tau}\right)^2\right] \prod_{i=1}^{N-1} \frac{dx_i}{\sqrt{2\pi\Delta\tau}} \end{aligned}$$

Taking the limit as $N \rightarrow \infty$, $\Delta\tau \rightarrow 0$

$$\exp\left[-\sum_{i=0}^N \frac{\Delta\tau}{2} \left(\frac{x_{i+1} - x_i}{\Delta\tau}\right)^2\right] \rightarrow \exp\left(-\int_0^\beta \frac{1}{2} \dot{x}^2 d\tau\right)$$

Then we heuristically obtain the *path integral representation*

$$\langle y_1 | e^{-\beta H} | y_0 \rangle = \int_{\mathcal{C}_\beta[y_1, y_0]} e^{-\int_0^\beta \frac{1}{2} \dot{x}^2 d\tau} \mathcal{D}x$$

where $\mathcal{C}_\beta[y_1, y_0]$ is the space of all continuous maps $x : [0, \beta] \rightarrow \mathbb{R}$ such that $x(0) = y_0$, $x(\beta) = y_1$. Note we can show the derivation also works for

$$H = \frac{p^2}{2} + V(x)$$

in which case

$$S = \int \frac{1}{2} \dot{x}^2 + V(x) d\tau$$

The **partition function** $Z(\beta)$ is closely related to the heat kernel

$$\begin{aligned} Z(\beta) &= \text{Tr}_{\mathcal{H}} (e^{-\beta H}) = \int_{\mathbb{R}} \langle y | e^{-\beta H} | y \rangle dy \\ &= \int \left[\int_{\mathcal{C}_{\beta}[y,y]} e^{-S[x]} \mathcal{D}x \right] dy \\ &= \int_{\mathcal{C}_{S^1}} e^{-S[x]} \mathcal{D}x \end{aligned}$$

Example 4.1. Here we go through in detail the example of a free quantum particle on a circle **DO THIS**.

Path integrals for fermions

We now want to construct the idea of a path integral for fermions. To start, we define fermionic coherent states

$$\begin{aligned} |\eta\rangle &= e^{\hat{\psi}\eta} |0\rangle \\ \langle\bar{\eta}| &= \langle 0| e^{\hat{\psi}\bar{\eta}} \end{aligned}$$

Proposition 4.2. These states satisfy

- $\hat{\psi} |\eta\rangle = \eta |\eta\rangle$ and $\langle\bar{\eta}| \hat{\bar{\psi}} = \langle\bar{\eta}| \bar{\eta}$
- $1_{\mathcal{H}} = \int e^{-\bar{\eta}\eta} |\eta\rangle \langle\bar{\eta}| d^2\eta$
- $\text{Tr}(\hat{A}) = \int \langle -\bar{\eta} | \hat{A} | \eta \rangle e^{-\bar{\eta}\eta} d^2\eta$
- $S \text{Tr}(\hat{A}) = \text{Tr}_{\mathcal{H}} \left((-1)^F \hat{A} \right) = \int \langle \bar{\eta} | \hat{A} | \eta \rangle e^{-\bar{\eta}\eta} d^2\eta$

Using these and following the same procedure as for bosons we have

$$\langle \bar{\chi}' | e^{-\beta H} | \chi \rangle = \int \langle \bar{\chi}' | e^{-\Delta\tau H} | \eta_{N-1} \rangle \dots \langle \bar{\eta}_1 | e^{-\Delta\tau H} | \chi \rangle \prod_{i=1}^{N-1} e^{-\bar{\eta}_i \eta_i} d^2\eta_i$$

Let's order the Hamiltonian so that all $\hat{\psi}$'s appear to the RHS of all $\hat{\bar{\psi}}$'s. Then

$$\begin{aligned} \lim_{\Delta\tau \rightarrow 0} \langle \bar{\eta}_{i+1} | e^{-\Delta\tau H} | \eta_i \rangle &= \langle \bar{\eta}_{i+1} | 1 - \Delta\tau H(\hat{\bar{\psi}}, \hat{\psi}) | \eta_i \rangle \\ &= \langle \bar{\eta}_{i+1} | 1 - \Delta\tau H(\bar{\eta}_{i+1}, \eta_i) | \eta_i \rangle \end{aligned}$$

so

$$\langle \bar{\eta}_{i+1} | e^{-\Delta\tau H(\hat{\bar{\psi}}, \hat{\psi})} | \eta_i \rangle = e^{-\Delta\tau H(\bar{\eta}_{i+1}, \eta_i)} e^{\bar{\eta}_{i+1} \eta_i}$$

using this we have

$$\begin{aligned}
\langle \bar{\chi}' | e^{-\beta H} | \chi \rangle &= \lim_{N \rightarrow \infty} \int \exp \left(\sum_{k=1}^N \bar{\eta}_k \eta_{k-1} - \Delta \tau H(\bar{\eta}_k, \eta_{k-1}) \right) \prod_{k=1}^{N-1} e^{-\bar{\eta}_k \eta_k} d^2 \eta_k \\
&= \lim_{N \rightarrow \infty} \int \exp \left(- \sum_{k=1}^N \left[\bar{\eta}_k \frac{\eta_k - \eta_{k-1}}{\Delta \tau} + H(\bar{\eta}_k, \eta_{k-1}) \right] \Delta \tau \right) e^{\bar{\eta}_N \eta_N} \prod_{k=1}^{N-1} d^2 \eta_k \\
&= \int e^{-S[\bar{\eta}, \eta]} e^{\bar{\eta}(\beta) \eta(\beta)} \mathcal{D} \eta \mathcal{D} \bar{\eta}
\end{aligned}$$

where $\eta(0) = \chi$, $\eta(\beta) = \chi'$, and

$$S[\bar{\eta}, \eta] = \int_0^\beta \bar{\eta} \dot{\eta} + H(\bar{\eta}, \eta) d\tau$$

so

$$\begin{aligned}
Z(\beta) &= \text{Tr}_{\mathcal{H}}(e^{-\beta H}) = \int \langle -\bar{\chi}' | e^{-\beta H} | \chi \rangle e^{-\bar{\chi} \chi} d^2 \chi \\
&= \underbrace{\int \exp(-S[\bar{\psi}, \psi]) \mathcal{D} \psi \mathcal{D} \bar{\psi}}_{\text{antiperiodic boundary conditions}}
\end{aligned}$$

and

$$\begin{aligned}
S \text{Tr}(e^{-\beta H}) &= \text{Tr}_{\mathcal{H}}((-1)^F e^{-\beta H}) = \int \int \langle \bar{\chi}' | e^{-\beta H} | \chi \rangle e^{-\bar{\chi} \chi} d^2 \chi \\
&= \underbrace{\int \exp(-S[\bar{\psi}, \psi]) \mathcal{D} \psi \mathcal{D} \bar{\psi}}_{\text{periodic boundary conditions}}
\end{aligned}$$

4.2 SQM with a potential

Take a worldline theory of a single bosonic field $x(t)$ and a single \mathbb{C} -fermion $\psi(t)$, and choose the action

$$S[x, \psi, \bar{\psi}] = \int \left[\frac{1}{2} \dot{x}^2 + \frac{i}{2} (\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi) - \frac{1}{2} (\partial h)^2 - \bar{\psi} \psi \partial^2 h \right] dt$$

where $h = h(x(t))$ plays the role of a potential.

Proposition 4.3. *S is invariant under*

$$\begin{aligned}
\delta x &= \epsilon \bar{\psi} - \bar{\epsilon} \psi \\
\delta \psi &= \epsilon (i\dot{x} + \partial h) \\
\delta \bar{\psi} &= \bar{\epsilon} (-i\dot{x} + \partial h)
\end{aligned}$$

Proof. Explicitly calculating we get

$$\begin{aligned}
\delta S &= \int \left[\dot{x} \left(\epsilon \dot{\bar{\psi}} - \bar{\epsilon} \dot{\psi} \right) + \frac{i}{2} \left\{ \bar{\epsilon}(-i\dot{x} + \partial h) \dot{\psi} + \bar{\psi} \epsilon (i\ddot{x} + \dot{x} \partial^2 h) - \bar{\epsilon}(-i\ddot{x} + \dot{x} \partial^2 h) \psi - \dot{\bar{\psi}} \epsilon (i\dot{x} + \partial h) \right\} \right. \\
&\quad \left. - (\partial h)(\partial^2 h)(\epsilon \bar{\psi} - \bar{\epsilon} \psi) - \bar{\epsilon}(-i\dot{x} + \partial h) \psi \partial^2 h - \bar{\psi} \epsilon (i\dot{x} + \partial h) \partial^2 h - \bar{\psi} \psi (\partial^3 h)(\epsilon \bar{\psi} - \bar{\epsilon} \psi) \right] dt \\
&= \int \left[\epsilon \left\{ \dot{x} \dot{\bar{\psi}} - \frac{i}{2} (i\ddot{x} + \dot{x} \partial^2 h) \bar{\psi} + \frac{i}{2} (i\dot{x} + \partial h) \dot{\bar{\psi}} - (\partial h)(\partial^2 h) \bar{\psi} + (i\dot{x} + \partial h)(\partial^2 h) \bar{\psi} \right\} \right. \\
&\quad \left. + \bar{\epsilon} \left\{ -\dot{x} \dot{\psi} + \frac{i}{2} (-i\dot{x} + \partial h) \dot{\psi} - \frac{i}{2} (-i\ddot{x} + \dot{x} \partial^2 h) \psi + (\partial h)(\partial^2 h) \psi - (-i\dot{x} + \partial h)(\partial^2 h) \psi \right\} \right] dt \\
&= \frac{1}{2} \int \left[\epsilon \frac{d}{dt} \{ (\dot{x} + i\partial h) \bar{\psi} \} + \bar{\epsilon} \frac{d}{dt} \{ (-\dot{x} + i\partial h) \psi \} \right] dt \\
&= \frac{1}{2} \int \left\{ \frac{d}{dt} [\epsilon (\dot{x} + i\partial h) \bar{\psi} + \bar{\epsilon} (-\dot{x} + i\partial h) \psi] - [\dot{\epsilon} (\dot{x} + i\partial h) \bar{\psi} + \dot{\bar{\epsilon}} (-\dot{x} + i\partial h) \psi] \right\} dt
\end{aligned}$$

As $\epsilon, \bar{\epsilon}$ are constant this is a total derivative and so assuming field decay in time (neccessary for a finite action) $\delta S = 0$. \square

Remark. This is a natural generalisation of the first model we considered

By the Noether procedure, promoting $\epsilon \rightarrow \epsilon(t)$, we now find that

$$\begin{aligned}
\delta S &= \int -\frac{1}{2} [\dot{\epsilon} (\dot{x} + i\partial h) \bar{\psi} + \dot{\bar{\epsilon}} (-\dot{x} + i\partial h) \psi] + \frac{i}{2} [\bar{\psi} \dot{\epsilon} (i\dot{x} + \partial h) - \dot{\bar{\epsilon}} (-i\dot{x} + \partial h) \psi] dt \\
&= \int \dot{\epsilon} \bar{\psi} \left[-\frac{1}{2} (\dot{x} + i\partial h) + \dot{x} - \frac{i}{2} (i\dot{x} + \partial h) \right] + \dot{\bar{\epsilon}} \psi \left[-\frac{1}{2} (-\dot{x} + i\partial h) - \dot{x} - \frac{i}{2} (-i\dot{x} + \partial h) \right] dt \\
&= -i \int (\dot{\epsilon} Q + \dot{\bar{\epsilon}} \bar{Q}) dt
\end{aligned}$$

where the charges are

$$\begin{aligned}
Q &= \bar{\psi} (i\dot{x} + \partial h) \\
\bar{Q} &= \psi (-i\dot{x} + \partial h)
\end{aligned}$$

Lemma 4.4. On shell $\{Q, \bar{Q}\} = -2i\partial_t$.

Proof. The charges obey

$$\begin{aligned}
\{Q, \bar{Q}\} x &= (Q\bar{Q} + \bar{Q}Q)x \\
&= -Q\psi + \bar{Q}\bar{\psi} \\
&= -(i\dot{x} + \partial h) + (-i\dot{x} + \partial h) \\
&= -2i\dot{x} \\
\{Q, \bar{Q}\} \psi &= \bar{Q}(i\dot{x} + \partial h) \\
&= -i\dot{\psi} - \psi \partial^2 h \\
&\approx -2i\dot{\psi} \quad \text{as on e.o.m } \dot{\psi} = -i\psi \partial^2 h \\
\{Q, \bar{Q}\} \bar{\psi} &\approx -2i\dot{\bar{\psi}}
\end{aligned}$$

(the symbol \approx here means "equals on-shell"). Then, up to fermionic e.o.m, the anticommutator of the supercharges generates time translation as $\{Q, \bar{Q}\} \approx -2i\partial_t$. \square

To canonically quantise, have

$$p = \frac{\delta L}{\delta \dot{x}} = \dot{x}$$

$$\pi = \frac{\delta L}{\delta \dot{\psi}} = i\bar{\psi}$$

so

$$H = p\dot{x} + \pi\dot{\psi} - L = \frac{1}{2}p^2 + (\partial h)^2 + \frac{1}{2}\partial^2 h(\bar{\psi}\psi - \psi\bar{\psi})$$

Remark. Note a choice of ordering has been made here, that is inequivalent quantum mechanically due to the choice $\pi = i\bar{\psi}$.

From quantisation, we have

$$[\hat{x}, \hat{p}] = i$$

$$\{\hat{\psi}, \hat{\pi}\} = i$$

(note we have set $\hbar = 1$). For x as usual take the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$ in which case $\hat{x}\Psi(x) = x\Psi(x)$ and $\hat{p}\Psi(x) = -i\frac{\partial\Psi}{\partial x}$. The relations $\{\hat{\psi}, \hat{\bar{\psi}}\} = 1$ are reminiscent of $[a, a^\dagger] = 1$ in a SHO, where the Hamiltonian is determined by the number operator $N = a^\dagger a$. As such let's make the following definition:

Definition 4.5. The *Fermionic number operator* is $\hat{F} = \hat{\bar{\psi}}\hat{\psi}$

Proposition 4.6. \hat{F} obeys the relations

$$[\hat{F}, \hat{\psi}] = -\hat{\psi}$$

$$[\hat{F}, \hat{\bar{\psi}}] = \hat{\bar{\psi}}$$

We also let the vacuum of the fermionic system be $|0\rangle$ defined by $\hat{\psi}|0\rangle = 0$. The 1st excited state is $\hat{\bar{\psi}}|0\rangle = |1\rangle$, but since $\{\hat{\bar{\psi}}, \hat{\bar{\psi}}\} = 0$, there are no other excited states. Hence

$$\mathcal{H} = L^2(\mathbb{R}, dx)|0\rangle \oplus L^2(\mathbb{R}, dx)|1\rangle$$

$$= \mathcal{H}_B \oplus \mathcal{H}_F$$

In the quantum theory, we have

$$\hat{Q} = \hat{\bar{\psi}}(i\hat{p} + \partial h)$$

$$\hat{\bar{Q}} = \hat{\psi}(-i\hat{p} + \partial h)$$

$$\hat{H} = \frac{1}{2}\hat{p}^2 + (\partial h)^2 + \frac{1}{2}\partial^2 h(\hat{\bar{\psi}}\hat{\psi} - \hat{\psi}\hat{\bar{\psi}})$$

We have immediately, (dropping hats from now on)

Lemma 4.7. *Our sQM has the algebra*

$$\begin{aligned}\{Q, Q\} &= 0 = \{\bar{Q}, \bar{Q}\} \\ \{Q, \bar{Q}\} &= 2H\end{aligned}$$

Proof. This first part follows immediately from the fact that $\{\psi, \psi\} = 0 = \{\bar{\psi}, \bar{\psi}\}$. The second part is an **exercise** \square

Corollary 4.8. $[H, Q] = 0 = [H, \bar{Q}]$.

Remark. *This is why we chose the particular ordering in H .*

4.3 Supersymmetric ground states and the Witten index

We start with the following result:

Proposition 4.9. $\langle \Psi | H | \Psi \rangle \geq 0$ with equality iff $Q | \Psi \rangle = 0 = \bar{Q} | \Psi \rangle$

Hence a ground state of zero energy in SQM must be supersymmetrically invariant. If we represent

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then the conditions for a ground state become

$$\begin{aligned}Q | \Psi \rangle &= \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + \partial h & 0 \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0 \\ \bar{Q} | \Psi \rangle &= \begin{pmatrix} 0 & -\frac{d}{dx} + \partial h \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0\end{aligned}$$

which gives, for our ground state $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} Ae^{-h(x)} \\ Be^{h(x)} \end{pmatrix}$$

We want a normalisable state, so

- $\lim_{|x| \rightarrow \infty} h(x) = \infty \Rightarrow B = 0 \Rightarrow |\Psi\rangle = \begin{pmatrix} Ae^{-h(x)} \\ 0 \end{pmatrix}$
- $\lim_{|x| \rightarrow \infty} h(x) = -\infty \Rightarrow A = 0 \Rightarrow |\Psi\rangle = \begin{pmatrix} 0 \\ Be^{h(x)} \end{pmatrix}$
- $\lim_{x \rightarrow \infty} h(x) = \pm\infty$ and $\lim_{x \rightarrow -\infty} h(x) = \mp\infty \Rightarrow A = 0 = B \Rightarrow |\Psi\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

In the third case there is no zero energy state, so the ground state will have higher energy and SUSY is **spontaneously broken**.

Excited states $E > 0$ come in pairs. If $\mathcal{H} = \bigoplus \mathcal{H}_n$ where

$$\forall |\Psi_n\rangle \in \mathcal{H}_n \quad H |\Psi_n\rangle = E_n |\Psi_n\rangle$$

then we can further split each \mathcal{H}_n into bosonic and fermionic states

$$\mathcal{H}_n = \mathcal{H}_{B,n} \oplus \mathcal{H}_{F,n}$$

Proposition 4.10. *We have*

$$\mathcal{H}_{F,n} \cong \mathcal{H}_{B,n} \text{ for } n > 0$$

Proof. We have in particular

$$\begin{aligned} Q : \mathcal{H}_{F,n} &\rightarrow \mathcal{H}_{B,n}, & Q(\mathcal{H}_{B,n}) &= 0 \\ \bar{Q} : \mathcal{H}_{B,n} &\rightarrow \mathcal{H}_{F,n}, & \bar{Q}(\mathcal{H}_{F,n}) &= 0 \end{aligned}$$

Thus, given $|b\rangle \in \mathcal{H}_{B,n}$ we have

$$2E_n |b\rangle = (Q\bar{Q} + \bar{Q}Q) |b\rangle = Q(\bar{Q}|b\rangle)$$

For $E_n > 0$ we have

$$|b\rangle = \frac{1}{2E_n} Q\bar{Q} |b\rangle = Q |f\rangle \text{ where } |f\rangle = \frac{\bar{Q}|b\rangle}{2E_n} \in \mathcal{H}_{F,n}$$

Similarly, every state in $\mathcal{H}_{F,n}$ with $n > 0$ can be written as $\bar{Q}|g\rangle$ for some $|g\rangle \in \mathcal{H}_{B,n}$. Thus

$$\mathcal{H}_{F,n} \cong \mathcal{H}_{B,n} \text{ for } n > 0$$

and each excited state comes in pairs with a bosonic and fermionic partner. □

Hence we can make the following definition:

Definition 4.11. *The **Witten Index** is*

$$I_w = \dim \mathcal{H}_{B,0} - \dim \mathcal{H}_{F,0} = \text{Tr}_{\mathcal{H}}(-1)^F = \text{Tr}_{\mathcal{H}}((-1)^F e^{-\beta H})$$

where the last two expression follows because excited states come in pairs so cancel out. The final expression is thus independent of β . We include it to regularise the trace and to make the connection to the path integral.

We can now use our fermionic path integral to write the Witten index as

$$\int \exp(-S_E[\bar{\psi}, \psi]) \mathcal{D}\psi \mathcal{D}\bar{\psi}$$

where

$$S_E = \oint \left[\frac{1}{2} \dot{x}^2 + \bar{\psi} \dot{\psi} + \frac{1}{2} (\partial h)^2 + \partial^2 h \bar{\psi} \psi \right] d\tau$$

is the Euclidean action, that is invariant under the SUSY transform

$$\begin{aligned} \delta x &= \epsilon \bar{\psi} - \bar{\epsilon} \psi \\ \delta \psi &= \epsilon (-\dot{x} + \partial h) \\ \partial \bar{\psi} &= \bar{\epsilon} (\dot{x} + \partial h) \end{aligned}$$

Remark. *These look different to before as now we are using the $\cdot = \frac{d}{d\tau}$ which is Wick rotated.*

Note these transformations only make sense globally on S^1 , since $(x, \psi, \bar{\psi})$ are all periodic. and $\epsilon, \bar{\epsilon}$ constants.

Let's now compute I_w using the path integral. As in $d = 0$ consider rescaling $h \rightarrow \lambda h$ for $\lambda \in \mathbb{R}_{>0}$. Then

$$\frac{d}{d\lambda} I_w(\lambda) = - \int_P \left[\oint_{S^1} \lambda (\partial h)^2 + \partial^2 h \bar{\psi} \psi d\tau \right] e^{-S_E[x, \psi, \bar{\psi}]} \mathcal{D}x \mathcal{D}\psi \mathcal{D}\bar{\psi}$$

However

$$\begin{aligned} Q_\lambda \oint \partial h \psi d\tau &= \oint_{S^1} \left[\lambda (\partial h)^2 + \partial^2 h \bar{\psi} \psi - \frac{\partial h}{\partial x} \frac{dx}{d\tau} \right] d\tau \\ &= \oint_{S^1} \lambda (\partial h)^2 + \partial^2 h \bar{\psi} \psi d\tau - \underbrace{\oint_{S^1} dh}_{=0} \end{aligned}$$

so this insertion is Q-exact and we conclude

$$\frac{dI_w}{d\lambda}(\lambda) = 0$$

as expected from the canonical calculation. In particular, as $\lambda \rightarrow \infty$ the term $\exp(-\frac{\lambda}{2} \oint (\partial h)^2 d\tau)$ suppresses all maps $x : S^1 \rightarrow \mathbb{R}$ except in a neighbourhood of constant maps to critical point of h . Near such critical points, we expand $x(\tau) = x_* + \delta x(\tau)$. Then to quadratic order

$$S_E^{(2)} = \oint \left[\frac{1}{2} \delta x \left(-\frac{d^2}{d\tau^2} + h''(x_*) \right) \delta x + \bar{\psi} \left(\frac{d}{d\tau} + h''(x_*) \right) \psi \right] d\tau$$

Since $\delta x(\tau)$ and the fermions must each be periodic, can expand as a Fourier series

$$\begin{aligned} \delta x(\tau) &= \sum_{n \in \mathbb{Z}} \delta x_n \exp \frac{2\pi i n \tau}{\beta} \\ \psi(\tau) &= \sum_{n \in \mathbb{Z}} \psi_n \exp \frac{2\pi i n \tau}{\beta} \end{aligned}$$

we $\delta x_n = (\delta x_n)^*$ since $\delta x \in \mathbb{R}$. We now find near a critical point

$$\begin{aligned} \int e^{S_E^{(2)}} \mathcal{D}\delta x \mathcal{D}\psi \mathcal{D}\bar{\psi} &= \frac{\det(\partial_\tau + h''(x_*))}{\sqrt{\det(-\partial_\tau^2 + h''(x_*)^2)}} \\ &= \frac{\prod_{n \in \mathbb{Z}} \left(\frac{2\pi i n}{\beta} + h''(x_*) \right)}{\sqrt{\prod_{n \in \mathbb{Z}} \left(\left(\frac{2\pi n}{\beta} \right)^2 + h''(x_*)^2 \right)}} \\ &= \frac{h''(x_*)}{|h''(x_*)|} \text{ (only } n = 0 \text{ terms don't cancel)} \end{aligned}$$

Hence

$$I_w = \sum_{x_* : h'(x_*)=0} \frac{h''(x_*)}{|h''(x_*)|}$$

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³A zero Witten index indicates that your symmetry is spontaneously broken (why?)

5 Non-Linear Sigma Models

We now let our bosonic field x define a map $x : M \rightarrow N$ from our worldline M to a compact n -dimensional Riemannian manifold (N, g) . We call N the **target space**. We often let x^a be coordinates on $U \subset N$ open and $x^a(\tau)$ be the corresponding field. Choose the following action:

Definition 5.1. *The Non-Linear Sigma Model (NLSM) action is*

$$S[x] = \int_M \frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b d\tau$$

where we pullback the metric g .

This is an interacting worldline QFT. Varying $S[x]$ we get the following proposition:

Proposition 5.2. *The Euler-Lagrange equations for the NLSM actions are the geodesic equations on N*

$$\frac{d^2 x^a}{d\tau^2} + \Gamma_{bc}^a \dot{x}^b \dot{x}^c = 0$$

where

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} [\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}]$$

Proof.

$$\begin{aligned} \delta S &= \int_M [g_{ab} \dot{x}^a \frac{d(\delta x^b)}{d\tau} + \frac{1}{2} \partial_c g_{ab} \dot{x}^a \dot{x}^b \delta x^c] d\tau \\ &= \int_M \left[-\frac{d}{d\tau} (g_{ab} \dot{x}^a) + \frac{1}{2} \partial_c g_{ab} \dot{x}^b \dot{x}^c \right] \delta x^c d\tau + g_{ab}(x) \dot{x}^a \delta x^b |_{\delta M} \end{aligned}$$

The equation of motion are thus the geodesic equations □

To quantise take

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab} \dot{x}^b$$

and then we have the usual canonical quantisation relations

$$[\hat{x}^a, \hat{p}_b] = i\delta_b^a$$

We can choose the Hilbert space to be $\mathcal{H} = L^2(N, \sqrt{g} d^n x)$ using the Riemannian volume element $\sqrt{g} d^n x$ on N . However, there's no preferred choice of Hamiltonian. Classically, as usual

$$H = p_a \dot{x}^a - L = \frac{1}{2} g^{ab}(x) p_a p_b$$

but there's an ordering ambiguity in turning this into a quantum operator on $L^2(N, \sqrt{g} d^n x)$. It is reasonable to require

- \hat{H} should be generally covariant.
- \hat{H} should reduce to $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$ in the case $(N, g) = (\mathbb{R}^n, d^n x)$.
- \hat{H} should contain ≤ 2 derivatives when acting on either $\Psi \in \mathcal{H}$ or g .

There's a 1 parameter family of such \hat{H} s given by

$$\begin{aligned}\hat{H} &= -\frac{1}{2} \underbrace{\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a} \left(g^{ab} \sqrt{g} \frac{\partial}{\partial x^b} \right)}_{\text{usual Laplacian in curved space}} + \alpha R[g] \quad \text{for } \alpha \in \mathbb{R} \\ &= -\frac{1}{2} \nabla^a \nabla_a + \alpha R\end{aligned}$$

where $R[g] = R_g = R$ is the Ricci scalar. There's no preferred choice of α and different ways to regularise the path integral \Rightarrow different values of α .

5.1 Supersymmetric NLSM

We will now build a supersymmetric theory with $n = \dim N$ fermions $\psi^a \in \Pi\Omega^0(M, x^*T_N)$ with action

$$S[x, \psi] = \int_M \left[\frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b + i g_{ab} \bar{\psi}^a (\nabla_\tau \psi)^b - \frac{1}{2} R_{abcd} \psi^a \bar{\psi}^b \psi^c \bar{\psi}^d \right] d\tau$$

where

- $M = [0, \beta]$ or S^1 .
- $\Omega^0(M, \dots)$ are the functions on the worldline
- x^* is the tangent space target index
- $\nabla_\tau \psi^a = \frac{d\psi^a}{d\tau} + \Gamma_{bc}^a \frac{dx^b}{d\tau} \psi^c$ is the pullback of the connection on N .

By now we know what to do when we have got an action, we show it is invariant under a SUSY:

Proposition 5.3. *The action is invariant under the SUSY transform*

$$\begin{aligned}\delta x^a &= \epsilon \bar{\psi}^a - \bar{\epsilon} \psi^a \\ \delta \psi^a &= \epsilon (i \dot{x}^a - \Gamma_{bc}^a \bar{\psi}^b \psi^c) \\ \delta \bar{\psi}^a &= \bar{\epsilon} (-i \dot{x}^a - \Gamma_{bc}^a \bar{\psi}^b \psi^c)\end{aligned}$$

generated by the Noether charges Q, \bar{Q} with

$$\begin{aligned}Q &= i \bar{\psi}^a (g_{ab} \dot{x}^b + i g_{bc} \bar{\psi}^b \Gamma_{ad}^c \psi^d) \\ \bar{Q} &= -i \psi^a (g_{ab} \dot{x}^b + i g_{bc} \psi^b \Gamma_{ad}^c \bar{\psi}^d)\end{aligned}$$

It is also invariant under $\psi^a \rightarrow e^{i\alpha}\psi^a$, $\bar{\psi}^a \rightarrow e^{-i\alpha}\bar{\psi}^a$, generated by the charge $F = g_{ab}\psi^a\bar{\psi}^b$. Conservation of F in the quantum theory implies no fermionic excitations created/ destroyed by time evolution.

We now want to quantise by calculating

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab}\dot{x}^b + ig_{bc}\bar{\psi}^b\Gamma_{ad}^c\psi^d$$

$$\pi_a = \frac{\delta L}{\delta \dot{\psi}^a} = ig_{ab}\bar{\psi}^b$$

and imposing the canonical commutation relations

$$[\hat{x}^a, \hat{p}_b] = i\delta_b^a$$

$$\left\{ \hat{\psi}^a, \hat{\bar{\psi}}^b \right\} = \delta^{ab}$$

with all others trivial. For the bosonic field, choose $\mathcal{H} = L^2(N, \sqrt{g}dx)$ with $\hat{p}_a \rightarrow -i\frac{\partial}{\partial x^a}$. For the fermions, again choose $\bar{\psi}$ to be raising operators and the ψ to be lowering operators. Pick a vacuum state $|0\rangle$ defined by $\forall a \psi^a |0\rangle = 0$, then all other states of the fermionic system are generated by acting with $\bar{\psi}$ on $|0\rangle$. Each $\bar{\psi}^a$ can only act once, since $\{\bar{\psi}^a, \bar{\psi}^b\} = 0$. We can thus interpret these as *forms* on N

$$|0\rangle \leftrightarrow 1$$

$$\bar{\psi}^a |0\rangle \leftrightarrow dx^a$$

$$\bar{\psi}^a \bar{\psi}^b |0\rangle \leftrightarrow dx^a \wedge dx^b$$

$$\dots \leftrightarrow \dots$$

$$\bar{\psi}^1 \dots \bar{\psi}^n |0\rangle \leftrightarrow dx^1 \wedge \dots \wedge dx^n$$

⁴ Altogether, the Hilbert space of SUSY QM is

$$\mathcal{H} = \Omega^*(N) = \bigoplus_{p=0}^n \Omega^p(N)$$

i.e.

$$\Psi(x, \bar{\psi}) = f(x) + \alpha_a(x)\bar{\psi}^a + \beta_{ab}(x)\bar{\psi}^a\bar{\psi}^b + \dots + \omega_{1\dots n}(x)\bar{\psi}^1 \dots \bar{\psi}^n$$

$$\leftrightarrow f(x) + \alpha_a(x)dx^a + \beta_{ab}(x) \wedge dx^a dx^b + \dots + \omega_{1\dots n}(x)dx^1 \wedge \dots \wedge dx^n$$

⁴Note that the antisymmetry of the $\bar{\psi}$ s corresponds to the antisymmetry of the wedge product on one forms

Acting on this space

$$\begin{aligned}\hat{x}^a &\rightarrow x^a \times \\ \hat{p}_a &\rightarrow -i \frac{\partial}{\partial x^a} \\ \bar{\psi}^a &\rightarrow dx^a \wedge \\ \psi^a &\rightarrow g^{ab} \frac{\partial}{\partial x^b}\end{aligned}$$

Remark. To see this last assignment observe

$$\begin{aligned}\psi^e \left(\underbrace{\bar{\psi}^a \bar{\psi}^b \dots \bar{\psi}^c}_{\text{odd number}} \right) |0\rangle &= \left\{ \psi^e, \bar{\psi}^a \bar{\psi}^b \dots \bar{\psi}^c \right\} |0\rangle \\ &= \left(\left\{ \psi^e, \bar{\psi}^a \right\} \bar{\psi}^b \dots \bar{\psi}^c + \bar{\psi}^a \left\{ \psi^e, \bar{\psi}^b \right\} \dots \bar{\psi}^c + \dots + \bar{\psi}^a \bar{\psi}^b \dots \left\{ \psi^e, \bar{\psi}^c \right\} \right) |0\rangle \\ &= \left(g^{ea} \bar{\psi}^b \dots \bar{\psi}^c - g^{eb} \bar{\psi}^a \dots \bar{\psi}^c + \dots + g^{ec} \bar{\psi}^a \bar{\psi}^b \dots \right) |0\rangle\end{aligned}$$

This is just what we get from

$$g^{ef} \frac{\partial}{\partial x^f} (dx^a \wedge dx^b \wedge \dots \wedge dx^c)$$

Definition 5.4. Define the **Hodge star** operator $\star : \Omega^p(N) \rightarrow \Omega^{n-p}(N)$ by acting on

$$\omega = \frac{1}{p!} \omega_{a_1 \dots a_p} dx^{a_1} \wedge \dots \wedge dx^{a_p}$$

$$\star \omega = \frac{1}{p!(n-p)!} \sqrt{g} \epsilon^{a_1 \dots a_p}_{b_{p+1} \dots b_n} \omega_{a_1 \dots a_p} dx^{b_{p+1}} \wedge \dots \wedge dx^{b_n}$$

Define an inner product on \mathcal{H} by

$$\langle \alpha | \beta \rangle = \int_N \bar{\alpha} \wedge \star \beta$$

where

$$\begin{aligned}\alpha, \beta \in \Omega^p(N) &\Rightarrow \int_N \bar{\alpha} \wedge \star \beta = \int_N \bar{\alpha}^{a_1 \dots a_p} \beta_{a_1 \dots a_p} \sqrt{g} d^n x \\ \alpha \in \Omega^p(N), \beta \notin \Omega^p(N) &\Rightarrow \int_N \bar{\alpha} \wedge \star \beta = 0 \quad \text{by definition}\end{aligned}$$

This gives a natural interpretation to the supercharges

$$\begin{aligned}Q &= i \bar{\psi}^a p_a \rightarrow d \\ \bar{Q} &= -i \psi^a p_a \rightarrow d^\dagger = (-1)^{n(p+1)+1} \star d \star\end{aligned}$$

Here the adjoint is with respect to inner product on \mathcal{H} defined by

$$\begin{aligned}d^\dagger : \Omega^p(N) &\rightarrow \Omega^{p-1}(N) \\ \alpha \in \Omega^p, \beta \in \Omega^{p-1} &\langle \alpha, d^\dagger \beta \rangle = \langle d\alpha, \beta \rangle\end{aligned}$$

5.2 SQM Non-linear Sigma Models

We can deal with the ordering ambiguity by demanding the SUSY algebra

$$2\hat{H} = \left\{ \hat{Q}, \hat{\bar{Q}} \right\}$$

still holds in the quantum theory. This fixes

$$H = \frac{1}{2} (d^\dagger d + dd^\dagger)$$

Definition 5.5. Define the **Laplacian** $\Delta : \Omega^p(N) \rightarrow \Omega^p(N)$ as

$$\Delta = dd^\dagger + d^\dagger d$$

Lemma 5.6. For $f \in \Omega^0(N)$, $\Delta f = d^\dagger df$.

Proof. $\Delta f = \underbrace{dd^\dagger f}_{=0} + d^\dagger df = d^\dagger df$ □

Proposition 5.7. For $f \in \Omega^0(N)$,

$$\Delta f = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b f)$$

Proof. We directly calculate

$$\begin{aligned} \Delta f &= d^\dagger (\partial_a f dx^a) \\ &= \star d(\star df) \\ &= \star d \left(\frac{\sqrt{g}}{(n-1)!} g^{ab} \partial_a f \epsilon_{bc\dots d} \underbrace{dx^c \wedge \dots \wedge dx^d}_{n-1} \right) \\ &= \star \left[\frac{1}{(n-1)!} \partial_m (\sqrt{g} g^{ab} \partial_a f) \epsilon_{bc\dots d} \underbrace{dx^m \wedge dx^c \wedge \dots \wedge dx^d}_n \right] \\ &= \frac{1}{g} \partial_a (g^{ab} \sqrt{g} \partial_b f) \star (dx^1 \wedge \dots \wedge dx^n) \\ &= \frac{1}{\sqrt{g}} \partial_a (g^{ab} \partial_b f \sqrt{g}) \end{aligned}$$

□

Definition 5.8. A form ω is called **harmonic** if $\Delta\omega = 0$.

Acting on any form ω

$$\begin{aligned} 2 \langle \omega | \hat{H} | \omega \rangle &= \langle \omega | dd^\dagger | \omega \rangle + \langle \omega | d^\dagger d | \omega \rangle \\ &= ||d^\dagger | \omega \rangle ||^2 + ||d | \omega \rangle ||^2 \geq 0 \end{aligned}$$

So supersymmetric ground states are in 1-1 correspondence with harmonic forms, of any degeneracy on N .

Let's decompose the space of all harmonic forms as

$$\text{Harm}^*(N) = \bigoplus_{p=0}^n \text{Harm}^p(N)$$

Lemma 5.9. $\forall \omega \in \text{Harm}^p$, ω must be **closed** (i.e. $d\omega = 0$) and **coclosed** (i.e. $d^\dagger \omega = 0$).

Definition 5.10. The **de-Rham cohomology** is

$$\begin{aligned} H_{dR}^p(N) &= \ker(d : \Omega^p \rightarrow \Omega^{p+1}) / \text{Im}(d : \Omega^{p-1} \rightarrow \Omega^p) \\ &= \{ \omega \in \Omega^p : d\omega = 0 \} / \{ \omega \in \Omega^p : \exists \alpha \in \Omega^{p-1}, \omega = d\alpha \} \end{aligned}$$

Remark. Note this is a genuine object we can consider as Ω^p is a vector space, and as $d^2 = 0$ we know $\text{Im } d$ is a subspace of $\ker d$.

We now state an important theorem from Riemannian geometry:

Theorem 5.11 (Hodge's Theorem). *Hodge's theorem states*

$$\text{Harm}^p(N) \cong H_{dR}^p(N)$$

Note in the de Rham cohomology $\forall \alpha \ \omega \sim \omega + d\alpha$. The role of the coclosure condition is to select a unique representation

$$d\omega = 0 = d^\dagger \omega \Rightarrow d^\dagger d\alpha = 0$$

is the only freedom of choice of α , and the only solution is 0.⁵

5.3 Witten index of a NLSM

A consequence of de-Rham's theorem is that the space of SUSY ground states $\cong H_{dR}^*(N)$, with states with p fermions living in $H_{dR}^p \Rightarrow$ the Witten index

$$\begin{aligned} I_w &= \text{Tr}((-1)^F e^{-\beta H}) \\ &= n_B - n_F \\ &= \sum_{p=0}^n (-1)^p \dim H_{dR}^p(N) \\ &= \chi(N) \end{aligned}$$

χ the **Euler characteristic** of N .

Now, suppose C_p is a p -cycle in N with no boundary, and $\omega \in \Omega^p(N)$. Stokes' theorem gives

$$\int_{D_{p+1}} d\omega = \int_{C_p} \omega$$

⁵This is analogous to choosing $d^\dagger A = \partial^\mu A_\mu = 0$ in EM

where $C_p = \partial D_{p+1}$. If $\exists D_{p+1}$ ⁶ and $\omega \in H_{dR}^p$ then $d\omega = 0$ and

$$\Rightarrow \int_{C_p} \omega = 0$$

Further $\forall \alpha \in \Omega^{p-1}(N)$.

$$\int_{C_p} \omega + d\alpha = \int_{D_{p+1}} d\omega$$

This motivates the following definition and theorem:

Definition 5.12. *The p^{th} homology group is*

$$\begin{aligned} H_p(N) &= \{p\text{-cycles with no boundary}\} / \{p\text{-cycles that are the boundary of a } (p+1)\text{-cycle}\} \\ &= \{C_p : \partial C_p = 0\} / \{C_p : \exists D_{p+1}, C_p = \partial D_{p+1}\} \end{aligned}$$

Theorem 5.13 (de Rham's Theorem). *de Rham's theorem states*

$$H_{dR}^p(N) \cong H_p(N)$$

Example 5.14. *It is true that*

$$\dim H_{dR}^p(S^n) = \begin{cases} 1 & p = 0, n \\ 0 & p \neq 0, n \end{cases}$$

Consider a general genus g surface Σ_g . We have

$$\dim H_{dR}^p(\Sigma_g) = \begin{cases} 1 & p = 0, 2 \\ 2g & p = 1 \end{cases}$$

It is also known

$$\begin{aligned} \chi(S^n) &= 1 + (-1)^n = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \\ \chi(\Sigma_g) &= 2 - 2g \end{aligned}$$

Now from the path integral

$$\chi(N) = \int e^{-S[x, \psi]} \mathcal{D}x \mathcal{D}\psi \mathcal{D}\bar{\psi}$$

where all fields are periodic with period β .

Proposition 5.15. *the whole action is supersymmetrically trivial, specifically*

$$\begin{aligned} S &= \oint \left(\frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b + g_{ab} \bar{\psi}^a \nabla_\tau \psi^b - \frac{1}{2} R_{abcd} \bar{\psi}^a \psi^b \bar{\psi}^c \psi^d \right) d\tau \\ &= \bar{Q} \left[\oint \frac{g_{ab} \bar{\psi}^a}{2} \left(i \dot{x}^b + \Gamma_{cd}^b \bar{\psi}^c \psi^d \right) \right] d\tau \end{aligned}$$

⁶This is where the condition that C_p has no boundary is necessary, as the boundary of a boundary is empty, so for such a D_{p+1} to exist it is necessary $\partial C_p = \emptyset$.

Proof. **Exercise** □

Corollary 5.16. *The path integral is independent of \hat{p} , and so independent of the circumference β of S^1 .*

In particular, if we expand

$$x^a(\tau) = x_0^a + \delta x^a(\tau)$$

with

$$\oint \delta x^a(\tau) d\tau = 0$$

then as $\beta \rightarrow 0$ all the $\psi^a(\tau) = \psi_0^a + \delta\psi^a(\tau)$ contributions from the non-zero modes are highly suppressed, e.g.

$$\delta x^a(\tau) = \sum_{k \neq 0} \delta_k^a(\tau) e^{\frac{2\pi i k \tau}{\beta}}$$

the derivatives being as $\frac{1}{\beta} \rightarrow \infty$. In fact the contributions from $\delta x, \delta\psi$ precisely cancel one another, leaving us just with an integral over the zero modes, that is the path integral localises to constant maps $x : S^1 \rightarrow N$, but there's no preferred point in N so we still need to integrate over N . Thus

$$\begin{aligned} \chi(N) &= \int e^{-S[x_0, \psi_0]} d^n x_0 d^n \psi_0 d^n \bar{\psi}_0 \\ &= \int \exp \left[-\frac{1}{2} R_{abcd}(x_0) \bar{\psi}_0^a \bar{\psi}_0^b \psi_0^c \psi_0^d \right] d^n x_0 d^n \psi_0 d^n \bar{\psi}_0 \\ &= \int_N \text{Tr} (R \wedge R \wedge \cdots \wedge R) \quad \text{where } R^a_b = R_{cd}{}^a_b dx^c \wedge dx^d \text{ is the curvature 2 form} \end{aligned}$$

This is the **Gauss-Bonnet formula**

6 Atiyah Singer Index Theorem

We take $\dim N = n = 2m$. We can then restrict ψ^a to be real (i.e. $\psi = \bar{\psi}$), giving the action

$$S[x, \psi] = \oint \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b + \frac{1}{2} g_{ab} \psi^a \nabla_\tau \psi^b d\tau$$

since $R_{a[bcd]} = 0$ (this is the algebraic Bianchi identity).

Proposition 6.1. *This action is invariant under SUSY transforms with $\epsilon = -\bar{\epsilon}$ and*

$$\begin{aligned} \delta x^a &= \epsilon \psi^a \\ \delta \psi^a &= -\epsilon \dot{x}^a \end{aligned}$$

(This is sometimes called $\mathcal{N} = \frac{1}{2}$ SUSY). We have momenta

$$\begin{aligned} p_a &= \frac{\delta L}{\delta \dot{x}^a} = g_{ab} \dot{x}^b + \frac{i}{2} \psi_c \Gamma_{ab}^c \psi^b \\ \pi_a &= \frac{\delta L}{\delta \dot{\psi}^a} = g_{ab} \psi^b \end{aligned}$$

Thus we impose canonical commutation relations $[\hat{x}^a, \hat{p}_b] = i\delta_b^a, \{\hat{\psi}^a, \hat{\psi}^b\} = 2g^{ab}$.

6.1 Spinors in n=2m Dimensions

The Dirac γ matrices obey $(\gamma^i)^\dagger = \gamma^i$ and $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. We construct $m = \frac{n}{2}$ raising and lowering operators over \mathbb{C} by

$$\gamma_\pm^I = \frac{1}{2} (\gamma^{2I} \pm i\gamma^{2I-1})$$

Lemma 6.2. *These raising and lowering operators obey*

$$\begin{aligned} \{\gamma_+^I, \gamma_-^J\} &= \delta^{IJ} \\ \{\gamma_+^I, \gamma_+^J\} &= 0 = \{\gamma_-^I, \gamma_-^J\} \end{aligned}$$

Proof. **exercise** □

Starting from a spinor χ that obeys $\forall I \gamma_-^I \chi = 0$ we construct a basis of the space S of spinors by acting with any combination of the γ_+^I . Since $(\gamma_+^I)^2 = 0$, each component can act at most once, so $\dim S = 2^{\frac{n}{2}} = 2^m$. The group $Spin(n)$, the double cover of $SO(n)$, acts on these spinors via the generators $\Sigma^{ij} = -\frac{1}{4} [\gamma^i, \gamma^j]$.

Lemma 6.3. *The generators Σ^{ij} have the commutation relations*

$$[\Sigma^{ij}, \Sigma^{kl}] = i (\delta^{ik} \Sigma^{jl} + \delta^{jl} \Sigma^{ik} - \delta^{jk} \Sigma^{il} - \delta^{il} \Sigma^{jk})$$

Proposition 6.4. *This representation is not irreducible.*

Proof. To see this let $\gamma^{n+1} = i^{\frac{n}{2}} \gamma^1 \gamma^2 \dots \gamma^n$. This obeys

$$\begin{aligned} (\gamma^{n+1})^2 &= 1 \\ \{\gamma^{n+1}, \gamma^i\} &= 0 \\ [\gamma^{n+1}, \Sigma^{ij}] &= 0 \end{aligned}$$

We can thus decompose $S = S^+ \oplus S^-$ where S^\pm are the ± 1 eigenspaces of γ^{n+1} , and correspond to states constructed from an even/odd number of raising operators. □

The Dirac operator $\not{\partial}$ anticommutes with γ^{n+1} and thus decomposes into

$$\not{\partial} = \begin{pmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{pmatrix}$$

where $\partial^\pm : S^\pm \rightarrow S^\mp$. Note that ∂^\pm annihilates S^\mp , so $(\partial^\pm)^2 = 0$.

Definition 6.5. *The **index** of the chiral Dirac operator ∂^+ is*

$$\text{index}(\partial^+) = \dim \ker(\partial^+) - \dim \ker(\partial^-)$$

Remark. *You may, like me, be wondering why we are not asking what the index of $\not{\partial}$ is. It turns out that general theory says that it is always zero, so it is not too interesting, but in even dimensions this splitting of the spinor bundle into chiral parts lets us say something interesting about ∂^\pm .*

6.2 Susy QM

In our quantisation of $S = \int \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b + \frac{i}{2} g_{ab} \psi^a \nabla_\tau \psi^b d\tau$ the Hilbert space is thus naturally $L^2(S(N), \sqrt{g} d^n x)$ (square integrable section of the spin bundle on N), and the supercharge

$$Q = \psi^a \left(i g_{ab} \dot{x}^b + \underbrace{\psi_c \Gamma_{ab}^c \psi^b}_{= \frac{1}{2} \Gamma_{ab}^c g_{cd} [\psi^c, \psi^d]} \right)$$

corresponds to the covariant Dirac operator $i\nabla$. The Witten index $\text{Tr}((-1)^F e^{-\beta H}) = \dim \ker(\nabla^+) - \dim \ker(\nabla^-)$. The path integral is again independent of circumference β , and it can be shown in the limit $\beta \rightarrow 0$ it is localised at constant fields. Hence we expand perturbatively

$$\begin{aligned} x^a(\tau) &= x_0^a + \delta x^a(\tau) \\ \psi^a(\tau) &= \psi_0^a + \delta \psi^a(\tau) \end{aligned}$$

with $\oint \delta x dt = 0$. We use Riemann normal coordinate near $x_0 \in N$ to write

$$\begin{aligned} g_{ab}(x) &= \delta_{ab} - \frac{1}{3} R_{abcd}(x_0) \delta x^c \delta x^d + \mathcal{O}(\delta x^3) \\ \Gamma_{bc}^a(x) &= \partial_d \Gamma_{bc}^a(x_0) \delta x^d + \mathcal{O}(\delta x^2) = -\frac{1}{3} [R^a_{bcd}(x_0) + R^a_{cbd}(x_0)] \delta x^d + \mathcal{O}(\delta x^2) \end{aligned}$$

To second order in the fluctuations, the actions becomes

$$S^{(2)}[x_0, \psi_0, \delta x, \delta \psi] = \oint -\frac{1}{2} \delta x_a \frac{d^2}{dt^2} \delta x^a + \frac{1}{2} \delta \psi_a \frac{d}{d\tau} \delta \psi^a - \frac{1}{4} R_{abcd} \psi_0^a \psi_0^b \delta x^c \delta x^d d\tau$$

(raising and lowering indices with the flat metric). For any fixed values of (x_0, ψ_0) this is a free action, so the path integral over fluctuations gives

$$\int e^{-S[x_0, \psi_0, \delta x, \delta \psi]} \mathcal{D}(\delta x) \mathcal{D}(\delta \psi) = \frac{\sqrt{\det'(\partial_\tau \delta_b^a)}}{\sqrt{\det'(-\partial_\tau^2 + \mathcal{R}^a_b(x_0, \psi_0) \partial_\tau)}} = \frac{1}{\sqrt{\det'(-\partial_\tau \delta_b^a + \mathcal{R}^a_b)}}$$

where $\mathcal{R}^a_b(x_0, \psi_0) = R^a_{bcd}(x_0) \psi_0^c \psi_0^d$ and \det' indicates the determinant without the zero modes. The matrix \mathcal{R}^a_b is an antisymmetric $n \times n$ matrix, and $n = 2m$. We decompose the tangent space $T_{x_0} N$ into 2-dimensional subspaces on which

$$\mathcal{R}^a_b|_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix} \quad i = 1, \dots, m$$

Let $-D_i$ be the restriction of $-\delta_b^a \partial_\tau + \mathcal{R}^a_b$ to the i^{th} subspace. We expand

$$\delta x^a(t) = \sum_{k \neq 0} \delta x_k^a e^{2\pi i k \tau}$$

Then the eigenvalues of $-D_i$ on this subspace are $-2\pi i k \pm \omega_i$ for $k \in \mathbb{Z} \setminus \{0\}$. Therefore

$$\begin{aligned} \det'(-D_i) &= \prod_{k \neq 0} (-2\pi i k + \omega_i)(-2\pi i k - \omega_i) \\ &= \prod_{k \neq 0} [-(2\pi k)^2 - \omega_i^2] \\ &= \prod_{k=1}^{\infty} (2\pi k)^4 \prod_{k=1}^{\infty} \left[1 + \frac{\omega_i^2}{(2\pi k)^2} \right]^2 \end{aligned}$$

This is clearly divergent. We can regularise this by using zeta function regularisation

$$\prod_{k=1}^{\infty} (2\pi k)^4 = (4\pi^2)^{\zeta(0)} e^{-2\zeta'(0)} = 1$$

Hence the important factor is

$$\prod_{k=1}^{\infty} \left[1 + \frac{\omega_i^2}{(2\pi k)^2} \right]^2$$

we recall

$$\sinh(z) = z \prod_{k=1}^{\infty} \left[1 + \frac{z^2}{\pi^2 k^2} \right]$$

so we have

$$\sqrt{\det'(-D_i)} = \frac{\sinh(\omega_i/2)}{(\omega_i/2)}$$

Theorem 6.6. *The Atiyah-Singer Index Theorem says that*

$$\text{index}(D^+) = \int_N \det \left(\frac{\mathcal{R}/2}{\sinh \mathcal{R}/2} \right)$$

where D^+ is the chiral Dirac operator on N and $\mathcal{R}^a_b = R^a_{bcd} dx^c \wedge dx^d$

Proof. The whole regularised path integral gives

$$\begin{aligned} I_w = \text{index}(D^+) &= \int \prod_{i=1}^m \frac{(\omega_i/2)}{\sinh(\omega_i/2)} \\ &= \int \det \left(\frac{\mathcal{R}^a_b(x_0, \psi_0)/2}{\sinh(\mathcal{R}^a_b(x_0, \psi_0)/2)} \right) d^n x_0 d^n \psi_0 \\ &= \int_N \det \left(\frac{\mathcal{R}/2}{\sinh \mathcal{R}/2} \right) \end{aligned}$$

□

7 Supersymmetric QFT

If we have a d dimensional theory that is Lorenz invariant, we must complete the supersymmetry algebra $\{Q, Q^\dagger\} = 2H$. The Hamiltonian is part of the d -momentum multiplet of P_μ , i.e. we need further supercharges. If we want to preserve $Q^\dagger = (Q)^\dagger$ then these supercharges must have the same spin and so must each have spin $\frac{1}{2}$. Specifically, the SUSY algebra in d -dimensions is

$$\{Q_\alpha, Q_\beta^\dagger\} = 2\gamma_{\alpha\beta}^\mu P_\mu$$

where α, β are spinor indices and γ^μ is a Dirac γ matrix. We'll mostly be concerned with $d = 2$ where Dirac spinors have $2^{\frac{d}{2}} = 2$ complex components $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$. With coordinates $(t, s) \in \mathbb{R}^2$ and Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1)$, we can represent the Dirac γ s

$$\gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma^s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

These obeys $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The action for a free massless Dirac spinor in $d = 2$ is

$$S[\psi] = \frac{1}{2\pi} \int_{\mathbb{R}^2} i\bar{\psi} \not{\partial} \psi d^2x$$

where $\not{\partial} = \gamma^\mu \partial_\mu$ and $\bar{\psi} = \psi^\dagger \gamma^t$. So

$$S[\psi] = \frac{1}{2\pi} \int_{\mathbb{R}^2} i\bar{\psi}_- (\partial_t + \partial_s) \psi_- + i\bar{\psi}_+ (\partial_t - \partial_s) \psi_+ dt ds$$

Classically,

$$(\partial_t + \partial_s) \psi_- = 0 \Rightarrow \psi_-(t, s) = f(t - s) \quad (\text{right - moving}) \\ (\partial_t - \partial_s) \psi_+ = 0 \Rightarrow \psi_+(t, s) = g(t + s) \quad (\text{left - moving})$$

Under a $SO(1, 1)$ transformation $\psi_\pm \mapsto e^{\pm \frac{\gamma}{2}} \psi_\pm$, $\bar{\psi}_\pm \mapsto e^{\pm \frac{\gamma}{2}} \bar{\psi}_\pm$, so

$$\begin{pmatrix} t \\ s \end{pmatrix} \mapsto \begin{pmatrix} \cosh \gamma & \sinh \gamma \\ \sinh \gamma & \cosh \gamma \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix}$$

8 Superspace in d=2

Let $\mathbb{R}^{2/4}$ denote the superspace with coordinate

$$(x^0 = t, x^1 = s, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$$

Under an $SO(1, 1)$ transformation

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \rightarrow \begin{pmatrix} \cosh \gamma & \sinh \gamma \\ \sinh \gamma & \cosh \gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \\ \theta^\pm \rightarrow e^{\pm \frac{\gamma}{2}} \theta^\pm \\ \bar{\theta}^\pm \rightarrow e^{\pm \frac{\gamma}{2}} \bar{\theta}^\pm$$

We introduce fermionic derivatives

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i\bar{\theta}^{\pm} \frac{\partial}{\partial x^{\pm}}$$

$$\bar{Q}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} + i\theta^{\pm} \frac{\partial}{\partial x^{\pm}}$$

where $\partial_{\pm} = \frac{\partial}{\partial x^{\pm}} = \frac{1}{2} \left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right)$. These derivatives obey the anticommutation relations

$$\{Q_{\pm}, \bar{Q}_{\pm}\} = -2i\partial_{\pm}$$

so they represent our supersymmetry algebra on $\mathbb{R}^{2/4}$.

Supersymmetry transformations act geometrically on $\mathbb{R}^{2/4}$, being generated by

$$\delta = \epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+ \bar{Q}_- + \bar{\epsilon}_- \bar{Q}_+$$

where we note that the parameters $\epsilon_{\pm}, \bar{\epsilon}_{\pm}$ must themselves be spinors in order for $\Phi \rightarrow \Phi + \delta\Phi$.

A superfield \mathcal{F} is simply a function on $\mathbb{R}^{2/4}$. A generic superfield thus has an expansion

$$\mathcal{F}(x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm}) = f_0(x^{\pm}) + \theta^+ f_+(x^{\pm}) + \theta^- f_-(x^{\pm}) + \bar{\theta}^+ g_+(x^{\pm}) + \bar{\theta}^- g_-(x^{\pm}) + \dots + \theta^+ \bar{\theta}^+ \theta^- \bar{\theta}^- D(x^{\pm})$$

having $2^4 = 16$ components.

Notice that under a SUSY transformation

$$\mathcal{F} \rightarrow \mathcal{F} + \delta\mathcal{F},$$

the highest component field $D(x^{\pm})$ can change by at most a bosonic derivative.

8.1 Chiral Superfield

It's often useful to have smaller superfields that are constrained in some way. For this purpose introduce

$$D_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - i\bar{\theta}^{\pm} \partial_{\pm}$$

$$\bar{D}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} + i\theta^{\pm} \partial_{\pm}$$

These derivatives obey

$$\{D_{\pm}, \bar{D}_{\pm}\} = 2i\partial_{\pm} \quad (\text{all others } 0)$$

and canonically

$$\{D, Q\} = 0 = \{\bar{D}, Q\} \quad (\text{for all choice } +/-)$$

A **chiral superfield** Φ is then a superfield which obeys

$$\bar{D}_{\pm} \Phi = 0$$

These can depend on $(x^\pm, \theta^\pm, \bar{\theta}^\pm)$ only through (y^\pm, θ^\pm) where

$$y^\pm = x^\pm - i\theta^\pm \bar{\theta}^\pm$$

i.e. $(\bar{D}_\pm y^\pm = 0, \bar{D}_\pm y^\mp = 0)$ // We can expand a chiral superfield as

$$\Phi = \phi(y^\pm) + \theta^+ \psi_+(y^\pm) + \theta^- \psi_-(y^\pm) + \theta^+ \theta^- F(y^\pm)$$

Notice that the produce $\Phi_1 \Phi_2$ of two chiral superfield is again chiral, while the conjugate $\bar{\Phi}$ of a chiral superfield boeys

$$D_\pm \bar{\Phi} = 0$$

and is called **antichiral**

Under a supersymmetry transformation

$$\Phi \rightarrow \Phi + \delta\Phi$$

but since all $\{Q, D\} = 0$

$$\bar{D}_\pm(\delta\Phi) = \delta(\bar{D}_\pm\Phi) = 0$$

so supersymmetry transofmrailton preserve chirality.

To work out the susy transformation on component fields, first note

$$\begin{aligned} Q_\pm &= \frac{\partial}{\partial\theta^\pm}|_{x,\theta} + i\bar{\theta}^\pm \frac{\partial}{\partial x^\pm|_{\theta,\bar{\theta}}} \\ &= \frac{\partial}{\partial\theta^\pm}|_{\theta,\bar{\theta}} + \underbrace{\frac{\partial y^\pm}{\partial\theta^\pm}|_{x,\theta}}_{=-i\bar{\theta}^\pm} \frac{\partial}{\partial y^\pm}|_{\theta,\bar{\theta}} + i\bar{\theta}^\pm \frac{\partial}{\partial y^\pm}|_{\theta,\bar{\theta}} \\ &= \frac{\partial}{\partial\theta^\pm}|_{y,\theta} \end{aligned}$$

and similarly

$$\bar{Q}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm}|_{y,\theta} - 2i\theta^\pm \frac{\partial}{\partial y^\pm}|_{\theta,\bar{\theta}}$$

Using this , one find the component transformations

$$\begin{aligned} \delta\phi &= \epsilon_+ \psi_- - \epsilon_- \psi_+ \\ \delta\psi_\pm &= \epsilon_\pm F \pm \bar{\epsilon}_\mp \partial_\pm \phi \\ \delta F &= -2i\bar{\epsilon}_+ \partial_- \psi_+ - 2i\bar{\epsilon}_- \partial_+ \psi_- \end{aligned}$$

Again, note the SUSY transform of the θ^\pm term F is a bosonic total derivative.

8.2 Supersymmetric Invariant Action

The fact that the D term of a generic superfield and F term of a chiral superfield vary only by total derivatives allows to readily construct SUSY invariants.

Let

$$K(F_i, \Phi^a, \bar{\Phi}^a) \quad (\text{Kähler potential})$$

be any \mathbb{R} smooth function of \mathbb{R} superfield

$$F_i(x^\pm, \theta^\pm \bar{\theta}^\pm)$$

and chiral superfields

$$\Phi^a(x^\pm + i\theta^\pm \bar{\theta}^\pm, \theta^\pm)$$

Then

$$\int_{\mathbb{R}^{2/4}} K(F_i, \Phi^a, \bar{\Phi}^a) d^2 d^2 \theta d^2 \bar{\theta}$$

is SUSY invariant provided the component fields behave appropriately as $|x^\pm| \rightarrow \infty$. Likewise, suppose $W(\Phi^a)$ (the **superpotential**) is a holomorphic function of Φ^a . Then

$$\bar{D}_\pm W(\Phi^a) = 0$$

and so

$$\int_{\mathbb{R}^{2/4}} W(\Phi^a) d^2 y d^2 \theta$$

is again a SUSY invariant.

8.3 Wess Zumino Model

Let's consider now the cimplets case of a single chiral superfield Φ and its conjugate $\bar{\Phi}$. We take

$$K(\Phi, \bar{\Phi}) = \bar{\Phi}\Phi$$

and keep W general.

$$S[\Phi, \bar{\Phi}] = \int_{\mathbb{R}^{2/4}} \bar{\Phi}\Phi d^2 x d^4 \theta + \int_{\mathbb{R}^{2/2}} W(\Phi) d^2 y d^2 \theta + \int_{\mathbb{R}^{2/2}} \bar{W}(\bar{\Phi}) d^2 \bar{y} d^2 \bar{\theta}$$

where $W(\Phi)$ is holomorphic function of chiral superfield $\Phi = \phi(y^\pm) + \theta^+ \psi_+(y^\pm) + \theta^- \psi_-(y^\pm) + \theta^+ \theta^- F(y^2)$ and $y^\pm = x^\pm - i\theta^\pm \bar{\theta}^\pm$. We have

$$W(\Phi)|_{\theta^\pm \bar{\theta}^\pm} = F \partial W(\phi) - \psi_+ \psi_- \frac{\partial^2 W}{\partial \phi^2}(\phi)$$

For the Kähler potential $|\Phi|_{\theta^4}^2$ we need to write

$$\begin{aligned}\Phi(x^\pm, \theta^\pm, \bar{\theta}^\pm) &= \underbrace{\phi(y^\pm)}_{(1)} + \theta^+ \psi_+(y^\pm) + \theta^- \psi_-(y^\pm) + \theta^+ \theta^- F(y^\pm) \\ &= \underbrace{\phi(x^\pm) - i\theta^+ \bar{\theta}^+ \partial_+ \phi(x^\pm) - i\theta^- \bar{\theta}^- \partial_- \phi(x^\pm) - \theta^+ \bar{\theta}^+ \theta^- \bar{\theta}^- \partial_+ \partial_- \phi(x^\pm)}_{(1)} + \theta^+ \psi_+(x^\pm) - i\theta^+ \theta^- \bar{\theta}^- \partial_- \psi_+(x^\pm) +\end{aligned}$$

as a function on non chiral superspace. Similarly,

$$\bar{\Phi}(x^\pm, \theta^\pm, \bar{\theta}^\pm) = \bar{\phi}(x^\pm) + i\theta^+ \bar{\theta}^+ \partial_+ \bar{\phi} + i\theta^- \bar{\theta}^- \partial_- \bar{\phi} - \theta^+ \bar{\theta}^+ \theta^- \bar{\theta}^- \partial_+ \partial_- \bar{\phi} - \bar{\theta}^+ \bar{\psi}_+(x^\pm) - i\bar{\theta}^+ \theta^- \bar{\theta}^- \partial_- \bar{\psi}_+ - \bar{\theta}^- \bar{\psi}_- - i\bar{\theta}^- \theta^+ \bar{\theta}^+ \partial_+ \bar{\psi}_-$$

we need to extract the $\theta^2 \bar{\theta}^2$ terms from $\bar{\Phi}\Phi$. We have

$$\bar{\Phi}\Phi|_{\theta^4} = -\bar{\phi}\partial_+\partial_-\phi + \partial_+\bar{\phi}\partial_-\phi + \partial_-\bar{\phi}\partial_+\phi - \partial_+\partial_-\bar{\phi}\phi + i\bar{\psi}_+\partial_-\psi_+ - i\partial_-\bar{\psi}_+\psi_+ + i\bar{\psi}) = \partial_+\psi_- - i\partial_+\bar{\psi}_-\psi_- + |F|^2$$

Combining everything together, we find a component action

$$S[\phi, \psi, F] = \int_{\mathbb{R}^2} \left[\partial^\mu \bar{\phi} \partial_\mu \phi + i\bar{\psi}_+ \partial_+ \psi_- + i\bar{\psi}_- \partial_- \psi_+ + |F|^2 + FW'(\phi) - \psi_+ \psi_- W''(\phi) + \bar{F}\bar{W}'(\bar{\phi}) - \bar{\psi}_- \bar{\psi}_+ \bar{W}''(\bar{\phi}) \right] d^2x$$

with the kinetic terms coming from $|\Phi|_{\theta^4}^2$ nad the potential terms coming form $W(\Phi)$ and $\bar{W}(\bar{\Phi})$. The field F is auxiliary (i.e. its e.o.m are purely algebraic), so we can eliminate it using its e.o.m

$$\begin{aligned}F + \bar{W}'(\bar{\phi}) &= 0 \\ \Rightarrow F &= -\frac{\partial \bar{W}}{\partial \bar{\phi}}\end{aligned}$$

This gives us the interactions

$$\int -|W'(\phi)|^2 d^2x = \int V(\phi) d^2x$$

giving a potential $V(\phi) = |W'(\phi)|^2$ for the scalars.

8.4 Symmetries of the WZ model

By construction the model is invariant under supersymmetry transformations. The Noether currents for the supersymmetry are G_\pm^μ

$$\begin{aligned}G_\pm^0 &= 2\partial_\pm \bar{\psi} \psi_\pm \pm i\bar{\psi}_\pm F \\ G_\pm^i &= \mp 2\partial_\pm \bar{\phi} \psi_\pm + i\bar{\psi}_\mp F\end{aligned}$$

and similarly for \bar{G}_\pm^μ . The corresponding charge is

$$Q_\pm = \int_{\mathbb{R}^1} G_\pm^0 dx^1$$

Notice that the G_{\pm}^{μ} have spin $\frac{3}{2}$, so the charge are both spin $\frac{1}{2}$, i.e. $Q_{\pm} \rightarrow e^{\mp\frac{1}{2}} Q_{\pm}$. Consider the $U(1)_A$ transformation acting as

$$\Phi(x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm}, \bar{\theta}^{\pm}) \rightarrow \text{bar}\Phi(x^{\pm}, e^{\mp i\alpha} \theta^{\pm}, e^{\pm i\alpha} \bar{\theta}^{\pm})$$

which leaves $\theta^+ \theta^-$ invariant. Then $W(\Phi)|_{\theta^2}$ is likewise invariant, as is $\bar{\Phi}\Phi|_{\theta^4}$, so these transformations are also symmetries. In terms of the component fields we can equivalently think of these as

$$\begin{aligned}\phi &\rightarrow \phi \\ \psi_{\pm} &\rightarrow e^{\mp i\alpha} \psi_{\pm} \\ F &\rightarrow F\end{aligned}$$

The corresponding Noether charge is

$$F_A = \int_{\mathbb{R}^1} [\bar{\psi}_+ \psi_+ - \bar{\psi}_- \psi_-] dx^1$$

Now consider the $U(1)_V$ transformations $\Phi(x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm}) \rightarrow \Phi(x^{\pm}, e^{-i\beta} \theta^{\pm}, e^{i\beta} \bar{\theta}^{\pm})$ where θ^+, θ^- transform together, and we allow the whole superfield to have charge q . In this case, $\theta^+ \theta^- \rightarrow e^{-2i\beta} \theta^+ \theta^-$ whereas $\theta^2 \bar{\theta}^2 \rightarrow \theta^2 \bar{\theta}^2$, so the Kähler term is invariant for any q , whereas the superpotential term will only be invariant if W itself has charge 2 ($W \rightarrow e^{2-\beta} W$). In particular, for monomial $W(\Phi) = c\Phi^k$, we have $U(1)_V$ symmetry iff we assign charge $q = \frac{2}{k}$ to Φ . At the level of the component fields, these transformations can be taken to act as

$$\begin{aligned}\phi &\rightarrow e^{\frac{2i}{k}\beta} \phi \\ \psi_{\pm} &\rightarrow e^{(\frac{2}{k}-1)i\beta} \psi_{\pm} \\ F &\rightarrow e^{(\frac{2}{k}-1)i\beta} F\end{aligned}$$

Let

$$W(\Phi, M, \Lambda) = \frac{1}{2} M \Phi^2 + \frac{1}{3} \Lambda \Phi^3$$

W_{eff} must

- be Holomorphic
- have $U_V(1)$ charge 2 and be invariant under $\Phi \rightarrow e^{i\alpha} \Phi, M \rightarrow e^{-2i\alpha} M, \Lambda \rightarrow e^{-3i\alpha} \Lambda$.
- reduce to W as $\Lambda \rightarrow 0$

This first two conditions fix

$$W_{eff}(\Phi, M, \Lambda) = M \Phi^2 f\left(\underbrace{\frac{\Phi \Lambda}{M}}_{=t}\right)$$

where $f(t)$ must be holomorphic in t , and regular as $t \rightarrow 0$ and $\frac{f(t)}{t}$ regular as $t \rightarrow \infty$. Thus we must have $f(t) = a + b(t)$ and the final condition fixed $a = \frac{1}{2}, b = \frac{1}{3}$, hence

$$W_{eff}(\Phi, M, \Lambda) = \frac{1}{2}M\Phi^2 + \frac{1}{3}\Lambda\Phi^3$$

Finally, we freeze the superfield (M, Λ) so their VEVs (m, λ) be sending $\epsilon \rightarrow 0$ in the kinetic terms

$$\frac{1}{\epsilon}\overline{M}M + \frac{1}{\epsilon}\overline{\Lambda}\Lambda$$

The value of ϵ also can't affect W_{eff} because we can promote $\frac{1}{\epsilon}$ to a real superfield and the superpotential can't depend on real superfields (only chiral super fields). More generally, the quantum superpotential is always independent of couplings appearing only in the Kähler potential $K(\Phi, \overline{\Phi})$. Hence

$$W_{eff}(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}\lambda\Phi^3 = W(\Phi)$$

so receives no quantum corrections. However, the Kähler potential *does* generically get quantum corrections. In particular the kinetic terms can receive correction, so there can be non-trivial wavefunction renormalisation

$$\Phi_r = Z_\Phi^{\frac{1}{2}}\Phi$$

i.e.

$$\partial^\mu \overline{\phi} \partial_\mu \phi \rightarrow Z_\phi \partial^\mu \overline{\phi} \partial_\mu \phi = \partial^\mu \overline{\phi}_r \partial_\mu \phi_r$$

In terms of the renormalised fields, we have

$$W_{eff}(\Phi_r) = \frac{1}{2}m_r\Phi_r^2 + \frac{1}{3}\lambda_r\Phi_r^3$$

where

$$m_r = Z_\Phi^{-1}m$$

$$\lambda_r = Z_\Phi^{-\frac{1}{2}}$$

8.5 Kähler Geometry

A **Kähler manifold** is a manifold M with three compatible structures:

- Riemannian metric g
- Positive symplectic form ω
- Complex structure J

where

- A 2 form $\omega \in \Omega^2(M)$ is **symplectic** if

- $d\omega = 0$, i.e. if $\omega = \omega_{ij}(x)dx^i \wedge dx^j$ then $\partial_{[i}\omega_{jk]} = 0$
- ω is non-degenerate, i.e. $\forall X \neq 0, \exists Y \text{ s.t. } \omega(X, Y) \neq 0$

- An **almost complex structure** J is a map

$$J : TM \rightarrow TM$$

s.t. $J^2 = -1$. e.g. on \mathbb{R}^2 $J = \epsilon_{ab}$. We define the holomorphic and antiholomorphic **tangent bundles** on M by

$$T^{(1,0)}M = \{ X \in TM \otimes \mathbb{C} : \frac{1}{2}(1 - iJ)X = X \}$$

$$T^{(0,1)}M = \{ X \in TM \otimes \mathbb{C} : \frac{1}{2}(1 + iJ)X = X \}$$

An almost complex structure J is said to be **integrable** if

$$\forall X, Y \in TM \otimes \mathbb{C}, \frac{1 + iJ}{2} \left[\frac{1 - iJ}{2}X, \frac{1 - iJ}{2}Y \right] = 0$$

i.e. the Lie bracket of two holomorphic vector fields is holomorphic. Taking the real and imaginary parts of these equations, it's equivalent to

$$N(X, Y) = -J^2([X, Y]) + J([JX, Y] + [X, JY]) - [JX, JY] = 0$$

Here $N(X, Y)$ is known as the **Nijenhuis Tensor**. Note

$$N(fX, gY) = fgN(X, Y)$$

J is a complex structure iff it is an integrable almost complex structure.

Theorem 8.1 (Newlander Nirenberg). *If any real manifold has $N(X, Y) = 0$ then \exists complex coordinates $x^i \rightarrow (z^a, \bar{z}^{\bar{a}})$ on any patch $U \subset M$, and transition function on overlaps are purely holomorphic.*

If J is a complex structure, we can split

$$TM \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M$$

globally. Similarly, we split

$$T^*M \otimes \mathbb{C} = T^{*(1,0)}M \oplus T^{*(0,1)}M$$

where $T_p^{*(1,0)}M$ is the dual vector space to $T_p^{(1,0)}M$ for each $p \in M$. Likewise we can split

$$\Omega^k(M, \mathbb{C}) = \bigoplus_{k=p+q} \Omega^{(p,q)}(M)$$

where

$$\eta \in \Omega^{(p,q)}(M) \Leftrightarrow \eta(z, \bar{z}) = \eta_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q}(z, \bar{z}) dz^{a_1} \wedge \dots \wedge dz^{a_p} \wedge d\bar{z}^{\bar{b}_1} \wedge \dots \wedge d\bar{z}^{\bar{b}_q}$$

We also have $d : \Omega^k \rightarrow \Omega^{k+1}$ and so on a ocmplex manifold split as $d = \partial + \bar{\partial}$ where

$$\begin{aligned}\partial : \Omega^{(p,q)} &\rightarrow \Omega^{(p+1,q)} \\ \bar{\partial} : \Omega^{(p,q)} &\rightarrow \Omega^{(p,q+1)}\end{aligned}$$

Also,

$$0 = d^2 = \partial^2 + (\partial\bar{\partial} + \bar{\partial}\partial) + \bar{\partial}^2$$

As we must have $\partial^2 = 0 = \bar{\partial}^2$, $\partial\bar{\partial} + \bar{\partial}\partial$ separately.