

Algebraic Topology Example Sheet 2

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1 Question 1

1.1 (i) and (ii)

Proposition 1.1.

$$H_n(X \sqcup Y) \cong H_n(X) \oplus H_n(Y)$$

Proof. Recall that we have universal properties defining the direct sum of A, B abelian groups and the disjoint union of X, Y topological spaces described by the diagrams:

$$\begin{array}{ccc} A & \xrightarrow{i_A} & A \oplus B \xleftarrow{i_B} B \\ & \searrow \phi & \downarrow \phi \oplus \psi \swarrow \psi \\ & & C \end{array} \quad \begin{array}{ccc} X & \xrightarrow{i_X} & X \sqcup Y \xleftarrow{i_Y} Y \\ & \searrow f & \downarrow f \sqcup g \swarrow g \\ & & Z \end{array}$$

Applying the map H_n , known to be a functor, to the right diagram gives

$$\begin{array}{ccccc}
H_n(X) & \xrightarrow{(i_X)_*} & H_n(X \sqcup Y) & \xleftarrow{(i_Y)_*} & H_n(Y) \\
& \searrow f_* & \downarrow (f \sqcup g)_* & \swarrow g_* & \\
& & H_n(Z) & &
\end{array}$$

As homology groups are abelian groups, by the universal property we are done. \square

Moreover, as a result of the uniqueness of the maps in the universal property we have by setting $\phi = f_*, \psi = g_*$,

$$(f \sqcup g)_* = f_* \oplus g_*$$

1.2 (iii)

Now with the notation, recall

$$\ker(\phi \oplus \psi) = \{(a, b) \in A \oplus B \mid \phi(a) + \psi(b) = 0_C\}$$

Suppose ψ is an isomorphism, then we may write

$$\ker(\phi \oplus \psi) = \{(a, \psi^{-1}(-\phi(a)))\}$$

which given an immediate isomorphism between A and $\ker(\phi \oplus \psi)$.

1.3 (iv)

Let $X_+ = X \sqcup \{+\}$.

Proposition 1.2.

$$\tilde{H}_n(X_+) = H_n(X)$$

Proof. We have now developed the machinery to calculate this simply. We define

$$\tilde{H}_n(X_+) = \ker(\gamma_*^{X_+} : H_n(X_+) \rightarrow H_n(+))$$

where $\gamma^{X_+} : X_+ \rightarrow \{+\}$ is the unique such continuous map. By our universal property we must have

$$\begin{aligned}
\gamma^{X_+} &= \gamma^X \sqcup \gamma^+ \\
\Rightarrow \gamma_*^{X_+} &= \gamma_*^X \oplus \gamma_*^+
\end{aligned}$$

Now $\gamma^+ = \text{id}_+ \Rightarrow \gamma_*^+ = \text{id}_{H_n(+)}$, an isomorphism, so

$$\tilde{H}_n(X_+) \cong H_n(X)$$

\square

2 Question 2

Note the definition of a Moore space

2.1 (i)

Example 2.1. S^n is a Moore space of type (\mathbb{Z}, n)

Example 2.2. $S^n \cup e^{n+1}$, a CW complex where the $n+1$ -cell is attached along a map of degree k is a Moore space of type (\mathbb{Z}_k, n) .

2.2 (ii)

To construct a general Moore space $M(G, n)$ for finitely generated G , we may use the following construction:

1. Take $\{g_1, \dots, g_i, g_{i+1}, \dots, g_{i+j}\}$ a generating set for G such that

$$\text{ord}(g_k) = \begin{cases} \infty & k = 1, \dots, i \\ d_{k-i} & k = i+1, \dots, i+j \end{cases}$$

That is,

$$G \cong \mathbb{Z}^i \oplus \mathbb{Z}_{d_1} \oplus \dots \oplus \mathbb{Z}_{d_j}$$

Note this is always possible of finitely generated abelian groups (with maybe i or j being zero).

2. Recall $\tilde{H}_n(X \vee Y) = \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$
3. Hence construct

$$X = (\vee_{a=1}^i S^n) \vee (S^n \cup_{f_1} e^{n+1}) \vee \dots \vee (S^n \cup_{f_j} e^{n+1})$$

where $f_k : S^n \rightarrow S^n$ are attaching maps with $\deg f_k = d_k$

The constructed X is Moore of type (G, n) .

2.3 (iii)

For this question I do not have a correct answer, but effectively I know I somehow want to find an exact sequence like

$$\dots \longrightarrow \tilde{H}_n(M(K, n)) \xrightarrow{\phi_*} \tilde{H}_n(M(G, n)) \longrightarrow \tilde{H}_n(C_\phi) \longrightarrow \tilde{H}_{n-1}(M(K, n)) \longrightarrow \dots$$

3 Question 3

3.1 (i)

We want to calculate $H_n(\mathbb{RP}^2)$, which we will do in two ways:

3.1.1 a)

Note that $\mathbb{RP}^2 = S^1 \cup_f e^2$ where the attaching map $f : S^1 \rightarrow S^1$ has degree 2. We immediately see from previous results in cellular homology that \mathbb{RP}^2 is a Moore space of type $(\mathbb{Z}_2, 1)$. As such

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

3.1.2 b)

Using the decomposition provided of $\mathbb{RP}^2 = U \cup V$, let $A = \overline{U}, B = \overline{V}$ and consider the Mayer Vietoris sequence for reduced homology. $C = A \cap B$ is the dashed circle, so $\tilde{H}_n(C) = \tilde{H}_n(S^1)$, A deformation retracts onto S^1 , so $\tilde{H}_n(A) = \tilde{H}_n(S^1)$, and finally B is contractible so $\tilde{H}_n(B) = 0$. Hence the sequence looks like

$$\dots \longrightarrow \tilde{H}_n(S^1) \xrightarrow{i_*} \tilde{H}_n(S^1) \xrightarrow{j_*} \tilde{H}_n(\mathbb{RP}^2) \xrightarrow{\partial} \tilde{H}_{n-1}(S^1) \longrightarrow \dots$$

Where $i : C \rightarrow A \rightarrow S^1, j : S^1 \rightarrow A \rightarrow \mathbb{RP}^2$ are both inclusion composed with deformation retraction (or its inverse). This sequence is only interesting in the case $n = 1$ in which we have

$$\mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \xrightarrow{j_*} \tilde{H}_1(\mathbb{RP}^2) \xrightarrow{\partial} 0$$

Then we have

$$\begin{aligned} \text{Im } j_* &= \ker \partial = \tilde{H}_1(\mathbb{RP}^2) \\ \ker j_* &= \text{Im } i_* = 2\mathbb{Z} \end{aligned}$$

where the second follows from the fact that our i ends up being degree 2, as the deformation retract onto a is degree 2. We then have the same result as before (after 1st iso thm application) that

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

3.2 (ii)

We want to calculate $H_n(K)$, which we will do in two ways:

3.2.1 a)

Note that with the given construction we can write

$$K = S^1 \vee (S^1 \cup_f e^2)$$

as the attaching map to a has degree 0, and f is the attaching map corresponding to f , which is of degree 2. As such K is a Moore space of type $(\mathbb{Z} \oplus \mathbb{Z}_2, 1)$, so

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

3.2.2 b)

By following similar procedure as in part i), we get the only interesting part of the Mayer Vietoris sequence to be

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \tilde{H}_1(K) \longrightarrow 0$$

This replicates our result from above.