Algebraic Topology:
All the results - None of the proofs

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Contents

1 Introduction

In every mathematical physicist's life there comes a point when they need to know some algebraic topology, and there are many great resources to learn this from. These will be my personal notes which will accumulate many resources, which I will try to reference, though I doubt I will give when each one was used. The current list is

- *Diferential Forms in Algebraic Topology* (Bott, Tu)
- nlab
- *Algebraic Topology* and *Vector Bundles and K-Theory* (Hatcher)
- often wikipedia
- this blog post

I hope to come back some day and fill in all the proofs, but in the name of current expedience I will avoid this.

2 Preliminaries

This section will contain small, relatively self-contained bits of knowledge which will be useful to know throughout.

2.1 General Topology

Definition 2.1. *An **open cover** of a space X is a collection of open sets $U_\alpha \subset X$ s.t. $X = \bigcup_\alpha U_\alpha$*

Definition 2.2. *A map of topological spaces $f : X \to Y$ is **proper** if $\forall K \subset Y$ compact, $f^{-1}(K) \subset X$ is compact.*

Proposition 2.3. *The image of a proper map in a locally-compact Hausdorff space is closed*

Proposition 2.4. *A compact subspace of a Hausdorff space is closed.*

2.2 Differential Topology

Definition 2.5. *Given a manifold M with atlas $\{(U_\alpha, \phi_\alpha)\}$ the **cocycle condition** is that on triple intersect $U_\alpha \cap U_\beta \cap U_\gamma$*

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

Definition 2.6. *A **critical point** of a smooth map of manifolds $f : M \to N$ is $p \in M$ s.t. $(f_*)_p : T_pM \to T_{f(p)}N$ is not surjective. A **critical value** is the image of a critical point*

Theorem 2.7 (Sard)**.** *The set of critical values of a smooth map has measure 0.*

Definition 2.8. *A **good cover** of an n-dimensional manifold is an open cover where all finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ are diffeomorphic to $\mathbb{R}^n$. A manifold with a good cover is said to be of **finite type**.*

Theorem 2.9. *Every manifold is of finite type, and moreover if it is compact, the cover can be chosen to be finite.*

Proof. Use a cover provided by taking geodesic balls at each point. The second point follows from the definition of compact. □

Definition 2.10. *A **partition of unity** on a manifold M is a collection of non-negative C^∞ functions $\{\rho_\alpha\}$ s.t.*

- *Each $p \in M$ has a neighbourhood where $\sum \rho_\alpha$ is a finite sum*
- *$\sum \rho_\alpha$*

Remark. *It can be, on occasion, useful to know at least one bump function which can be argued can be put into a partition of unity. My favourite is $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} e^{\frac{-1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

Definition 2.11. *Given a manfiold with open cover $\{U_\alpha\}$, a partition of unity $\{\rho_\alpha\}$ s.t $\text{supp}(\rho_\alpha) \subset U_\alpha$ is called **subordinate** to $\{U_\alpha\}$.*

Proposition 2.12. *Given a manfiold with open cover $\{U_\alpha\}$:*

- *$\exists$ a partition of unity subordinate to it.*
- *$\exists$ a partition of unity $\{\rho_\beta\}$ s.t. each ρ_β has compact support and $\exists \alpha$ s.t. $\text{supp}(\rho_\beta) \subset U_\alpha$*

Proposition 2.13. *Every manifold is paracompact.*

2.3 Category Theory

The category preliminary is contained in my algebraic geometry notes.

2.4 Orientation and Integration

We may use partitions of unity to define the integral of a top form over a manifold M.

Theorem 2.14 (Stokes' Threom)**.** *If ω is an $(n-1)$-form with compact support on an oriented n-dimensional manifold M*

$$\int_M d\omega = \int_{\partial M} \omega$$

Proposition 2.15. *Given a diffeomorphim $T : \mathbb{R}^n \to \mathbb{R}^n$ and $\omega \in \Omega^n(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} T^*\omega = \text{sgn}\, J(T) \int_{\mathbb{R}^n} \omega$$

where $J(T)$ is the determinant of the Jacobian of the transform.

Remark. *Note the above prop makes sense, as if T is invertible, then $\text{sgn}\, J(T)$ never changes sign as it cannot be 0.*

Definition 2.16. *A diffeomorphism for which $\text{sgn}\, J(T) = 1$ is call **orientation preserving***

Definition 2.17. *Let M be a smooth manifold with atlas $\{(U_\alpha, \phi_\alpha)\}$. We call the atlas **oriented** if all the transition functions $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ are orientation preserving. A manifold is called **orientable** if it has an oriented atlas.*

Proposition 2.18. *An n-dimensional manifold is orientable iff it has a global nowhere-vanishing n-form.*

Example 2.19. *$\mathbb{R}^n$ is orientable as it has the global non-vanishing n-form $dx_1 \wedge \cdots \wedge dx_n$ where $\{x_i\}$ are cartesian coordinates.*

2.5 Bundle Theory

Definition 2.20. *Given manifolds M, N, fibre bundle $E \to M$, and map $f : N \to M$, we define the* ***pullback bundle of E by f*** *to be*

$$f^*E \equiv \{(n, e) \mid f(n) = \pi(e)\} \subset N \times E$$

the bundle with base N, with the natural projection onto the first component p_1.

Lemma 2.21. *The pullback bundle is the unique maximal subset of $N \times E$ s.t.*

$$\begin{tikzcd} f^*E \arrow[r, "p_2"] \arrow[d, "p_1"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

commutes.

Exercise 2.22. *Look at how this relates to the categorical concept of a pullback.*

Example 2.23. *The pullback by the identity map is isomorphic to the bundle itself, i.e.*

$$\mathrm{id}^* E \leftrightarrow E$$
$$(\pi(e), e) \leftrightarrow e$$

Proposition 2.24. *If $\{g_{\alpha\beta}\}$ are the transition functions for $E \to M$ wrt to the cover $\{U_\alpha\}$ of M, then $\{f^*g_{\alpha\beta} = g_{\alpha\beta} \circ f\}$ are the transition functions for $f^*E \to N$ wrt to the cover $\{f^{-1}U_\alpha\}$*

Corollary 2.25. *The pullback of an oriented vector bundle is oriented.*

Lemma 2.26. *Given $g : M'' \to M'$, $f : M' \to M$, $(f \circ g)^*E = g^*(f^*E)$.*

Lemma 2.27. *The pullback of a trivial bundle is trivial. i.e. If $E = F \times M$, then for $f : N \to M$, $f^*E = N \times F$.*

Remark. *If we let $\mathrm{Vect}_k(M)$ be the isomorphism classes of rank-k real vector bundles, and $\mathrm{Vect}_k(f) = f^*$ be the pullback of vector bundles along f, then we get a contravariant functor from manifolds with smooth maps to pointed sets with basepoint preserving maps, where the basepoint of $\mathrm{Vect}_k(M)$ is the trivial bundle over M.*

Proposition 2.28. *If $f, g : M \to N$ are homotopic then f^*E and g^*E are isomorphic.*

Remark. *This result holds true more generally for a paracompact topological space M. As all manifolds are paracompact, this holds in our case.*

Corollary 2.29. *A bundle with contractible base is trivial.*

Proof. Suppose we have

$$M \underset{g}{\overset{f}{\rightleftarrows}} *$$

s.t. $g \circ f$ is homotopic to id_M. Then

$$E \cong (g \circ f)^* E = f^* (g^* E)$$

As g^*E is a bundle over a point it is necessarily trivial, and so $f^*(g^*E)$ is also. ☐

Example 2.30. *We want to work out* $\mathrm{Vect}_k(S^1)$*. Intuition might tell us that* $Vect_1(S^1) = \mathbb{Z}$ *(number of twists of a mobius like band, where 0 is the trivial bundle) so we have a starting point. This will not turn out to be the case.*
To specify a isomorphism class of rank k-vector bundles, we can take the cover of S^1 *with only two open sets and then an element of* $\mathrm{Vect}_k(S^1)$ *is specified by two elements* $g, h \in GL(k, \mathbb{R})$ *up to conjugation. We can use a reduction of structure bundle to only ask about elements of* $O(k)$*.*

Definition 2.31. *Given* $S \subset M$ *a* k*-dimsneionsal submanifold of an oriented* n*-dimensional manifold* M*, a* ***tubular neighbourhood*** *of* S *in* M *is* U *an open neighbourhood of* S *in* M *diffeomorphic to a rank-*$(n-k)$ *vector bundle over* S*.*

Example 2.32. *The 'namesake' example is when* S *is a line in* $M = \mathbb{R}^3$*, then we can attach an open disk to each point on* S *so the disk is normal to* S*. The union of these disks forms a 'tube' whose centre line is* S*.*

Definition 2.33. *Given* $S \subset M$ *a submanifold, the* ***normal bundle*** *of* S *in* M *is* $N = N_{S/M}$ *defined by the exact sequence*

$$0 \to TS \to TM|_S \to N \to 0$$

We may also notate the normal bundle as $T^{\perp}S$*.*

Example 2.34. *The normal bundle to the bases of a vector bundle (injected via the zero section) is the bundle itself.*

Theorem 2.35. *Every submanifold has a tubular neighbourhood* T *and* T *is diffeomorphic to the normal bundle.*

We can give every real vector bundle $E \to M$ a Riemmanian metric as follows: Let $\{U_\alpha\}$ be an open cover of M which trivialises E, on each U_α choose a frame of $E|_{U_\alpha}$ and declare it to be orthonormal (givng a Riemannian metric $\langle \cdot, \cdot \rangle_\alpha$ here) and then given a partition of unity subordinate to U_α, take

$$\langle \cdot, \cdot \rangle = \sum_\alpha \langle \cdot, \cdot \rangle_\alpha$$

3 de-Rham Theory

3.1 General Cohomology

We will start with some basic definitions and results, with very few proofs for now.

Definition 3.1. *A direct sum of vector spaces* $C = \oplus_{k \in \mathbb{Z}} C^k$ *is called a* ***differential complex*** *if there are homomorphisms*

$$\cdots \to C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \to \dots$$

s.t. $d^2 = 0$

Definition 3.2. *The* ***cohomology*** *of* C *is* $H(C) = \oplus_{k\in\mathbb{Z}} H^k(C)$ *where*

$$H^k(X) = {}^{\left(\ker d \cap C^k\right)}\big/_{\left(\operatorname{Im} d \cap C^k\right)}$$

Definition 3.3. *A (linear) map* $f : A \to B$ *between two differential complexes is called a* ***chain map*** *if it commutes with the differential operator, i.e.* $f \circ d_A = d_B \circ f$.

Proposition 3.4. *A short exact sequence of chain maps*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

induces a long exact sequence of cohomology

$$\begin{tikzcd}
\dots \arrow[r] & H^k(A) \arrow[r, "f^*"] & H^k(B) \arrow[r, "g^*"] & H^k(C) \arrow[dll, rounded corners, "d^*"] \\
& H^{k+1}(A) \arrow[r] & \dots &
\end{tikzcd}$$

Proof. Consider the diagram obtained

$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^{k+1} \arrow[u] \arrow[r, "f"] & B^{k+1} \arrow[u] \arrow[r, "g"] & C^{k+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A^k \arrow[u, "d_A"] \arrow[r, "f"] & B^k \arrow[u, "d_B"] \arrow[r, "g"] & C^k \arrow[u, "d_C"] \arrow[r] & 0 \\
& \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

f induces a well defined map on the cohomology f^* as $f(a + d_A\omega) = f(a) + (f \circ d_A)(\omega) = f(a) + (d_B \circ f)(\omega)$, so $[f(a)] = [f(a + d_A\omega)]$ in $H^\bullet(B)$. Likewise for g.
Pick $c \in C^k$, then by surjectivity of g, $\exists b \in B^k$, $g(b) = c$. If $dc = 0$, we can say $g(db) = dg(b) = dc = 0$, so by exactness $db = f(a)$ for some $a \in A^{k+1}$. As such we define $d^* : H^k(C) \to H^{k+1}(A)$ by $d^*[c] = [a]$. ☐

3.2 The de-Rham complex

I have seen a lot of de-Rham definitions in the past, so I will come back and fill this in when I have time.

Definition 3.5. *The* ***de-Rham complex*** $\Omega^\bullet_{dR}$ *is a differential complex where* $C^k = \Omega^k_{dR}$ *are the k-forms and d is the exterior derivative.*

Remark. *The de-Rham complex for Euclidean spaces admits a functorial description as* $\Omega^\bullet$ *is the unique contravariant functor from Euclidean spaces with smooth maps to commutative differential graded algebras s.t.* Ω^0 *is the pullback of functions. The fact that the de-Rham complex admits this functorial description tells us that pullback commutes with the exterior derivative. This definition can be extended to the category of differentiable manifolds.*

Remark. *As there a no k forms on a manifold M when $k > \dim M$, $H^k_{dR}(M) = 0$.*

Example 3.6. *The cohomology of the de-Rham complex is sometimes denoted as H^k_{dR}. When the context makes clear that we are consider the de-Rham cohomology we omit the dR.*
Consider the point space $*$. Functions on $*$ are specified by points in $\mathbb{R}$, and are all constant, so closed. There can be no cohomology higher than the dimension of the space, so we get

$$H^k_{dR}(*) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

Closed 0-forms on $\mathbb{R}$ are again constant functions. Further, any one form $\omega = g(x)dx$ can be written as df for $f(x) = \int_0^x g(u)du$ so we get

$$H^k_{dR}(\mathbb{R}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

If $U \subset \mathbb{R}$ is a union of m dijoint open intervals in $\mathbb{R}$ we have

$$H^k_{dR}(U) = \begin{cases} \mathbb{R}^m & k = 0 \\ 0 & k > 0 \end{cases}$$

Definition 3.7. *The **de-Rham complex with compact support** is the differential complex of the k-forms with compact support, denoted $\Omega^\bullet_c(M)$. The cohomology is denotes $H^\bullet_c(M)$.*

Proposition 3.8. *If M is compact, $H^k_c(M) = H^k_{dR}(M)$.*

Example 3.9. *We can consider the same cohomologies as above to get*

$$H^k_c(*) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

as all functions on $*$ are constant.
There are no constant functions on $\mathbb{R}$ with compact support except for the zero map, so $H^0_c(\mathbb{R}) = 0$. The only other non-trivial cohomolgy is $H^1_c(\mathbb{R})$. Our previous construction (starting the integral at $-\infty$ which we can do as g has compact support) of an f s.t. $df = g(x)dx$ works iff f gets compact support, and this happens where the integral $\int_{\mathbb{R}} g(x)\,dx = 0$ so

$$H^1_c(\mathbb{R}) = \Omega^1_c(\mathbb{R}) \Big/ \ker \int_{\mathbb{R}}$$

Remark. *$\Omega^\bullet_c$ also admits a functorial description, but we must restrict from all smooth maps as pullbacks of functions with compact support might not have compact support. $\Omega^\bullet_c$ can be made either as*

- *a contravariant functor when maps are restricted to be proper*
- *a covariant functor when maps are restricted to be inclusions of open sets.*

3.3 The Poincaré lemma

We will now build up a bit of theory culminating in a full understanding of de-Rham cohomology of $\mathbb{R}^n$. We start by considering the maps

$$\mathbb{R}^n \times \mathbb{R} \underset{s}{\overset{\pi}{\rightleftarrows}} \mathbb{R}^n$$

given by $\pi(x,t) = x$, $s(x) = (x,0)$. Under the functor $\Omega^\bullet$ we get

$$\Omega^\bullet(\mathbb{R}^n \times \mathbb{R}) \underset{s^*}{\overset{\pi^*}{\leftrightarrows}} \Omega^\bullet(\mathbb{R}^n)$$

Proposition 3.10. *The induced maps on cohomology*

$$H^k_{dR}(\mathbb{R}^n \times \mathbb{R}) \underset{s^*}{\overset{\pi^*}{\leftrightarrows}} H^k_{dR}(\mathbb{R}^n)$$

are inverse isomorphisms.

Proof. Certainly $\pi \circ s = \mathrm{id}_{\mathbb{R}^n} \Rightarrow s^* \circ \pi^* = \mathrm{id}_{\Omega^\bullet(\mathbb{R}^n)} \Rightarrow s^* \circ \pi^* = \mathrm{id}_{H^\bullet_{dR}(\mathbb{R}^n)}$. It remains to show $\pi^* \circ s^* = \mathrm{id}$ in cohomoloy. Note that it is not the identity on the complex, as for example $(\pi^* \circ s^*)f(x,t) = f(x,0)$. However it is sufficient to show $\exists K : \Omega^k(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{k-1}(\mathbb{R}^n \times \mathbb{R})$

$$1 - \pi^* \circ s^* = \pm(d \circ K \pm K \circ d)$$

As the RHS maps closed forms to exact ones, it induces 0 on the cohomology. finish constructing K. $\square$

Corollary 3.11 (Poincaré lemma)**.** *We have*

$$H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

Corollary 3.12. *Applying the proposition to an atlas for a manifold M we get*

$$H^k_{dR}(M \times \mathbb{R}) \cong H^k(M) \Rightarrow H^k(M \times \mathbb{R}^n) \cong H^k(M)$$

Corollary 3.13. *Homotopic maps induce the same cohomology map.*

Corollary 3.14. *The de-Rham cohomology is homotopy invariant.*

A similar result holds for compact de-Rham cohomology, namely

Lemma 3.15. $H^{k+1}_c(M \times \mathbb{R}) \cong H^k_c(M) \Rightarrow H^{k+l}(M \times \mathbb{R}^l) \cong H^k(M)$

Corollary 3.16 (Poincaré lemma for compact support)**.** *We have*

$$H^k_c(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \textit{otherwise} \end{cases}$$

3.4 Mayer-Vietoris

Write $M = U \cup V$ where M is a manifold and $U, V \subset M$ are open. We then get the inclusions

$$M \longleftarrow U \coprod V \underset{i_U}{\overset{i_V}{\leftleftarrows}} U \cap V$$

Applying the functor $\Omega^\bullet$ to this yields

$$\Omega^\bullet(M) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \underset{i_V^*}{\overset{i_U^*}{\rightrightarrows}} \Omega^\bullet(U \cap V)$$

Definition 3.17. *The **Mayer-Vietoris sequence** is the that obtained using the difference of the above two maps, that is*

$$0 \to \Omega^\bullet(M) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V) \to 0$$
$$(\omega, \tau) \mapsto \tau - \omega$$

Proposition 3.18. *The Mayer-Vietoris sequence is exact*

Proof. do this eventually, requires partitions of unity □

Proposition 3.19. *The Mayer-Vietoris sequence induces a long exact sequence of cohomology*

$$\begin{tikzcd} \dots \arrow[r] & H^k(M) \arrow[r] & H^k(U) \oplus H^k(V) \arrow[r] & H^k(U \cap V) \arrow[dll, "d^*"] \\ & H^{k+1}(M) \arrow[r] & \dots & \end{tikzcd}$$

Let us now see some uses of the machinery we have just developed:

Example 3.20. *Consider a circle S^1 and cover it with two open sets (north and south, slightly overlapping). The part of the sequence we care about is*

$$\begin{tikzcd} 0 \arrow[r] & H^0(S^1) \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[dll, "d^*"] \\ & H^1(S^1) \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Calling the map $\mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$ δ, we can see under δ, $(\omega, \tau) \mapsto (\omega - \tau, \omega - \tau)$. Hence $\dim \operatorname{Im} \delta = 1 \Rightarrow \dim \ker \delta = 1$. Counting dimensions of the maps we see

$$H^0(S^1) \cong \mathbb{R} \cong H^1(S^1)$$

We can extend this to calculate for any sphere. Covering S^n with U, V the north/south hemisphere respectively extended so they cover the equator, we get $U \cap V$ is homotopic to S^{n-1}, and U, V are contractible. Hence in the Mayer-Vietoris we get a sequence

$$\begin{tikzcd} \dots \arrow[r] & H^k(S^n) \arrow[r] & (\delta_{k0})(\mathbb{R} \oplus \mathbb{R}) \arrow[r] & H^k(S^{n-1}) \arrow[dll, "d^*"] \\ & H^{k+1}(S^n) \arrow[r] & \dots & \end{tikzcd}$$

This means that we have for $k > 0$, $H^k(S^{n-1}) \cong H^{k+1}(S^n)$ and

$$\begin{tikzcd} 0 \arrow[r] & H^0(S^n) \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & H^0(S^{n-1}) \arrow[dll, "d^*"] \\ & H^1(S^n) \arrow[r] & 0 & \end{tikzcd}$$

We can then prove by induction that for $n \geq 1$ $H^k(S^n) = \mathbb{R}$ if $k = 0, n$ and 0 otherwise.

Proposition 3.21. *If a manifold has a finite good cover, then its cohomology is finite dimensional.*

Proof. We will use proof by induction on the cardinality of the good cover, noting that if M is diffeomorphic to $\mathbb{R}^n$ then we have a cover given by M, and the result is true by the Poincarlemma.
We now note from the part of the Mayer-Vietoris sequence for $U \cup V$

$$\dots \to H^{k-1}(U \cap V) \xrightarrow{d^*} H^k(U \cup V) \xrightarrow{r} H^k(U) \oplus H^k(V) \to \dots$$

that

$$H^k(U \cup V) \cong \ker r \oplus \operatorname{Im} r \cong \operatorname{Im} d^* \oplus \operatorname{Im} r$$

(this is just the first isomorphisms theorem and exactness). Hence, if $H^k(U), H^k(V)$, and $H^k(U \cap V)$ are f.d. then so is $H^k(U \cup V)$.
Now suppose M has good cover $\{U_0, \dots, U_p\}$. Then $(U_0 \cup \dots \cup U_{p-1}) \cap U_p$ has a good cover

$$\{U_0 \cap U_p, \dots, U_{p-1} \cap U_p\}$$

By the induction hypothesis $(U_0 \cup \dots \cup U_{p-1}) \cap U_p$ has f.d. cohomology, and so does M from the Mayer-Vietoris (taking $U = U_0 \cup \dots \cup U_{p-1}$, $V = U_p$). $\square$

With this results, we can define a related quantity

Definition 3.22. *On an n-dimensional manifold with f.d. cohomology, the* ***Euler characterstic*** *of M is*

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k_{dR}(M)$$

We can also build a Mayer-Vietoris sequence for the functor $\Omega^\bullet_c$ taken to be covariant when restricted to inclusions. The image of the inclusion $j : U \hookrightarrow M$ under the functor is $j_* : \Omega^\bullet_c(U) \to \Omega^\bullet_c(M)$ which extends a form by 0. This gives the sequence

$$\Omega^\bullet_c(M) \xleftarrow{\text{sum}} \Omega^\bullet_c(U) \oplus \Omega^\bullet_c(V) \xleftarrow{-j_* \oplus j_*} \Omega^\bullet_c(U \cap V)$$

Proposition 3.23. *The Mayer-Vietoris sequence with compact support*

$$0 \longleftarrow \Omega^\bullet_c(M) \longleftarrow \Omega^\bullet_c(U) \oplus \Omega^\bullet_c(V) \longleftarrow \Omega^\bullet_c(U \cap V) \longleftarrow 0$$

is exact

Remark. *This is in the opposite direction to our other Mayer-Vietories sequence for standard de-Rham cohomology. The difference is from the functor being covariant.*

Proposition 3.24. *The Mayer-Vietoris sequence with compact support induces a long exact sequence of cohomology*

$$\begin{tikzcd}
\dots \arrow[r] & H^k_c(U \cap V) \arrow[r] & H^k_c(U) \oplus H^k_c(V) \arrow[r] & H^k_c(M) \arrow[dll, "d^*"] \\
& H^{k+1}_c(U \cap V) \arrow[r] & \dots &
\end{tikzcd}$$

Example 3.25. *We can use this Mayer-Vietoris sequence to calculte $H^\bullet_c(S^1)$, which we can check against $H^\bullet_{dR}(S^1)$ as they must be the same.*
Using the same cover as before we get the same part of the sequence

$$0 \to H^0_c(S^1) \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1_c(S^1) \to 0$$

and again the image and kernel in $\mathbb{R} \oplus \mathbb{R}$ are 1 dimensional.

Many of our other results have compact support analogues, e.g.

Proposition 3.26. *If a manifold has a finite good cover, then its compact-support cohomology is finite dimensional.*

We can say more about the relation between cohomology using that integration descends to cohomology, giving on oriented n-dimensional manifolds M a pairing

$$\int : H^k(M) \otimes H_c^{n-k}(M) \to \mathbb{R}$$

given by $(\omega, \tau) \mapsto \int_M \omega \wedge \tau$

Lemma 3.27. *The two Mayer-Vietoris sequences may be paired to gether to form the diagram*

$$\begin{tikzcd}
{} \arrow[r] & H^k(U\cup V) \arrow[r] & H^k(U)\oplus H^k(V) \arrow[r] & H^k(U\cap V) \arrow[r] & H^{k+1}(U\cup V) \arrow[r] & {} \\
 & \otimes & \otimes & \otimes & \otimes & \\
{} & H_c^{n-k}(U\cup V) \arrow[l] \arrow[d, "\int_{U\cup V}"] & H_c^{n-k}(U)\oplus H_c^{n-k}(V) \arrow[l] \arrow[d, "\int_U + \int_V"] & H_c^{n-k}(U\cap V) \arrow[l] \arrow[d, "\int_{U\cap V}"] & H_c^{n-k-1}(U\cup V) \arrow[l] \arrow[d, "\int_{U\cup V}"] & {} \arrow[l] \\
 & \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} &
\end{tikzcd}$$

sign-commutative in the sense that

$$\int_{U\cap V} \omega \wedge d_* \tau = \pm \int_{U\cup V} (d^*\omega) \wedge \tau$$

Remark. *The above lemma is equivalent to saying we get the sign-commutative diagram*

$$\begin{tikzcd}
{} \arrow[r] & H^k(U\cup V) \arrow[r] \arrow[d] & H^k(U)\oplus H^k(V) \arrow[r] \arrow[d] & H^k(U\cap V) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \left(H_c^{n-k}(U\cup V)\right)^* \arrow[r] & \left(H_c^{n-k}(U)\right)^* \oplus \left(H_c^{n-k}(V)\right)^* \arrow[r] & \left(H_c^{n-k}(U\cap V)\right)^* \arrow[r] & {}
\end{tikzcd}$$

Proposition 3.28 (Poincaré duality)**.** *If M is an n-dimensional orientable manifold and has a finite good cover,*

$$H^k(M) \cong \left(H_c^{n-k}(M)\right)^*$$

Proof. Again proceed by induction on the size of the good cover, noting it is true for $\mathbb{R}^n$. The above lemma used with the five lemma gives that if Poincaré duality holds for $U, V, U\cap V$ then it holds for $U \cup V$. ☐

Remark. *This result can be extended to any orientable manofolds*

Corollary 3.29. *The Euler characteristic any odd-dimensional, compact, orientable manifold is 0*

Proof. If M is compact orientable n-dimensional, then

$$\dim H^k(M) = \dim H_c^{n-k}(M) = \dim H^{n-k}(M)$$

so if n odd

$$\begin{aligned}
\chi(M) &= \sum_{k=0}^{n}(-1)^k \dim H^k(M) \\
&= \sum_{k=0}^{\frac{n-1}{2}}(-1)^k \dim H^k(M) + \sum_{k=\frac{n+1}{2}}^{n} (-1)^k \dim H^{n-k}(M) \\
&= \sum_{k=0}^{\frac{n-1}{2}}(-1)^k \dim H^k(M) + \sum_{k=\frac{n-1}{2}}^{0} (-1)^{n-k} \dim H^{k}(M) \\
&= [1+(-1)^n]\sum_{k=0}^{\frac{n-1}{2}}(-1)^k \dim H^k(M)
\end{aligned}$$

As n odd, $1+(-1)^n = 0$. □

3.5 Cohomology of Bundles

Proposition 3.30. *Let M be a manifold and $\{U_\alpha\}$ a collection of open subsets. Then*

$$H^k_{dR}\left(\coprod_\alpha U_\alpha\right) = \prod_\alpha H^k_{dR}(U_\alpha)$$
$$H^k_c\left(\coprod_\alpha U_\alpha\right) = \bigoplus_\alpha H^k_c(U_\alpha)$$

Proposition 3.31 (Künneth Formula)**.** $H^k(M \times F) = \oplus_{p+q=k} H^p(M) \otimes H^q(F)$

With this result we can start to develop more specialised results for cohomology on bundles. For important definitions on bundles look at my EKC of Gauge theory notes.
The Künneth formula has a specialisation for fibre bundles.

Theorem 3.32 (Leray-Hirsch)**.** *Let $E \to M$ be a fibre bundle with fibre F. If there are global cohomology classes $e_1, \ldots, e_r$ on E which, restricted to each fibre, freely generate the cohomology of F, then $H^k(M)$ is a free module over $H^k(M)$ i.e*

$$H^k(E) \cong H^k(M) \otimes \mathbb{R}\{e_1, \ldots, e_r\} \cong H^k(M) \otimes H^k(F)$$

Proposition 3.33. *If $E \to M$ is a vector bundle, then $H^k(E) \cong H^k(M)$.*

Proof. Deformation retract onto the zero section of the bundle and then use homotopy invariance of cohomolgy □

Remark. *how does this agree with Leray Hirsch?*

Proposition 3.34. *Let $E \to M$ be a rank-k vector bundle, where E, M are orientable and of finite type. Then $H^{p+k}_c(E) \cong H^p_c(M)$.*

Proof. Let $\dim M = n$. Then

$$\begin{aligned} H^p_c(E) &\cong \left(H^{n+k-p}(E)\right)^* \quad \text{(Poincaré duality)} \\ &\cong \left(H^{n+k-p}(M)\right)^* \quad \text{(homotopy invariance)} \\ &\cong H^{p-k}_c(M) \quad \text{(Poincaré duality)} \end{aligned}$$

□

Remark. *This result can be generalised to remove the orientability assumption on M, using more machinery.*

In vector bundles there is an additional type of de-Rham cohomology we can look at

Definition 3.35. *The* ***de-Rham complex with compact vertical support*** *is the differential complex $\Omega^\bullet_{cv}$. of k-forms with compact support in the fibres. The associated cohomology is notated as $H^\bullet_{cv}$*

Definition 3.36. *We define the* ***integration along the fibre*** *map for a rank-k vector bundle $\pi : E \to M$ to be*

$$\begin{aligned} \pi_* : \Omega^{p+k}_{cv}(E) &\to \Omega^p(M) \\ (\pi^*\phi) \wedge f(x,t)dt_{i_1} \wedge \cdots \wedge dt_{i_r} &\mapsto 0 \quad (r < k) \\ (\pi^*\phi) \wedge f(x,t)dt_1 \wedge \cdots \wedge dt_n &\mapsto \phi \int_{\mathbb{R}^n} f(x,t)dt_1 \dots dt_n \end{aligned}$$

where t_i are coordinates on the fibre, and $\phi \in \Omega^\bullet(M)$, so $\pi^*\phi$ is the pullback to E.

Proposition 3.37 (Projection Formula)**.** *Let $\pi : E \to M$ be an oriented rank-k vector bundle, $\tau \in \Omega^p(M)$ and $\omega \in \Omega^q_{cv}(E)$. Then*

$$\pi_* \left((\pi^*\tau) \wedge \omega\right) = \tau \wedge \pi_*\omega$$

Moreover, if $p + q = \dim E$, then

$$\int_E (\pi^*\tau) \wedge \omega = \int_M \tau \wedge \pi_*\omega$$

Theorem 3.38 (Thom Isomorphism)**.** *If $E \to M$ is an orientable rank-k vector bundle with base manifold of finite type then*

$$H^{p+k}_{cv}(E) \cong H^p(M)$$

where the isomorphism $H^{p+k}_{cv}(E) \to H^p(M)$ is π_*, and the isomorphism $\mathcal{T} : H^p(M) \to H^{p+k}_{cv}(E)$ is called the ***Thom isomorphism***.

Remark. *The theorem is actually true for arbitrary manifolds.*

Definition 3.39. *The image of the constant function $1 \in H^0(M)$ under the Thom isomorphism $\mathcal{T} : H^p(M) \xrightarrow{\cong} H^{p+k}_{cv}(E)$ is called the* ***Thom class*** *of the oriented vector bundle E. We write it as $\Phi = \Phi(E)$.*

Lemma 3.40. *The Thom isomorphism is given explicitly by*

$$\mathcal{T}\omega = \pi^*\omega \wedge \Phi$$

for $\omega \in H^p(M)$.

Proof. Using the projection formula and that $\pi_*\Phi = 1$ we get

$$\pi_*\left((\pi^*\omega) \wedge \Phi\right) = \omega \wedge \pi_*\Phi = \omega$$

$\square$

Proposition 3.41. *The Thom class Φ of a rank-k vector bundle E can be uniquely characterised as the cohomology class in $H^n_{cv}(E)$ which restricts to the generator of $H^k_c(F)$ on each fibre.*

Proof. Since $\pi_*\Phi = 1$, $\Phi|_F$ is a bump form that integrates to 1, and any such Φ satisfies

$$\pi_*\left((\pi^*\omega) \wedge \Phi\right) = \omega \wedge \pi_*\Phi = \omega$$

$\square$

Proposition 3.42. *If we have two oriented vector bundles $E, F \to M$ with projections π_E, π_F then*

$$\Phi(E \oplus F) = \pi_E^*\Phi(E) \wedge \pi_F^*\Phi(F)$$

Proof. Note

$$H^{k+l}_c(\mathbb{R}^k \times \mathbb{R}^l) \cong H^k_c(\mathbb{R}^k) \otimes H^l_c(\mathbb{R}^l)$$

and this isomorphism is given by the wedge products of the generators. $\square$

3.6 The Poincaré Dual and Thom Isomorphism.

The concept of Poinaré duality can be extended to the idea of a Poincaré dual:

Definition 3.43. *Given M an n-dimensional oriented manifold, and $i : S \hookrightarrow M$ a closed k-dimensional oriented submanifold, the* ***(closed) Poincaré dual*** *to S is $[\eta_S] \in H^{n-k}(M)$ given by*

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S$$

for any $\omega \in H^k_c(M)$.

Let us unpack this definition: Given such an ω, $\mathrm{supp}(\omega|_S) = \mathrm{supp}(\omega) \cap S$ is closed and compact, and as pullback an d commute we know $\int_S i^*\omega$ indeed exists and is well defined. Then by Poincar'e duality the map $H^k_c(M) \to \mathbb{R}$, $,\omega \mapsto \int_S i^*\omega$ (which is a linear functional) corresponds to a unique element of $H^{n-k}(M)$.
We can define the related notion

Definition 3.44. *Given M an n-dimensional oriented manifold, and $i : S \hookrightarrow M$ a compact k-dimensional oriented submanifold, the* ***(compact) Poincaré dual*** *to S is $[\eta'_S] \in H^{n-k}_c(M)$ given by*

$$\int_S i^*\omega = \int_M \omega \wedge \eta'_S$$

for any $\omega \in H^k(M)$.

Remark. *As manifolds are Hausdorff, any compact submanifold is also closed. Hence a compact submanifold has both an associated closed Poincaré dual and an associated compact Poincaré dual. These are in general different and so need specifying.*

Example 3.45. *Consider $M = \mathbb{R}^n$ with compact submanifold $P = *$. Note $H^n(\mathbb{R}^n) = 0$ so $[\eta_P] = [0]$. Contrastingly, $[\eta'_P]$ must generate $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$, and as closed 0-forms are constant functions, all we require is that $\int_{\mathbb{R}^n} \eta'_P = 1$, which can be achieved by a normalised bump function.*

We can now make the following link between a the Poincaré dual and the Thom isomorphism:

Proposition 3.46. *Let M be an orientable n-dimensional manifold, and $S \subset M$ and orientable k-dimensional submanifold. Then*

$$\eta_S = j_* \Phi(N_S)$$

where $j : N_S \hookrightarrow M$ is the injection so j_* is extension of forms by 0.

Proof. Consider the diagram

$$H^p(S) \xrightarrow{\mathcal{T}} H^{p+n-k}_{cv}(T) \xrightarrow{j_*} H^{p+n-k}(M)$$

where T is a tubular □

Corollary 3.47. *If $E \to M$ is an oriented vector bundle over an oriented bsae then*

$$\eta_{s_0(M)} = j_* \Phi(E)$$

where $s_0 : M \to E$ is the zero section.

Corollary 3.48 (Localisation Principle)**.** *The support of η_S for $S \subset M$ can be shrunk to any tubular neighbourhood of S.*

Proof. Any tubular neighbourhood is diffeomorphic to the normal bundle. □

Proposition 3.49. *Under Poincarduality, transversal intersection of oriented submanifolds corresponds to the wedge product of forms, i.e*

$$\eta_{R \cap S} = \eta_R \cap \eta_S$$

Proof. If R, S are oriented submanifolds that intersect transversally, then $N_{R \cap S} = N_R \oplus N_S$, and so

$$\begin{aligned} \eta_{R\cap S} &= j_* \Phi(N_{R\cap S}) \\ &= j_* \Phi\left(N_R \oplus N_S\right) \\ &= j_* \left(\pi_R^* \Phi(N_R) \wedge \pi_S^* \Phi(N_S)\right) \\ &= \eta_R \wedge \eta_S \end{aligned}$$

□

Proposition 3.50. *If $f : N \to M$ is an orientation-preserving map of oriented manifolds, $S \subset M$ a closed oriented submanifold s.t. $f^{-1}(S) \subset N$ is closed, then*

$$\eta_{f^{-1}(S)} = f^* \eta_S$$

Proof. Suppose $\dim S = k$, $\dim M = m$, $\dim N = n$. Consider the diagram

$$\begin{tikzcd} H^p(S) \arrow[r, "\mathcal{T}"] \arrow[d, "f^*"'] & H^{p+m-k}_{cv}(T) \arrow[r, "j_*"] \arrow[d, "f^*"'] & H^{p+m-k}(M) \arrow[d, "f^*"'] \\ H^p(f^{-1}S) \arrow[r, "\mathcal{T}"] & H^{p+m-k}_{cv}(f^{-1}T) \arrow[r, "j_*"] & H^{p+n-k}(N) \end{tikzcd}$$

where T is a tubular neighbourhood of S. This diagram commutes, and so following the image of the generator 1 in $H^0(f^{-1}S)$ gives the result. □

3.7 Euler Class

We now want to go through the proces of calculating a Thom class explicitly. We will start by constructing a generator of $H^n_c(\mathbb{R}^n)$ from $H^{n-1}(S^{n-1})$ from:

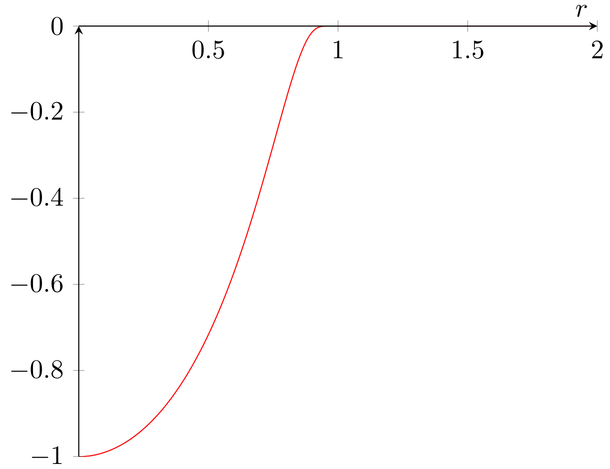
Definition 3.51. *Given an orientation on an n-dimensional manifold M specified by $\omega \in \Omega^n(M)$, $\tau \in \Omega^n(M)$ is called **positive** if $[\omega] = [\tau] \in H^n(M)$.*

We define positive orientation on S^{n-1} to be compatible with the standard orientation on $\mathbb{R}^n$, that is the generator $[\sigma] \in H^{n-1}(S^{n-1})$ is positive if $dr \wedge \pi^*\sigma$ is positive on $\mathbb{R}^n \setminus \{0\}$. Here, r is the radial coordinate, and $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is the deformation retraction onto the sphere along radius.

Example 3.52. *If we consider S^1, then using $\pi : \mathbb{R}^2 \setminus \{0\} \to S^1$ we have $\pi^*(d\theta) = d\theta$, and as $dx \wedge dy = rdr \wedge d\theta$. the form $d\theta$ is positive giving orientation to S^1.*

Definition 3.53. *Letting $[\sigma] \in H^{n-1}(S^{n-1})$ be the positive generator, the form $\psi \in \pi^*\sigma$ is called the **angular form** on $\mathbb{R}^n \setminus \{0\}$.*

Given the angular form, and a function $\rho(r)$ s.t $d\rho = \rho'(r)dr$ is a bump funcion, integral 1, $\rho(0) \neq 0$ (we will choose $\rho(0) = -1$, see below), then we can take $[d\rho \wedge \psi]$ as a generator of $H^n_c(\mathbb{R}^n)$.



Remark. *Note we can write*

$$d\rho \wedge \psi = d(\rho \wedge \psi)$$

as ψ is closed, but because $\rho(0) \neq 0$ this is not a global form so $d\rho \wedge \psi$ isn't exact.

Definition 3.54. *Given an oriented rank-k vector bunle over M, let E^0 be the complement to the zero section. The **global angular form** is global form on E_0 whose restriction to each fibre is that angular form on $\mathbb{R}^k \setminus \{0\}$*

We now specialise to the case of a rank-2 vector bundle $\pi : E \to M$ over an n-dimensional base. Taking an open cover $\{U_\alpha\}$ of M with local coordinates $x^1_\alpha, \dots, x^n_\alpha$ we can take local coordinates on E^0 given by $r_\alpha, \theta_\alpha, \pi^*x^1_\alpha, \dots, \pi^*x^n_\alpha$. On $U_\alpha \cap U_\beta$ define $\varphi_{\alpha\beta}$ by

$$\theta_\beta = \theta_\alpha + \pi^*\varphi_{\alpha\beta}$$

Lemma 3.55. *On triple intersections* $U_\alpha \cap U_\beta \cap U_\gamma$

$$\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma} \in 2\pi\mathbb{Z}$$
$$\Rightarrow d\varphi_{\alpha\beta} + d\varphi_{\beta\gamma} - d\varphi_{\alpha\gamma} = 0$$

Proof. This follows immediately from the cocyle condition. ☐

Lemma 3.56. $\exists \xi_\alpha \in \Omega^1(U_\alpha)$ *s.t.*

$$\frac{1}{2\pi} d\varphi_{\alpha\beta} = \xi_\beta - \xi_\alpha$$

Proof. Take $\{\rho_\alpha\}$ to be a partition of unity subordinate to $\{U_\alpha\}$ and let $\xi_\alpha = -\sum_\gamma \rho_\gamma d\varphi_{\alpha\gamma}$ Then

$$\begin{aligned}\xi_\beta - \xi_\alpha &= -\sum_\gamma \rho_\gamma (d\varphi_{\beta\gamma} - d\varphi_{\alpha\gamma}) \\ &= \sum_\gamma \rho_\gamma d\varphi_{\alpha\beta} = d\varphi_{\alpha\beta}\end{aligned}$$

☐

Lemma 3.57. *The* $d\xi_\alpha$ *combine to make a global closed two form* $e = e(E)$ *on* M *that is not necessarily exact.*

Proof. On $U_\alpha \cap U_\beta$

$$d\xi_\alpha - d\xi_\beta = d(\xi_\alpha - \xi_\beta) = -\frac{1}{2\pi} d^2\varphi_{\alpha\beta} = 0$$

Closed-ness is clear, as on each patch it is exact, but in general the ξ_α do not combine to make a global one form. ☐

Definition 3.58. *The cohomology* $[e] \in H^2(M)$ *is called the* ***Euler class*** *of the oriented vector bundle* E.

Example 3.59. *The Euler class of the trivial bundle is 0.*

Proposition 3.60. *The Euler class does not depend on the choice of* ξ *in the definition.*

Proof. If $\{\overline{\xi}\}$ is another such set then $\overline{\xi}_\alpha - \xi_\alpha$ makes a global one form so the difference in e is an exact form. ☐

Proposition 3.61. $d\psi = -\pi^* e$

Remark. *Note that on an oriented rank-2 vector bundle the transition functions can be taken to have values in* $SO(2) \cong U(1)$.

Proposition 3.62. *We have the formula that*

$$e(E)|_{U_\alpha} = -\frac{1}{2\pi i}\sum_\gamma d\left(\rho_\gamma d\log g_{\gamma\alpha}\right)$$

Corollary 3.63. *The Euler class if functorial, i.e. given $f : N \to M$ covered by an orientation preserving map and $E \to M$ a rank-2 oriented vector bundle*

$$e(f^*E) = f^*e(E)$$

Proposition 3.64. $\Phi(E) = d\left(\rho \wedge \psi\right) = d\rho \wedge \psi - \rho\pi^* e(E)$

Corollary 3.65. *The pullback of the Thom class to M by the zero section is the Euler class*

Proof. We explicitly calculate, calling $s : M \to E$ the zero section

$$\begin{aligned} s^*\Phi &= d(\rho(0)) \wedge s^*\psi - \rho(0)s^*\pi^* e \\ &= (\pi \circ s)^* e = e \end{aligned}$$

□

4 Chern Classes

We start by recalling a definition:

Definition 4.1. *A **complex line bundle** is a complex vector bundle of rank 1.*

Remark. *Analogously to how real vector bundles have reduction of structure group $GL(r, \mathbb{R}) \to O(r)$, complex vector bundles have reduction $GL(r, \mathbb{C}) \to U(\mathbb{C})$ 6*

Lemma 4.2. *There is a bijection between complex line bundles and oriented rank-2 real vector bundles.*

Proof. Every rank-r $\mathbb{C}$-vector space E corresponds to a rank-$2r$ $\mathbb{R}$-vector space $E_\mathbb{R}$ by forgetting the complex structure. Then as $U(1) \cong SO(2)$ each complex line bundle in the case $r = 1$ this is a bijection if we give an orientation to the real bundle, which picks out $SO(2) \subset O(2)$. □

Definition 4.3. *The **first Chern class** of a complex line bundle L with base M is the Euler class of $L_\mathbb{R}$, that is*

$$c_1(L) = e(L_\mathbb{R}) \in H^2(M)$$

Remark. *We can come up with a very categorical definition of the Chern classes, which I would like to explain now (glossing over much of the detail). We will need a few ingredients:*

- *The functor $\mathrm{Vect}_{n,\mathbb{C}}$ sending spaces to isomorphism classes of complex rank-n vector bundles,*
- *The functor $H^\bullet(\cdot, \mathbb{Z})$ sending spaces to their integer-valued singular cohomology ring.*

A fact we will need is that $\mathrm{Vect}_{n,\mathbb{C}}$ is a representable functor:

- *rank-n vector bundles over X are principal $GL_n(\mathbb{C})$ bundles over X, and as such are classified by homotopy classes of maps $X \to BGL_n(\mathbb{C})$. It is known that $BGL_n(\mathbb{C}) = Gr_n(\mathbb{C}^\infty)$.*

It is aactually a general fact that homotopy classes of maps $X \to K(G, n)$ are in bijection with $H^n(X, G)$, where $K(G, n)$ is an Eilenberg-Maclan space, so our second functor is also represnetable, but we will not want to use this.
The Yoneda lemma then says that to give a Chern class - i.e. a natural map $c : \mathrm{Vect}_{n,\mathbb{C}} \to H^\bullet(X, \mathbb{Z})$ - is equivalent to giving an element of the cohomology ring $H^\bullet(Gr_n(\mathbb{C}^\infty), \mathbb{Z})$. If we require the map c to obey the axioms

- *$c = 1$ on the trivial bundle*
- *c takes values only in H^{2k}, $k \leq n$ on rank-n bundles*
- *for vector bundles E, E', $c(E \oplus E') = c(E)c(E')$*
- *$c \in H^2(\mathbb{CP}^1, \mathbb{Z})$ is a generator for the tautological line bundle.*

then it turns out the Chern class is fully determined