Lie Algebras and Representation Theory

Linden Disney-Hogg

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Contents

1   Introduction

Here we will be reviewing the Lie theory necessary for work on affine Toda, and cementing notation. We will work from [1], folding in my SFP and Kac-Moody notes. Notation is uniformised where possible with Harry Braden, see my Affine Toda notes.

2   Useful Facts

The following are some small results which are useful in calculations. These may depend on definitions made later in the text, a format I usually disagree with, but this will give continuity later.

2.1   Miscellaneous

Theorem 2.1. *If X is a complex square matrix then*

$$\det e^X = e^{\operatorname{Tr} X}$$

Theorem 2.2 (Baker-Campbell-Hausdorff Formula)**.** *Let $\mathfrak{g}$ be a Lie algebra, and $X, Y \in \mathfrak{g}$. Then under sufficient existence conditions $e^X e^Y = e^Z$ where*

$$Z = X + Y + \frac{1}{2}\left[X, Y\right] + \frac{1}{12}\left(\left[X, [X, Y]\right] + \left[Y, [Y, X]\right]\right) + \dots$$

Corollary 2.3 (Lie Product Formula)**.** *For $X, Y \in \mathfrak{g}$, a Lie algebra*

$$e^{X+Y} = \lim_{N \to \infty} \left(e^{\frac{X}{N}} e^{\frac{Y}{N}}\right)^N$$

Fact 2.4. *Rotation matrices corresponding to a rotation θ degrees about an axis n have the form*

$$R_{ij} = \cos\theta \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta \epsilon_{ijk} n_k$$

2.2 Pauli Matrices

Definition 2.5 (Pauli Matrices)**.** *The **Pauli matrices** are*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note they are all Hermitian and traceless.

Fact 2.6.
$$\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk}\sigma_k \tag{2.2.1}$$

2.3 Extra Definitions

These definitions are necessary from other areas of maths, but do not naturally sit in any other section:

Definition 2.7. *If V is a n-dimensional vector space a **flag** in V i a chain of subspaces $0 = V_0 \leq V_1 \leq \cdots \leq V_n = V$ s.t. $\dim V_i = i$.*

Definition 2.8. *$x \in \mathrm{End}(V)$ is **semisimple** if the roots of its minimal polynomial are distinct. (i.e is diagonalisable if k is algebraically closed).*

Part I
Basic Concepts and Definitions

3 Lie Groups

3.1 Basics

Definition 3.1 (Lie Group)**.** *A **Lie group** is a group that has a smooth manifold structure such that the group operations are smooth functions on the manifold.*

Idea. *Lie groups are introduced here to give a foundation to the idea of a continuous symmetry group, which will act in some way. It is a consequence of the Peter-Weyl theorem that every compact Lie groups is isomorphic to a subgroup of $GL(n, \mathbb{C})$ for some n so often one can restrict their thought process to matrix Lie groups.*

Idea. *When showing that something is a structure is a Lie group, typically the hardest step is showing that it is a manifold. One approach is to find an explicit parametrisation of the manifold, often a good approach if the group has very obvious parameters upon which it depends (e.g. the Heisenberg group). Alternatively, one can use the preimage theorem, useful if the group is defined by some constraints (e.g. the orthogonal group).*

Theorem 3.2 (Closed Subgroup Theorem)**.** *If G is a Lie group and $H \leq G$ is closed, then H is an embedded Lie group with the subspace topology.*

4 Lie Algebras

4.1 Algebras

Throughout we will take k to be a field.

Definition 4.1. *An **algebra** is a triple (A, m, i) of*

- *a k-vector space A*

- *a linear map $m : A \otimes A \to A$*

- *an element $i : k \to A$*

satisfying associativity and unitality.

Notation. *For $a, b \in A$ we will denote $m(a, b) = a \cdot b$.*

Remark. *Linearity of m gives distributivity of the multiplication over k.*

Proposition 4.2. *If a unit exists for $(A, \cdot)$, it is unique*

Proof. Let $1, 1' \in A$ be the units. Then

$$1 = 1 \cdot 1' = 1'$$

$\square$

Example 4.3. *Some examples of algebras are*

- *The base field k*

- *polynomials over k, $k[X]$.*

- *$\mathrm{End}(V)$ where V is a vector space, with multiplication given by composition*

Example 4.4. *The **free algebra** $k \langle x_1, \ldots, x_n \rangle$ is the vector space consisting formally of all possible combinations of the x_i in order to make it a vector space, namely*

$$k \langle x_1, \ldots, x_n \rangle = \bigoplus_{m=0}^{\infty} k \cdot \prod_{1 \leq j_i \leq n} x_{j_1} \cdots x_{j_m}$$

Example 4.5. *Given a group G we have the **group algebra** $A \equiv kG$ with*

- *basis $\{ x_g \, | \, g \in G \}$*

- *multiplication $x_g \cdot x_h = x_{gh}$*

- *unit x_{e_G}*

Definition 4.6. $(A, \cdot)$ *is **commutative** if* $\forall a, b \in A,\ a \cdot b = b \cdot a.$

Example 4.7. *kG is abelian iff G is abelian.*

Definition 4.8. *A homomorphism of algebras $f : A \to B$ is a linear map of vector spaces compatible with $\cdot$ s.t.*

- $\forall a, b \in A,\ f(a \cdot b) = f(a) \cdot f(b)$

- $(f(1_A) = 1_B)$

Definition 4.9. *A **derivation** is a map $\partial : A \to A$ satisfying $\forall a, b, \in A,\ \partial(ab) = a\partial(b) + \partial(a)b.$ We denote the vector space of derivations as $\mathrm{Der}(A)$.*

Remark. *In general a derivation can be defined on any k-algebra.*

4.2 Basics

Definition 4.10 (Lie Algebra)**.** *A **Lie algebra** is a vector space, equipped with a bracket that:*

- *is bilinear*

- *is antisymmetric*

- *obeys the Jacobi identity*

Example 4.11. *Given an associative algebra A, we can make A a Lie algebra using*

$$[a, b] = ab - ba$$

for $a, b \in A$.

Example 4.12. *$\mathrm{Der}(A)$ is a Lie algebra under commutation.*

Idea. *Given a matrix Lie group G, it has a natural associated Lie algebra $\mathfrak{g} = T_I G$ with bracket given by the matrix commutator. This will be the prototypical Lie algebra to think of. The Lie algebra corresponding to a Lie group is useful to work with as we can take a basis, which will commute under addition, rather than work with a difficult presentation of the group. Using the exponential map, we can map back into the original Lie group. This is not generally injective or surjective, but map injectively onto the connected component containing the identity.*

Fact 4.13. *The dimension of the tangent space to a manifold is equal to the dimension of the manifold. Hence, a Lie group and its associated Lie algebra have the same dimension*

Definition 4.14. *The **general linear algebra** corresponding to vector space V is $\mathfrak{gl}(V) = \mathrm{End}(V)$ with the bracket given by commutation. If V is an n-dimensional k-vector space we can write this as $\mathfrak{gl}_n(k)$*

Definition 4.15. *A **Lie subalgebra** is a vector subspace $\mathfrak{h} \le \mathfrak{g}$ s.t. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.*

Definition 4.16 (Structure Constants)**.** *Given a basis $\{T^a\}$ of a Lie algebra, the **structure constants** f_c^{ab} are defined such that*

$$\left[T^a, T^b\right] = f_c^{ab} T^c$$

Definition 4.17 (Complexification and Real Form)**.** *Given a real Lie algebra $\mathfrak{g}_{\mathbb{R}} \equiv \mathrm{Span}_{\mathbb{R}}\{T^a : a = 1, \ldots, D\}$ the **complexification** is $\mathfrak{g}_{\mathbb{C}} \equiv \mathrm{Span}_{\mathbb{C}}\{T^a : a = 1, \ldots, D\}$. Conversely, given such a $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{R}}$ is called a **real form** of **realification**.*

4.3 Ideals, Homomorphisms, and Simplicity

Definition 4.18. *A **Lie algebra homomorphism** is a linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ s.t.*

$$\forall X, Y \in \mathfrak{g},\ [\phi(X), \phi(Y)] = \phi\left([X, Y]\right)$$

The definition extends naturally to isoomrphisms.

Definition 4.19. *A **automorphism** is a Lie algebra isomorphisms $\mathfrak{g} \to \mathfrak{g}$. We denote the set (which forms a group) of all such as $\mathrm{Aut}(\mathfrak{g})$.*

Definition 4.20 (ideal)**.** *An **ideal** is a subalgebra $\mathfrak{h} \leq \mathfrak{g}$ s.t.*

$$\forall X \in \mathfrak{g},\ \forall Y \in \mathfrak{h},\ [X, Y] \in \mathfrak{h} \Leftrightarrow [\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$$

Proposition 4.21. *Given a homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$, $\ker \phi$ is an ideal in $\mathfrak{g}$, and $\mathrm{Im}\,\phi$ is a subalgebra of $\mathfrak{g}'$.*

Definition 4.22 (Centre)**.** *The **centre** of a Lie algebra $\mathfrak{g}$ is*

$$\zeta(\mathfrak{g}) = \{\, X \in \mathfrak{g} : \forall Y \in \mathfrak{g}\ [X, Y] = 0 \,\}$$

Proposition 4.23. *The centre is an ideal.*

Proposition 4.24. *If $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$ are ideals then so is $\mathfrak{h} + \mathfrak{h}'$.*

Definition 4.25 (Simple Lie Algebra)**.** *A Lie Algebra is **simple** if it non-abelian and contains no non-trivial ideals.*

Idea. *An ideal is to a Lie algebra what a normal subgroup is to a Lie group. Hence simple Lie algebras are the analogy of simple Lie groups.*

Definition 4.26 (Semi-Simple Lie Algebra)**.** *A Lie Algebra is **semi-simple** if it contains no non-trivial abelian ideals.*

4.4 Adjoint Map

Definition 4.27 (Adjoint Map)**.** *The **adjoint map** of an element $X \in \mathfrak{g}$ is $ad_X : \mathfrak{g} \to \mathfrak{g}$ defined such that for $Y \in \mathfrak{g}$*

$$ad_X\left(Y\right) = [X, Y]$$

Lemma 4.28.

$$[ad_X, ad_Y] = ad_{[X,Y]}$$

Proof. Act on an element $Z \in \mathfrak{g}$ and apply the Jacobi identity. $\qquad\square$

Definition 4.29. *$X \in \mathfrak{g}$ is **ad-nilpotent** (or just nilpotent if context is clear) if $\exists N > 0$ s.t. $(\mathrm{ad}_X)^N = 0$.*

Proposition 4.30. *$X \in \mathfrak{g}$ nilpotent $\Rightarrow \exp(\mathrm{ad}_X) \in \mathrm{Aut}(\mathfrak{g})$*

Proposition 4.31. *$\forall X \in \mathfrak{g}$, ad_X is a derivation.*

Definition 4.32. *A derivation* $\partial : \mathfrak{g} \to \mathfrak{g}$ *is called **inner** if* $\exists X \in \mathfrak{g}$ *s.t.* $\partial = \mathrm{ad}_X$.

Remark. *4.30 holds if* ad_X *is replaced by any nilpotent derivation.*

Proposition 4.33. *If* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *is an arbitrary Lie algebra and* $X \in \mathfrak{g}$ *is nilpotent then*

$$\forall Y \in \mathfrak{g}, \ (\exp X)\, Y\, (\exp X)^{-1} = \exp\left(\mathrm{ad}_X\right) Y$$

Definition 4.34 (ad-diagonalisable)**.** *An element* $X \in \mathfrak{g}$ *is **ad-diagonalisable** if the linear map* ad_X *is diagonalisable.*

4.5   Solvability and Nilpotency

Definition 4.35 (Derived Algebra)**.** *The **derived algebra** of a Lie algebra* $\mathfrak{g}$ *is*

$$i(\mathfrak{g}) = \mathrm{Span}\left\{\, [X,Y] : X, Y \in \mathfrak{g} \,\right\}$$

Note this will sometimes be notated as $[\mathfrak{g}, \mathfrak{g}]$.

Proposition 4.36. *The derived algebra is an ideal*

Definition 4.37. *The **derived series** of* $\mathfrak{g}$ *is the sequence of ideals* $\mathfrak{g}^{(i)}$ *defined by*

- $\mathfrak{g}^{(0)} = \mathfrak{g}$

- $\mathfrak{g}^{(i+1)} = \left[\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}\right]$

Remark. $\mathfrak{g}^{(1)} = i(\mathfrak{g})$

Definition 4.38. *A Lie algebra* $\mathfrak{g}$ *is **solvable** if* $\exists n$ *s.t.* $\mathfrak{g}^{(n)} = 0$.

Proposition 4.39. *Let* $\mathfrak{g}$ *be a Lie algebra:*

1. $\mathfrak{g}$ *solvable* $\Rightarrow$ *all subalgebras and homomorphic images of* $\mathfrak{g}$ *solvable.*

2. *If* $I \subset \mathfrak{g}$ *is a solvable ideal s.t.* $\mathfrak{g}/I$ *is solvable, so is* $\mathfrak{g}$.

3. *If* $I, H \subset \mathfrak{g}$ *are solvable ideals, so is* $I + J$.

Corollary 4.40. $\exists !$ *maximal solvable ideal of* $\mathfrak{g}$

Definition 4.41. *We denote the unique maximal solvable ideal by* $\mathrm{Rad}(\mathfrak{g})$.

Proposition 4.42. $\mathrm{Rad}\, \mathfrak{g} = 0 \Leftrightarrow \mathfrak{g}$ *semisimple*

Proposition 4.43. $\mathfrak{g}/\mathrm{Rad}\, \mathfrak{g}$ *is semisimple*

Having developed this definition of solvable, we can make a similar looking, but notably differnt, one:

Definition 4.44. *The **descending central series** of* $\mathfrak{g}$ *is the sequence of ideals* $\mathfrak{g}^i$ *defined by*

- $\mathfrak{g}^0 = \mathfrak{g}$

- $\mathfrak{g}^{i+1} = \left[\mathfrak{g}, \mathfrak{g}^i\right]$

Definition 4.45. *A Lie algebra* $\mathfrak{g}$ *is* **nilpotent** *if* $\exists n$ *s.t.* $\mathfrak{g}^n = 0$.

Proposition 4.46. *Nilpotent* $\Rightarrow$ *solvable.*

We can again make a similar large prop:

Proposition 4.47. *Let* $\mathfrak{g}$ *be a Lie algebra:*

1. $\mathfrak{g}$ *nilpotent* $\Rightarrow$ *all subalgebras and homomorphic images of* $\mathfrak{g}$ *nilpotent.*

2. *If* $\mathfrak{g}/\zeta(\mathfrak{g})$ *is nilpotent, so is* $\mathfrak{g}$.

3. *If* $\mathfrak{g}$ *is nilpotent and non-zero, then* $\zeta(\mathfrak{g}) \neq 0$.

Theorem 4.48 (Engel)**.** *If all elements of* $\mathfrak{g}$ *are ad-nilpotent, then* $\mathfrak{g}$ *is nilpotent.*

Proof. Note the direction nilpotent $\Rightarrow$ all elements ad-nilpotent is clear, as to be nilpotent is to say $\exists n$ s.t

$$\forall X_1, \ldots, X_n, Y \in \mathfrak{g}, \; \mathrm{ad}_{X_1} \ldots \mathrm{ad}_{X_n} Y = 0$$

so take all $X_i = X$ equal to see X ad-nilpotent. $\qquad\square$

4.6 Cartan Subalgebras

Definition 4.49. *A non-zero subalgebra consisting of semisimple elements is called a* **toral subalgebra***.*

Definition 4.50. *Suppose* $\mathfrak{g}$ *is semisimple. Then* $\mathfrak{g}$ *has a toral subalgebra.*

Lemma 4.51. *A toral subalgebra is abelian.*

Definition 4.52 (Cartan Subalgebra)**.** *A* **Cartan subalgebra** $\mathfrak{h} \leq \mathfrak{g}$ *is a maximal abelian subalgebra containing only ad-diagonalisable elements. It is typically written as* $\mathfrak{h} = \mathrm{Span}\left\{H^i\right\}$. *Write* $\dim \mathfrak{h} = \mathrm{rank}\, \mathfrak{g} = r$

Fact 4.53. *All Cartan subalgebras of a given Lie algebra have the same dimension. Though Cartan subgalgebras are not unique, they are all conjugate.*

Idea. *It is a fact that Cartan subalgebras correspond to maximal toral subgroups of the corresponding Lie group. This gives a route to visualisation of what the subalgebra might be. A Cartan subalgebra is defined such that representations of the basis* $\left\{H^i\right\}$ *are simultaneously diagonalisable.*

5 Representations

5.1 Basics

Definition 5.1 (Representations)**.** *A* **representation** *of a Lie group* G *is a smooth group homomorphism*

$$D : G \to GL(V)$$

where V *is a n-dimensional vector space called the* **representation space***. In analogy a representation of a Lie algebra is a Lie algebra homomorphism*

$$d : \mathfrak{g} \to \mathfrak{gl}(V)$$

Definition 5.2 (Dimension)**.** *The **dimension** of a representation is the dimension of the representation space.*

Definition 5.3 (Trivial Representation)**.** *The **trivial representation** is the unique one dimensional representation that sends every element to the identity.*

Definition 5.4 (Fundamental Representation)**.** *For a matrix Lie group $G \leq GL(V)$ the **fundamental representation** is the identity map $R = id_{GL(V)}$. Note $dim\,(R) = dim\,(V)$.*

Definition 5.5 (Adjoint Representation)**.** *Given a Lie algebra $\mathfrak{g}$ the **adjoint representation** is $d_{adj} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $d_{adj}\,(X) = ad_X$. Note $dim\,(d_{adj}) = dim\,(\mathfrak{g})$*

Proposition 5.6. $\ker d_{adj} = \zeta(\mathfrak{g})$

Definition 5.7 (Faithful Representation)**.** *A representation is **faithful** if it is injective.*

Definition 5.8 (Isomorphic Representations)**.** *Two representations R_1, R_2 of a Lie algebra $\mathfrak{g}$ are **isomorphic**, written $R_1 \cong R_2$ if $\exists S$ an invertible matrix such that*

$$\forall X \in \mathfrak{g} \quad R_2\,(X) = S R_1\,(X)\,S^{-1}$$

Definition 5.9. *Given $v \in V$ the **minimal subrep** containing v is*

$$A \cdot V = \{w \in V \mid \exists a \in A,\, w = a \cdot v\}$$

Definition 5.10. *Let V_1, V_2 be reps of A. Then a homomorphism of reps (an **intertwiner**) is linear map $\phi : V_1 \to V_2$ s.t. $\forall v \in V_1$, $a \in A$, with $\rho_i : A \to \mathrm{End}(V_i)$ we have*

$$
\begin{array}{ccc}
V_1 & \xrightarrow{\ \phi\ } & V_2 \\
{\scriptstyle \rho_1(a)}\downarrow & & \downarrow{\scriptstyle \rho_2(a)} \\
V_1 & \xrightarrow[\ \phi\ ]{} & V_2
\end{array}
$$

commutes, i.e. $\phi(a \cdot v) = a \cdot \phi(v)$

Proposition 5.11. *Let $f : V \to W$ be an intertwiner. Then*

- $\ker f \subset V$ *is a subrep*

- $\mathrm{Im}\, f \subset W$ *is a subrep*

Proposition 5.12. *Let $\rho : \mathfrak{g} \to GL(V)$ be a rep of semi-simple $\mathfrak{g}$ with generators $\{T^a\}$. Then $\mathrm{Tr}\, \rho(T^a) = 0$.*

Proof. As $\mathfrak{g}$ is semisimple, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ (as $\mathfrak{g}\big/_{[\mathfrak{g},\,\mathfrak{g}]}$ is semisimple and abelian), so each T^a is a sum of commutators. Then $\rho(T^a)$ is also a commutator so traceless. □

5.2 Constructing Representations

Theorem 5.13. *A representation D of a Lie group G induces a representation on the the corresponding Lie algebra $\mathfrak{g}$ as such: For $X \in \mathfrak{g}$ let $g : (-\epsilon, \epsilon) \to G$ be a curve in G defined such that $g(0) = e$ the identity of G and $g'(0) = X$. This curve necessarily exists for some $\epsilon > 0$. Then define d by*

$$d(X) \equiv \frac{d}{dt} D\left(g(t)\right)\big|_{t=0}$$

Conversely, a representation of a Lie algebra $\mathfrak{g}$ induces a representation in some neighbourhood of e in G given by

$$D(\exp X) \equiv \exp d(X)$$

Definition 5.14 (Direct Sum Representation)**.** *Given representations R_1, R_2 with representation spaces V_1, V_2 the **direct sum representation** is $R_1 \oplus R_2$ with representation space $V_1 \oplus V_2$ acting by*

$$(R_1 \oplus R_2)(X)(v_1 \oplus v_2) = (R_1(X)v_1) \oplus (R_2(X)v_2)$$

Definition 5.15 (Tensor Product Representation)**.** *Given representations R_1, R_2 with representation spaces V_1, V_2 the **tensor product representation** is $R_1 \otimes R_2$ with representation space $V_1 \otimes V_2$ with*

$$(R_1 \otimes R_2)(X) = R_1(X) \otimes I_{V_2} + I_{V_1} \otimes R_2(X)$$

Idea. *Considering a representation as a visualisation of the action of the algebra induced by a group on a representation space, then $R_1 \oplus R_2$ can be viewed as two non-interacting systems, whereas $R_1 \otimes R_2$ can be viewed as two entangled systems.*

5.3 Reducibility

Definition 5.16 (Invariant Subspace)**.** *Given a representation R of Lie algebra $\mathfrak{g}$ with representation space V, an **invariant subspace** is $U \leq V$ such that*

$$\forall X \in \mathfrak{g}, \ \forall u \in U \quad R(X)u \in U$$

Definition 5.17 (Irreducible Representation)**.** *A representation is **irreducible** if it has no invariant subspaces.*

Definition 5.18 (Full Reducible)**.** *A representation is **fully reducible** if it can be expressed as the direct sum of irreducible representations.*

Fact 5.19. *If R_i are finite dimensional irreducible representations of a simple Lie algebra for $i = 1, \ldots, m$ then*

$$R_1 \otimes R_2 \otimes \cdots \otimes R_m$$

is fully reducible.

Lemma 5.20 (Schur)**.** *Let V_1, V_2 be two A-reps and let $f : V_1 \to V_2$ be a non-zero intertwiner. Then*

- *V_1 irreducible $\Rightarrow f$ is injective*

- *V_2 irreducible $\Rightarrow f$ is surjective*

Definition 5.21. *A representation is **indecomposable** is when $V = V_1 \oplus V_2$, either $V_1 = 0$ or $V_2 = 0$*

Proposition 5.22. *Any irreducible rep is indecomposable*

Proof. V_1, V_2 are subreps of $V_1 \oplus V_2$. □

Remark. *The converse to the above is not true,*

5.4 Universal Enveloping Algebras

Definition 5.23. *Let $\{x_i\}$ be a basis of $\mathfrak{g}$ a Lie algebra and suppose the bracket is specified by*

$$[x_i, x_j] = \sum_k c_{ij}^k x_k$$

*for some **structure constants** c_{ij}^k. Then the **universal enveloping algebra** $\mathcal{U}_\mathfrak{g}$ is the associative algebra generated by the x_i with the relations*

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k$$

i.e.

$$\mathcal{U}_\mathfrak{g} = {}^{k \langle x_1, \ldots, x_n \rangle} \Big/ \Big\langle x_i x_j - x_j x_i - \sum_k c_{ij}^k x_k \Big\rangle$$

We get the map

$$\iota : \mathfrak{g} \to \mathcal{U}_\mathfrak{g}$$
$$x_i \mapsto x_i$$

The above definition involves a choice of basis of $\mathfrak{g}$. In general we want to remove this to give a universal property of $\mathcal{U}_\mathfrak{g}$.

Proposition 5.24. *For any associative algebra A, and Lie algebra map $\mathfrak{g} \xrightarrow{f} A^{Lie}$, there exists a unique map of associative algebras $\mathcal{U}_\mathfrak{g} \xrightarrow{\mathcal{U}(f)} A$ s.t.*

commutes

Exercise 5.25. *Prove that $\mathcal{U}_\mathfrak{g}$ is uniquely defined (up to isomorphism) by this universal property.*

Proof. Consider

$$\mathcal{U}_\mathfrak{g} = \left[\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} \right] \Big/ \langle x \otimes y - y \otimes x - [x, y] \rangle$$

□

Proposition 5.26. *If $\mathcal{U}_{\mathfrak{g}}$ satisfies the universal property then there is a bijection*

$$\{ rep \ of \ Lie \ algebra \ \mathfrak{g} \} \leftrightarrow \{ reps \ of \ associative \ algebra \ \mathcal{U}_{\mathfrak{g}} \}$$

5.5   Casimir elements

Definition 5.27. *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation, β a symmetric non-degenerate associative bilinear form on L. Let $\{X_i\}$ be a basis of $\mathfrak{g}$ and $\{Y_i\}$ a basis dual wrt β. Then*

$$c_\rho(\beta) = \sum_i \rho(X_i)\rho(Y_i)$$

*is called a **Casimir element** of ρ.*

Proposition 5.28. $[c_\rho, \rho(\mathfrak{g})] = 0$.

Corollary 5.29. *If ρ is irreducible, c_ρ is a scalar.*

5.6   Weights and Character

Definition 5.30 (Weights). *Given a representation R of a Lie algebra with Cartan subalgebra $\mathfrak{h} = \mathrm{Span}\{H^i\}$ the **weights** of the representation are the eigenvalues $S_R = \{\lambda : \exists v \in V \ R(H^i)v = \lambda^i v\}$*

Notation. *We denote the multiplicity of weights as if they were eigenvalues, i.e. $m(\lambda)$*

Fact 5.31. *Weights can be viewed as maps in $\mathfrak{h}^*$ given by*

$$\lambda : \mathfrak{h} \to \mathbb{C}$$
$$\lambda : e_i H^i \mapsto e_i \lambda^i$$

Thus

$$R(H)v = \lambda(H)v$$

Idea. *This idea is made more explicit in the concept of roots, which are the weights of the adjoint representation.*

Fact 5.32. *Given weights set S_R, if $\dim V < \infty$ then*

$$V = \bigoplus_{\lambda \in S_R} V_\lambda$$

where $V_\lambda = \{v \in V : R(H^i)v = \lambda^i v\}$

Definition 5.33. *The **group ring** corresponding to a representation R is $\mathbb{Z}[R]$ the free $\mathbb{Z}$-module with basis elements $\{e(\lambda) \,|\, \lambda \in S_R\}$ with multiplication given $e(\lambda)e(\mu) = e(\lambda + \mu)$.*

Definition 5.34. *The **formal character** of a representation is $ch_R = \sum_{\lambda \in S_R} m(\lambda)e(\lambda)$ as a formal element of the group ring.*

Proposition 5.35. *The character gives a ring homomorphism between the group ring and the ring of representations, i.e.*

- $ch_{R_1 \oplus R_2} = ch_{R_1} + ch_{R_2}$

- $ch_{R_1 \otimes R_2} = ch_{R_1} \cdot ch_{R_2}$

6 SU(2)

6.1 Definition and Basic Properties

Definition 6.1 (U(n)). *The **unitary group** of dimension n is*

$$U(n) = \{\, U \in GL(n, \mathbb{C}) : U^\dagger U = I \,\}$$

It is a path connected, compact, Lie group.

Definition 6.2 (SU(n)). *The **special unitary group** of dimension n is*

$$SU(n) = \{\, U \in U(n) : \det U = 1 \,\}$$

Through the det homomorphism, it can be seen $SU(n)$ is a closed subgroup of $U(n)$, hence also a Lie group. The corresponding Lie algebra is

$$\mathfrak{su}(n) = \{\, Z \in GL(n, \mathbb{C}) : Z^\dagger + Z = 0, \operatorname{Tr} Z = 0 \,\}$$

The real dimension is

$$\underbrace{2 \times \frac{1}{2}(n-1)n}_{\text{off diagonal elements}} + \underbrace{n}_{\text{diagonal elements}} - \underbrace{1}_{\text{trace constraint}} = n^2 - 1$$

In the case $n = 2$ we have

$$SU(2) = \{\, \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \, |\alpha|^2 + |\beta|^2 = 1 \,\}$$

This can be expressed as, for $A \in SU(2)$

$$A = a_0 I + i\boldsymbol{a} \cdot \boldsymbol{\sigma}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, and $a_0^2 + |\boldsymbol{a}|^2 = 1$. Hence $SU(2) \cong S^3$. In addition, by parametrising $SU(2)$ by the a_i, it can be seen that $\{\, i\sigma_i \,\}$ forms a basis of $\mathfrak{su}(2)$. It is typical to normalise this basis to $\{\, T^a = -\frac{1}{2} i\sigma_a \,\}$.

Fact 6.3. *The structure constants in this basis $\{\, T^a \,\}$ are*

$$f_c^{ab} = \epsilon_{abc}$$

Fact 6.4. *The function*

$$R : SU(2) \to SO(3)$$

$$R(A)_{ij} = \frac{1}{2} \operatorname{Tr}\left(\sigma_i A \sigma_j A^\dagger \right)$$

Is a double cover of $SO(3)$, with $R(A) = R(-A)$, and inverse given by

$$A = \pm \frac{I + \sigma_i R(A)_{ij} \sigma_j}{2\sqrt{1 + \operatorname{Tr} R(A)}}$$

Hence $SO(3) \cong {}^{SU(2)}\!/_{\mathbb{Z}_2}$

6.2 Representations

In this subsection we will discuss only representations of $\mathfrak{su}_2$:

Definition 6.5 (Cartan-Weyl basis of $\mathfrak{su}_\mathbb{C}(2)$). $\{\,H, E_\pm\,\}$, where $H = \sigma_3$ and $E_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, is the **Cartan-Weyl basis** for $\mathfrak{su}_\mathbb{C}(2)$. The elements satisfy the commutation relations

$$[H, E_\pm] = \pm 2E_\pm$$
$$[E_+, E_-] = H$$

Fact 6.6. H spans a Cartan subalgebra of $\mathfrak{su}_\mathbb{C}(2)$, ad-diagonalisable w.r.t the Cartan Weyl basis, with eigenvalues $0, \pm 2$ respectively.

Lemma 6.7. If $v \in V_\lambda$ for some representation, then $E_\pm \cdot v \in V_{\lambda \pm 2}$

Corollary 6.8. $\exists \lambda \in S_R$ s.t. $V_{\lambda+2} = 0$. An non-zero $v \in V_\lambda$ is called a **maximal vector**.

Theorem 6.9. Finite dimensional irreducible representations of $\mathfrak{su}_\mathbb{C}(2)$ with $R(H)$ diagonalisable are determined by the highest weight $\Lambda \in \mathbb{N}_0$. Such a representation is called R_Λ. It has weight set

$$S_\Lambda = \{\,\Lambda - 2n : n = 0, \dots, \Lambda\,\}$$

The eigenvectors of $R_\Lambda(H)$ are $v_{\Lambda-2n}$ defined such that

$$R_\Lambda(H)v_\lambda = \lambda v_\lambda$$
$$R_\Lambda(E_+)v_\Lambda = 0$$
$$\left(R_\Lambda(E_-)\right)^n v_\Lambda = v_{\Lambda-2n}$$
$$R_\Lambda(E_+)v_{\Lambda-2n} = r_n v_{\Lambda-2n+2}$$

with

$$r_n = (\Lambda + 1 - n)n$$

Note that $\dim(R_\Lambda) = \Lambda + 1$, so

- R_0 is the trivial representation d_0.

- R_1 is the fundamental representation.

- R_2 is the adjoint representation.

6.2.1 Young Tableaux

We will now want to show how Young Tableaux can be used to identify tensor products of irreducible $\mathfrak{su}(N)$ reps as direct sums.

Definition 6.10. A **Young diagram** is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order. A **Young tableau** is obtained by filling in the boxes of the Young diagram with symbols taken from some alphabet (usually taken to be the natural numbers for brevity).

Part II
Semi-Simple Lie Algebras

7 Lie's Theorem and Cartan's Theorem

Notation. *Throughout we will take k to be an algebraically closed field of characteristic 0.*

Theorem 7.1. *Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$, where V is f.d. Then if $V \neq 0$, it contains a common eigenvector for all endomorphisms in $\mathfrak{g}$.*

Corollary 7.2 (Lie's Theorem). *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable subalgebra where V is f.d. Then $\mathfrak{g}$ stabilises some flag in V (i.e. the matrices of L are upper triangular wrt a suitable basis of V).*

Corollary 7.3. *If $\mathfrak{g}$ of dim n is solvable then $\exists\, 0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ a chain of ideals s.t. $\dim \mathfrak{g}_i = i$.*

Corollary 7.4. *$\mathfrak{g}$ solvable $\Rightarrow i(\mathfrak{g})$ nilpotent.*

<mark>Return to this section</mark>

8 Killing Form

Definition 8.1 (Killing Form). *The **Killing form** is a map $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined such that for $X, Y \in \mathfrak{g}$*

$$\kappa\left(X, Y\right) = \operatorname{Tr}\left(ad_X \circ ad_Y\right)$$

In components

$$\kappa(X, Y) = \kappa^{ab} X_a Y_b$$
$$\kappa^{ab} = f_d^{ac} f_c^{bd}$$

It is symmetric and bilinear

Remark. <mark>*It would seem to me that this definition needs that the algebra be f.d. Look into this*</mark>

Theorem 8.2. *Let $\mathfrak{g}$ be a Lie algebra and κ the Killing form. Then*

$$\forall X, Y, Z \in \mathfrak{g} \quad \kappa\left(\left[Z, X\right], Y\right) + \kappa\left(X, \left[Z, Y\right]\right) = 0$$

Proof. Expand out using $ad_{[X,Y]} = [ad_X, ad_Y]$. $\qquad\square$

Lemma 8.3. *Let $I \subset \mathfrak{g}$ be an ideal. Then $\kappa_I = \kappa|_{I \times I}$*

Theorem 8.4 (Cartan). *Let $\mathfrak{g}$ be a Lie algebra with Killing form κ. Then*

$$\kappa \text{ non-degenerate} \Leftrightarrow \mathfrak{g} \text{ semi-simple}$$

Proof. The direction non-degenerate $\Rightarrow$ semi-simple is as follows. Assume $\mathfrak{g}$ not semi-simple. Let $\{T^a\}$ be a basis of a abelian ideal of $\mathfrak{g}$, and extend with $\{T^i\}$ to a basis of the full algebra. Then

$$\left[T^a, T^b\right] = 0 \Rightarrow f_\alpha^{ab} = 0$$
$$\left[T^a, T^\alpha\right] \in \text{Span}\left\{T^b\right\} \Rightarrow f_i^{a\alpha} = 0$$

so

$$\begin{aligned}\kappa^{ab} &= f_\beta^{a\alpha} f_\alpha^{b\beta} \\ &= f_\beta^{ai} f_i^{b\beta} \\ &= f_j^{ai} f_i^{bj} = 0\end{aligned}$$

so κ degenerate. $\qquad\qquad\square$

Theorem 8.5. *Let $\mathfrak{g}$ be semisimple. Then $\exists \mathfrak{g}_1, \ldots, \mathfrak{g}_t$ simple ideals s.t. $\mathfrak{g} = \bigoplus_{i=1}^{t} \mathfrak{g}_i$. Every simple ideal of $\mathfrak{g}$ coincides with a $\mathfrak{g}_i$*

Corollary 8.6. *If $\mathfrak{g}$ is semisimple then $\mathfrak{g} = i(\mathfrak{g})$ and all ideals and homomorphic images of $\mathfrak{g}$ are semisimple.*

Theorem 8.7. *$\mathfrak{g}$ semisimple $\Rightarrow d_{adj}(\mathfrak{g}) = \text{Der}(\mathfrak{g})$.*

9 Cartan Classification

Élie Cartan classified finite dimensional, simple, complex Lie algebras in 1894. This sections will therefore restrict to treatment of such Lie algebras. Note that $\mathfrak{su}_\mathbb{C}(2)$ will be a prototypical such Lie algebra.

9.1 Cartan-Weyl Basis

Definition 9.1 (Cartan-Weyl Basis). *Given a Lie algebra $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h} = \text{Span}\left\{H^i : i = 1, \ldots, r\right\}$, let $\left\{E^\alpha : \alpha \in \Phi\right\}$ be the simultaneous eigenvectors of ad_{H^i} s.t.*

$$ad_{H^i}\left(E^\alpha\right) = \alpha^i E^\alpha$$

Then
$$B = \left\{H^i : i = 1, \ldots, r\right\} \cup \left\{E^\alpha : \alpha \in \Phi\right\}$$

*is the **Cartan-Weyl basis** for $\mathfrak{g}$.*

Definition 9.2 (Roots). *The **roots** of a Lie Algebra are $\alpha \in \Phi$. These are the weights corresponding to the adjoint representation. The roots can be seen as elements of the dual space to the Cartan subalgebra $\mathfrak{h}^*$ by*

$$\alpha : \mathfrak{h} \to \mathbb{R}$$
$$\alpha(H^i e_i) = \alpha^i e_i$$

Fact 9.3. *The roots are all non-degenerate i.e. the eigenspace is 1 dimensional.*

Theorem 9.4. *Let κ be the Killing form on $\mathfrak{g}$. Then,*

- $\forall H \in \mathfrak{h}, \forall \alpha \in \Phi, \; \kappa\left(H, E^\alpha\right) = 0$

- $\forall \alpha, \beta \in \Phi, \alpha + \beta \neq 0, \; \kappa\left(E^\alpha, E^\beta\right) = 0$

- $\forall H \in \mathfrak{h}, \exists H' \in \mathfrak{h}, \; \kappa\left(H, H'\right) \neq 0$

- $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$ *and* $\kappa\left(E^\alpha, E^{-\alpha}\right) \neq 0$

Hence $\kappa|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate, so invertible.

Fact 9.5. *Given $\alpha \in \Phi$, $\{\lambda : \lambda\alpha \in \Phi\} = \{\pm 1\}$. Note one direction of this inclusion follows from the previous result.*

Definition 9.6 (κ^{ij}). *Given a Cartan subalgebra $\mathfrak{h} = \mathrm{Span}\{H^i\}$ and Killing form κ the notation will be used κ^{ij} for the matrix elements of $\kappa|_{\mathfrak{h} \times \mathfrak{h}}$,*

$$\kappa^{ij} = \kappa(H^i, H^j)$$

Definition 9.7 (Compact Type). *A real Lie algebra $\mathfrak{g}_\mathbb{R}$ is of **compact type** if $\exists$ a basis in which $\kappa^{ij} = -K\delta^{ij}$ for some $K \in \mathbb{R}^+$.*

Fact 9.8. *Every complex semi-simple Lie algebra of finite dimension has a real form of compact type.*

Definition 9.9 (Dual Space Inner Product). *The Killing form gives the dual space $\mathfrak{h}^*$ a natural inner product $(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ defined by*

$$(\alpha, \beta) = \left(\kappa^{-1}\right)_{ij} \alpha^i \beta^j = k^{ij} \alpha_i \beta_j$$

Defining $\alpha_i = \left(\kappa^{-1}\right)_{ij} \alpha^j$. Note that it is immediate that the bracket is symmetric and bilinear. Positive definiteness follows from the fact

$$(\alpha, \beta) = \sum_{\delta \in \Phi}(\alpha, \delta)(\beta, \delta)$$

(proven by noting the trace of an operator is the sum over its eigenvalues, so $k^{ij} = \sum_{\delta \in \Phi} \delta^i \delta^j$)

$$(\alpha, \alpha) = \sum_{\delta \in \Phi}(\alpha, \delta)^2$$

Note that the above definition shows that we actually have an inner product on the span of the roots. The following lemma closes that hole

Lemma 9.10.

$$\mathfrak{h}^* = \mathrm{Span}_\mathbb{C} \Phi$$

Hence r roots may be chosen to form a basis $\{\alpha_{(i)} : i = 1, \ldots, r\}$

Proof. Suppose not. Then $\exists \lambda \in \mathfrak{h}^*$ s.t.

$$\forall \alpha \in \Phi, \quad (\lambda, \alpha) = 0$$

now let $H_\lambda = \lambda_i H^i$, where $\lambda^i = \lambda(H^i)$. Then

$$
\begin{aligned}
\forall H \in \mathfrak{h}, \quad &[H_\lambda, H] = 0 \\
\forall \alpha \in \Phi, \quad &[H_\lambda, E^\alpha] = (\lambda, \alpha) E^\alpha \\
\Rightarrow \forall X \in \mathfrak{g}, \quad &[H_\lambda, X] = 0
\end{aligned}
$$

Hence $\mathfrak{g}$ has the non-trivial ideal $\mathrm{Span}_{\mathbb{C}}\{H_\lambda\}$ and so $\mathfrak{g}$ is not simple. Contradiction. $\qquad \square$

Definition 9.11 (Coroots). *Given a root $\alpha \in \Phi$ the **coroot** is*

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)} \alpha = \epsilon_\alpha \alpha, \ \epsilon_\alpha = \frac{2}{(\alpha, \alpha)}$$

Note $(\alpha^\vee)^\vee = \alpha$. We denote $\Phi^\vee = \{\alpha^\vee \,|\, \alpha \in \Phi\}$.

Lemma 9.12. $[\mathfrak{h}, [E^\alpha, E^{-\alpha}]] = 0$

Corollary 9.13. $[E^\alpha, E^{-\alpha}] \in \mathfrak{h}$

Definition 9.14 (t^α). *Define*

$$t^\alpha = \frac{[E^\alpha, E^{-\alpha}]}{\kappa \left(E^\alpha, E^{-\alpha}\right)}$$

In components

$$t^\alpha = \left(\kappa^{-1}\right)_{ij} \alpha^j H^i$$

Theorem 9.15 (Cartan-Weyl Basis Algebra). *The algebra of the Cartan-Weyl basis is*

$$
\begin{aligned}
\left[H^i, H^j\right] &= 0 \\
\left[H^i, E^\alpha\right] &= \alpha_i E^\alpha \\
\left[E^\alpha, E^\beta\right] &= \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta} & \alpha + \beta \in \Phi \\ \kappa \left(E^\alpha, E^{-\alpha}\right) t^\alpha & \alpha + \beta = 0 \\ 0 & \textit{otherwise} \end{cases}
\end{aligned}
$$

Fact 9.16. $\forall H \in \mathfrak{h}$, $\alpha, \beta \in \Phi$

- $\kappa \left(t^\alpha, H\right) = \alpha(H)$

- $\kappa \left(t^\alpha, E^\beta\right) = 0$

- $[t^\alpha, H] = 0$

- $\left[t^\alpha, E^\beta\right] = (\alpha, \beta) E^\beta$

Theorem 9.17. *Letting*

$$e^\alpha = \sqrt{\frac{2}{(\alpha,\alpha)\kappa\left(E^\alpha, E^{-\alpha}\right)}} E^\alpha$$

$$h^\alpha = \frac{2}{(\alpha,\alpha)} t^\alpha$$

yields the algebra

$$\left[h^\alpha, h^\beta\right] = 0$$

$$\left[h^\alpha, e^\beta\right] = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} e^\beta$$

$$\left[e^\alpha, e^\beta\right] = \begin{cases} n_{\alpha,\beta} e^{\alpha+\beta} & \alpha + \beta \in \Phi \\ h^\alpha & \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 9.18 $(sl(2)_\alpha)$**.** *For $\alpha \in \Phi$ there is a $\mathfrak{su}_\mathbb{C}(2)$ subalgebra*

$$sl(2)_\alpha = \mathrm{Span}\left\{\, h^\alpha, e^{\pm\alpha} \,\right\}$$

9.2 Root Geometry

Definition 9.19 (α-string through β)**.** *The α-**string through** β is*

$$S_{\alpha,\beta} = \{\, \beta + \rho\alpha \in \Phi : \rho \in \mathbb{Z} \,\}$$

The corresponding subspace of $\mathfrak{g}$ is

$$V_{\alpha,\beta} = \left\{ e^{\beta+\rho\alpha} : \beta + \rho\alpha \in S_{\alpha,\beta} \right\}$$

*The **length** of the string is*

$$l_{\alpha,\beta} = |S_{\alpha,\beta}|$$

Fact 9.20. *$V_{\alpha,\beta}$ is a representation space of the adjoint representation $R = d_{adj}$ of $sl(2)_\alpha$. The non-degeneracy of the roots ensures that the weight set*

$$S_R = \left\{ \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2\rho : \beta + \rho\alpha \in S_{\alpha,\beta} \right\}$$
$$= \{\, -\Lambda, -\Lambda + 2, \dots, \Lambda \,\} = S_\Lambda$$

Hence $R_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

Fact 9.21. *Writing*

$$S_{\alpha,\beta} = \{\, \beta + \rho\alpha \in \Phi : n_- \leq \rho \leq n_+ \,\}$$

with $n_+ \geq 0$ and $n_- \leq 0$ yields

- $l_{\alpha,\beta} = n_+ - n_- + 1$

- $R_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -(n_+ + n_-) \in \mathbb{Z}$

Lemma 9.22.
$$\forall \alpha, \beta \in \Phi, \ (\alpha,\beta) \in \mathbb{R}$$

Proof. Write

$$\frac{(\alpha,\alpha)}{2} R_{\alpha,\beta} = \sum_{\delta \in \Phi} \left[\frac{(\alpha,\alpha)}{2} R_{\alpha,\delta} \right] \left[\frac{(\beta,\beta)}{2} R_{\beta,\delta} \right]$$

$$\Rightarrow \frac{2}{(\beta,\beta)} R_{\alpha,\beta} = \sum_{\delta \in \Phi} R_{\alpha,\delta} R_{\beta,\delta} \in \mathbb{R}$$

$$\Rightarrow (\beta,\beta) \in \mathbb{R}$$

$$\Rightarrow (\alpha,\beta) = \frac{(\alpha,\alpha)}{2} R_{\alpha,\beta} \in \mathbb{R}$$

$\square$

Definition 9.23 ($\mathfrak{h}_{\mathbb{R}}^*$)**.** *Given a basis of roots* $\{\,\alpha_{(i)} : i = 1, \ldots, r\,\}$, $\mathfrak{h}_{\mathbb{R}}^*$ *is defined as*

$$\mathfrak{h}_{\mathbb{R}}^* = \operatorname{Span}_{\mathbb{R}} \{\,\alpha_{(i)} : i = 1, \ldots, r\,\}$$

Fact 9.24. $\Phi \subset \mathfrak{h}_{\mathbb{R}}^*$

Theorem 9.25. $(\cdot,\cdot) : \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ *is a Euclidean inner product.*

Definition 9.26 (Length and angle between roots)**.** *Given* $\alpha \in \Phi$ *the* **length** *of* α *is defined as*

$$|\alpha| = (\alpha,\alpha)^{\frac{1}{2}}$$

Given also $\beta \in \Phi$, *the* **angle** *between* α *and* β *is* ϕ *defined such that*

$$(\alpha,\beta) = |\alpha||\beta| \cos \phi$$

Theorem 9.27.
$$\cos \phi = \pm \frac{\sqrt{n}}{2}$$

for $n \in \{\,0,1,2,3,4\,\}$

Lemma 9.28. *Let* $\alpha, \beta \in \Phi$ *be non-proportional. Then*

$$(\alpha,\beta) > 0 \Rightarrow \alpha - \beta \in \Phi$$
$$(\alpha,\beta) < 0 \Rightarrow \alpha + \beta \in \Phi$$

Definition 9.29 (Positive roots)**.** *Given a hyperplane* $H \leq \mathfrak{h}^*$ *s.t.* $H \cap \Phi = \emptyset$ *and* $\mathfrak{h}^*$ *separated into* $\mathfrak{h}_{\pm}^*$ *let*
$$\Phi_{\pm} = \Phi \cap \mathfrak{h}_{\pm}^*$$

Define the **positive roots** *to be* $\alpha \in \Phi_+$

Fact 9.30. *Let $\Phi_\pm$ be as defined.*

- $\alpha \in \Phi_\pm \Rightarrow -\alpha \in \Phi_\mp$

- $\alpha, \beta \in \Phi_\pm, \alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Phi_\pm$

Definition 9.31 (Simple Roots). *A roots δ is **simple** if*

- $\delta \in \Phi_+$

- $\nexists \alpha, \beta \in \Phi_+ \; s.t. \; \delta = \alpha + \beta$

*The set of simple roots, Δ, is called the **base**.*

Notation. *We will write $\epsilon_i = \epsilon_{\alpha_{(i)}}$ for $\alpha_{(i)} \in \Delta$.*

Theorem 9.32 (Properties of simple roots). *Let $\alpha, \beta \in \Delta$, $\alpha \neq \beta$. Then*

- $\alpha - \beta \notin \Phi$

- $l_{\alpha,\beta} = 1 - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$

- $(\alpha, \beta) \leq 0$

- $\Phi_+ \subset \mathrm{Span}_{\mathbb{N}_0} \Delta$

- *The simple roots are linearly independent.*

- $|\Delta| = r$

Hence $\Delta = \{\, \alpha_{(i)} : i = 1, \ldots, r \,\}$ is a basis for $\mathfrak{h}_\mathbb{R}^*$

Definition 9.33. *A root system is **reducible** if $\exists \Phi_1, \Phi_2$, s.t. $\Phi = \Phi_1 \cup \Phi_2$ and $(\Phi_1, \Phi_2) = 0$. If not we say it is **irreducible**.*

Definition 9.34 (Height). *Write $\beta \in \Phi$ as $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$ Then the **height** of the root is*

$$ht : \Phi \to \mathbb{Z}$$
$$ht(\beta) = \sum_{\alpha \in \Delta} n_\alpha$$

We define a partial order on $\mathbb{R}^r$ by defining

$$\beta \prec \alpha \Leftrightarrow \alpha - \beta \in \mathrm{Span}_{\mathbb{N}_0} \Delta$$

9.3   The Weyl Group

Definition 9.35 (Weyl Reflections and Group). *For $\alpha \in \Phi$, the **Weyl reflection** in the hyperplane orthogonal to α is*

$$w_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$$
$$w_\alpha(x) = x - \frac{2(\alpha, x)}{(\alpha, \alpha)} \alpha$$

Noting

- $(w_\alpha)^2 = id$

- $w_\alpha(\alpha) = -\alpha$

- $(\alpha, x) = 0 \Rightarrow w_\alpha(x) = x$

- $(w_\alpha(x), w_\alpha(y)) = (x, y)$

It can be seen w_α is indeed a reflection in the hyperplane orthogonal to α, denoted P_α, and hence is an isometry. The **Weyl group** *is*

$$W = \{\, w_\alpha : \alpha \in \Phi \,\}$$

with composition as the group operation. Using the previous results it can be seen w_α restricts to an isometry on Φ. Hence W acts via permutation on Φ, and so $W \leq S_\Phi$

Remark. *It is possible to take an axiomatic approach to root systems using the concept of the Weyl group. We say $\Phi \subset \mathbb{R}^r$ is a* **root system** *if it satisfies:*

1. *Φ is finite, $\operatorname{Span} \Phi = \mathbb{R}^r$, $0 \notin \Phi$*

2. *$\alpha \in \Phi \Rightarrow \{\lambda \mid \lambda\alpha \in \Phi\} = \{\pm 1\}$*

3. *$\alpha \in \Phi \Rightarrow w_\alpha(\Phi) = \Phi$*

4. *$\alpha, \beta \in \Phi \Rightarrow R_{\alpha,\beta} \in \mathbb{Z}$*

We have seen that these axioms apply to the roots of our Lie algebras, and indeed these are sufficient to give a 1-to-1 correspondence.

Lemma 9.36. $(\alpha, \beta) = 0 \Rightarrow w_\alpha w_\beta = w_\beta w_\alpha$.

Definition 9.37. *The connected components of $\mathbb{R}^r \setminus \bigcup_\alpha P_\alpha$ are called the* **Weyl chambers** *of $\mathbb{R}^r$.*

Proposition 9.38. *Weyl chambers are in natural 1-to-1 correspondence with choices of base Δ.*

Proof. Each root in $\mathbb{R}^r \setminus \bigcup_\alpha P_\alpha$ specifies a hyperplane splititng $\mathfrak{h}$, giving a base, and each such root lies in exactly one Weyl chamber. $\qquad\square$

Definition 9.39. *The Weyl chamber specified by a choice of base, $\mathfrak{C} = \mathfrak{C}(\Delta)$ is called the* **fundamental Weyl chamber** *relative to Δ.*

Proposition 9.40. *W is generated by the $r_i = w_{\alpha_{(i)}}$ with the relations*

- $r_i^2 = 1$

- $(r_i r_j)^{m_{ij}} = 1$ *or equivalently* $\underbrace{r_j r_i r_j \ldots}_{m_{ij}} = \underbrace{r_i r_j r_i \ldots}_{m_{ij}}$

where

m_{ij}	2	3	4	6	∞
$a_{ij}a_{ji}$	0	1	2	3	≥ 4

and $a_{ij} = R_{\alpha_{(i)}, \alpha_{(j)}}$

Remark. *A group of this form is called a **Coxeter group**.*

Proposition 9.41. *Let Φ be irreducible. Then*

1. *wrt $\prec$ $\exists! \theta \in \Phi$ maximal,*

2. *W acts irreducibly on $\mathbb{R}^r$ (i.e. the W-orbit of a root spans $\mathbb{R}^r$)*

3. *At most two root lengths in Φ, and roots of the same length are conjugate under W*

4. *the maximal root θ is long.*

Definition 9.42. *The **Coxeter number** and **dual Coxeter number** are respectively*

$$h = 1 + \sum_{\alpha \in \Delta} n_\alpha$$

$$g = h^\vee = 1 + \sum_{\alpha \in \Delta} n_\alpha^\vee$$

where

$$\theta = \sum_{\alpha \in \Delta} n_\alpha \alpha = \epsilon_\theta^{-1} \sum_{\alpha \in \Delta} n_\alpha^\vee \alpha^\vee$$

Note this implies $n_\alpha^\vee = \epsilon_\theta \epsilon_\alpha^{-1} n_\alpha$

Remark. *It is sometimes common to normalise the roots by setting $\epsilon_\theta = 1$.*

9.4 Cartan Matrix and Dynkin Diagrams

Definition 9.43 (Cartan Matrix)**.** *The **Cartan matrix** A is an $r \times r$ matrix with elements*

$$A^{ij} = \frac{2(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})}$$

Definition 9.44 (Chevalley Basis)**.** *For each $\alpha_{(i)} \in \Delta$ the basis of $sl(2)_{\alpha_{(i)}}$*

$$\left\{ h^i = h^{\alpha_{(i)}}, e_\pm^i = e^{\pm \alpha_{(i)}} \right\}$$

*is the **Chevalley basis***

Theorem 9.45. *In the Chevalley basis*

- $\left[h^i, h^j \right] = 0$

- $\left[h^i, e_\pm^j \right] = \pm A^{ji} e_\pm^j$

- $\left[e_+^i, e_-^j \right] = \delta_{ij} h^i$

Theorem 9.46 (Serre Relation)**.** *The **Serre relation** is*

$$\left(ad_{e_\pm^i}\right)^{1-A^{ji}} e_\pm^j = 0$$

Idea. *The previous two results show that the Cartan Matrix completely determines a finite dimensional, simple, complex Lie algebra.*

Theorem 9.47 (Properties of the Cartan Matrix)**.** *The Cartan matrix satisfies*

- $\forall i \; A^{ii} = 2$

- $A^{ij} = 0 \Rightarrow A^{ji} = 0$

- *for $i \neq j \; A^{ij} \in \mathbb{Z}_{\leq 0}$*

- $|A^{ij}| \leq 4$

- $\det A > 0$

- *A irreducible (for simple Lie algebras)*

Definition 9.48 (Dynkin Diagrams)**.** *The **Dynkin diagrams** are constructed from the Cartan matrix as follows:*

- *Draw a node for each simple root.*

- *Connect the ith and j^{th} root with $\max\{\; |\; A^{ij}|, |A^{ji}|\;\}$ lines.*

- *If the roots have different length, draw an arrow from the longer root to the shorter.*

Proposition 9.49. *The number of connected components of the Dynkin diagram is equal to the number of irreducible components of Φ.*

Theorem 9.50. *The Dynkin diagrams clsasify finite dimensional semi-simple Lie algebras, and the correspondence is*

$$A_n \leftrightarrow \mathfrak{sl}_{n+1}$$
$$B_n \leftrightarrow \mathfrak{so}_{2n+1}$$
$$C_n \leftrightarrow \mathfrak{sp}_{2n}$$
$$D_n \leftrightarrow \mathfrak{so}_{2n}$$

10 Representations of Classical Lie Algebras

10.1 Lattices

Definition 10.1 (Root Lattice)**.** *Given a Lie algebra $\mathfrak{g}$ with simple roots $\Delta = \{\, \alpha_{(i)} : i = 1, \ldots, r \,\}$ the **root lattice** is*

$$\mathcal{L}[\mathfrak{g}] = \left\{ \sum_{i=1}^{r} m_i \alpha_{(i)} : m_i \in \mathbb{Z} \right\}$$

Definition 10.2 (Coroot Lattice). *The **coroot lattice** is given by*

$$\mathcal{L}^{\vee}[\mathfrak{g}] = \left\{ \sum_{i=1}^{r} m_i \alpha_{(i)}^{\vee} : m_i \in \mathbb{Z} \right\}$$

Definition 10.3 (Quantisation Condition). *Given a representation R of the $sl(2)_\alpha$ subalgbera, and $v \in V_\lambda$ for λ a weight of the representation,*

$$R(h^\alpha)v = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} v$$

Hence as the weights of $sl(2)_\alpha$ are integers

$$\frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z}$$

Definition 10.4 (Weight Lattice). *The **weight lattice**, $\mathcal{L}_W[\mathfrak{g}]$, is the dual of the coroot lattice, defined as*

$$\mathcal{L}_W[\mathfrak{g}] = \left(\mathcal{L}^{\vee}[\mathfrak{g}] \right)^* = \left\{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* : \forall \mu \in \mathcal{L}^{\vee}[\mathfrak{g}] \, (\lambda, \mu) \in \mathbb{Z} \right\}$$

Theorem 10.5. *Note*

$$(\lambda, \alpha_{(i)}^{\vee}) = \frac{2(\alpha_{(i)}, \lambda)}{(\alpha_{(i)}, \alpha_{(i)})} \in \mathbb{Z}$$

Hence all weights Lie in the weight lattice.

Definition 10.6 (Fundamental Weights). *The dual to the coroot basis, i.e. the weights $\omega_{(i)}$ such that*

$$(\alpha_{(i)}^{\vee}, \omega_{(j)}) = \delta_{ij}$$

*are the **fundamental weights***

Fact 10.7. *Writing $\omega_{(i)} = \sum_{j=1}^{r} B^{ij} \alpha_{(j)}$ yields $\delta_{ij} = A^{ki} B_{jk}$. Hence*

$$\alpha_{(i)} = \sum_{j=1}^{r} A^{ij} \omega_{(j)}$$

where A^{ij} is the Cartan matrix.

10.2   Highest Weight Representations

Definition 10.8 (Highest Weight). *Given a finite dimensional representation, the **highest weight** is Λ such that*

$$\forall v_\Lambda \in V_\Lambda, \forall \alpha \in \Phi_+ \quad R(E^\alpha) v_\Lambda = 0$$

Note that a highest weight must exist if the rep is f.d. If the representation is irreducible, all other weight are generated as product of $R(E^{-\alpha})$ for $\alpha \in \Phi_+$.

Theorem 10.9 (Theorem of the Highest Weight). *Every finite dimension, irreducible, representation has a highest weight. Moreover, if two such representation have the same highest weight, they are isomorphic. Call the representation R_Λ.*

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Fact 10.10. *Given a representation R, for $v \in V_\lambda$,*

$$\lambda + \alpha \in S_R \Rightarrow R(E^\alpha)v \in V_{\lambda+\alpha}$$

Hence if $R = R_\Lambda$ all weights are of the form

$$\lambda = \Lambda - \mu$$

for $\mu = \sum_{i=1}^r \mu_i \alpha_{(i)}$, $\mu_i \in \mathbb{N}_0$. Hence writing

$$\lambda = \sum_{i=1}^r \lambda^i \omega_{(i)}$$

gives that

$$\forall m_{(i)} \in \mathbb{Z}, \ 0 \le m_{(i)} \le \lambda^i, \quad \lambda - m_{(i)}\alpha_{(i)} \in S_R$$

Theorem 10.11 (Fruedenthal's Formula)**.** *Given rep R_Λ and weight λ, the multiplicity of λ, $m(\lambda)$ is given recursively by*

$$[(\Lambda + \delta, \Lambda + \delta) - (\lambda + \delta, \lambda + \delta)] \, m(\lambda) = 2 \sum_{\alpha \succ 0} \sum_{i=1}^\infty m(\lambda + i\alpha)(\lambda + i\alpha, \alpha)$$

where we take $m(\mu) = 0$ if μ is not a weight, and $\delta = \frac{1}{2}\sum_{\alpha \succ 0} \alpha$.

Lemma 10.12. *If $R_\Lambda, R_{\Lambda'}$ are two representations with weights λ, λ' respectively, then $\lambda + \lambda'$ is a weight of $R_\Lambda \otimes R_{\Lambda'}$*

10.3 Characters

Definition 10.13. *The **universal Casimir element** of $\mathfrak{g}$ is*

$$c_\mathfrak{g} = \sum_\mu X_\mu Y_\mu \in \mathcal{U}_\mathfrak{g}$$

where $\{X_\mu\} = \{H \cdot E^\alpha\}$ is the Cartan-Weyl basis and Y_μ are dual wrt the Killing form.

Lemma 10.14. $c_\mathfrak{g} \in \zeta(\mathcal{U}_\mathfrak{g})$

Lemma 10.15. *Take the highest weight rep $R = R_\Lambda$ and denote $\mathbb{Z}[R] = \mathbb{Z}(\Lambda)$. Then if $z \in \zeta(\mathcal{U}_\mathfrak{g})$ and $v \in \mathbb{Z}(\Lambda)$ is maximal then so is $z \cdot v$. Hence $z \cdot v \propto v$*

Definition 10.16. *The **character** of the highest weight rep R_Λ is the k-algebra homomorphism $\chi_\Lambda : \zeta(\mathcal{U}_\mathfrak{g}) \to k$ given by*

$$z \cdot c = \chi_\Lambda(z)v$$

Definition 10.17. *Two weights λ, μ are **linked** (denoted $\lambda \sim \mu$) if $\lambda + \delta$ and $\mu + \delta$ are conjugate under the Weyl group.*

Theorem 10.18 (Harish-Chandra)**.** $\lambda \sim \mu \Leftrightarrow \chi_\lambda = \chi_\mu$

References

[1] James E. Humphreys. *Introduction to Lie algebras and representation theory*, vol. 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. ISBN 0-387-90053-5. Second printing, revised.