The Eisenhart Lift

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1 The Eisenhart Lift

1.1 The metric and equations of motion

Consider the $(d+2)$-dimensional line element,

$$ds^2 = \hat{g}_{\mu\nu}\, dx^\mu\, dx^\nu = h_{ij}\, dx^i\, dx^j + 2dt\left(dv - \Phi dt + N_i dx^i\right), \tag{1.1.1}$$

where $i, j = 1, \ldots, d$, $x^{d+1} = t$, $x^{d+2} = v$ and Φ, N_i and h_{ij} are independent of the coordinate v. Then $\xi = \partial_v$ is a Killing vector. We have

$$\hat{g} = \begin{pmatrix} h_{ij} & N_i & 0 \\ N_j & -2\Phi & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{g}^{-1} = \begin{pmatrix} h^{ij} & 0 & -h^{ik}N_k \\ 0 & 0 & 1 \\ -h^{jk}N_k & 1 & 2\Phi + N_i h^{ij} N_j \end{pmatrix},$$

where h^{ij} is the inverse of h_{ij}. The geodesic Lagrangian is

$$\mathcal{L} = \frac{1}{2}\hat{g}_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu = \frac{1}{2}h_{ij}\, \dot{x}^i\, \dot{x}^j + \dot{t}\dot{v} - \Phi\dot{t}^2 + N_i\, \dot{x}^i \dot{t} := \tilde{L} + \dot{t}\dot{v},$$

where $\dot{x}^\mu = dx^\mu/d\lambda$ for an affine geodesic parameter λ ($\tilde{L}$ is defined below). Calculating the equations of motion from $\mathcal{L}$ enables a simple determination of (appropriate combinations of) the Christoffel symbols for $\hat{g}$. Recall

$$\hat{\Gamma}^\mu_{\nu\rho} = \frac{1}{2}\hat{g}^{\mu\delta}\left(\hat{g}_{\delta\nu,\rho} + \hat{g}_{\delta\rho,\nu} - \hat{g}_{\nu\rho,\delta}\right) := \hat{g}^{\mu\delta}[\nu\rho, \delta]_{\hat{g}}.$$

and the equations of motion are

$$0 = \ddot{x}^\mu + \hat{\Gamma}^\mu_{\nu\rho}\dot{x}^\mu\dot{x}^\rho.$$

Setting

$$A := A_\mu dx^\mu = N_i dx^i - \Phi dt, \qquad F = dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu = \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

we get

$$\begin{aligned} F_{ij} &= \partial_i N_j - \partial_j N_i = -F_{ji} \\ F_{it} &= -(\partial_t N_i + \partial_i \Phi) = -F_{ti} \end{aligned}$$

and using that the equations of motion for v, x^i and t (from $\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu}$) yield

$$0 = \frac{d}{d\lambda}\dot{t} = \ddot{t}$$

(for v) then

$$\begin{aligned}\frac{1}{2}(\partial_i h_{jk})\dot{x}^j\dot{x}^k - (\partial_i\Phi)\dot{t}^2 + (\partial_i N_j)\dot{x}^j\dot{t} &= \frac{d}{d\lambda}\left(h_{ij}\dot{x}^j + N_i\dot{t}\right) \\ &= h_{ij}\ddot{x}^j + (\partial_k h_{ij})\dot{x}^j\dot{x}^k + (\partial_t h_{ij})\dot{x}^j\dot{t} + (\partial_j N_i)\dot{x}^j\dot{t} + (\partial_t N_i)\dot{t}^2\end{aligned}$$

(for $\dot{x}^i$) and

$$\begin{aligned}\frac{1}{2}(\partial_t h_{ij})\dot{x}^i\dot{x}^j - (\partial_t\Phi)\dot{t}^2 + (\partial_t N_i)\dot{x}^i\dot{t} &= \frac{d}{d\lambda}\left(\dot{v} - 2\Phi\dot{t} + N_i\dot{x}^i\right) \\ &= \ddot{v} - 2(\partial_i\Phi)\dot{t}\dot{x}^i - 2(\partial_t\Phi)\dot{t}^2 - 2\Phi\ddot{t} + (\partial_j N_i)\dot{x}^i\dot{x}^j + (\partial_t N_i)\dot{x}^i\dot{t} + N_i\ddot{x}^i\end{aligned}$$

we get (collating them together)

$$\begin{aligned}0 &= \ddot{t}, \\ 0 &= h_{ij}\,\ddot{x}^j + [jk,i]_h\,\dot{x}^j\dot{x}^k + (\partial_t h_{ij} + \partial_j N_i - \partial_i N_j)\,\dot{t}\dot{x}^j + (\partial_i\Phi + \partial_t N_i)\,\dot{t}^2, \\ &= h_{ij}\,\ddot{x}^j + [jk,i]_h\,\dot{x}^j\dot{x}^k + (\partial_t h_{ij} - F_{ij})\,\dot{t}\dot{x}^j + F_{ti}\dot{t}^2, \\ 0 &= \ddot{v} + N_i\ddot{x}^i + \left[\frac{1}{2}\left(\partial_j N_i + \partial_i N_j\right) - \frac{1}{2}\partial_t h_{ij}\right]\,\dot{x}^i\dot{x}^j - 2\partial_i\Phi\,\dot{t}\dot{x}^i - \partial_t\Phi\,\dot{t}^2, \\ &= \ddot{v} + \left[\frac{1}{2}\left(\partial_j N_i + \partial_i N_j\right) - N_k\Gamma^k_{ij} - \frac{1}{2}\partial_t h_{ij}\right]\,\dot{x}^i\dot{x}^j + \left[-N^k(\partial_t h_{ki} - F_{ki}) - 2\partial_i\Phi\right]\,\dot{t}\dot{x}^i + \left(-\partial_t\Phi + N^i F_{it}\right)\,\dot{t}^2\end{aligned}$$

where we have substituted F in the latter, and use the notation

$$[jk,i]_h = \frac{1}{2}\left(\partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk}\right)$$

Note that to raise the index of N has required we recognise that

$$N^i = \hat{g}^{ij}N_j = h^{ij}N_j$$

1.2 Equivalence of equations of motion

The canonical momenta are given by $p_\mu = \partial\mathcal{L}/\partial\dot{x}^\mu = \hat{g}_{\mu\nu}\dot{x}^\nu$ giving

$$p_v = \dot{t}, \qquad p_i = h_{ij}\dot{x}^j + N_i\dot{t}, \qquad p_t = \dot{v} - 2\Phi\dot{t} + N_i\dot{x}^i,$$

and so

$$\dot{t} = p_v, \qquad \dot{x}^i = h^{ij}(p_j - N_j p_v), \qquad \dot{v} = p_t - N^i p_i + [2\Phi + N^2]\,p_v.$$

Likewise, the geodesic Hamiltonian is

$$\mathcal{H} = p_\mu\dot{x}^\mu - \mathcal{L} = \frac{1}{2}\hat{g}^{\mu\nu}\,p_\mu\,p_\nu = \frac{1}{2}h^{ij}\,(p_i - N_i p_v)(p_j - N_j p_v) + p_t p_v + \Phi\,p_v^2.$$

The equations of motion are

$$\frac{dt}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_t} = p_v, \qquad \frac{dv}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_v}, \qquad \frac{dx^i}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_i} = h^{ij}\,(p_j - N_j p_v),$$
$$\frac{dp_t}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial t}, \qquad \frac{dp_v}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial v} = 0, \qquad \frac{dp_i}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^i}.$$

Because v is a cyclic coordinate its conjugate momentum p_v is conserved along geodesics: thus $p_v = m$ is a constant and we may write

$$\mathcal{H} := H + m\,p_t, \qquad H := \frac{1}{2} h^{ij}\,(p_i - mN_i)(p_j - mN_j) + m^2\,\Phi.$$

We observe that we have the geodesics have the conserved quantities,

$$\frac{1}{2}\hat{g}^{\mu\nu}\,p_\mu p_\nu = m\left[\frac{p^i p_i}{2m} - N^i p_i + mN^i N_i + p_t + m\Phi\right] := -mE_0,$$
$$\hat{g}^{\mu\nu}\,p_\mu \xi_\nu = p_v = m.$$

Following the identifications of [2] we view $p_v = m$ as the mass, $-p_t = E$ as the energy, E_0 as the internal energy, and $m\Phi = V$ as the potential energy. Taking the internal energy to vanish in the nonrelativistic limit the null geodesics of $\hat{g}$ may be identified with the motion in the d-dimensional space with potential energy V. We note that two conformally related metrics have the same null geodesics, and so the d-dimensional world lines will be the same. For $m \neq 0$ the equations of motion for t then give $dt/d\lambda = m$, whence $dt = m\,d\lambda$ and we may eliminate the affine geodesic parameter λ for t. The equations of motion are then precisely those coming from the standard mechanical system

$$\tilde{L} = \frac{1}{2} h_{ij}\,\dot{x}^i\,\dot{x}^j + N_i\,\dot{x}^i - \Phi$$

where $\dot{x}^i$ is now the standard dx^i/dt (and $\dot{t} = 1$). Now

(a) in the case of a non-null geodesic, if we parameterised the curve by arc length, $\lambda = s$ and $t = ms$, then from (1.1.1) we have

$$\frac{dv}{dt} = \frac{1}{2m^2} - \tilde{L}.$$

The equations of motion for v follow from this and

$$v = \frac{t}{2m^2} - \int \tilde{L}\,dt + b.$$

(b) in the case of a null geodesics we have

$$\frac{dv}{dt} = -\tilde{L}, \qquad v = -\int \tilde{L}\,dt + b.$$

Thus we have for each $m \neq 0$ and b a bijection between the geodesics of $\hat{g}$ and the equations of motion of $\tilde{L}$.

1.3 Connection and Curvature

From the equations of motion we read that the nonvanishing Christoffel symbols for $\hat{g}$ are

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk}, \qquad \hat{\Gamma}^i_{jt} = -\frac{1}{2}F^i{}_j + \frac{1}{2}h^{ik}\partial_t h_{kj}, \qquad \hat{\Gamma}^i_{tt} = h^{ik}\left(\partial_t N_k + \partial_k \Phi\right) = -F^i{}_t,$$

$$\hat{\Gamma}^v_{tt} = -\partial_t\Phi + N^k F_{kt}, \quad \hat{\Gamma}^v_{ij} = \frac{1}{4}\left[\nabla^{(h)}_i N_j + \nabla^{(h)}_j N_i - \partial_t h_{ij}\right], \quad \hat{\Gamma}^v_{ti} = -\frac{1}{2}N^k(\partial_t h_{ki} - F_{ki}) - \partial_i\Phi.$$

Recall now the equation for the Riemann tensor

$$\hat{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho\hat{\Gamma}^\mu_{\nu\sigma} - \partial_\sigma\hat{\Gamma}^\mu_{\nu\rho} + \hat{\Gamma}^\mu_{\rho\lambda}\hat{\Gamma}^\lambda_{\nu\sigma} - \hat{\Gamma}^\mu_{\sigma\lambda}\hat{\Gamma}^\lambda_{\nu\rho}$$

We immediately notice

$$\hat{R}^i{}_{jkl} = R^i{}_{jkl} + \hat{\Gamma}^i_{kt}\hat{\Gamma}^t_{jl} - \hat{\Gamma}^i_{lt}\hat{\Gamma}^t_{jk} + \hat{\Gamma}^i_{kv}\hat{\Gamma}^v_{jl} - \hat{\Gamma}^i_{lv}\hat{\Gamma}^v_{jk} = R^i{}_{jkl}$$

as there are non-vanishing Christoffel symbols with v as lower index, or t as an upper index. Further, as all Christoffel symbols are independent of v (as the metric is) we can then say that $\hat{R}^\mu{}_{\nu v\sigma} = 0$. As such $\hat{R}^\mu{}_{\nu\rho\sigma} = 0$ if any of $\nu, \rho, \sigma = v$. We can also see that $\hat{R}^t{}_{\nu\rho\sigma} = 0$ by the formula, so we now need only determine

1. $R^v{}_{ijk}$
2. $R^v{}_{itk}$
3. $R^v{}_{tjk}$
4. $R^v{}_{ttk}$

1.4 The Frame

Given the metric (1.1.1) we define the frame $\{\hat{e}^A\}$,

$$ds^2 = \hat{g}_{\mu\nu}\,dx^\mu\,dx^\nu = h_{ij}\,dx^i\,dx^j + 2dt\left(dv - \Phi dt + N_i dx^i\right) = \hat{\eta}_{AB}\,\hat{e}^A\hat{e}^B = \eta_{ab}\,e^a e^b + \hat{e}^+\hat{e}^- + \hat{e}^-\hat{e}^+.$$

Here $A \in \{+,-,a,b,\ldots\}$, $\hat{\eta}_{+-} = \hat{\eta}_{-+} = 1$, and we take

$$\hat{e}^+ := dt, \qquad \hat{e}^- := dv - \Phi\,dt + N_i\,dx^i, \qquad \hat{e}^a := \hat{e}^a_\mu dx^\mu = e^a_i dx^i = e^a,$$

and

$$e^a_i\eta_{ab}e^b_j = h_{ij}.$$

The coframe $\{\hat{E}_A\}$ with $\hat{e}^A(\hat{E}_B) = \delta^A_B$ is given by

$$\hat{E}_+ := \partial_t + \Phi\,\partial_v, \qquad \hat{E}_- := \partial_v, \qquad \hat{E}_a := E_a - N_a\,\partial_v,$$

where $N_a = N_i E^i_a$ and similarly

$$e^a(E_b) = \delta^a_b, \qquad E_b = E^i_b\,\partial_i.$$

We emphasise that N, ϕ and e^a may depend on x^i *and* t.

Denoting the structure constants $[\hat{E}_B, \hat{E}_C] = c^A{}_{BC}\hat{E}_A$ we have from

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$$

for a one-form α, then for the torsion free connection

$$d\hat{e}^A = -\hat{\omega}^A{}_B \wedge e^B = \hat{\omega}^A{}_{BC} e^B \wedge e^C$$

we have

$$d\hat{e}^A(\hat{E}_B, \hat{E}_C) = \hat{\omega}^A{}_{BC} - \hat{\omega}^A{}_{CB} = -\hat{e}^A([\hat{E}_B, \hat{E}_C]) = -c^A{}_{BC},$$

from which

$$\hat{\omega}^A{}_{BC} = \frac{1}{2}\hat{\eta}^{AF}(c_{CFB} + c_{BFC} - c_{FBC}).$$

The v-independence of the metric means that

$$[\hat{E}_-, \hat{E}_B] = 0, \qquad c^A{}_{-B} = 0$$

while

$$\begin{aligned}
[\hat{E}_+, \hat{E}_a] &= \partial_t E_a - (\partial_t N_a)\partial_v - (E_a \Phi)\partial_v \\
&= (\partial_t E^j_a)[\partial_j - N_j \partial_v] - E^j_a)[\partial_t N_j + \partial_j \Phi]\partial_v \\
&= (\partial_t E^j_a\, e^b_j)\, \hat{E}_b + F_{at}\, \hat{E}_-
\end{aligned}$$

and

$$\begin{aligned}
[\hat{E}_a, \hat{E}_b] &= [E_a, E_b] - (E_a N_b - E_b N_a)\, \partial_v \\
&= c^f{}_{ab}\, \hat{E}_f - F_{ab}\, \hat{E}_-
\end{aligned}$$

giving the (possibly) non-vanishing structure constants as

$$c^f{}_{ab}, \quad c^-{}_{ab} = -F_{ab}, \quad c^b{}_{+a} = (\partial_t E^j_a\, e^b_j), \quad c^-{}_{+a} = F_{at}.$$

Now

$$\begin{aligned}
d\hat{e}^+ &= 0, \\
d\hat{e}^- &= \frac{1}{2}F_{ab}\, e^a \wedge e^b + F_{it}\, dx^i \wedge dt = \frac{1}{2}F_{ab}\, \hat{e}^a \wedge \hat{e}^b + F_{at}\, \hat{e}^a \wedge \hat{e}^+, \\
d\hat{e}^a &= d\left(\hat{e}^a_\mu dx^\mu\right) = e^a_i dx^i = e^a = (\partial_j e^a_i)\, dx^j \wedge dx^i + (\partial_t e^a_i)\, dt \wedge dx^i \\
&= \omega^a{}_{bc}\, e^b \wedge e^c - (E^i_b\, \partial_t e^a_i)\, dt \wedge e^b = \omega^a{}_{bc}\, e^b \wedge e^c + (\partial_t E^i_b\, e^a_i)\, dt \wedge e^b,
\end{aligned}$$

from which we see

$$\begin{aligned}
\hat{\omega}^a{}_{BC}\, \hat{e}^B \wedge \hat{e}^C &= \omega^a{}_{bc}\, e^b \wedge e^c + (\partial_t E^i_b\, e^a_i)\, \hat{e}^+ \wedge \hat{e}^b, \\
\hat{\omega}^-{}_{BC}\, \hat{e}^B \wedge \hat{e}^C &= \frac{1}{2}F_{bc}\, \hat{e}^b \wedge \hat{e}^c + F_{at}\, \hat{e}^a \wedge \hat{e}^+.
\end{aligned}$$

Set

$$\alpha^a{}_b := e^a_i\, \partial_t E^i_b = c^a{}_{+b}, \quad \alpha_{ab} = -\alpha_{ba},$$

Using the antisymmetry of the connection then $0 = \hat{\omega}_{++A} = \hat{\omega}^-{}_{+A}$ and so

$$\hat{\omega}^-{}_{a+} = F_{at}, \quad \hat{\omega}^-{}_{ab} = \frac{1}{2}F_{ab}, \quad \hat{\omega}^a{}_{bc} = \omega^a{}_{bc}, \quad \hat{\omega}_{ab+} = -\frac{1}{2}F_{ab} - \frac{1}{2}\left[\partial_t E^i_a\, E_{ib} - \partial_t E^i_b\, E_{ia}\right] = -\frac{1}{2}F_{ab} + \alpha_{ab}.$$

1.5 Bargman Structures

A Bargmann structure $(B, \hat{g}, \xi)$ is a principal bundle $\pi : B \to M$, where $\dim B = \dim M + 1$, equipped with a Lorentzian metric $\hat{g}$ and nowhere vanishing null vector field ξ such that with respect to the usual Levi-Civita connection $\hat{\nabla}\xi = 0$. Then $M := B/\mathbb{R}\xi$ is equipped with a Newton-Cartan geometry (M, K, θ, ∇) where

$$K = \pi_* \hat{g}^{-1}, \qquad \hat{g}(\xi) = \pi^* \theta,$$

K is degenerate and $\pi^*\theta$ generates $\ker K$.

In our setting we have a metric of Brinkmann form

$$\hat{g} = h + dt \otimes \omega + \omega \otimes dt, \quad \omega = dv - \Phi(x,t)\, dt + N_i(x,t)\, dx^i, \quad h = h_{ij}(x,t) dx^i \otimes dx^j.$$

Then $\xi = \partial_v$, $\theta = dt$.

2 Introduction

Let us start with a bit of back story, so we can develop and go further. This will be built off of [1].

2.1 Galilei and Newton Structures

We start with some more classical work.

Definition 2.1 (Galilei group)**.** *The* ***Galilei group*** *is the matrix group*

$$G = \left\{ \begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \mid R \in SO(d), \ , b, c \in \mathbb{R}^n, \ e \in \mathbb{R} \right\} \leq GL_{d+2}(\mathbb{R})$$

We think of G as acting on $(\boldsymbol{x}, t, 1)$ s.t.

$$\begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Rx + tb + c \\ t + e \\ 1 \end{pmatrix}$$

with this action we see:

1. R are rotations in space
2. b are boosts
3. c, e are translations in space and time respectively

With this interpretation we have

Definition 2.2. *The* ***Homogeneous Galilei group/Euclidean group*** H *is the group of Galilean transformations that preserve the spatio-temporal origin* $(\mathbf{0}, 0, 1)$.

Proposition 2.3. *H consists of matrices of the form*

$$\begin{pmatrix} R & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Moreover $H \cong SO(d) \ltimes \mathbb{R}^d$ as a Lie group (not a as a Lie transformation group [3]) is faithfully represented by matrices of the form

$$\begin{pmatrix} R & b \\ 0 & 1 \end{pmatrix} \in GL_{d+1} .$$

Proof. See my CQIS notes for a more built up discussion of this fact. □

We now recall the following def:

Definition 2.4. *The* ***frame bundle*** *of a k-dimensional smooth manifold M is $GL(M)$, the GL_k-principal fibre bundle with fibres at $x \in M$ given by the space of ordered bases of T_xM.*

Definition 2.5. *A* ***proper Galilei structure*** *$H(M)$ is a reduction of structure group of the frame bundle of a $(d+1)$-dimensional M via $H \hookrightarrow GL_{d+1}$.*

References

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