An example of a Lax pair

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June 2020

1 1 dimension

Consider the $\mathfrak{su}(2)$ algebra generated by $\{H, E, F\}$ with the relations

$$\begin{aligned}[H, E] &= 2E \\ [H, F] &= -2F \\ [E, F] &= H\end{aligned}$$

Let us re-write this by introducing $X = E + F$, $Y = E - F$, which now have the commutation relations

$$\begin{aligned}[H, X] &= 2Y \\ [H, Y] &= 2X \\ [X, Y] &= -2H\end{aligned}$$

Consider the Hamiltonian system with phase space $q, p \in \mathbb{R}$ and Hamiltonian

$$\mathcal{H} = \frac{1}{2}p^2 + V(q)$$

Proposition 1.1. *The algebra elements*

$$\begin{aligned}L(\zeta) &= pH + W(q)X + \zeta Y, \quad \zeta \in \mathbb{C} \\ M &= \frac{1}{2}W_q Y\end{aligned}$$

for a Lax pair for the system if $V_q = WW_q$ *(i.e.* $V = \frac{1}{2}W^2 + c$*) and* $\cdot_q = \partial_q \cdot$*.*

Proof. Hamilton's equations for this system are

$$\begin{aligned}\dot{q} &= p \\ \dot{p} &= -V_q\end{aligned}$$

so

$$\dot{L} = -V_q H + pW_q X$$

whereas

$$[L, M] = pW_q X - WW_q H$$

□

Now we want to think of generalisations of this Lax pair. A generic Hamiltonian for a $2d$ phase space can be written in the form

$$\mathcal{H} = \frac{p^2}{2\lambda(q)} + V(q)$$

Proposition 1.2. *The algebra elements*

$$\begin{aligned}L(\zeta) &= \frac{1}{\sqrt{\lambda}}pH + W(q)X + \zeta Y, \quad \zeta \in \mathbb{C} \\ M &= \frac{1}{2\sqrt{\lambda}}W_q Y\end{aligned}$$

form a Lax pair for the system.

Proof. We repeat a similar calculation:

$$\dot{q} = \frac{p}{\lambda}$$
$$\dot{p} = -V_q + \frac{p^2\lambda_q}{2\lambda^2}$$

so

$$\begin{aligned} \dot{L} &= \frac{1}{\sqrt{\lambda}}\left[\left(-V_q + \frac{p^2\lambda_q}{2\lambda^2}\right) - \frac{p^2\lambda_q}{2\lambda^2}\right]H + \frac{pW_q}{\lambda}X \\ [L, M] &= \frac{pW_q}{\lambda}X - \frac{WW_q}{\sqrt{\lambda}}H \end{aligned}$$

□

Now if, like me, you do not see the spitting obvious thing that scaling $p \to \frac{p}{\sqrt{\lambda}}$ is a sensible thing to do, how might you approach this? Start by supposing a more general form related to our original pair

$$\begin{aligned} L(\zeta) &= f(q,p)H + g(q,p)X + \zeta Y \\ M &= h(q,p)Y \end{aligned}$$

We then get that for L, M to be a Lax pair we have the equations

$$\begin{aligned} f_q\left(\frac{p}{\lambda}\right) + f_p\left(-V_q + \frac{p^2\lambda_q}{2\lambda^2}\right) &= -2gh \\ g_q\left(\frac{p}{\lambda}\right) + g_p\left(-V_q + \frac{p^2\lambda_q}{2\lambda^2}\right) &= 2fh \end{aligned}$$

Let's make the ansatz that g, h are functions of q only. Then we get from the second equation

$$g_q \cdot \frac{p}{\lambda} = 2fh$$

Equation the order of p on each side we must get $f(q,p) = pF(q)$ and then

$$g_q = 2Fh\lambda$$

Substituting our new form of f into the first equation gives

$$\frac{p^2F_q}{\lambda} + F\left(-V_q + \frac{p^2\lambda_q}{2\lambda^2}\right) = -2gh$$

Again equating orders of p we have

$$\begin{aligned} \frac{F_q}{\lambda} + \frac{F\lambda_q}{2\lambda^2} = 0 &\Rightarrow F_q\lambda^{\frac{1}{2}} + \frac{1}{2}F\lambda^{-\frac{1}{2}}\lambda_q = 0 \\ &\Rightarrow \left(F\lambda^{\frac{1}{2}}\right)_q = 0 \\ &\Rightarrow F = \frac{\alpha}{\sqrt{\lambda}}, \quad \alpha \in \mathbb{R} \end{aligned}$$

Subbing this back into the first equation gives

$$-\alpha\lambda^{-\frac{1}{2}}V_q = -2gh$$

so

$$gg_q = \frac{\alpha\lambda^{-\frac{1}{2}}V_q}{2h}2\alpha h\lambda^{\frac{1}{2}} = \alpha^2 V_q$$

and

$$h = \frac{\lambda^{-\frac{1}{2}}g_q}{2\alpha}$$

We recognise taking $g = W$, $\alpha = 1$, this is the solution from above.

2 2 dimensions

We want to now try to see if we can expand upon this result. Note that our first systems has a natural generalisation of the following form:

Proposition 2.1. *The algebra elements*

$$L(\boldsymbol{\zeta}) = \sum_{i=1}^{n} p_i H_i + W_i X_i + \zeta_i Y_i$$
$$M = \frac{1}{2}(\partial_i W_i) Y_i$$

form a Lax pair for the evolution of the Hamiltonian

$$\mathcal{H} = \sum_{i=}^{n} \frac{1}{2} p_i^2 + V_i(q^i)$$

where $\langle H_i, X_i, Y_i \rangle$ are distinct copies of the previous algebra that commute with each other.

This, however, covers only a small class of Hamiltonians, so we might try the next simplest non-trivial case; that of the 2d in Liouville form.

Definition 2.2. *An n-dimensional **Liouville system** is one whose Hamiltonian is of the form*

$$\mathcal{H} = \frac{1}{\lambda}\left[\sum_{i=1}^{n} \frac{1}{2}\sigma_i p_i^2 + V_i\right]$$

where $\lambda = \sum_{i=1}^{n} \lambda_i$ and λ_i, σ_i, V_i are functions of q^i only.