Proof of the Arnol'd Liouville Theorem - Required Concepts

Symplectic Manifold

Let M be a $2n$-dimensional manifold and ω^2 a differential 2-form on M that is closed and non-degenerate. Then the pair (M, ω^2) is called a **symplectic manifold**.

I

On any symplectic manifold there is a natural isomorphism between the cotangent space and tangent space at a point $x \in M$, $I : T_x^*M \to T_xM$, given by for $\omega^1 \in T_x^*M$, $\forall v \in T_xM$

$$\omega^1(v) = \omega^2(v, I\omega^1)$$

It can be seen that I is a linear map as ω^2 is, and that injectivity follows from the non-degeneracy of ω^2. By counting dimensions we then know I is an isomorphism.

Phase Flow

To $F : M \to \mathbb{R}$ we may associate the 1-parameter group of diffeomorphism $\phi_F^t : M \to M$ defined by

$$\left.\frac{d}{dt}\right|_{t=0} \phi_F^t x = (IdF)(x)$$

Such a 1-parameter group is called the **phase flow**.

Poisson Bracket

Given $F, G : M \to \mathbb{R}$ we may define the **Poisson bracket** $\{F, G\} : M \to \mathbb{R}$ by

$$\begin{aligned}\{F, G\}(x) &= \left.\frac{d}{dt}\right|_{t=0} F(\phi_G^t x) \\ &= dF(IdG(x)) \quad \text{by the chain rule} \\ &= \omega^2(IdF, IdG)(x)\end{aligned}$$

Canonical Sympletic Form

Given coordinates $(\boldsymbol{q}, \boldsymbol{p})$ on a manifold, the **canonical symplectic form** is

$$\omega^2 = d\boldsymbol{p} \wedge d\boldsymbol{q}$$