Ueber die algebraischen Curven von den Geschlechtern $p = 4$, 5 und 6, welche eindeutige Transformationen in sich besitze

Written by A. Wiman.
Translated by Linden Disney-Hogg, Andrew Beckett, Jacob Bradley, and Isabella Deutsch.

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Preface from the translators

This is an attempt at a translation of the Wiman paper,

Wiman, Anders. "Ueber die algebraischen Curven von den Geschlechtern $p = 4$, 5 und 6, welche eindeutigen Transformationen in sich besitzen." Bihang till Kongl. Svenska vetenskaps-akademiens handlingar 21 (1895): 1-41.

which translates as

About the algebraic curves of genus $p = 4$, 5 and 6, which contain unique transformations,

an influential paper (for an incomplete list, see the Google Scholar list of citing articles here) written in German.
Care was taken to try and faithfully recreate the formatting of the original article, but this was not always possible. Importantly, footnotes are numbered within pages in the original article. As we do not make sure that pages line up exactly in this translation, we leave footnotes being numbered within the document. Further, sections are made more distinct in this version, and the corresponding page of the start of the section is included. Margin notes are included to anchor the pages of the original document to this translation.
LDH performed the original transcription and translation of the paper, and maintains the GitHub copy where errata may be reported. AB, JB, and ID contributed equally in refining the translation and providing proofreading.
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The Paper

Introduction

The present treatise is intended to be a continuation of a work I have already published,[1] which, among other things, treated the unambiguous correspondences [unique symmetries?] on the non-hyperelliptical curves of genus $p = 3$. For the sake of convenience, we take the following equation from it, which was constructed using a theorem given by Mr. ZEUTHEN:

$$\text{(A)} \qquad 2(p-1) = 2n(p'-1) + \sum \frac{n}{n_i}(n_i - 1)\,.$$

Here, on a curve of genus p, there should be an unambiguous correspondence [unique symmetry?] of period n [order n?]; the summation is over the coincidences where n related points coincide in $\frac{n}{n_i}$ by n_i [the points where n_i sheets comes together in $\frac{n}{n_i}$ groups]; finally, p' denotes the genus of a curve, the points of which are in a one-to-one relationship with the point groups of n points that belong together through the correspondence.[2]
Since we refrain from the hyperelliptical case, which is easy to handle, we can bring the relevant structures to the projectively invariant normal curve C_{2p-2} in the $(p-1)$-dimensional space, whose coordinates are proportional to the ABEL integrands of the first kind. From this arises the advantage for us that the unambiguous correspondences [unique symmetries] are represented by collineations of the curve.[3]

1 p.4

start p.4

For $p = 4$ we get a normal curve of the sixth order in ordinary three-dimensional space. This curve must be located on a surface of the second degree F_2, which, however, can also degenerate into a cone K_2.[4] The generatrices [generators?] of this surface are formed by the trisecants of the curve.
CAYLEY has given a general theory for the singularities of a twisted curve,[5], and we do not find it inappropriate to quote some of his results for our case, since the characteristic numbers of the invariant normal curve also designate properties of the general algebraic structures of the relevant genus.
The C_6 mentioned has no real double points or peaks [spitzen, perhaps cusps], since they can never appear on the φ-normal curve. The only possible essential singularities are *stationary tangents* Θ and double osculating planes Δ; but those must be generators of the surface F_2, because every other degree has at

[1] This is cited as "Bihang till K. Vet.Akad. Handlingar, Bd. 21, Afd. I. N:r 1."
[2] Refer once again to a work by Mr. Hurwitz (Math. Ann., Bd. 32, S. 290).
[3] See the 5th section of my previously-mentioned work.
[4] Noted in an essay by Mr. WEBER, Math. Ann., Bd. 13, S.35.
[5] See the related presentation by SALMON-FIEDLER, *Analytische Geometrie des Raumes* II, 3. Auflage, S. 105.

most two points with F_2 and thus the curve is common. We shall find that these singularities appear in connection with the collineations of the curve.

Furthermore, let m be the *order* of the curve;

r order of the developable surface or class of the perspective cone [one word, Perspektivekegel] formed by the tangents (*rank* of the curve);

n *class* of the curve;

α the number of *stationary planes*;

h the number of *apparent double points*;

g number of lines in two osculating planes which lie in a given plane;

x order of the *double curve*;

y the class of the double-touching developable surface.

Two systems of equations are obtained, the first by using PLÜCKER's formulas for an arbitrary perspective cone over the given curve. For order, class, double points, points, inflexions, double tangents one gets the related numbers: m, r, h, o, $n+\Theta$, y. So given start p.5

$$m = 6\,, \quad p = 4$$

in advance, one finds,

$$r = 18\,, \quad h = 6\,, \quad n = 36 - \Theta\,, \quad y = 96\,.$$

So consider the cross-sectional curve of the developable surface. Here are the PLÜCKER character numbers: r, n, x, $m+\Theta$, α, $g+\Delta$. So

$$x = 126 - \Theta\,, \quad \alpha = 60 - 2\Theta\,, \quad g = 531 - \frac{65}{2}\Theta + \frac{\Theta^2}{2} - \Delta\,.$$

The points of contact of the α-planes have a special meaning as those places of the structure of genus p (4), in which a unique algebraic function of lower than the $(p+1)$th (5th) order becomes infinite without becoming infinite in other places. These places have been considered by many authors and given for their number $(p-1)p(p+1)$, so in our case 60; this is also confirmed if only the Θ-points are calculated twice.
The developable surface of order 18 and surface F_2 intersect in a curve of order 36; this can only be calculated twice from the cuspidal curve and consist of a number of common generators. Each generating system of F_2 delivers 12 of these, which can also be seen from the fact that each F_2-generator has to divide the developable into 18 points, namely three double points on the curve and twelve common generators of the other system. Two of these 24 common straight lines move together in a stationary tangent that may appear, which gives the upper value 12 for Θ, which is actually reached.

In the meantime we have found 2 point groups with projective invariant properties, which can only be interchanged with a collineation of the curve: the 60 contact points of the stationary planes and the 24 contact points of the common generators. In addition, one could take the 24 points that cut the latter straight line from the curve. But this does not give a new group in two really occurring cases: if all common generators are stationary tangents or if all tangents in the above-mentioned cut-out points yield generators of the other F_2-system. start p.6
CAYLEY also treated other singularities by means of the theory of reciprocal surfaces.[6] Let us only give the following results. Let:

γ' be the number of osculating planes which also contain a tangent;

t the number of points through which three tangents pass;

t' the number of planes containing three tangents.

The following formulas apply here if only Θ and Δ occur from essential singularities:

$$\begin{aligned}\gamma' &= rn + 12r - 14n - 6m - 8\Theta - 4\Delta\,;\\ t &= \frac{1}{6}\left(r^3 - 3r^2 - 58r - 3r\,(n + 3m + 3\Theta) + 42n + 78m + 78\Theta\right)\,;\\ t' &= \frac{1}{6}\left(r^3 - 3r^2 - 58r - 3r\,(m + 3n + 3\Theta) + 42m + 78n + 78\Theta\right)\,,\end{aligned}$$

or in our case:

$$\gamma' = 324 - 12\Theta - 4\Delta\,;\quad t = 480 - 12\Theta\,;\quad t' = 120\,.$$

Here we have found new invariant point groups. The t' in particular have attracted attention in the general theory of curves, namely in the question of the adjoint φ-curves, which each simply touch the base curve at $p-1$ points.[7] For the number of solutions one has $2^{p-1}(2^p - 1)$, which for $p = 4$ agrees with the one already obtained.

2 p.6

The collineations of C_6 must also transform the F_2 into them. The two systems of generators can either be transformed into themselves or exchanged with one another. Such a special transformation can also occur which leaves all generators of a species individually fixed; two points on each are fixed, which together satisfy two fixed generators of the other kind, and the transformation has a period of three, because the points of the curve on the fixed generators must be cyclically exchanged; the six points on the two generators of the other kind start p.7
mentioned remain fixed and must have stationary tangents.

[6]See, for example, the presentation given by SALMON-FIELDER (o. a. A. S. 660)
[7]See CLEBSCH-LINDEMANN, *Vorlesungen über Geometrie* I, S. 847.

It can be seen from this that *no new simple group can appear in the transformation groups of a structure [gebildes, happy with structure?] of genus $p = 4$. Either the generating systems can be exchanged, and then half of the operations yield a distinctive subgroup which transforms each into itself, or, if exchange does not occur, the group is holohedrally isomorphic with one of the known finite binary groups, unless that every generator of a kind is transformed into itself by a cyclic G_3, but where this G_3 must have excellent properties.* (The latter also applies to the case where the surface F_2 merges into a cone, and so the two systems coincide.)

If one uniquely assigns the ratios $x_1 : x_2$ or $y_1 : y_2$ to the two sets to be generated, then it is initially evident that C_6 can be found by a homogeneous equation $F(x_1, x_2, y_1, y_2) = 0$, which is grade 3 in both x_i and y_i. This parameter setting is achieved by choosing F_2 $xz = yw$ for the equation and then through

$$x : y : z : w = x_2 y_1 : x_1 y_1 : x_1 y_2 : x_2 y_2 \tag{1}$$

the ratios $x_1 : x_2$ and $y_1 : y_2$ are determined. From the point $x = y = z = 0$ or $x_1 = y_1 = 0$ you can now project the area and thus the curve onto the plane $w = 0$, whereby the image curve has the equation $F(y, x, y, z) = 0$ must satisfy.[8]

A collineation of the first kind can be replaced with two binary substitutions as far as the area is concerned:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2\,, \quad & x_2' &= a_{21}x_1 + a_{22}x_2\,; \\ y_1' &= b_{11}y_1 + b_{12}y_2\,, \quad & y_2' &= b_{21}y_1 + b_{22}y_2\,. \end{aligned}$$

Two generators of each kind merge into one another, and their 4 points of intersection remain fixed, but if one substitution is identical, all points of the fixed generators of the other kind remain fixed. If you choose the basic elements,[9] start p.8
you get the simpler form:

$$x_1' = a_1 x_1\,, \quad x_2' = a_2 x_2\,, \quad y_1' = b_1 y_1\,, \quad y_2' = b_2 y_2\,. \tag{2}$$

The collineation in space is now determined from (1) and (2):

$$x' = a_2 b_1 x_1\,, \quad y' = a_1 b_1 y\,, \quad z' = a_1 b_2 z\,, \quad w' = a_2 b_2 w\,. \tag{3}$$

In a collineation of the second kind, where the generating sets are exchanged, one can initially start from substitutions of the following form:

$$\begin{aligned} x_1' &= \alpha_{11}y_1 + \alpha_{12}y_2\,, \quad & x_2' &= \alpha_{21}y_1 + \alpha_{22}y_2\,; \\ y_1' &= \beta_{11}x_1 + \beta_{12}x_2\,, \quad & y_2' &= \beta_{21}x_1 + \beta_{22}x_2\,. \end{aligned}$$

It is easy to see that through the requirements:

$$x_1' : x_2' = x_1 : x_2\,, \quad y_1' : y_2' = y_1 : y_2\,,$$

[8] see SALMON-FIELDER, for example o.a. A. S. 521 or CLEBSCH-LINDEMANN, Vorlesungen über Geometrie II 1, S. 422.

[9] It is well known that coincident basic elements do not occur in finite groups.

two points fixed on the surface during collineation can be determined, unless these conditions are met in an infinite number of points. We choose generators through fixed points as basic elements and then get collineations:

$$x_1' = \alpha_1 y_1\,, \quad x_2' = \alpha_2 y_2\,, \quad y_1' = \beta_1 x_1\,, \quad y_2' = \beta_2 x_2\,. \tag{4}$$

The collineation of space now results from (1) and (4):

$$x' = \alpha_2 \beta_1 z\,, \quad y' = \alpha_1 \beta_1 y\,, \quad z' = \alpha_1 \beta_2 x\,, \quad w' = \alpha_2 \beta_2 w\,. \tag{5}$$

If this operation is repeated, a collineation of the first kind is obtained:

$$x' = \alpha_1 \alpha_2 \beta_1 \beta_2 x\,, \quad y' = \alpha_1^2 \beta_1^2 y\,, \quad z' = \alpha_1 \beta_1 \alpha_2 \beta_2 z\,, \quad w' = \alpha_2^2 \beta_2^2 w\,, \tag{6}$$

but not arbitrary, since all points of the straight line $y = w = 0$ remain fixed with the same.
If we also emphasize that the equation (A) for $p = 4$ only allows the prime number solutions 2, 3, 5, then we now have the means to identify the possible collineation groups of a structure of genus $p = 4$ as quickly as possible.

3 p.9

start p.9

We now want to visit these collineations. The equations of the structures in question are written with the smallest possible number of constants, which means that the arbitrary modules are also given.
For a cyclic group of period 2, either the generatrices can be swapped or not. In the first case one gets an infinite number of fixed points, the intersections of two mutually exchanging generators. These fulfill a conic section, where the pole of the plane of the conic section with respect to the F_2 provides the center of perspective. The equation of the curve is obtained by choosing two fixed points on the curve for $x_1 = y_1 = 0$ and $x_2 = y_2 = 0$ and the simultaneous interchangeability of x_1, y_1 and x_2, y_2 takes into account the following:

$$\begin{aligned} &x_1^2 y_1^2 (x_1 y_2 + x_2 y_1) + x_1 y_1 (x_1^2 y_2^2 + x_2^2 y_1^2 + a x_1 x_2 y_1 y_2) + \\ &+ b(x_1^3 y_2^3 + x_2^3 y_1^3) + c x_1 x_2 y_1 y_2 (x_1 y_2 + x_2 y_1) + \\ &+ x_2 y_2 (d(x_1^2 y_2^2 + x_2^2 y_1^2) + e x_1 x_2 y_1 y_2) + f x_2^2 y_2^2 (x_1 y_2 + x_2 y_1) = 0 \end{aligned} \tag{1}$$

If we go to the coordinates of space, we find (according to Eq. (5) §2, where $\alpha_1 = \alpha_2 = \beta_1 = \beta_2$) for the perspective plane $x - z = 0$ and for the perspective center $y = w = x + z = 0$. The curve must be on a cone of the 3rd order, whose apex is the center, so that every two points on the same generatrix are interchanged by the collineation. The tangents of the 6 points in the perspective plane pass through the center and the associated planes are stationary. The planes of the 9 turning points of the cone are doubly osculating, thus forming 9 Δ-planes.

Now the cyclic G_2 may not cause any permutation of the generator sets. You can bring it to the form:

$$x_1' = -x_1\,,\quad y_1' = -y_1\,,\quad x_2' = x_2\,,\quad y_2' = y_2\,.$$

The equation can be written in two equivalent ways, from which we choose the following:

$$(2)\qquad x_1^3y_1^3 + x_1y_1x_2y_2(ax_1y_1 + bx_2y_2) + cx_2^3y_2^3 + x_1^3y_1y_2^2 + x_1x_2^2y_1^3 + \\ + dx_2^3y_1^2y_2 + ex_1^2x_2y_2^3 = 0\,.$$

Two points of the curve are transformed into themselves: $x_1 = y_2 = 0$ and $x_2 = y_1 = 0$. If one takes account of the equations (2) and (3) of the previous section, one finds that in this collineation every point of the straight lines $x = z = 0$ and $y = w = 0$ remains fixed. Any straight line that intersects these two lines must merge into itself. The connecting lines of corresponding points on the curve thus form a ruled surface, for which those two straight lines are guidelines, and it is easily seen that this ruled surface is of fifth order and $y = w = 0$ as triple $x = z = 0$ as a double directrix. start p.10

The possible location of the C_6 on a 3rd order cone or a 5th order ruled surface thus characterizes the two different types of involutory-unique correspondences on the curve. Since only 6 or 5 modules are included in (1) and (2) and 9 modules are available for the general C_6, 3 or 4 conditions are fulfilled there. The cone K_3, or the ruled surface R_5, designates a structure of the genus $p' = 1$, or 2, to which the C_6 is 1-2-uniquely related. No other rectilinear surfaces can simply contain the curve if at the same time the generatrices are supposed to be bisecants.

In the case of equation (2) we examine the bisecant ruled surface whose directrix is $y = w = 0$, which itself contains 2 points of the curve. From each point of this straight line there are 5 bisecants (except $y = w = 0$ itself) and in each plane through it there are 6, so that the order 11 is obtained. The ruled surface thus consists of the mentioned R_5 and a R_6, for which the C_6 is a double curve. The generatrices of this R_6, which start from a point of the directrix, merge into each other through the transformation G_2; but since every plane through the straight line remains invariant, those generators must also lie in the same plane through this straight line. One can therefore call $y = w = 0$ a double tangent line for the R_6, since the cross-sectional curves there receive tangent nodes.[10]
This R_6 is usually of genus $p' = 2$, but splits into 2 K_3 when

$$(3)\qquad d = e\,.$$

This equation says that the two points of C_6 on $y = w = 0$ have the same plane $y + dw = 0$. The transformation group of the curve in itself is in this case a *Klein four-group.* Two G_2 interchanging the generating sets appear: start p.11

$$x_1' = \pm y_1\,,\quad y_1' = \pm x_1\,,\quad x_2' = y_2\,,\quad y_2' = x_2\,.$$

[10] I have dealt with this R_6 in my writing »Klassifikation af regelytorna af 6. graden »S. 58 (Diss. Lund 1892)

The associated perspective levels are $x \mp z = 0$ and the perspective centers $y = w = x \pm z = 0$.
However, one can also obtain a *Klein four-group of a different kind*, which leaves the generating groups uninterchanged at all. We start again from Equation (2), but at first we think of the ratios of all 8 coefficients as completely arbitrary. Due to the remaining operations of the Klein four-group, the two generators of a family that are fixed at G_2 must be exchanged with each other. By means of suitable normalization, the two new G_2 can be given the following form:

$$x_1' = \pm x_2\,, \quad x_2' = x_1\,, \quad y_1' = \pm y_2\,, \quad y_2' = y_1\,.$$

The equation for the curve is:

$$\begin{aligned}(4) \qquad & x_1^3y_1^3 + x_2^3y_2^3 + ax_1y_1x_2y_2(x_1y_1 + x_2y_2) + by_1y_2(x_1^3y_2 + x_2^3y_1) + \\ & + cx_1x_2(x_1y_2^3 + x_2y_1^3) = 0\,.\end{aligned}$$

A cyclic G_4 must have a G_2 as a subgroup, which leaves the generating groups uninterchanged. A look back at equation (A) teaches us that the two fixed points of C_6 at G_2 must also maintain their fixed position at G_4. In the collineations of period 4, the systems must be exchanged; this is already evident from the fact that the necessary number of points for the purpose of cyclic permutation are missing on the fixed generators occurring in this case. So we start from equation (2) under the most general assumption possible with regard to the coefficients. A collineation of period 4, which swaps the systems and keeps the points $x_1 = y_2 = 0$ and $x_2 = y_1 = 0$ fixed, can be brought to the following form:

$$x_1' = y_2\,, \quad y_2' = -ix_1\,, \quad x_2' = iy_1\,, \quad y_1' = x_2\,.$$

The generator for which $x_1 = x_2$ can be chosen arbitrarily. We can therefore start p.12
state that in the equation we are looking for, e.g. B. [erroneous B?] the coefficients for $x_1^3y_1y_2^2$ and $x_2^3y_1^2y_2$ become equal. Then we get equation (4) again, only with the specialization that

$$(5) \qquad c = -b\,.$$

However, the curve determined in this way also contains the two groups of four belonging to cases (3) and (4), so that the complete collineation group is of the nature of a *dihedral* G_8.
One now easily proves that the 5 groups discussed are the only possible ones without collineations of higher prime number period.

4 p.12

We now add the cyclic groups of period 3. With such a group, either each generator of a group can merge into itself or not. In the latter case, the collineation can be created in the following form:

$$x_1' = j^2x_1\,, \quad x_2' = x_2\,, \quad y_1' = jy_1\,, \quad y_2' = y_2\,, \quad \left(j = e^{\frac{2\pi i}{3}}\right)$$

or in the coordinates of space (according to (2) and (3) §2):

$$x' = jx\,, \quad y' = y\,, \quad z' = j^2 z\,, \quad w' = w\,.$$

The equation of a C_6 that merges into itself through this transformation can be written with only 3 modules:

$$x_1^3 y_1^3 + x_2^3 y_2^3 + a(x_1^3 y_2^3 + x_2^3 y_1^3) + b x_1^2 y_1^2 x_2 y_2 + c x_1 y_1 x_2^2 y_2^2 = 0\,, \tag{6}$$

whereby, of course, one disregards structures with double points and thus lower genus. But the curve (6) also has three perspective G_2 in itself:

$$x_1' = y_1(jy_1, j^2 y_1)\,, \quad y_1' = x_1\,, \quad x_2' = y_2\,, \quad y_2' = x_2(jx_2, j^2 x_2)\,.$$

or (according to (4) and (5) §2):

$$x' = z(j^2 z, jz)\,, \quad y' = y\,, \quad z' = x(jx, j^2 x)\,, \quad w' = w\,.$$

The relevant perspective planes are:

$$x - z = 0\,, \quad x - j^2 z = 0\,, \quad x - jz = 0\,,$$

and the perspective centra are obtained as the intersections of the straight lines start p.13
$y = w = 0$ with $x + z = 0$, $x + j^2 z = 0$ and $x + jz = 0$.

The transformation group of the curve (6) is thus a *dihedral group* G_6. Six related points lie in a plane through $y = w = 0$. The straight lines connecting the corresponding points at the G_2 form 3 K_3, the vertices of which are the above-mentioned perspective centers. These K_3 are cyclically exchanged by the G_3. One can now easily deduce that a plane through $y = w = 0$ must either touch all or none of K_3. There are 6 planes of that kind, which must also touch the curve three times; the points of contact belong to stationary planes, and their tangents each pass through one of the three vertices.

We now think of the G_3 as a subgroup of a higher group that does not swap systems. This group cannot be a cyclic G_6, since the non-fixed points permute each other cyclically to six on such a fixed generator, and not an octahedral group, because the cyclic G_4 required for this do not occur according to the previous section. The possible cases to be discussed here are thus dihedral G_6 and the tetrahedral group.

In the case of the other collineations of a linear dihedral group, those elements which remain fixed in the cyclic subgroup must be exchanged. In order for this exchange to be possible, a condition must be fulfilled in equation (6), for which we can set

$$b = c\,. \tag{7}$$

The complete collineation group of the curve is then of the type of a *dihedral* G_{12}. As subgroups we mention the dihedral G_6, which has the general curve

(6), the dihedral G_6, which does not commute the generatrices, whose three collineations of period 2 are the following:

$$x_1' = x_2(jx_2, j^2x_2)\,, \quad x_2' = x_1\,, \quad y_1' = y_2(j^2y_2, jy_2)\,, \quad y_2' = y_1\,,$$

and finally the cyclic G_6, which is formed by repeating the operation:

$$x_1' = jy_2\,, \quad y_2' = x_1\,, \quad x_2' = y_1\,, \quad y_1' = j^2x_2\,.$$

We now assume that the subgroup which does not interchange the systems is start p.14
a tetrahedral group. As an excellent [normal?] subgroup, we have a Klein four-group here. We are therefore looking for the further conditions which the coefficients of equation (4) must satisfy in this case. The fixed elements in the individual operations of the Klein four-group are to be cyclically exchanged in the collineations of period 3 of the tetrahedron group. This collineation of period 3 can now easily be represented in the following way:

$$x_1' = \pm i(x_1 \pm x_2)\,, \quad x_2' = x_1 \pm x_2\,, \quad y_1' = \pm i(y_1 \pm y_2)\,, \quad y_2' = y_1 \pm y_1\,,$$

and thus obtains for the equation of the curve merging into itself:

$$(x_1y_1 + x_2y_2)^3 + b(x_1y_1 - x_2y_2)(x_1y_2 - x_2y_1)(x_1y_2 + x_2y_2) = 0\,. \tag{8}$$

The complete collineation group of curve (8) in itself turns out to be holohedrally isomorphic with an *octahedral group*. The level $x_1y_1 + x_2y_2 = 0$ or $y + w = 0$ always merges into itself. The pole of this plane in relation to the area F_2 must therefore remain fixed for all 24 collineations. The curve lies on 6 cones K_3 whose vertices are in the invariant plane.

5 p.14

We now assume that each generator of the y family merges into itself at a G_3. The G_3 can be generated by the collineation:

$$x_1' = jx_1\,, \quad x_2' = x_2\,, \quad y_1' = y_1\,, \quad y_2' = y_2\,,$$

and the equation of an associated curve is:

$$x_1^3 f_3(y_1, y_2) + x_2^3 \varphi_3(y_1, y_2) = 0\,. \tag{9}$$

This curve has 3 independent modules.
We now consider the case that f_3 and φ_3 contain a common transformation group. The same must of course also hold for their HESSEian [Hessian] covariants, and we first assume that these do not coincide. There are two possibilities
here: either the points of one covariant can be base points of a G_2, which swaps start p.15
those of the other, or the points of both covariants are swapped by a G_2, whose base points must be supplied by its simultaneous quadratic covariant; but that case turns out to be impossible, because in order for the three points $f_3 = 0$

(resp. $\varphi_3 = 0$) to merge into each other through a G_2, one of them must remain fixed and thus provide a basic point.
But one also comes to higher groups if $f_3 = 0$ and $\varphi_3 = 0$ can be exchanged with each other. Because the 6 points $f_3\varphi_3 = 0$ have to be ordered cyclically into one or more series, the period of a collineation can only be 6 or 2. By the same way of reasoning for the HESSE [HESSE'schen, so perhaps Hessian] covariants, one obtains 2 as the only possible period. The equation of the curve can be found in the following form:

$$\begin{aligned} x_1^3(y_1 - y_2)(y_1 - ay_2)(y_1 - by_2)+ \\ +x_2^3(y_1 + y_2)(y_1 + ay_2)(y_1 + by_2) = 0\,. \end{aligned} \tag{10}$$

The transformation group is *a dihedral* G_6 whose collineations from period 2 are the following:

$$x_1' = x_2(jx_2, j^2x_2)\,, \quad x_2' = x_1(j^2x_1, jx_1)\,, \quad y_1' = y_1\,, \quad y_2' = -y_2\,.$$

Since the points of HESSE's covariant can be paired in two ways, one has the possibility to consider that $f_3 = 0$ and $\varphi_3 = 0$ are also swapped in two ways. But then $f_3 = 0$ and $\varphi_3 = 0$ must each contain a transformation different from the identity, which can be obtained by combining two permutations. The equation of the curve can be brought into the following form:

$$x_1^3y_1(y_1^2 + ay_2^2) + x_2^3y_2(ay_1^2 + y_2^2) = 0\,. \tag{11}$$

The collineation group of this curve is a *dihedral* G_{12}. A generating operation of the cyclic subgroup G_6 is found in:

$$x_1' = -jx_1\,, \quad x_2' = x_2\,, \quad y_1' = -y_1\,, \quad y_2' = y_2\,.$$

The 6 usual collineations are the following:

$$x_1' = \pm x_2(\pm jx_2, \pm j^2x_2)\,, \quad x_2' = x_1\,, \quad y_1' = \pm y_2\,, \quad y_2' = y_1\,.$$

We now examine the case where the HESSE covariants of f_3 and φ_3 coincide. start p.16
The equation of the curve can easily be brought into the form:

$$x_1^3y_1^3 + x_1^3y_2^3 + x_2^3y_1^3 + a^3x_2^3y_2^3 = 0\,. \tag{12}$$

This curve has 12 stationary tangents whose tangents are cut out in threes each by the generators $x_1 = 0$ $x_2 = 0$, $y_1 = 0$ and $y_2 = 0$. Here one has a G_3 for each system, in which each generator remains fixed:

$$\begin{aligned} x_1' &= jx_1\,, \quad x_2' = x_2\,, \quad y_1' = \;\; y_1\,, \quad y_2' = y_2\,; \\ x_1' &= \;\; x_1\,, \quad x_2' = x_2\,, \quad y_1' = jy_1\,, \quad y_2' = y_2\,. \end{aligned}$$

By combining these G_3 you get a G_9, which consists of 4 G_3 and is distinguished in the overall group. If one adds a collineation that exchanges the generators $x_1 = 0$ and $x_2 = 0$ or $y_1 = 0$ and $y_2 = 0$:

$$x_1' = ax_2\,, \quad x_2' = x_1\,, \quad y_1' = y_2\,, \quad y_2' = \frac{1}{a}y_1\,,$$

and combines this with G_6, so one gets 9 collineations from the period 2. So you have a G_{18}, which converts every generator system into itself and is obviously holohedrally isomorphic with the collineation group of a general planar C_3. There are also 18 collineations that swap systems. You get these by combining the G_{18} with any of them, like:

$$x_1' = y_1\,, \quad y_1' = x_1\,, \quad x_2' = y_2\,, \quad y_2' = x_2\,.$$

Of these 18 new collineations, 12 have period 6 and 6 have period 2.
With the mentioned 36 collineations of the curve (12), 2 straight lines remain fixed: the straight lines connecting the points $x_1 = y_1 = 0$ and $x_2 = y_2 = 0$ or $x_1 = y_2 = 0$ and $x_2 = y_1 = 0$, or (Eq. (1) §2) $x = z = 0$ and $y = w = 0$. The transformation groups of these straight lines are dihedral G_6 and thus obtained with G_{36} from the combination of these two G_6, and by combining subgroups of G_6 subgroups of G_{36} arise. The vertices of three K_3 containing the curve lie on each of the two fixed straight lines. The mentioned dihedral start p.17
G_6 arise. [erroneous full stop?] by summarizing all possible permutations of these vertices. The G_{36} is therefore to be regarded as a subgroup of the group of all possible permutations of 6 things, namely the above-mentioned G_{18} that does not permute the systems is obtained as the subgroup of G_{36}, in which permutations occur in an even number.
The question arises whether the curve (12) can contain further collineations. The generator $x_1 = 0$ can only be exchanged with the other three generators containing Θ-points: $x_2 = 0$, $y_1 = 0$, $y_2 = 0$. So if $x_1 = 0$ merges into itself at n collineations, then the order of the group is $4n$. The number n is obtained by combining G_3, which leaves every point of the imaginary generator unchanged, and the common transformation group of f_3 and φ_3 in itself. However, the latter group is usually a G_3 if the HESSE covariates coincide, but a dihedral group of order 6 if f_3 and φ_3 are mutual third-degree covariates. In this case, where $a^3 = -1$ and thus the equation of the curve:

$$x_1^3 y_1^3 + x_1^3 y_2^3 + x_2^3 y_1^3 - x_2^3 y_2^3 = 0\,, \tag{13}$$

36 new collineations appear. You get this by combining the G_{36} with any new one, e.g

$$x_1' = x_2\,, \quad x_2' = x_1\,, \quad y_1' = y_1\,, \quad y_2' = -y_2\,.$$

Of the 36 collineations thus added, those which interchange the staves [systems?] are of period 4; of the rest, 12 have period 6 and 6 have period 2. Those lines fixed at G_{36} are swapped at the 36 new collineations, and one can consider the G_{72} as a 6-thing-swapping group, where about the three first and three last things set the same Form 2 interchangeable groups. The G_{72} has three distinct subgroups of order 36: the collineation group of the general curve (12), the subgroup that does not interchange the systems, and finally a group of even permutations of the 6 K_3 vertices. The latter is holohedrally isomorphic with the collineation group of the plane harmonic C_3. As mentioned, the curve (12) start p.18
lies on 6 K_3, the vertices of which are occupied three times each on the two

distinguished straight lines. The same curve also lies on 9 R_3, whose triple directrix connects two cone vertices. The special curve (13) also lies on 6 other R_5 whose triple directrixes each connect two Θ-points; these lines are double tangent straight lines of 6 R_6 for which (13) is double curve.
In general, 72 points of the curve (13) permute at G_{72} as a closed system. But the 12 Θ points only permute among themselves, and the same applies to the system of the 36 α-points, and finally also to 18 points in the 9 straight lines connecting two K_3 vertices, which the Form points of contact of 9 double-osculating planes. Each of these points remains fixed at a G_6, or G_2 or G_4.

6 p.18

It remains to consider the case where the curve is transformed into itself by a G_5. With an odd number of periods, of course, the generating systems cannot be interchanged. Since the points of the curve on the 4 fixed generators cannot be cyclically permuted to 5 each, they must be occupied in the 4 fixed points of intersection. However, this can only be achieved by simply touching the curve in each of the 4 points. It is now easy to see through experiments that the equation of a curve of the type considered here with a collineation of period 5 can be brought into the following form:

$$x_1^3 y_1^2 y_2 + x_1^2 x_2 y_2^3 + x_1 x_2 y_2^3 + x_1 x_2^2 y_1^3 + a^5 x_2^3 y_1 y_2^2 = 0\,, \tag{14}$$

where if $\varepsilon^5 = 1$, the collineations are represented as follows:

$$x_1' = \varepsilon x_1\,, \quad x_2' = \varepsilon^4 x_2\,, \quad y_1' = \varepsilon^2 y_1\,, \quad y_2' = \varepsilon^3 y_2\,.$$

However, the curve (14) also has 5 collineations of period 2:

$$x_1' = a^2 \varepsilon^4 x_2\,, \quad x_2' = \frac{\varepsilon}{a} x_1\,, y_1' = \varepsilon^3 y_2\,, \quad y_2' = \frac{\varepsilon^2}{a} y_1\,,$$

so that the complete group of its collineations is a *dihedral* G_{10}. One easily finds start p.19
that no collineation of period 10 can transform the curve (14) into itself; because either all 4 fixed points of G_5 would also remain fixed here, which contradicts equation (A), or 2 of them would be swapped with each other, which would be geometrically inconsistent. Collineations with a higher number of periods are also impossible for the same reasons. One only has to consider the case in which a generatrix is transformed into itself by an icosahedron group. Because the G_3 occurring here must be of the type dealt with in the 4th section, and the systems were always exchanged there, the complete group must be of order 120. With such a curve with 120 transformations in itself, the 24 tangents, which are also generators of F_2, form 6 quadrilaterals corresponding to the 6 G_5 within the icosahedron group, whose corners lie in the points of contact.
If an exchange of the systems for the curve (14) is to be possible, the quadrilateral on which it is based must be converted into itself if one does not want to assume several such quadrilaterals, and thus also several G_5 and one G_{120}. One

easily finds the condition $a^5 = -1$; then there are 5 cyclic G_4 whose generating operations are the following:

$$x_1' = \varepsilon^2 y_1\,, \quad y_1' = -\varepsilon^4 x_2\,, \quad x_2' = \varepsilon^3 y_2\,, \quad y_2' = \varepsilon x_1\,,$$

where ε is a fifth root of unity.
Meanwhile Mr. GORDAN recognized[11] that the curve obtained in this way:

$$x_1^3 y_1^2 y_2 + x_1^2 x_2 y_2^3 + x_1 x_2^2 y_1^3 - x_2^3 y_1 y_2^2 = 0 \tag{15}$$

allows 120 collineations. Mr. KLEIN calls this curve *Bring's curve*, for the following reason. If one introduces pentahedron coordinates z_1, z_2, z_3, z_4, z_5 with the identical relation

$$\sum z_i = 0\,,$$

one obtains the same as an intersection curve of two surfaces, the main surface start p.20
and the diagonal surface:

$$\sum z_i^2 = 0\,, \quad \sum z_i^3 = 0\text{[12]}\,. \tag{15a}$$

The 120 collineations of the curve result from the permutations of the z_i, and the 60 straight permutations leave the generating systems unchanged.
The nature of the individual collineations is briefly indicated here. By cyclic permutation of all z_i one gets 24 collineations of period 5. Twenty collineations of the type:

$$z_1' = z_2\,, \quad z_2' = z_3\,, \quad z_3' = z_1\,, \quad z_4' = z_5\,, \quad z_5' = z_4\,,$$

have the period 6. The resulting 10 cyclic G_6 have subgroups of period 3, which yield 20 new collineations, E.g.:

$$z_1' = z_2\,, \quad z_2' = z_3\,, \quad z_3' = z_1\,, \quad z_4' = z_4\,, \quad z_5' = z_5\,;$$

In addition, there are the subgroups from period 2:

$$z_1' = z_2\,, \quad z_2' = z_2\,, \quad z_3' = z_3\,, \quad z_4' = z_5\,, \quad z_5' = z_4\,.$$

These 10 collineations are perspective; the perspective level [plane?] in the chosen example is $z_4 - z_3 = 0$ and the perspective center $z_1 = z_2 = z_3 = z_4 + z_5$. The 60 points of contact of the stationary planes are in the perspective planes. The 10 straight lines $z_i = z_k = 0$ each go through three perspective centers; then the straight line $z_1 = z_2 = 0$ goes through the centra [center?] assigned to the planes $z_3 = z_4$, $z_4 = z_5$ and $z_5 = z_3$ and stays with the permutations of z_3, z_4, z_5 resulting dihedral G_6. Thirty collineations have period 4; in these, one pentahedron plane remains fixed and the 4 others are cyclically interchanged.

[11] See P. GORDAN, *über die Auflösung der Gleichungen vom fünften Grade* (Math. Ann. Bd. 13) or F. KLEIN, *Vorlesungeng über das Ikosaeder*, S.194 u. f.

[12] One can consider the coordinates of a point on the curve as the roots of an equation of the 5th degree, which has been brought into BRING's normal form by Tschirnhaus transformation.

By repeating one of these, 15 collineations of period 2 *with axes* arise. As an example we choose:

$$z_1' = z_2\,, \quad z_2' = z_1\,, \quad z_3' = z_4\,, \quad z_4' = z_3\,, \quad z_5' = z_5\,.$$

The one axis $z_1 + z_2 = z_3 + z_4 = z_5 = 0$ goes through 2 of the aforementioned start p.21
perspective centers: $z_1 + z_2 = z_3 = z_4 = 0$ and $z_3 + z_4 = z_1 = z_2 = z_5 = 0$, and in fact each of the 15 connecting lines is such an axis. According to the discussion in Equation (2) of section 3, each such axis passes through the tangents of a doubly osculating plane.

The higher subgroups within the G_{120} are: the icosahedron groups, which leave the systems uninterchanged, 5 octahedron groups, in each of which one pentahedron plane remains fixed, 6 G_{20} of the kind previously discussed, 10 dihedral G_{12}, in which the pentahedron planes permute in 2 systems of 2 and 3, and finally dihedral groups of the order 10, 8, 6 and 4, which, however, form only subgroups of the subgroups discussed.

Corresponding to the 10 perspective G_2 and the 15 G_2 with axes, the BRING curve lies at 10 K_3 and 15 R_5. Each turning plane of this K_3 osculates the curve twice. However, 15 turning points, the mentioned axes, belong to 2 K_3. Sixty others are obtained, so that the total number of planes doubly osculating the curve is 75. Of those planes γ' which touch the curve at one point and osculate at another, 300 are absorbed (according to a formula at the end of the 1st section) by the 75 Δ-planes; the remaining 24 osculate in the points whose tangents are generatrices of F_2, because the other generatrix through such a point also contains a tangent of the curve. One could easily draw out several such remarkable properties of BRING's curve.

The 60 α-points, the 30 points of tangency of the system of 15 Δ-planes and the 24 points in which the tangents are generatrices of the F_2 form the only systems of less than 120 points that are permute closed at G_{120}.

7 p.21

Now that we have found all possible collineation groups of a structure of genus $p = 4$ whose normal curve of the 6th order lies on a real surface of the 2nd degree, the results obtained are summarized here. M denotes the number of independent modules of the entities concerned. start p.22

a. $M = 6$.

   1) G_2, which swaps the generating systems of the surface F_2. Equation (1).

b. $M = 5$.

   2) G_2 without swapping the systems. Eq. (2).

c. $M = 4$.

3) Klein four-group with swapping of the systems. Eq. (3).

d. $M = 3$.

4) Klein four-group without swapping of the systems. Eq. (4).
5) Diedric G_6 with exchange of systems. Eg. (6).
6) G_3. Eq. (9).

e. $M = 2$.

7) Diedric G_8 with exchange of systems. Eq. (5).
8) Diedric G_{12} with exchange of systems. Eq. (7).
9) Diedric G_6 without swapping of the systems. Eq. (10).

f. $M = 1$.

10) Octahedral group. Eq. (8).
11) Diedric G_6 without swapping of the systems. Eq. (11).
12) Dihedral group of order 10 without interchanging the systems. Eq. (14).
13) Group of Order 36. Eq. (12).

g. $M = 0$.

14) Group of Order 72. Eq. (13).
15) Group of Order 120. Eq. (15).

8 p.22

We now consider the case in which the normal curve of a structure of genus $p = 4$ sits on a cone of the 2nd degree K_2. This special case has attracted the attention of Messrs. NOETHER and SCHOTTKY in the literature. As a strange property of the curve, it has been recognized that the same curve (in the typical form of a C_9 with 8 triple points) can be a coincidence curve with a unique transformation of the plane in itself. One also has in itself a more general class of unique transformations of the plane, that »with 8 points », which is characterized by a unique correspondence among the points of such a C_9. For start p.23
this reason Mr. S. KANTOR, in his investigations of the isolated types of unique periodic transformations of the plane, looked for the cyclic transformations of the above structures with $p = 4$ in themselves;[13] but not the complete groups, because that was not necessary for the problem to be solved. It is to this more general task that we shall concern ourselves here. We want to use a method that differs somewhat from KANTOR's.
If the curve is projected onto a plane from a point of the cone, the image curve is

[13] Acta Mathematatica, Bd. 19, S. 164.

a C_6 with 2 infinitely close triple points, which come from the branches located on the generatrix of the cone through the starting point.[14] So the equation of the image curve can be brought to the following form:

$$z^3y^3 + z^2y^2f_2(x,y) + zyf_4(x,y) + f_6(x,y) = 0\,,$$

where $y = x = 0$ forms a triple point with $y = 0$ as the common tangent of the 3 branches. The generatrices of the cone are mapped by the ray pencil $x = \lambda y$, and the points of the curve occupied on one generatrice [generatrix?] are represented by the equation:

$$z^3 + z^2f_2(\lambda,1) + zf_4(\lambda,1) + f_6(\lambda,1) = 0 = \varphi_3(z)\,,$$

where $z = \infty$ is the apex of the cone or the common point of all referred to as generators. The polars of the pole $z = \infty$ for $\varphi_3(z)$ are of course invariant structures. Here we consider the linear polar $3z + f_2(\lambda,1) = 0$ whose location satisfies the conic $3zy + f_2(x,y) = 0$. But this conic section is the image of a cross-section of the cone. From now on we choose the center of projection on this cross-section; the conic section then turns into a straight line, which we can denote by $z = 0$, and for the general form of the equation of the image curve we get:

$$\text{(I)}\qquad z^3y^3 + zyf_4(x,y) + f_6(x,y) = 0\,.$$

The collineation group of the normal curve occupied on the cone must be of such a nature that both a point , the apex of the cone, remains fixed and a plane, start p.24
the location of the linear polars mentioned. The question arises whether, with a collineation belonging here, each generator can remain fixed; the 3 points of the curve must be cyclically permuted on a generatrix, so that the only possibility is a G_3. So we can state the theorem:

The collineation group of a normal curve of genus $p = 4$ located on a cone K_2 is either a linear group or a combination of a linear group with a G_3.

If we now start from the projected curve (I), we have to consider a group of quadratic transformations instead of a collineation group. However, the distinguished straight line $z = 0$ must always merge into itself, and 2 fundamental points must also lie in the infinitely neighboring triple points. The equation of a related transformation can be brought into the form:

$$\text{(a)}\qquad x' : y' : z' = (a_1x + b_1y)(a_2x + b_2y) : (a_2x + b_2y)^2 : zy\,.$$

However, it can be seen here that the substitution:

$$x' : y' = (a_1x + b_1y) : (a_2x + b_2y)$$

must convert the functions f_6 and f_4 into itself. One has therefore first to look for the common transformation group of these two functions.

[14]See CLEBSCH-LINDEMANN, *Vorlesungeng* II, S. 431.

The generators of the cone, which are fixed for a single spatial collineation, can be chosen for $x = 0$ and $y = 0$. The 4 fixed points in the collineation are the apex of the cone and 3 points in the aforesaid [aforementioned] fixed plane, 2 of which lie on those fixed generatrices, and the third is the pole of their connecting line with respect to the cross-section of the cone. But there are cases where all the points of a line joining two of these points, or of a plane through three of them, remain fixed. The planar transformation (a) turns into a collineation here:

$$x' : y' : z' = \alpha x : \beta y : \gamma z \,. \tag{a'}$$

This collineation transforms $f_6(x', y')$ into $k{\cdot}f_6(x, y)$ and $f_4(x', y')$ into $l{\cdot}f_4(x, y)$. The individual terms of (I) are to be multiplied by the same number k. From this we get the conditions: start p.25

$$\beta^3\gamma^3 = \beta\gamma l = k\,; \quad \gamma = \frac{k}{\beta l}\,; \quad k^2 = l^3\,. \tag{b}$$

Projecting the curve from the fixed point on the generatrix $x = 0$, the equation of the image curve is:

$$z^3x^3 + zxf_4(x, y) + f_6(x, y) = 0\,,$$

and the collineation is transformed into the following plane

$$x' : y' : z = \alpha x : \beta y : \gamma_1 z\,; \quad \gamma_1 = \frac{k}{\alpha l} = \frac{\beta}{\alpha}\gamma\,. \tag{a_1'}$$

If one sets $\alpha = \beta$ in (a'), each generator is transformed into itself; one gets $k = \alpha^6$, $l = \alpha^4$ and $\gamma = \alpha$. The collineation is thus an identical one [identity]. However, other conditions arise when f_4 vanishes identically, which, however, is initially impossible.

If $\beta = \gamma$, each point of the generator $x = 0$ is transformed into itself. The same applies to the generator $y = 0$ if $\alpha = \gamma_1$. If these relations are valid at the same time, one gets $\alpha^2 = \beta^2$ or $\alpha = -\beta$. The spatial collineation is then a harmonious perspectivity [harmonische Perspectivität] whose plane of perspective goes through the solid generators.

If you take $\alpha = \gamma$, then $\beta = \gamma_1$. The straight line with nothing but fixed points in the spatial collineation is here the polar line of the plane through the fixed generatrices.

By means of the above indications, the properties of the spatial collineations can be deduced by considerations in the plane.

9 p.25

First, we assume that f_4 does not vanish identically. The collineation group of the normal curve itself is obtained from the joint group of f_4 and f_6, which satisfies condition (b) $k^2 = l^2$. The curve is touched by the 12 generators, $27f_6^2 + 4f_4^3 = 0$, of the cone. So that the curve does not have a double point, no 2 of those 12 straight lines may coincide; unless f_6 and f_4 share common factors that include stationary tangents; but f_6 must not contain a double or multiple factor, which also goes into f_4. Here is a brief list of the different types. start p.26

1) Perspective collineation of period 2: $x' : y' : z' = -x : y : z$.

$$(1) \qquad z^3y^3 + zy(ax^4 + bx^2y^2 + cy^4) + dx^6 + ex^4y^2 + fx^2y^4 + gy^6 = 0\,.$$

2) Collineation »with axes »of period 2 : $x' : y' : z' = x : -y : z$.

$$(2) \qquad z^3y^2 + z(ax^4 + bx^2y^2 + cy^4) + x(dx^4 + ex^2y^2 + fy^4) = 0\,.$$

Here the order of the image curve is reduced because the center of projection is a point on the curve. The axes are the straight lines connecting two fixed points of the curve in the invariant plane and their polar line.

3) Klein four-group, consisting of 2 operations of type 1) and one of type 2). You get the equation:

$$(3) \qquad z^3y^2 + z[a(x^4 + y^4) + bx^2y^2] + x[c(x^4 + y^4) + dx^2y^2] = 0\,.$$

Here f_6 and f_4 merge into themselves when x and y are swapped.

4) Klein four-group, which consists of nothing but operations of the type 2). Since 2 points of $f_6 = 0$ remain fixed for each substitution, f_6 must be the JAKOBI covariant of f_4.

$$(4) \qquad z^3y^2 + z[a(x^4 + y^4) + bx^2y^2] + x(x^4 - y^4) = 0\,.$$

5) Diedric G_8, which contains as subgroups a Klein four-group of each of the two previous species.

$$(5) \qquad z^3y^2 + azx^2y^2 + x(x^4 + y^4) = 0\,.$$

The distinguished cyclic subgroup derives from the repetition of the operation, $x'y' : z' = x : iy : -z$.

The 5 species thus obtained fully correspond in sequence to the 5 given in the 3rd section. The following species, however, is an essentially new one

6) cyclic G_4 containing a G_2 of type 1) as a subgroup.

$$(6) \qquad z^3y^2 + z(x^4 + ay^4) + y(bx^4 + cy^4) = 0\,.$$

$x' : y' : z' = ix : y : z$. start p.27
The generator $x = 0$ remains fixed at each point. In addition, one point of the curve moves into the projection center on $y = 0$, which causes the order of the image curve to decrease. The case follows from (1) if there $b = d = f = 0$.

7) Diedric G_6.

$$z^3y^3 + azy^3x^2 + x^6 + bx^3y^3 + y^6 = 0\,. \tag{7}$$

The generating operation of the cyclic G_3 is the following: $x' : y' : z' = x : jy : z$. With this G_3 every point of the polar line remains fixed. The 3 G_2 are of type 1) and are obtained by swapping x and y. If one has $b = 0$ in 7), the transformation group is a

8) dihedral G_{12}.

$$z^3y^3 + azy^3x^2 + x^6 + y^6 = 0\,. \tag{8}$$

The generating operation of the cyclic subgroup G_6 is: $x' : y' : z' = -x : jy : z$.

Cases 7) and 8) correspond exactly to cases (6) and (7) of section 4.

9) Cyclic G_3 where the points of a generator remain fixed.

$$z^3y^3 + zy^2(ax^3 + by^3) + x^6 + cx^3y^3 + dy^6 = 0\,. \tag{9}$$

$x' : y' : z' = jx : y : z$.

The points of $x = 0$ remain fixed here. If f_6 is also the JAKOBI covariant of f_4, the transformation group consists of 4 such G and a Klein four-group of the kind 4), which together form a

10) tetrahedral group.

$$z^3y^3 + azy^2(x^3 + y^3) + x^6 + 20x^3y^3 - 8y^6 = 0\,. \tag{10}$$

One also gets the equation under the form 4) if only $b = 2ai\sqrt{3}$, by which condition the eqvian harmonic [äqvianharmonische?] invariant of f_4 vanishes. Special case of both (1) and (9) is the following.

11) Cyclic G_6

$$z^3y^3 + azy^5 + x^6 + by^6 = 0\,. \tag{11}$$

$x' : y' : z' = -jx : y : z$.

Also with this G_6 all points of the generatrix $x = 0$ remain fixed. If you have $b = 0$ here, the group is a start p.28

12) cyclic G_{12}.

$$z^3y^3 + zy^5 + x^6 = 0\,. \tag{12}$$

$x' : y' : z' := -jx : iy : -iz$.

13) Cyclic G_5.

$$z^3y^2 + azy^4 + x^5 + by^5 = 0\,. \tag{13}$$

$x' : y' : z' = \varepsilon x : y : z$.
The points of $x = 0$ remain fixed. If $b = 0$, the group is a

14) cyclic G_{10}. $x' : y' : z' = \varepsilon x : -y : z$.

$$z^3y^2 + zy^4 + x^5 = 0\,. \tag{14}$$

The subgroup G_2 here is of the type 2).

In the remaining cases, f_4 vanishes identically. The normal curve always merges into itself through a perspective G_3, which leaves each generator unchanged. In general,

15) this G_3, $x' : y' : z' = x : y : jz$, is also the complete group of the curve

$$z^3y^3 = f_6(x, y)\,. \tag{15}$$

But if f_6 has a linear transformation group in itself of order μ, one obtains the group of the curve by combining this G_μ with G_3, from which the order 3μ results. Within this $G_{3\mu}$ the perspective G_3 is marked.
The possible groups of f_6 in itself have already been compiled in our earlier discussion.[15] We found there in the 4th section that the same are the following: G_2, Klein four-group, dihedral G_6, G_5, dihedral G_{12} and group of octahedron [octahedral group]. We can therefore immediately write down the following normal equations:

$$z^3y^3 = x^6 + ax^4y^2 + bx^2y^4 + y^6\,; \tag{16}$$
$$z^3y^2 = x(x^4 + ax^2y^2 + y^4)\,; \tag{17}$$
$$z^3y^3 = x^6 + ax^3y^3 + y^6\,; \tag{18}$$
$$z^3y^2 = x^5 + y^5\,; \tag{19}$$
$$z^3y^3 = x^6 + y^6\,; \tag{20}$$
$$z^3y^2 = x(x^4 + y^4)\,. \tag{21}$$

The group of (16) is a cyclic $G_6 : x' : y' : z' = -x : y : jz$, and that of (19) start p.29
is a cyclic $G_{15} : x' : y' : z' = \varepsilon x : y : jz$. The group of (17) is of order 12 and consists of 3 cyclic G_6, all of which have the perspective G_3 as a subgroup. Of these G_6, 2 are of the same nature as that of Curve (16); but the third, $x' : y' : z' = x : -y : jz$, has a G_2 *with axes* as a subgroup. We do not dwell on the groups of curves (18) and (20) of the respective orders 18 and 36. The group of order 72 of curve (21) has 3 cyclic G_{12}, which are assigned to the cyclic G_4 of an octahedron group. One G_{12} is transformed into the plane collineation,

[15]Bihang till K. Sv. Vet.-Akad. Handl. Band 21, Afd. I, N:o 1

$x' : y' : z' = x : iy : -jz$.
Regarding the cyclic groups, our results differ in a few points from those given by Mr. KANTOR.[16] So did Mr. KANTOR does not mention the G_2 »with axes »(our case 2) and the cyclic G_6 belonging to the curve (17), which has a perspective G_3 and a G_2 »with axes »as subgroups. We have found 2 types of cyclic G_{12} here: one belongs to the curve (12) and contains as subgroups a perspective G_2 and a G_3 with a generator fixed at each point, the other to the curve (21) and contains a perspective G_3 and a G_2 »with axes ». Mr. KANTOR considers these two cases (designated 12 and 16 in his account) to be equivalent, which is not necessarily true. On the other hand, KANTOR's forms 6 and 15, 9 and 10 are equivalent to each other (and to our cases 6 and 9), but it should be noted that KANTOR's forms 6, 15 and 10 are not given with the full number of terms.

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We now turn to the treatment of structures of genus $p = 5$. The plane curve of the lowest order, into which a general structure of this genus can be uniquely transformed, is known to be a C_6 with 5 double points; but if the structure has a special group g_3^1, it can be transformed into a C_5 with a double point.[17] In the start p.30
latter case, the pencil of rays intersects the g_3^1 through the double point, and the adjoint φ-curves are conics through the same point. If one now goes to the φ-normal-curve in 4-dimensional space, one finds that it must lie on a third-order ruled surface, which is the image of our original plane, with the rays from the double point in generatrix and the double point itself in be transformed into a simple guide line of the ruled surface. Each generatrix is thus a trisecant and the directrix a bisecant of the normal curve. From this it can be seen that the ruled surface or the plane of the C_5 plays an excellent [ausgezeichnete, distinguished?] role. Consequently, the unique transformations of the C_5 must be birational in themselves, because no quadratic or higher collineations are possible. The latter is concluded from the fact that the system of adjoint conics must also be transformed into itself in such a birational transformation; but within this system there is an excellent [auszeichnete, distinguished?] split subdivision, and the associated conics each consist of a straight line through the double point and an arbitrary straight line; the system of straight lines must therefore be converted into itself, which only occurs with a collineation. The possible collision groups of C_5 must be of a very trivial nature. One descends from the total group to the distinguished subgroup, in which the two branches in the double point remain uninterchanged, and from this to a (cyclic) distinguished subgroup, in which each straight line merges into itself through the double point. However, these distinguished groups may consist of the total group or just identity.
According to a theorem by Mr. WEBER, an algebraic structure with any p is characterized by $\frac{1}{2}(p-2)(p-3)$ linearly independent quadratic relations

[16]a. o. O., S. 166.
[17]See CLEBSCH-LINDEMANN, *Vorlesungen über Geometrie*, I, S. 709

between the φ-functions.[18] However, these relations are not sufficient to define the structure if a g_3^1 exists.[19] So for $p = 5$ we have three quadratic relations, $\Phi_1^{(2)} = 0$, $\Phi_2^{(2)} = 0$, $\Phi_3^{(2)} = 0$, the normal curve C_8 can generally be considered as its intersection curve;[20] but the $\Phi^{(2)} = 0$ in the case of the g_3^1 have a common surface, namely the previously mentioned ruled surface of the third order. start p.31

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From now on we stick to the general case. The normal curve is determined in the 4-dimensional space by the three quadratic relations:

$$F_1(x_1, x_2, x_3, x_4, x_5) = 0\,, \quad F_2 = 0\,, \quad F_3 = 0\,. \tag{1}$$

We take any linear combination of F_i:

$$\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0 \tag{2}$$

and look for its discriminant:

$$\Delta_5(\lambda_1, \lambda_2, \lambda_3) = 0\,, \tag{3}$$

which can be written in the known way in the form of a determinant with 5 rows and 5 rows [Zeilen, columns]. *The nature of the covariant curve $\Delta_5 = 0$ in the λ plane is of fundamental importance for our normal curve.* Equation (3) determines those combinations of λ_i for which the left term of (2) is a function of only 4 variables, say x_1', x_2', x_3', x_4'. The location of the vertices $x_1' = x_2' = x_3' = x_4' = 0$ clearly corresponds to the curve Δ_5. However, this location is determined by the determinant group

$$\begin{vmatrix} F_1^1 & F_1^2 & F_1^3 & F_1^4 & F_1^5 \\ F_2^1 & F_2^2 & F_2^3 & F_2^4 & F_2^5 \\ F_3^1 & F_3^2 & F_3^3 & F_3^4 & F_3^5 \end{vmatrix} = 0\,, \tag{4}$$

where $F_i^k = \frac{\partial F_i}{\partial x_k}$. The system (4) represents a curve of order 10 in the 4-dimensional space,[21] which we will denote by D_{10}. The normal curve cannot have a double point because then its genus is degraded [reduced?]. However, as is easy to find, both equations (1) and (4) should hold for a double point. The necessary condition follows that the curve D_{10} must not separate the normal curve. Thus, no double points of D_{10} and Δ_5 can come from such intersections. But the curve Δ_5 can have double points, namely in those points of the λ-plane, for which the left term of (2) as a function of only 3 variables, say x_1', x_2', start p.32

[18] Math. Ann., Bd. 13.

[19] Based on a sentence by L. KRAUS, Math. An.., Bd. 16. It should be noted here that the relations $\Phi^{(2)} = 0$ are not sufficient to define the structure even in a case without g_3^1, namely for the plane C_5 without double points.

[20] See a paper by Mr. NOETHER, Math. Ann., Bd. 26.

[21] See SALMON-FIEDLER, *Algebra der linearen Transformationen* 2 Auflage [2nd edition] (1877), S.368

x_3', can be written; such a double point corresponds to a straight line on D_{10}, $x_1' = x_2' = x_3' = 0$. The relations $F_i = 0$ between 3 variables correspond to just as many systems of *infinitely many Abelian root forms*. L. KRAUS now wanted to show that there are at most 3 such systems;[22] but his proof is incorrect, and we shall find that as many as 10 such systems can occur. In this case, Δ_5 has 10 double points, which is only possible if it consists of 5 lines, and D_{10} consists of 10 lines, which intersect 4 at 5 points, these lines and multiple points of D_{10} correspond to double points and straight lines of Δ_5. But we shall come back to this case later.

According to BRILL-NOETHER's law of reciprocity, the special groups g_4^1 of the structures of genus $p = 5$ are assigned to each other in pairs. The curve D_{10} has a simple connection with the system of g_4^1. If one projects the normal curve from a point of D_{10}, one obtains a curve C_8 as its image in 3-dimensional space, which extends to a surface of the 2nd degree $F(x_1', x_2', x_3', x_4') = 0$, where $F = 0$ gives the corresponding relation with only 4 variables. The generating systems of this surface cut out 2 reciprocal g_4^1 to each other. But if that relation were between only 3 antecedents, the projected curve lies on a second-degree cone, so that the two sets g_4^1 coincide, and the only g_4^1 is its own reciprocal; the infinitely many associated ABEL root forms are assigned to the tangent planes of the cone. Thus the system of g_4^1 is 1-2-uniquely related to the curve D_{10}, as we later want to illustrate geometrically in a different way. With a collineation of the normal start p.33
curve in itself, the system of relations $\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0$ must of course also be converted into itself. The λ_i thus experience linear substitutions, in which the covariant curve Δ_5 is necessarily transformed. This is not contradicted by the fact that a collineation of the Δ_5 is not always the result of a collineation of the normal curve. On the other hand, the normal curve can undergo unique transformations in which each point of Δ_5 remains fixed, and therefore each relation (2) is retained unchanged. With such a collineation, the curve D_{10} must also merge point by point, if one disregards straight lines that correspond to double points of Δ_5; the remaining curve D_{10} can consequently not be an actual 4-dimensional curve, but can either be composed of a point and a 3-dimensional curve or of a straight line and a flat curve, depending on a point in the collineation and the points of a space or the points of a straight line and a plane remain fixed. As a necessary condition for this one immediately obtains that each relation (1) $F_i = 0$ can be written either in the form

$$\text{(a)} \qquad a_i x_1^2 + f_i(x_2, x_3, x_4, x_5) = 0 \quad (i = 1, 2, 3)$$

or in the form

$$\text{(b)} \qquad \varphi_i(x_1, x_2) + \psi_i(x_3, x_4, x_5) = 0\,.$$

In the first case one has the collineation:

$$(\alpha) \qquad x_1' = -x_1\,, \quad x_i' = x_i \quad (i = 2, 3, 4, 5)$$

[22] a. o. O.

with 8 fixed points of the curve in space $x_1 = 0$; in the latter case the collineation:

$$(\beta) \qquad x_1' = -x_1\,, \quad x_2' = -x_2\,, \quad x_i' = x_i \quad (i = 3, 4, 5)$$

without fixed points of the curve. The straight lines connecting corresponding points in cases (a) and (b) thus form (according to equation (A) introduction) a ruled surface of genus $p' = 1$ or $p' = 3$. However, symmetric transformations with the assigned genus $p' = 2$ can also occur, but without transforming every point of the curve Δ_5 into itself. In case (a) the discriminant of equation (2) breaks down into a linear and a biquadratic factor and in case (b) into a start p.34 quadratic and a cubic factor, so that the curve Δ_5 consists of a straight line and a C_4 rez [respectively], a conic section and a C_3. But these curves can also decay [decompose], so that the following 7 typical cases are obtained.

1) Δ_5 is an actual C_3.

2) Δ_5 breaks down into a straight line and a C_4. The intersection points of the straight lines with C_4 correspond to 4 systems of infinitely many ABEL root forms. But here, as in the previous case, other such systems can appear, corresponding to the possible double points of C_4 or C_5.

3) Δ_5 splits into a conic section and a C_3. The 6 points of intersection of these curves correspond to 6 systems of ABEL root forms.

4) Δ_5 breaks down into 2 straight lines and a C_3. Any relation $F_i = 0$ has the form

   $$\text{(c)} \qquad a_i x_1^2 + b_i x_2^2 + \psi_i(x_3, x_4, x_5) = 0\,.$$

   Here you have 2 collineations of type (α) and one of type (β). Seven systems of ABEL root forms always occur.

5) Δ_5 splits into a straight line and 2 conics.

   $$\text{(d)} \qquad F_i = a_i x_1^2 + \varphi_i(x_2, x_3) + \psi_i(x_4, x_5) = 0\,.$$

   A collineation of type (α) and 2 of type (β). Eight systems of ABEL root forms.

6) Δ_5 breaks down into 3 lines and a conic section.

   $$\text{(e)} \qquad F_i = a_i x_1^2 + b_i x_2^2 + c_i x_3^2 + \psi_i(x_4, x_5) = 0\,.$$

   There are 3 collineations of type (α) and 4 of type (β). Nine systems of ABEL root forms.

7) Δ_5 is divided into 5 straight lines.

   $$\text{(f)} \qquad F_i = a_i x_1^2 + b_i x_2^2 + c_i x_3^3 + d_i x_4^4 + e_i x_5^2 = 0\,.$$

   In this case one has 5 collineations of type (α) and 10 of type (β). From the 3 independent relations $F_i = 0$ one can eliminate any pair x_k^2 and x_i^2 and thus obtain 10 relations between 3 variables, which correspond to just as many systems of ABEL root forms.

The question can now be raised as to how one can arrive at the structure in 4-dimensional space given Δ_5. First it should be emphasized that both a general C_5 and a structure of the genus $p = 5$ have twelve modules that are indestructible by unique transformation, so that a finite number of solutions can be expected. It resulted in start p.35

$$\Delta_5(\lambda) = \sum \pm u_{11}u_{22}u_{33}u_{44}u_{55}$$

in determinant form, where the $u_{ik} = u_{kl}$ are linear functions of the λ. The unique transformation connecting the curves Δ_5 and D_{10} has the form

$$x_1 : x_2 : x_3 : x_4 : x_5 = U_{i_1} : U_{i_2} : U_{i_3} : U_{i_4} : U_{i_5}$$

or

$$x_i x_k = \rho U_{ik}\,,$$

where U_{ik} is a subdeterminant from Δ_5 to u_{ik}. From these equations one can produce the λ_1^4, $\lambda_1^3\lambda_2$, ... as a linear function of the x_1^2, x_1x_2, ... According to this, the intersections of Δ_5 with all fourth-order curves correspond to the intersections of D_{10} with the manifolds represented by the quadratic relations; *in particular, the intersections of D_{10} with the doubly counted 3-dimensional spaces correspond to a quadruple infinite system of fourth-order tangent curves at Δ_5.* This system results from changing the determinant $\Delta 5$. So one would have to look for that class of touch [contact] systems by which Δ_5 can be written in the form of a symmetric determinant.[23] The formations in the 4-dimensional space originating from different contact systems are generally not collinearly related, because for this it would be necessary for the systems to also be related in this way. It should also be noted that a collineation of the Δ_5 must also convert the underlying contact system into itself, so that a corresponding collineation exists for the normal curve with $p = 5$. The touch systems in question must be adjoint and pass through the double points of Δ_5, which is evident from the fact that start p.35
the double points of the Δ_5 correspond to straight lines of D_{10}.

12 p.36

A C_7 in the usual 3-dimensional space and a flat C_6 can always, if their genus $p = 5$, come out as projections of the normal curve, because the ∞^2 planes resp. ∞^2 straight lines cut out special groups in the two cases mentioned. Now the question arises how, given a plane C_6 with 5 double points, one can determine the two points of the curve which are to be regarded as successive projection centers. These points, together with any pair united in a double point, should form a group G_4, which belongs to a special group g_4^1. However, this condition is only satisfied by those points which are cut out by a conic section drawn through the 5 double points.

[23]Cf. HESSE's investigations on the contact conic sections of a C_3 and on the systems of the *first* kind of C_3 in contact with a C_4, CRELLE's Journal, vol. 36, 49.

The C_7 projected from the various points of the normal curve each reveal the peculiarities of the system of special groups g_4^1 through their *trisecant ruled surface.* The 3 points which, together with the projection center, form a group G_4 of such a special group must be projected in a straight line. *Hence the generatrices of the ruled surface and the arrays g_4^1 correspond uniquely.* The point groups of the reciprocal g_4^1 are cut from the planes through the trisecant associated to the given g_4^1, and in particular 2 generators corresponding to reciprocal g_4^1 intersect each other, cutting through their common plane the pixel [Bildpunkt?] of the projection center goes. The locus of the intersections of two such reciprocal generators is the projected image of the curve D_{10}. If a g_4^1 and consequently the corresponding trisecant is reciprocal to itself, the trisecant must be a double torsale [Torsale?] such that the torsale plane contains the tangents of the 3 points lying on the torsale of the C_7. The C_7 is the fivefold curve and the D_{10} double curve of the trisecant end ruled surface. A generatrix thus intersects 12 others in the 3 points of C_7 and one in the point of D_{10}, and the degree of the ruled surface is consequently 15. For the genus P of a cross-section one obtains according to PLUCKER's formulas:

$$P = 11 - \delta ,$$

where δ means the number of the above-mentioned double torsals with the same plane. Depending on the nature of Δ_5 and D_{10} one has (corresponding to the typical cases of the previous section) the following 7 cases. start p.37

1) Δ_5 is an actual C_5, which can have a maximum of 6 double points. The genus of R_{15} is usually 11, but can be lowered to 5 by δ.

2) $\Delta_5 = C_4 + C_1$. $R_{15} = R_{12} + K_3$. The genus of R_{12} alternates between 7 and 4, with the humiliation [reduction] being effected by double points of C_4. K_3 is of genus 1, and the apex contains that point of C_7 which corresponds to the center of projection in the transformation (α) of the previous section. The generatrices of the cone connect the points corresponding to this transformation. K_3 and R_{12} intersect in C_7, counted quadruple, touching each other along the 4 double torsals, which correspond to the intersections of the constituent parts of Δ_5. Also in the following cases, in which R_{15} decays, the sectional curves only consist of C_7 and double torsals.

3) $\Delta_5 = C_3 + C_2$. $R_{15} = R_9 + R_6$. R_9 has genus 4 or (through a double point of C_3) 3. R_6 is always of genus $P = 2$.[24]

4) $\Delta_5 = C_3 + 2C_1$. $R_{15} = R_9 + 2K_3$.

5) $\Delta_5 = 2C_2 + C_1$. $R_{15} = 2R_6 + K_3$.

6) $\Delta_5 = C_2 + 3C_1$. $R_{15} = R_6 + 3K_3$.

[24]Regarding this R_6 see my writing, *Klassifikation af regelytorna af 6. graden* S. 57 (Lund 1892. Diss.)

7) $\Delta_5 = 5C_1$. $R_{15} = 5K_3$. The C_7 lies here on $5K_3$, each pair of which touches each other along the connecting line of the vertices.

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In order to determine the complete collineation group of a given normal curve, one first has to consider the covariant curve Δ_5. You classify them in one of the 7 cases of the 11th section and then look for their collineation group. From this group one has to remove the subgroup to which collineations of the normal curve also belong. A given linear substitution in the λ_i corresponds to the transposed substitution in the F_i due to the equation $\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 = 0$. Consequently, one can search for the typical periodic collineations in the 4-dimensional space from the outset, through which a linear transformation is effected under 3 quadratic relations whose intersection curve *has no double point.* start p.38
If we transfer the resultant of such an investigation to the λ-level [plane], it follows that only the collineations from period 2 and the non-perspective G_3, G_4 and G_5 need to be considered.
The finite planar and *simple* collineation groups are (apart from the cyclic ones with a prime number period) the icosahedron group, the G_{168} known from the theory of a special C_4 and a G_{360}.[25] The first two are known not to have an invariant form of 5th order; the same must be true for G_{360} because it contains groups of icosahedrons [icosahedral groups].[missing full stop?] If one then also takes the results of section 10 regarding the structures with a g_3^1 into account, we have the following theorem:
All groups occurring within the non-hyperelliptic structures of genus $p = 5$ can be broken down into a series of purely cyclic groups.
If Δ_5 now has a group of order μ, to which collineations of the normal curve also belong, then one has the following numbers for the order of the total group of the latter, depending on the typical case of Δ_5: μ, 2μ, 2μ, 4μ, 4μ, 8μ, 16μ.
Finally, those curves are listed here in their normal equations whose moduli are completely determined by the condition of admitting certain collineation groups.

1) *Group of Order 192.*

$$F_1 = x_1^2 + x_4^2 + x_5^2 = 0\,,$$
$$F_2 = x_2^2 + x_4^2 - x_5^2 = 0\,,$$
$$F_3 = x_3^2 + x_4 x_5 = 0\,.$$

One has (with the identity) 8 collineations by sign changes from the x_i, where each F_i is unchanged. Then one has to notice that the form $x3x_4 \cdot (x_4^2 + x_5^2) \cdot (x_4^2 - x_5^2)$ is an octahedron form broken down into its 3 distinct quadratic factors. One uses the group of this binary form with start p.39

[25]This G_{360} was discovered by Mr. VALENTINE (Kong. Dansk. Vid. Selsk. Skrift. (6) V, 1889), but otherwise seems to be unknown.

suitable exchanges [permutations] and substitutions of x_1, x_2, x_3 and then by combining it with G_8 the complete G_{192} is obtained.[26] Eliminating x_5 gives the equations:

$$x_1^2 + x_2^2 + 2x_4^2 = 0\,, \quad x_4^2(x_2^2 + x_4^2) - x_3^4 = 0\,.$$

The curve Δ_5 consists of a conic section and 3 straight lines, which form a polar triangle in relation to the conic section. In the 3 following cases, Δ_5 consists of 5 straight lines, which are also in a special position.

2) *Group of order 64.*

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 &= 0\,, \\ x_1^2 + ix_2^2 - x_3^2 - ix_4^2 &= 0\,, \quad (i^2 = -1.) \\ x_1^2 - x_2^2 + x_3^3 - x_4^2 &= 0\,. \end{aligned}$$

The G_{64} is created by combining the character changes with the cyclic permutation of $x_1,$, x_2, x_3, x_4.

3) *Group of order 96.*

$$\begin{aligned} x_1^2 + x_4^2 + x_5^2 &= 0\,, \\ x_2^2 + jx_4^2 + j^2x_5^2 &= 0\,, \quad (j^3 = 1.) \\ x_3^2 + j^2x_4^2 + jx_5^2 &= 0\,. \end{aligned}$$

The G_{96} is obtained by combining the character changes and swapping x_1, x_2 and x_3 on the one hand, and x_4 and x_5 on the other.

4) *Group of order 160.*

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 &= 0\,, \\ x_1^2 + \varepsilon x_2^2 + \varepsilon^2 x_3^2 + \varepsilon^3 x_4^2 + \varepsilon^4 x_5^2 &= 0\,, \quad (\varepsilon^5 = 1.) \\ \varepsilon^4 x_1^2 + \varepsilon^3 x_2^2 + \varepsilon^2 x_3^2 + \varepsilon x_4^2 + x_5^2 &= 0\,. \end{aligned}$$

Here the allowed permutations [erroneous comma?] of the x_i form a dihedral G_{10}. Namely, the array $(x_1\, x_2\, x_3\, x_4\, x_5)$ can both be cyclically shifted into itself and reversed into the array $(x_5\, x_4\, x_3\, x_2\, x_1)$. The G_{160} is created by combining this G_{10} with the character changes. start p.40

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Finally, a few hints about the formations of the $p = 6$ family are permitted. Apart from the hyperelliptical case, there are 3 essentially different types, which we want to treat here in series.

[26]In the theory of elliptic modular functions, this G_{192} belongs to the eighth order main congruence group. See KLEIN-FRICKE, *Modulfunctionen* I, S. 338 u.f.

1) The curve has a special group g_5^2, i.e. it can be transformed into a plane C_5. With a unique transformation of C_5 in itself, the point groups G_5 of g_5^2 must form a closed invariant system. Such a transformation must consequently be a collineation, and according to a theorem in the previous section composed entirely of cyclic groups.

2) The curve has a special family g_3^1. Such a curve is easily converted into a C_6 with a triple and a double point. In the case of a group of unique transformations of the curve in itself, the point groups G_3 of the special family must be permuted in a closed manner, which of course can only be effected by one of the finite linear groups; but not by an icosahedron group, because the number of G_3 containing 2 coincident points is 16, and no 16 points in this group form a closed system. In addition, the transformation group can still be composed of a cyclic group of period 3, which cyclically exchanges the points of each G_3.

3) In the general case, the curve has 5 g_4^1 and can be unambiguously converted into a C_6 with 4 double points, whereby the point groups of the g_4^1 are cut out by the straight line pencils by a double point each and the conic pencil by all 4 δ. With a unique transformation of the curve in itself, these g_4^1 can either be permuted in themselves or in some way. One also gets a case where all 120 permutations of the 5 g_4^1 transform the curve into itself: start p.41

$$2\sum(x^4yz + y^3z^3) - 2\sum x^4y^2 + \sum x^3y^2z - 6x^2y^2z^2 = 0\,.$$

The double points are here in the coordinate corners and in the point $x = y = z$.

Finally, as a result of our investigations, we can cite the following general propositions, which Mr. DYCK has already given $p = 0, 1, 2, 3$ for the genera.[27]
The only simple and finite groups of unique transformations of a curve in itself within the genus $p = 0, \ldots, 6$ are the cyclic groups of prime order, the icosahedron group and a G_{168} (at $p = 3$). The composite groups within our genera, with 3 exceptions, can be broken down into a sequence of merely cyclic groups. The exceptions are: Bring's curve ($p = 4$) and the just mentioned curve with $p = 6$, whose groups are holohedrally isomorphic with the group of 120 permutations of 5 things, as well as a hyperelliptic curve[28] of genus $p = 5$, whose group is also of order 120; namely, within these groups, an icosahedron group is distinguished.

[27] *Über reguläre Riemann'sche Flächen*, Math. Ann., Bd. 17, S. 474

[28] The equation of this curve is $y^2 = x(x^{10} + 11x^5 - 1)$. In the theory of elliptic module [modular] functions, it supplies the modules belonging to a distinguished congruence subgroup of the 10th order. See KLEIN-FRICKE, *Modulfunctionen* I, S.651