KP-II EQUATION WITH SOLITON SEGMENT INITIAL CONDITIONS

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We consider the initial value problem for the KP-II equation

$$(1)\qquad \begin{aligned} &(u_t + uu_x + u_{xxx})_x = -u_{yy}, \quad (x,y) \in \mathbb{R}^2, \quad t > 0 \\ &u(x,y,0) = \begin{cases} \operatorname{sech}^2\left(\frac{x}{\sqrt{12}}\right) & -w/2 < y < +w/2 \\ 0 & \text{else} \end{cases}, \quad (x,y) \in \mathbb{R}^2, \end{aligned}$$

that consists of a line soliton segment, i.e., set to zero, outside of an area of width w. Due to scaling, rotation, and Galilean symmetries, this data can be transformed to a truncated soliton of arbitrary slope, amplitude, and background. The data for (1) is shown in Fig. 1. Note the initial condition also contains an odd reflection of the soliton, thus the initial data satisfy the constraint

$$(2)\qquad \int_{-L_x}^{L_x} u_{yy}\,\mathrm{d}\,x = 0.$$

Line soliton solutions of (1) with amplitude a and slope q take the form

$$(3)\qquad u(x,y,t) = a\operatorname{sech}^2\left(\sqrt{\frac{a}{12}}(x + qy - ct)\right), \quad c = \frac{a}{3} + q^2.$$

First, we numerically evolve the initial data–appropriately smoothed in the y direction–using an approach similar to that given in *Kodama et al.* The result is shown in Fig. 2. By using the odd reflection, the background mean remains $\mathcal{O}\left(10^{-8}\right)$ throughout the experiment.

Next, we use the x-independent KP-Whitham equations in the soliton limit, which, in the zero mean case, take the Riemann invariant form

$$(4)\qquad \begin{aligned} &\partial_t R_\pm + V_\pm \partial_y R_\pm = 0, \\ &R_\pm = \sqrt{a} \pm q, \quad V_\pm = 2q \pm \frac{2}{3}\sqrt{a}. \end{aligned}$$

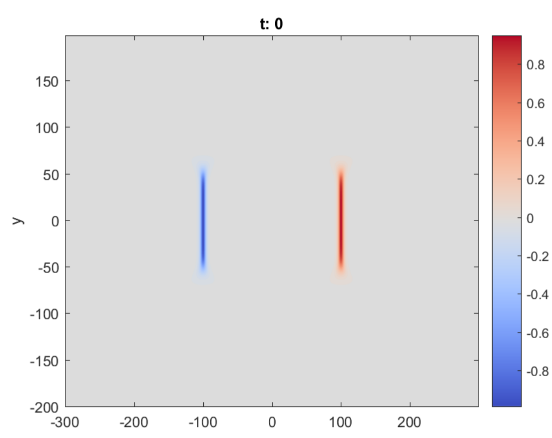

FIGURE 1. Truncated line soliton.

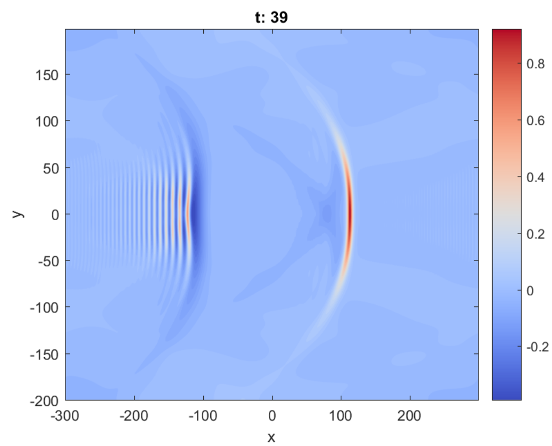

FIGURE 2. Numerical evolution to $t = 75$ of smoothed initial value problem in (1).

The initial value problem (1) becomes eq. (4) with the data

$$a(y,0) = \begin{cases} 1 & -w/2 < y < w/2 \\ 0 & \text{else} \end{cases}, \quad q(y,0) = \begin{cases} 0 & -w/2 < y < w/2 \\ q_{hi} & y > w/2 \\ q_{lo} & y < w/2 \end{cases}. \tag{5}$$

The upper half of this solution evolves according to the previous result found for the half soliton, with

$$a(y,t) = \begin{cases} 0 & y - w/2 > 2t \\ \frac{9}{64}(2 - (y - w/2)/t)^2 & -\frac{2}{3}t < y - w/2 < 2t \\ 1 & y - w/2 < -\frac{2}{3}t \end{cases}, \quad q(y,t) = \begin{cases} 1 & y - w/2 > 2t \\ 1 - \sqrt{a(y,t)} & -\frac{2}{3}t < y - w/2 < 2t \\ 0 & y - w/2 < -\frac{2}{3}t \end{cases}. \tag{6}$$

We use similar reasoning for the lower half of the solution The soliton slope q_{lo} in the absence of soliton amplitude is undefined (vacuum). In order to enforce a well-posed problem, we seek a simple wave solution in the form $R_- = \mathit{const}$ and $V_+ = y/t$. This solution implies $q_{lo} = -1$ and

$$a(y,t) = \begin{cases} 0 & y + w/2 < -2t \\ \frac{9}{64}(2 + (y + w/2)/t)^2 & -2t < y + w/2 < \frac{2}{3}t \\ 1 & y + w/2 > \frac{2}{3}t \end{cases}, \quad q(y,t) = \begin{cases} 1 & y + w/2 < -2t \\ -1 + \sqrt{a(y,t)} & 2t < y + w/2 < \frac{2}{3}t \\ 0 & y + w/2 > \frac{2}{3}t \end{cases}. \tag{7}$$

Matching these solutions at $y = 0$, then inserting these results into eq. (3) and plotting, yields Fig. 3. The two solutions depicted in Figs. 2 and 3 look very similar.

A more quantitative description involves looking at the characteristic speeds $V \in \{\pm 2/3, \pm 2\}$. We can again extract these quantity from the numerical simulation. The results are shown in Fig. 4.

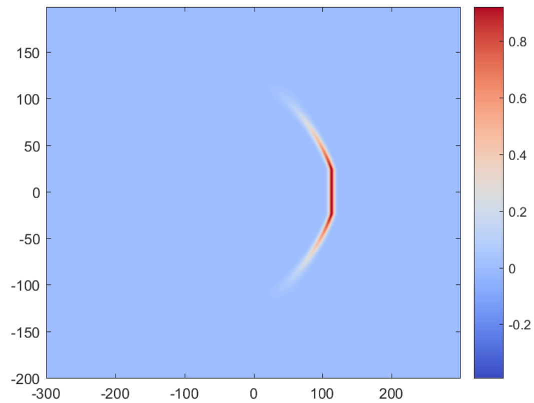

FIGURE 3. Line soliton (3) modulated by the Whitham simple wave solution (6) at $t = 75$.

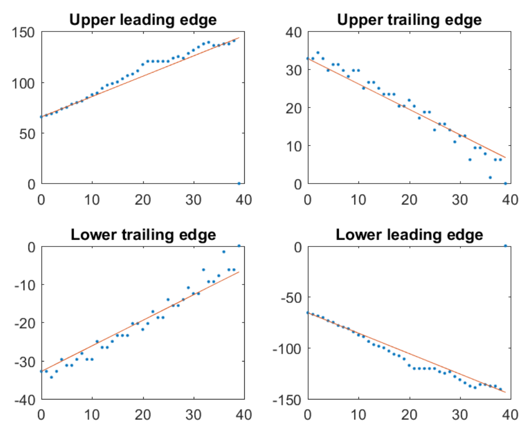

FIGURE 4. An extraction of the curve $y_{-}(t)$ from numerical solution of (1) (dots), revealing the predicted simple wave result $y_{-}(t) = -2t/3$.