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On the evolution of perturbations to solutions of the Kadomtsev–Petviashvilli equation using the Benney–Luke equation

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Abstract

The Benney–Luke equation, which arises as a long wave asymptotic approximation of water waves, contains the Kadomtsev–Petviashvilli (KP) equation as a leading-order maximal balanced approximation. The question analyzed is how the Benney–Luke equation modifies the so-called web solutions of the KP equation. It is found that the Benney–Luke equation introduces dispersive radiation which breaks each of the symmetric soliton-like humps well away from the interaction region of the KP web solution into a tail of multi-peaked oscillating profiles behind the main solitary hump. Computation indicates that the wave structure is modified near the center of the interaction region. Both analytical and numerical techniques are employed for working with non-periodic, non-decaying solutions on unbounded domains.

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1. Introduction

In recent years, theoretical [5, 6] and experimental evidence [18] have appeared concerning special classes of soliton solutions to the Kadomtsev–Petviashvilli (KP) equation. These types of solutions are often called web solutions and are generalizations of the now classic Y-junction, two-soliton, and *N*-soliton solutions to the KP equation as discussed in [1, 14, 16]. These solutions are described by a series of rays that intersect in what is termed an interaction region; see figure 1. Far enough away from the interaction region, the rays are well described by classic line soliton solutions to the Korteweg–de Vries (KdV) equation.

It is well known that both the Benney–Luke (BL) equation [4] and, contained within it as a maximally balanced limit, the KP equation are long wave, shallow water asymptotic approximations to the Euler surface–water wave equations in inviscid, irrotational water [1, 3].

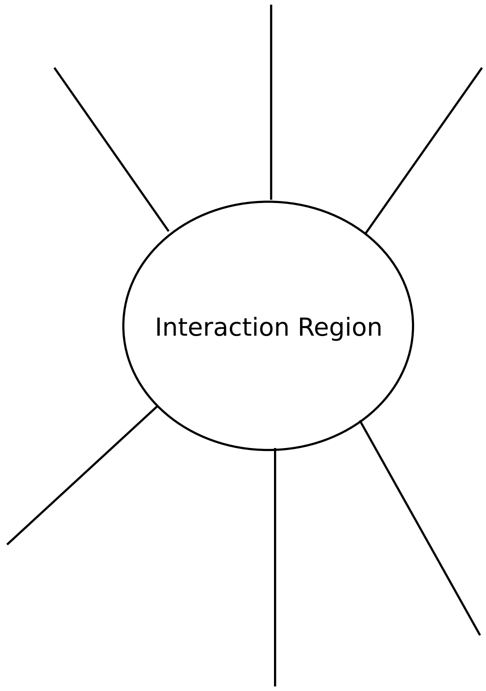


Figure 1. Web solution.

From the web KP solutions so far obtained [5, 6], we now know that these special solutions to KP can exhibit a variety of exotic and intriguing phenomena. The issue addressed here is to analyze the behavior of web solutions to KP in the context of the BL equation as a prototype approximation to the Euler equations. This can provide insight into the realistic behavior of the complex KP web solution patterns mentioned above.

The Benney–Luke (BL) equation is considered in the form [3]

$$q_{tt} - \tilde{\Delta} q + \alpha\epsilon\tilde{\Delta}^2 q + \epsilon(\partial_t|\tilde{\nabla} q|^2 + q_t\tilde{\Delta} q) = 0, \tag{1}$$

where small dispersion, slow transverse variation and weak nonlinearity are all balanced via q the velocity potential, $\tilde{\nabla} = \langle\partial_x, \epsilon^{1/2}\partial_y\rangle$, $\tilde{\Delta} = \partial_x^2 + \epsilon\partial_y^2$, $|\epsilon| \ll 1$, and $\alpha = \tilde{\sigma} - \frac{1}{3}$, where $\tilde{\sigma}$ is related to the surface tension. Web solutions arise when $\tilde{\sigma}$ is small, which is the case for water waves. As shown in [3], one may derive the KP equation by introducing the coordinates $\xi = x - t$ and $\tau = \epsilon t$, taking terms only of order ϵ, and then making the substitution $w = q_\xi$. In these coordinates, KP takes the form

$$2w_{\tau\xi} - \alpha w_{\xi\xi\xi\xi} + w_{yy} + (3ww_\xi)_\xi = 0. \tag{2}$$

It is also worth pointing out that in these coordinates the surface height η is

$$\eta = q_\xi - \epsilon\left(q_\tau + q_\xi^2\right) + \mathcal{O}(\epsilon^2) = w + \mathcal{O}(\epsilon), \tag{3}$$

and thus, as stated, a solution to KP gives a leading-order description of the surface height profile in shallow water. In order to get a deeper understanding of the wave height profile, we then use the solutions to KP as leading-order solutions in a formal series solution to the BL equation.

As we show, the BL equation introduces a number of what can be called slow speeds that alter the way in which the rays of the web solutions to KP propagate. Combining this with the work of [6], in which an algorithmic method for determining the position of the rays is given, one can therefore determine the far field influence of the BL equation for any web solution. It should also be pointed out that another effect of BL in the far field is that the symmetric pulse shapes of the rays of the KP solutions develop multi-peaked, asymmetric profiles, i.e. behind each main pulse is an oscillating tail.

In order to study what happens in and around the finite interaction region, numerical results are employed; they also help provide intuition. In this paper, we analyze in detail the special and interesting case of the Y-junction solutions with small surface tension. As can be seen in both the far-field and interaction region results, the most significant contribution of the BL equation appears to be the addition of small amplitude dispersive tails to the leading-order KP solutions. This oscillatory phenomena is not symmetric, but instead radiates in a preferential direction. Thus, a great deal of fine structure missed at the level of KP appears in the BL equation. Finally, our numerical results reveal that there is linear growth in the wave height (an increasing dip in the wave) of the first-order perturbation in the interaction region. This growth does not appear to arise from either numerical or physical instabilities. It seems likely that the linear growth is developing into an asymptotic secularity and as such indicates that after long enough time the linearized first-order perturbation theory will need modification. However, our work also shows that the Y-junction appears to be stable since the growth and shape of the perturbation, while changing the wave structure at the center of the Y-junction, do not greatly distort the leading-order approximation. The numerical results obtained here, on the perturbed equation, complement earlier numerical investigations found in [9] and [11].

2. Perturbation scheme for Benney–Luke

To study perturbations of solutions to the KP equation we first transform the Benney–Luke equation (1) into the appropriate moving coordinates of (2). Then we will derive the system of linear equations that will determine the behavior of perturbed solutions.

So, introducing the coordinates $\xi = x - t$ and $\tau = \epsilon t$ gives

$$\partial_\xi = \partial_x, \qquad \partial_t = -\partial_\xi + \epsilon\partial_\tau.$$

Thus, (1) becomes

$$-2\partial_{\xi\tau}q - \partial_y^2 q + \alpha\partial_\xi^4 q - 3q_\xi q_{\xi\xi} + \epsilon\left(q_{\tau\tau} + 2\alpha q_{\xi\xi yy} - \partial_\xi\left(q_y^2\right) + \partial_\tau\left(q_\xi^2\right) - q_\xi q_{yy} + q_\tau q_{\xi\xi}\right) + \mathcal{O}(\epsilon^2) = 0.$$

In order to simplify matters in later sections, we introduce the coordinate change $w = Au(\beta\xi, \gamma y, \delta\tau)$, with

$$A = (-8\alpha)^{1/5}, \qquad \beta = \frac{2}{A^2}, \qquad \gamma^2 = \frac{6}{A^3}, \qquad \delta = -\frac{2}{A},$$

so that if w solves (2), then u solves the canonical KP equation

$$(-4u_\tau + u_{\xi\xi\xi} + 6uu_\xi)_\xi + 3u_{yy} = 0. \tag{4}$$

Introducing the transformation $q = \frac{A}{\beta}\phi(\beta\xi, \gamma y, \delta\tau)$, we get (1) in the form

$$4\partial_{\xi\tau}\phi - 3\partial_y^2\phi - \partial_\xi^4\phi - 6\phi_\xi\phi_{\xi\xi} + \epsilon A\left(2\phi_{\tau\tau} - 3\phi_{\xi\xi yy} - 3\partial_\xi\left(\phi_y^2\right) - 2\partial_\tau\left(\phi_\xi^2\right) - 3\phi_\xi\phi_{yy} - 2\phi_\tau\phi_{\xi\xi}\right) + \mathcal{O}(\epsilon^2) = 0.$$

For reasons that will be made clear later, we define the slower time $\nu = \epsilon\tau$ and the expansion

$$\phi(\xi, y, \tau) = \phi_0(\xi, y, \tau, \nu) + \epsilon\phi_1(\xi, y, \tau) + \cdots,$$

with $\phi_0(\xi, y, 0, 0) = \phi(\xi, y, 0)$ and $\phi_j(\xi, y, 0) = 0$. Letting $\phi_\xi = w + \epsilon w_1 + \cdots$, we see that w and w_1 must satisfy

$$(-4w_\tau + w_{\xi\xi\xi} + 6ww_\xi)_\xi + 3w_{yy} = 0$$
$$- 4w_{1,\xi\tau} + 3w_{1,yy} + \partial_\xi^4 w_1 + 6\partial_\xi^2(ww_1) = \partial_\xi\left(F(\phi_0) + 4w_\nu\right), \qquad (5)$$

with

$$F(\phi_0) = A(2\phi_{0,\tau\tau} - 3\phi_{0,\xi\xi yy} - 3\partial_\xi((\phi_{0,y})^2)$$
$$- 2\partial_\tau((\phi_{0,\xi})^2) - 3\phi_{0,\xi}\phi_{0,yy} - 2\phi_{0,\tau}\phi_{0,\xi\xi}).$$

Thus, we see that by knowing w_1, we have from (3) that the surface height η is given by

$$\eta = A(w + \epsilon(w_1 + A\phi_{0,\tau} - A(w)^2) + \mathcal{O}(\epsilon^2)).$$

Therefore, we clearly see that the leading-order behavior of the surface height is determined by KP and that knowing w_1 gives the next-order correction from the Benney–Luke equation.

3. Web solutions of KP and far-field asymptotics of Benney–Luke

We will use more detailed information about the web solutions of KP that will act as the leading-order approximation to solutions of the BL equation. These web solutions are, ignoring the slow time $\nu = \epsilon\tau$ for the moment, of the form [6]

$$w(\xi, y, \tau) = 2\partial_\xi^2 \log(v(\xi, y, \tau)),$$

where

$$v(\xi, y, \tau) = \begin{vmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(N-1)} \\ f_2 & f_2^{(1)} & \cdots & f_2^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ f_N & f_N^{(1)} & \cdots & f_N^{(N-1)} \end{vmatrix}$$

with $f_i^{(k)} = \partial_\xi^k f_i$. The functions f_i have the particular form

$$f_i = \sum_{j=1}^{M} b_{ij}\, \mathrm{e}^{\theta_j},$$

with b_{ij} constants $N < M$, and

$$\theta_j(\xi, y, \tau) = -k_j\xi + k_j^2 y - k_j^3\tau,$$

where the values k_j are distinct for all j. In [6], solutions of this form are fully categorized in the case that the $N \times M$ matrix B of coefficients b_{ij} satisfies the following two properties.

(i) The determinant of every $N \times N$ minor of B is non-negative.
(ii) In its row-reduced-echelon form (RREF), B has N pivot columns, and B is irreducible in the sense that the RREF of B has a non-zero entry in each non-pivot column and each row has a non-zero entry besides the pivot.

We also use the convention in [6] that $k_j > k_i$ if $j > i$.

With the assumptions on the values k_j and the coefficient matrix B, it was proved in [6] that as $y \to \infty$, there are N lines with slopes $c_{lj} = k_l + k_j$, $j > l$, such that

$$w \sim \frac{1}{2}(k_j - k_l)^2 \mathrm{sech}^2\left(\frac{\theta_j - \theta_l + \theta_{lj}}{2}\right).$$

Here, the phase θ_{ij} is a constant depending upon the coefficients b_{il}, b_{ij}, k_l and k_j. Likewise, as $y \to -\infty$ there are $M - N$ analogous lines. This result also shows that as $|y|$ becomes large, in given regions of the real plane, pairs of phase functions θ_j balance against one another in the sense that only the terms e^{θ_l} and e^{θ_j} are relevant along a given line.

Next we consider these general web solutions as the leading-order solutions to the Benney–Luke equation. To do this, we modify the definition of the phase functions θ_j such that

$$\theta_j(\xi, y, \tau, \nu) = -k_j\xi + k_j^2 y - k_j^3\tau + \theta_j^0(\nu),$$

where $\theta_j^0(0) = 0$. Note that this is equivalent to defining $\tilde{b}_{ij}(\nu) = b_{ij}\, e^{\theta_j(\nu)}$, and so some care must be taken in ensuring that the new coefficient matrix $\tilde{B}(\nu)$ still satisfies the assumptions stated above. This is straightforward to show though since we have only multiplied B by a positive $M \times M$ diagonal matrix. Therefore, if B satisfies the required conditions, then so must $\tilde{B}$.

Assuming the phases θ_l and θ_j balance against one another as mentioned above, then as $|y| \to \infty$, we introduce the local coordinates

$$X_{lj} = \theta_j - \theta_l + \theta_{lj}^{(0)}$$
$$Y_{lj} = y + (k_j + k_l)\xi,$$

where $\theta_{lj}^{(0)} = \theta_j^0 - \theta_l^0$.

Treating derivatives with respect to Y_{lj} as asymptotically negligible for large $|y|$, to the leading order one has

$$\partial_\xi = -(k_j - k_l)\partial_{X_{lj}}$$
$$\partial_y = \left(k_j^2 - k_l^2\right)\partial_{X_{lj}}$$
$$\partial_\tau = -\left(k_j^3 - k_l^3\right)\partial_{X_{lj}} + \partial_\tau,$$

and therefore, dropping the subscripts on X_{lj} and calling $s \equiv w_1$ for clarity, and then dividing through by $k_{lj} = (k_j - k_l)$, equation (5) becomes, after integrating once in X,

$$4\partial_\tau s - k_{lj}^3\partial_X s + k_{lj}^3\partial_X^3 s + 6k_{lj}\partial_X(ws) = -F(\phi^0) - 4w_\nu. \qquad (6)$$

We now define $\theta_{lj}^0(\nu) = \theta_j^0 - \theta_l^0$ and then note that the inhomogeneity in the above equation is independent of τ and that as $|y|$ becomes large we have that $F \sim \partial_X G(X)$ where

$$G(X) = A\left(-2\frac{\left(k_j^3 - k_l^3\right)^2}{k_j - k_l}w + 3k_{lj}^3(k_j + k_l)^2\partial_X^2 w + \left(\frac{9}{2}k_{lj}(k_j + k_l)^2 + 3\left(k_j^3 - k_l^3\right)\right)(w)^2\right).$$

Calling $q(X) = 2\operatorname{sech}^2(X)$ and using the transformation $X \to \frac{X}{2}$ and $\tau \to \frac{k_{lj}^3}{32}\tau$, equation (6) becomes

$$s_\tau - 4\partial_X s + \partial_X^3 s + 6\partial_X(qs) = -\partial_X\left(\frac{4}{k_{lj}^3}G + \frac{4\partial_\nu\theta_{lj}^{(0)}}{k_{lj}}q\right).$$

In order to solve (6), we write $s(X,\tau) = h(X) + u(X,\tau)$ with $u(X,0) = -h(X)$, where h and u respectively solve

$$-4h + \partial_X^2 h + 6qh = -\left(\frac{4}{k_{lj}^3}G + \frac{4\partial_\nu\theta_{lj}^{(0)}}{k_{lj}}q\right) \qquad (7)$$

$$u_\tau - 4\partial_X u + \partial_X^3 u + 6\partial_X(qu) = 0. \qquad (8)$$

We now solve equations (7) and (8).

4. Solving the inhomogeneous problem

In this section, we wish to solve the inhomogeneous second-order problem (7). Given that q_X represents the kernel of the differential operator in (7), we might try to find $\partial_\nu \theta_{lj}^{(0)}$ by way of the Fredholm alternative which would require

$$\int_{-\infty}^{\infty} \left(\frac{4}{k_{lj}^3} G + \frac{4\partial_\nu \theta_{lj}^{(0)}}{k_{lj}} q \right) \frac{\mathrm{d}q}{\mathrm{d}X} \, \mathrm{d}X = 0.$$

However, the above integral is identically zero for any value of $\partial_\nu \theta_{lj}^{(0)}$. Thus, we expect that there is a bounded solution to (7). But at this point we do not have enough information to determine the evolution of the slow phases introduced in the previous section. This issue is addressed in the next section where a secularity condition emerges when trying to solve (8).

Next we solve the general inhomogeneous problem (7). To do this, following [2], we introduce the transformation $\zeta = \tanh(X)$, so that we get the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} \left((1-\zeta^2) \frac{\mathrm{d}h}{\mathrm{d}\zeta} \right) + \left(12 - \frac{4}{1-\zeta^2} \right) h = -M_{lj}(\zeta), \tag{9}$$

with

$$M_{lj}(\zeta) = M_{lj}^0 + 8 \frac{\partial_\nu \theta_{lj}^{(0)}}{k_{lj}} + M_{lj}^1 \zeta^2,$$

where

$$M_{lj}^0 = \frac{2A}{k_{lj}} \left(-2 \frac{\left(k_j^3 - k_l^3\right)^2}{k_j - k_l} - \frac{3}{2} k_{lj}^3 (k_j + k_l) + \frac{k_{lj}^2}{2} \left(\frac{9}{2} k_{lj} (k_j + k_l)^2 + 3\left(k_j^3 - k_l^3\right) \right) \right)$$

and

$$M_{lj}^1 = \frac{2A}{k_{lj}} \left(\frac{9}{2} k_{lj}^3 (k_j + k_l) - \frac{k_{lj}^2}{2} \left(\frac{9}{2} k_{lj} (k_j + k_l)^2 + 3\left(k_j^3 - k_l^3\right) \right) \right).$$

The left-hand side of (9) is a Legendre equation with homogeneous solution $\zeta(1-\zeta^2)$. Thus, we are interested in finding solutions to the inhomogeneous Legendre equation with right-hand sides 1 and ζ^2. For the right-hand side equal to unity, using reduction of order, we get the particular solution $h_p(\zeta) = h_{p,0}(\zeta) + h_{p,1}(\zeta)$, with

$$h_{p,0}(\zeta) = \frac{1}{48} \zeta (1-\zeta^2)(33\zeta - 40\zeta^3 + 15\zeta^5 - 15(-1+\zeta^2)^3 \tanh^{-1}(\zeta))$$

$$h_{p,1}(\zeta) = \frac{1}{96} (1-\zeta^2)^2 \left(16 - 50\zeta^2 + 30\zeta^4 + 15\zeta(-1+\zeta^2)^2 \log\left(\frac{1-\zeta}{1+\zeta} \right) \right).$$

For the Legendre equation with inhomogeneity ζ^2, we correspondingly get $h_p^q = h_{p,0}^q(\zeta) + h_{p,1}^q(\zeta)$, with $p(\zeta) = 1 - \zeta^2$:

$$h_{p,0}^q(\zeta) = \frac{\zeta p(\zeta)}{192} (-15\zeta + 59\zeta^3 - 105\zeta^5 + 45\zeta^7 + 15 p^3(\zeta)(1+3\zeta^2) \tanh^{-1}(\zeta)),$$

and

$$h_{p,1}^q(\zeta) = \frac{p^2(\zeta)}{384} (1+3\zeta^2) \left(16 - 50\zeta^2 + 30\zeta^4 - 15\zeta p^2(\zeta) \log\left(\frac{1-\zeta}{1+\zeta} \right) \right).$$

Therefore, we have found the solution h as

$$h(X) = C q_X - \left(M_{lj}^0 + 8 \frac{\partial_\nu \theta_{lj}^{(0)}}{k_{lj}} \right) h_p(\tanh(X)) - M_{lj}^1 h_p^q(\tanh(X)), \tag{10}$$

where C is an arbitrary constant.

5. Solving linear KdV

Having solved the steady state, inhomogeneous problem, we now solve (8). To do this, we first note that (8) is the linearization of

$$q_\tau - 4q_X + q_{XXX} + 6qq_X = 0. \tag{11}$$

This version of KdV has the corresponding Lax pair

$$\begin{aligned}\psi_{XX} + q(X)\psi &= \lambda\psi,\\ \psi_\tau &= (\gamma + q_X)\psi - (4(\lambda - 1) + 2q)\psi_X,\end{aligned}$$

where γ is an arbitrary parameter. We see that by separation of variables, $\psi(X,\tau) = f(X)T(\tau)$, and we can reduce the Schrödinger equation in the Lax pair for KdV to an ordinary differential equation in f. We are only interested in finding the scattering data associated with the Lax pair, and so we can set $\lambda = -k^2$, and using machinery from the inverse scattering solution technique for the 1-soliton of KdV, we find the scattering solution

$$f_+(X;k) = \frac{1}{1-\mathrm{i}k}(-\mathrm{i}k + \tanh(X))\,\mathrm{e}^{\mathrm{i}kX}.$$

Given that the 1-soliton solution to KdV corresponds to a reflectionless potential for the Schrödinger equation, the back scattering solution $f_-(X;k) \approx \mathrm{e}^{-\mathrm{i}kX}$ as $X \to -\infty$ is given by

$$f_-(X;k) = a(k)f_+(X;-k).$$

This shows that $a(k)$, or the reciprocal of the transmission coefficient $T(k)$, is

$$a(k) = -\frac{1+\mathrm{i}k}{1-\mathrm{i}k}.$$

We now define

$$\psi_+(X,t;k) = f_+(X;k)T_+(\tau;k)$$

and

$$\psi_-(X,\tau;k) = f_-(X;k)T_-(\tau;k).$$

Substituting these solutions into the time-dependent part of the Lax pair and then taking the appropriate limits give

$$\begin{aligned}T_+(\tau;k) &= \mathrm{e}^{(\gamma_1+\mathrm{i}4(k^2+1)k)\tau},\\ T_-(\tau;k) &= \mathrm{e}^{(\gamma_2-\mathrm{i}4(k^2+1)k)\tau},\end{aligned}$$

where γ_1 and γ_2 are values we can choose arbitrarily. In order that we may build solutions to linear KdV out of products of ψ_+ and ψ_-, we need a modification of a lemma found in [15].

Lemma 1. *If ψ_1 and ψ_2 are two solutions to the Schrödinger equation and*

$$R(\psi_i) = \partial_\tau\psi_i + \partial_{XXX}\psi_i - 3(-q(X) + k^2)\partial_X\psi_i - 4\partial_X\psi_i = 0$$

for $i = 1, 2$, then $\partial_X(\psi_1\psi_2)$ solves the linearized KdV equation (8).

Using then that we need $R(\psi_\pm) = 0$, one can show that $\gamma_1 = \gamma_2 = 0$. Then, defining

$$g(X,\tau) = \frac{1}{\mathrm{i}}\partial_k\left(\psi_-(X,\tau;k) - \psi_+(X,\tau;k)\right)|_{k=i}\ ,$$

it follows that $R(g) = 0$. From here, define the functions

$$\begin{aligned}W(X) &= f_+^2(X;\mathrm{i}),\\ H(X,\tau) &= \omega\psi_+(X,\tau;\mathrm{i})g(X,\tau),\end{aligned}$$

where ω is chosen such that $\int_{-\infty}^{\infty} H(X,0)\partial_X W(X) dX = 1$. Finally, we have from [15] that the general solution to the linear KdV problem with the initial condition $u(X,0) = -h(X)$ is

$$u(X,\tau) = \int_{-\infty}^{\infty} \frac{1}{4\pi \mathrm{i} k a^2(k)} \left[\partial_X \psi_+^2(X,k,\tau)\hat{h}_-(k) - \partial_X \psi_-^2(X,k,\tau)\hat{h}_+(k)\right] \mathrm{d}k$$
$$- \int_{-\infty}^{\infty} [H(y,0)\partial_X W(X) - W(y)\partial_X H(X,\tau)]\, h(y)\, \mathrm{d}y,$$

where

$$\hat{h}_\pm(k) = -\int_{-\infty}^{\infty} f_\pm^2(y;k) h(y)\, \mathrm{d}y.$$

The term involving $a(k)$ can be simplified. Using $f_-(X,k) = a(k) f_+(X,-k)$ and observing that $f_+(X;-k) = f_+^*(X;k)$, where the asterisk denotes complex conjugation, we can rewrite the solution $u(X,\tau)$ as

$$u(X,\tau) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi k} \mathrm{Im}[\mathrm{e}^{8\mathrm{i}k(k^2+1)\tau} \partial_X f_+^2(X;k) f_+^2(y;-k)] h(y)\, \mathrm{d}y\, \mathrm{d}k$$
$$- \int_{-\infty}^{\infty} [H(y,0)\partial_X W(X) - W(y)\partial_X H(X,\tau)]\, h(y)\, \mathrm{d}y. \tag{12}$$

Next we establish the long-time behavior of the solution $u(X,\tau)$. To do this, we first define the function

$$u_{\mathrm{osc}}(X,\tau) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi k} \mathrm{Im}[\mathrm{e}^{8\mathrm{i}k(k^2+1)\tau} \partial_X f_+^2(X;k) f_+^2(y;-k)] h(y)\, \mathrm{d}y\, \mathrm{d}k.$$

Letting

$$K(X,\tau,k,y) = \frac{1}{k} \mathrm{Im}[\mathrm{e}^{8\mathrm{i}k(k^2+1)\tau} \partial_X f_+^2(X;k) f_+^2(y;-k)]$$

and

$$\tilde{p}_1(X,\tau,k,y) = \frac{2\sin(8k(k^2+1)\tau + 2kX - 2ky)}{(k^2+1)^2},$$
$$\tilde{p}_1(X,\tau,k,y) = \frac{2\cos(8k(k^2+1)\tau + 2kX - 2ky)}{(k^2+1)^2}$$

one has that

$$K(X,\tau,k,y) =$$
$$\frac{\tilde{p}_1}{k}\{\tanh(X)(2k^2 + \mathrm{sech}^2(X))(-k^2 + \tanh^2(y)) - 2k^2(1 - k^2 - 2\mathrm{sech}^2(X))\tanh(y)\}$$
$$+\tilde{p}_2\{(1 - k^2 - 2\mathrm{sech}^2(X))(-k^2 + \tanh^2(y)) + 2\tanh(X)(2k^2 + \mathrm{sech}^2(X))\tanh(y)\}.$$

Then we consider u_{osc} when $\tau \to \infty$ for fixed X. Given that $h(y)$ is a rapidly decaying function, and thus the Fourier transforms of $h(y)$, $\tanh(y)h(y)$, and $\tanh^2(y)h(y)$ should all be as well, any terms in k that do not include $\frac{1}{k}$ after cancellations should contribute nothing in the long-time limit by the Riemann–Lebesgue lemma. So for fixed X and large τ we get that

$$u_{\mathrm{osc}}(X,\tau) \sim -\frac{2}{\pi} \tanh(X)\mathrm{sech}^2(X) \int_0^{\infty} \frac{\sin(8k(k^2+1)\tau + 2kX)}{k(k^2+1)^2} \tilde{h}(k)\, \mathrm{d}k,$$

where

$$\tilde{h}(k) = \int_{-\infty}^{\infty} \cos(2ky) \tanh^2(y) h(y)\, \mathrm{d}y,$$

since $h(y)$ is always even, and we have also used the fact that everything under the integrand is an even function in k. At this point we assume $\tilde{h}(0) \neq 0$. Otherwise, we have that $\frac{\tilde{h}(k)}{k} \in L_1(R)$ and therefore we may apply the Riemann–Lebesgue lemma immediately to conclude that $u_{\text{osc}} \to 0$ as $\tau \to \infty$ [8].

Proceeding with our assumption then, using basic trigonometric identities we have that

$$\frac{\sin(8k(k^2+1)\tau + 2kX)}{k(k^2+1)^2} = \frac{\sin(8k(k^2+1)\tau)}{k(k^2+1)^2}\cos(2kX) + 2X\cos(8k(k^2+1)\tau)\frac{\text{sinc}(2kX)}{(k^2+1)^2},$$

and so we can, again by Riemann–Lebesgue, ignore the $\cos(8k(k^2+1)\tau)$ term for fixed X. Therefore, we get that

$$u_{\text{osc}}(X,\tau) \sim -\frac{2}{\pi}\tanh(X)\text{sech}^2(X)\int_0^\infty \frac{\sin(8k(k^2+1)\tau)}{k(k^2+1)} J(k;X)\,\text{d}k,$$

with

$$J(k;X) = \frac{\cos(2kX)\tilde{h}(k)}{(k^2+1)}.$$

By using $\int_0^\infty = \int_0^a + \int_a^\infty$, we can ignore the second integral for large time since $\frac{J(k;X)}{k} \in L_1([a,\infty))$. For the integral over $[0,a]$, introduce the transformation $\sigma = 8k(k^2+1)$. Therefore, the leading-order behavior is given by

$$u_{\text{osc}}(X,\tau) \sim -\frac{2}{\pi}\tanh(X)\text{sech}^2(X)\int_0^{8a(a^2+1)} \frac{\sin(\sigma\tau)}{\sigma}\tilde{J}(\sigma;X)\,\text{d}\sigma,$$

where

$$\tilde{J}(\sigma;X) = \frac{\cos(2k(\sigma)X)\tilde{h}(k(\sigma))}{(k^2(\sigma)+1)(3k^2(\sigma)+1)}.$$

Given that J is smooth in k, and k is smooth in σ, we can Taylor expand $\tilde{J}$ around $\sigma = 0$, which corresponds to $k = 0$, and thus $\tilde{J}(\sigma;X) = \tilde{h}(0) + \mathcal{O}(\sigma)$. The higher order polynomial terms will eliminate the singularity at the origin which then allows us to apply Riemann–Lebesgue again. Therefore, using the transformation $\kappa = \sigma t$, we can now show that the true leading-order behavior for $u_{\text{osc}}(X,\tau)$ as $\tau \to \infty$ for fixed X is

$$u_{\text{osc}}(X,\tau) \sim -\frac{2}{\pi}\tilde{h}(0)\tanh(X)\text{sech}^2(X)\int_0^{8a(a^2+1)\tau} \frac{\sin(\kappa)}{\kappa}\,\text{d}\kappa.$$

Using the fact that

$$\lim_{s\to\infty}\int_0^s \frac{\sin(\kappa)}{\kappa}\,\text{d}\kappa = \frac{\pi}{2},$$

we see that

$$\lim_{\tau\to\infty} u_{\text{osc}}(x,\tau) = -\tilde{h}(0)\tanh(X)\text{sech}^2(X).$$

We see then that the contribution of the double integral in equation (12) is bounded for all time.

However, from the readily established fact that

$$\partial_X H(X,\tau) = \omega\left((X - 8\tau)\tanh(X) - \frac{3}{2}\right)\text{sech}^2(X),$$

we see that there will be unbounded growth in solution (12) as $\tau \to \infty$. Therefore, we add the secularity condition

$$\int W(y)h(y)\,\mathrm{d}y = 0. \tag{13}$$

Note, as our work up to this point shows, that this condition will always be the secularity condition for reasonably well-behaved solutions of the forced problem (7). Given that it emerges from the presence of discrete spectra in the forward scattering problem, it is reminiscent of the perturbation theory of solitons [10].

In order to find $\partial_\nu\theta_{ij}^{(0)}$, we use the solution to the forced stationary problem from (10)

$$h(X) = Cq_X - \left(M_{lj}^0 + 8\frac{\partial_\nu\theta_{lj}^{(0)}}{k_{lj}}\right)h_p(\tanh(X)) - M_{lj}^1 h_p^q(\tanh(X)),$$

where C is arbitrary. Given that $W(y) = \frac{1}{4}\mathrm{sech}^2(y)$, one immediately has that $\int W(y)q_y(y)\,\mathrm{d}y = 0$. Thus, we can use (13) to determine the phase speed $\partial_\nu\theta_{lj}^{(0)}$ as

$$\partial_\nu\theta_{lj}^{(0)} = -\frac{k_{lj}}{8}\left(M_{lj}^0 + M_{lj}^1\frac{\langle W, h_p^q(\tanh(X))\rangle}{\langle W, h_p(\tanh(X))\rangle}\right), \tag{14}$$

where $\langle\, ,\rangle$ denotes the $L_2(R)$ inner product.

Having now determined the slow phase speeds $\partial_\nu\theta_{ij}^{(0)}(\nu)$ by way of the secularity constraint above, and given the fact that $h(y)$ is even in our case, and $H(y,0)$ is odd, one has that $u(X,\tau)$ in (12) is equal to u_{osc}. Thus, we now see that

$$\lim_{\tau\to\infty} u(X,\tau) = -\tilde{h}(0)\tanh(X)\mathrm{sech}^2(X).$$

Further, since u_{osc} is an oscillatory integral, then on intermediate time scales of τ, $u(X,\tau)$ leads to decaying oscillatory behavior behind the main solitary wave. We can also now write down the far-field perturbation as

$$s(X,\tau) = \tilde{h}(X) + \tilde{u}(X,\tau),$$

where

$$\tilde{h}(X) = -\left(M_{lj}^0 + 8\frac{\partial_\nu\theta_{lj}^{(0)}}{k_{lj}}\right)h_p(\tanh(X)) - M_{lj}^1 h_p^q(\tanh(X))$$

and $\tilde{u}(X,0) = -\tilde{h}(X)$. Note that we can choose $C = 0$ for Cq_X because the said term is a stationary solution of linear KdV. Thus, requiring $u(X,0) = -h(X)$ gives $u(X,\tau) = -Cq_X + \tilde{u}(X,\tau)$. Therefore, one must have $s = h(X) + u(X,\tau) = \tilde{h} + \tilde{u}(X,\tau)$, since the term Cq_X cancels out.

6. Numerical investigations

6.1. One-dimensional perturbations

First we look at Benney–Luke reduced to one dimension, which, in terms of the scalings introduced at the beginning of the paper, is

$$4\partial_{\xi\tau}\phi - \partial_\xi^4\phi - 6\phi_\xi\phi_{\xi\xi} + 2\epsilon A\left(\phi_{\tau\tau} - \partial_\tau\left(\phi_\xi^2\right) - \phi_\tau\phi_{\xi\xi}\right) = 0.$$

Let $X = \xi + c(\epsilon)\tau$ with $c(\epsilon) = 1 + \epsilon c_1 + \mathcal{O}(\epsilon^2)$, and let $\phi(X,\tau) = \phi_0(X) + \epsilon\phi_1(X,\tau) + \mathcal{O}(\epsilon^2)$. Setting $w = \phi_{0,X} = 2\mathrm{sech}^2(X)$ and $s = \phi_{1,X}$, one gets the equation for the perturbation as

$$4s_\tau + 4s_X - s_{XXX} - 6(ws)_X = -\partial_X\left(2A\left(w - \tfrac{3}{2}w^2\right) - 4c_1 w\right)s(X,0) = 0. \tag{15}$$

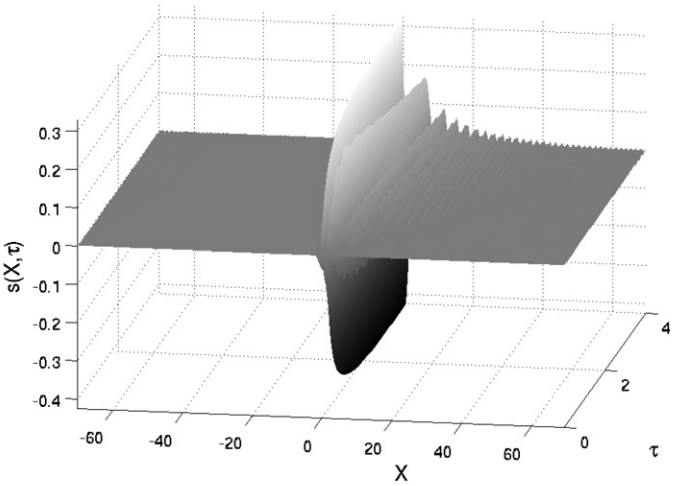


Figure 2. 1D perturbation: $0 \leqslant \tau \leqslant 4$.

Repeating the analysis from above would give an exact form for $s(X, \tau)$ and also provide a secularity condition for finding c_1. Choosing $M = N = 1$ and $k_1 = -1$, $k_2 = 0$, and $k_3 = 1$, which corresponds to perturbing around $w(X) = 2\mathrm{sech}^2(X)$, one can show that

$$c_1 = 2A\left(-1 + \frac{3}{2}\frac{\langle h_p^q, W\rangle}{\langle h_p, W\rangle}\right). \tag{16}$$

Using a pseudo-spectral method in space with 257 modes and exponential time-differencing fourth-order Runge–Kutta (ETDRK4) [7, 13] with a time step of 10^{-2} for integration in time, we generated figure 2, which shows the evolution of the perturbation $s(X, \tau)$ for $0 \leqslant \tau \leqslant 4$.

This one-dimensional argument, requiring no distinction between near or far field, shows that Benney–Luke introduces oscillatory tails onto the leading-order approximate solution of Benney–Luke.

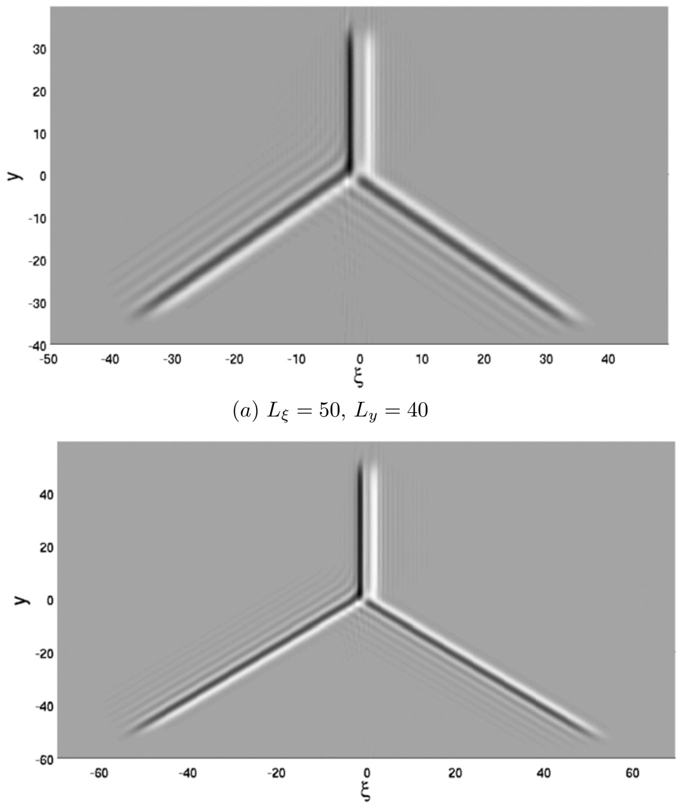
6.2. Two-dimensional perturbations

Now that we have an understanding of the impact of the Benney–Luke equation in the large $|y|$ limit, and we now wish to see what happens in the interaction region. Again, solving (5) analytically is difficult, and thus we solve it numerically. For the solution to KP, w, we choose $N = 1$, $M = 3$, with $b_{1j} = 1$, $k_1 = -1$, $k_2 = 0$, and $k_3 = 1$. This choice corresponds to the well-known Y-junction solution, which, as shown in [14], represents the resonant interaction of two intersecting solitary wave solutions to KP that intersect at any angle less than a certain critical one. One can show that

$$w^0 = 2\mathrm{sech}^2\left(\xi + \tau - \frac{\theta_{13}^0}{2}\right) + \mathcal{O}(\mathrm{e}^{-y}), \quad y \to \infty,$$

$$w^0 = \frac{1}{2}\mathrm{sech}^2\left(\frac{1}{2}(\chi + \tau + \theta_{23}^0\right) + \mathcal{O}(\mathrm{e}^{y}), \quad y + \xi = \chi, \quad y \to -\infty,$$

$$w^0 = \frac{1}{2}\mathrm{sech}^2\left(\frac{1}{2}(\mu - \tau + \theta_{12}^0)\right) + \mathcal{O}(\mathrm{e}^{y}), \quad y - \xi = \mu, \quad y \to -\infty,$$



(a) $L_\xi = 50$, $L_y = 40$

(b) $L_\xi = 70$, $L_y = 60$

Figure 3. $\tau = 1$.

and we clearly see the relevant directions. It is also straightforward to compute the following quantities:

$$M_{13}^0 = 8A, \quad M_{13}^1 = -12A$$
$$M_{23}^0 = \tfrac{1}{2}A, \quad M_{23}^1 = \tfrac{3}{2}A$$
$$M_{12}^0 = \tfrac{13}{2}A, \quad M_{12}^1 = -\tfrac{33}{2}A.$$

Then, choosing $A = 1.209$, corresponding to $\tilde{\sigma} = 0.01$ or surface tension close to zero, we get using (14) and numerical quadrature that $\partial_\nu\theta_{13}^{(0)} = 0.0579$, $\partial_\nu\theta_{12}^{(0)} = 0.0207$, and $\partial_\nu\theta_{23}^{(0)} = 0.0028$. Finally, we choose the magnitude of the perturbation ϵ to be 10^{-1}.

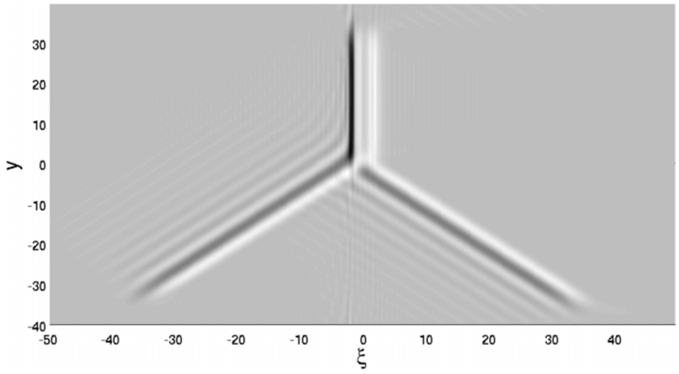
In order to cope with the fact that we are on an infinite domain, we first introduce a smooth, compactly supported function $V(y)$, with support in $y \in [-L_y, L_y]$, where on its support, $V(y)$ is a super exponential function [12] given by

$$V(y) = \mathrm{e}^{\ln(\tilde{\epsilon})|\frac{y}{L_y}|^{20}},$$

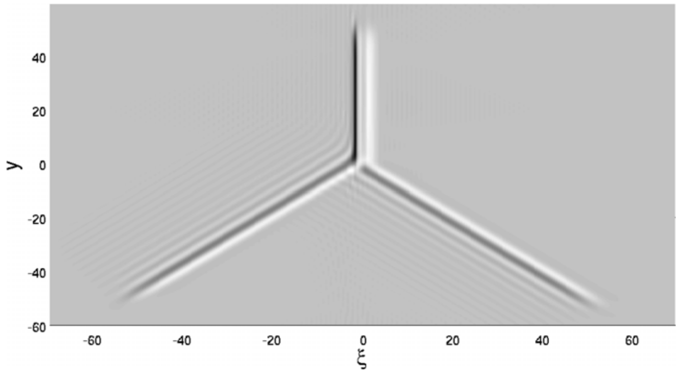
where $\tilde{\epsilon}$ is machine precision, which for our purposes is on the order of 10^{-16}.

Writing the linearized equation (5) as

$$s_\tau + \mathcal{L}s = \tilde{F} \tag{17}$$

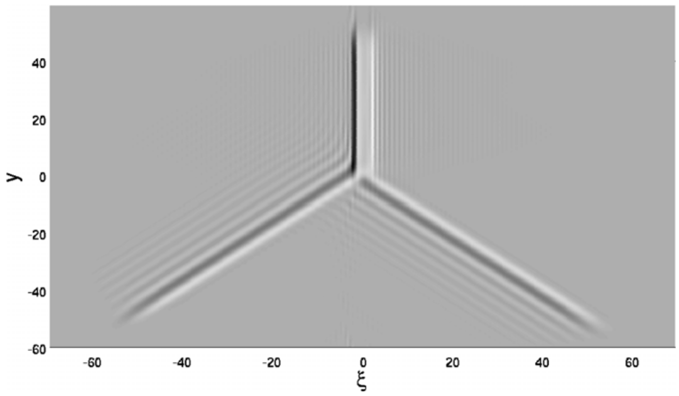


$(a)\ L_\xi = 50,\ L_y = 40$



$(b)\ L_\xi = 70,\ L_y = 60$

Figure 4. $\tau = 2$.

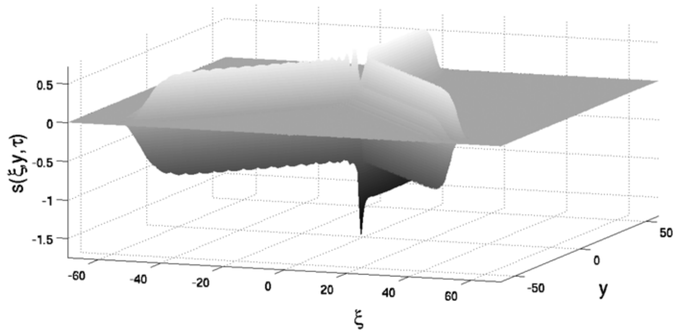


$(a)\ L_\xi = 70,\ L_y = 60$

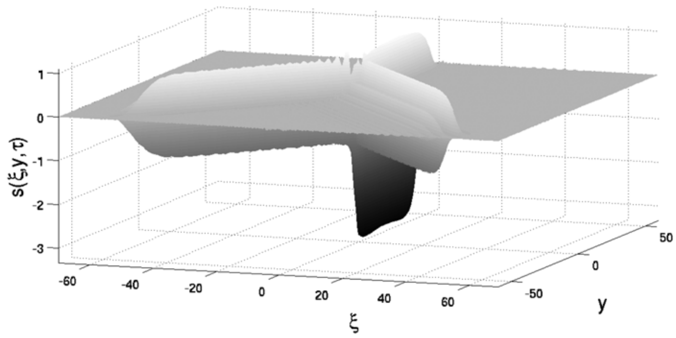
Figure 5. $\tau = 3$.

if we write $s = s_n + s_f$, where $s_n = V(y)s$ and $s_f = (1 - V(y))s$, we see outside the support of V that

$$\partial_\tau s_f + \mathcal{L}s_f = (1 - V)\tilde{F},$$



$(a)\ \tau = 1,\ L_\xi = 70,\ L_y = 60$



$(b)\ \tau = 2,\ L_\xi = 70,\ L_y = 60$

Figure 6. $\tau = 1, 2$.

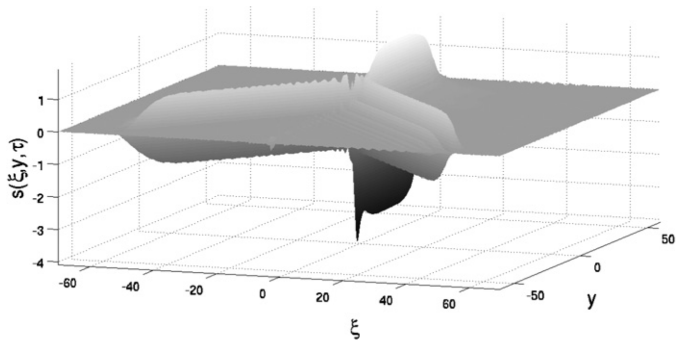


Figure 7. $\tau = 3$, $L_\xi = 70$, $L_y = 60$.

and therefore for very large L_y, we can find a reasonable approximation to s_f outside of the support of V by way of the asymptotic analysis from the previous sections. We denote this approximation as $s_{f,\text{asym}}$ so that $s \approx s_n + (1 - V)s_{f,\text{asym}}$, where again we have chosen $Vs_f = 0$. Therefore, substituting the approximation to v back into (17) gives the problem

$$\partial_\tau s_n + \mathcal{L}s_n = V\tilde{F}.$$

Our approach is similar to the windowing techniques used in [17] and [12], but now we do not attempt to smooth out the approximation at the boundaries. This will have the effect of

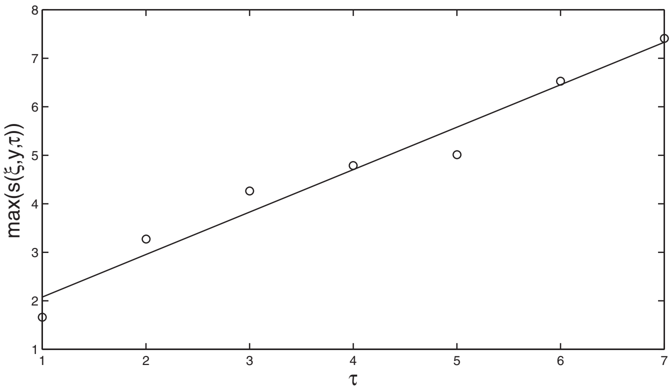
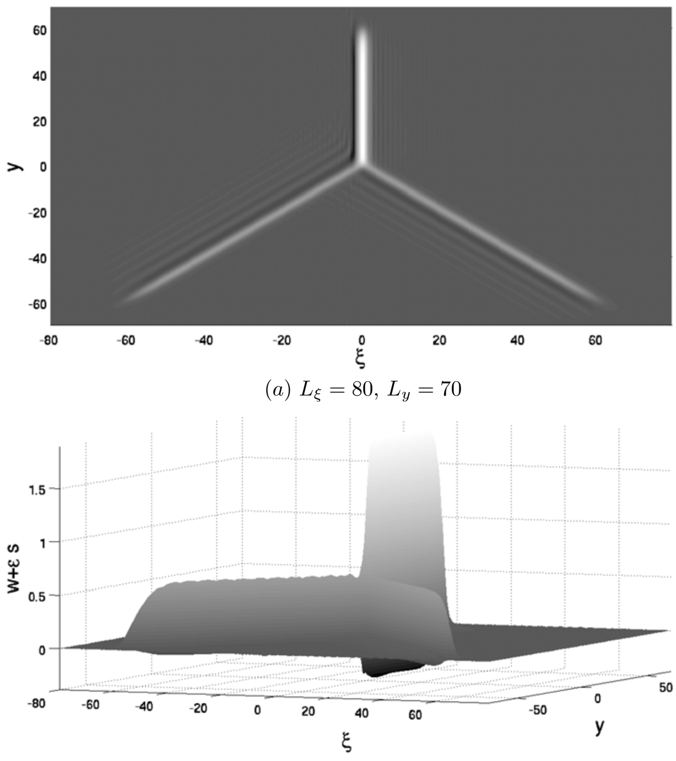


Figure 8. Amplitude comparison: $1 \leqslant \tau \leqslant 7$.

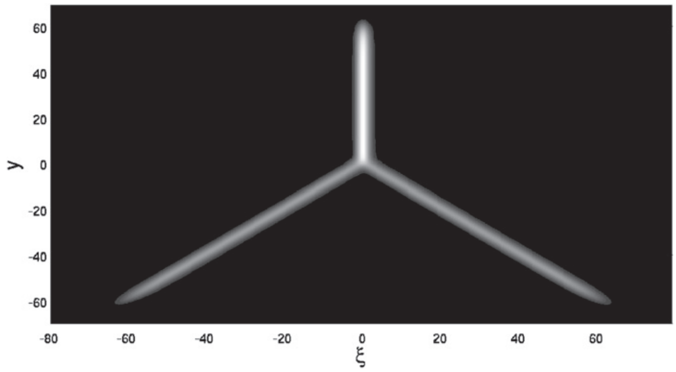


(a) $L_\xi = 80,\ L_y = 70$

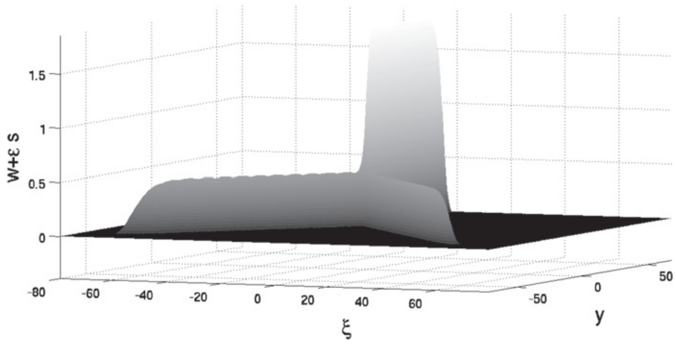
(b) $L_\xi = 80,\ L_y = 70$

Figure 9. $\tau = 4, \epsilon = 0.1$.

introducing a jump across the boundary with our choice for $s_{f,\text{asym}}$. However, $s_{f,\text{asym}}$ never becomes very large, and any impact it might have on the interaction region should be nominal over reasonable lengths of time, especially if L_y is kept very large. Therefore, the results appearing in the interior, far from the boundary of the domain, are believable.



(a) $L_\xi = 80,\ L_y = 70$



(b) $L_\xi = 80,\ L_y = 70$

Figure 10. $\tau = 4, \epsilon = 0.01$.

Given that the forcing $\tilde{F}$ will, along the ξ-axis, be zero far away from the interaction region, for short enough times, we can assume periodic boundary conditions in ξ, and likewise we have periodic boundary conditions $s_n(x, -L_y) = s_n(x, L_y) = 0$. Note, since the forcing should have dominant contributions along the lines $y = \pm\xi$, we introduce the length L_ξ with $L_\xi \gg L_y \gg 1$, so that $s_n(-L_\xi, y, t) = s_n(L_\xi, y, t)$ and $s_n(\xi, -L_y, t) = s_n(\xi, L_y, t) = 0$.

Also some care must be taken in introducing the operator ∂_ξ^{-1} in turning (5) into an evolution equation. Thankfully, the only questionable term is $\partial_\xi^{-1}\partial_y^2$, which means that the inverse derivative will be well defined if the average of v in ξ is independent of y. Looking at (5), if one assumes that s and its derivatives will decay to zero as $|\xi| \to \infty$, which means v will be close to zero along the boundary, assuming the ξ average of v is independent of y is reasonable.

Using then a pseudo-spectral method in space with 257 modes along each axis and ETDRK4 for time integration with a time step of 10^{-2}, we get the results in the following figures. In each figure, we have also transformed into the moving coordinate $\xi \to \xi + \tau$. Figures 3 and 4 then show our results at $\tau = 1$ and $\tau = 2$ on domains of two different sizes. The results appear identical in each panel. Finally, figure 5 shows our results for $\tau = 3$, which in this view only introduces more dispersive radiation. Given that our numerical scheme for the one-dimensional problem is essentially the same as in two dimensions, with the exception of windowing, it appears that the dispersive radiation is not a result of windowing and so cannot be considered spurious.

However, figures 6 and 7 show that the magnitude of the waveforms in the figures grows as time increases. Figure 8 shows a least-squares fitting of the maximum of the numerical solution from $\tau = 1$ to $\tau = 7$. As can be seen, the growth is well modeled by a linear function, and so the growth cannot be attributed to numerical or physical instabilities, which grow exponentially. Thus, the Benney–Luke equation introduces depressive perturbations to the interior of the Y-junction on time scales of order $\frac{1}{\epsilon}$. It appears that this is leading to a secularity whereby after long enough time the linearized equation will no longer remain valid. However, in figures 9 and 10 we have plotted the sum of the leading-order approximation and the perturbation at $\tau = 4$ for $\epsilon = 10^{-1}$ and $\epsilon = 10^{-2}$, respectively. As can be seen from the figures, the unaccounted for secularity does introduce a significant dip and dispersive radiation at $\epsilon = 10^{-1}$. However, this all but vanishes to the eye when ϵ is reduced by an order of magnitude. Thus, it appears that once one properly accounts for the secularity, the Y-junction will be stable within Benney–Luke.

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