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A New Class of Soliton Solutions for the (Modified) Kadomtsev – Petviashvili Equation

By WALTER RENGER of Columbia, Missouri

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Abstract. We construct solutions of the Kadomtsev – Petviashvili equation and its counterpart, the modified Kadomtsev – Petviashvili equation, with an infinite number of solitons by a careful examination of the limits of N – soliton solutions as $N \to \infty$. We give sufficient conditions to ensure that these limits exist and satisfy the (modified) Kadomtsev – Petviashvili equation.

1. Introduction

In this note we study solutions of the Kadomtsev – Petviashvili (KP) equation [16] and the modified Kadomtsev – Petviashvili (mKP) equation [6], [18], [20] with an infinite number of solitons. We start with the well - known N – soliton solutions (cf. [1] – [6], [8] – [10], [14], [15], [18] – [23], [26], [27], [29], [31] – [33], [35], [36]) and construct the limits of these solutions for $N \to \infty$. After briefly summarizing the N – soliton solutions for the (m)KP – equation in Section 2 we will prove under appropriate conditions that the limits of these N – soliton solutions for $N \to \infty$ exist. Furthermore, we prove that they actually are solutions of the (m)KP – equation. In Section 3 this is done for the KP – equation, in Section 4 for the mKP – equation.

We mention that similar results are already known for other soliton equations, namely the KdV – equation [11], [12], the Toda – Lattice [13], [37]; see also [7], [17], [24], [25], [28], [30], [38]. To the best of our knowledge, the current investigation is the first of this kind for 2+1 dimensional completely integrable evolution equations.

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2. N – soliton solutions

The KP equation

$$V_t - 6VV_x + V_{xxx} + 3\int_{\infty}^{x} dx'\, V_{yy} \;=\; 0 \tag{2.1}$$

has N – soliton solutions of the form (cf., e. g., [9], [10])

$$V \;=\; V_N + \lambda\,, \tag{2.2}$$

where

$$V_N(t,x,y) \;=\; -2\partial_x^2 \ln \det[1 + \Lambda_N(t,x,y)]\,, \tag{2.3}$$

$$\Lambda_N(t,x,y) \;=\; \left\{ c_j(t,y)c_l(t,y)\,\frac{\exp[-(p_j+q_l)x]}{p_j+q_l} \right\}_{j,l=1}^{N}, \tag{2.4}$$

$$c_j(t,y) \;=\; c_j \exp\left[\frac{\varepsilon}{2}\,(p_j^2 - q_j^2)y + \big(2(p_j^3+q_j^3) - 3\lambda(p_j+q_j)\big)t\right], \tag{2.5}$$

for all $(t,x,y) \in \mathbb{R}^3$, $p_j,\, q_j > 0$, $c_j \in \mathbb{R}\backslash\{0\}$, $j = 1,\dots,N$, $\lambda \in \mathbb{R}$, $\varepsilon = \pm 1$. The N – soliton solutions ϕ_N of the mKP equation

$$\phi_t - 6\phi^2\phi_x + \phi_{xxx} + 3\int_{\infty}^{x} dx'\,\phi_{yy} + 6\varepsilon\phi_x \int_{\infty}^{x} dx'\,\phi_y \;=\; 0\,, \quad \varepsilon \;=\; \pm 1\,, \tag{2.6}$$

are given by (cf. [10])

$$\phi_N(t,x,y) \;=\; -\kappa + \partial_x \ln\left\{ \frac{\det\big[1 + \hat{\Lambda}_N(t,x,y)\big]}{\det\big[1 + \Lambda_N(t,x,y)\big]} \right\}, \tag{2.7}$$

with

$$\hat{\Lambda}_N(t,x,y) \;=\; \left\{ \frac{\kappa - p_j}{\kappa + q_j}\, c_j(t,y)c_l(t,y)\,\frac{\exp[-(p_j+q_l)x]}{p_j+q_l} \right\}_{j,l=1}^{N}, \qquad \kappa^2 \;=\; \lambda\,, \quad (t,x,y) \in \mathbb{R}^3\,, \tag{2.8}$$

and the other quantities defined as in (2.5). (The Matrix $\hat{\Lambda}_N(t,x,y)$ is slightly different from the one used in [10], but the two are related by a similarity transformation, so the corresponding expressions for $\phi_N(t,x,y)$ are actually identical.)

The matrices $\Lambda_N(t,x,y)$ and $\hat{\Lambda}_N(t,x,y)$ act as operators in $\mathbb{C}^N$. However, it will be more convenient to embed $\mathbb{C}^N$ in the Hilbert space of square summable sequences, $\ell^2(\mathbb{N})$, in the canonical way. We extend $\Lambda_N(t,x,y)$ to an operator in this space by setting all other components equal to zero.

We consider both $\Lambda_N(t,x,y)$ and $\hat{\Lambda}_N(t,x,y)$ as operators on $\ell^2(\mathbb{N})$. All entries, except for the explicitly given first $N \times N$ ones, are defined to be zero. In the same manner, the operator 1 will always be understood as the identity operator on $\ell^2(\mathbb{N})$.

With this convention one obtains

$$\hat{\Lambda}_N(t,x,y) = D_\infty \Lambda_N(t,x,y), \tag{2.9}$$

$$D_\infty = \left\{ \frac{\kappa - p_j}{\kappa + q_j} \delta_{j,l} \right\}_{j,l \in \mathbb{N}}. \tag{2.10}$$

($\delta_{j,l} = 1$ for $j = l$ and 0 otherwise.)

For later reference we introduce the quantities

$$\begin{aligned} A_N(t,x,y) &= \partial_x \ln \det[1 + \Lambda_N(t,x,y)] \\ &= \mathrm{Tr}\big[[1 + \Lambda_N(t,x,y)]^{-1} \partial_x \Lambda_N(t,x,y)\big], \end{aligned} \tag{2.11}$$

$$\begin{aligned} B_N(t,x,y) &= \partial_x \ln \det[1 + \hat{\Lambda}_N(t,x,y)] \\ &= \mathrm{Tr}\big[[1 + \hat{\Lambda}_N(t,x,y)]^{-1} \partial_x \hat{\Lambda}_N(t,x,y)\big]. \end{aligned} \tag{2.12}$$

Thus we can write

$$V_N(t,x,y) = -2\partial_x A_N(t,x,y), \tag{2.13}$$

$$\phi_N(t,x,y) = -\kappa + B_N(t,x,y) - A_N(t,x,y). \tag{2.14}$$

3. Soliton limits for the KP-equation

The strategy we want to employ is the following: We show that $\lim_{N\to\infty} V_N(t,x,y)$ is well defined and finite, denote it by $V_\infty(t,x,y)$. Simultaneously, we will prove that the derivatives of this limit are given by the limit of the derivatives of $V_N(t,x,y)$ and that we can also interchange limit and integration. Then we can conclude that the new function $V = V_\infty(t,x,y) + \lambda$ is indeed a solution of the KP-equation.

To make this argument work we will have to impose some restrictions on the sequences $\{p_j\}_{j\in\mathbb{N}}$, $\{q_j\}_{j\in\mathbb{N}}$, and $\{c_j\}_{j\in\mathbb{N}}$, by which $V_\infty(t,x,y)$ is defined (see (2.3) – (2.5)). First of all we note that the N-soliton solutions described by (2.2) – (2.5) are not necessarily regular, they can have singularities. A convergence argument will be impossible unless we exclude such cases (compare [10]). In order to keep the limit bounded, we will also need boundedness of the sequences $\{p_j\}_{j\in\mathbb{N}}$, $\{q_j\}_{j\in\mathbb{N}}$. The condition $\{c_j\}_{j\in\mathbb{N}}$ has to fulfill is not so clear. A natural condition would be to assume that the trace norm of $\Lambda_N(t,x,y)$ is bounded independently of N, because then we can define the Fredholm-determinant $\det_1[1+\Lambda_N(t,x,y)]$ in the limit $N \to \infty$. However, in general, $\Lambda_N(t,x,y)$ is not self-adjoint and thus certainly not a positive operator; there seems to be no simple way to calculate its trace norm. It turns out that it is sufficient to assume (3.2) given below, which implies boundedness of the trace and of the Hilbert-Schmidt norm of $\Lambda_N(t,x,y)$ with respect to N.

Hypothesis A. The sequences $\{p_j\}_{j\in\mathbb{N}}$, $\{q_j\}_{j\in\mathbb{N}}$ are bounded, $0 < p_j, q_j < \kappa_0$ for all $j \in \mathbb{N}$ and some $\kappa_0 > 0$, and satisfy

$$(p_j - p_l)(q_j - q_l) \geq 0, \quad \text{for all} \quad j, l \in \mathbb{N}. \tag{3.1}$$

The sequence $\{c_j\}_{j\in\mathbb{N}}$, $c_j \in \mathbb{R}\backslash\{0\}$, $j \in \mathbb{N}$, satisfies

$$\sum_{j\in\mathbb{N}} \frac{c_j^2}{p_j} < \infty, \quad \sum_{j\in\mathbb{N}} \frac{c_j^2}{q_j} < \infty. \tag{3.2}$$

We define

$$\Lambda_\infty(t,x,y) = \left\{c_j(t,y)c_l(t,y)\,\frac{\exp[-(p_j+q_l)x]}{p_j+q_l}\right\}_{j,l\in\mathbb{N}}. \tag{3.3}$$

We want to show convergence of $A_N(t,x,y)$ as defined in (2.11) and of its derivatives. To achieve this, we first want to estimate the trace norm $\|\partial_x\Lambda_N(t,x,y)\|_1$ of the matrix $\partial_x\Lambda_N(t,x,y)$. For later reference we also include a similar estimate on the Hilbert-Schmidt norm $\|\Lambda_N(t,x,y)\|_2$ of the matrix $\Lambda_N(t,x,y)$ itself.

Lemma 3.1. *Assume Hypothesis A. Then for all $\mathcal{M} \in \mathbb{N}\cup\{\infty\}$, the matrix $\partial_x\Lambda_{\mathcal{M}}(t,x,y)$ is trace class with trace norm bounded by*

$$\|\partial_x\Lambda_{\mathcal{M}}(t,x,y)\|_1 \le \sup_{j,l\in\mathbb{N}} \exp[-(v_j+w_l)x]\sum_{j=1}^{\infty} c_j(t,y)^2\exp(-2u_jx) < \infty, \tag{3.4}$$

where $u_j = \min(p_j,q_j)$, $v_j = p_j - u_j$, $w_j = q_j - u_j$ for all $j \in \mathbb{N}$. $\Lambda_{\mathcal{M}}(t,x,y)$ Hilbert-Schmidt with Hilbert-Schmidt norm bounded by

$$\begin{aligned}\|\Lambda_{\mathcal{M}}(t,x,y)\|_2 \le& \left\{\sum_{j=1}^{\mathcal{M}}\frac{c_j(t,y)^2}{2p_j}\exp(-2u_jx)\sum_{l=1}^{\mathcal{M}}\frac{c_l(t,y)^2}{2q_l}\exp(-2q_lx)\right\}^{\frac12}\\ &< \infty.\end{aligned} \tag{3.5}$$

Proof. By definition (cf. (2.4) resp. (3.3)),

$$\begin{aligned}&\partial_x\Lambda_{\mathcal{M}}(t,x,y)\\ &= -\{c_j(t,y)c_l(t,y)\exp[-(p_j+q_l)x]\}_{j,l=1}^{\mathcal{M}}\\ &= -\{\exp(-v_jx)\delta_{j,k}\}_{j,k=1}^{\infty}\{c_k(t,y)c_m(t,y)\exp[-(u_k+u_m)x]\}_{k,m=1}^{\mathcal{M}}\\ &\quad\times\{\exp(-w_lx)\delta_{m,l}\}_{m,l=1}^{\infty}.\end{aligned} \tag{3.6}$$

Here the matrix

$$\{c_k(t,y)c_m(t,y)\exp[-(u_k+u_m)x]\}_{k,m=1}^{\mathcal{M}} =: C_{\mathcal{M}}(t,x,y) \tag{3.7}$$

is positive, i.e., for all $\varphi \in \ell^2(\mathbb{N})$, $(t,x,y) \in \mathbb{R}^3$,

$$(\varphi, C_{\mathcal{M}}(t,x,y)\varphi) = \left|\sum_{j=1}^{\mathcal{M}} c_j(t,y)\exp(-u_jx)\varphi_j\right|^2 \ge 0. \tag{3.8}$$

Thus its trace norm is simply the trace

$$\|C_{\mathcal{M}}(t,x,y)\|_1 = \sum_{j=1}^{\mathcal{M}} c_j(t,y)^2 \exp(-2u_j x) \le \sum_{j=1}^{\infty} c_j(t,y)^2 \exp(-2u_j x). \tag{3.9}$$

Therefore,

$$\begin{aligned} \|\partial_x \Lambda_{\mathcal{M}}(t,x,y)\|_1 \le & \left\| \{\exp(-v_j x)\delta_{j,k}\}_{j,k=1}^{\infty} \right\| \\ & \times \|C_{\mathcal{M}}(t,x,y)\|_1 \left\| \{\exp(-w_l x)\delta_{m,l}\}_{m,l=1}^{\infty} \right\| \end{aligned} \tag{3.10}$$

proves (3.4).

(3.5) follows from a straightforward calculation:

$$\begin{aligned} \|\Lambda_{\mathcal{M}}(t,x,y)\|_2^2 &= \operatorname{Tr}[\Lambda_{\mathcal{M}}(t,x,y)^* \Lambda_{\mathcal{M}}(t,x,y)] \\ &= \sum_{j,l=1}^{\mathcal{M}} \frac{c_j(t,y)^2 c_l(t,y)^2}{(p_j+q_l)^2} \exp[-2(p_j+q_l)x] \\ &\le \sum_{j=1}^{\mathcal{M}} \frac{c_j(t,y)^2}{2p_j} \exp(-2p_j x) \sum_{l=1}^{\mathcal{M}} \frac{c_l(t,y)^2}{2q_l} \exp(-2q_l x) \\ &< \infty \end{aligned} \tag{3.11}$$

by (3.2). □

We can also extend this lemma to higher order derivatives.

Lemma 3.2. *Suppose Hypothesis A holds. For all $\alpha, \beta, \gamma \in \mathbb{N}_0$, let a be the multi-index $a = (\alpha, \beta, \gamma)$, $\partial^a = \partial_x^\alpha \partial_y^\beta \partial_t^\gamma$. Then for all $\mathcal{M} \in \mathbb{N} \cup \{\infty\}$,*

$$\|\partial^a \partial_x \Lambda_{\mathcal{M}}(t,x,y)\|_1 \le \text{const.}(a)\, \|\partial_x \Lambda_{\mathcal{M}}(t,x,y)\|_1 \le \text{const.}(a,t,x,y), \tag{3.12}$$

$$\|\partial^a \Lambda_{\mathcal{M}}(t,x,y)\|_2 \le \text{const.}(a)\, \|\Lambda_{\mathcal{M}}(t,x,y)\|_2 \le \text{const.}(a,t,x,y), \tag{3.13}$$

with the constants independent of $\mathcal{M}$ and locally uniform with respect to $(t,x,y) \in \mathbb{R}^3$ (i. e., uniform in all compact subsets of $\mathbb{R}^3$). Furthermore,

$$\|\partial^a \partial_x \Lambda_\infty(t,x,y) - \partial^a \partial_x \Lambda_N(t,x,y)\|_1 \xrightarrow[N\to\infty]{} 0, \tag{3.14}$$

$$\|\partial^a \Lambda_\infty(t,x,y) - \partial^a \Lambda_N(t,x,y)\|_2 \xrightarrow[N\to\infty]{} 0 \tag{3.15}$$

locally uniformly with respect to $(t,x,y) \in \mathbb{R}^3$.

Proof. To prove (3.12) we use an induction argument: Since the estimate (3.4) is independent of $\mathcal{M}$ and locally uniform with respect to $(t,x,y) \in \mathbb{R}^3$, we have already proved it for $\alpha = \beta = \gamma = 0$. Assume it holds for α, β, $\gamma \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|\partial_x \partial^\alpha \partial_x \Lambda_{\mathcal{M}}(t,x,y)\|_1 &= \left\| \{-(p_j+q_l)\{\partial^\alpha \partial_x \Lambda_{\mathcal{M}}(t,x,y)\}_{j,l}\}_{j,l=1}^{\mathcal{M}} \right\|_1 \\
&\le \left\| \{p_j \delta_{j,k}\}_{j,k=1}^{\infty} \right\| \|\partial^\alpha \partial_x \Lambda_{\mathcal{M}}(t,x,y)\|_1 \\
&\quad + \|\partial^\alpha \partial_x \Lambda_{\mathcal{M}}(t,x,y)\|_1 \left\| \{q_l \delta_{m,l}\}_{m,l=1}^{\infty} \right\| \\
&\le_{,} \mathit{const.}\, \|\partial^\alpha \partial_x \Lambda_{\mathcal{M}}(t,x,y)\|_1
\end{aligned} \tag{3.16}$$

proves (3.12) for $\alpha+1$, β, γ. The induction steps $\beta \to \beta+1$ and $\gamma \to \gamma+1$ are done in a similar manner. There we get more complicated factors, but we can still expand them to get an expression analogous to (3.16).

(3.13) is proved by exactly the same induction argument, starting from (3.5) for $\alpha = \beta = \gamma = 0$.

Now let P_N be the projection from $\ell^2(\mathbb{N})$ to $\mathbb{C}^N$, $Q_N = 1 - P_N$. Then for all positive self-adjoint trace class operators F on $\ell^2(\mathbb{N})$,

$$\begin{aligned}
\|F - P_N F P_N\|_1 &= \|Q_N F + P_N F Q_N\|_1 \\
&\le 2\,\|Q_N F\|_1 \\
&\le 2 \left\| Q_N F^{\frac{1}{2}} \right\|_2 \left\| F^{\frac{1}{2}} \right\|_2 \\
&= 2\,\|Q_N F Q_N\|_1^{\frac{1}{2}}\, \|F\|_1^{\frac{1}{2}} \xrightarrow[N\to\infty]{} 0\,.
\end{aligned} \tag{3.17}$$

The matrices $\partial_x \Lambda_\infty(t,x,y)$ and $Q_N \partial_x \Lambda_\infty(t,x,y) Q_N$ are both positive, so by (3.4) we can use (3.17) to get

$$\|\partial_x \Lambda_\infty(t,x,y) - \partial_x \Lambda_N(t,x,y)\|_1 \xrightarrow[N\to\infty]{} 0\,. \tag{3.18}$$

Because the bound in (3.12) is locally uniform with respect to $(t,x,y) \in \mathbb{R}^3$, this convergence is locally uniform, too. So we have (3.14) for $\alpha = \beta = \gamma = 0$ and can use an induction argument as in the proof of (3.12) to infer the statement for general α, β, and γ.

The proof of (3.15) is similar. For $\alpha = \beta = \gamma = 0$,

$$\begin{aligned}
&\|\Lambda_\infty(t,x,y) - \Lambda_N(t,x,y)\|_2^2 \\
&= \operatorname{Tr}\big[(Q_N \Lambda_\infty(t,x,y) + P_N \Lambda_\infty(t,x,y) Q_N)^* \\
&\quad \times (Q_N \Lambda_\infty(t,x,y) + P_N \Lambda_\infty(t,x,y) Q_N)\big] \\
&= \operatorname{Tr}\big[\Lambda_\infty(t,x,y) Q_N \Lambda_\infty(t,x,y) + Q_N \Lambda_\infty(t,x,y) P_N \Lambda_\infty(t,x,y) Q_N\big] \\
&\le 2 \sum_{j=1}^{\infty} \sum_{l=N+1}^{\infty} c_j(t,y)^2 c_l(t,y)^2 \frac{\exp[-2(p_j+q_l)x]}{(p_j+q_l)^2} \xrightarrow[N\to\infty]{} 0
\end{aligned} \tag{3.19}$$

by (3.2) locally uniformly with respect to $(t,x,y) \in \mathbb{R}^3$. Induction in α, β, and γ completes the proof. □

Getting $[1 + \Lambda_N(t,x,y)]^{-1}$ under control turns out to be a bit more involved. We

Lemma 3.3. *Suppose Hypothesis A holds. Then for all $\mathcal{M} \in \mathbb{N} \cup \{\infty\}$,*

$$\left\| [1 + \Lambda_{\mathcal{M}}(t,x,y)]^{-1} \right\| \leq const.(t,x,y)\,, \tag{3.20}$$

with the constant independent of $\mathcal{M}$ and locally uniform with respect to $(t,x,y) \in \mathbb{R}^3$. Furthermore,

$$\left\| [1 + \Lambda_{\infty}(t,x,y)]^{-1} - [1 + \Lambda_N(t,x,y)]^{-1} \right\|_2 \xrightarrow[N\to\infty]{} 0 \tag{3.21}$$

locally uniformly with respect to $(t,x,y) \in \mathbb{R}^3$.

Proof. If we consider $\Lambda_N(t,x,y)$ as an operator on $\mathbb{C}^N$, we can calculate its determinant (see Pólya, Szegö [34], p. 92)

$$\begin{aligned} \det \Lambda_N(t,x,y) &= \exp\left[-\sum_{j=1}^{N} (p_j + q_j)x \right] \prod_{j=1}^{N} c_j^2 \left[\prod_{r,s=1}^{N} (p_r + q_s) \right]^{-1} \\ &\quad \times \prod_{\substack{l,k=1 \\ k>l}}^{N} (p_l - p_k)(q_l - q_k) \\ &\geq 0 \end{aligned} \tag{3.22}$$

by (3.1). By expanding, we find

$$\begin{aligned} \det[1 + \Lambda_N(t,x,y)] &= 1 + \det[\Lambda_N(t,x,y)] + \sum_{j_1=1}^{N} \det\left[\Lambda_N(t,x,y)^{j_1}\right] \\ &\quad + \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{N} \det\left[\Lambda_N(t,x,y)^{j_1,j_2}\right] + \cdots \\ &\quad + \sum_{\substack{j_1,j_2,\dots,j_{N-1}=1 \\ j_1<j_2<\dots<j_{N-1}}}^{N} \det\left[\Lambda_N(t,x,y)^{j_1,j_2,\dots,j_{N-1}}\right], \end{aligned} \tag{3.23}$$

where $\Lambda_N(t,x,y)^{j_1,j_2,\dots,j_k}$ denotes the $(N-k)\times(N-k)$ matrix obtained by deleting the $j_1,\dots,j_k$ rows and columns of $\Lambda_N(t,x,y)$. Each of these matrices is of the same form as the original matrix $\Lambda_N(t,x,y)$, thus all the terms in (3.23) are positive and

$$\det[1 + \Lambda_N(t,x,y)] \geq 1\,. \tag{3.24}$$

So we know that for all finite N the operator $1 + \Lambda_N(t,x,y)$ is invertible,

$$\left\| [1 + \Lambda_N(t,x,y)]^{-1} \right\| \leq const.(N,t,x,y)\,. \tag{3.25}$$

We can choose the constant uniform on each compact subset $K \subset \mathbb{R}^3$, but at this point we do not know whether it is uniform in N. While our hypothesis guarantees that $\Lambda_\infty(t,x,y)$ has finite trace, this does not necessarily imply that its trace norm is finite,

so the Fredholm determinant $\det_1[1+\Lambda_\infty(t,x,y)]$ might not be well defined. How
$\Lambda_\infty(t,x,y)$ is Hilbert–Schmidt and we can work with the regularized determin
$\det_2[1+\Lambda_\infty(t,x,y)]$ (see, e.g., SIMON [39], p. 106). For finite N

$$\begin{aligned}\det{}_2[1+\Lambda_N(t,x,y)] &= \det[1+\Lambda_N(t,x,y)]\exp[-\mathrm{Tr}(\Lambda_N(t,x,y))]\\ &\geq const.(t,x,y)\\ &> 0\end{aligned} \tag{3.26}$$

by (3.24) and (3.2). But by (3.15) and Theorem 9.2 of [39] this implies

$$\det{}_2[1+\Lambda_\infty(t,x,y)] = \lim_{N\to\infty}\det{}_2[1+\Lambda_N(t,x,y)] \geq const.(t,x,y) > 0 \tag{3.27}$$

Again by [39], Theorem 9.2, this guarantees that $1+\Lambda_\infty(t,x,y)$ is invertible. Fo
$(t,x,y)\in\mathbb{R}^3$ fixed,

$$\left\|[1+\Lambda_\infty(t,x,y)]^{-1}\right\| = c(t,x,y) < \infty, \tag{3.28}$$

with some constant $c(t,x,y)$. If (t,x,y) and (τ,ξ,η) are in $\mathbb{R}^3$, then we can calculate
the Hilbert–Schmidt norm

$$\begin{aligned}&\|\Lambda_\infty(t,x,y)-\Lambda_\infty(\tau,\xi,\eta)\|_2^2\\ &= \sum_{j,l=1}^{\mathcal{M}}\left|\frac{c_j(t,y)c_l(t,y)}{p_j+q_l}\exp[-(p_j+q_l)x]-\frac{c_j(\tau,\eta)c_l(\tau,\eta)}{p_j+q_l}\exp[-(p_j+q_l)\xi]\right|^2\\ &\xrightarrow[(\tau,\xi,\eta)\to(t,x,y)]{} 0\end{aligned} \tag{3.29}$$

by the Weierstrass test. (The sum converges by (3.2) and in the limit $(\tau,\xi,\eta)\to$
(t,x,y) each term converges to 0.) Therefore there is an $\varepsilon>0$ such that for all (τ,ξ,η)
with $|(\tau,\xi,\eta)-(t,x,y)|<\varepsilon$, $\|\Lambda_\infty(t,x,y)-\Lambda_\infty(\tau,\xi,\eta)\|_2<\frac{1}{2c(t,x,y)}$, which implies

$$\begin{aligned}&\left\|[1+\Lambda_\infty(\tau,\xi,\eta)]^{-1}\right\|\\ &\leq \left\|[1+\Lambda_\infty(t,x,y)]^{-1}\right\|\\ &\quad\times\left\|\left\{1+[1+\Lambda_\infty(t,x,y)]^{-1}[\Lambda_\infty(\tau,\xi,\eta)-\Lambda_\infty(t,x,y)]\right\}^{-1}\right\|\\ &\leq c(t,x,y)\left\{1-\left\|[1+\Lambda_\infty(t,x,y)]^{-1}\right\|\,\|[\Lambda_\infty(\tau,\xi,\eta)-\Lambda_\infty(t,x,y)]\|\right\}^{-1}\\ &\leq 2c(t,x,y).\end{aligned} \tag{3.30}$$

If $K\subset\mathbb{R}^3$ is compact, we can cover it with a finite number of such ε-neighborhoods
and $c_K:=\sup_{(t,x,y)\in K}[c(t,x,y)]$ is finite. By (3.15) there is an $N_K\in\mathbb{N}$ such that for
all $\mathcal{M}\in\mathbb{N}\cup\{\infty\}$, $\mathcal{M}\geq N_K$, and for $(t,x,y)\in K$, $\|\Lambda_\infty(t,x,y)-\Lambda_{\mathcal{M}}(t,x,y)\|\leq\frac{1}{2c_K}$.
Then for such $\mathcal{M}\geq N_K$ and $(t,x,y)\in K$,

$$\left\|[1+\Lambda_{\mathcal{M}}(t,x,y)]^{-1}\right\| \leq 2c_K \tag{3.31}$$

...an estimate analogous to (3.30). Together with (3.25) this proves (3.20). (3.21) is ...an immediate consequence of (3.20) and (3.15), since

$$
\begin{aligned}
&\left\|[1+\Lambda_\infty(t,x,y)]^{-1}-[1+\Lambda_N(t,x,y)]^{-1}\right\|_2 \\
(3.32)\quad &\leq \left\|[1+\Lambda_\infty(t,x,y)]^{-1}\right\| \, \|[\Lambda_\infty(t,x,y)-\Lambda_N(t,x,y)]\|_2 \, \left\|[1+\Lambda_N(t,x,y)]^{-1}\right\| \\
&\xrightarrow[N\to\infty]{} 0
\end{aligned}
$$

locally uniformly with respect to $(t,x,y)\in\mathbb{R}^3$. □

From these three lemmas we can conclude the convergence of $A_N(t,x,y)$ in the limit $N\to\infty$.

Lemma 3.4. *Suppose Hypothesis A holds and define*

$$
(3.33)\qquad A_\infty(t,x,y) \;=\; \mathrm{Tr}\big[[1+\Lambda_\infty(t,x,y)]^{-1}\partial_x\Lambda_\infty(t,x,y)\big]\,.
$$

For all α, β, $\gamma\in\mathbb{N}_0$, let $\partial^a=\partial_x^\alpha\partial_y^\beta\partial_t^\gamma$. Then

$$
(3.34)\qquad \partial^a A_\infty(t,x,y) \;=\; \lim_{N\to\infty}\partial^a A_N(t,x,y)\,,
$$

the convergence being locally uniform with respect to $(t,x,y)\in\mathbb{R}^3$.

Proof. (3.33) is well defined because of Lemmas 3.1 and 3.3.
Convergence of $A_N(t,x,y)\xrightarrow[N\to\infty]{}A_\infty(t,x,y)$ then follows from

$$
\begin{aligned}
&|A_\infty(t,x,y)-A_N(t,x,y)| \\
(3.35)\quad &\leq \left\|[1+\Lambda_\infty(t,x,y)]^{-1}-[1+\Lambda_N(t,x,y)]^{-1}\right\| \, \|\partial_x\Lambda_\infty(t,x,y)\|_1 \\
&\quad+\left\|[1+\Lambda_N(t,x,y)]^{-1}\right\| \, \|[\partial_x\Lambda_\infty(t,x,y)-\partial_x\Lambda_N(t,x,y)]\|_1 \xrightarrow[N\to\infty]{} 0
\end{aligned}
$$

by (3.4), (3.14), (3.20), (3.21). All estimates are locally uniform with respect to $(t,x,y)\in\mathbb{R}^3$, so we have (3.34) for $\alpha=\beta=\gamma=0$.

To prove the convergence for general multi-indices $a=(\alpha,\beta,\gamma)$ we note that $\partial^a A_N(t,x,y)$ is of the form

$$
(3.36)\qquad \partial^a A_N(t,x,y) \;=\; \sum_{k=1}^{n(a)}\mathrm{Tr}[K^a_{N,k}(t,x,y)]\,,
$$

with $n(a)\in\mathbb{N}$ and matrices

$$
(3.37)\qquad
\begin{aligned}
K^a_{N,k}(t,x,y) \;=\; &\pm[1+\Lambda_N(t,x,y)]^{-1}\partial^{a_1}\Lambda_N(t,x,y)[1+\Lambda_N(t,x,y)]^{-1} \\
&\times\cdots\times[1+\Lambda_N(t,x,y)]^{-1}\partial^{a_{s(k)}}\partial_x\Lambda_N(t,x,y)
\end{aligned}
$$

for some $s(k)\in\mathbb{N}_0$ and multi-indices $a_1,\ldots,a_{s(k)}$, $a_1+\cdots+a_{s(k)}=a$. By Lemmas 3.2 and 3.3 $\left\|K^a_{N,k}(t,x,y)\right\|_1\leq const.(a,k,t,x,y)$ with the constant locally uniform with respect to $(t,x,y)\in\mathbb{R}^3$ and independent of N. Thus

$$
(3.38)\qquad K^a_{\infty,k}(t,x,y) \;:=\; \lim_{N\to\infty}K^a_{N,k}(t,x,y)
$$

exists and is trace class. Then

$$
\begin{aligned}
&\big|\mathrm{Tr}[K^{a}_{\infty,k}(t,x,y)] - \mathrm{Tr}[K^{a}_{N,k}(t,x,y)]\big| \\
&\le \big|\mathrm{Tr}\big[\{[1+\Lambda_\infty(t,x,y)]^{-1} - [1+\Lambda_N(t,x,y)]^{-1}\}\partial^{a_1}\Lambda_\infty(t,x,y) \\
&\qquad \times [1+\Lambda_\infty(t,x,y)]^{-1} \times \cdots \times [1+\Lambda_\infty(t,x,y)]^{-1}\partial^{a_{s(k)}}\partial_x\Lambda_\infty(t,x,y)\big]\big| \\
&\quad + \big|\mathrm{Tr}\big[[1+\Lambda_N(t,x,y)]^{-1}\{\partial^{a_1}\Lambda_\infty(t,x,y) - \partial^{a_1}\Lambda_N(t,x,y)\} \\
&\qquad \times [1+\Lambda_\infty(t,x,y)]^{-1} \times \cdots \times [1+\Lambda_\infty(t,x,y)]^{-1}\partial^{a_{s(k)}}\partial_x\Lambda_\infty(t,x,y)\big]\big| \\
&\quad + \\
&\quad \vdots \\
&\quad + \big|\mathrm{Tr}\big[[1+\Lambda_N(t,x,y)]^{-1}\partial^{a_1}\Lambda_N(t,x,y)[1+\Lambda_N(t,x,y)]^{-1} \\
&\qquad \times \cdots \times [1+\Lambda_N(t,x,y)]^{-1}\{\partial^{a_{s(k)}}\partial_x\Lambda_\infty(t,x,y) - \partial^{a_{s(k)}}\partial_x\Lambda_N(t,x,y)\}\big]\big| \\
&\le \big\|[1+\Lambda_\infty(t,x,y)]^{-1} - [1+\Lambda_N(t,x,y)]^{-1}\big\| \\
&\qquad \times \big\|\partial^{a_1}\Lambda_\infty(t,x,y)[1+\Lambda_\infty(t,x,y)]^{-1} \\
&\qquad \times \cdots \times [1+\Lambda_\infty(t,x,y)]^{-1}\partial^{a_{s(k)}}\partial_x\Lambda_\infty(t,x,y)\big\|_1 \\
&\quad + \big\|[1+\Lambda_N(t,x,y)]^{-1}\big\| \, \|\partial^{a_1}\Lambda_\infty(t,x,y) - \partial^{a_1}\Lambda_N(t,x,y)\|_2 \\
&\qquad \times \big\|[1+\Lambda_\infty(t,x,y)]^{-1} \times \cdots \times [1+\Lambda_\infty(t,x,y)]^{-1}\partial^{a_{s(k)}}\partial_x\Lambda_\infty(t,x,y)\big\|_1 \\
&\quad + \\
&\quad \vdots \\
&\quad + \big\|[1+\Lambda_N(t,x,y)]^{-1}\partial^{a_1}\Lambda_N(t,x,y)[1+\Lambda_N(t,x,y)]^{-1} \\
&\qquad \times \cdots \times [1+\Lambda_N(t,x,y)]^{-1}\big\| \\
&\qquad \times \|\partial^{a_{s(k)}}\partial_x\Lambda_\infty(t,x,y) - \partial^{a_{s(k)}}\partial_x\Lambda_N(t,x,y)\|_1 \xrightarrow[N\to\infty]{} 0
\end{aligned}
\tag{3.39}
$$

locally uniformly with respect to $(t,x,y) \in \mathbb{R}^3$ by (3.12) - (3.15), (3.20), and (3.21) □

Now we are prepared to prove our first main result.

Theorem 3.5. *Assume Hypothesis A. Then the function*

$$V(t,x,y) \;=\; V_\infty(t,x,y) + \lambda \;:=\; -2\partial_x A_\infty(t,x,y) + \lambda \tag{3.40}$$

is a solution of the KP-equation (2.1).

Proof. Applying the Weierstrass test to (3.5) shows that, independently of $\mathcal{M} \in \mathbb{N} \cup \{\infty\}$, $\|\Lambda_{\mathcal{M}}(t,x,y)\|_2 \xrightarrow[x\to+\infty]{} 0$. Thus for large enough x,

$$\big\|[1+\Lambda_{\mathcal{M}}(t,x,y)]^{-1}\big\| \;\le\; \{1 - \|\Lambda_{\mathcal{M}}(t,x,y)\|\}^{-1} \xrightarrow[x\to+\infty]{} 1\,. \tag{3.41}$$

In a similar fashion by (3.4) $\|\partial_x\Lambda_{\mathcal{M}}(t,x,y)\|_1 \xrightarrow[x\to+\infty]{} 0$, by (3.12) the same holds for its derivatives, so $\partial_y^2 A_{\mathcal{M}}(t,x,y) \xrightarrow[x\to+\infty]{} 0$ for all $\mathcal{M} \in \mathbb{N} \cup \{\infty\}$. Boundedness of all

...rivatives guarantees that we can interchange the order of differentiation, so

$$
\begin{aligned}
\int_{\infty}^{x} dx' \partial_y^2 V_\infty(t,x',y) &= -2 \int_{\infty}^{x} dx' \partial_x \partial_y^2 A_\infty(t,x',y) \\
&= -2\partial_y^2 A_\infty(t,x,y) \\
&= \lim_{N\to\infty} \left(-2\partial_y^2 A_N(t,x,y)\right) \\
&= \lim_{N\to\infty} \int_{\infty}^{x} dx' \partial_y^2 V_N(t,x',y) .
\end{aligned}
\tag{3.42}
$$

By Lemma 3.4 we can also interchange the limits $N \to \infty$ with derivatives, thus we have similar convergence results for all the other terms in (2.1) and the theorem follows from the fact that $V = V_N(t,x,y) + \lambda$ solves the KP – equation. □

Remark 3.6. Presumably, our hypotheses are not optimal; however, they are general enough to cover a large class of solutions. In particular, we remark that the only restriction on the sequences $\{p_j\}_{j\in\mathbb{N}}$, $\{q_j\}_{j\in\mathbb{N}}$ is the boundedness condition and the requirement of having the N – soliton solutions regular. The only condition which is technical and could perhaps be weakened is the one imposed on the sequence $\{c_j\}_{j\in\mathbb{N}}$.

4. Soliton limits for the mKP – equation

We follow the same steps we used for the KP – equation; we start with the N – soliton solutions given by (2.7). Hypothesis A is not sufficient to ensure that these solutions are regular (cf. Rmk. 4.4). We need the following additional hypothesis (cf. (2.8)).

Hypothesis B. Suppose that either $\kappa = \lambda^{\frac{1}{2}} \geq \sup_{j\in\mathbb{N}}(p_j)$ or $\kappa < \inf_{j\in\mathbb{N}}(-q_j)$.

Now we can extend the lemmas from the previous Section 3 to $\hat{\Lambda}_N(t,x,y)$, respectively, $B_N(t,x,y)$.

Lemma 4.1. *Assume Hypotheses A and B. Let a be a multi – index $a = (\alpha,\beta,\gamma)$, α, β, $\gamma \in \mathbb{N}_0$ arbitrary, and abbreviate $\partial^a = \partial_x^\alpha \partial_y^\beta \partial_t^\gamma$. Then for all $\mathcal{M} \in \mathbb{N}\cup\{\infty\}$,*

$$
\left\| \partial^a \partial_x \hat{\Lambda}_{\mathcal{M}}(t,x,y)\right\|_1 \leq const.(a)\ \|\partial_x \Lambda_{\mathcal{M}}(t,x,y)\|_1 \leq const.(a,t,x,y)\,, \tag{4.1}
$$

$$
\left\| \partial^a \hat{\Lambda}_{\mathcal{M}}(t,x,y)\right\|_2 \leq const.(a)\ \|\Lambda_{\mathcal{M}}(t,x,y)\|_2 \leq const.(a,t,x,y)\,, \tag{4.2}
$$

with the constants independent of $\mathcal{M}$ and locally uniform with respect to $(t,x,y) \in \mathbb{R}^3$. Moreover,

$$
\left\| \partial^a \partial_x \hat{\Lambda}_\infty(t,x,y) - \partial^a \partial_x \hat{\Lambda}_N(t,x,y)\right\|_1 \xrightarrow[N\to\infty]{} 0\,, \tag{4.3}
$$

$$
\left\| \partial^a \hat{\Lambda}_\infty(t,x,y) - \partial^a \hat{\Lambda}_N(t,x,y)\right\|_2 \xrightarrow[N\to\infty]{} 0 \tag{4.4}
$$

locally uniformly with respect to $(t,x,y) \in \mathbb{R}^3$. *Furthermore,*

$$\left\| [1+\hat{\Lambda}_{\mathcal{M}}(t,x,y)]^{-1} \right\| \le \mathit{const.}(t,x,y)\,, \tag{4.5}$$

with the constant locally uniform with respect to $(t,x,y) \in \mathbb{R}^3$, *and*

$$\left\| [1+\hat{\Lambda}_{\infty}(t,x,y)]^{-1} - [1+\hat{\Lambda}_{N}(t,x,y)]^{-1} \right\|_2 \xrightarrow[N\to\infty]{} 0 \tag{4.6}$$

locally uniformly with respect to $(t,x,y) \in \mathbb{R}^3$.

Proof. (4.1) – (4.4) follow from Lemma 3.2 and (2.9). (The matrix $D_\infty(t,x,y)$ in (2.9) is independent of t, x, y and bounded by Hypotheses A and B.) In order to prove (4.5), we note that we can rewrite (2.9) as $\hat{\Lambda}_N(t,x,y) = D_N\Lambda_N(t,x,y)$, where D_N, the restriction of D_∞ to $\mathbb{C}^N$, is an $N\times N$ matrix. Then

$$\det\hat{\Lambda}_N(t,x,y) = \prod_{j=1}^{N} \frac{\kappa-p_j}{\kappa+q_j}\det\Lambda_N(t,x,y) \ge 0 \tag{4.7}$$

by Hypothesis B and (3.22). Thus we can follow the proof of Lemma 3.3 to infer (4.5) and (4.6). □

Lemma 4.2. *Suppose Hypotheses A and B hold. Define*

$$B_\infty(t,x,y) = \mathrm{Tr}\big[[1+\hat{\Lambda}_\infty(t,x,y)]^{-1}\partial_x\hat{\Lambda}_\infty(t,x,y)\big]\,. \tag{4.8}$$

For all α, β, $\gamma \in \mathbb{N}_0$, *let* $\partial^a = \partial_x^\alpha\partial_y^\beta\partial_t^\gamma$. *Then*

$$\partial^a B_N(t,x,y) \xrightarrow[N\to\infty]{} \partial^a B_\infty(t,x,y) \tag{4.9}$$

locally uniformly with respect to $(t,x,y) \in \mathbb{R}^3$.

Proof. Because of the boundedness of D_∞ (cf. (2.9) resp. (2.10)) we can follow the proof of Lemma 3.4 step by step. □

By Lemmas 3.4 and 4.2

$$\phi_N(t,x,y) \xrightarrow[N\to\infty]{} \phi_\infty(t,x,y) = -\kappa + B_\infty(t,x,y) - A_\infty(t,x,y) \tag{4.10}$$

and $\phi_\infty(t,x,y)$ satisfies the differentiated version of the mKP – equation. However, in order to prove that it satisfies the mKP – equation in integral form (2.6) we need a slight strengthening of Hypothesis A, namely

Hypothesis C. The sequence $\{c_j\}_{j\in\mathbb{N}}$, $c_j \in \mathbb{R}\backslash\{0\}$, $j\in\mathbb{N}$, satisfies

$$\sum_{j=1}^{\infty} \frac{c_j^2}{\min(p_j,q_j)} < \infty\,. \tag{4.11}$$

With this hypothesis we finally obtain

Theorem 4.3. *Suppose Hypotheses A, B, and C hold. Then the function*

$$\phi_\infty(t,x,y) = -\kappa + B_\infty(t,x,y) - A_\infty(t,x,y) \tag{4.12}$$

satisfies the mKP - equation (2.6).

Proof. We know that $\phi_N(t,x,y)$ solves the mKP - equation for every $N \in \mathbb{N}$. Lemmas 3.4 and 4.2 allow us to exchange the limit $N \to \infty$ with derivatives. So we just have to check whether we can interchange the limit with the integral, i. e., we have to prove

$$\int_\infty^x dx' \partial_y^\alpha \phi_N(t,x,y) \xrightarrow[N\to\infty]{} \int_\infty^x dx' \partial_y^\alpha \phi_\infty(t,x,y), \quad \alpha = 1,2. \tag{4.13}$$

For that purpose we want to rely on dominated convergence. The integrand converges and is bounded by

$$|\partial_y^\alpha \phi_N(t,x,y)| \le \left|\partial_y^\alpha B_N(t,x,y)\right| + \left|\partial_y^\alpha A_N(t,x,y)\right|, \quad \alpha = 1,2. \tag{4.14}$$

Here

$$\begin{aligned} &|\partial_y A_N(t,x,y)| \\ =\ & \Big|\mathrm{Tr}\big[\{-[1+\Lambda_N(t,x,y)]^{-1}\partial_y\Lambda_N(t,x,y)[1+\Lambda_N(t,x,y)]^{-1} \\ &+ [1+\Lambda_N(t,x,y)]^{-1}\partial_y\}\partial_x\Lambda_N(t,x,y)\big]\Big| \\ \le\ & \Big\{\big\|[1+\Lambda_N(t,x,y)]^{-1}\big\| \, \|\partial_y\Lambda_N(t,x,y)\| \, \big\|[1+\Lambda_N(t,x,y)]^{-1}\big\| \\ &+ \big\|[1+\Lambda_N(t,x,y)]^{-1}\big\| \, const.\Big\} \, \|\partial_x\Lambda_N(t,x,y)\|_1 \\ \le\ & const.(t,x,y) \, \|\partial_x\Lambda_N(t,x,y)\|_1 \, . \end{aligned} \tag{4.15}$$

The constant is independent of N and locally uniform with respect to $(t,x,y) \in \mathbb{R}^3$. Since $\|\partial_y\Lambda_N(t,x,y)\| \xrightarrow[x\to+\infty]{} 0$ (apply the Weierstrass test to (3.5) and use (3.13)) and $\big\|[1+\Lambda_N(t,x,y)]^{-1}\big\| \xrightarrow[x\to+\infty]{} 1$ (by (3.41)), this constant converges in the limit $x \to +\infty$, so we can choose it uniform for x on a half - axis $[a_0,\infty)$ and (t,y) in a compact subset of $\mathbb{R}^2$. The same argument can be applied to $\partial_y^2 A_N(t,x,y)$, $\partial_y B_N(t,x,y)$, and $\partial_y^2 B_N(t,x,y)$. Therefore it is sufficient to estimate (using (3.4) and the notation introduced in Lemma 3.1),

$$\begin{aligned} &\int_\infty^x dx' \, \|\partial_x\Lambda_\infty(t,x',y)\|_1 \\ \le\ & \sup_{j,l\in\mathbb{N}} \exp[-(v_j+w_l)x] \int_\infty^x dx' \sum_{j=1}^\infty c_j^2 \exp(-2u_j x') \\ =\ & \sup_{j,l\in\mathbb{N}} \exp[-(v_j+w_l)x] \sum_{j=1}^\infty \frac{c_j^2}{2u_j} \exp(-2u_j x) \\ <\ & \infty \end{aligned} \tag{4.16}$$

by Hypothesis C. (The interchange of the integral with the sum is justified bec
all terms in the sum are positive.) Thus $|\partial_y \phi_N(t,x,y)|$ and $|\partial_y^2 \phi_N(t,x,y)|$ are b
bounded by an integrable function and (4.13) follows by the dominated converge
theorem.

Remark 4.4. Hypothesis B is necessary in the sense that if there is a p_{j_0} w
$p_{j_0} > \kappa \geq 0$ or an q_{j_0} with $-q_{j_0} < \kappa \leq 0$, then for this index j_0, $\frac{\kappa - p_{j_0}}{\kappa + q_{j_0}} < 0$.
following a procedure similar to the proof of [10], Prop. 3.2 we can then show t
there is a choice for the sequence $\{c_j\}_{j\in\mathbb{N}}$ such that $\det[1+\hat{\Lambda}_N(t,x,y)]$ changes s
at least once. Hypothesis B also excludes the case $\kappa = \inf_{j\in\mathbb{N}}(-q_j)$, but this is d
for the more technical reason of keeping D_∞ bounded.

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Department of Mathematics
University of Missouri
Columbia, MO 65211
U.S.A.

Current address:
Institut für Mathematik
Technische Universität Clausthal
Erzstraße 1
D – 38678 Clausthal – Zellerfeld
Germany
e – mail:
renger@math.tu – clausthal.de