Compiled Notes on the KP-Whitham Modulation Equations

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1 Preliminaries

We consider the Kadomtsev-Petviashvili (KP) equation

$$(u_t + uu_x + u_{xxx})_x + \lambda u_{yy} = 0. \tag{1}$$

When $\lambda = -1$, it is called the KPI equation. When $\lambda = +1$, it is KPII. In what follows, we study the Whitham modulation equations derived by Ablowitz, Biondini, and Wang in [1]. In what follows, we summarize their results.

Note that Ablowitz et al considered the "IST-friendly" normalization with the uu_x term replaced with $6uu_x$. They also considered the small dispersion, hydrodynamic scaling that yields a small parameter in front of the u_{xxx} term. Here, we are considering the "physics-friendly" normalization without the 6 and where order one length and time scales are considered fast.

Equation (1) can also be written in the form

$$u_t + uu_x + u_{xxx} + \lambda v_y = 0, \quad v_x - u_y = 0. \tag{2}$$

Equation (2) admits cnoidal periodic traveling wave solutions in the form

$$\begin{aligned} u(\theta) &= r_1 - r_2 + r_3 + 2(r_2 - r_1)\mathrm{cn}^2\left(2\mathrm{K}(m)\theta, m\right), \\ m &= \frac{r_2 - r_1}{r_3 - r_1}, \\ \theta &= kx + ly - \omega t - \theta_0. \end{aligned} \tag{3}$$

This solution exhibits five independent parameters (sans a shift of the origin in θ_0): (r_1, r_2, r_3, k, l). The wave's frequency ω is

$$\omega = kV + \lambda\frac{l^2}{k}, \quad V = \frac{1}{3}(r_1 + r_2 + r_3). \tag{4}$$

Ablowitz et al consider the parameter set (r_1, r_2, r_3, q, p) where $q = l/k$ and $p = \overline{v} - q\overline{u}$. The parameters $\overline{u}$, $\overline{v}$ are period averages over the periodic solution (3).

According to Whitham's prescription, we now consider these five parameters to formally depend on slow space and time variables. By period-averaging conservation laws, period-averaging the Langrangian, or using multiple scales, one can obtain partial differential equations for the modulation variables. The result is [1]

$$\frac{\partial r_j}{\partial t} + (V_j + \lambda q^2)\frac{\partial r_j}{\partial x} + 2\lambda q \frac{\mathrm{D} r_j}{\mathrm{D} y} + \lambda v_j \frac{\mathrm{D} q}{\mathrm{D} y} + \lambda \frac{\mathrm{D} p}{\mathrm{D} y} = 0, \quad j = 1,2,3, \tag{5}$$

$$\frac{\partial q}{\partial t} + (V_2 + \lambda q^2)\frac{\partial q}{\partial x} + 2\lambda q \frac{\mathrm{D} q}{\mathrm{D} y} + v_{4.1}\frac{\mathrm{D} r_1}{\mathrm{D} y} + v_{4.3}\frac{\mathrm{D} r_3}{\mathrm{D} y} = 0, \tag{6}$$

$$\frac{\partial p}{\partial x} - (1-\alpha)\frac{\mathrm{D} r_1}{\mathrm{D} y} - \alpha \frac{\mathrm{D} r_3}{\mathrm{D} y} + v_5 \frac{\partial q}{\partial x} = 0, \tag{7}$$

where

$$\frac{\mathrm{D}}{\mathrm{D} y} = \frac{\partial}{\partial y} - q\frac{\partial}{\partial x}, \tag{8}$$

$$V_1 = V - \frac{2}{3}(r_2 - r_1)\frac{\mathrm{K}(m)}{\mathrm{K}(m) - \mathrm{E}(m)}, \tag{9}$$

$$V_2 = V - \frac{2}{3}(r_2 - r_1)\frac{(1-m)\mathrm{K}(m)}{\mathrm{E}(m) - (1-m)\mathrm{K}(m)}, \tag{10}$$

$$V_3 = V + \frac{2}{3}(r_2 - r_1)\frac{(1-m)\mathrm{K}(m)}{m\mathrm{E}(m)}, \tag{11}$$

$$v_1 = V + \frac{4}{m}(r_2 - r_1)\frac{(1+m)\mathrm{E}(m) - \mathrm{K}(m)}{\mathrm{K}(m) - \mathrm{E}(m)}, \tag{12}$$

$$v_2 = V + \frac{4}{m}(r_2 - r_1)\frac{(1-m)^2\mathrm{K}(m) - (1-2m)\mathrm{E}(m)}{\mathrm{E}(m) - (1-m)\mathrm{K}(m)}, \tag{13}$$

$$v_3 = V + \frac{4}{m}(r_2 - r_1)\frac{(2-m)\mathrm{E}(m) - (1-m)\mathrm{K}(m)}{\mathrm{E}(m)}, \tag{14}$$

$$v_4 = \frac{2m\mathrm{E}(m)}{\mathrm{E}(m) - (1-m)\mathrm{K}(m)}, \tag{15}$$

$$v_{4.1} = \frac{2}{3} - \frac{v_4}{6}, \quad v_{4.3} = \frac{1}{3} + \frac{v_4}{6}, \quad v_5 = r_1 - r_2 + r_3, \quad \alpha = \frac{\mathrm{E}(m)}{\mathrm{K}(m)}. \tag{16}$$

1.1 Soliton reduction

The KP-Whitham equations in the limit $r_2 \to r_3$, so that $m \to 1$ become [1]

$$\frac{\partial r_1}{\partial t} + (r_1 + \lambda q^2)\frac{\partial r_1}{\partial x} + 2\lambda q \frac{\mathrm{D} r_1}{\mathrm{D} y} + \lambda r_1 \frac{\mathrm{D} q}{\mathrm{D} y} + \lambda \frac{\mathrm{D} p}{\mathrm{D} y} = 0, \tag{17}$$

$$\frac{\partial r_3}{\partial t} + (\frac{1}{3}r_1 + \frac{2}{3}r_3 + \lambda q^2)\frac{\partial r_3}{\partial x} + 2\lambda q \frac{\mathrm{D} r_3}{\mathrm{D} y} + \lambda \frac{4r_3 - r_1}{3}\frac{\mathrm{D} q}{\mathrm{D} y} + \lambda \frac{\mathrm{D} p}{\mathrm{D} y} = 0, \tag{18}$$

$$\frac{\partial q}{\partial t} + (\frac{1}{3}r_1 + \frac{2}{3}r_3 + \lambda q^2)\frac{\partial q}{\partial x} + 2\lambda q \frac{\mathrm{D} q}{\mathrm{D} y} + \frac{1}{3}\frac{\mathrm{D} r_1}{\mathrm{D} y} + \frac{2}{3}\frac{\mathrm{D} r_3}{\mathrm{D} y} = 0, \tag{19}$$

$$\frac{\partial p}{\partial x} - \frac{\mathrm{D} r_1}{\mathrm{D} y} + r_1 \frac{\partial q}{\partial x} = 0. \tag{20}$$

The cnoidal wave (3) limits to the line soliton solution

$$u(x,y,t) = r_1 + 2(r_2 - r_1)\,\mathrm{sech}^2\left[\sqrt{\frac{r_3 - r_1}{6}}(x + qy - (V + \lambda q^2)t)\right]. \tag{21}$$

2 Soliton-mean flow equations

We consider the soliton reduction with the physical parameters $(\overline{u}, \overline{v}, q, a)$ where, in terms of the parameters $(r_1, r_2 = r_3, p, q)$ in Ablowitz et al, we have

$$\overline{u} = r_1, \tag{22}$$
$$\overline{v} = p + qr_1, \tag{23}$$
$$a = 2(r_3 - r_1). \tag{24}$$

Then the soliton solution (21) can be written in the form

$$u(x,y,t) = \overline{u} + a\,\mathrm{sech}^2\left[\sqrt{\frac{a}{12}}(x + qy - ct)\right], \quad c = \overline{u} + \frac{a}{3} + \lambda q^2. \tag{25}$$

$\overline{u}$ is the far field mean. a is the soliton amplitude. $q = \tan\varphi$ is the slope or angle of propagation.

The variable transformation in (22)-(24) inserted into equation (20) yields

$$\overline{v}_x - \overline{u}_y = 0. \tag{26}$$

The transformation and (26) transform eq. (17) into

$$\overline{u}_t + \overline{u}\overline{u}_x + \lambda\overline{v}_y = 0. \tag{27}$$

We immediately recognize eqs. (27) and (26) as the dispersionless KP equation. The transformation (22)-(24) and (26) take eq. (18) to the soliton amplitude equation

$$a_t + (\overline{u} + \frac{a}{3} - \lambda q^2)a_x + 2\lambda q a_y + \frac{2}{3}a\overline{u}_x + \frac{4}{3}\lambda a(q_y - qq_x) = 0. \tag{28}$$

Finally, equation (19) for the soliton propagation slope q is

$$q_t + (\overline{u} + \frac{a}{3} - \lambda q^2)q_x + 2\lambda q q_y + \overline{u}_y - q\overline{u}_x + \frac{1}{3}(a_y - qa_x) = 0. \tag{29}$$

Equations (26), (27), (28), and (29) constitute the soliton-mean flow modulation equations. The mean flow equations decouple from the soliton evolution.

2.1 Riemann invariant form for $\partial_y \to 0$

We now consider the soliton-mean flow modulation equations with no y dependence so that $\overline{v}$ is constant. Note that the soliton geometry, i.e., its slope q, still remains so the problem is two-dimensional. Equations (26), (27), (28), and (29) become

$$\begin{bmatrix} \overline{u} \\ a \\ q \end{bmatrix}_t + \begin{bmatrix} \overline{u} & 0 & 0 \\ \frac{2}{3}a & \overline{u} + \frac{a}{3} - \lambda q^2 & -\frac{4}{3}\lambda a q \\ -q & -\frac{1}{3}q & \overline{u} + \frac{a}{3} - \lambda q^2 \end{bmatrix} \begin{bmatrix} \overline{u} \\ a \\ q \end{bmatrix}_x = 0. \tag{30}$$

The eigenvalues of the coefficient matrix in (30) are [Note that $\lambda_\pm$ are named based on associated Riemann invariants]

$$\lambda_{\overline{u}} = \overline{u}, \quad \lambda_+ = \overline{u} + \frac{a}{3} - \lambda q^2 - \frac{2}{3}\sqrt{\lambda a q^2}, \quad \lambda_- = \overline{u} + \frac{a}{3} - \lambda q^2 + \frac{2}{3}\sqrt{\lambda a q^2}. \tag{31}$$

Since $a \geq 0$, we immediately obtain that KPI with $\lambda = -1$ yields complex characteristic velocities. This is a manifestation of the transverse instability of line solitons in KPI.

If $\lambda = 1$ (KPII), then the characteristic velocities are all real. In what follows, we will assume, without loss of generality that $q \geq 0$. Since $q(x,t)$ defines the shape of the line soliton, if q changes sign then it is necessarily a multivalued function of x.

The ordering of the characteristic velocities is determined by the relative magnitudes of a and q. In particular

$$q^2 = a/9 \quad \Rightarrow \lambda_{\overline{u}} = \lambda_+, \tag{32}$$
$$q = 0 \quad \Rightarrow \lambda_+ = \lambda_-, \tag{33}$$
$$q^2 = a \quad \Rightarrow \lambda_{\overline{u}} = \lambda_-, \tag{34}$$

i.e., eqs. (30) appear to be *non-strictly hyperbolic* for KPII **(NEEDS FURTHER COMPARISON WITH RIEMANN INVARIANT FORM)**. We therefore consider this case only in the what remains.

The left eigenvectors associated to each eigenvalue are

$$\mathbf{l}_{\overline{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{l}_+ = \begin{bmatrix} \frac{2\sqrt{a}}{\sqrt{a}+q} \\ 1 \\ 2\sqrt{a} \end{bmatrix}, \quad \mathbf{l}_- = \begin{bmatrix} \frac{2\sqrt{a}}{\sqrt{a}-q} \\ 1 \\ -2\sqrt{a} \end{bmatrix}. \tag{35}$$

We recognize $R_{\overline{u}} = \overline{u}$ as one Riemann invariant for the modulation system (30). Taking the dot product of eq. (30) with $\mathbf{l}_+$ yields the characteristic form

$$\mathrm{d}\overline{u} + \frac{\sqrt{a}+q}{2\sqrt{a}}\mathrm{d}a + (\sqrt{a}+q)\mathrm{d}q = 0. \tag{36}$$

This can be written in the form

$$\mathrm{d}\overline{u} + \frac{1}{2}\mathrm{d}\left(\sqrt{a}+q\right)^2 = 0, \tag{37}$$

which can be integrated to give another Riemann invariant

$$R_+ = \overline{u} + \frac{1}{2}(\sqrt{a}+q)^2. \tag{38}$$

The dot product of (30) with $\mathbf{l}_-$ gives the other Riemann invariant

$$R_- = \overline{u} + \frac{1}{2}(\sqrt{a}-q)^2. \tag{39}$$

Note that we have the ordering [Isn't this dependent on $\operatorname{sgn} q$?]

$$R_{\overline{u}} \leq R_- \leq R_+, \tag{40}$$

and

$$q^2 = a \quad \Longleftrightarrow \quad R_{\overline{u}} = R_- \tag{41}$$
$$q = 0 \quad \Longleftrightarrow \quad R_+ = R_-. \tag{42}$$

[Proposed correction to above, allowing for possibility of negative q:]

$$\sqrt{a} = q \quad \Longleftrightarrow \quad R_{\overline{u}} = R_- \tag{43}$$
$$\sqrt{a} = -q \quad \Longleftrightarrow \quad R_{\overline{u}} = R_+ \tag{44}$$
$$q = 0 \quad \Longleftrightarrow \quad R_+ = R_-. \tag{45}$$

The mapping from the Riemann invariants R_j, $j = 1, 2, 3$ to the modulation parameters $\overline{u}$, a, and q is multivalued:

$$\overline{u} = R_1, \tag{46}$$
$$a = -R_1 + \frac{1}{2}\left(R_2 + R_3 \pm 2\sqrt{(R_2 - R_1)(R_3 - R_1)}\right), \tag{47}$$
$$q^2 = -R_1 + \frac{1}{2}\left(R_2 + R_3 \mp 2\sqrt{(R_2 - R_1)(R_3 - R_1)}\right). \tag{48}$$

The branching of this multivalued function occurs precisely when

$$a = q^2 \quad \text{or} \quad R_1 = R_3. \tag{49}$$

WHAT IS SPECIAL ABOUT $a = q^2$?

The system (30) written in Riemann invariant form is

$$\frac{\partial R_j}{\partial t} + \lambda_j \frac{\partial R_j}{\partial x} = 0, \quad j = 1, 2, 3, \tag{50}$$

where

$$\lambda_{\overline{u}} = R_1, \tag{51}$$
$$\lambda_+ = \frac{5}{3}R_1 - \frac{2}{3}\left(R_2 + 2\sigma\sqrt{(R_2 - R_1)(R_3 - R_1)}\right), \tag{52}$$
$$\lambda_- = \frac{5}{3}R_1 - \frac{2}{3}\left(R_3 + 2\sigma\sqrt{(R_2 - R_1)(R_3 - R_1)}\right), \tag{53}$$

where

$$\sigma = \operatorname{sgn}(q^2 - a). \tag{54}$$

2.2 Riemann invariant form for $\overline{u} = 0$

When $\overline{u} = 0$ throughout, the soliton modulation equations (28) and (29) become

$$\begin{bmatrix} a \\ q \end{bmatrix}_t + \begin{bmatrix} \frac{1}{3}a - q^2 & -\frac{4}{3}aq \\ -\frac{1}{3}q & \frac{1}{3}a - q^2 \end{bmatrix} \begin{bmatrix} a \\ q \end{bmatrix}_x + \begin{bmatrix} 2q & \frac{4}{3}a \\ \frac{1}{3} & 2q \end{bmatrix} \begin{bmatrix} a \\ q \end{bmatrix}_y = 0. \tag{55}$$

The matrices for the x and y-dependence above have the same eigenvectors. Then denoting the characteristic speeds for the x-matrix $U_\pm$ and those for the y-matrix $V_\pm$, we have

$$\begin{aligned} U_\pm &= \tfrac{1}{3}a - q^2 \mp \tfrac{2}{3}q\sqrt{a}, & V_\pm &= 2q \pm \tfrac{2}{3}\sqrt{a}, \\ \vec{l}_\pm &= \begin{bmatrix} \pm\frac{1}{2\sqrt{a}} \\ 1 \end{bmatrix} & \vec{r}_\pm &= \begin{bmatrix} \pm 2\sqrt{a} \\ 1 \end{bmatrix} \end{aligned} \tag{56}$$

Applying the left eigenvectors to the system (55) yields

$$\mathrm{d}\sqrt{a} \pm \mathrm{d}q = 0, \tag{57}$$

so that the corresponding Riemann invariants are

$$R_\pm = \sqrt{a} \pm q. \tag{58}$$

Note the conversion back to physical parameters is easily found

$$a = ((R_+ + R_-)/2)^2 \quad q = (R_+ - R_-)/2 \tag{59}$$

Then, equations (55) in Riemann invariant form are

$$\frac{\partial R_\pm}{\partial t} + U_\pm \frac{\partial R_\pm}{\partial x} + V_\pm \frac{\partial R_\pm}{\partial x} = 0, \tag{60}$$

where

$$U_\pm = \frac{2}{3}R_\pm R_\mp - \frac{1}{3}R_\pm^2, \quad V_\pm = \pm\frac{2}{3}R_\pm \mp \frac{4}{3}R_\mp \tag{61}$$

Then $U_+ = U_-$ if and only if $R_+ = R_-$ if and only if $q = 0$, representing a reduction of the system of equations to the Hopf equation $a_t + \frac{1}{3}aa_x = 0$ so that the equations are strictly hyperbolic. Note that $V_+ = V_-$ would require $\sqrt{a} \leq 0$ and is thus unphysical. [Except we use $a = 0$ in our vertical half-soliton problem, so there might be an issue here?]

2.3 Leveraging Invariances

The KPII equation (1) admits the following symmetries [1] [Changed their a's to b's for later readability, verified with Mathematica]

$$\begin{aligned} u(x,y,t) &\mapsto u(x-x_0, y-y_0, t-t_0) && \text{(space/time translations)} \\ u(x,y,t) &\mapsto b + u(x-bt, y, t) && \text{(Galilean)} \\ u(x,y,t) &\mapsto b^2 u(bx, b^2 y, b^3 t) && \text{(scaling)} \\ u(x,y,t) &\mapsto u(x+by-b^2t, y-2bt, t) && \text{(pseudo-rotations)} \end{aligned}$$

Each of these symmetries generates a corresponding symmetry for the KP-Whitham system [eqref]. For the system written as $(\overline{u}, \overline{v}, a, q)$, the space/time invariance is trivial. The other corresponding transformations can be derived as follows: [Need to complete MMA verification]

Galilean transformations:

$$\begin{aligned}
\overline{u}(x,y,t) &\mapsto b + \overline{u}(x-bt,y,t) \\
\overline{v}(x,y,t) &\mapsto \overline{v}(x-bt,y,t) \\
a(x,y,t) &\mapsto b + a(x-bt,y,t) \\
q(x,y,t) &\mapsto q(x-bt,y,t)
\end{aligned}$$

scaling transformations:

$$\begin{aligned}
\overline{u}(x,y,t) &\mapsto b^2\overline{u}(bx,b^2y,b^3t) \\
\overline{v}(x,y,t) &\mapsto b^3\overline{v}(bx,b^2y,b^3t) \\
a(x,y,t) &\mapsto b^2a(bx,b^2y,b^3t) \\
q(x,y,t) &\mapsto bq(bx,b^2y,b^3t)
\end{aligned}$$

pseudo-rotations

$$\begin{aligned}
\overline{u}(x,y,t) &\mapsto \overline{u}(x+by-b^2t,y-2bt,t) \\
\overline{v}(x,y,t) &\mapsto \overline{v}(x+by-b^2t,y-2bt,t) + b\overline{u}(x+by-b^2t,y-2bt,t) \\
a(x,y,t) &\mapsto a(x+by-b^2t,y-2bt,t) \\
q(x,y,t) &\mapsto b + q(x+by-b^2t,y-2bt,t)
\end{aligned}$$

2.4 Riemann Problem for $\overline{u} = 0$

Consider the full Riemann problem for the line soliton described above, with a jump in a and q over a line of slope m:

$$a(x,y,0) = \begin{cases} \tilde{a}_1, & y < mx \\ \tilde{a}_2, & y > mx \end{cases}, \quad q(x,y,0) = \begin{cases} \tilde{q}_1, & y < mx \\ \tilde{q}_2, & y > mx \end{cases}. \tag{62}$$

The angle at which the jump occurs does not affect the overall solution [Insert Mark's notes on this.] Thus the problem can be rescaled via Sec. 2.3 to [Can also rescale $a_2 = 1$ and $q_2 = 0$, but I think this obscures our results.]

$$a(y,0) = \begin{cases} a_1, & y < 0 \\ a_2, & y > 0 \end{cases}, \quad q(y,0) = \begin{cases} q_1, & y < 0 \\ q_2, & y > 0 \end{cases}. \tag{63}$$

We can set $q_1 \geq 0$, as the solution for $q_1 < 0$ can be found by taking $y \mapsto -y$. Then we seek a regime where the initial condition (63) evolves into the two constant regions separated by two rarefaction waves and a middle constant state (a_0, q_0), with corresponding characteristic speeds $V_{\pm,0}$ and Riemann invariants $R_{\pm,0}$.

Since $q_1 \geq 0$, $R_{-,1} \leq R_{+,1}$, so we expect the $(-)$-wave to develop below the middle state and the $(+)$-wave to develop above the middle state. Thus we can determine the middle state (a_0, q_0) by equating the Riemann invariants

$$\begin{array}{rr} R_{-,0} = R_{-,2}, & R_{+,0} = R_{+,1}. \\ \sqrt{a_0} - q_0 = \sqrt{a_2} - q_2, & \sqrt{a_0} + q_0 = \sqrt{a_1} + q_1, \\ \sqrt{a_0} = \left(\sqrt{a_2} + \sqrt{a_1}\right)/2 + (q_1 - q_2)/2, & q_0 = (q_1 + q_2)/2 + \left(\sqrt{a_1} - \sqrt{a_2}\right)/2. \end{array} \tag{64}$$

Note if either side matches the middle state, then only one of the two waves will manifest. For example, if $(a_0, q_0) = (a_1, q_1)$, then only the $(+)$-wave will manifest.

We expect the solution to be of the form

$$R_-(y,t) = \begin{cases} R_{-,1}, & y \leq W_1 t \\ f(y/t), & W_1 t < y < W_2 t \\ R_{-,2}, & y \geq W_2 t \end{cases} \quad R_+(y,t) = \begin{cases} R_{+,1}, & y \leq W_3 t \\ g(y/t), & W_3 t < y < W_4 t \\ R_{+,2}, & y \geq W_4 t \end{cases}, \tag{65}$$

where the speeds are

$$\begin{array}{ll} W_1 = V_-\left(R_{-,1}, R_{+,1}\right) & W_2 = V_-\left(R_{-,2}, R_{+,1}\right) \\ W_3 = V_+\left(R_{-,2}, R_{+,1}\right) & W_4 = V_+\left(R_{-,2}, R_{+,2}\right) \end{array} \tag{66}$$

Then we can find the conditions that give rise to rarefaction waves by checking for monotonicity of $R_\pm$ and by restricting to $W_1 \leq W_2$ and $W_3 \leq W_4$. The monotonicity requirement is

$$\begin{aligned} \operatorname{sgn} V'_\pm(R_\pm(y/t), \overline{R_\mp}) &= \operatorname{sgn} R_\pm(y/t) \\ \operatorname{sgn}(\pm 2/3) &= \operatorname{sgn}\left(R_{\pm,2} - R_{\pm,1}\right) \\ \sqrt{a_2} - \sqrt{a_1} \pm (q_2 - q_1) &\gtrless 0 \end{aligned}$$

Thus $(-)$-rarefaction waves occur when

$$\sqrt{a_2} - \sqrt{a_1} < q_2 - q_1 \tag{67}$$

and $(+)$-rarefaction waves occur when

$$\sqrt{a_2} - \sqrt{a_1} > -(q_2 - q_1) \tag{68}$$

thus the conditions under which solely rarefaction waves occur are

$$|a_2 - a_1| \leq q_2 - q_1. \tag{69}$$

Outside of these regions, we assume shock formation. Figure 1 shows these results in parameter space. All that remains is to determine f and g. We seek a simple wave solution in the form $R_\mp = const = \overline{R_\mp}$ and $V_\pm = y/t$, or

$$\pm\frac{2}{3}R_\pm \mp \frac{4}{3}\overline{R_\mp} = y/t. \tag{70}$$

Then we can solve to find

$$f(y/t) = \tfrac{3}{2}\left(\tfrac{4}{3}\overline{R_-} + y/t\right), \quad g(y/t) = \tfrac{3}{2}\left(\tfrac{4}{3}\overline{R_+} - y/t\right) \tag{71}$$

And the solution can be converted back to physical parameters via Eq. (59).

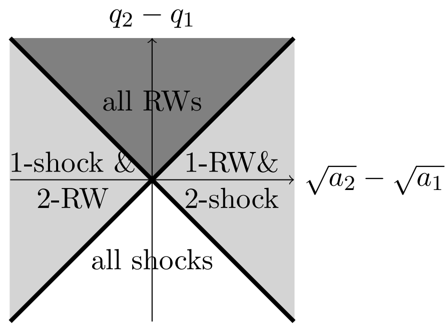

Figure 1: Allowable region of parameter space for all-rarefaction wave solutions for the y-variation case.

2.5 Riemann invariant form for $\overline{u} \neq 0$

TO DO

References

[1] Mark J Ablowitz, Gino Biondini, and Qiao Wang. Whitham modulation theory for the Kadomtsev–Petviashvili equation. *Proc. R. Soc. A*, 473(2204):20160695, 2017.