Discrete Hydrodynamics

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1 High-Level Goal

Identify the role discretization plays in nonlinear hydrodynamic-like behavior

2 Sub-Project 1: Classification of the Riemann problem for the Toda lattice

Nondimensionalized equation

$$\partial_t^2 \tilde{x}_n = \exp\left(\tilde{x}_{n-1} - \tilde{x}_n\right) - \exp\left(\tilde{x}_n - \tilde{x}_{n+1}\right), \quad n \in \mathbb{Z} \tag{1}$$

At equilibrium,

$$x_n(0) = cn, \quad \partial_t x_n(0) = 0 \tag{2}$$

2.1 Literature Review

- Gerald Teschl's Overview of the Toda Lattice equations, with emphasis on integrability, soliton solutions Paper
- Original Toda paper
- Original IST treatment by Teschl
  G. Teschl, Inverse scattering transform for the Toda hierarchy, Math. Nach. 202, 163–171 (1999).

2.2 [Jessica] Numerical Code with appropriate boundary conditions

2.3 [Michelle & Nevil] Determine/Identify the Riemann Problem for this system

2.4 [Michelle] Symmetries, conservation laws

2.4.1 Conservation laws

The following is from Consider the tranformation of the Toda equation used in its IST formulation

$$\begin{aligned} \dot{a}(t) &= a(t)\left(b^{+}(t) - b(t)\right) \\ \dot{b}(t) &= 2\left(a(t)^2 - a^{-}(t)^2\right), \end{aligned} \tag{3}$$

where

$$f^{\pm}(n) = f(n \pm 1) \tag{4}$$

Then the Lax pair for Eq. (3) is

$$\begin{aligned} H(t) &= a(t)S^{+} + a^{-}(t)S^{-} + b(t) \\ P(t) &= a(t)S^{+} - a^{-}(t)S^{-}, \end{aligned} \tag{5}$$

where $S^{\pm}$ are the shift operators. Then an infininte number of conseervation laws can be found by taking traces of $H(t)^j$, i.e.

$$\operatorname{tr}\left(H(t)^j - H_0^j\right), \quad j \in \mathbb{N}, \tag{6}$$

are conserved quantities, where H_0 is the operator corresponding to the constant solution $(a_0, b_0) = (1/2, 0)$. The equations for conservation of the total momentum and the total energy are

$$\begin{aligned} \operatorname{tr}\left(H(t) - H_0\right) &= \sum_{n\in\mathbb{Z}} b(n,t) = -\frac{1}{2}\sum_{n\in\mathbb{Z}} p(n,t), \\ \operatorname{tr}\left(H(t)^2 - H_0^2\right) &= \sum_{n\in\mathbb{Z}} b(n,t)^2 + 2\left(a(n,t)^2 - \frac{1}{4}\right) = \frac{1}{2}\mathcal{H}(p,q), \end{aligned} \tag{7}$$