SIAM J. APPL. MATH.
Vol. 52, No. 4, pp. 909–928, August 1992

001

DISPERSIVE REGULARIZATION OF THE WHITHAM EQUATION FOR THE TODA LATTICE*

A. M. BLOCH†‡ AND Y. KODAMA†§

Abstract. In this paper the averaged (Whitham) equations for the slow modulations of multiphase wavetrains for the Toda lattice are studied. Our main result is showing how the solution with step initial conditions may be regularized by choosing an appropriate Riemann surface on which the Whitham equation is defined.

Key words. Toda lattice, Whitham equation, shocks, regularization

AMS(MOS) subject classifications. 34A, 35L, 70F

1. Introduction. The Toda lattice system on the line, a set of particles interacting under an exponential potential, is of great interest from a mathematical and physical point of view, and there has been much research on various aspects of this problem. In this paper, we study the asymptotic behavior of the infinite Toda lattice subject to a step-like initial condition by analyzing the Whitham (averaged) equations for this system. This enables us to characterize the essential structure of the solution even in the presence of shocks that correspond to the slow modulations of multiphase wavetrains for the Toda lattice.

In [6] the shock waves arising from such an initial condition were studied from a numerical point of view. It was observed that different behavior corresponding to different magnitudes of the step input occurred. These observations are borne out by our analysis. Similar results have recently been obtained independently by Venakides, Deift, and Oba [17].

Our approach is adopted from the study of the weak dispersion limit of the Korteweg–de Vries (KdV) equation. This problem has been studied in detail over the past two decades by using modulation theory (see [4] and the earlier work of Whitham [18]) and by using asymptotics and inverse scattering (see [12], [16], and [5]). These approaches link the averaged dynamics of g-phase wavetrains with the evolution of Abelian differentials on a Riemann surface of genus g.

In [2] and [7] the classical zero dispersion limit of Toda was studied (corresponding to a Riemann surface of genus 0). In [2] it was shown how shocks could develop in finite time from smooth initial data for this system.

Here we consider a step-like initial condition and analyze the Whitham equations of genus g and their Riemann invariants following the work of Levermore [13]. We show that regularity of the solutions fixes the minimal genus. The predicted qualitative behavior of the solutions agrees with the numerical results of [6]. Our approach is different from the recent work of Venakides, Deift, and Oba [17], who remarkably have given an explicit asymptotic solution of the Toda equations subject to step-like

*Received by the editors September 24, 1990; accepted for publication (in revised form) August 5, 1991.

†Department of Mathematics, The Ohio State University, Columbus, Ohio 43210.

‡This author's research was supported in part by grants DMS-9002136 and DMS-9157556 from National Science Foundation and by a Seed grant from The Ohio State University.

§This author's research was supported in part by National Science Foundation grant DMS-9109041.

initial conditions. This then fixes the genus of the solution.

We would like to emphasize that the Whitham equations only describe the asymptotic behavior of the system as it moves away from the shock point. Immediately around the shock point, the averaging process may break down, and, to analyze the behavior there, we must study the original Toda lattice. This analysis can be found in [17]. However, the Whitham equation analysis gives a clear geometric picture of the macroscopic behavior of the Toda lattice (away from the shock point) and does not involve the intricate analysis needed to study the microscopic behavior in the neighborhood of the shock.

The outline of the paper is as follows. In §2 we write the conservation laws for the Toda lattice and hence derive the Whitham equations from averaging. In §3 we discuss the hyperbolic structure of the Whitham equations. In §4 we discuss regularization of the Whitham equations for different step inputs. Finally, in §5 we summarize the results and explain their physical significance.

2. The averaged (Whitham) equations for the Toda lattice. To obtain the Whitham equations that are given by the averaged conservation laws, we first briefly summarize the isospectral theory of the Toda lattice equation and derive the conservation laws.

2.1. The Lax pair and the conservation laws for the Toda lattices. Recall that, in (essentially) Flaschka's form [3], [14], the Toda lattice equations may be written as

$$\dot{a}_k = b_{k+1} - b_k, \quad \dot{b}_k = b_k(a_k - a_{k-1}), \quad k \in \mathbb{Z}. \tag{2.1}$$

This asymmetrical form of the Toda lattice equations (see [11]) is more convenient for our purposes than the original symmetrical form of Flaschka.

Equations (2.1) can be derived from the compatibility conditions for the spectral equations

$$L\psi = \lambda\psi, \qquad \psi_t = B\psi, \tag{2.2}$$

where the operators L and B, called the Lax pair, are given by

$$L = \Delta + u_0 + u_1\Delta^{-1}, \qquad B = \Delta + u_0. \tag{2.3}$$

Here Δ is the unit shift operator $\exp(\partial/\partial k)$, i.e., $\Delta f(k,t) = f(k+1,t)$, $f : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$, and $u_0(k,t) = a_k(t)$, $u_1(k,t) = b_k(t)$. In terms of these variables, (2.1) becomes

$$\frac{\partial u_0}{\partial t} = (\Delta - 1)u_1, \qquad \frac{\partial u_1}{\partial t} = u_1(1 - \Delta^{-1})u_0. \tag{2.4}$$

The full Toda hierachy is easily obtained in analogous fashion (see [7]).

Note that $B = (L)_+ = L - (L)_-$, where $(\ )_+$ denotes the polynomial part in Δ of L. Let Δ and $(L)_-$ be formal series in L

$$\Delta = L - \sum_{i=0}^{\infty} G_i L^{-i}, \tag{2.5}$$

$$(L)_- = \sum_{i=1}^{\infty} F_i L^{-i}, \tag{2.6}$$

where $F_i = F_i(u_0, u_1)$ and $G_i = G_i(u_0, u_1)$. Then we have the following result.

LEMMA 2.1 (see [11]). *The conservation laws for system* (2.4) *are given by*

$$\begin{aligned}\frac{\partial}{\partial t}\ln\left(\frac{\Delta\psi}{\psi}\right) &= \frac{\partial}{\partial t}\ln(\lambda - \sum_{i=0}^{\infty} G_i\lambda^{-i}) \\ &= (\Delta - 1)\left(\frac{\psi_t}{\psi}\right) = (\Delta - 1)(\lambda - \sum_{n=1}^{\infty} F_i\lambda^{-i}).\end{aligned} \tag{2.7}$$

Proof. It is a calculation that

$$\frac{\partial}{\partial t}\ln\left(\frac{\Delta\psi}{\psi}\right) = \frac{\partial}{\partial t}(\Delta\ln\psi - \ln\psi) = (\Delta - 1)\frac{\psi_t}{\psi}. \tag{2.8}$$

Now, from (2.5),

$$\begin{aligned}\Delta\psi &= (L - \Sigma G_i L^{-i})\psi = (\lambda - \Sigma G_i\lambda^{-i})\psi, \\ \frac{\partial\psi}{\partial t} &= B\psi = (L - (L)_-)\psi = (\lambda - \Sigma F_i\lambda^{-i})\psi.\end{aligned}$$

□

It remains to calculate the coefficients G_i and F_i in terms of u_0 and u_1. This can be done as follows. From (2.3) and (2.5), we have that

$$\begin{aligned}\Delta =& L - G_0 - G_1 L^{-1} - G_2 L^{-2}\cdots \\ =& \Delta + u_0 + u_1\Delta^{-1} - G_0 - G_1(\Delta + u_0 + u_1\Delta^{-1})^{-1} \\ & - G_2(\Delta + u_0 + u_1\Delta^{-1})^{-2}\cdots \\ =& \Delta + (u_0 - G_0) + (u_1 - G_1)\Delta^{-1} + (G_1 u_0^{(-1)} - G_2)\Delta^{-2} \\ & + (-G_1(u_0^{(-1)}u_0^{(-2)} - u_1^{(-1)}) + G_2(u_0^{(-1)} + u_0^{(-2)}) - G_3)\Delta^{-3}\cdots,\end{aligned}$$

where $u^{(-k)} = \Delta^{-k}u$. Hence equating coefficients of the powers of Δ gives

$$\begin{aligned}G_0 &= u_0, \\ G_1 &= u_1, \\ G_2 &= u_1 u_0^{(-1)}, \\ G_3 &= u_1(u_0^{(-1)})^2 + u_1 u_1^{(-1)}, \\ &\vdots\end{aligned} \tag{2.9}$$

Now, from (2.3) and (2.6), we have that

$$(L)_- = u_1\Delta^{-1} = \Sigma F_i L^{-i}.$$

Inverting Δ^{-1} in this equation gives

$$\begin{aligned}F_1 &= u_1, \\ F_2 &= u_1 u_0^{(-1)}, \\ F_3 &= u_1 u_1^{(-1)} + u_1(u_0^{(-2)})^2, \\ &\vdots\end{aligned} \tag{2.10}$$

The conserved density equation (2.7) may thus be written as

$$\begin{aligned}\ln\left(\frac{\Delta\psi}{\psi}\right) &= \ln(\lambda - \Sigma G_i\lambda^{-i}) \\ &= \ln\lambda - \frac{1}{\lambda}\left(G_0 + \frac{G_1}{\lambda} + \frac{G_2}{\lambda^2}\cdots\right) - \frac{1}{2\lambda^2}\left(G_0 + \frac{G_1}{\lambda} + \frac{G_2}{\lambda^2}\cdots\right)^2\cdots \\ &= \ln\lambda - \frac{G_0}{\lambda} - \frac{1}{\lambda^2}\left(G_1 + \frac{G_0^2}{2}\right) - \frac{1}{\lambda^3}(G_2 + G_0G_1 + \frac{1}{3}G_0^3)\cdots .\end{aligned}$$

Thus, setting $\ln(\Delta\psi/\psi) = \ln\lambda - P_1/\lambda - P_2/\lambda_2\cdots$, we obtain the conserved densities for the Toda lattice equation (2.4)

$$\begin{aligned}P_1 &= u_0\,, \\ P_2 &= u_1 + \tfrac{1}{2}u_0^2\,, \\ P_3 &= u_1u_0^{(-1)} + u_0u_1 + \tfrac{1}{3}u_0^3\,, \\ &\ \vdots\end{aligned} \tag{2.11}$$

and the conservation laws are given by

$$\frac{\partial P_i}{\partial t} = (\Delta - 1)F_i. \tag{2.12}$$

2.2. The Whitham equations. We recall briefly here the averaging procedure. The solutions of the Toda equations are assumed to be slowly modulating g-phase wavetrains. Since the oscillations of the wavetrain are fast compared to the modulations, the idea is to average out these fast oscillations. The procedure for the $g = 1$ case of the KdV equation is given in [18], and the general case is analyzed in [4].

To obtain the Whitham equations, we introduce two timescales and then average the flux and density over the fast variables in the conservation laws.

Suppose that F is a flux or density. We denote by $\langle F\rangle$ the spatial average over the fast variables of F (keeping the slow variables fixed). In the continuous (e.g., KdV) case,

$$\langle F\rangle := \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} F(x,t)dx. \tag{2.13}$$

For the discrete Toda system,

$$\langle F\rangle := \lim_{N\to\infty}\frac{1}{2N}\sum_{n=-N}^{N} F(n,t). \tag{2.14}$$

Since these systems are integrable, there exist infinitely many commuting flows. For the case of g-phase wavetrains, these flows may be viewed as translations on the g-torus, and, generically, each flow will cover the surface of the torus ergodically (if the characteristic frequencies are incommensurate).

Hence we can replace the spatial average by an integral over the g-torus as follows:

$$\langle F\rangle = \frac{1}{(2\pi)^g}\int_0^{2\pi}\cdots\int_0^{2\pi} F(\theta_1,\cdots,\theta_g)d\theta_1\cdots d\theta_g. \tag{2.15}$$

The integral in the θ_i variables may then be replaced by the integral over the cycles in an associated Riemann surface, and we can thus show that the averaged quantities are Abelian differentials on a Riemann surface.

Now, to obtain the averaged (Whitham) equations, we break up the dynamics into slow and fast scales. Let t_0 and n_0 denote the fast time and spatial variables and let $T=\epsilon t$ and $X=\epsilon n$ be the slow variables.

Then the shift operator Δ and the time derivative in (2.4) may be given, respectively, by

$$\Delta = e^{\partial/\partial n} = e^{\partial/\partial n_0+\varepsilon(\partial/\partial X)} \sim e^{\partial/\partial n_0}\left(1+\varepsilon\frac{\partial}{\partial X}\right) \tag{2.16}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial T}. \tag{2.17}$$

Substituting into (2.12) and averaging over the fast variables, from the terms of order ε, we obtain the Whitham equations

$$\frac{\partial}{\partial T}\langle P_i\rangle = \frac{\partial}{\partial X}\langle F_i\rangle. \tag{2.18}$$

We now wish to write the Whitham equations in terms of the Abelian differentials on a Riemann surface of genus g (see [4] for the KdV equation and [10] for the KP equation).

Let $y^2=R_g(\lambda)$ be the Riemann surface of genus g

$$R_g(\lambda) = \prod_{i=1}^{2g+2}(\lambda-\lambda_i), \tag{2.19}$$

where the branch points λ_i are real and are assumed to satisfy $\lambda_1<\lambda_2<\cdots<\lambda_{2g+2}$.

Krichever [9] showed that a solution of (2.2) was given by the Baker–Akhiezer function

$$\psi(n,t;\lambda) = r_0\exp(it\int_{\lambda_0}^{\lambda}\omega_1 + in\int_{\lambda_0}^{\lambda}\omega_0)F(\theta_1,\cdots,\theta_g), \tag{2.20}$$

where ω_0 and ω_1 are the normalized Abelian differentials of the third and second kinds, respectively, with singularity at infinity

$$\begin{aligned}\omega_0(\lambda) &= \frac{\prod_{i=1}^{g}(\lambda-\alpha_i)}{\sqrt{R_g(\lambda)}}d\lambda,\\ &\sim \frac{d\lambda}{\lambda}+O\left(\frac{d\lambda}{\lambda^2}\right) \quad \text{as } \lambda\to\infty,\end{aligned} \tag{2.21}$$

$$\begin{aligned}\omega_1(\lambda) &= \frac{1}{2}d\lambda + \frac{\lambda^{g+1}-\frac{1}{2}\sigma_1\lambda^g+\gamma_1\lambda^{g-1}+\cdots+\gamma_g}{2\sqrt{R_g(\lambda)}}d\lambda,\\ &\sim d\lambda+O\left(\frac{d\lambda}{\lambda^2}\right) \quad \text{as } \lambda\to\infty,\end{aligned} \tag{2.22}$$

with the normalization

$$\oint_{a_j} \omega_0 = \oint_{a_j} \omega_1 = 0, \qquad j = 1, \cdots, g, \tag{2.23}$$

the contour integral along the a_j-cycle, the canonical cycle over the region $[\lambda_{2j}, \lambda_{2j+1}]$. The coefficients α_i and γ_i in (2.21) and (2.22) are uniquely determined by (2.23). Here the coefficient σ_1 in (2.22), and, similarly, σ_2 and σ_3, to be used later, are defined, respectively, as

$$\sigma_1 = \sum_{i=1}^{2g+2} \lambda_i, \quad \sigma_2 = \sum_{i<j} \lambda_i \lambda_j, \quad \sigma_3 = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k. \tag{2.24}$$

$F(\theta_1, \cdots, \theta_g)$ is a quasi-periodic function given by the Riemann theta function on the Riemann surface. Note that, in fact, (2.20) is the solution to the symmetric form of (2.2), but all the following calculations are the same for either form.

Hence the differential of the log of (2.20) on the Riemann surface is given by

$$d \ln \psi = it\omega_1 + in\,\omega_0 + \frac{1}{F}\frac{\partial F}{\partial \lambda} d\lambda, \tag{2.25}$$

from which we have that

$$\omega_0 = -i\frac{\partial}{\partial n}(d \ln \psi) + i\frac{\partial}{\partial n}\left(\frac{dF}{F}\right), \tag{2.26}$$

$$\omega_1 = -i\frac{\partial}{\partial t}(d \ln \psi) + i\frac{\partial}{\partial t}\left(\frac{dF}{F}\right). \tag{2.27}$$

Now, averaging, using the prescription given earlier, gives

$$\omega_0 = -i\frac{\partial}{\partial X}\langle d \ln \psi \rangle, \tag{2.28}$$

$$\omega_1 = -i\frac{\partial}{\partial T}\langle d \ln \psi \rangle, \tag{2.29}$$

and hence the Whitham equation may be written as the integrability condition of (2.28) and (2.29) as follows:

$$\frac{\partial \omega_0}{\partial T} = \frac{\partial \omega_1}{\partial X}. \tag{2.30}$$

Note that (2.30) is an averaged form of the conservative law (2.7).

Thus, expanding (2.30) for λ large, we have, from our previous expressions for the averaged quantities, the following theorem.

THEOREM 2.2. *The averaged quantities $\langle P_i \rangle$ and $\langle F_i \rangle$ may be characterized in terms of the Riemann surface by*

$$\omega_0(\lambda) = \frac{d\lambda}{\lambda}\left(1 + \frac{\langle P_1 \rangle}{\lambda} + \frac{2\langle P_2 \rangle}{\lambda^2} + \cdots\right), \tag{2.31}$$

$$\omega_1(\lambda) = d\lambda\left(1 + \frac{\langle F_1 \rangle}{\lambda^2} + \frac{2\langle F_2 \rangle}{\lambda^3} + \cdots\right), \tag{2.32}$$

where

$$\langle P_1\rangle = \langle u_0\rangle = \tfrac{1}{2}\sigma_1 - \mu_1\,,$$
$$\langle P_2\rangle = \langle u_1 + \tfrac{1}{2}u_0^2\rangle = \tfrac{3}{16}\sigma_1^2 - \tfrac{1}{4}\sigma_2 - \tfrac{1}{4}\sigma_1\mu_1 + \tfrac{1}{2}\mu_2,$$
$$\vdots \tag{2.33}$$
$$\langle F_1\rangle = \langle u_1\rangle = \tfrac{1}{16}\sigma_1^2 - \tfrac{1}{4}\sigma_2 + \tfrac{1}{2}\gamma_1,$$
$$\langle F_2\rangle = \langle u_0u_1\rangle = \tfrac{1}{16}\sigma_1^3 - \tfrac{1}{4}\sigma_1\sigma_2 + \tfrac{1}{4}\sigma_3 + \tfrac{1}{4}\sigma_1\gamma_1 + \tfrac{1}{4}\gamma_2\,,$$
$$\vdots \tag{2.34}$$

Here σ_1, σ_2, and σ_3 are defined in (2.24), *and μ_1 and μ_2 are given by*

$$\mu_1 = \sum_{k=1}^{g}\alpha_k \quad \textit{and} \quad \mu_2 = \sum_{k<\ell}\alpha_k\alpha_\ell. \tag{2.35}$$

3. Hyperbolic structure of the Whitham equations. Bikbaev and Novokshenov [1] and Levermore [13] showed that the rich structure of the KdV equation enabled us to analyze in detail the behavior of the solution of the Whitham equation. We can adapt their analysis to the Toda equations. We first note that the $2g+2$ branch points of the Riemann surface $y^2 = R_g$ of (2.19), $\lambda_1, \lambda_2, \cdots, \lambda_{2g+2}$, give the Riemann invariants for the Whitham equation. We can see this as follows.

Define

$$s(\lambda) = \frac{\lambda^{g+1} - \frac{1}{2}\sigma_1\lambda^g + \gamma_1\lambda^{g-1} + \cdots + \gamma_g}{2\prod_{i=1}^{g}(\lambda-\alpha_i)}, \tag{3.1}$$

where the coefficients are defined by (2.21) and(2.22), i.e., $s(\lambda) = \tilde{\omega}_1(\lambda)/\omega_0(\lambda)$, where

$$\tilde{\omega}_1(\lambda) = \frac{\lambda^{g+1} - \frac{1}{2}\sigma_1\lambda^g + \gamma_1\lambda^{g-1} + \cdots + \gamma_g}{2\sqrt{R_g(\lambda)}}. \tag{3.2}$$

Then we have the following result.

LEMMA 3.1. *The branch points satisfy the Riemann invariant equation*

$$\frac{\partial\lambda_k}{\partial T} = s(\lambda_k)\frac{\partial\lambda_k}{\partial X}, \tag{3.3}$$

where the characteristic velocity $s(\lambda_k)$ is given by (3.1) *at $\lambda = \lambda_k$,*

$$s(\lambda_k) = \frac{\tilde{\omega}_1(\lambda_k)}{\omega_0(\lambda_k)}. \tag{3.4}$$

Proof. Following [4], from (2.21) and (2.22), we see that

$$\frac{\partial\omega_0}{\partial T} = -\sum_{i=1}^{g}\frac{\omega_0}{\lambda-\alpha_i}\frac{\partial\alpha_i}{\partial T} + \sum_{k=1}^{2g+2}\frac{\omega_0}{2(\lambda-\lambda_k)}\frac{\partial\lambda_k}{\partial T}\,,$$
$$\frac{\partial\omega_1}{\partial X} = \sum_{k=1}^{2g+2}\frac{\widetilde{\omega}_1}{2(\lambda-\lambda_k)}\frac{\partial\lambda_k}{\partial X} + \frac{d\lambda}{2\sqrt{R_g}}\left(-\frac{1}{2}\frac{\partial\sigma_1}{\partial X}\lambda^g + \frac{\partial\gamma_1}{\partial X}\lambda^{g-1} + \cdots + \frac{\partial\gamma_g}{\partial X}\right).$$

Evaluating the residue at $\lambda = \lambda_k$ in (2.30) gives the result. $\square$

We also have the following lemma.

LEMMA 3.2. *The characteristic velocities satisfy*

$$s(\lambda_j) > s(\lambda_k), \qquad \text{for } \lambda_j > \lambda_k, \tag{3.5}$$

and

$$\frac{\partial s(\lambda_k)}{\partial \lambda_k} > 0, \quad \forall k. \tag{3.6}$$

Proof. The proof is as in [1] and [13] once we observe that

(i) $s(\lambda) \to \lambda + O(1/\lambda)$ as $\lambda \to \infty$;

(ii) $\lambda_{2i} < \alpha_i < \lambda_{2i+1}$;

(iii) $\lambda^{g+1} - \frac{1}{2}\sigma_1\lambda^g + \cdots + \gamma_g$ has at least one zero in each gap $[\lambda_{2k}, \lambda_{2k+1}]$, $k = 1, \cdots g$.

Here (ii) and (iii) follow from the normalization conditions for ω_0 and ω_1. Conditions (i)–(iii) then give the result from observing the sign of rational function $s(\lambda)$. Note that the sign of $s(\lambda)$ is opposite that of Levermore. $\square$

The following is an immediate corollary of Lemma 3.2.

COROLLARY 3.3. *For monotonically decreasing smooth initial data, the λ_k will not develop shocks.*

Of course, we have discontinuous initial data, but the result still holds. We prove existence and nonexistence of shocks by explicit computation of velocities using Corollary 3.3 as a guide.

4. Regularization of shock waves. In this section we show how to regularize shock waves by going to a higher-genus solution.

We consider a canonical example, where the initial data is

$$\begin{aligned} u_0(X,0) &= \begin{cases} -2a, & X < 0, \\ 0, & X = 0, \\ 2a, & X > 0, \end{cases} \\ u_1(X,0) &= 1, \ \ \forall X, \end{aligned} \tag{4.1}$$

which was suggested originally by Holian, Flaschka, and McLaughlin [6]. In terms of the original Toda lattice variables, the position of the nth particle, this corresponds to taking the velocity of the nth particle to be a when $n < 0$, $-a$ when $n > 0$, and 0 when $n = 0$, while the displacement of each particle is initially zero.

The basic idea is then to show, by studying the behavior of the Riemann invariants, that the $g = 0$ solution may develop a shock, while the higher-genus solution does not. Other initial conditions with step profile may be treated in an analogous fashion.

We see that the nature of the solution is quite different for the parameter values $a > 1$, $0 < a < 1$, $-1 < a < 0$, and $a < -1$. Note that the sound speed in the lattice is unity. We also note that the solutions for $a > 0$ and $t > 0$ are related to the solutions for $a < 0$ and $t < 0$ because of the symmetry of (2.4). We discuss this further in §5. We now consider each of the four cases separately.

4.1. The case where $a > 1$. First, we have the following lemma.

LEMMA 4.1. *The genus $g = 0$ equations develop a shock for the initial data* (4.1) *with $a > 1$.*

Proof. From (2.23) and (2.24) with $g = 0$, we have that

$$u_0 = \tfrac{1}{2}(\lambda_1 + \lambda_2), \qquad u_1 = \tfrac{1}{16}(\lambda_1 - \lambda_2)^2. \tag{4.2}$$

Hence for the given initial conditions, λ_1 and λ_2 may be given by

$$\begin{aligned} \lambda_1 &= -2a - 2\,, & X &< 0\,, \\ &= 2a - 2\,, & X &> 0\,, \\ \lambda_2 &= -2a + 2\,, & X &< 0\,, \\ &= 2a + 2\,, & X &> 0\,, \end{aligned} \tag{4.3}$$

as illustrated in Fig. 1. Now from (3.4) the characteristic velocities of the Riemann invariants λ_1 and λ_2 are given by

$$\begin{aligned} s(\lambda_1) &= -\sqrt{u_1} = \tfrac{1}{4}(\lambda_1 - \lambda_2), \\ s(\lambda_2) &= +\sqrt{u_1} = -\tfrac{1}{4}(\lambda_1 - \lambda_2). \end{aligned} \tag{4.4}$$

We can now see that λ_1 and λ_2 develop shocks as follows. First, from (4.4) we see that λ_1 moves to the right and that λ_2 moves to the left. An explicit calculation of the velocities at the corners of λ_1 and λ_2 gives

$$\begin{aligned} s(\lambda_2^+) &= -s(\lambda_1^-) = 1 + a\,, \\ s(\lambda_2^-) &= -s(\lambda_1^+) = 1\,, \end{aligned} \tag{4.5}$$

where $+$ denotes the limiting velocity at 0 from the right, and $-$ that from the left. Now, since $s(\lambda_i^+) > s(\lambda_i^-)$, $i = 1, 2$, a shock develops. □

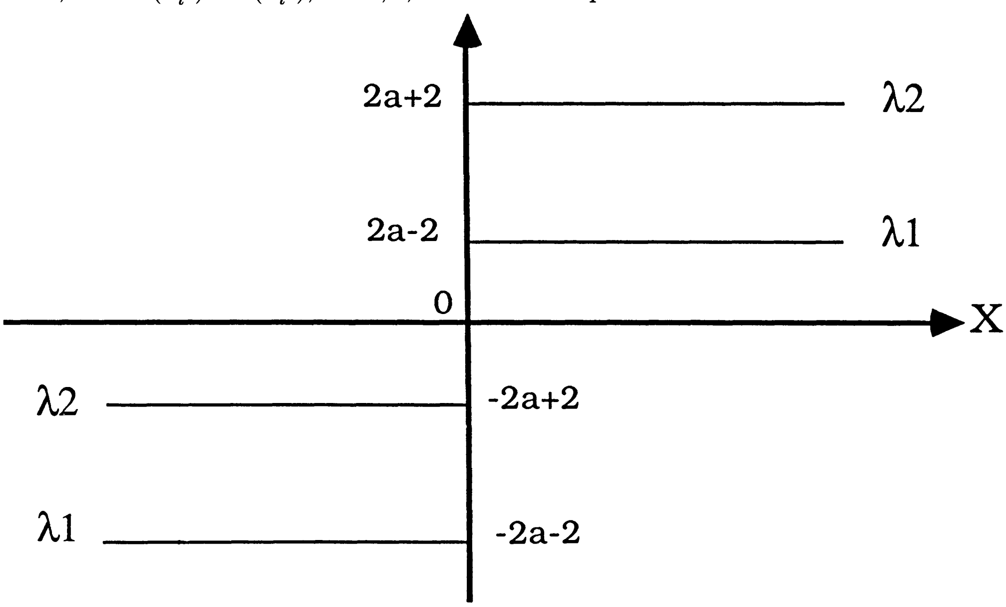

FIG. 1. *Genus $g = 0$ initial data for $a > 1$.*

We now show that regularization of the equations for $a > 1$ requires genus $g = 1$. The basic idea is to note that the initial data (4.1) can also be considered as the degenerate data of the higher-genus Whitham equation. We discuss the physical significance of this and the following results in the last section.

We have the following theorem.

THEOREM 4.2. *For $a > 1$ the genus $g = 1$ Whitham equations with the initial data* (4.1) *do not develop a shock.*

Proof. For the given set of initial data, we calculate first that λ_i, $i = 1, \cdots, 4$, satisfying (2.33) and (2.34) with $g = 1$, are given by

$$\begin{aligned} \lambda_1 &= -2a-2\,, && \forall X\,,\\ \lambda_2 &= -2a+2\,, && X<0\,,\\ &= -2a-2=\lambda_1\,, && X>0\,,\\ \lambda_3 &= 2a+2=\lambda_4\,, && X<0\,,\\ &= 2a-2\,, && X>0\,,\\ \lambda_4 &= 2a+2\,, && \forall X\,, \end{aligned} \tag{4.6}$$

as illustrated in Fig. 2.

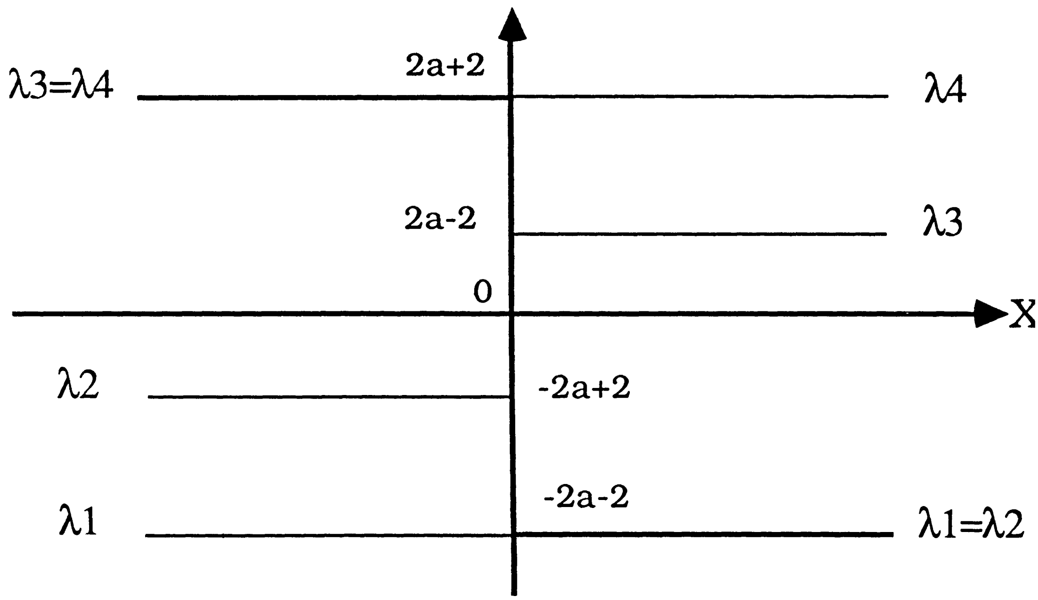

FIG. 2. *Degenerate $g = 1$ initial data for $a > 1$.*

It should be noted that, for fixed g, the λ_i are determined uniquely since they must satisfy the relationship between the λ_i and the first four conserved densities $\langle P_1\rangle, \cdots, \langle P_4\rangle$. Initially, of course, $\langle P_3\rangle$ and $\langle P_4\rangle$ are linearly dependent on $\langle P_1\rangle$ and $\langle P_2\rangle$ since the $g = 0$ and $g = 1$ solution match at $t = 0$. In fact, we see that $\lambda_3 = \lambda_4$ ($X < 0$) implies that $\alpha_1 = \lambda_3 = \lambda_4$ and $\gamma_1 = \lambda_3/2(\lambda_1 + \lambda_2)$, and $\lambda_1 = \lambda_2$ ($X > 0$) implies $\alpha_1 = \lambda_1 = \lambda_2$ and $\gamma_1 = \lambda_1/2(\lambda_3 + \lambda_4)$, and for these values ω_0 and ω_1 reduce to the Abelian differentials for the $g = 0$ case.

Note also that the λ_i are all monotonic decreasing and, from the Levermore theory, the no-shock condition for smooth initial data is given by $\partial s(\lambda_i)/\partial\lambda_i > 0$. We now prove the equivalent statement for our nonsmooth initial data by explicit calculation of the velocities.

We now calculate explicitly the characteristic speeds $s_i = s(\lambda_i)$ of (3.3),

$$s(\lambda) = \frac{\lambda^2 - \frac{1}{2}\sigma_1\lambda + \gamma_1}{2(\lambda - \alpha_1)}, \tag{4.7}$$

where α_1 and γ_1 are determined by the normalization (2.23), i.e.,

$$I_1^1 - \alpha_1 I_1^0 = 0\,, \tag{4.8}$$

$$I_1^2 - \frac{1}{2}\sigma_1 I_1^1 + \gamma_1 I_1^0 = 0\,, \tag{4.9}$$

with

$$I_1^\ell(\lambda_1, \cdots, \lambda_4) = \int_{\lambda_2}^{\lambda_3} \frac{\lambda^\ell d\lambda}{\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)}}. \tag{4.10}$$

Thus (4.7) becomes

$$s(\lambda) = -\frac{1}{4}\sigma_1 + \frac{\lambda^2 - I_1^2/I_1^0}{2(\lambda - I_1^1/I_1^0)}. \tag{4.11}$$

We can find a more convenient form for I_1^ℓ by making the successive changes of variables $\sqrt{\lambda - \lambda_2} = \xi$, and then $\xi = \sqrt{\lambda_3 - \lambda_2}\sin\theta$. This yields

$$I_1^\ell(\lambda_1, \cdots, \lambda_4) = \int_0^{\pi/2} \frac{\{\lambda_3 - (\lambda_3 - \lambda_2)\cos^2\theta\}^\ell d\theta}{\sqrt{\{(\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_2)\sin^2\theta\}\{(\lambda_4 - \lambda_2) - (\lambda_3 - \lambda_2)\sin^2\theta\}}}\,.$$

We must now compute $s(\lambda_3^-)$, the limit of $s(\lambda_3)$ from the left (as $X \to 0^-$), as well as $s(\lambda_3^+)$, $s(\lambda_2^-)$, and $s(\lambda_2^+)$. We give the calculation for $s(\lambda_3^-)$ in detail. Note that, to obtain the correct values of $\lambda_i^\pm$ to substitute into $s(\lambda)$, we must observe that, from (3.5), λ_2 moves to the right relative to λ_3. (A similar observation must be made for subsequent calculations.)

For $X < 0$, using the values of λ_i in (4.6), the integrals I_1^ℓ are given by

$$\begin{aligned} I_1^0 &= \frac{1}{4\sqrt{a}} \int_0^{\pi/2} \frac{d\theta}{\cos\theta\sqrt{1 + a\sin^2\theta}}, \\ I_1^1 &= \lambda_3 I_1^0 - \sqrt{a} \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{1 + a\sin^2\theta}}, \\ I_1^2 &= -\lambda_3^2 I_1^0 + 2\lambda_3 I_1^1 + 4a\sqrt{a} \int_0^{\pi/2} \frac{\cos^3\theta}{\sqrt{1 + a\sin^2\theta}}\, d\theta, \end{aligned} \tag{4.12}$$

from which

$$\alpha_1 = \frac{I_1^1}{I_1^0} = \lambda_3 - \frac{\sqrt{a}}{I_1^0} \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{1 + a\sin^2\theta}}\,, \tag{4.13}$$

$$\frac{I_1^2}{I_1^0} = \lambda_3^2 - \frac{2\lambda_3\sqrt{a}}{I_1^0} \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{1 + a\sin^2\theta}} + \frac{4a\sqrt{a}}{I_1^0} \int_0^{\pi/2} \frac{\cos^3\theta}{\sqrt{1 + a\sin^2\theta}} d\theta\,. \tag{4.14}$$

Hence from (4.11) the characteristic velocity $s(\lambda_3^-)$ is given by

$$s(\lambda_3^-) = 1 + 2a - 2a\frac{\int_0^{\pi/2} \frac{\cos^3\theta}{\sqrt{1+a\sin^2\theta}}d\theta}{\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1+a\sin^2\theta}}d\theta}$$

$$= \frac{\sqrt{a}\sqrt{1+a}}{\log|\sqrt{a}+\sqrt{1+a}|} > 1. \tag{4.15}$$

In the above calculation, note that the I_1^ℓ are actually infinite, but $s(\lambda_3^-)$ is well defined.

Note that the wavefront speed is greater than unity, which is the sound speed in the lattice, reflecting shock formation. This is discussed later in §5. Similarly, we can show that $s(\lambda_2^+) = -s(\lambda_3^-)$. The velocities $s(\lambda_3^+)$ and $s(\lambda_2^-)$ may be found in terms of elliptic integrals as follows:

$$s(\lambda_2^-) = -s(\lambda_3^+) = (-2a+2)(1-\Gamma) < 0, \tag{4.16}$$

where

$$\Gamma = \frac{\int_0^{\pi/2} \frac{\sin^4\theta}{\sqrt{(a\cos^2\theta+\sin^2\theta)(a\sin^2\theta+\cos^2\theta)}}d\theta}{\int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{(a\cos^2\theta+\sin^2\theta)(a\sin^2\theta+\cos^2\theta)}}d\theta} < 1\,.$$

From these velocity calculations, we can see that the $g = 1$ Whitham equation has a regular single-phase solution, and no shock occurs, as shown in Fig. 3. □

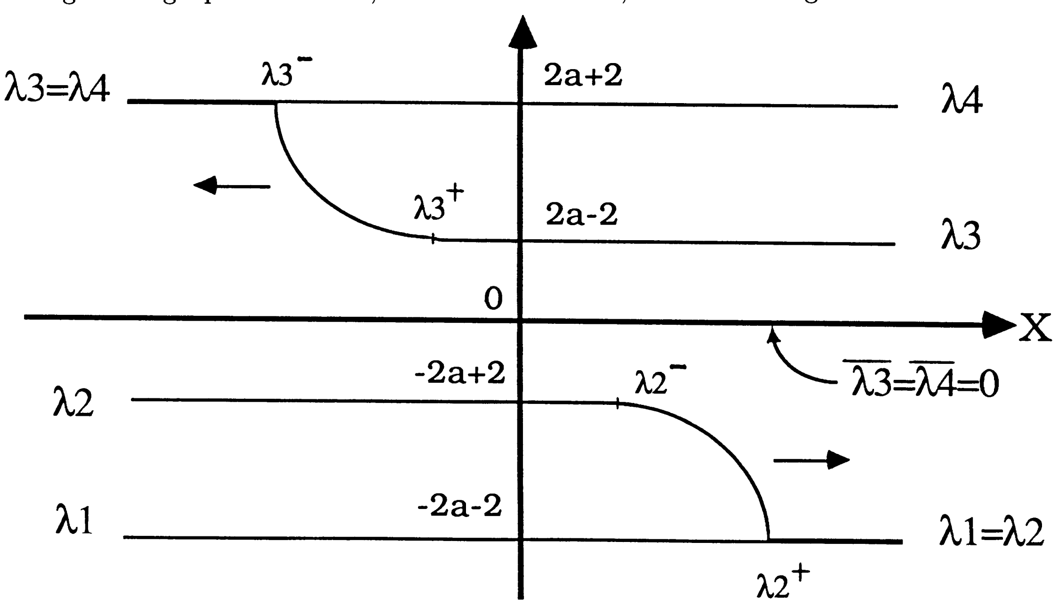

FIG. 3. *Degenerate $g = 2$ solution for $a > 1$.*

Note also that, for example, in the region $s(\lambda_3^+) \le -X/T \le s(\lambda_3^-)$, the unique solution λ_3, with all other λ_i fixed, can be given in hodograph form (for the general case, see [7], [8], and [15])

$$X + s(\lambda_3)T = 0. \tag{4.17}$$

Since λ_3 is monotonic, we can invert (4.17) to find λ_3 as a function of X and T. This solution may be connected to the steady $g = 1$ solution in $0 < -X/T < s(\lambda_3^+)$ and the $g = 0$ solution in $-X/T \geq s(\lambda_3^-)$ (outside the domain of influence). We can also see from (4.17) that for fixed T, $\partial X/\partial\lambda_3 \to 0$ as $\lambda_3 \to \lambda_3^-$ and $\partial X/\partial\lambda_3 \to \infty$ as $\lambda_3 \to \lambda_3^+$. In Fig. 4 we show the time evolution of the solution (4.17) in the $X - T$ plane. The shaded region in Fig. 4 corresponds to the transient $g = 1$ solution of the Whitham equation, and the unshaded region, including $X = 0$ to the steady state $g = 1$ solution.

We wish to remark that we have chosen the solution here of minimum genus. We can choose a solution of any higher genus with degeneracy in the spectrum since, if any two λ_i become degenerate, the corresponding factors in the Abelian differentials cancel out, as the spectrum is real. Hence the velocity calculations for the higher-genus case remain precisely the same. Such a degeneracy is, in fact, needed to explain the detailed behavior of the original Toda lattice, while the solution of the $g = 1$ Whitham equations describes the averaged behavior of the original lattice. As pointed out to us by Venakides, to satisfy the boundary conditions at $n = 0$ in the original lattice, we need a degenerate genus-2 Riemann surface with two branch points coinciding at $\lambda = 0$. The above calculations are then precisely the same, except that now we have the additional characteristics $\overline{\lambda}_3 = \overline{\lambda}_4 = 0$ and the total ordering $\lambda_1 \leq \lambda_2 \leq \overline{\lambda}_3 = \overline{\lambda}_4 \leq \lambda_3 \leq \lambda_4$, as illustrated in Fig. 3. This ensures that the particle at $X = 0$ (i.e., $n = 0$ in the lattice) remains fixed. The solution near $X = 0$ then corresponds to the binary oscillations of the lattice behind the shock as observed in [6] and as predicted in [17] (see §5).

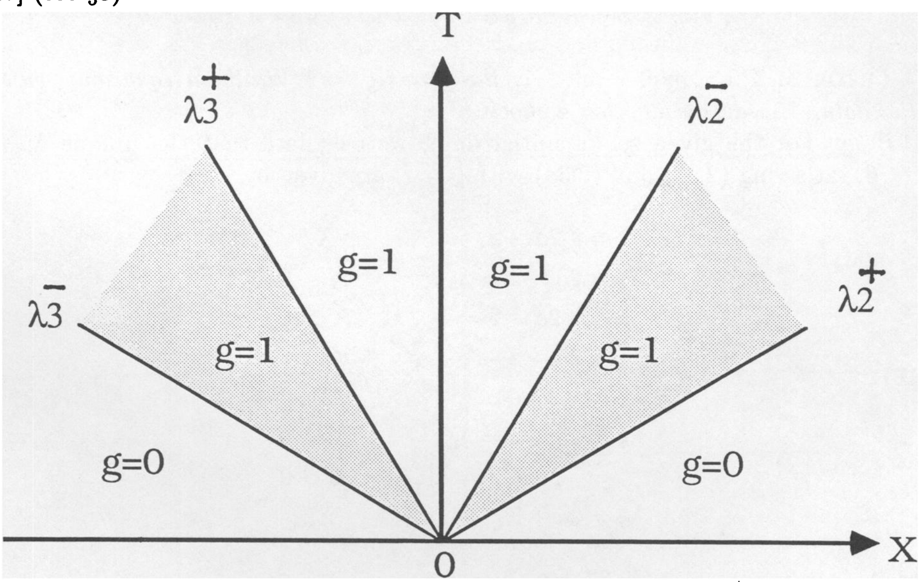

FIG. 4. *$X - T$ plane for the solution for $a > 1$.*

4.2. The case where $0 < a < 1$. We first note that the $g = 0$ solution develops a shock for the same reasons as in the case where $a > 1$. In this case, we need genus $g = 2$ to regularize the solutions for the following reason. We can get monotonic decreasing λ_i by choosing them as in (4.6). However, for $0 < a < 1$, these coincide along the X-axis, as in Fig. 5. This figure is only instantaneously correct. Immediately afterward, the characteristics intersect, requiring a reordering of the λ_i, or, effectively, change to a genus $g = 2$ solution. Thus the $g = 1$ solution does not preserve the ordering of the λ_i as time evolves, even though the λ_i are monotone decreasing. The $g = 2$ solution does preserve the ordering and gives a consistent solution for all time $T > 0$. We have, in fact, the following theorem.

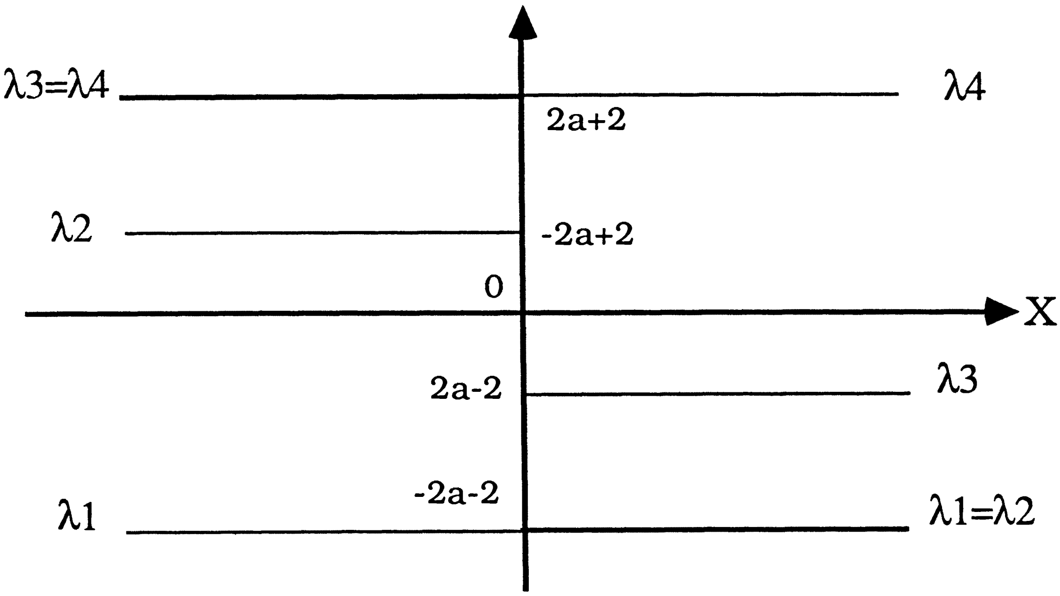

FIG. 5. *Degenerate $g = 1$ initial data for $0 < a < 1$.*

THEOREM 4.3. *For $0 < a < 1$, the genus $g = 2$ Whitham equations with the initial data* (4.1) *do not develop a shock.*

Proof. For the given set of initial data, we calculate that the unique λ_i, $i = 1, \cdots, 6$, satisfying (2.33) and (2.34) with $g = 2$, are given by

$$\begin{aligned}
\lambda_1 &= -2a - 2\,, && \forall X\,,\\
\lambda_2 &= 2a - 2 = \lambda_3\,, && X < 0\,,\\
&= -2a - 2 = \lambda_1\,, && X > 0\,,\\
\lambda_3 &= 2a - 2\,, && \forall X\,,\\
\lambda_4 &= -2a + 2\,, && \forall X\,,\\
\lambda_5 &= 2a + 2 = \lambda_6\,, && X < 0\,,\\
&= -2a + 2 = \lambda_4\,, && X > 0\,,\\
\lambda_6 &= 2a + 2\,, && \forall X\,.
\end{aligned} \tag{4.18}$$

Here the coefficients α_i in (2.33) are given by

$$\begin{aligned}
\alpha_1 &= 2a - 2 = \lambda_3\,, && X < 0\,,\\
&= -2a - 2 = \lambda_1\,, && X > 0\,,\\
\alpha_2 &= 2a + 2 = \lambda_6\,, && X < 0\,,\\
&= -2a + 2 = \lambda_4\,, && X > 0\,.
\end{aligned}$$

Since these are all monotonic decreasing again by Lemma 3.2, we expect no shock to develop.

We now calculate the limiting velocities $s(\lambda_5^-)$, etc. The calculation proceeds as follows. Note that

$$s(\lambda) = \frac{\lambda^3 - \frac{1}{2}\sigma_1\lambda^2 + \gamma_1\lambda + \gamma_2}{2(\lambda-\alpha_1)(\lambda-\alpha_2)}. \tag{4.19}$$

The normalization conditions for ω_0 and ω_1 (2.23)

$$\oint_{a_1} \omega_i = \oint_{a_2} \omega_i = 0, \qquad i = 0,1, \tag{4.20}$$

give the equations

$$\begin{aligned} &I_k^2 - (\alpha_1+\alpha_2)I_k^1 + \alpha_1\alpha_2 I_k^0 = 0\,, \qquad k=1,2\,,\\ &I_k^3 - \frac{1}{2}\sigma_1 I_k^2 + \gamma_1 I_k^1 + \gamma_2 I_k^0 = 0\,, \qquad k = 1,2\,, \end{aligned} \tag{4.21}$$

where

$$I_k^\ell(\lambda_1,\cdots,\lambda_6) = \int_{\lambda_{2k}}^{\lambda_{2k+1}} \frac{\lambda^\ell d\lambda}{\sqrt{R_g}}\,.$$

Hence we can solve for α_1, α_2, γ_1, and γ_2 in terms of the I_k^ℓ. The following form of I_1^ℓ and I_2^ℓ (obtained by using similar transformations to those discussed earlier) is useful for the velocity calculations:

$$I_1^\ell = \int_0^{\pi/2} \frac{(\lambda_2+(\lambda_3-\lambda_2)\sin^2\theta)^\ell\, d\theta}{\sqrt{(\lambda_2-\lambda_1+(\lambda_3-\lambda_2)\sin^2\theta)(\lambda_4-\lambda_2-(\lambda_3-\lambda_2)\sin^2\theta)(\lambda_5-\lambda_2-(\lambda_3-\lambda_2)\sin^2\theta)(\lambda_6-\lambda_2-(\lambda_3-\lambda_2)\sin^2\theta)}},$$

$$I_2^\ell = \int_0^{\pi/2} \frac{(\lambda_4+(\lambda_5-\lambda_4)\sin^2\theta)^\ell\, d\theta}{\sqrt{(\lambda_4-\lambda_1+(\lambda_5-\lambda_4)\sin^2\theta)(\lambda_4-\lambda_2+(\lambda_5-\lambda_4)\sin^2\theta)(\lambda_4-\lambda_3+(\lambda_5-\lambda_4)\sin^2\theta)(\lambda_6-\lambda_4-(\lambda_5-\lambda_4)\sin^2\theta)}}.$$

Computing the coefficients in terms of the integrals as $X \to 0^-$, we can show, after lengthy computation, that $s(\lambda_5)$ at $X = 0^-$ is given by

$$s(\lambda_5^-) = \frac{\sqrt{a}\sqrt{a+1}}{\log(\sqrt{a}+\sqrt{a+1})} \tag{4.22}$$

(i.e., (4.15) in the genus-1 case) and

$$s(\lambda_2^+) = -s(\lambda_5^-), \tag{4.23}$$

while $s(\lambda_5)$ at $X = 0^+$ and $s(\lambda_2)$ at $X = 0^-$ are given by

$$s(\lambda_5^+) = 1 - a = -s(\lambda_2^-). \tag{4.24}$$

This proves the theorem. □

In Figs. 6 and 7 we illustrate the initial data (4.18) and corresponding solution for $T > 0$.

Again, the solution between $s(\lambda_5^+) \le -X/T \le s(\lambda_5^-)$ can be obtained from the hodograph equation, as in §4.1.

We now consider step initial data with $a < 0$. The result is that we again must distinguish two cases: $-1 < a < 0$ and $a < -1$.

4.3. The case where $-1 < a < 0$. In this case we have the following theorem.

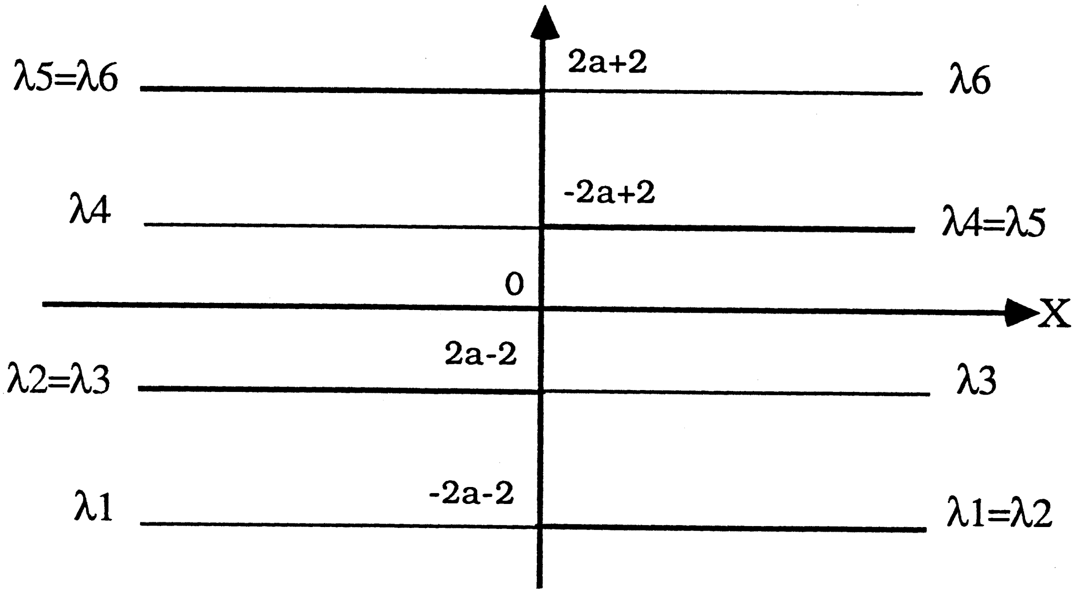

FIG. 6. *Degenerate* $g = 2$ *initial data for* $0 < a < 1$.

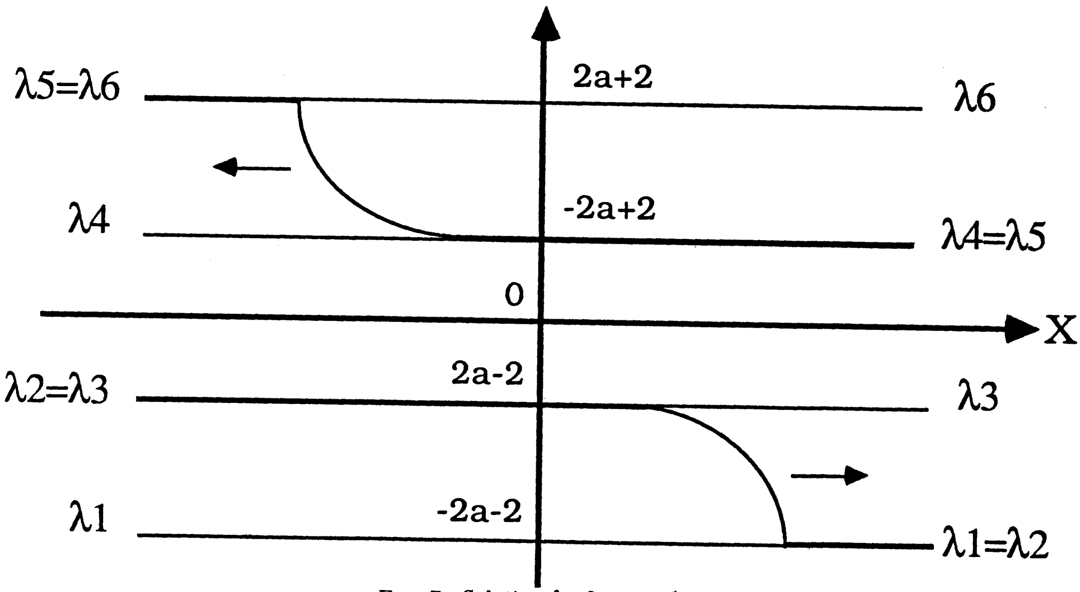

FIG. 7. *Solution for* $0 < a < 1$.

THEOREM 4.4. *For* $-1 < a < 0$, *the genus* $g = 0$ *Whitham equations with initial data* (4.1) *do not develop a shock.*

Proof. From (4.2) we find that λ_1 and λ_2 are precisely as in Lemma 4.1. However, for $-1 < a < 0$, λ_1 and λ_2 are monotone *decreasing*, and we expect no shock. In fact, from explicit calculation, we can show that

$$s(\lambda_1^-) = -s(\lambda_2^+) = 1, \qquad s(\lambda_1^+) = -s(\lambda_2^-) = 1 + a. \tag{4.25}$$

Hence $s(\lambda_i^+) < s(\lambda_i^-)$, $i = 1, 2$, and indeed no shock occurs. $\square$

In Figs. 8 and 9 we show the initial data (4.18) and the solutions λ_1 and λ_2 for $T > 0$.

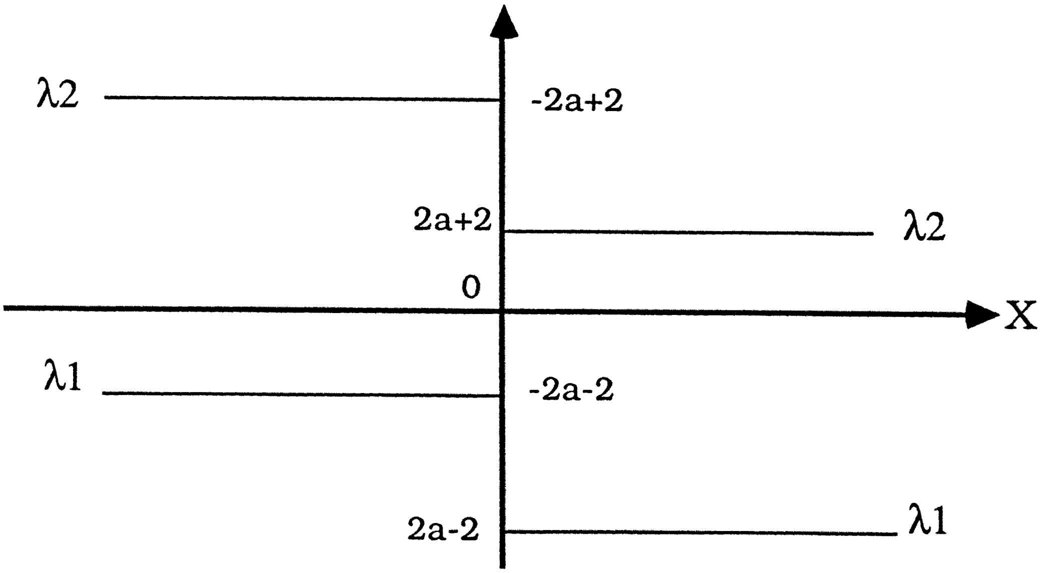

FIG. 8. *Genus* $g = 0$ *initial data for* $-1 < a < 0$.

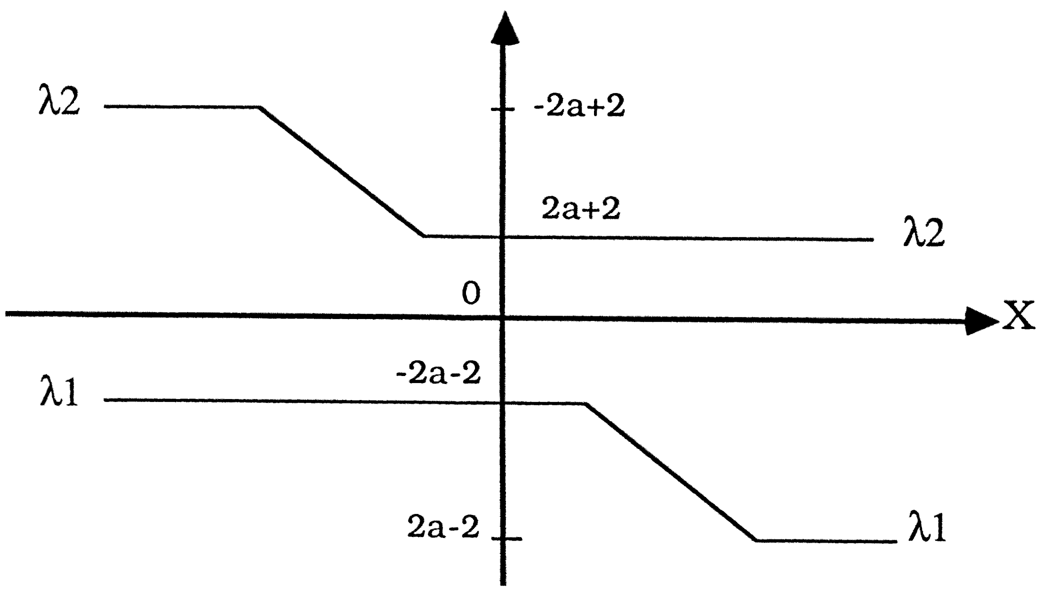

FIG. 9. *Solution for* $-1 < a < 0$.

Note that for $g = 0$ the solutions λ_1 and λ_2 are linear in X.

4.4. The case where $a < -1$. Here we have the following theorem.

THEOREM 4.5. *For $a < -1$, the genus $g = 1$ Whitham equations with the initial data* (4.1) *do not develop a shock.*

Proof. For the given set of initial data (4.1), the λ_i, $i = 1, \cdots, 4$, satisfying (2.33) and (2.34) with $g = 1$ are found to be

$$(4.26)\qquad \begin{aligned} \lambda_1 &= 2a+2\,, && X<0\,,\\ &= 2a-2 = \lambda_2\,, && X>0\,,\\ \lambda_2 &= 2a+2\,, && \forall X\,,\\ \lambda_3 &= -2a-2\,, && \forall X\,,\\ \lambda_4 &= -2a+2\,, && X<0\,,\\ &= -2a-2 = \lambda_3\,, && X>0\,. \end{aligned}$$

Since these are all monotonically decreasing, we expect no shocks to develop. Indeed, calculating as in the previous theorems gives

$$(4.27)\qquad s(\lambda_4^-) = -s(\lambda_1^+) = 1\,, \qquad s(\lambda_4^+) = -s(\lambda_1^-) = 0\,.$$

Hence no shocks develop. □

Figures 10 and 11 show the initial data (4.26) and solutions for $T > 0$.

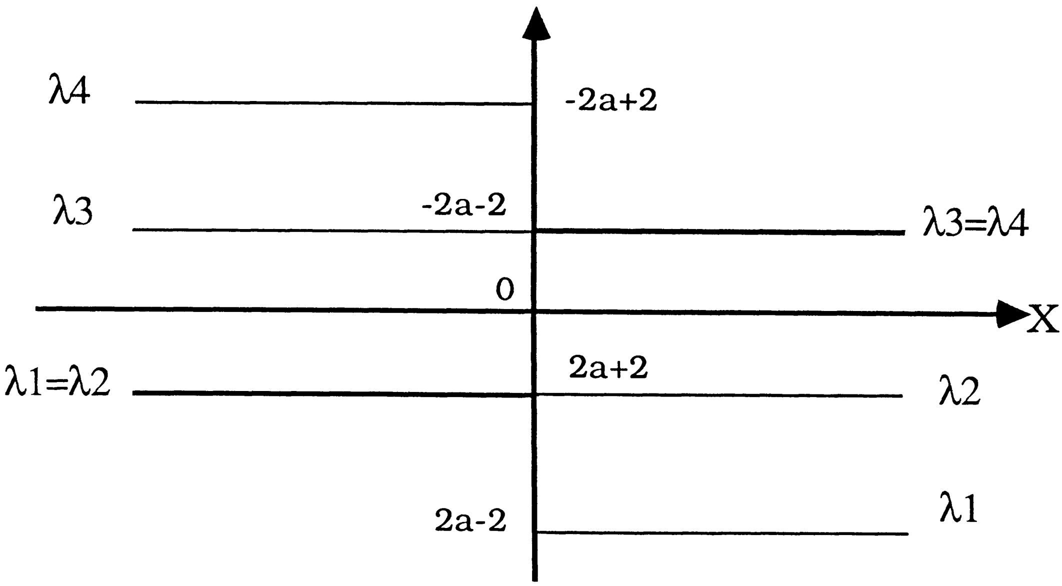

FIG. 10. *Degenerate $g = 1$ initial data for $a < -1$.*

5. Summary and discussion. We have shown how regularity of solutions dictates the choice of Whitham equations for the Toda lattice with step initial data.

We remark that the problem with smooth initial data may be solved by the generalized hodograph method [7], [8], [15], with similarity variables. This method is similar to that used in [5] for the KdV case. In a future communication, we hope to give a method for connecting the different genus solutions before and after the shock.

We remark finally on the physical significance of our solutions. The physical behavior predicted by these solutions agrees with the numerical studies of Holian, Flaschka, and McLaughlin [6] for $a > 0$. Their initial condition may be viewed as being imparted by a "piston" moving with velocity a. They then discuss three regions of the shock profile—the shock front, the rear of the shock profile near the piston, and

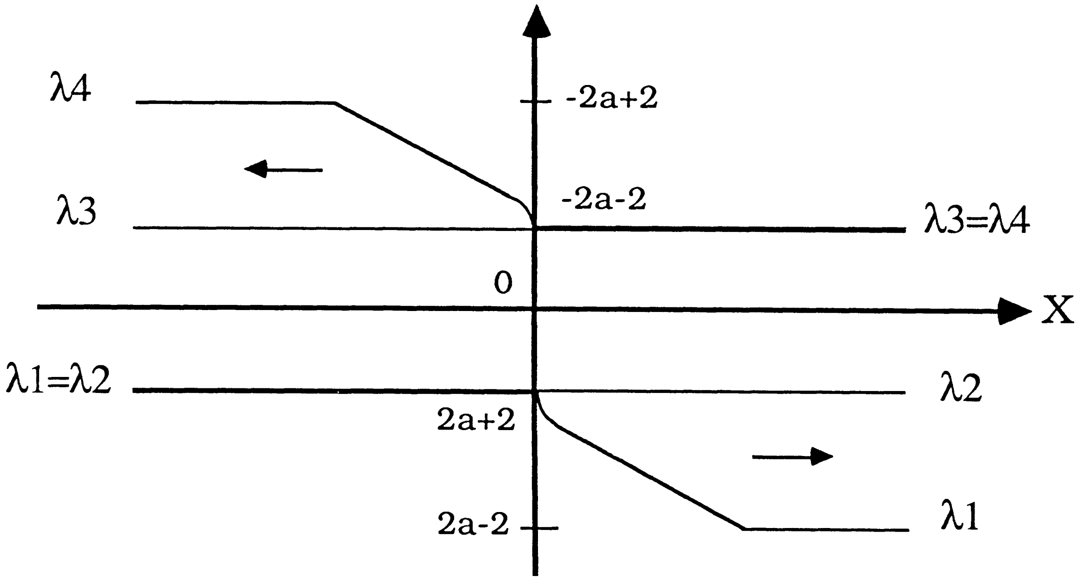

FIG. 11. *Solution for $a < -1$.*

a transition region between front and rear. For the two cases, a less than and above the critical value 1 (the sound speed in the lattice), we obtain quite different behavior.

For $a > 1$, we observe regular periodic behavior in the rear of the shock region corresponding to "binary" oscillation—displacement of particles being equal and opposite and thus oscillating at the highest frequency the lattice can support. This is consistent with the behavior given by the degenerate $g = 2$ solution (with $\overline{\lambda}_3 = \overline{\lambda}_4 = 0$) illustrated in Figs. 3 and 4.

For $0 < a < 1$, we observe essentially zero oscillation in the rear of the shock and nonzero oscillation elsewhere. Again, this is what we predict from the degenerate $g = 2$ solution as the interior exhibits zero gap (Fig. 7).

Note that $a > 0$ corresponds to the lattice contracting, which can generate shock waves; $a < 0$, on the other hand, corresponds to the lattice expanding, which can generate rarefaction waves. Again, for the latter, we obtain different behavior for $-1 < a < 0$ and $a < -1$.

For $-1 < a < 0$, since the velocity of the expansion is less than the sound velocity, no shock occurs, and the $g = 0$ solution provides a good description for the lattice behavior (Fig. 9).

For $a < -1$, the expansion velocity is greater than the sound velocity, and the degenerate $g = 1$ solution is required to describe the behavior. However, the gap never opens, and, in this case, u_0 and u_1 near $X = 0$ never become stationary except at $T = \infty$. Physically, this implies the lattice is continually expanding (Fig. 11).

We can also give the following interpretation of the above results. Note that (2.4) is invariant under the transformation $u_0 \to -u_0$, $t \to -t$. Hence considering a solution for $a < 0$ and $0 < T < \infty$ is equivalent to considering a solution for $a > 0$ and $-\infty < T < 0$. Thus, for fixed $a > 0$, we have given the entire solution for $-\infty < T < \infty$ with the given step data at $T = 0$. The solution is asymmetric about $T = 0$ due to the shock.

Acknowledgments. The authors thank R. Brockett, P. Deift, and S. Venakides for some valuable conversations.

REFERENCES

[1] R. F. Bikbaev and V. Yu. Novokshenov, *Self-similar solutions of the Whitham equations, and* KdV *equations with* f.-g. *boundary conditions*, in Proc. of the III Intern. Workshop, Kiev, 1987, Vol. 1, 1988, pp. 32–35.

[2] R. W. Brockett and A. M. Bloch, *Sorting with the dispersionless limit of the Toda lattice*, in Proc. of the CRM Workshop on Hamiltonian Systems, Transformation Groups and Spectral Transform Methods, J. Harnad and J. E. Marsden, eds., Publications CRM, Montreal, Quebec, Canada, 1990, pp. 103–112.

[3] H. Flaschka, *The Toda lattice*, Phys. Rev. B, 9 (1976), pp. 1924–1925.

[4] H. Flaschka, M. G. Forest, and D. W. McLaughlin, *Multiphase averaging and the inverse spectral solution of the Korteweg–de Vries equation*, Comm. Pure Appl. Math., 33 (1980), pp. 739–784.

[5] A. V. Gurevich and L. D. Pitaevskii, *Nonstationary structure of noncolliding shock waves*, JETP Lett., 38 (1974), p. 291.

[6] B. L. Holian, H. Flaschka, and D. W. McLaughlin, *Shock waves in the Toda lattice: Analysis*, Phys. Rev. A, 24 (1981), pp. 2595–2623.

[7] Y. Kodama, *Solutions of the dispersionless Toda equation*, Phys. Lett. A, 147 (1990), pp. 477–482.

[8] Y. Kodama and J. Gibbons, *Integrability of the dispersionless* KP *hierarchy*, in Proc. of the Workshop on Nonlinear and Turbulent Processes in Physics, Kiev, World Scientific, Singapore, 1990, pp. 160–180.

[9] I. M. Krichever, *Algebraic curves and nonlinear difference equations*, Russian Math. Surveys, 33 (1978), pp. 255–256.

[10] ———, *Method of averaging for two-dimensional "integrable" equations*, Funct. Anal. Appl., 22 (1988), pp. 37–52.

[11] B. A. Kupershmidt, *Discrete Lax Equations and Differential Difference Calculus*, Asterique, 23, Soc. Math. France, Paris, 1985.

[12] P. D. Lax and C. D. Levermore, *The small dispersion limit of the Korteweg–de Vries equation* I, II, III, Comm. Pure Appl. Math. 30 (1983), pp. 253–290, 571–593, 809–829.

[13] C. D. Levermore, *The hyperbolic nature of the zero dispersion* KdV *limit*, Comm. Partial Differential Equations, 13 (1988), pp. 495–514.

[14] J. Moser, *Finitely many mass points on the line under the influence of an exponential potential*, Battelles Recontres, Springer Notes in Physics, New York, 1974, pp. 417–497.

[15] S. P. Tsarev, *On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type*, Soviet Math. Dokl., 31 (1985), pp. 488–491.

[16] S. Venakides, *The generation of modulated wavetrains in the solution of the Korteweg–de Vries equation*, Comm. Pure Appl. Math., 38 (1985), pp. 833–909.

[17] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. Pure Appl. Math., 43 (1991), pp. 1171–1242.

[18] G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley, New York, 1974.