Lecture 4: The Bellman Operator

Dynamic Programming

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Infinite horizon, $t \to \infty$

- We know

$$
\begin{aligned}
V^0(M_t) &= \text{whatever} \\
V^1(M_t) &= \max_{C_t \in \mathcal{C}(M_t)} \left\{ u(M_t, C_t) + \beta V^0(M_{t+1}) \right\} \\
V^2(M_t) &= \max_{C_t \in \mathcal{C}(M_t)} \left\{ u(M_t, C_t) + \beta V^1(M_{t+1}) \right\} \\
V^3(M_t) &= \max_{C_t \in \mathcal{C}(M_t)} \left\{ u(M_t, C_t) + \beta V^2(M_{t+1}) \right\} \\
&\cdots \\
\lim_{n \to \infty} V^n(M_t) &= \max_{C_t \in \mathcal{C}(M_t)} \left\{ u(M_t, C_t) + \beta V^{n-1}(M_{t+1}) \right\} ?
\end{aligned}
$$

where $M_{t+1} = \Gamma(M_t, C_t)$

- **Does the limit exist?**

Operator notation

- Write the **Bellman equation** on the following general form

$$V^n(M_t) = \max_{C_t \in \mathcal{C}(M_t)} u(M_t, C_t) + V^{n-1}(\Gamma(M_t, C_t)) \text{ for all } M_t \in \mathcal{M}$$

- **Alternatively** in operator form

$$V^n(M_t) = J(V^{n-1})(M_t) \text{ for all } M_t \in \mathcal{M}$$

- A **fixed point** is a *function* V such that

$$V(M_t) = J(V)(M_t) \text{ for all } M_t \in \mathcal{M}$$

- **Is there always a fixed point, and is it unique?**

Contraction mapping requirement

- Let $\mathcal{F}(\mathcal{M})$ be the space of bounded continuous functions

Theorem

Assume $u(M_t, C_t)$ is real-valued, continuous and bounded, $0 < \beta < 1$ and the constraint set, $\mathcal{C}(M_t)$ is non-empty, compact-valued and continuous, then J has a unique fixed point $V \in \mathcal{F}(\mathcal{M})$, and for all $V_0 \in \mathcal{F}(\mathcal{M})$

$$|J^n(V_0) - V| \leq \beta^n |V_0 - V|, \quad n = 0, 1, 2, 3, \ldots$$

- **Full proof:** Lucas and Stokey (1989), theorem 4.6
- **Main idea:** Apply *Blackwell's contraction mapping theorem* requiring that J is
  1. Monotone
  2. Discounted

Monotone (requirement 1)

$$\begin{aligned} V(M_t) &\geq Q(M_t),\ \forall M_t \in \mathcal{M} \Rightarrow \\ J(V)(M_t) &\geq J(Q)(M_t),\ \forall M_t \in \mathcal{M} \end{aligned}$$

$$C_V^\star(M_t) \equiv \arg\max_{C_t \in \mathcal{C}(M_t)} u(M_t, C_t) + \beta V(\Gamma(M_t, C_t))$$

$$C_Q^\star(M_t) \equiv \arg\max_{C_t \in \mathcal{C}(M_t)} u(M_t, C_t) + \beta Q(\Gamma(M_t, C_t)$$

- **Insert into** $J(V)(M_t)$

$$\begin{aligned} J(V)(M_t) &= \max_{C_t \in \mathcal{C}(M_t)} u(M_t, C_t) + \beta V(\Gamma(M_t, C_t)) \\ &= u(M_t, C_V^\star(M_t)) + \beta V(\Gamma(M_t, C_V^\star(M_t)) \\ &\geq u(M_t, C_Q^\star(M_t)) + \beta V(\Gamma(M_t, C_Q^\star(M_t)) \\ &\geq u(M_t, C_Q^\star(M_t)) + \beta Q(\Gamma(M_t, C_Q^\star(M_t)) \\ &= J(Q)(M_t) \end{aligned}$$

Discounted (requirement 2)

$$\exists \gamma \in (0,1) : J(V+k)(M_t) \leq J(V)(M_t) + \gamma k.$$

- **We have**

$$\begin{aligned}
J(V+k)(M_t) &= \max_{C_t \in \mathcal{C}(M_t)} u(M_t, C_t) + \beta(V(\Gamma(M_t, C_t)) + k) \\
&= \max_{C_t \in \mathcal{C}(M_t)} u(M_t, C_t) + \beta V(\Gamma(M_t, C_t)) + \beta k \\
&= J(V)(M_t) + \beta k \\
&\leq J(V)(M_t) + \gamma k \text{ for } \gamma = \beta \in (0,1)
\end{aligned}$$

- **What could break down here?**

Summarize

1. The **uniqueness** of the value function can be proven
2. **Iteration** on the value function can be proven to converge at a rate of β
3. Further properties:
   1. **Monotonicity** in states expanding the choice set
   2. **Concavity** if choice set is convex and u is concave
   3. **Differentiability** (e.g. Benvenste and Scheinkman (1979), Clausen and Strub (2016))
4. **Unique policy function** typically requires that the choice set is convex and u is strictly concave
5. **Boyd's Weighted Contraction Mapping Theorem** can be used if returns are unbounded (see Carroll (2012))

Value function iteration (VFI)

Algorithm 11: Find the fixed point V

input : tol $= 1.0e - 10$
output: $V[\bullet]$
$C^\star[\bullet]$

for $i_M = 1$ **to** $\#_M$ **do**
  $V[i_M] = 0$ (initialize)
while $?$ **do**
  $V_-[\bullet] = V[\bullet]$
  for $i_M = 1$ **to** $\#_M$ **do**
    $V[i_M], C^\star[i_M] =$ find_V$(V[\bullet])$
  $\delta = \max(|V_-[:] - V[:]|)$

Relation to Finite Horizon: Simulation

- Solution to *infinite* horizon is basically the solution to the *first-period finite* horizon problem with very large T

$T < \infty$	$T = \infty$
$\{V_t^\star, C_t^\star\}_{t=1}^T$: $\#_M \times T$	$\{V^\star, C^\star\}$: $\#_M \times 1$

- Simulating is similar. Start with M_1 resources:

$$T < \infty$$

$$\begin{array}{ccccc} M_1 & \rightarrow & C_1^\star(M_1) & = & C_1 \\ M_2 = \Gamma(M_1, C_1) & \rightarrow & C_2^\star(M_2) & = & C_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

$$T = \infty$$

$$\begin{array}{ccccc} M_1 & \rightarrow & C^\star(M_1) & = & C_1 \\ M_2 = \Gamma(M_1, C_1) & \rightarrow & C^\star(M_2) & = & C_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Policy iteration

- Think of **step** n in VFI where we for all $M_t \in \mathcal{M}$ set

$$\begin{aligned} V^n(M_t) &= u(M_t, C^{\star n}(M_t)) + \beta \mathbb{E}_t \left[V^{n-1}(\Gamma(M_t, C^{\star n}(M_t)) \right] \\ C^{\star n}(M_t) &= \arg\max_{C_t} u(M_t, C_t) + \beta \mathbb{E}_t \left[V^{n-1}(\Gamma(M_t, C_t)) \right] \end{aligned}$$

- **Alternative:** *Simulate* forward for k periods using $C^{\star n}(\bullet)$ as decision rule, and update by

$$\begin{aligned} V^n(M_t) &= \sum_{j=0}^{k} \beta^j u(M_{t+j}, C^{\star n}(M_{t+j})) \\ &\quad + \beta^{k+1} V^{n-1}(\Gamma(M_{t+k+1}, C^{\star}_{t+k+1})) \\ C^{\star n+1}(M_t) &= \arg\max_{C_t} u(M_t, C_t) + \beta \mathbb{E}_t \left[V^n(\Gamma(M_t, C_t)) \right] \end{aligned}$$

- **Better convergence?** *Yes*, in terms of speed. *No*, in terms of pool of attraction (VFI i *globally* convergent)
- **Everything is discrete:** The simulation can be replaced by inversion of a matrix! [Bertel will show you]

Guess and verify

- Consider the **neoclassical growth model**

$$
\begin{aligned}
V(K_t) &= \max_{C_t} \log C_t + \beta V(K_{t+1}) \\
&\text{s.t.} \\
K_{t+1} &= AK_t^{\alpha} - C_t
\end{aligned}
$$

- **Assume** that $V(K_t) = a + b \log K_t$ such that

$$
a + b \log K_t = \max_{K_{t+1}} \log(AK_t^{\alpha} - K_{t+1}) + \beta(a + b \log K_{t+1})
$$

- The **FOC** then is

$$
\frac{1}{AK_t^{\alpha} - K_{t+1}} = \frac{\beta b}{K_{t+1}} \leftrightarrow K_{t+1} = \frac{\beta b}{1+\beta b} AK_t^{\alpha}
$$

- **Insert FOC** and **solve for** a **and** b (independent of K_t)

$$
a + b \log K_t = \log(AK_t^{\alpha} - \frac{\beta b}{1+\beta b} AK_t^{\alpha}) + \beta(a + b \log(\frac{\beta b}{1+\beta b} AK_t^{\alpha})
$$

Projection methods

- Guess and verify is only possible for **very special models**
- Value and policy functions might, however, be well **approximated by parametric functions** (typically polynomials, Weierstrass theorem)
- **Solve for the parameters numerically** instead of solving the maximization problems (relying on the first-order conditions instead)

The Bellman equation

- **The model:**
  1. A household gets utility from consumption and disutility from labor
  2. The household's income dependent on whether it works or not
  3. The household accumulates human capital by working
  4. It can save in an account with an interest rate of r
- **Task:** Write up the Bellman equation on a post-it for your choice of utility function, wage process and human capital accumulation equation. Put it on the white board.

Until next

- **Ensure that you understand:**
  - Algorithm 11
  - How to set up a Bellman equation
- Go to **PadLet** and ask or answer a question (https://padlet.com/thomas_jorgensen1/DP)
- **Think about:** What is the problem with having respectively:
  1. Multiple states
  2. Multiple choices
  3. Multiple shocks