

# Lecture 3: Numerical Integration + Simulation

Dynamic Programming

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**Algorithm 5:** Find all  $V_t^*$  (algorithm 9 inserted)

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**input :**  $\mathcal{G}_M$  (grid for  $M$  with  $\#_M$  elements)  
 $\mathcal{G}_C$  (grid for  $C$  (as a share of  $M$ ) with  $\#_C$  elements in  $(0, 1)$ )

**output:**  $V_t^*[\bullet]$  for all  $t$   
 $C_t^*[\bullet]$  for all  $t$

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1  for  $i_M = 1$  to  $\#_M$  do
2     $V_T^*[i_M] = \sqrt{\mathcal{G}_M[i_M]}$  (initialize terminal period)
3  for  $t = T - 1$  to 1 do
4    for  $i_M = 1$  to  $\#_M$  do
5       $V_t^*[i_M] = -\infty$ 
6       $M_t = \mathcal{G}_M[i_M]$ 
7      for  $i_C = 1$  to  $\#_C$  do
8         $C_t = \mathcal{G}_C[i_C] M_t$ 
9         $EV_{t+1} = \pi \text{interp}(M_t - C_t + 1, \mathcal{G}_M, V_{t+1}^*)$ 
           $+ (1 - \pi) \text{interp}(M_t - C_t, \mathcal{G}_M, V_{t+1}^*)$ 
10        $V = \sqrt{C_t} + \beta EV_{t+1}$ 
11       if  $V > V_t^*[i_M]$  then
12          $V_t^*[i_M] = V$ 
13          $C_t^*[i_M] = C_t$ 

```



# Continuous stochastic shocks

- **Example:** Consumption-saving model with **Gaussian income shocks**

$$V_t(M_t) = \max_{C_t} \sqrt{C_t} + \beta \mathbb{E}_t [V_{t+1}(M_{t+1})]$$

s.t.

$$A_t = M_t - C_t$$

$$M_{t+1} = R \cdot A_t + Y_{t+1}$$

$$Y_{t+1} = \exp(\xi_{t+1})$$

$$\xi_{t+1} \sim \mathcal{N}(0, \sigma_\xi^2)$$

$$A_t \geq 0$$

- How can we **evaluate**

$$\mathbb{E}_t [V_{t+1}(M_{t+1})] = \mathbb{E}_t [V_{t+1}(RA_t + Y_{t+1})] \text{ for known } A_t?$$



# Numerical integration

- **General problem:** How can we calculate

$$\mathbb{E}(f(x)) = \int f(x)dg(x)$$

- $f : \mathbb{R} \rightarrow \mathbb{R}$  some function
- $g(x)$  is the cumulative distribution function for  $x$
- **General solution:** Turn it into a discrete sum

$$\mathbb{E}(f(x)) \approx \sum_{i=1}^S \omega_i f(x_i)$$

- **How to choose  $S$  and the *nodes* ( $x_i$ ) and *weights* ( $\omega_i$ )?**

Three standard methods:

- ① Monte Carlo integration
- ② Equiprobable integration
- ③ Gaussian quadrature



# 1. Monte Carlo integration

- Draw  $S$  (pseudo-)random  $x_i$  from  $g(x)$  indexed by  $i$
- The integral is approximated by

$$\mathbb{E}(f(x)) \approx \sum_{i=1}^S \frac{1}{S} f(x_i)$$

- Can you imagine a potential **drawback** of this method?



## 2. Equiprobable points

- 1 Construct a grid of  $S + 1$  **equally spaced** nodes  $\in [0, 1]$ :

$$\pi = \{0, \frac{1}{S}, \frac{2}{S}, \dots, 1\}$$

- 2 Calculate  $z_i = g^{-1}(\pi_i)$  for  $i \in \{1, 2, \dots, S\}$
- 3 Find the **weighted mid-points** (only once!)

$$x_i = \int_{z_{i-1}}^{z_i} x dg(x) \cdot S \text{ for } i \in \{1, 2, \dots, S\}$$



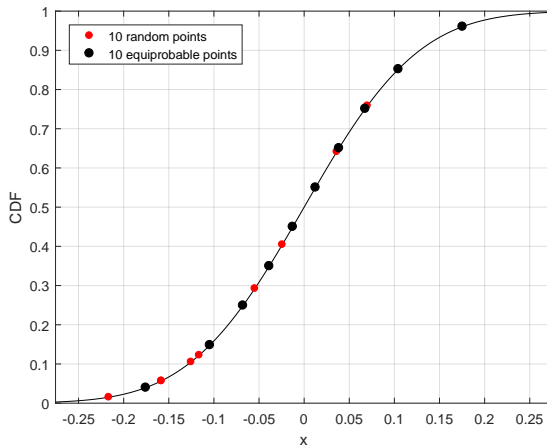
- 4 The integral is approximated by

$$\mathbb{E}(f(x)) \approx \sum_{i=1}^S \frac{1}{S} f(x_i)$$

- Can you imagine a potential **drawback** of this method?



## 2. Equiprobable points - illustration



### 3. Gaussian quadrature

- There are **formulas** for the sequences of  $x_i$  and  $\omega_i$  for *exact* integration of certain polynomials
- Formulas are **domain dependent**:
  - $[a, b] \rightarrow$  Gauss-Chebyshev or Gauss-Legendre quadrature
  - $[0, \infty] \rightarrow$  Gauss-Laguerre quadrature (e.g. exponential distribution)
  - $[-\infty, \infty] \rightarrow$  **Gauss-Hermite quadrature**
    - $S$  points can correctly integrate polynomials of degree  $2S - 1$





### 3. Gauss-Hermite

- **Gauss-Hermite** quadrature uses that

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \sum_{i=1}^S \omega_i f(x_i) + \frac{S! \sqrt{\pi}}{s^S (2S)!} f^{(2S)}(\epsilon)$$

for some  $\epsilon$  and where the  $(x_i, \omega_i)$ 's can be easily found

- **Well behaved function:** For  $S \rightarrow \infty$  we have

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx \approx \sum_{i=1}^S \omega_i f(x_i) \quad (1)$$

- **Random normal variable:**  $Y \sim \mathcal{N}(\mu, \sigma^2)$  so that

$$\begin{aligned} \mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^S \omega_i f(\sqrt{2}\sigma x_i + \mu) \end{aligned}$$



### 3. Gauss-Hermite: Derivation

Use the change of variables

$$x = \frac{y - \mu}{\sqrt{2}\sigma}$$

and insert  $y = \sqrt{2}\sigma x + \mu$  to get

$$\begin{aligned}\mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(y) e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(\sqrt{2}\sigma x + \mu) e^{-\frac{(\sqrt{2}\sigma x + \mu - \mu)^2}{2\sigma^2}} d(\sqrt{2}\sigma x + \mu) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(\sqrt{2}\sigma x + \mu) e^{-x^2} \sqrt{2}\sigma dx \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\sqrt{2}\sigma x + \mu) e^{-x^2} dx \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^S \omega_i f(\sqrt{2}\sigma x_i + \mu)\end{aligned}$$

where the last line is from the Gauss-Hermite formula, eq. (1)



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**Algorithm 10:** Find  $V_t^*$  given  $V_{t+1}^*$  [find\_V]

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**input :**  $V_{t+1}^*[\bullet]$  and Gauss-Hermite nodes and weights:  $x[\bullet]$ ,  
 $\omega[\bullet]$   
 $\mathcal{G}_M$  (grid for  $M$  with  $\#_M$  elements)  
 $\mathcal{G}_C$  (grid for  $C$  with  $\#_C$  elements in  $(0,1)$ )  
**output:**  $V_t^*[\bullet]$  (value of optimal choice)  
 $C_t^*[\bullet]$  (optimal choice)

```

1  for  $i_M = 1$  to  $\#_M$  do
2       $V_t^*[i_M] = -\infty$ 
3       $M_t = \mathcal{G}_M[i_M]$ 
4      for  $i_c = 1$  to  $\#_C$  do
5           $C_t = \mathcal{G}_C[i_c]M_t$ 
6           $EV_{t+1} = 0$ 
7          for  $i = 1$  to  $S$  do
8               $M_{t+1} = R(M_t - C_t) + \exp(\sqrt{2}\sigma_{\xi}x[i])$ 
9               $EV_{t+1} = EV_{t+1} + \frac{\omega[i]}{\sqrt{\pi}} \text{interp}(M_{t+1}, \mathcal{G}_M, V_{t+1}^*)$ 
10          $V = \sqrt{C_t} + \beta EV_{t+1}$ 
11         if  $V > V_t^*[i_M]$  then
12              $V_t^*[i_M] = V$ 
13              $C_t^*[i_M] = C_t$ 

```



# Comparing methods

**Table:** Integration of  $f(x) = x^2$  with  $x \sim \mathcal{N}(0, 1)$ .

MC (10)	MC (50000)	Equi (10)	Equi (50)	Hermite (10)
0.8824	0.9959	0.9590	0.9947	1.0000

- **Monte Carlo** is imprecise even when using many points
- **Equiprobable points** are much more accurate
- **Gaussian quadrature rules!** But also smooth polynomial, so not surprising



# Multi-dimensional integration I

$$\mathbb{E}(f(x_1, x_2)) \approx \sum_i^{S_1} \sum_j^{S_2} w_{1i} w_{2j} f(x_{1i}, x_{2j})$$

- ① **Monte-Carlo:** The same
- ② **Equiprobable:** Somewhat more complicated (not unique)
- ③ **Quadrature:** Simplest with a tensor product

- For  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathcal{N}(\mu, \Omega)$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ \rho_{12} & \sigma_2 \end{bmatrix}}_{\Omega^{\frac{1}{2}}} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$\eta_1, \eta_2 \sim \mathcal{N}(0, 1)$$

- $\Omega^{\frac{1}{2}}$  is **not unique** (here *lower cholesky*. Matlab has `sqrtm`)
- Tensor product uses a lot of points  $\rightarrow$  **sparse grids**



# Multi-dimensional integration II

- ① Let  $(y_i, \omega_i^y)$  and  $(z_j, \omega_j^z)$  be two sets of Gauss-Hermite nodes
- ② For all  $i$  and  $j$  calculate

$$\tilde{y}_i = \sqrt{2}y_i$$

$$\omega_i^{\tilde{y}} = \omega_i^y / \sqrt{\pi}$$

$$\tilde{z}_j = \sqrt{2}z_j$$

$$\omega_j^{\tilde{z}} = \omega_j^z / \sqrt{\pi}$$

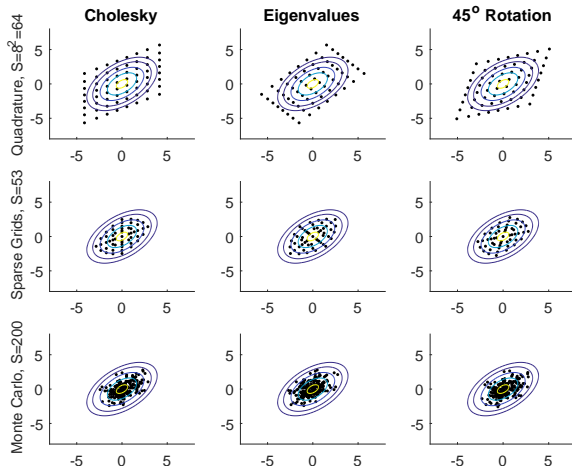
- ③ Then for  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathcal{N}(\mu, \Omega)$

$$\mathbb{E}(f(x_1, x_2)) \approx \sum_i^{S_1} \sum_j^{S_2} \omega_i^{\tilde{y}} \omega_j^{\tilde{z}} f(\sigma_1 \tilde{y}_i + \mu_1, \rho_{12} \tilde{y}_i + \sigma_2 \tilde{z}_j + \mu_2)$$

$$\text{where } \begin{bmatrix} \sigma_1 & 0 \\ \rho_{12} & \sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ \rho_{12} & \sigma_2 \end{bmatrix}^T = \Omega$$



# Multi-dimensional integration - illustration



$$\Omega = [1, 0.5; 0.5, 1]$$

the lines are contour lines for the pdf



## 4. Discretization a la Tauchen

- Assume that  $x_t$  follows the **stationary process** ( $\rho < 1$ )

$$x_{t+1} = \rho x_t + \xi_{t+1}, \quad \xi_{t+1} \sim \mathcal{N}(0, \sigma_\xi^2)$$

- Discretize**  $x_t$  into  $S$  nodes and use a **transition matrix** such that

$$\Pr[x_{t+1} = x_j | x_t = x_i] = \omega_{ij}$$

- Originally:** Tauchen (1986), Tauchen and Hussey (1991), Rouwenhorst (1995)
- Inaccurate if  $\rho$  close to 1** (solutions in Galindev and Lkhagvasuren (2009) and Kpoecky and Suen (2009))
- Extension to **non-Gaussian** processes recently developed by Civalle-Diaz-Fazilet (2017)
- New methods for estimating a **fully flexible Markov process** directly on data, De Nardi et. al. (2016) and Druedahl and Munk-Nielsen (2017)





# Counter-factual analysis I

- We can now **solve**

$$V_t(M_t) = \max_{C_t} \sqrt{C_t} + \beta \mathbb{E}_t [V_{t+1}(M_{t+1})]$$

s.t.

$$A_t = M_t - C_t$$

$$M_{t+1} = R \cdot A_t + Y_{t+1}$$

$$Y_{t+1} = \exp(\xi_{t+1})$$

$$\xi_{t+1} \sim \mathcal{N}(0, \sigma_\xi^2)$$

$$A_t \geq 0$$

and find  $C_t^*(M_t)$  for all  $M_t$

- **How would you measure the cost of income risk?**



# Counter-factual analysis II

- **How would you measure the cost of income risk?**

- ① **Solve** the model for **various values of  $\sigma_{\xi}$** , i.e. find  $C_t^*(M_t; \sigma_{\xi})$
- ② **Simulate**  $N$  individuals for  $T$  periods for each  $\sigma_{\xi}$   
(same seed and draws up to scaling with  $\sigma_{\xi}$ )

$$C_{it} = C_t^*(M_{it}; \sigma_{\xi})$$

$$M_{it+1} = R(M_{it} - C_{it}) + Y_{it+1}$$

$$C_{it+1} = C_{t+1}^*(M_{it+1}; \sigma_{\xi})$$

$$M_{it+2} = R(M_{it+1} - C_{it+1}) + Y_{it+2}$$

$$\vdots$$

- ③ **Compare the average value-of-life**

$$V(\sigma_{\xi}) = \frac{1}{N} \sum_{i=1}^N \sqrt{C_{i1}} + \beta \sqrt{C_{i2}} + \cdots + \beta^T \sqrt{C_{iT}}$$

- **Consumption equivalent:** Let the utility function be  $\sqrt{C_t + k}$ , and use  $k$  to negate changes in the average value-of-life when changing  $\sigma_{\xi}$



## Envelope theorem

$$F(x) = \max_{z \in \mathcal{Z}} f(x, z) = f(x, z^*(x))$$

$$z^*(x) = \arg \max_{z \in \mathcal{Z}} f(x, z)$$

If differentiability is not a problem then

$$\begin{aligned} F'(x) &= f'_x(x, z^*(x)) + f'_z(x, z^*(x))z^{*\prime}(x) \\ &= f'_x(x, z^*(x)) \end{aligned}$$



# FOC and Euler-equation

- Consider the problem:

$$\begin{aligned} V_t(M_t) &= \max_{C_t} u(C_t) + \beta V_{t+1}(M_{t+1}) \\ &= u(C_t^*(M_t)) + \beta V_{t+1}(R(M_t - C_t^*(M_t)) + Y_{t+1}) \end{aligned}$$

- Using the **envelope theorem** we get

$$\begin{aligned} V_t'(M_t) &= \beta R V_{t+1}'(R(M_t - C_t^*(M_t)) + Y_{t+1}) \\ &= \beta R V_{t+1}'(M_{t+1}) \end{aligned}$$

- FOC for optimal  $C_t$ ,  $\frac{\partial u(C_t) + \beta V_{t+1}(R(M_t - C_t) + Y_{t+1})}{\partial C_t}$ :

$$\begin{aligned} u'(C_t) &= \beta R V_{t+1}'(M_{t+1}) = V_t'(M_t) \leftrightarrow \\ u'(C_t) &= \beta R u'(C_{t+1}) \end{aligned}$$

- This is the **Euler-equation** for interior optimal choices



# Variational approach I

- **Optimal plan**  $C_t^*, C_{t+1}^*, C_{t+2}^* \dots$  but
  - set  $C_t = C_t^* + \Delta$  today (require  $0 < C_t^* < M_t$ )
  - set  $C_{t+1} = C_{t+1}^* - R\Delta$  tomorrow
  - so that  $M_{t+1} = R(M_t - (C_t^* + \Delta))$
  - and  $M_{t+1} - C_{t+1} = R(M_t - C_t^*) - C_{t+1}^*$  (no  $\Delta$ )
- We have

$$V_t^*(M_t) = u(C_t^*) + \beta u(C_{t+1}^*) + \beta^2 u(C_{t+2}^*) + \dots$$

$$V_t(M_t) = u(C_t) + \beta u(C_{t+1}) + \beta^2 u(C_{t+2}) + \dots$$

- So that

$$\begin{aligned} V_t^*(M_t) - V_t(M_t) &= [u(C_t^*) - u(C_t)] + \beta[u(C_{t+1}^*) - u(C_{t+1})] \\ &= [u(C_t^*) - u(C_t^* + \Delta)] \\ &\quad + \beta[u(C_{t+1}^*) - u(C_{t+1}^* - R\Delta)] \end{aligned}$$



# Variational approach II

- Finally, we have

$$\lim_{\Delta \rightarrow 0} V^*(M_t) - V(M_t) = 0 \Leftrightarrow$$

$$\lim_{\Delta \rightarrow 0} \frac{V^*(M_t) - V(M_t)}{\Delta} = 0 \Leftrightarrow$$

$$\lim_{\Delta \rightarrow 0} \frac{[u(C_t^*) - u(C_t^* + \Delta)] + \beta[u(C_{t+1}^*) - u(C_{t+1}^* - R\Delta)]}{\Delta} = 0 \Leftrightarrow$$

$$\lim_{\Delta \rightarrow 0} \frac{u(C_t^*) - u(C_t^* + \Delta)}{\Delta} - \beta R \lim_{\Delta \rightarrow 0} \frac{u(C_{t+1}^*) - u(C_{t+1}^* - R\Delta)}{-R\Delta} = 0 \Leftrightarrow$$

$$u'(C_t^*) - \beta R u'(C_{t+1}^*) = 0 \Leftrightarrow$$

$$u'(C_t^*) = \beta R u'(C_{t+1}^*)$$



# Euler-residuals I

- ① Solve a deterministic model
- ② **Simulate** consumption paths over  $T$  periods for  $N$  individuals indexed by  $i$
- ③ Calculate the **Euler-residual** in each period for each individual

$$\mathcal{E}_{it} \equiv u'(C_{it}) - \beta R u'(C_{it+1})$$

- ④ Calculate the **average absolute Euler-error**

$$\frac{1}{\sum_{t=1}^T \sum_{i=1}^N \mathbf{1}_{\{0 < C_t < M_{it}\}}} \sum_{t=1}^T \sum_{i=1}^N \mathbf{1}_{\{0 < C_t < M_{it}\}} |\mathcal{E}_{it}|$$



# Euler-residuals II

- ① Solve a stochastic model.
- ② **Simulate** consumption paths over  $T$  periods for  $N$  individuals indexed by  $i$
- ③ Calculate the **Euler-residual** in each period for each individual

$$\begin{aligned}\mathcal{E}_{it} &\equiv u'(C_{it}) - \beta R \mathbb{E}_t [u'(C_{t+1}^*(R(M_{it} - C_{it}) + Y_{it+1})))] \\ &\approx u'(C_{it}) - \beta R \sum_{j=1}^S \omega_j [u'(C_{t+1}^*(R(M_{it} - C_{it}) + Y_{jt+1})))]\end{aligned}$$

- ④ Calculate the **average absolute Euler-error**

$$\frac{1}{\sum_{t=1}^T \sum_{i=1}^N \mathbf{1}_{\{0 < C_t < M_{it}\}}} \sum_{t=1}^T \sum_{i=1}^N \mathbf{1}_{\{0 < C_t < M_{it}\}} |\mathcal{E}_{it}|$$





# Euler-residuals III

- Instead of

$$\frac{1}{\sum_{t=1}^T \sum_{i=1}^N \mathbf{1}_{\{0 < C_t < M_{it}\}}} \sum_{t=1}^T \sum_{i=1}^N \mathbf{1}_{\{0 < C_t < M_{it}\}} |\mathcal{E}_{it}|$$

- It is **common** to use

$$\frac{1}{\sum_{t=1}^T \sum_{i=1}^N \mathbf{1}_{\{0 < C_t < M_{it}\}}} \sum_{t=1}^T \sum_{i=1}^N \log_{10}(|\mathcal{E}_{it}|/C_{it}) \mathbf{1}_{\{0 < C_t < M_{it}\}}$$

so that an error of  $-2$  means that the agent is making a mistake equal to \$1 for every \$100 consumed, while an error of  $-3$  means that the agent is making a mistake equal to \$0.1 for every \$100 consumed etc,



# Until next

- **Ensure that you understand:**
  - ① The principle in one-dimensional **numerical integration** and algorithm 10
  - ② The principle in **counter-factual simulation**
  - ③ The use of avg. Euler errors as an **accuracy measure**
- Go to **PadLet** and ask or answer a question  
([https://padlet.com/thomas\\_jorgensen1/DP](https://padlet.com/thomas_jorgensen1/DP))
- **Think about:** What happens when  $t \rightarrow \infty$ ?

