Lecture 9: Structural Estimation

Dynamic Programming

Thomas Jørgensen

Structural estimation

- We know how to **solve dynamic programming models**
- **How can we estimate them?** We need
  1. Data on (some) *states*
  2. Data on (some) *choices*
- Two **standard approaches**
  1. Maximum likelihood (ML)
  2. General Method of Moments (GMM)
- **Simulated versions:**
  1. Maximum Simulated Likelihood (MSL, SML)
  2. Method of Simulated Moments (MSM, SMM)
- **Example model:** Life-cycle buffer-stock model
  - States: M_{it}, P_{it}
  - Choice: C_{it}
- **Parameters** to estimate: $\theta = \{\beta, \rho\}$
  - Calibration: G, σ_ψ, σ_ξ, R, and λ ("known")

Maximum likelihood estimation (MLE)

- Assume that observed log-consumption is contaminated with mean-zero i.i.d. normal **measurement error**

$$\epsilon_{it}(\theta) \equiv \log C_{it} - \log C_t^\star(M_{it}, P_{it}; \theta) \sim \mathcal{N}(0, \sigma_\xi^2)$$

- The **likelihood** of observing the data then is

$$\Pr(M, P|\theta) = \Pi_{i=1}^{N} \Pi_{t=1}^{T_d} \phi(\epsilon_{it}(\theta))$$

where $M = \{M_{it}\}_{1,1}^{N,T_d}$ and $P = \{P_{it}\}_{1,1}^{N,T_d}$ and

$$\phi(\epsilon_{it}) = \frac{1}{\sqrt{2\pi\sigma_\xi^2}} \exp\left(-\frac{{\epsilon_{it}}^2}{2\sigma_\xi^2}\right)$$

is the Gaussian density function

- **MLE** then is

$$\hat{\theta} \quad = \quad \arg\min_{\theta} -\frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T_d} \log(\phi(\epsilon_{it}(\theta)))$$

Note: *We need to resolve the model for each new guess of θ*

Maximum Simulated Likelihood (MSL)

- MLE Requires that we observe *permanent* income, P_{it}!
- Integrate it out to get the marginal likelihood:

$$\begin{aligned}\Pr(M|\theta) &= \mathbb{E}[\Pr(M,P|\theta)|M,\theta] \\ &= \int_{P_{T_d}} \int_{P_{T-1}} \cdots \int_{P_1} \Pr(M,P|\theta) dP_T dP_{T-1} \cdots dP_1\end{aligned}$$

- Drawing S draws of $P_1, P_2, \ldots, P_{T_d}$ could be used to get

$$\widehat{\Pr(M|\theta)} = \frac{1}{S}\sum_{s=1}^{S} \Pr(M, P^{(s)}|\theta)$$

- and the MSL estimator is then

$$\hat{\theta} \quad = \quad \arg\min_{\theta} -\log\left(\widehat{\Pr(M|\theta)}\right)$$

- ***We would need extremely many draws to approximate this T-dimensional integral: Remember Jensen's Inequality!***
- *(We could actually use the Kalman Filter in the current model!)*

Method of Simulated Moments (MSM)

- Let Λ^d be some **moments in the data**
  - Could be avg., var, cov, regression-coefs, etc.
- Let $\Lambda_s(\theta)$ be the **same moments** calculated on data **simulated** from the model solved with parameters θ
- **MSM** then is

$$\hat{\theta} = \arg\min_{\theta} \left(\Lambda^d - \Lambda^m(\theta)\right)' W \left(\Lambda^d - \Lambda^m(\theta)\right)$$

where we average across simulations

$$\Lambda^m(\theta) \equiv \frac{1}{S} \sum_{s=1}^{S} \Lambda_s(\theta)$$

and W is a positive-definite **weighting matrix**.

- We still approximate the T_d-dimensional integral with S simulations but we do not suffer from Jensen's inequality

Weighting matrix

- Typical choices are

  1. **Theoretically optimal** (see Adda and Cooper for formula)
     Can cause problems in finite samples
  2. **Diagonal matrix** with **inverse** of (bootstrapped) empirical **variances of the moments** (scaled appropriately)
  3. **Freely chosen** to focus on fitting some specific dimensions of the data

Indirect inference / minimum distance

- Many different names for very similar approaches
  - McFadden (1989): Method of Simulated Moments (MSM)
  - Duffie and Singleton (1993): Simulated Minimum Distance (SMD)
  - Gourieroux, Monfort and Renault (1993) + Smith (1993): Indirect Inference (II)
- SMD/II rely on an **auxillary statistical model**
  - Let Λ^d be the parameters of the auxillary model when estimated on the *actual* data
  - Let $\Lambda_s(\theta)$ be the parameters of the auxillary model when estimated on *simulated* data
- **Note:** The auxillary statistical model is *misspecified* and its parameters are thus typically *not interpretable*

Simulation Pitfalls

- **FIX the seed (or draws!)**
- **Flat** objective function!
  - Discrete choices: Taking a mean of an **indicator function**
- **Gradient** based numerical optimization will likely FAIL!
  - Use, e.g., `fminsearch` (Nelder-Mead)
  - Or some smoothing device (e.g. Logit)
- As $N, S \to \infty$ this problem vanishes
- The problem is also less severe around θ_0
- Continuous outcomes do not have this problem

Asymptotics

- **MSM** is **consistent** and **asymptotically normal** under standard assumptions

$$\sqrt{N}(\hat{\theta} - \theta_0) \to \mathcal{N}(0, (1 + S^{-1})V)$$

where θ_0 are the true parameters

- **Standard formulas for V:** See Adda and Cooper
Remember: *Standard errors are large if large changes in θ imply small changes in the objective function*
- **Computational limitations:** To compute standard errors we need to resolve the model

Identification

- **Is there enough variation in the data to identify θ?**
  Very hard to *prove* anything because the model is typically strongly non-linear
- **MSM:** At least the same number of moments as parameters
- **Problems:**
  1. The objective function might have multiple minima
  2. The objective function could be very flat in some directions
- **Graphical inspection is useful:** Plot the objective function in the neighborhood of the found optimum
  Systematic approaches are being developed (Andrews et al (2017))
- **Use more data**
  1. **Quantitatively:** More agents, more time periods
  2. **Qualitative:** New types of data, e.g natural experiments around policy changes

Estimation experiment

1. **Solve** the buffer-stock model and **simulate** a full panel
2. Construct a **data set** from the simulated data

   Likelihood: Log-consumption at age 45 with measurement error
   MSM: Average wealth for each age between 40 and 55
3. Try to **estimate** $\theta = \{\beta, \rho\}$

Methods
Reduced form estimation
Examples
Additional methods
Until next

Implementation, $\hat{\theta}_{MSM} = \arg\min_{\theta} Q(\theta)$

For Λ^d and a given value of θ, $Q(\theta)$:

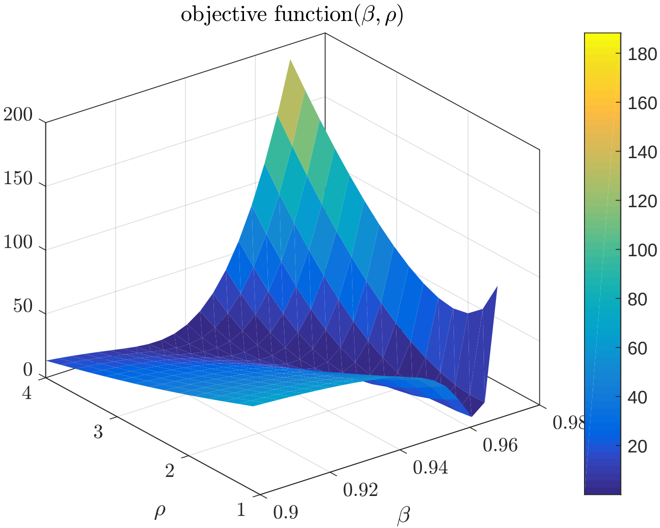
1. Solve model to get $c_t^\star(m;\theta)$ on a grid of m
2. For $s = 1, \dots, S$:
   1. Simulate N agents for T periods to get

$$
\begin{aligned}
C_{it}^{(s)}(\theta) &= P_{it}^{(s)} \cdot \check{c}_t^\star(M_{it}^{(s)}(\theta)/P_{it}^{(s)};\theta) \\
M_{it}^{(s)}(\theta) &= RA_{it-1}^{(s)}(\theta) + Y_{it}^{(s)} \\
A_{it-1}^{(s)}(\theta) &= M_{it-1}^{(s)}(\theta) - C_{it-1}^{(s)}(\theta) \\
Y_{it}^{(s)} &= P_{it}^{(s)} \xi_{it}^{(s)} \\
P_{it}^{(s)} &= GP_{it-1}^{(s)} \psi_{it}^{(s)}
\end{aligned}
$$

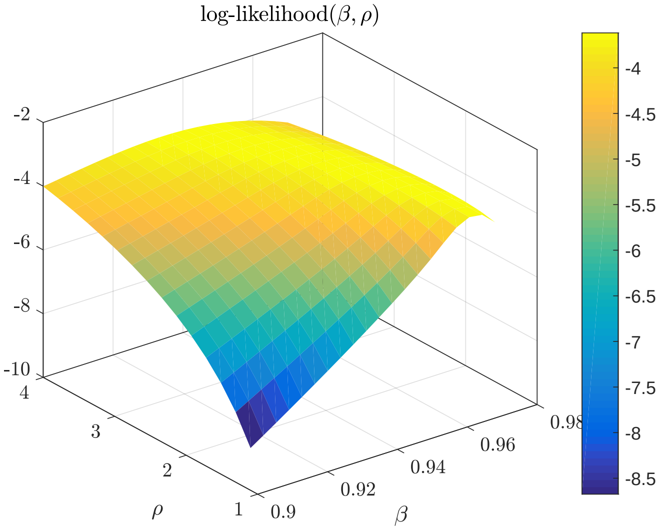
   for some initial A_{i0} and P_{i0} and draws of **?**$\xi_{it}^{(s)}$ and $\psi_{it}^{(s)}$.
   2. Calculate the moments using this simulated data, $\Lambda_s(\theta)$ $(\{\frac{1}{N}\sum_{i=1}^{N} A_{it}^{(s)}(\theta)\}_{t=40}^{55})$
3. Calculate the objective function

$$
Q(\theta) = \left(\Lambda^d - \frac{1}{S}\sum_{s=1}^{S}\Lambda_s(\theta)\right)' W \left(\Lambda^d - \frac{1}{S}\sum_{s=1}^{S}\Lambda_s(\theta)\right)
$$

Buffer-stock: MSM



Buffer-stock: Likelihood



Robustness

- **Curse of dimensionality and lack of identification**
  $\Rightarrow$ we cannot estimate all the parameters of the model
  $\Rightarrow$ *first step calibration is necessary*
  1. Calculations on own data (e.g. exogenous processes)
  2. References to previous estimates
  3. Standard choices
- **Robustness:** Can we vary the calibration choices without changing the result substantially?
  - **Or the opposite:** When does the result break down?
  - Approach being developed by yours truly..
- **Calibration** is also important for
  1. Gaining intuition for how the model work
  2. Initial guesses for estimation algorithm

Reduced form estimation

- Critic of structural estimation: **Requires many assumptions**
- **Alternative:** Estimate reduced form equations "derived" from the model
- **My (and others) claim:** To turn reduced form parameter estimates into policy advice *a lot of assumptions are often implicitely required*

  > *"All econometric work relies heavily on a priori assumptions. The main difference between structural and experimental (or "atheoretic") approaches is not in the number of assumptions but the extent to which they are made explicit."* (Keane, 2012)

- **The beauty of models:**
  1. Ensure *consistent* world view
  2. Allow us to combine *heterogenous facts* and extrapolate from a myriad of past experiences
  3. Better models are clearly defined – even if we never find *the* true model we can make *progress*
- **Frontier:** Combine the two and use exogenous variation to estimate structural model.

The Lucas critique

- **The Lucas critique:** *Behavioral rules change with policy*
  $\Rightarrow$ policy advice can not rely on estimated behavioral rules
  $\Rightarrow$ we need to estimate *structural parameters*

  > *"Invariance of parameters in an economic model is not, of course, a property which can be assured in advance, but it seems reasonable to hope that neither tastes nor technology vary systematically with variations in counter-cyclical policies." (Lucas, 1977)*

- **Other stuff might be approximately invariant**
- **Rigourous microfoundations:**
  1. **Mathematically:** Based on (boundedly) rational behavior derived as a solution to a formal optimization problem
  2. **Economically:** The assumptions are realistic

Examples

- **Gourinchas and Parker (2002):** First structural estimation of buffer-stock consumption model
  - **Method:** MSM with a lot of first stage calibrations
  - **Data:** Cross-sectional consumption data from CEX
- **Two other examples:**
  1. **Cagetti (2003):** MSM matching of *median* wealth profiles
  2. **Druedahl and Jørgensen (2017):**
     MSM Monte Carlo of misspecifying the income process
  3. **Druedahl and Jørgensen (2018):**
     MSM estimation of an extended Buffer-Stock model with learning

Gourinchas and Parker (2002) I

TABLE III

STRUCTURAL ESTIMATION RESULTS

MSM Estimation	Robust Weighting	Optimal Weighting
Discount Factor (β)	0.9598	0.9569
S.E.(A)	(0.0101)	
S.E.(B)	(0.0179)	(0.0150)
Discount Rate $(\beta^{-1}-1)(\%)$	4.188	4.507
S.E.(A)	(1.098)	
S.E.(B)	(1.949)	(1.641)
Risk Aversion (ρ)	0.5140	1.3969
S.E.(A)	(0.1690)	
S.E.(B)	(0.1707)	(0.1137)
Retirement Rule:		
γ_0	0.0015	$5.68\,10^{-6}$
S.E.(A)	(3.84)	
S.E.(B)	(3.85)	(16.49)
γ_1	0.0710	0.0613
S.E.(A)	(0.1215)	
S.E.(B)	(0.1244)	(0.0511)
χ^2(A)	175.25	
χ^2(B)	174.10	185.67

Note: MSM estimation for entire group. Standard errors calculated without (A) and with (B) correction for first stage estimation. Cell size is 36,691 households. The last row reports a test of the overidentifying restrictions distributed as a Chi-squared with 36 degrees of freedom. The critical value at 5% is 50.71. Efficient estimates are calculated with a weighting matrix $\widehat{\Omega}$ computed from the robust estimates.

Gourinchas and Parker (2002) II

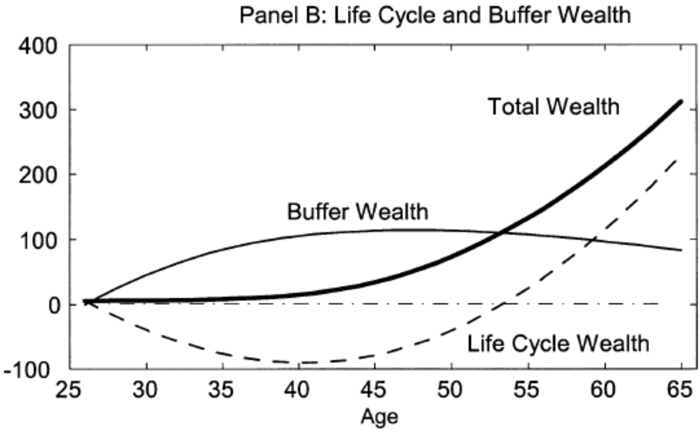


FIGURE 7.—The role of risk in saving and wealth accumulation.

Mathematical Programming with Equilibrium Constraints (MPEC)

- **Idea:** Do not solve the model, treat it as a constraint
- **Example:** Infinite horizon buffer-stock consumption model

$$\begin{aligned} \hat{\theta}, \hat{c}_1, \dots, \hat{c}_\# &= \arg \max_{\theta, c_1, \dots, c_\#} \mathcal{L}(\theta) \\ &\text{s.t.} \\ 0 &\leq c_j \leq m_j \\ 0 &\geq \mathcal{E}_j \\ 0 &= (m_j - c_j)\mathcal{E}_j \end{aligned}$$

where $\mathcal{E}_j$ is the j'th Euler-residual

$$\mathcal{E}_j \equiv \beta R \mathbb{E}_t[(G\psi_{t+1} c_{t+1}(\frac{1}{G\psi_{t+1}} R a_i + \xi_{t+1}))^{-\rho}] - c_j^{-\rho}$$

and $c_{t+1}(\bullet)$ is interpolated using $c_1, c_2, \dots, c_\#$

- **See Jørgensen (2013) + Will see this later again**

Until next

- **Ensure that you understand:**
  1. Maximum likelihood estimation
  2. Method of simulated moments
  3. How to discuss identification