Exercise Set 1

Lectures 1-5, Weeks 1-3

Dynamic Programming, Spring 2018

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This exercise set contains exercises for the theory and tools part of the course. In brackets after each exercise it is stated which lecture the exercise is based on; L1 e.g. means lecture 1. For most exercises there is some ex ante code available at *Absalon,* which should help you get started. The plan is that you should work on these exercises for the first three weeks of the course.

Exercise 1 [L1]: The Simplest Consumption Model

Consider the simplest consumption-saving model:

$$\begin{aligned}
V^{\star}(M) &= \max_{C_1,C_2,\dots,C_T} \left\{ \sqrt{C_1} + \beta\sqrt{C_2} + \beta^2\sqrt{C_3} + \cdots + \beta^T\sqrt{C_T} \right\} \\
&\text{s.t.} \\
M &= C_1 + C_2 + \cdots + C_T \\
C_t &\in \mathbb{N}
\end{aligned}$$

for $T = 3$, $\beta = 0.90$ and $M = 5$.

1. Solve the model using the *brute force* (algorithm 3 from lecture 1).
2. Vary β and M and check that the results fit with your intuition. *If $\beta = 0$ then everything should e.g. be consumed in the first period.*
3. Solve the model using *backwards induction* (algorithm 4+5 from lecture 1).
4. Check that the solutions from 1) and 3) are the same.
5. Solve the model for $T = 10$ using a method of your choice.
6. Simulate $C_1, C_2, \dots, C_{10}$ using the policy function found in 5) and $M = 5$.

Exercise 2 [L2]: Discrete shock

Consider the simplest consumption-saving model, but with a discrete shock:

$$\begin{aligned}
V_t(M_t) &= \max_{C_t}\left\{\sqrt{C_t}+\beta\mathbb{E}_t[V_{t+1}(M_{t+1})]\right\} \\
&\text{s.t.} \\
M_{t+1} &= \begin{cases} M_t - C_t + 1 & \text{with probability } \pi \in (0,1) \\ M_t - C_t & \text{else} \end{cases} \\
C_t &\leq M_t \\
C_t &\in \mathbb{N}
\end{aligned}$$

for $T = 10$, $\beta = 0.90$, $\pi = 0.5$ and $M_1 = 10$.

1. Solve the model using *backwards induction* (see algorithm 6 from lecture 2).
2. Plot $V_1(M_t)$ (the value function in period 1) and $C_1^\star(M_t)$ (the consumption function in period 1). *Ensure that both function are increasing in M_t.*
3. Vary π and check that both $V_1(M_t)$ and $C_1^\star(M_1)$ move weakly upwards in π.

Exercise 3 [L2]: Interpolation and continuous choice

Consider the following MATLAB code to create an linear interpolant:

```
f = @(x) (x-3).^2;

x_true  = linspace(0,6,100);
f_true  = f(x_true);
x_known = [1 2 3 4 5];
f_known = f(x_known);

f_linear_interp_func = griddedInterpolant(x_known,f_known,'linear');
f_linear_interp      = f_linear_interp_func(x_true);
```

1. Plot the true function, the known points and the linearly interpolated function in a single figure.

Consider the model:

$$
\begin{aligned}
V_t(M_t) &= \max_{C_t} \left\{ \sqrt{C_t} + \beta \mathbb{E}_t [V_{t+1}(M_{t+1})] \right\} \\
&\text{s.t.} \\
M_{t+1} &= \begin{cases} M_t - C_t + 1 & \text{with probability } \pi \in (0,1) \\ M_t - C_t & \text{else} \end{cases} \\
C_t &\leq M_t \\
C_t &\in \mathbb{R}
\end{aligned}
$$

for $T = 10$, $\beta = 0.90$, $\pi = 0.5$, $M_1 = 5$.

1. Solve the model using *backwards induction*, *grid search*, and *linear interpolation* (see algorithm 9 from lecture 2).
2. Plot $V_1(M_t)$ (the value function in period 1) and $C_1^\star(M_t)$ (the consumption function in period 1). *Ensure that both functions are increasing in M_t.*
3. Solve the model using *backwards induction*, but by calling an *optimizer* (see ex ante code) instead of using grid search. *Ensure that the results are very similar to those in 1).*

Exercise 4 [L3]: Numerical Integration

Consider the numerical integration problem

$$\int x^2 dg(x), \;\; x \sim \mathcal{N}(0,1)$$

Note that we can analytically show that

$$\int f(x) dg(x) = 1$$

1. Approximate the integral using *Monte Carlo integration.*
2. Approximate the integral using *equieprobable integration* (see ex ante code for help).
3. Approximate the integral using Gauss-Hermite *integration* (see ex ante code for help).
4. Compare the three methods across various number of grid points. How few grid points do you need for Gauss-Hermite integration?
5. Change the function f and see what happens.

Exercise 5 [L3]: Gaussian shock

Consider the model

$$
\begin{aligned}
V_t(M_t) &= \max_{C_t} \left\{ \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}) \right] \right\} \\
& \text{s.t.} \\
M_{t+1} &= R(M_t - C_t) + Y_{t+1} \\
Y_{t+1} &= \exp(\xi_{t+1}) \\
\xi_{t+1} &= \mathcal{N}(0, \sigma_\xi^2) \\
A_t &\geq 0
\end{aligned}
$$

for $T = 10$, $\beta = 0.98$, $\rho = 0.5$, $R = 1.0/\beta$, and $\sigma_\xi = 0.2$.

1. Solve it using *Gauss-Hermite quadrature* for evaluating the expectation (see algorithm 10 in lecture 3)
[re-use your code from exercise 3].

2. Simulate a panel of N household for T periods and plot the mean of C_t. *Everybody should be initialized with $M_t = 1.5$.*

3. Check that the average of C_t is *weakly increasing* over time.

4. Check that if you temporarily set $\sigma_\xi = 0$ then average C_t is *constant* over time.

5. Calculate the Euler-error

$$
\frac{1}{\sum_{i=1}^{N} \mathbf{1}_{\{0<C_1<M_{i1}\}}} \sum_{i=1}^{N} \mathbf{1}_{\{0<C_1<M_{i1}\}} |\mathcal{E}_{i1}|
$$

where

$$
\mathcal{E}_{it} \approx u'(C_{it}) - \beta R \sum_{j=1}^{S} \omega_j \left[u'(C_{t+1}^{\star}(R(M_{it} - C_{it}) + Y_j)) \right]
$$

and ω_j are the Gauss-Hermite weights and Y_j are the associated income nodes.

6. Likewise calculate the normalized Euler-error

$$
\frac{1}{\sum_{i=1}^{N} \mathbf{1}_{\{0<C_1<M_{i1}\}}} \sum_{i=1}^{N} \log_{10}(\mathcal{E}_{it}/C_{it}) \mathbf{1}_{0<C_1<M_{i1}\}}
$$

and discuss what this implies for the accuracy of the solution.

7. Look at how the Euler-errors change when you vary the number of grid points. *Ensure that you are able to get an normalized Euler error smaller than at least* -3.0.

8. Plot the value and consumption functions for multiple t – do you see any pattern?

Exercise 6 [L4]: Infinite horizon

Consider the same model as in exercise 5, but with $R = 1.00$.

1. Solve the model for $t \to \infty$ for a tolerance of 10^{-2} (see algorithm 10, lecture 4).
2. Plot the converged value and policy functions.
3. Vary β and check that the number of periods until convergence is increasing in β. *What are the two reasons for this lower convergence?*

Exercise 7 [L5]: Function approximation

Consider the function

$$f(x) = \min\left\{\max\left\{-1, 4(x-0.2\right\}, 1\right\}$$

1. The ex ante code contains an example of how well linear interpolation can approximate this function. Re-do the analyse with:
   (a) Cubic spline
   (b) Schumacker spline (see SchumakerSpline.m)
   (c) Regression with regular polynomials of 4th order
   (d) Regression with Chebyvev polynomials (see Chebyshev.m)

Next, consider the function

$$f(x,z) = (x+1)^4 \cdot (z+1)^4 + \mathbf{1}_{zx>0.3}$$

2. Use the ex ante code to investigate how large the error from using linear interpolation is.

Exercise 8 [L5]: Taste-Shocks

Consider the model

$$\begin{aligned}
V_t(M_t, \varepsilon_t^0, \varepsilon_t^1) &= \max_{L_t \in \{0,1\}} \left\{ W_t(M_t, L_t) + \sigma_\varepsilon \varepsilon_t^{L_t} \right\} \\
\mathcal{V}_t(M_t, L_t) &= \max_{C_t} \frac{C_t^{1-\rho}}{1-\rho} - \lambda L_t + \beta \mathbb{E}_t \left[V_{t+1}(\bullet_{t+1}) \right] \\
&\text{s.t.} \\
M_{t+1} &= R(M_t - C_t) + W \cdot L_t
\end{aligned}$$

where $T = 10$, $\beta = 0.98$, $\rho = 0.5$, $\lambda = 1.3$, $R = 1.0/\beta$, $W = 1$, $\sigma_\varepsilon = 0.5$, andξ_t^0 and ξ_t^1 are *Extreme Value Type I* such that

$$\begin{aligned}\mathbb{E}[M_t, \xi_t^0, \xi_t^1 | M_t] &= \sigma_\varepsilon \log \left(\sum_{j\in\{0,1\}} \exp\left(\frac{\mathcal{V}_t(M_t, j)}{\sigma_\varepsilon} \right) \right) \\ &\equiv \text{logsum}(\mathcal{V}_t(M_t, \bullet))\end{aligned}$$

1. Solve the model using your prefered method.
2. Plot the choice-specific value functions in period 1 and the optimal choice of L_1 for $\xi_1^0 = \xi_1^1 = 0$.
3. Simulate from the model using that

$$\begin{aligned}\Pr(L_t = 1 | M_t) &= \Pr(\mathcal{V}_t(M_t, 1) - \mathcal{V}_t(M_t, 0) \geq \sigma_\varepsilon(\varepsilon_t^0 - \varepsilon_t^1)) \\ &= \frac{\exp(\mathcal{V}_t(M_t, 1)/\sigma_\varepsilon)}{\sum_{j\in\{0,1\}} \exp(\mathcal{V}_t(M_t, j)/\sigma_\varepsilon)}\end{aligned}$$

4. Plot the frequency of $L_t = 1$.
5. Redo exercise for increasing values of σ_ξ. What happens to the frequency of $L_t = 1$?

Exercise 9 [L5]: Time-iteration

Return to the model in exercise 5 and 6. Remember that optimal consumption must satisfy the Euler-equation

$$u'(C_t) = \mathbb{E}_t \left[u'(C_{t+1}^\star(\Gamma(M_t, C_t)) \right]$$

1. Redo exercise 5 using *time-iteration* (see lecture 5) instead of *value function iteration* but with $\rho = 2.0$.