Lecture 10: Discrete-Continuous Choice Models

Dynamic Programming

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Introduction

- **A lot of choices are inherently discrete**
  1. Full-time / half-time
  2. Retirement
  3. Industry
  4. Sector
  5. Durable purchases

  etc.
- **Today:** Method for efficiently solving discrete-continuous choice models
- **Example:** Consumption-saving model with a discrete retirement choice

Bellman equation

- **Value function** (Is retirement absorbing?, timing?)

$$V_t(m_t, z_t, \varepsilon_t^0, \varepsilon_t^1) = \max_{z_{t+1} \in \mathcal{Z}(z_t)} \left\{ v_t(m_t | z_{t+1}) + \sigma_\varepsilon \varepsilon_t^{z_{t+1}} \right\}$$

$$\mathcal{Z}(z_t) = \begin{cases} \{0,1\} & \text{if } z_t = 0 \\ \{1\} & \text{if } z_t = 1 \end{cases}$$

- **Choice-specific value functions**

$$\begin{aligned} v_t(m_t|z_{t+1}) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} - \alpha \mathbf{1}_{\{z_{t+1}=0\}} + \beta \mathbb{E}_t \left[V_{t+1}(\bullet_{t+1}) \right] \\ &\text{s.t.} \\ m_{t+1} &= R(m_t - c_t) + W \xi_{t+1} \mathbf{1}_{\{z_{t+1}=0\}} \\ c_t &\leq m_t \\ \log \xi_{t+1} &\sim \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\ \varepsilon_{t+1}^0, \varepsilon_{t+1}^1 &\sim \text{Extreme Value Type 1} \end{aligned}$$

Logsum and choice probabilities

- **Retired:** Simple perfect foresight problem - Why?
- **For working households**

$$v_t(m_t|z_{t+1}) = \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} - \alpha z_{t+1} + \beta \mathbb{E}_t \left[\mathcal{W}_{t+1}(m_{t+1}) \right]$$

where

$$\mathcal{W}_t(m_t) = \begin{cases} \sigma_\varepsilon \log \left(\sum_{j \in \{0,1\}} \exp \left(\frac{v_t(m_t|j)}{\sigma_\varepsilon} \right) \right) & \text{if } \sigma_\varepsilon > 0 \\ \max_{j \in \{0,1\}} v_t(m_t|j) & \text{if } \sigma_\varepsilon = 0 \end{cases}$$

- **Choice probilities** for working households

$$\Pr(z_{t+1} = z|m_t) = \begin{cases} \frac{\exp(v_t(m_t|z)/\sigma_\varepsilon)}{\sum_{j \in \{0,1\}} \exp(v_t(m_t|j)/\sigma_\varepsilon)} & \text{if } \sigma_\varepsilon > 0 \\ \mathbf{1}_{v_t(m_t|z) > v_t(m_t|j), \forall j \neq z} & \text{if } \sigma_\varepsilon = 0 \end{cases}$$

Closed form solution?

- For a particular parametrization, John Rust showed in Iskhakov, Jørgensen, Rust and Schjerning (2017):

324 Iskhakov, Jørgensen, Rust, and Schjerning Quantitative Economics 8 (2017)

and instantaneous utility is given by $u(c) = \log(c)$. *Then for* $\tau \in \{1, \dots, T\}$ *the optimal consumption rule in the worker's problem* (2)–(4) *is given by*

$$c_{T-\tau}(M) = \begin{cases} M & \text{if } M \le y/R\beta, \\ [M + y/R]/(1+\beta) & \text{if } y/R\beta \le M \le \overline{M}^{l_1}_{T-\tau}, \\ [M + y(1/R + 1/R^2)]/(1+\beta+\beta^2) & \text{if } \overline{M}^{l_1}_{T-\tau} \le M \le \overline{M}^{l_2}_{T-\tau}, \\ \dots & \dots \\ \left[M + y\left(\sum_{i=1}^{\tau-1} R^{-i}\right)\right]\left(\sum_{i=0}^{\tau-1} \beta^i\right)^{-1} & \text{if } \overline{M}^{l_{\tau-2}}_{T-\tau} \le M \le \overline{M}^{l_{\tau-1}}_{T-\tau}, \\ \left[M + y\left(\sum_{i=1}^{\tau} R^{-i}\right)\right]\left(\sum_{i=0}^{\tau} \beta^i\right)^{-1} & \text{if } \overline{M}^{l_{\tau-1}}_{T-\tau} \le M < \overline{M}^{r_{\tau-1}}_{T-\tau}, \\ \left[M + y\left(\sum_{i=1}^{\tau-1} R^{-i}\right)\right]\left(\sum_{i=0}^{\tau} \beta^i\right)^{-1} & \text{if } \overline{M}^{r_{\tau-1}}_{T-\tau} \le M < \overline{M}^{r_{\tau-2}}_{T-\tau}, \\ \dots & \dots \\ [M + y(1/R + 1/R^2)]\left(\sum_{i=0}^{\tau} \beta^i\right)^{-1} & \text{if } \overline{M}^{r_2}_{T-\tau} \le M < \overline{M}^{r_1}_{T-\tau}, \\ [M + y/R]\left(\sum_{i=0}^{\tau} \beta^i\right)^{-1} & \text{if } \overline{M}^{r_1}_{T-\tau} \le M < \overline{M}_{T-\tau}, \\ M\left(\sum_{i=0}^{\tau} \beta^i\right)^{-1} & \text{if } M \ge \overline{M}_{T-\tau}. \end{cases} \tag{7}$$

The segment boundaries are totally ordered with

$$y/R\beta < \overline{M}^{l_1}_{T-\tau} < \dots < \overline{M}^{l_{\tau-1}}_{T-\tau} < \overline{M}^{r_{\tau-1}}_{T-\tau} < \dots < \overline{M}^{r_1}_{T-\tau} < \overline{M}_{T-\tau}, \tag{8}$$

and the rightmost threshold $\overline{M}_{T-\tau}$, *given by*

$$\overline{M}_{T-\tau} = \frac{(y/R)e^{-K}}{1-e^{-K}}, \quad \text{where } K = \delta\left(\sum_{i=0}^{\tau} \beta^i\right)^{-1}, \tag{9}$$

defines the smallest level of wealth sufficient to induce the consumer to retire at age $t = T - \tau$.

The proof of Theorem 1—in particular, the expressions for the kink points $M^{l_i}_{T-\tau}$ and $M^{r_i}_{T-\tau}$—is available in a supplementary file on the journal website, http://qeconomics.

Euler-equation

- **Euler-equation** for interior choices

$$c_t^{-\rho} = \beta R \mathbb{E}_t \left[c_{t+1}^{-\rho} \right]$$

- **Necessary?** Yes, e.g. by variational argument
- **Sufficient ?** Only if the *value function is strictly concave*

  1. **Retired:** Yes, no discrete choices
     $\Rightarrow$ can be solved by *standard* EGM

  2. **Working:** Possibly no, due to discrete choices
     $\Rightarrow$ we need an *extended* EGM

- **Consumption functions:** $c_t(m_t|z_{t+1})$

  $c_t(m_t|0)$ optimal consumption if *working*
  $c_t(m_t|1)$ optimal consumption if *retiring/retired*

Recap: EGM

- **Prerequisites**
  1. **Inverted Euler-equation:** $c_t = \left[\beta R \mathbb{E}_t \left[c_{t+1}^{-\rho}\right]\right]^{-\frac{1}{\rho}}$
  2. **Next-period consumption functions**: $c_{t+1}(m_{t+1}|z_{t+2})$
  3. **Asset grid:** $\mathcal{G}_a = \{a_1, a_2, \dots, a_\#\}$ with $a_1 = 10^{-6}$
- **Algorithm:** For each $a_i \in \mathcal{G}_a$
  1. Find consumption (using $\Pr(z_{t+2} = z|m_{t+1})$)

$$c_i = \left(\beta R \mathbb{E}_t \left[c_{t+1}(Ra_i + W\xi_{t+1}\mathbf{1}_{z_{t+1}=0}|z_{t+2})^{-\rho}\right]\right)^{-\frac{1}{\rho}}$$

  2. Find endogenous state

$$m_i = a_i + c_i$$

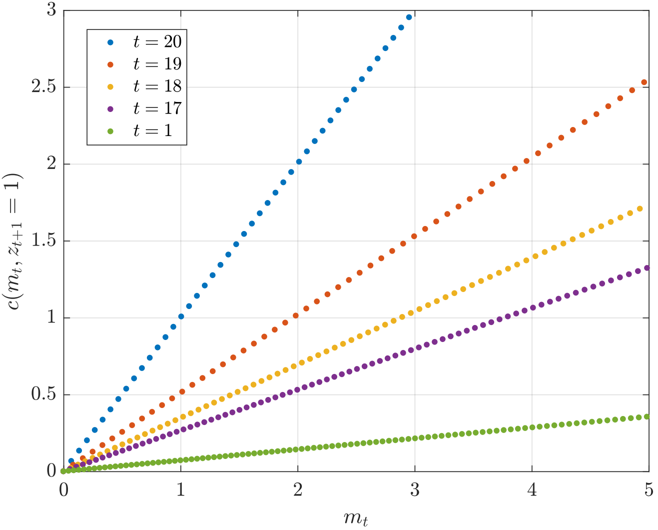
- The **consumption function**, $c_t(m_t|z_{t+1})$, is given by

$$\mathcal{G}_c = \{0, c_1, c_2, \dots, c_\#\} \text{ for } \mathcal{G}_m = \{0, m_1, m_2, \dots, m_\#\}$$

Parameters

- **Demograhics:** $T = 20$
- **Preferences:** $\beta = 0.96$, $\rho = 2$, $\alpha = 0.75$
- **Taste shocks:** $\sigma_{\varepsilon} = 0$
- **Income:** $W = 1$, $\sigma_{\xi} = 0$
- **Assets:** $R = 1.04$

Retired: Consumption function



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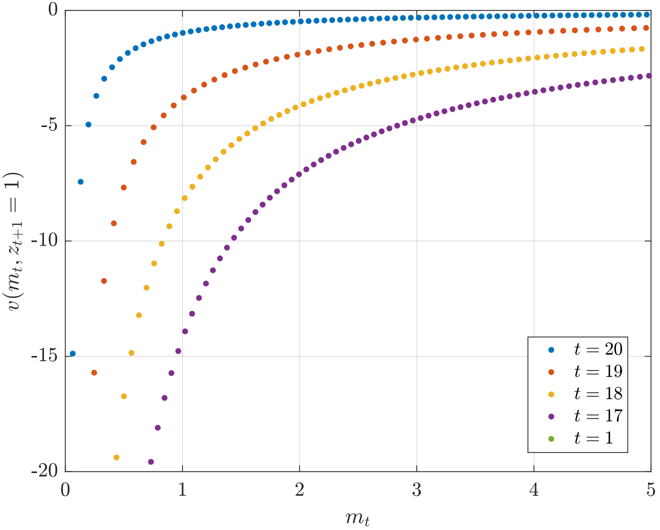
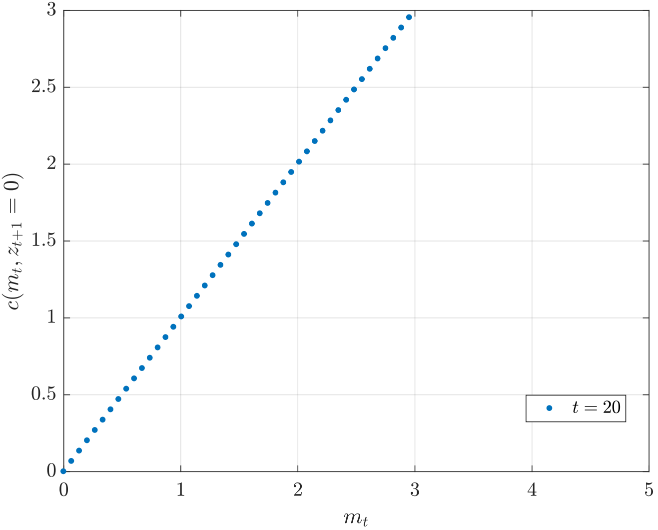
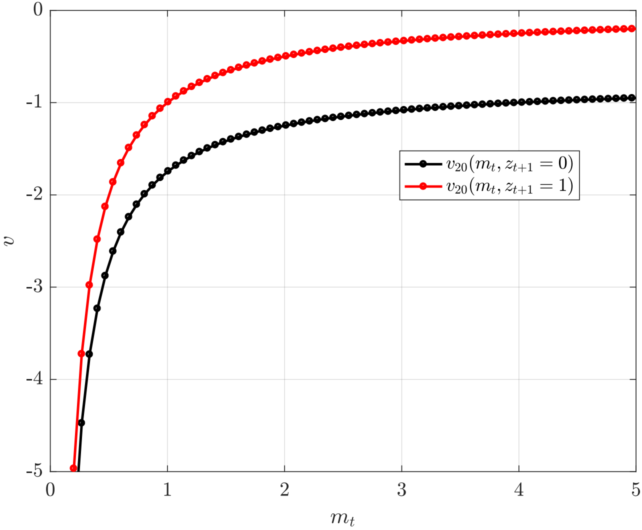
Retired: Value function



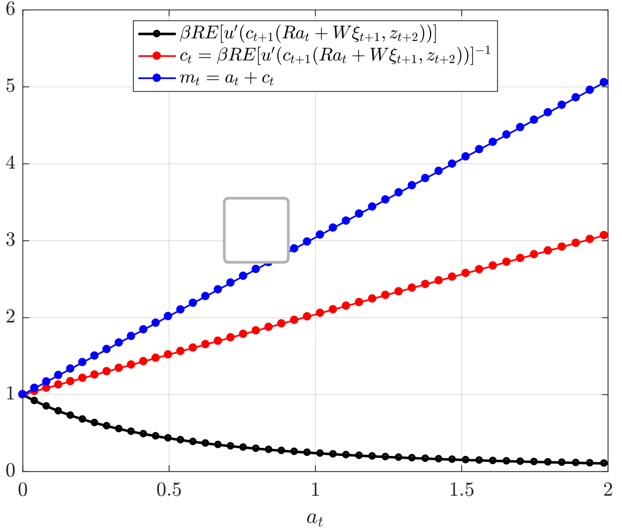
Working: Consumption, $t = T$



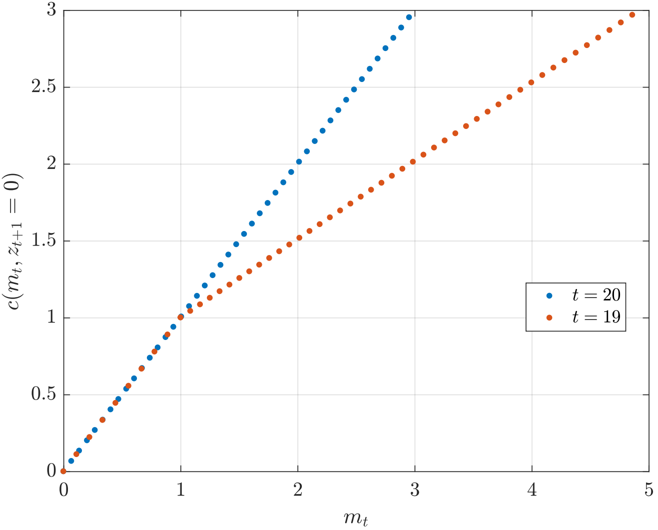
Choice-specific value functions, $t = T$



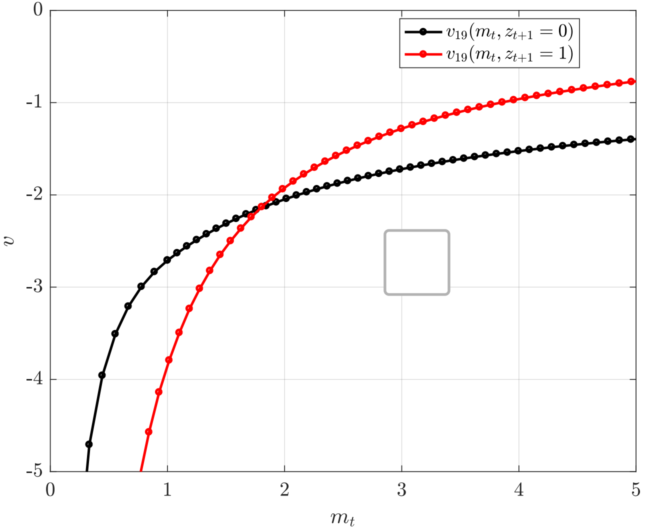
Working: EGM, $t = T - 1$ (a_t-space)



Working: Consumption, $t = T - 1$



Choice-specific value functions, $t = T - 1$



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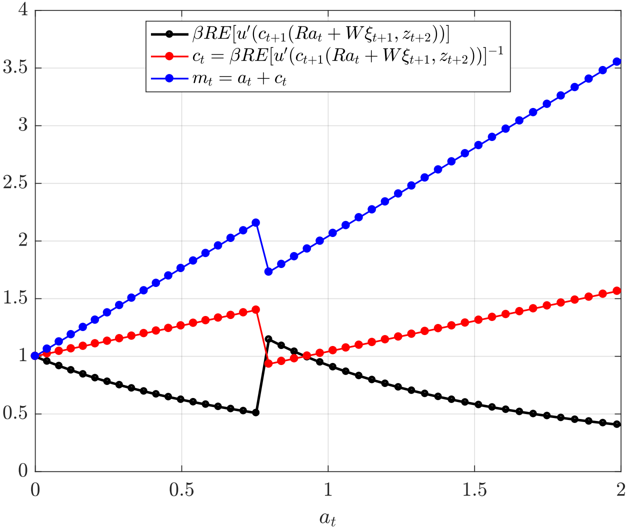
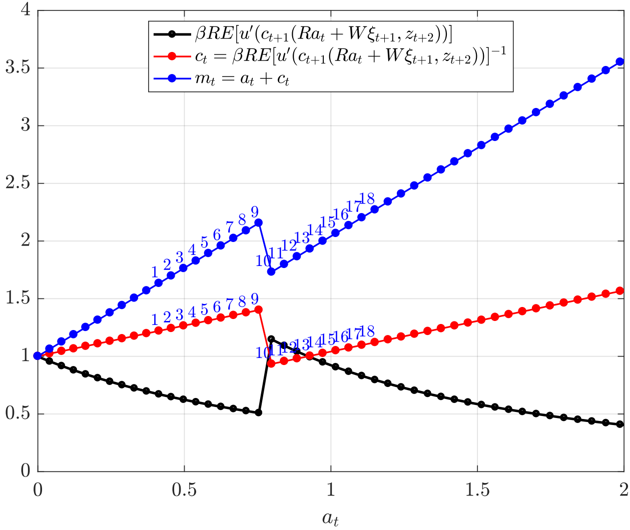
Working: EGM, $t = T - 2$ (a_t-space)



Working: EGM, $t = T - 2$ (a_t-space)



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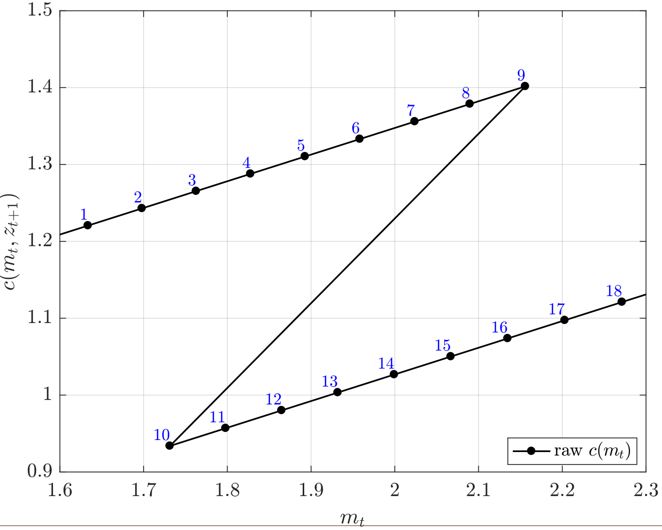
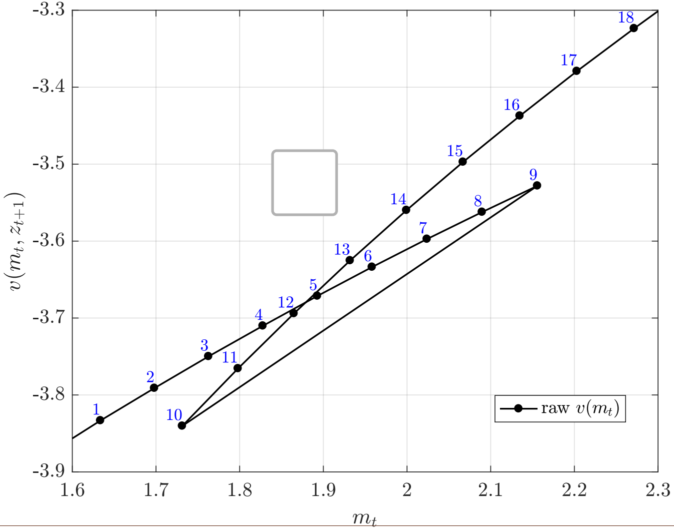
Raw consumption, $t = T - 2$ (m_t-space, working)



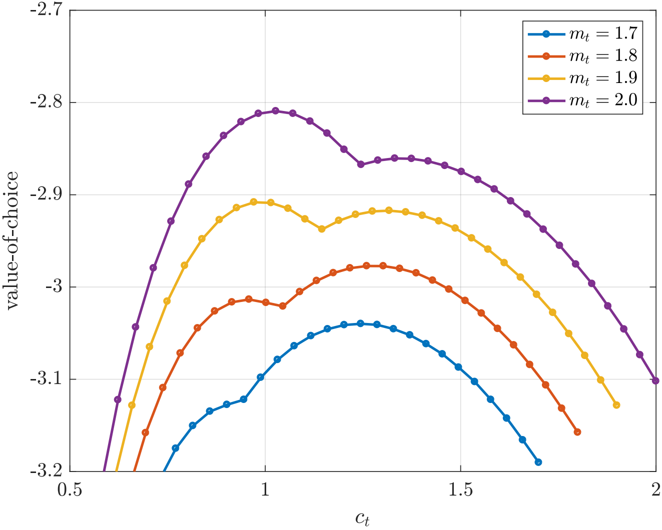
Raw values-of-choice, $t = T - 2$ (m_t-space, working)



Summary

1. **For high m_{T-1} retirement is optimal**
   - con retirement: *lower consumption due to no income*
   - pro retirement: *no disutility of labor*
2. Implies **kink in the value function** in period $T-1$
3. Implies *downward* **jump in optimal level of consumption** in $T-1$ at the kink
4. From $T-2$ implies *upward* jump in the **marginal utility of consumption** at some a_{T-2}
5. In the EGM c_{T-2} is then **not a montonic function** of a_{T-2}
6. For given m_t there are **multiple solutions to the Euler-equation**
   - the EGM finds all of them! (potentially in closed form)
   - **Q:** How do we find the optimum!? (what is the difficulty?)

Local optima, $u(c_t) + \beta \mathbb{E}_t \left[\mathcal{W}_{t+1} \right], t = T - 2$



- VFI in trouble: multi-start, grid search (even more expensive)

G^2EGM 1D algorithm for $z_{t+1} = 0$

1. **Apply EGM** to get $\mathcal{G}_m = \{m_i\}_1^\#$ and $\mathcal{G}_c = \{c_i\}_1^\#$
2. Find the **values-of-choice** $\mathcal{G}_v = \{v_i\}_1^\#$ with elements
$$v_i = u(c_i) + \beta \mathbb{E}_t \left[\mathcal{W}_{t+1}(R(m_i - c_i) + W\xi_{t+1}) \right]$$
3. Let $\mathcal{I}$ denote a **reordering** making $\mathcal{G}_m$ **strictly increasing**
$$\overline{\mathcal{G}}_m \equiv \mathcal{G}_m(\mathcal{I}) = \{\overline{m}_i\}_1^\#,\ \overline{\mathcal{G}}_c \equiv \mathcal{G}_c(\mathcal{I}) = \{\overline{c}_i\}_1^\#,\ \overline{\mathcal{G}}_v = \mathcal{G}_v(\mathcal{I}) = \{\overline{v}_i\}_1^\#$$
4. Loop through $i \in \{1, 2, \ldots, \# - 1\}$
   Loop through $j \in \{1, 2, \ldots, \#\}$
   If $m_i < \overline{m}_j < m_{i+1}$ then **interpolate**
$$\begin{aligned} c_{ij} &= \frac{c_{i+1} - c_i}{m_{i+1} - m_i}(\overline{m}_j - m_i) \\ v_{ij} &= u(c_{ij}) + \beta \mathbb{E}_t \left[\mathcal{W}_{t+1}(R(\overline{m}_j - c_{ij}) + W\xi_{t+1}) \right] \end{aligned}$$
   and if $v_{ij} > \overline{v}_j$ then **update** $\overline{v}_j = v_{ij}$ and $\overline{c}_j = c_{ij}$
5. **Extend** $\overline{\mathcal{G}}_m$, $\overline{\mathcal{G}}_c$ and $\overline{\mathcal{G}}_v$ with points on the borrowing constraint where m is between 0 and $\overline{m}_1$ and $c = m$

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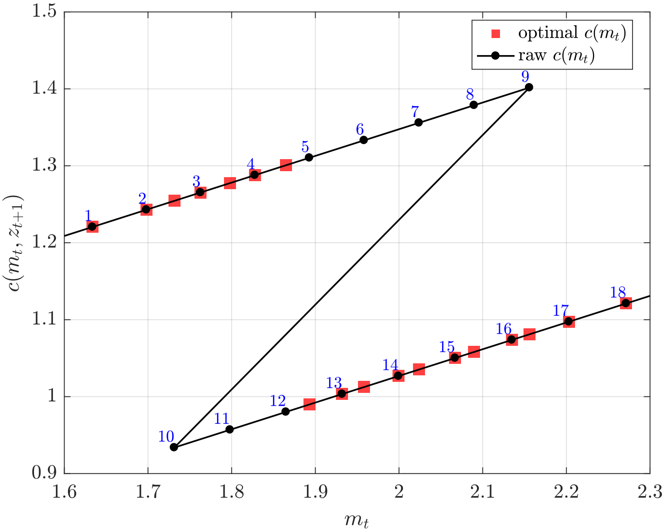
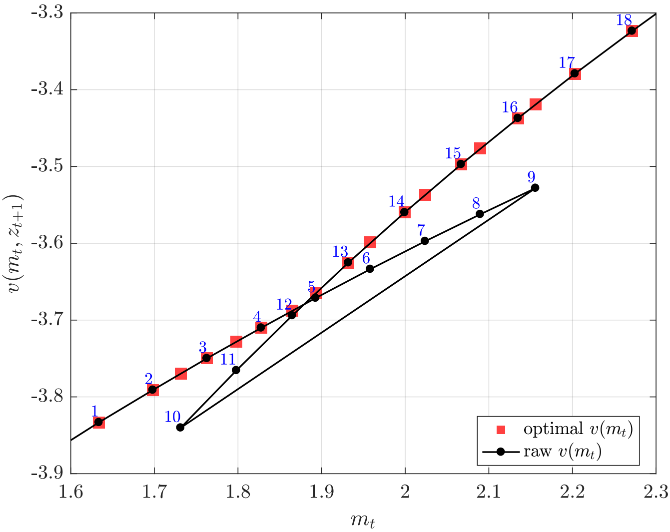
Upperenvelope, c_t, $t = T - 2$



Upperenvelope, v_t, $t = T - 2$



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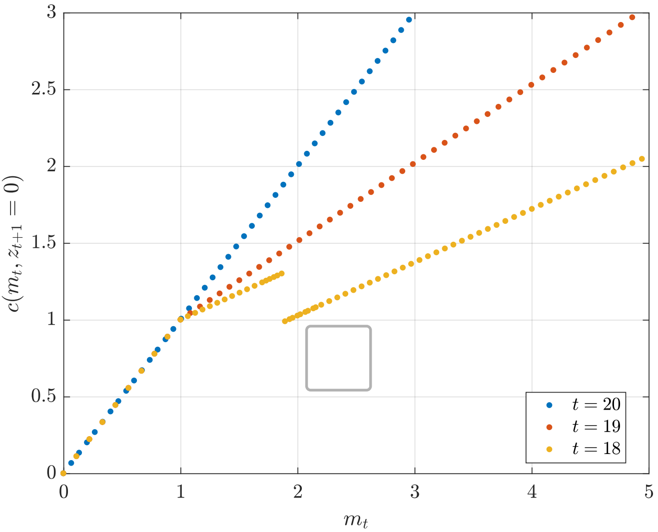
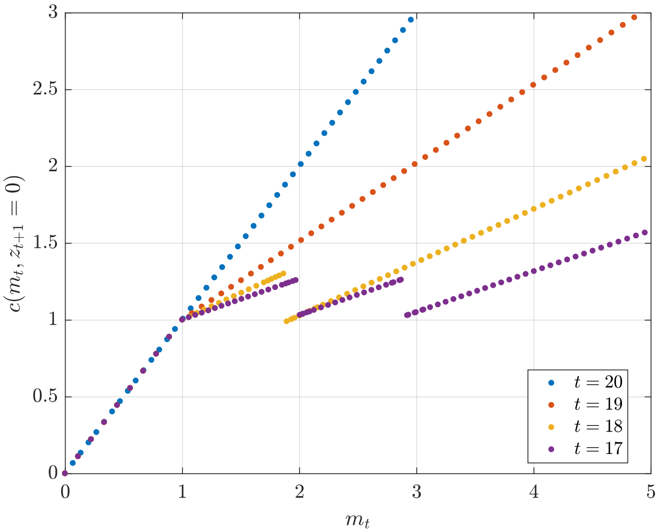
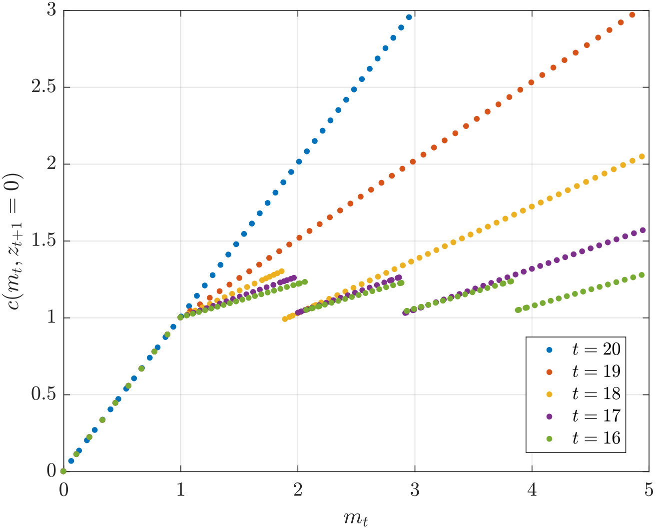
Working: Consumption, $t \geq 18$



Working: Consumption, $t \geq 17$



Working: Consumption, $t \geq 16$



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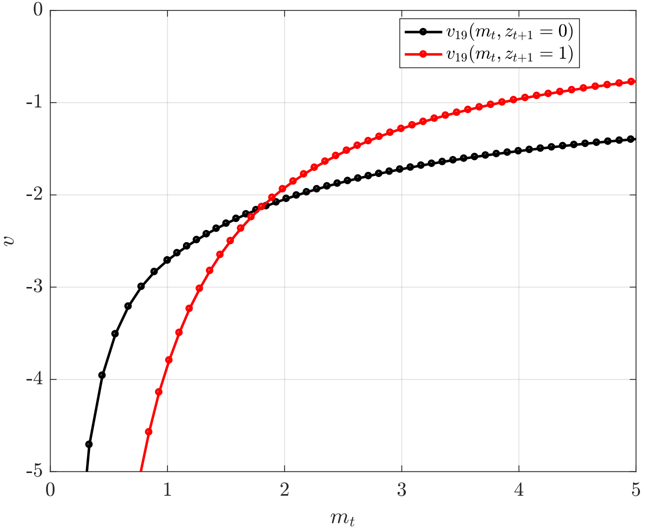
$\sigma_\varepsilon = 0.05$: Choice-specific value function



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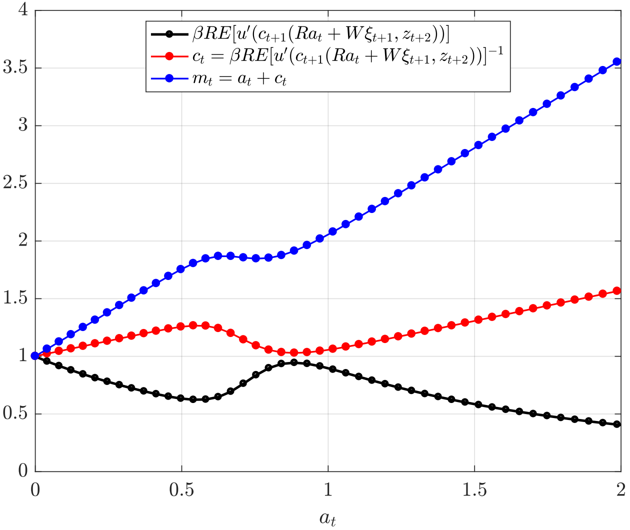
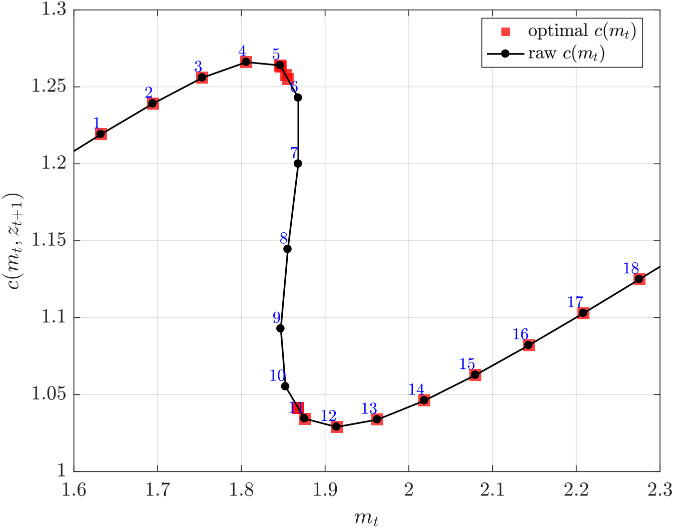
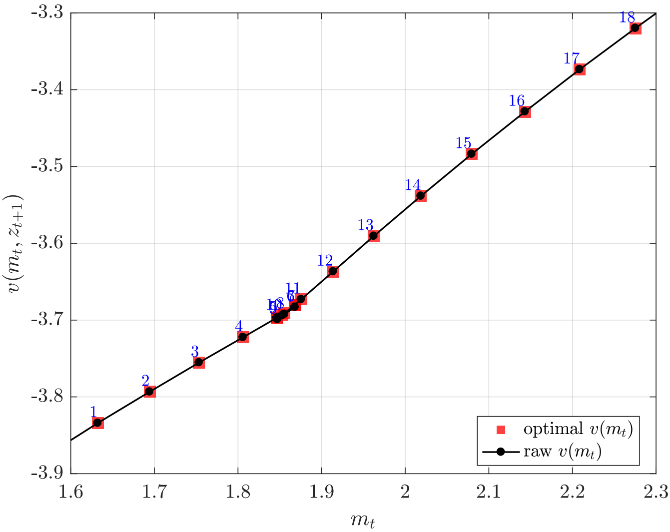
$\sigma_\varepsilon = 0.05$: EGM



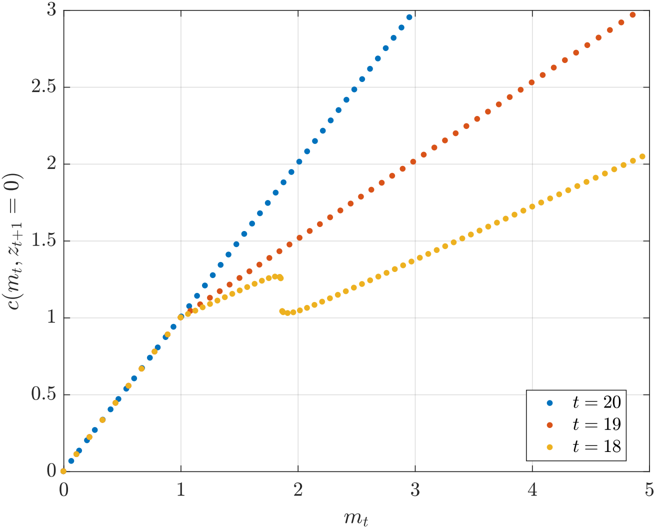
$\sigma_\varepsilon = 0.05$: Upperenvelope, c_t



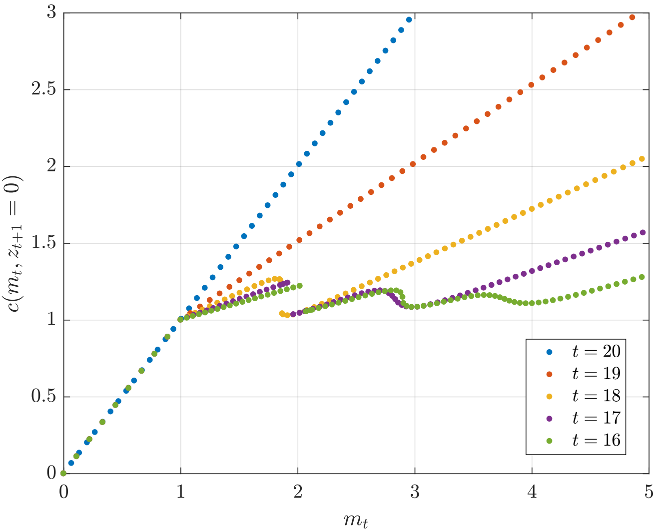
$\sigma_\varepsilon = 0.05$: Upperenvelope, v_t



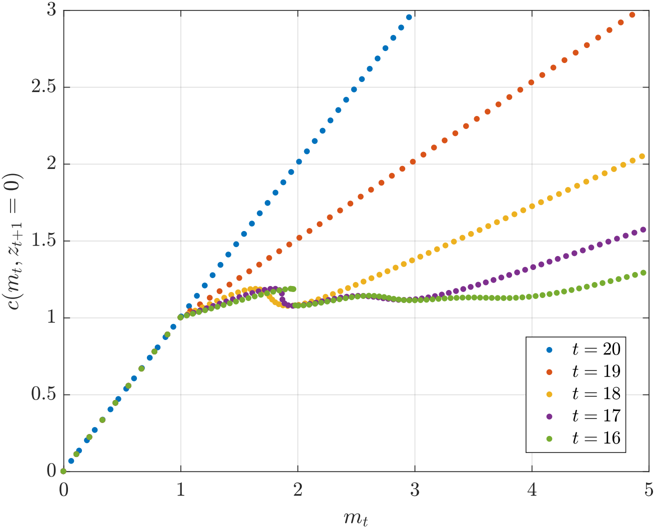
$\sigma_\varepsilon = 0.05$: Consumption, $t \geq 18$



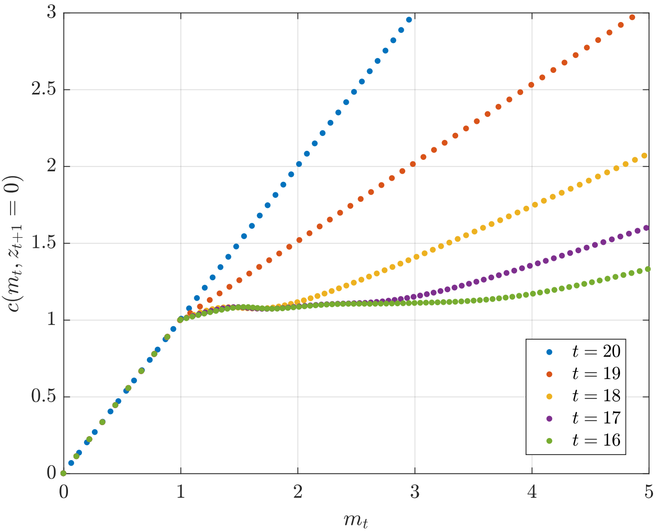
$\sigma_\varepsilon = 0.05$: Consumption, $t \geq 16$



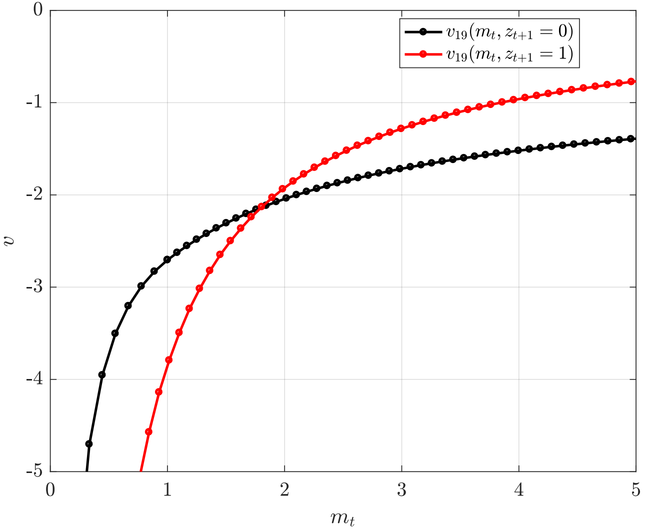
$\sigma_\varepsilon = 0.10$: Consumption, $t \geq 16$



$\sigma_\varepsilon = 0.20$: Consumption, $t \geq 16$



$\sigma_\varepsilon = 0.20$: Choice-specific value function



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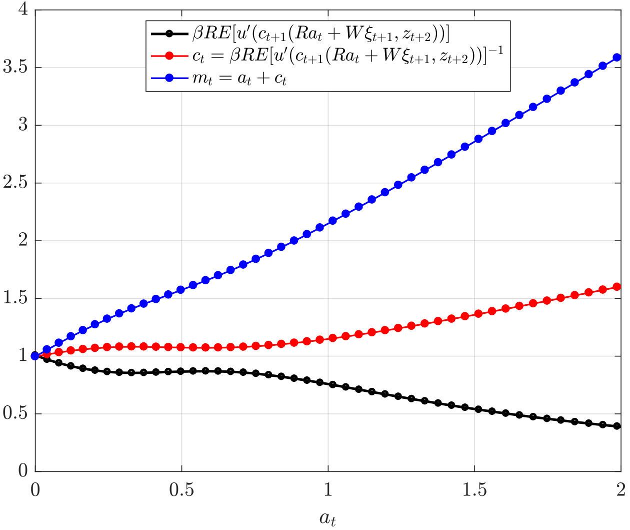
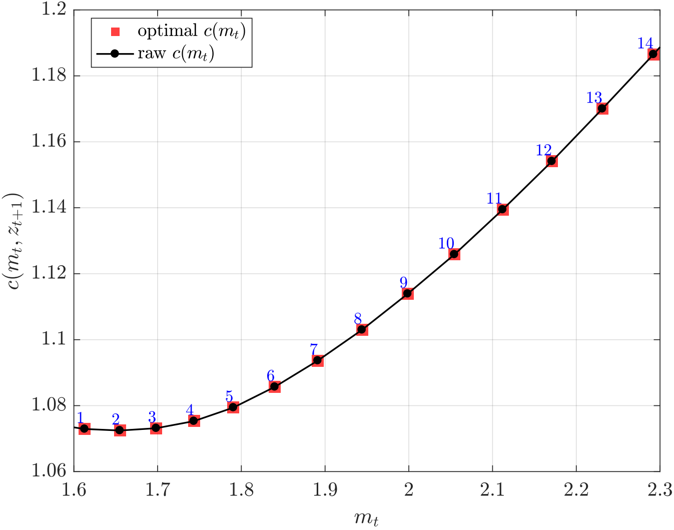
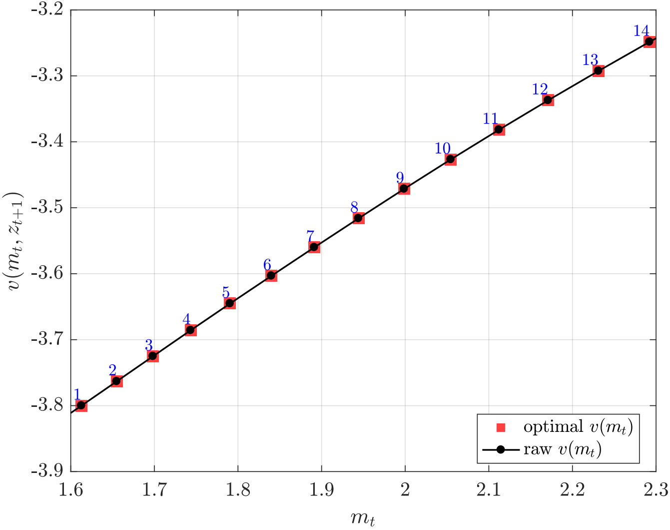
$\sigma_\varepsilon = 0.20$: EGM



$\sigma_\varepsilon = 0.20$: Upperenvelope, c_t



$\sigma_\varepsilon = 0.20$: Upperenvelope, v_t



Other extended EGMs

- **This version based on:** Druedahl and Jørgensen (2017)
- **Similar, but in my view more complicated**
  1. Fella (2014)
  2. Iskhakov, Jørgensen, Rust and Schjerning (2017)
- **Common:** Use EGM to *find all candidate solutions*
  Note: Any interior solution is a solution to the Euler-equation
- **Differences:** Different upper envelope algorithms used to *disgard non-optimal points*

Multi-dimensional EGM

- **Think of models with**
  1. Multiple continuous states and choices
  2. Multiple discrete states and choices
  3. Multiple potentially binding constraints
- **Problems for an EGM**
  1. Euler-equations only necessary (like in 1D)
  2. Endogenous grids are irregular – very costly to interpolate
  3. We do not know where the constraints are binding
- **Solution from Druedahl and Jørgensen (2017)**:
  Procede as we have done here but
  1. Choose the $\overline{\mathcal{G}}_m$ grid exogenously (*common grid*)
  2. Add a discrete choice determining which choices are respectively constrained and interior
- *No other EGM can handle multi-dimensional models with constraints and non-convexities such as discrete choices!*

Until next

- **Ensure that you understand:**
  1. That non-convexities can create multiple local optima
  2. The problems non-convexities cause for EGM
  3. How to apply G^2EGM for a one-dimensional problem
- **Prepare for next time:** *Look at the model section from a paper of your choice listed at Absalon*