Lecture 13: NPL

Dynamic Programming

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Outline for today

- Maximum Likelihood Estimation
- Infinite horizon
- Everything i discrete!
- “Swapping the nested fixed point”
- Aguirregabiria and Mira (2002)

Main Contributions of A&M (2002)

Nested Pseudo Likelihood (NPL) algorithm

- Solution of the DP problem in *choice probability space* (rather than value functions space)

Statistical and computational properties of the estimator.

- When NPL is initialized with consistent nonparametric estimates of conditional choice probabilities, successive iterations return a sequence of estimators of the structural parameters which we call *K-stage policy iteration estimators*.
- The sequence includes as extreme cases a Hotz-Miller (1993) estimator (for $K = 1$) and Rust's nested fixed point MLE estimator (in the limit when $K \to \infty$).

Monte Carlo experiments

- Monte Carlo based on Rust's bus replacement model.
- Reveal a trade-off between finite sample precision and computational cost in the sequence of policy iteration estimators.

Overall Purpose: Estimation

- **We want to estimate a model of a discrete choice** a
- Imagine, for example, that we have some empirical non-parametric choice-probabilities (i.e. shares), $\hat{P}(a|x)$
- **Data:** (a_i, x_i), $t = 1, ..., T_i$ and $i = 1, ..., n$

- **Likelihood function**

$$\ell_i^f(\theta) = \ell_i^1(\theta) + \ell_i^2(\theta)$$
$$= \sum_{t=2}^{T_i} log\left(P_\theta(a_{i,t}|x_{i,t})\right) + \sum_{t=2}^{T_i} log\left(f_\theta(x_{i,t}|x_{i,t-1}, a_{i,t-1})\right)$$

- **Two-Step-Estimator**
  - Consistent estimates of the conditional transition probability parameters θ_f can be obtained from transition data without having to solve the Markov decision model.
  - We focus on the estimation of $\alpha = (\theta_u, \theta_g)$ given initial consistent estimates of θ_f obtained from maximizing the partial log-likelihood $\ell^2(\theta) = \sum_i \ell_i^2(\theta)$

General Framework
From CCPs to Value Function
Fixed Point in Probability Space
NPL

Outline

The General Problem

$$V(s;\theta) = \max_{a \in \mathcal{A}(s)} \{u(s,a;\theta_u) + \beta \int V(s';\theta) p(s'|s,a;\theta_g,\theta_f) ds'\}$$

u and p: known up to a set of parameters, $\theta = (\theta_u, \theta_g, \theta_f)$

- **The agent's problem:** Maximize expected sum of current and future discounted utilities
  - a : Discrete control variable, $a \in \mathcal{A}(s) = \{1, 2, .., J\}$.
  - s : Current state, fully observed by agent
  - $s\prime$: Future state; possibly continuous and subject to uncertainty
- **The agents beliefs about $\mathbf{s}'$:**
  - Obeys a (controlled) Markov transition probability $p(s'|s,a;\theta_g,\theta_f)$
- **Model solution,** $V(s;\theta)$
  - Find the fixed point for the Bellman equation

Assumptions of Rust is maintained

Assumption (CI)

State variables, $s_t = (x_t, \varepsilon_t)$ obeys a (conditional independent) controlled Markov process with probability density

$$p(x_{t+1}, \varepsilon_{t+1}|x_t, \varepsilon_t, d_t, \theta_2, \theta_3) = q(\varepsilon_{t+1}|x_{t+1}, \theta_3)p(x_{t+1}|x_t, d_t, \theta_2)$$

Assumption (AS (additive separability))

$$U(s_t, d) = u(x_t, d; \theta_1) + \varepsilon_t(d)$$

Assumption (XV)

The unobserved state variables, ε_t are assumed to be multivariate iid. extreme value distributed (i.e. Gumbel)

Assumption (Discrete Observable State Variables)

$$x \in X = \{x^1, x^1, ..., x^m\}$$

Bellman Equation and Choice Probabilities

- Recall the *smoothed/expected value function* $V_\sigma(x) = \int V(x,\epsilon)g(\epsilon|x)d\epsilon$ where σ represents parameters that index the distribution of the $\epsilon's$.

- Under previous assumptions, we can summarize the solution by the *smoothed Bellman operator*, $\Gamma_\sigma(V_\sigma)$

$$V_\sigma(x) = \int \max_{a \in \mathcal{A}(x)} \left\{ u(x,a) + \epsilon(a) + \beta \sum_{x'} V_\sigma(x') f(x'|x,a) \right\} g(\epsilon|x)d\epsilon$$

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- The *conditional choice probability (CCP)*

$$P(a|x) = \int I\{a = \arg\max_{j \in \mathcal{A}(x)} \{v(j, x) + \epsilon(j)\} g(\epsilon|x) d\epsilon$$

where $v(a, x) = u(x, a) + \beta \sum_{x'} V_\sigma(x') f(x'|x, a)$ is the **choice-specific value function**

Outline

Re-formulation

- Similarly to the re-formulated the EV function, we write

$$\begin{aligned} V_\sigma(x) &= \mathbb{E}[V(x,\epsilon)|x] \\ &= \sum_{a\in\mathcal{A}} P(a|x)\cdot \mathbb{E}[V(x,\epsilon)|x,a] \\ &= \sum_{a\in\mathcal{A}} P(a|x)\cdot \{u(x,a) + \underbrace{e(a,x)}_{\equiv \mathbb{E}[\epsilon(a)|x,a]} + \beta\sum_{x'} V_\sigma(x')f(x'|x,a)\} \end{aligned}$$

- where ($\gamma = 0.57721566..$ is Eulers constant)

$$e(a,x) = \mathbb{E}[\epsilon(a)|x,a] = \gamma - \log P(a|x)$$

- We can therefore express this term as a function of CCPs:

$$e(a,P)$$

- Whenever I write "P", I mean $P(a|x)$

Re-formulation

- Rather than expressing the (smoothed) value function as a function of x, we can **stack all the M equations** such that

$$V_\sigma = \sum_{a \in \mathcal{A}} P(a) * \{u(a) + e(a, P) + \beta F(a) V_\sigma\}$$

  where

  - $*$ is element-by-element (Hadamard) product
  - $P(a)$ is a $M \times 1$ vector of conditional choice probabilities, $P(a|x)$
  - $u(a)$ is a $M \times 1$ vector of utilities, $u(a, x)$
  - $e(a, P)$ is a $M \times 1$ vector of conditional means, $\gamma - \log P(a|x)$
  - V_σ is a $M \times 1$ vector of smoothed value functions
  - $F(a)$ is a $M \times M$ transition matrix, $f(x'|x, a)$

- **Q: can you isolate V_σ as only a function of $P(a)$ here?**

Re-formulation

- Denote the unconditional choice probability as

$$F^U(P) = \sum_{a \in \mathcal{A}} P(a) * F(a)$$

- We can then write the smoothed value function as a function of P:

$$V_\sigma = \psi(P) = (1_{M\times M} - \beta F^U(P))^{-1} \sum_{a \in \mathcal{A}} P(a) * \{u(a) + e(a, P)\} \tag{1}$$

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- **BOOOOM!**
  We can now formulate a fixed point in *Probability space*
- Depends on parameters of the model through β, $u(a)$ and $F(a)$

Outline

Fixed Point in CCP space

- Recall that the choice probabilities are

$$P(a|x) = \int I\{a = \arg\max_{j \in \mathcal{A}(x)} \{v(j,x) + \epsilon(j)\}\, g(\epsilon|x) d\epsilon$$

where

$$\begin{aligned} v(j,x) &= u(x,a) + \beta \sum_{x'} V_\sigma(x') f(x'|x,a) \\ &= u(x,a) + \beta F(a) V_\sigma \\ &= u(x,a) + \beta F(a) \psi(P) \end{aligned}$$

- So we have a fixed point problem, $P = \Psi(P)(a|x)$:

$$P(a|x) = \int I\{a = \arg\max_{j \in \mathcal{A}(x)} \{u(x,a) + \beta F(a)\psi(P) + \epsilon(j)\}\, g(\epsilon|x) d\epsilon$$

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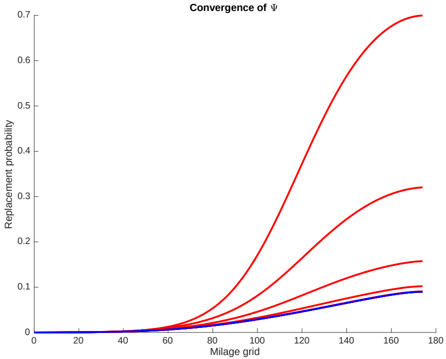
- If we have some initial guess on the CCP's (from data e.g.) we can use that guess to iterate on the choice-probabilities!
- Convergence of successive approximations in CPP-space is quite fast

$$P^{k+1} = \Psi(P^k)$$



Outline

Nested Pseudo-Likelihood (NPL)

- Recall that we have a consistent estimate of θ_f and focus on $\alpha = (\theta_u, \theta_g)$
- The log-likelihood function is thus

$$\alpha_{MLE} = \arg\max_{\alpha} \sum_{i=1}^{n} \sum_{t=2}^{T_i} \log P_{\alpha}(a_{i,t}|x_{i,t}) = \sum_{i=1}^{n} \sum_{t=2}^{T_i} \log[\Psi_{\alpha}(P)(a_{i,t}|x_{i,t})$$

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  if we have found the fixed point $P = \Psi_\alpha(P)$ for a given α
- This would have the same nesting as NFXP but replaced VFI with CCP-iteration
- However, Aguirregabiria and Mira (2002) suggests swapping the nesting as well!

Nested Pseudo-Likelihood (NPL)

A&M 2002 approach (Policy-Iteration est):

0: Initialize: Start with some guess of conditional choice probabilities, $P^0 \in [0,1]^{M \times J}$.
Often use sample frequencies (non-parametric estimates)

At iteration $K > 0$ apply the following two steps until convergence in P and α:

1: Obtain a new pseudo-likelihood estimate of α, $\hat{\alpha}^K$, as

$$\hat{\alpha}^K = \arg\max_{\alpha \in \Theta} \sum_{i=1}^{n} \sum_{t=2}^{T_i} \log \Psi_\alpha(P^{K-1})(a_{i,t}|x_{i,t}) \qquad (2)$$

where $\Psi_\alpha(P)(a|x)$ is the (a,x)'s element of $\Psi_\alpha(P)$.

2: Update P using the $\hat{\alpha}^K$ from step 1, i.e.

$$P^K = \Psi_{\hat{\alpha}^K}(P^{K-1}) \qquad (3)$$

Nested Pseudo-Likelihood (NPL): Fast?

- **Step 1** involves a numerical optimization over α
  But for each evaluation of the objective function we only need to evaluate

$$\int I\{a = \arg\max_{j \in \mathcal{A}(x)} \{u_\alpha(x,a) + \beta F_\alpha(a)\psi_\alpha(P) + \epsilon(j)\}\, g(\epsilon|x) d\epsilon \tag{4}$$

  where it is fast to calculate

$$\psi_\alpha(P) = (1_{M\times M} - \beta F_\alpha^U(P))^{-1} \sum_{a \in \mathcal{A}} P(a) * \{u_\alpha(a) + e(a,P)\}$$

  and with logit errors the integral in eq. (4) has closed form!

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  and with logit errors the integral in eq. (4) has closed form!
- **Step 2** involves a single evaluation of the CCP-fixed point in eq. (4)

$$P^{K} = \Psi_{\hat{\alpha}^{K}}(P^{K-1})$$

  which is the same as one evaluation of the objective function above with updated α.
  (this can be stored as output from the estimation in step 1)

Statistical Properties of NPL

For any K

- $\hat{\alpha}^K$ is asymptotically equivalent to MLE
- $\hat{\alpha}^K$ is $\sqrt{n}$ consistent
- $\hat{\alpha}^K$ is asymptotic normal with known variance-covariance matrix
  (A&M has an expression that accounts for first step estimation error)

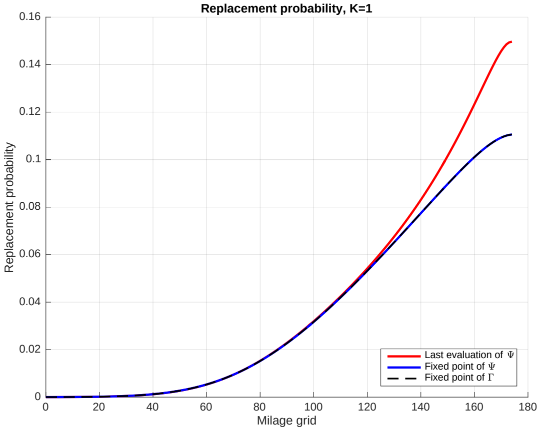
For $K = 1$

- $\hat{\alpha}^K$ encompasses Hotz-Miller (1993) estimator

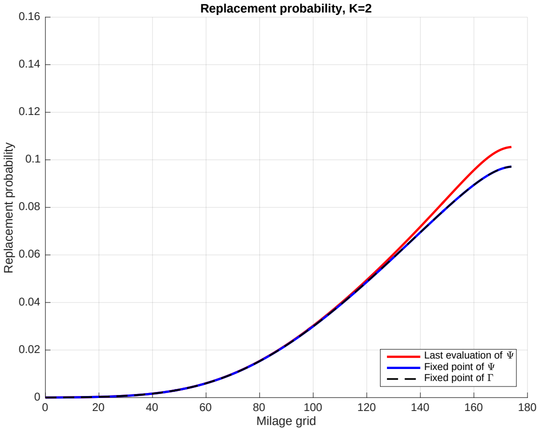
As $K \to \infty$

- $\hat{\alpha}^K$ converges to the MLE estimator obtained by NFXP
- Standard inference.

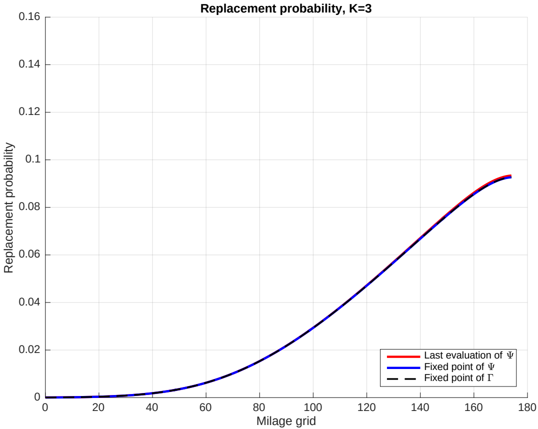
Replacement Prob, $K = 1$



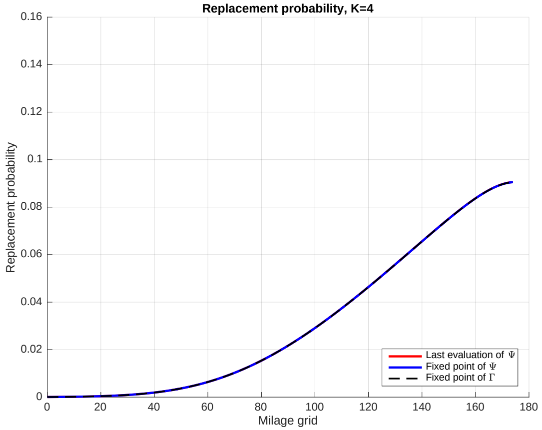
Replacement Prob, $K = 2$



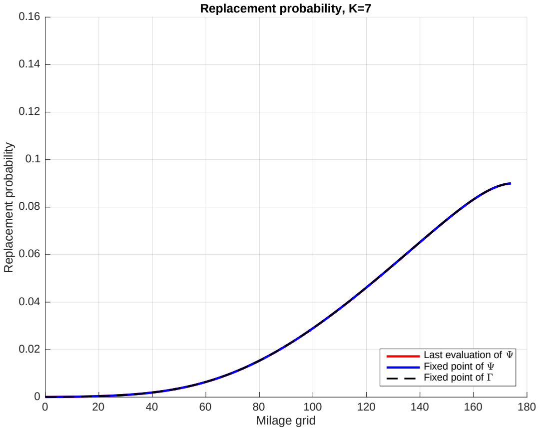
Replacement Prob, $K = 3$



Replacement Prob, $K = 4$



Replacement Prob, $K = 7$



Discussion: Hotz Miller 1993

- The CCP estimators of Hotz and Miller (1993) were defined as the values of α that solve systems of equations of the form

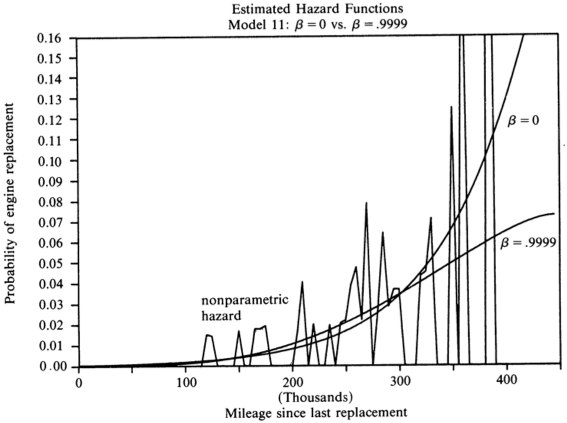
$$\arg\min_{\alpha\in\Theta} \sum_{i=1}^{N} \sum_{j=1}^{J} Z_i^j \left[I(a_i = j) - \tilde{P}_\alpha(P^0)(j|x) \right]$$

  where Z_i is are vectors of instrumental variables (e.g.) functions of x_i

- Can be shown that the 1-stage PI estimator is a CCP estimator with $Z_i = \partial\Psi(P^0)(a_i|x_i)/\partial\alpha$ is used as instrument.

Discussion: Small Sample problems

- Sometimes it can be hard to get a precise nonparametric estimate of the CCPs



Discussion: Precision of PI Estimators

TABLE I

MONTE CARLO EXPERIMENT

Experiment design	
Model:	Bus engine replacement model (Rust)
Parameters:	$\theta_0 = 10.47$; $\theta_1 = 0.58$; $\beta = 0.9999$
State space:	201 cells
Number observations:	1000
Number replications:	1000
Initial probabilities:	Kernel estimates

Monte Carlo distribution of MLE
(In parenthesis, percentages over the true value of the parameter)

	θ_0	θ_1
Mean Absolute Error:	2.08 (19.9%)	0.17 (29.0%)
Median Absolute Error:	1.56 (14.9%)	0.13 (22.7%)
Std. dev. estimator:	2.24 (21.4%)	0.16 (26.9%)
Policy iterations (avg.):	6.2	

Monte Carlo distribution of PI estimators (relative to MLE)
(All entries are 100* (*K*-PI statistic-MLE statistic)/MLE statistic)

		Estimators		
Parameter	Statistics	1-PI	2-PI	3-PI
θ_0	Mean AE	4.7%	1.6%	0.3%
	Median AE	14.2%	0.2%	−0.3%
	Std. dev.	6.8%	1.2%	0.3%
θ_1	Mean AE	18.7%	1.5%	0.2%
	Median AE	25.1%	0.7%	0.6%
	Std. dev.	11.0%	1.3%	0.2%

Discussion: Precision of PI Estimators

TABLE II

RATIO BETWEEN ESTIMATED STANDARD ERRORS AND STANDARD DEVIATION OF MONTE CARLO DISTRIBUTION

Parameters	Statistics	Estimators			
		1-PI	2-PI	3-PI	MLE
θ_0	Ratio	0.801	1.008	1.027	1.022
θ_1	Ratio	0.666	1.043	1.076	1.065

Regarding speed:

- For most problems the fixed point iterations (i.e., policy iterations) are much more expensive than likelihood and pseudo-likelihood "hill" climbing iterations.
- The size of the state space does not affect the number of policy iterations in any of the two algorithms.
- A&M found that NPL around 5 and 10 times faster than NFXP

Until next time

- **Ensure that you understand:**
  1. There is many (many) ways to try to reduce the computational time!
  2. NPL is one :)
- **Next time:** Life-cycle discrete choice problems