Lecture 3: Numerical Integration + Simulation

Dynamic Programming

Thomas Jørgensen

Algorithm 5: Find all $V_t^\star$ (algorithm 9 inserted)

input : $\mathcal{G}_M$ (grid for M with $\#_M$ elements)
$\mathcal{G}_C$ (grid for C (as a share of M) with $\#_C$ elements in $(0,1)$)
output: $V_t^\star[\bullet]$ for all t
$C_t^\star[\bullet]$ for all t

for $i_M = 1$ **to** $\#_M$ **do**
  $V_T^\star[i_M] = \sqrt{\mathcal{G}_M[i_M]}$ (initialize terminal period)

for $t = T-1$ **to** 1 **do**
  for $i_M = 1$ **to** $\#_M$ **do**
    $V_t^\star[i_M] = -\infty$
    $M_t = \mathcal{G}_M[i_M]$
    for $i_C = 1$ **to** $\#_C$ **do**
      $C_t = \mathcal{G}_C[i_C]M_t$
      $EV_{t+1} = \pi\text{interp}(M_t - C_t + 1, \mathcal{G}_M, V_{t+1}^\star)$
      $+ (1-\pi)\text{interp}(M_t - C_t, \mathcal{G}_M, V_{t+1}^\star)$
      $V = \sqrt{C_t} + \beta EV_{t+1}$
      if $V > V_t^\star[i_M]$ **then**
        $V_t^\star[i_M] = V$
        $C_t^\star[i_M] = C_t$

Continuous stochastic shocks

- **Example:** Consumption-saving model with **Gaussian income shocks**

$$\begin{aligned}
V_t(M_t) &= \max_{C_t} \sqrt{C_t} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}) \right] \\
&\text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= R \cdot A_t + Y_{t+1} \\
Y_{t+1} &= \exp(\xi_{t+1}) \\
\xi_{t+1} &\sim \mathcal{N}(0, \sigma_\xi^2) \\
A_t &\geq 0
\end{aligned}$$

- How can we **evaluate** $\mathbb{E}_t \left[V_{t+1}(M_{t+1}) \right] = \mathbb{E}_t \left[V_{t+1}(RA_t + Y_{t+1}) \right]$ for known A_t?

Numerical integration

- **General problem:** How can we calculate

$$\mathbb{E}(f(x)) = \int f(x) dg(x)$$

  - $f : \mathbb{R} \rightarrow \mathbb{R}$ some function
  - $g(x)$ is the cumulative distribution function for x

- **General solution:** Turn it into a discrete sum

$$\mathbb{E}(f(x)) \approx \sum_{i=1}^{S} \omega_i f(x_i)$$

- **How to choose S and the *nodes* (x_i) and *weights* (ω_i)?** Three standard methods:
  1. Monte Carlo integration
  2. Equiprobable integration
  3. Guassian quadrature

1. Monte Carlo integration

- Draw S (pseudo-)**random** x_i from $g(x)$ indexed by i
- **The integral is approximated by**

$$\mathbb{E}(f(x)) \approx \sum_{i=1}^{S} \frac{1}{S} f(x_i)$$

- Can you imagine a potential **drawback** of this method?

2. Equiprobable points

1. Construct a grid of $S+1$ **equally spaced** nodes $\in [0,1]$:

$$\pi = \{0, \frac{1}{S}, \frac{2}{S}, \ldots, 1\}$$

2. Calculate $z_i = g^{-1}(\pi_i)$ for $i \in \{1, 2, \ldots, S\}$
3. Find the **weighted mid-points** (only once!)

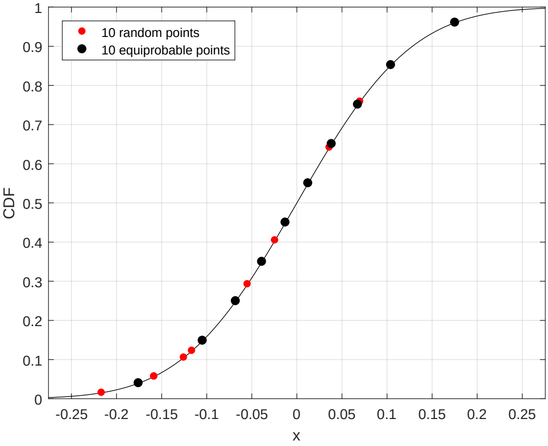
$$x_i = \int_{z_{i-1}}^{z_i} x \mathrm{d}g(x) \cdot S \quad \text{for } i \in \{1, 2, \ldots, S\}$$

4. **The integral is approximated by**

$$\mathbb{E}(f(x)) \approx \sum_{i=1}^{S} \frac{1}{S} f(x_i)$$

- Can you imagine a potential **drawback** of this method?

2. Equiprobable points - illustration



3. Gaussian quadrature

- There are **formulas** for the sequences of x_i and ω_i for *exact* integration of certain polynomials
- Formulas are **domain dependent**:
  - $[a, b] \rightarrow$ Gauss-Chebyshev or Gauss-Legendre quadrature
  - $[0, \infty] \rightarrow$ Gauss-Laguerre quadrature (e.g. exponential distribution)
  - $[-\infty, \infty] \rightarrow$ **Gauss-Hermite quadrature**
    - S points can correctly integrate polynomials of degree $2S - 1$

3. Gauss-Hermite

- **Gauss-Hermite** quadrature uses that

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx = \sum_{i=1}^{S} \omega_i f(x_i) + \frac{S!\sqrt{\pi}}{s^S(2S)!} f^{(2S)}(\epsilon)$$

for some ϵ and where the (x_i, ω_i)'s can be easily found

- **Well behaved function:** For $S \to \infty$ we have

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx \approx \sum_{i=1}^{S} \omega_i f(x_i) \tag{1}$$

- **Random normal variable:** $Y \sim \mathcal{N}(\mu, \sigma^2)$ so that

$$\begin{aligned} \mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(y)e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{S} \omega_i f(\sqrt{2}\sigma x_i + \mu) \end{aligned}$$

3. Gauss-Hermite: Derivation

Use the change of variables

$$x = \frac{y-\mu}{\sqrt{2}\sigma}$$

and insert $y = \sqrt{2}\sigma x + \mu$ to get

$$\begin{aligned}
\mathbb{E}[f(Y)] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} f(y)e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} f(\sqrt{2}\sigma x+\mu)e^{-\frac{(\sqrt{2}\sigma x+\mu-\mu)^2}{2\sigma^2}}d(\sqrt{2}\sigma x+\mu) \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} f(\sqrt{2}\sigma x+\mu)e^{-x^2}\sqrt{2}\sigma dx \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(\sqrt{2}\sigma x+\mu)e^{-x^2}dx \\
&\approx \frac{1}{\sqrt{\pi}}\sum_{i=1}^{S}\omega_i f(\sqrt{2}\sigma x_i+\mu)
\end{aligned}$$

where the last line is from the Gauss-Hermite formula, eq. (1)

Algorithm 10: Find $V_t^\star$ given $V_{t+1}^\star$ [find_V]

input : $V_{t+1}^\star[\bullet]$ and Gauss-Hermite nodes and weights: $x[\bullet]$, $\omega[\bullet]$
$\mathcal{G}_M$ (grid for M with $\#_M$ elements)
$\mathcal{G}_C$ (grid for C with $\#_C$ elements in $(0,1)$)

output: $V_t^\star[\bullet]$ (value of optimal choice
$C_t^\star[\bullet]$ (optimal choice)

for $i_M = 1$ **to** $\#_M$ **do**
- $V_t^\star[i_M] = -\infty$
- $M_t = \mathcal{G}_M[i_M]$
- **for** $i_c = 1$ **to** $\#_C$ **do**
  - $C_t = \mathcal{G}_C[i_C]M_t$
  - $EV_{t+1} = 0$
  - **for** $i = 1$ **to** S **do**
    - $M_{t+1} = R(M_t - C_t) + \exp(\sqrt{2}\sigma_\xi x[i])$
    - $EV_{t+1} = EV_{t+1} + \frac{\omega[i]}{\sqrt{\pi}}\text{interp}(M_{t+1}, \mathcal{G}_M, V_{t+1}^\star)$
  - $V = \sqrt{C_t} + \beta EV_{t+1}$
  - **if** $V > V_t^\star[i_M]$ **then**
    - $V_t^\star[i_M] = V$
    - $C_t^\star[i_M] = C_t$

Comparing methods

Table: Integration of $f(x) = x^2$ with $x \sim \mathcal{N}(0,1)$.

MC (10)	MC (50000)	Equi (10)	Equi (50)	Hermite (10)
0.8824	0.9959	0.9590	0.9947	1.0000

- **Monte Carlo** is imprecise even when using many points
- **Equiprobable points** are much more accurate
- **Gaussian quadrature rules!** But also smooth polynomial, so not surprising

Multi-dimensional integration I

$$\mathbb{E}(f(x_1, x_2)) \approx \sum_i^{S_1} \sum_j^{S_2} w_{1i} w_{2j} f(x_{1i}, x_{2j})$$

1. **Monte-Carlo:** The same
2. **Equiprobable:** Somewhat more complicated (not unique)
3. **Quadrature:** Simplest with a tensor product

- For $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathcal{N}(\mu, \Omega)$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ \rho_{12} & \sigma_2 \end{bmatrix}}_{\Omega^{\frac{1}{2}}} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$\eta_1, \eta_2 \sim \mathcal{N}(0, 1)$$

- $\Omega^{\frac{1}{2}}$ is **not unique** (here *lower cholesky*. Matlab has `sqrtm`)
- Tensor product uses a lot of points $\rightarrow$ **sparse grids**

Multi-dimensional integration II

1. Let (y_i, ω_i^y) and (z_j, ω_j^z) be two sets of Gauss-Hermite nodes
2. For all i and j calculate

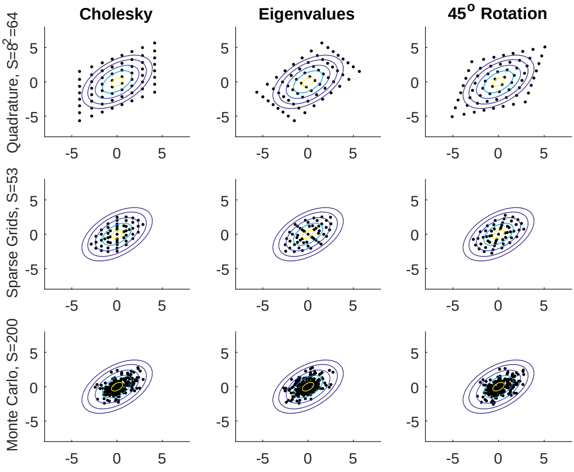
$$\begin{aligned}\tilde{y}_i &= \sqrt{2} y_i \\ \omega_i^{\tilde{y}} &= \omega_i^y / \sqrt{\pi} \\ \tilde{z}_j &= \sqrt{2} z_j \\ \omega_j^{z} &= \omega_j^z / \sqrt{\pi}\end{aligned}$$

3. Then for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathcal{N}(\mu, \Omega)$

$$\mathbb{E}(f(x_1, x_2)) \approx \sum_i^{S_1} \sum_j^{S_2} \omega_i^{\tilde{y}} \omega_j^{\tilde{z}} f(\sigma_1 \tilde{y}_i + \mu_1, \rho_{12} \tilde{y}_i + \sigma_2 \tilde{z}_j + \mu_2)$$

where $\begin{bmatrix} \sigma_1 & 0 \\ \rho_{12} & \sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ \rho_{12} & \sigma_2 \end{bmatrix}^T = \Omega$

Multi-dimensional integration - illustration



$\Omega = [1, 0.5; 0.5; 1]$
the lines are contour lines for the pdf

4. Discretization a la Tauchen

- Assume that x_t follows the **stationary process** ($\rho < 1$)

$$x_{t+1} = \rho x_t + \xi_{t+1},\ \ \xi_{t+1} \sim \mathcal{N}(0, \sigma_{\xi}^2)$$

- **Discretize** x_t into S nodes and use a **transition matrix** such that

$$\Pr[x_{t+1} = x_j | x_t = x_i] = \omega_{ij}$$

- **Originally:** Tauchen (1986), Tauchen and Hussey (1991), Rouwenhorst (1995)
- **Inaccurate if ρ close to** 1 (solutions in Galindev and Lkhagvasuren (2009) and Kpoecky and Suen (2009))
- Extension to **non-Gaussian** processes recently developed by Civale-Diaz-Fazilet (2017)
- New methods for estimating a **fully flexible Markov process** directly on data, De Nardi et. al. (2016) and Druedahl and Munk-Nielsen (2017)

Counter-factual analysis I

- We can now **solve**

$$
\begin{aligned}
V_t(M_t) &= \max_{C_t} \sqrt{C_t} + \beta \mathbb{E}_t \left[V_{t+1}(M_{t+1}) \right] \\
& \text{s.t.} \\
A_t &= M_t - C_t \\
M_{t+1} &= R \cdot A_t + Y_{t+1} \\
Y_{t+1} &= \exp(\xi_{t+1}) \\
\xi_{t+1} &\sim \mathcal{N}(0, \sigma_\xi^2) \\
A_t &\geq 0
\end{aligned}
$$

  and find $C_t^\star(M_t)$ for all M_t
- **How would you measure the cost of income risk?**

Counter-factual analysis II

- **How would you measure the cost of income risk?**
  1. **Solve** the model for **various values of** σ_ξ, i.e. find $C_t^\star(M_t;\sigma_\xi)$
  2. **Simulate** N individuals for T periods for each σ_ξ (same seed and draws up to scaling with σ_ξ)

$$
\begin{aligned}
C_{it} &= C_t^\star(M_{it};\sigma_\xi) \\
M_{it+1} &= R(M_{it} - C_{it}) + Y_{it+1} \\
C_{it+1} &= C_{t+1}^\star(M_{it+1};\sigma_\xi) \\
M_{it+2} &= R(M_{it+1} - C_{it+1}) + Y_{it+2} \\
&\vdots
\end{aligned}
$$

  3. **Compare the average value-of-life**

$$
V(\sigma_\xi) = \frac{1}{N}\sum_{i=1}^{N} \sqrt{C_{i1}} + \beta\sqrt{C_{i2}} + \cdots + \beta^T\sqrt{C_{iT}}
$$

- **Consumption equivalent:** Let the utility function be $\sqrt{C_t + k}$, and use k to negate changes in the average value-of-life when changing σ_ξ

Envelope theorem

$$F(x) = \max_{z \in \mathcal{Z}} f(x,z) = f(x, z^\star(x))$$
$$z^\star(x) = \arg\max_{z \in \mathcal{Z}} f(x,z)$$

If differentiability is not a problem then

$$\begin{aligned} F'(x) &= f'_x(x, z^\star(x)) + f'_z(x, z^\star(x)) z^{\star\prime}(x) \\ &= f'_x(x, z^\star(x)) \end{aligned}$$

FOC and Euler-equation

- **Consider the problem:**

$$\begin{aligned} V_t(M_t) &= \max_{C_t} u(C_t) + \beta V_{t+1}(M_{t+1}) \\ &= u(C_t^\star(M_t)) + \beta V_{t+1}(R(M_t - C_t^\star(M_t)) + Y_{t+1}) \end{aligned}$$

- Using the **envelope theorem** we get

$$\begin{aligned} V_t'(M_t) &= \beta R V_{t+1}'(R(M_t - C_t^\star(M_t)) + Y_{t+1}) \\ &= \beta R V_{t+1}'(M_{t+1}) \end{aligned}$$

- **FOC** for **optimal** C_t**,** $\frac{\partial u(C_t)+\beta V_{t+1}(R(M_t-C_t)+Y_{t+1})}{\partial C_t}$:

$$\begin{aligned} u'(C_t) &= \beta R V_{t+1}'(M_{t+1}) = V_t'(M_t) \leftrightarrow \\ u'(C_t) &= \beta R u'(C_{t+1}) \end{aligned}$$

- This is the **Euler-equation** for interior optimal choices

Variational approach I

- **Optimal plan** $C_t^\star, C_{t+1}^\star, C_{t+2}^\star \dots$ but
  - set $C_t = C_t^\star + \Delta$ today (require $0 < C_t^\star < M_t$)
  - set $C_{t+1} = C_{t+1}^\star - R\Delta$ tomorrow
  - so that $M_{t+1} = R(M_t - (C_t^\star + \Delta))$
  - and $M_{t+1} - C_{t+1} = R(M_t - C_t^\star) - C_{t+1}^\star$ (no Δ)
- We have

$$
\begin{aligned}
V_t^\star(M_t) &= u(C_t^\star) + \beta u(C_{t+1}^\star) + \beta^2 u(C_{t+2}^\star) + \dots \\
V_t(M_t) &= u(C_t) + \beta u(C_{t+1}) + \beta^2 u(C_{t+2}^\star) + \dots
\end{aligned}
$$

- So that

$$
\begin{aligned}
V_t^\star(M_t) - V_t(M_t) &= [u(C_t^\star) - u(C_t)] + \beta[u(C_{t+1}^\star) - u(C_{t+1})] \\
&= [u(C_t^\star) - u(C_t^\star + \Delta)] \\
&\quad +\beta[u(C_{t+1}^\star) - u(C_{t+1}^\star - R\Delta)]
\end{aligned}
$$

Variational approach II

- Finally, we have

$$\begin{aligned}
\lim_{\Delta\to 0} V^\star(M_t) - V(M_t) &= 0 \leftrightarrow \\
\lim_{\Delta\to 0} \frac{V^\star(M_t) - V(M_t)}{\Delta} &= 0 \leftrightarrow \\
\lim_{\Delta\to 0} \frac{[u(C_t^\star) - u(C_t^\star + \Delta)] + \beta[u(C_{t+1}^\star) - u(C_{t+1}^\star - R\Delta)]}{\Delta} &= 0 \leftrightarrow \\
\lim_{\Delta\to 0} \frac{u(C_t^\star) - u(C_t^\star + \Delta))}{\Delta} - \beta R \lim_{\Delta\to 0} \frac{u(C_{t+1}^\star) - u(C_{t+1}^\star - R\Delta)}{-R\Delta} &= 0 \leftrightarrow \\
u'(C_t^\star) - \beta R u'(C_{t+1}^\star) &= 0 \leftrightarrow \\
u'(C_t^\star) &= \beta R u'(C_{t+1}^\star)
\end{aligned}$$

Euler-residuals I

1. Solve a **deterministic** model
2. **Simulate** consumption paths over T periods for N individuals indexed by i
3. Calculate the **Euler-residual** in each period for each individual
$$\mathcal{E}_{it} \equiv u'(C_{it}) - \beta R u'(C_{it+1})$$
4. Calculate the **average absolute Euler-error**

$$\frac{1}{\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbf{1}_{\{0<C_t<M_{it}\}}}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbf{1}_{\{0<C_t<M_{it}\}}|\mathcal{E}_{it}|$$

Euler-residuals II

1. Solve a **stochastic** model.
2. **Simulate** consumption paths over T periods for N individuals indexed by i
3. Calculate the **Euler-residual** in each period for each individual

$$
\begin{aligned}
\mathcal{E}_{it} &\equiv u'(C_{it}) - \beta R \mathbb{E}_t \left[u'(C^\star_{t+1}(R(M_{it} - C_{it}) + Y_{it+1})) \right] \\
&\approx u'(C_{it}) - \beta R \sum_{j=1}^{S} \omega_j \left[u'(C^\star_{t+1}(R(M_{it} - C_{it}) + Y_{jt+1})) \right]
\end{aligned}
$$

4. Calculate the **average absolute Euler-error**

$$
\frac{1}{\sum_{t=1}^{T} \sum_{i=1}^{N} \mathbf{1}_{\{0<C_t<M_{it}\}}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathbf{1}_{\{0<C_t<M_{it}\}} |\mathcal{E}_{it}|
$$

Euler-residuals III

- Instead of

$$\frac{1}{\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbf{1}_{\{0<C_t<M_{it}\}}}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbf{1}_{\{0<C_t<M_{it}\}}|\mathcal{E}_{it}|$$

- It is **common** to use

$$\frac{1}{\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbf{1}_{\{0<C_t<M_{it}\}}}\sum_{t=1}^{T}\sum_{i=1}^{N}\log_{10}(|\mathcal{E}_{it}|/C_{it})\mathbf{1}_{\{0<C_t<M_{it}\}}$$

so that an error of -2 means that the agent is making a mistake equal to \$1 for every \$100 consumed, while an error of -3 means that the agent is making a mistake equal to \$0.1 for every \$100 consumed etc,

Until next

- **Ensure that you understand:**
  1. The principle in one-dimensional **numerical integration** and algorithm 10
  2. The principle in **counter-factual simulation**
  3. The use of avg. Euler errors as an **accuracy measure**
- Go to **PadLet** and ask or answer a question (https://padlet.com/thomas_jorgensen1/DP)
- **Think about:** What happens when $t \to \infty$?