# CHAPTER 2

# Modeling with Linear Programming

# Set 2.1a

**1**

(a) $x_2 - x_1 \geq 1$ or $-x_1 + x_2 \geq 1$

(b) $x_1 + 2x_2 \geq 3$ and $x_1 + 2x_2 \leq 6$

(c) $x_2 \geq x_1$ or $x_1 - x_2 \leq 0$

(d) $x_1 + x_2 \geq 3$

(e) $\frac{x_2}{x_1 + x_2} \leq .5$ or $.5x_1 - .5x_2 \geq 0$

**2**

(a) $(x_1, x_2) = (1, 4)$

$(x_1, x_2) \geq 0$

$6 \times 1 + 4 \times 4 = 22 < 24$

$1 \times 1 + 2 \times 4 = 9 \not\leq 6$ infeasible

(b) $(x_1, x_2) = (2, 2)$

$(x_1, x_2) \geq 0$

$$\left.\begin{array}{lll} 6 \times 2 + 4 \times 2 & = 20 & < 24 \\ 1 \times 2 + 2 \times 2 & = 6 & = 6 \\ -1 \times 2 + 1 \times 2 & = 0 & < 1 \\ 1 \times 2 & = 2 & = 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 2 + 4 \times 2 = \$18$

(c) $(x_1, x_2) = (3, 1.5)$

$x_1, x_2 \geq 0$

$$\left.\begin{array}{lll} 6 \times 3 + 4 \times 1.5 & = 24 & = 24 \\ 1 \times 3 + 2 \times 1.5 & = 6 & = 6 \\ -1 \times 3 + 1 \times 1.5 & = -1.5 & < 1 \\ 1 \times 1.5 & = 1.5 & < 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 3 + 4 \times 1.5 = \$21$

(d) $(x_1, x_2) = (2, 1)$

$$\left.\begin{array}{lll} x_1, x_2 & \geq 0 & \\ 6 \times 2 + 4 \times 1 & = 16 & < 24 \\ 1 \times 2 + 2 \times 1 & = 4 & < 6 \\ -1 \times 2 + 1 \times 1 & = -1 & < 1 \\ 1 \times 1 & = 1 & < 2 \end{array}\right\} \text{feasible}$$

$Z = 5 \times 2 + 4 \times 1 = \$14$

(e) $(x_1, x_2) = (2, -1)$

$x_1 \geq 0$, $x_2 < 0$, infeasible

Conclusion: (c) gives the best feasible solution

**3**

$(x_1, x_2) = (2, 2)$

Let $S_1$ and $S_2$ be the unused daily amounts of M1 and M2.

For M1: $S_1 = 24 - (6x_1 + 4x_2) = 4$ tons/day

For M2: $S_2 = 6 - (x_1 + 2x_2)$

$= 6 - (2 + 2 \times 2) = 0$ tons/day

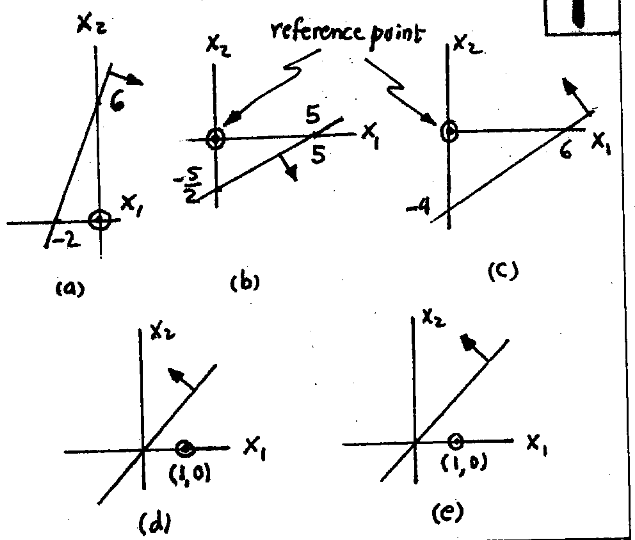
**4**

Quantity discount results in the following nonlinear objective function:

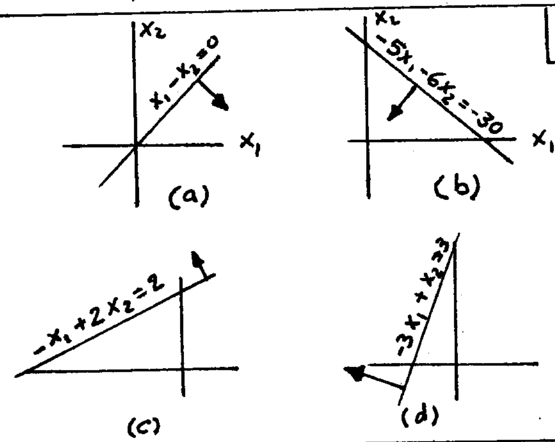
$$Z = \begin{cases} 5x_1 + 4x_2, & 0 \leq x_1 \leq 2 \\ 4.5x_1 + 4x_2, & x_1 > 2 \end{cases}$$

The situation cannot be treated as a linear program. Nonlinearity can be accounted for in this case using mixed integer programming (chapter 9).

## 1

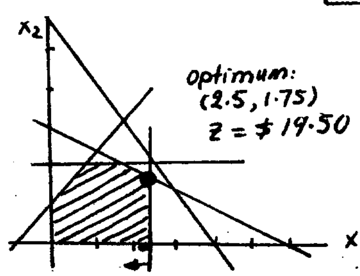



## 2

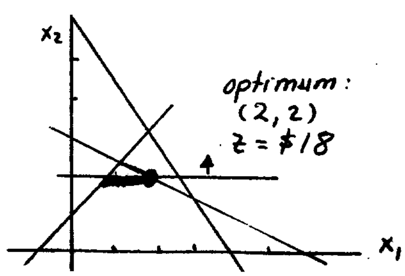



## 3

(a) $x_1 \leq 2.5$




(b) $x_2 \geq 2$

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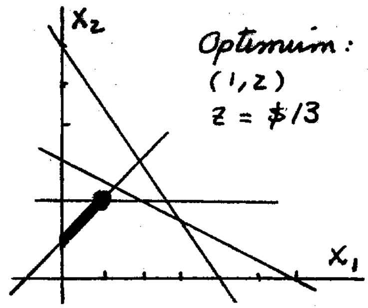
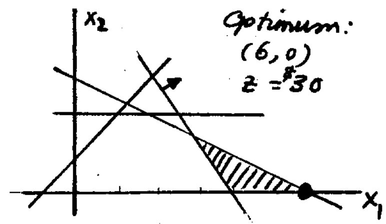
(c) $-x_1 + x_2 = 1$




(d) $6x_1 + 4x_2 \geq 24$




(e) No feasible space

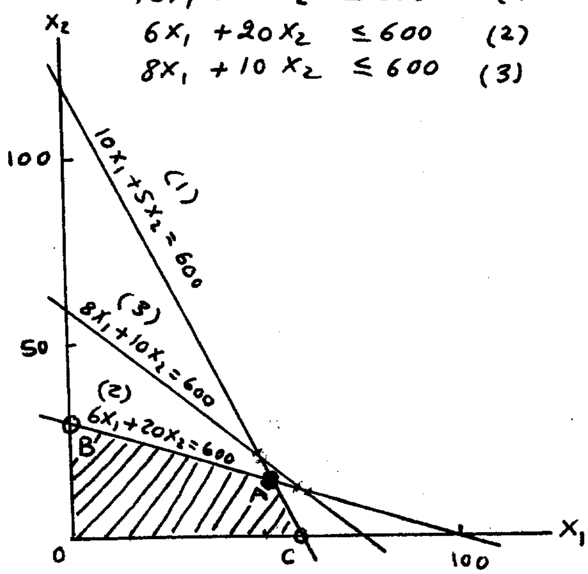
## 4

$x_1$ = daily units of product 1
$x_2$ = daily units of product 2

Maximize $z = 2x_1 + 3x_2$

s.t.

$$10x_1 + 5x_2 \leq 600 \quad (1)$$
$$6x_1 + 20x_2 \leq 600 \quad (2)$$
$$8x_1 + 10x_2 \leq 600 \quad (3)$$




Optimum occurs at A:

$x_1 = 52.94$
$x_2 = 14.12$
$z = \$148.24$

## Set 2.2a

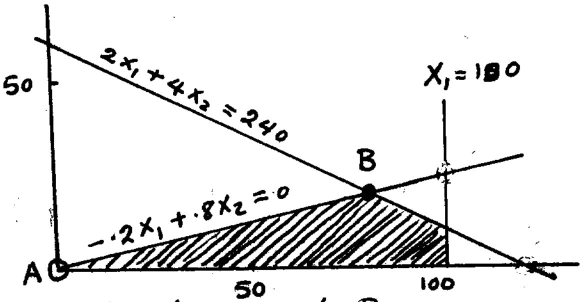
### 5

$x_1$ = number of units of A
$x_2$ = number of units of B

Maximize $Z = 20x_1 + 50x_2$

$$\frac{x_1}{x_1+x_2} \geq .8 \quad \text{or} \quad -.2x_1 + .8x_2 \leq 0$$

$$x_1 \leq 100$$
$$2x_1 + 4x_2 \leq 240$$
$$x_1, x_2 \geq 0$$




Optimal occurs at B:
$x_1$ = 80 units
$x_2$ = 20 units
$Z$ = \$2,600

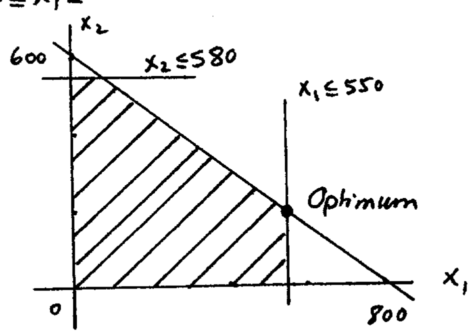
### 6

$x_1$ = number of sheets/day
$x_2$ = number of bars/day

Maximize $Z = 40x_1 + 35x_2$

s.t. $\frac{x_1}{800} + \frac{x_2}{600} \leq 1$

$0 \leq x_1 \leq 550, \quad 0 \leq x_2 \leq 580$



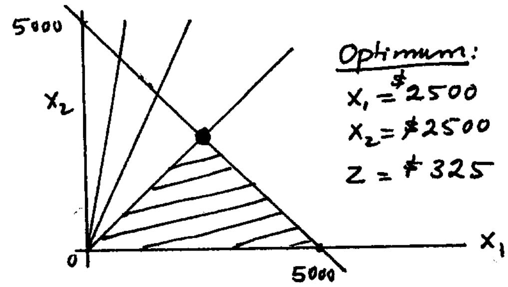

Optimum solution:
$x_1$ = 550 sheets
$x_2$ = 187.13 bars
$Z$ = \$28,549.40

### 7

$x_1$ = \$ invested in A
$x_2$ = \$ invested in B

Maximize $Z = .05x_1 + .08x_2$

s.t.
$$x_1 \geq .25(x_1+x_2)$$
$$x_2 \leq .5(x_1+x_2)$$
$$x_1 \geq .5x_2$$
$$x_1 + x_2 \leq 5000$$
$$x_1, x_2 \geq 0$$




Optimum:
$x_1$ = \$2500
$x_2$ = \$2500
$Z$ = \$325

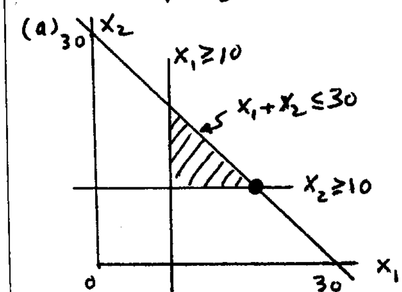
### 8

$x_1$ = number of practical courses
$x_2$ = number of humanistic courses

Maximize $Z = 1500x_1 + 1000x_2$

s.t.
$$x_1 + x_2 \leq 30$$
$$x_1 \geq 10$$
$$x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

(a)




Optimum:
$x_1 = 20$
$x_2 = 10$
$Z$ = \$40,000

(b) Change $x_1+x_2 \leq 30$ to $x_1+x_2 \leq 31$
Optimum $Z$ = \$41,500
$\Delta Z$ = \$41,500 − 40,000 = \$1500
Conclusion: Any additional course will be of the practical type.

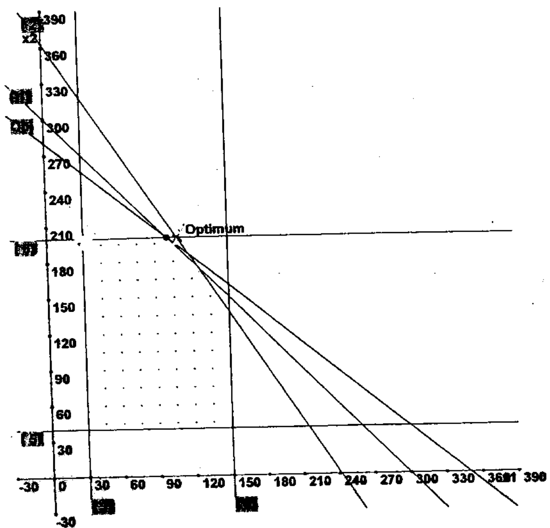
**9**

$x_1$ = units of solution A
$x_2$ = units of solution B

Maximize $z = 8x_1 + 10x_2$
subject to

$$\begin{aligned}
.5x_1 + .5x_2 &\le 150\\
.6x_1 + .4x_2 &\le 145\\
x_1 &\ge 30\\
x_1 &\le 150\\
x_2 &\ge 40\\
x_2 &\le 200\\
x_1, x_2 &\ge 0
\end{aligned}$$

Summary of Optimal Solution:
Objective Value = 2800.00
x1 = 100.00
x2 = 200.00

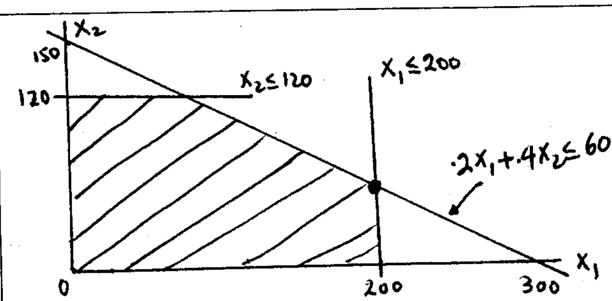



**10**

$x_1$ = nbr. of grano boxes
$x_2$ = nbr. of wheatie boxes

Maximize $Z = x_1 + 1.35x_2$
s.t. $.2x_1 + .4x_2 \le 60$
$x_1 \le 200$
$x_2 \le 120$
$x_1, x_2 \ge 0$

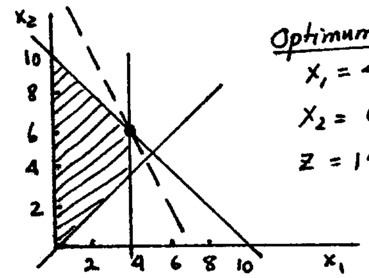
continued…




Optimum: $x_1 = 200$, $x_2 = 50$, $Z = \$267.50$
Area allocation: 67% grano, 33% wheatie

**11**

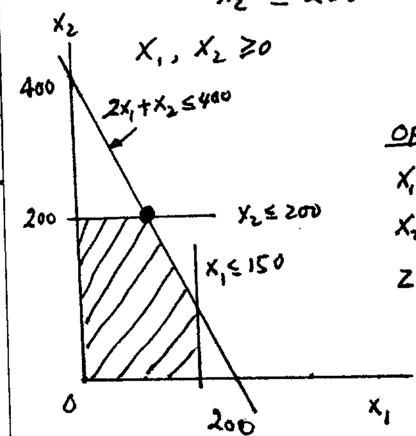
$x_1$ = play hours per day
$x_2$ = work hours per day
Maximize $Z = 2x_1 + x_2$
s.t.
$$\begin{aligned}
x_1 + x_2 &\le 10\\
x_1 - x_2 &\le 0\\
x_1 &\le 4\\
x_1, x_2 &\ge 0
\end{aligned}$$




Optimum solution:
$x_1 = 4$ hours
$x_2 = 6$ hours
$Z = 14$ "pleasarits"

**12**

$x_1$ = Daily nbr. of type 1 hat
$x_2$ = Daily nbr. of type 2 hat
Maximize $Z = 8x_1 + 5x_2$
s.t. $2x_1 + x_2 \le 400$
$x_1 \le 150$
$x_2 \le 200$
$x_1, x_2 \ge 0$




Optimum:
$x_1 = 100$ type 1
$x_2 = 200$ type 2
$Z = \$1800$

continued…

# Set 2.2a

**13**

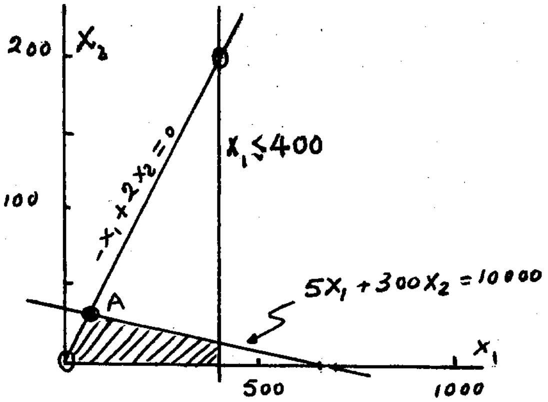
$x_1$ = radio minutes
$x_2$ = TV minutes

Maximize $z = x_1 + 25x_2$

s.t. $15x_1 + 300x_2 \le 10{,}000$

$\frac{x_1}{x_2} \ge 2$ or $-x_1 + 2x_2 \le 0$

$x_1 \le 400$, $x_1, x_2 \ge 0$




Optimum occurs at A:
$x_1 = 60.61$ minutes
$x_2 = 30.3$ minutes
$z = 818.18$

**14**

$x_1$ = tons of $C_1$ consumed per hour
$x_2$ = tons of $C_2$ consumed per hour

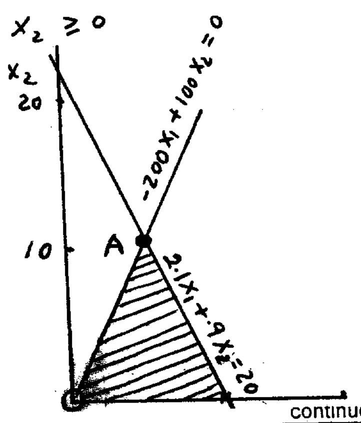
Maximize $z = 12000x_1 + 9000x_2$

s.t.
$1800x_1 + 2100x_2 \le 2000(x_1+x_2)$

or $-200x_1 + 100x_2 \le 0$

$2.1x_1 + .9x_2 \le 20$

$x_1, x_2 \ge 0$




continued...

(a) Optimum occurs at A:
$x_1 = 5.128$ tons per hour
$x_2 = 10.256$ tons per hour
$z = 153{,}846$ lb of steam

Optimal ratio $= \frac{5.128}{10.256} = .5$

(b) $2.1x_1 + .9x_2 \le (20+1) = 21$

Optimum $z = 161538$ lb of steam

$\Delta z = 161538 - 153846 = 7692$ lb

**15**

$x_1$ = Nbr. of radio commercials beyond the first
$x_2$ = Nbr. of TV ads beyond the first

Maximize $z = 2000x_1 + 3000x_2 + 5000 + 2000$

s.t. $300(x_1+1) + 2000(x_2+1) \le 20{,}000$

$300(x_1+1) \le .8 \times 20{,}000$

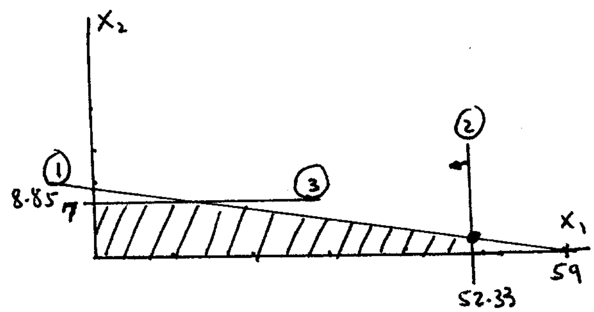
$2000(x_2+1) \le .8 \times 20{,}000$

$x_1, x_2 \ge 0$

or

Maximize $z = 2000x_1 + 3000x_2 + 7000$

s.t.
$300x_1 + 2000x_2 \le 17700$ (1)
$300x_1 \le 15700$ (2)
$2000x_2 \le 14000$ (3)
$x_1, x_2 \ge 0$




Optimum solution:

Radio commercials $= 52.33 + 1 = 53.33$

TV ads $= 1 + 1 = 2$

$z = 107666.67 + 7000 = 114666.67$

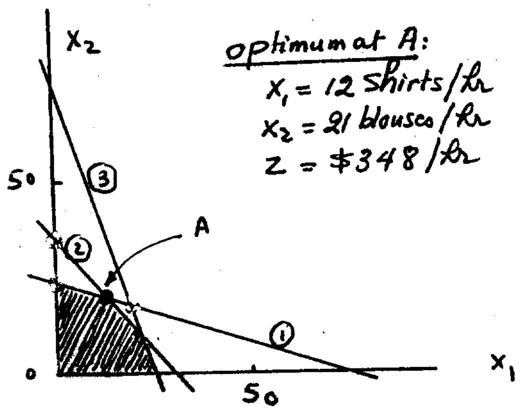
**16**

$x_1$ = number of shirts per hour
$x_2$ = number of blouses per hour

Maximize $Z = 8x_1 + 12x_2$

s.t.

$$20x_1 + 60x_2 \le 25 \times 60 = 1500 \quad (1)$$
$$70x_1 + 60x_2 \le 35 \times 60 = 2100 \quad (2)$$
$$12x_1 + 4x_2 \le 5 \times 60 = 300 \quad (3)$$
$$x_1, x_2 \ge 0$$



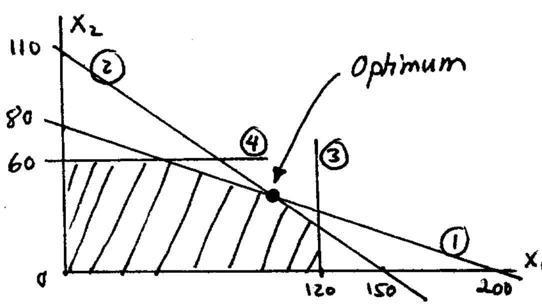

optimum at A:
$x_1 = 12$ shirts/hr
$x_2 = 21$ blouses/hr
$z = \$348$/hr

**17**

$x_1$ = Nbr. of desks per day
$x_2$ = Nbr. of chairs per day

Maximize $Z = 50x_1 + 100x_2$

$$\frac{x_1}{200} + \frac{x_2}{80} \le 1 \quad (1)$$
$$\frac{x_1}{150} + \frac{x_2}{110} \le 1 \quad (2)$$
$$x_1 \le 120, \quad x_2 \le 60 \quad (3,4)$$




Optimum:
$x_1 = 90$ desks
$x_2 = 44$ chairs
$z = \$8900$

**18**

$x_1$ = number of HiFi1 units
$x_2$ = number of HiFi2 units

Constraints:

$$6x_1 + 4x_2 \le 480 \times .9 = 432$$
$$5x_1 + 5x_2 \le 480 \times .86 = 412.8$$
$$4x_1 + 6x_2 \le 480 \times .88 = 422.4$$

or

$$6x_1 + 4x_2 + S_1 = 432$$
$$5x_1 + 5x_2 + S_2 = 412.8$$
$$4x_1 + 6x_2 + S_3 = 422.4$$

Objective function:

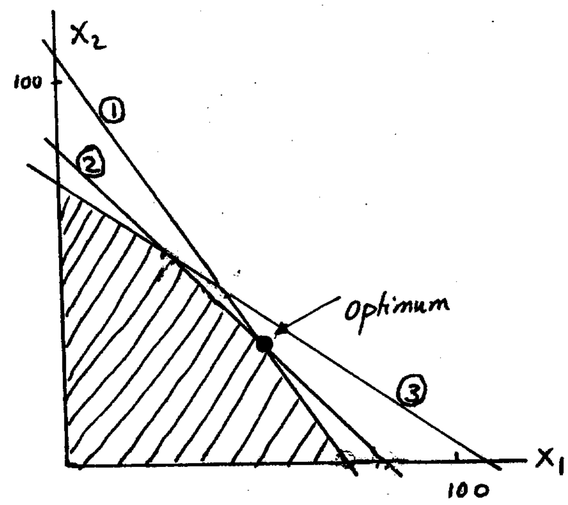
Minimize $S_1 + S_2 + S_3 = 1267.2 - 15x_1 - 15x_2$

Thus, $\min S_1 + S_2 + S_3 \equiv \max 15x_1 + 15x_2$

Maximize $Z = 15x_1 + 15x_2$

s.t.

$$6x_1 + 4x_2 \le 432 \quad (1)$$
$$5x_1 + 5x_2 \le 412.8 \quad (2)$$
$$4x_1 + 6x_2 \le 422.4 \quad (3)$$
$$x_1, x_2 \ge 0$$

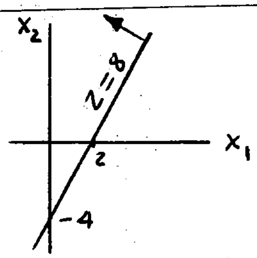



Optimum: (Problem has alternative optima)
$x_1 = 50.88$ units
$x_2 = 31.68$ units
$z = 1238.4$ minutes

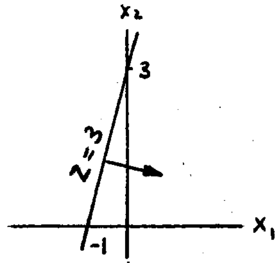
# Set 2.2b

**1**

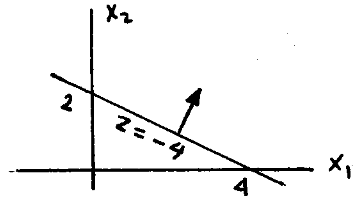
(a)




(b)

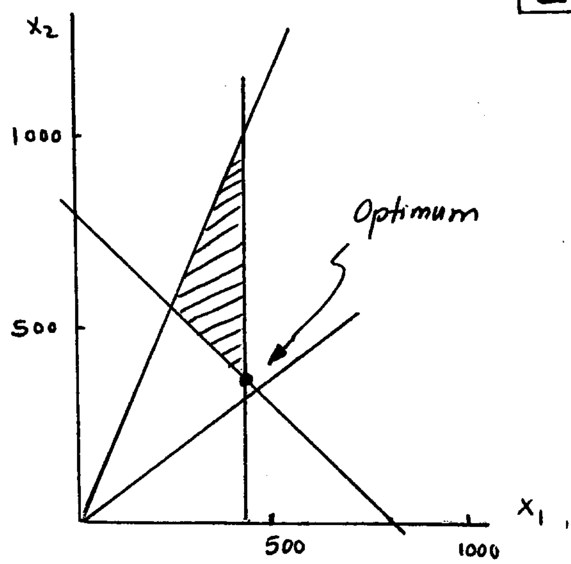



(c)




**2**

Additional constraint: $x_1 \le 450$




Optimum Solution:

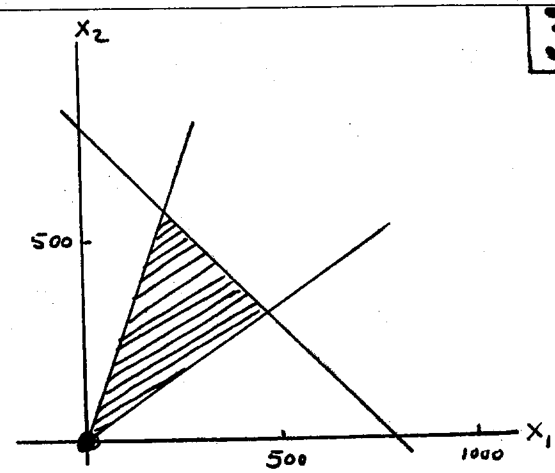
$x_1 = 450$ lb

$x_2 = 350$ lb

$z = \$450$

continued…

**3**




Optimum: $x_1 = 0$, $x_2 = 0$, $z = 0$, which is nonsensical

**4**

$x_1$ = number of hours/week in store 1

$x_2$ = number of hours/week in store 2

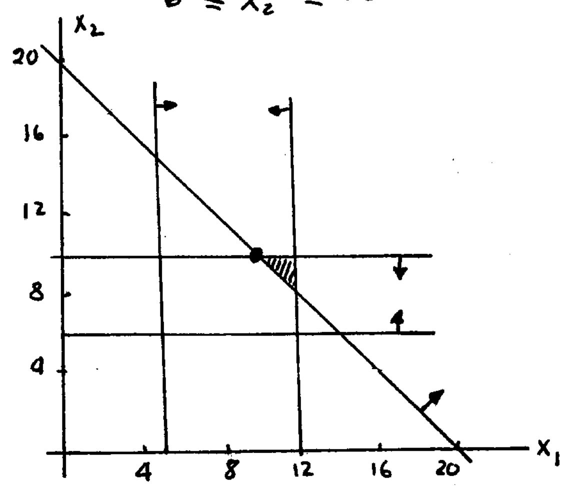
Minimize $z = 8x_1 + 6x_2$

s.t.

$$x_1 + x_2 \ge 20$$

$$5 \le x_1 \le 12$$

$$6 \le x_2 \le 10$$




Optimum:

$x_1 = 10$ hours

$x_2 = 10$ hours

$z = 140$ stress index

continued…

**5**

Let

$x_1 = 10^3$ bbl/day from Iran

$x_2 = 10^3$ bbl/day from Dubai

Refinery capacity $= x_1 + x_2$ $10^3$ bbl/day

Minimize $z = x_1 + x_2$

subject to

$$x_1 \geq .4(x_1 + x_2)$$

or

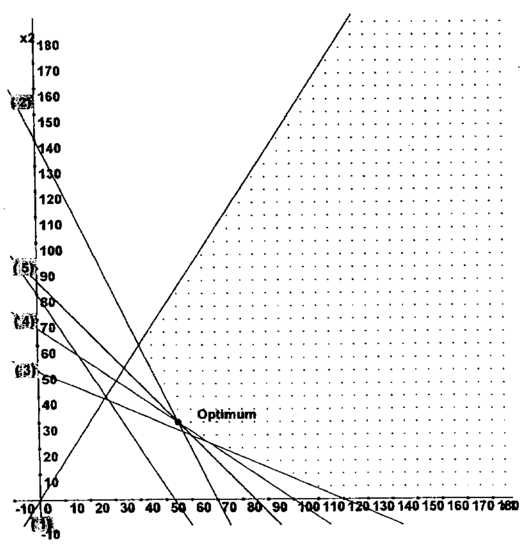
$$-.6x_1 + .4x_2 \leq 0$$
$$.2x_1 + .1x_2 \geq 14$$
$$.25x_1 + .6x_2 \geq 30$$
$$.1x_1 + .15x_2 \geq 10$$
$$.15x_1 + .1x_2 \geq 8$$
$$x_1, x_2 \geq 0$$

Optimum solution from TORA:

LINEAR PROGRAMMING – GRAPHICAL SOLUTION

Title: diet problem

Summary of Optimal Solution:
Objective Value = 85.00
x1 = 55.00
x2 = 30.00




**6**

Let

$x_1 = 10^3\$$ invested in blue chip stock

$x_2 = 10^3\$$ invested in high-tech stocks

Minimize $z = x_1 + x_2$

subject to

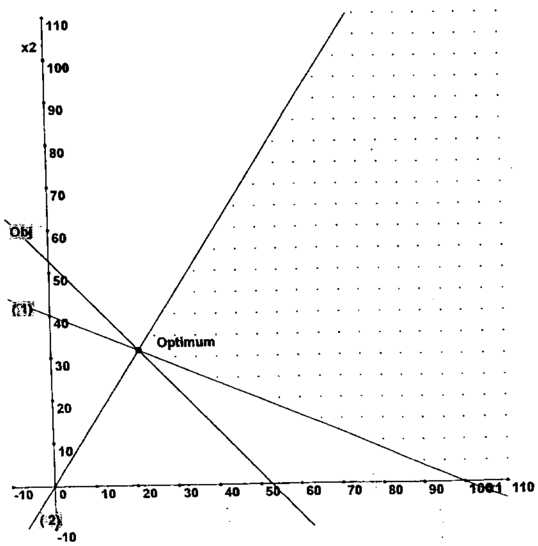
$$.1x_1 + .25x_2 \geq 10$$
$$.6x_1 - .4x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

TORA optimum solution:

LINEAR PROGRAMMING – GRAPHICAL SOLUTION

Title: diet problem

Summary of Optimal Solution:
Objective Value = 52.63
x1 = 21.05
x2 = 31.58

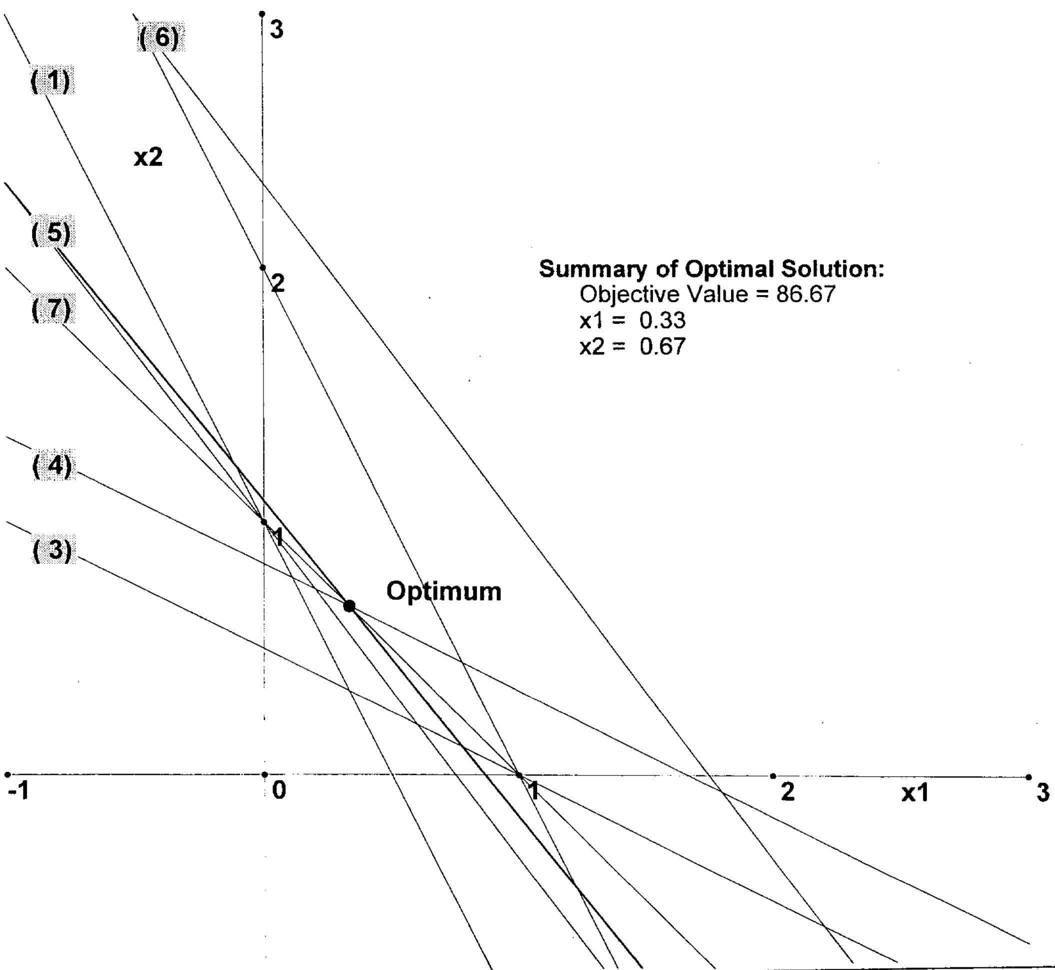


## Set 2.2b

**7**

$x_1$ = Ratio of scrap A in alloy
$x_2$ = Ratio of scrap B in alloy

| | x1 | x2 | | |
|---|---|---|---|---|
| Minimize | 100.00 | 80.00 | | |
| Subject to | | | | |
| ( 1) | 0.06 | 0.03 | >= | 0.03 |
| ( 2) | 0.06 | 0.03 | <= | 0.06 |
| ( 3) | 0.03 | 0.06 | >= | 0.03 |
| ( 4) | 0.03 | 0.06 | <= | 0.05 |
| ( 5) | 0.04 | 0.03 | >= | 0.03 |
| ( 6) | 0.04 | 0.03 | <= | 0.07 |
| ( 7) | 1.00 | 1.00 | = | 1.00 |




**Summary of Optimal Solution:**
Objective Value = 86.67
x1 = 0.33
x2 = 0.67

## 1

$x_e$ = Nbr. of efficiency apartments
$x_d$ = Nbr. of duplexes
$x_s$ = Nbr. of single-family homes
$x_r$ = Retail space in ft$^2$

Maximize $z = 600x_e + 750x_d + 1200x_s + 100x_r$

s.t. $x_e \le 500,\ x_d \le 300,\ x_s \le 250$

$$x_r \ge 10x_e + 15x_d + 18x_s$$

$$x_r \le 10000$$

$$x_d \ge \frac{x_e + x_s}{2}$$

$$x_e, x_d, x_s, x_r \ge 0$$

Optimal solution:

$z = 1,595,714.29$
$x_e = 207.14,\ x_d = 228.57$
$x_s = 250,\ x_r = 10,000$

LP does not guarantee integer solution. Use rounded solution or apply integer LP algorithm (Chapter 9).

## 2

$x_i$ = Acquired portion of property $i$

Each site is represented by a separate LP. The site that yields the smaller objective value is selected.

Site 1 LP:

Minimize $z = 25 + x_1 + 2.1x_2 + 2.35x_3 + 1.85x_4 + 2.95x_5$

s.t. $x_4 \ge .75$, all $x_i \ge 0,\ i = 1, 2, \dots, 5$
$20x_1 + 50x_2 + 50x_3 + 30x_4 + 60x_5 \ge 200$

Optimum: $z = 34.6625$ million \$
$x_1 = .875,\ x_2 = x_3 = 1,\ x_4 = .75,\ x_5 = 1$

Site 2 LP:

Minimize $z = 27 + 2.8x_1 + 1.9x_2 + 2.8x_3 + 2.5x_4$

s.t. $x_3 \ge .5,\ x_1, x_2, x_3, x_4 \ge 0$
$80x_1 + 60x_2 + 50x_3 + 70x_4 \ge 200$

Optimum: $z = 34.35$ million \$
$x_1 = x_2 = 1,\ x_3 = x_4 = .5$

Select site 2.

## 3

$x_{ij}$ = portion of project $i$ completed in year $j$

Maximize $z = .05(4x_{11} + 3x_{12} + 2x_{13}) + .07(3x_{22} + 2x_{23} + x_{24}) + .15(4x_{31} + 3x_{32} + 2x_{33} + x_{34}) + .02(2x_{43} + x_{44})$

s.t.

$$\sum_{j=1}^{3} x_{1j} = 1,\quad \sum_{j=3}^{4} x_{4j} = 1$$

$$.25 \le \sum_{j=2}^{5} x_{2j} \le 1,\quad .25 \le \sum_{j=1}^{5} x_{3j} \le 1$$

$$5x_{11} + 15x_{31} \le 3$$

$$5x_{12} + 8x_{22} + 15x_{32} \le 6$$

$$5x_{13} + 8x_{23} + 15x_{33} + 1.2x_{43} \le 7$$

$$8x_{24} + 15x_{34} + 1.2x_{44} \le 7$$

$$8x_{25} + 15x_{35} \le 7$$

Optimum:

$z = \$523,750$
$x_{11} = .6,\ x_{12} = .4$
$x_{24} = .225,\ x_{25} = .025$
$x_{32} = .267,\ x_{33} = .387,\ x_{34} = .346$
$x_{43} = 1$

## 4

$x_l$ = Nbr. of low income units
$x_m$ = Nbr. of middle income units
$x_u$ = Nbr. of upper income units
$x_p$ = Nbr. of public housing units
$x_s$ = Nbr. of school rooms
$x_r$ = Nbr. of retail units
$x_c$ = Nbr. of condemned homes

Maximize $z = 7x_l + 12x_m + 20x_u + 5x_p + 15x_r - 10x_s - 7x_c$

s.t. $100 \le x_l \le 200,\ 125 \le x_m \le 190$
$75 \le x_u \le 260,\ 300 \le x_p \le 600$
$0 \le x_s \le 2/.045$
$.05x_l + .07x_m + .03x_u + .025x_p + .045x_s + .1x_r \le .85(50 + .25x_c)$
$x_r \ge .023x_l + .034x_m + .046x_u + .023x_p + .034x_s$

continued...

$25x_s \geq 1.3x_l + 1.2x_m + .5x_u + 1.4x_p$

Optimum: $z = 8290.30$ thousand \$

$x_l = 100$, $x_m = 125$, $x_u = 227.04$

$x_p = 300$, $x_s = 32.54$, $x_r = 25$

$x_c = 0$

**5**

$x_1$ = Nbr. of single-family homes

$x_2$ = Nbr. of double-family homes

$x_3$ = Nbr. of triple-family homes

$x_4$ = Nbr. of recreation areas

Maximize $z = 10{,}000x_1 + 12000x_2 + 15000x_3$

s.t.

$2x_1 + 3x_2 + 4x_3 + x_4 \leq .85 \times 800$

$\dfrac{x_1}{x_1 + x_2 + x_3} \geq .5$ or $.5x_1 - .5x_2 - .5x_3 \geq 0$

$x_4 \geq \dfrac{x_1 + 2x_3 + 3x_3}{200}$ or $200x_4 - x_1 - 2x_2 - 3x_3 \geq 0$

$1000x_1 + 1200x_2 + 1400x_3 + 800x_4 \geq 100{,}000$

$400x_1 + 600x_2 + 840x_3 + 450x_4 \leq 200{,}000$

$x_1, x_2, x_3, x_4 \geq 0$

Optimum solution:

$x_1 = 339.15$ homes

$x_2 = 0$

$x_3 = 0$

$x_4 = 1.69$ areas

$z = \$3{,}391{,}521.20$

**6**

New land use constraint:

$2x_1 + 3x_2 + 4x_3 + x_4 \leq .85(800 + 100)$

New Optimum solution:

$z = \$3{,}815{,}461.35$

$x_1 = 381.54$ homes

$x_2 = x_3 = 0$

$x_4 = 1.91$ areas

$\Delta z = \$3{,}815{,}461.35 - 3{,}391{,}521.20$

$= \$423{,}940.35$

$\Delta z < \$450{,}000$, the purchasing cost of 100 acres. Hence, the purchase of the new acreage is <u>not</u> recommended.

**1**

The constraints remain unchanged, but the objective function is changed to

Maximize $z = y - \text{commission}$

where

$$\text{commission} = .001(\text{all transactions in } \$)$$

$$= .001\Big[(x_{12}+x_{13}+x_{14}+x_{15}) + \frac{1}{.769}(x_{21}+x_{23}+x_{24}+x_{25}) + \frac{1}{.625}(x_{31}+x_{32}+x_{34}+x_{35}) + \frac{1}{105}(x_{41}+x_{42}+x_{43}+x_{45}) + \frac{1}{.342}(x_{51}+x_{52}+x_{53}+x_{54})$$

Optimum solution:

| | Without | With |
|---|---|---|
| $z$ | 5.09032 | 5.06211 |
| $y$ | 5.09032 | 5.08986 |
| Return | 1.8064% | 1.2421% |

Commission = 5.08986 − 5.06211
= \$ 27,750

or, .555% of the original investment of \$5 million

**2**

Input $I$ in fund type $p$:

$$\sum_{\substack{j=1\\ j\neq p}}^{n} r_{jp}x_{jp} \longrightarrow \underset{\uparrow I}{(p)} \longrightarrow \sum_{\substack{j=1\\ j\neq p}}^{n} x_{pj}$$

Output $y$ in fund type $q$:

$$\sum_{\substack{j=1\\ j\neq q}}^{n} r_{jq}x_{jq} \longrightarrow \underset{\downarrow y}{(q)} \longrightarrow \sum_{\substack{j=1\\ j\neq q}}^{n} x_{qj}$$

For specific $p$ and $q$, the model below can be used to transform any fund to any other fund. In the present problem, $p = 1$ (\$) and $q = 2$ (€), 3 (£), 4 (¥), and 5 (KD).

General node $i$:

$$\sum_{\substack{j=1\\ j\neq i}}^{n} r_{ji}x_{ji} \longrightarrow (i) \longrightarrow \sum_{\substack{j=1\\ j\neq i}}^{n} x_{ij}$$

Maximize $z = y$

s.t.

$$I + \sum_{\substack{j=1\\ j\neq p}}^{n} r_{jp}x_{jp} = \sum_{\substack{j=1\\ j\neq p}}^{n} x_{pj}$$

$$\sum_{\substack{j=1\\ j\neq q}}^{n} r_{jq}x_{jq} = y + \sum_{\substack{j=1\\ j\neq q}}^{n} x_{qj}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{n} r_{ji}x_{ji} = \sum_{\substack{j=1\\ j\neq i}}^{n} x_{ij}, \quad i \neq p \text{ or } q$$

$$0 \le x_{ij} \le cap_i, \text{ all } i \text{ and } j$$

Note: Solver or AMPL is ideal for solving this problem interactively. See files solver2.3b-2.xls and ampl2.3b-2.txt.

Results: (No commission)

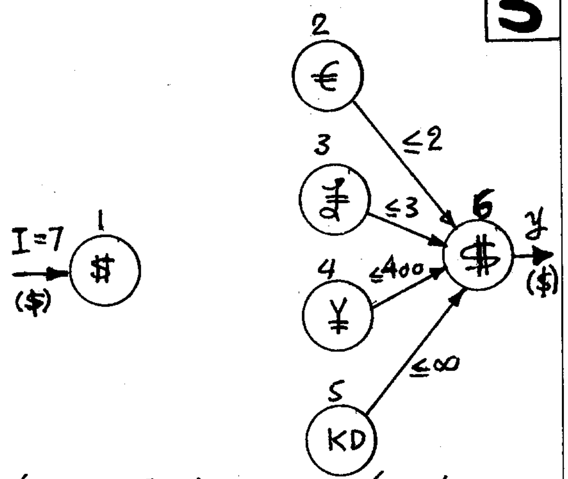
| p | q | Rate of return |
|---|---|---|
| \$ | \$ | 1.8064% |
| \$ | € | 1.7966% |
| \$ | £ | 1.8287% |
| \$ | ¥ | 2.8515% |
| \$ | KD | 1.0471% |

Wide discrepancy in ¥ and KD currencies may be attributed to the fact that their exchange rates may not be consistent with the remaining rates. Nevertheless, the problem shows that there may be advantages in targeting accumulation in different currencies.

continued…

## Set 2.3b

**3**




To formulate the objective function correctly, all output currencies are converted to a single currency (arbitrarily chosen to be \$). Thus

$$y = r_{21}x_{26} + r_{31}x_{36} + r_{41}x_{46} + r_{51}x_{56}$$

Maximize $z = y$

s.t. $x_{26} \le 2$, $x_{36} \le 3$, $x_{46} \le 400$; $x_{3j} \le 3.5$

$x_{1j} \le 5$, $x_{2j} \le 3$, $x_{4j} \le 100$, $x_{5j} \le 2.8$, all $j$

$$I + \sum_{i=2}^{5} x_{i1} r_{i1} = \sum_{j=2}^{5} x_{1j}$$

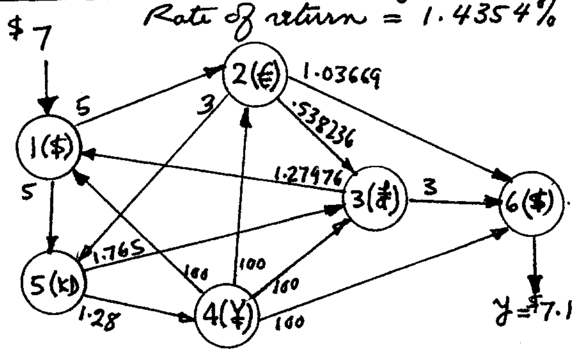
$$y = r_{21}x_{26} + r_{31}x_{36} + r_{41}x_{46} + r_{51}x_{56}$$

$$x_{i6} + \sum_{\substack{j=1 \\ j \ne i}}^{5} x_{ij} = \sum_{\substack{j=1 \\ j \ne i}}^{5} r_{ji}x_{ji}, \quad i = 2,3,4,5$$

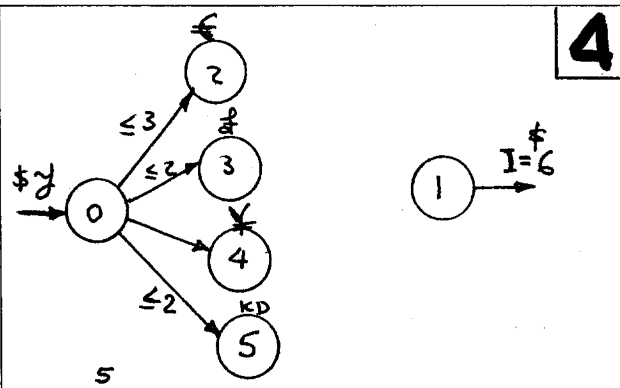
all $x_{ij} \ge 0$, $i \ne j$

Solution: Total accumulation $y = \$7.1$ million

Rate of return $= 1.4354\%$




**4**




$$y = \sum_{k=2}^{5} r_{k1}x_{0k} = r_{21}x_{02} + r_{31}x_{03} + r_{41}x_{04} + r_{51}x_{05}$$

Minimize $z = y$

s.t. $y = r_{21}x_{02} + r_{31}x_{03} + r_{41}x_{04} + r_{51}x_{05}$

$x_{02} \le 3$, $x_{03} \le 2$, $x_{05} \le 2$

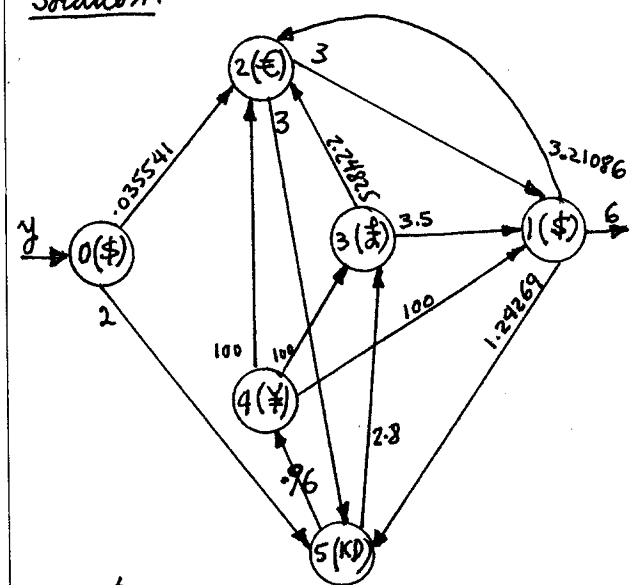
all $j$: $x_{1j} \le 5$, $x_{2j} \le 3$, $x_{4j} \le 3.5$, $x_{4j} \le 100$, $x_{5j} \le 2.8$

$$x_{0j} + \sum_{\substack{i=1 \\ i \ne j}}^{5} r_{ij}x_{ij} = \sum_{\substack{k=1 \\ k \ne j}}^{5} x_{jk}, \quad j = 2,3,4,5$$

$$\sum_{i=2}^{5} r_{i1}x_{i1} = \sum_{j=2}^{5} x_{1j} + I$$

all $x_{ij} \ge 0$

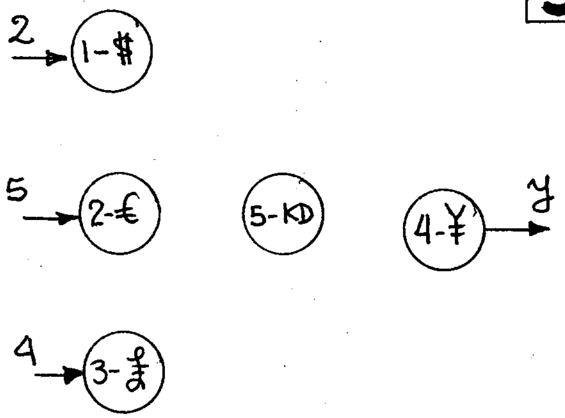
Solution:




$y = \$5.894170322$

Rate of return $= 1.7638\%$

**5**




Maximize $z = y$

s.t.

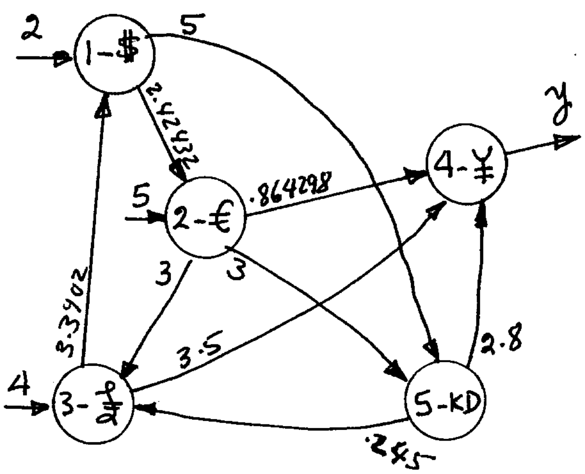
$$y = r_{14}x_{14} + r_{24}x_{24} + r_{34}x_{34} + r_{54}x_{54}$$

$$\sum_{\substack{i=1\\ i\neq j}}^{5} r_{ij}x_{ij} = \sum_{\substack{k=1\\ k\neq j}}^{5} x_{jk} - \begin{cases} 2, & j=1\\ 5, & j=2\\ 4, & j=3\\ -y, & j=4\\ 0, & j=5 \end{cases}$$

$$\left.\begin{array}{l} x_{ij} \leq C_i, \text{ for all } i \text{ and } j\\ x_{ij} \geq 0, \text{ for all } i \text{ and } j \end{array}\right\} i\neq j$$

<u>Solution</u>: $y = 1584.91$ million¥

Rate of return $= .8853\%$



**1**

(a) $x_i$ = Undertaken portion of project $i$

Maximize

$$z = 32.4x_1 + 35.8x_2 + 17.75x_3 + 14.8x_4 + 18.2x_5 + 12.35x_6$$

Subject to

$$10.5x_1 + 8.3x_2 + 10.2x_3 + 7.2x_4 + 12.3x_5 + 9.2x_6 \le 60$$
$$14.4x_1 + 12.6x_2 + 14.2x_3 + 10.5x_4 + 10.1x_5 + 7.8x_6 \le 70$$
$$2.2x_1 + 9.5x_2 + 5.6x_3 + 7.5x_4 + 8.3x_5 + 6.9x_6 \le 35$$
$$2.4x_1 + 3.1x_2 + 4.2x_3 + 5.0x_4 + 6.3x_5 + 5.1x_6 \le 20$$
$$0 \le x_j \le 1, \quad j = 1, 2, \dots, 6$$

TORA optimum solution:

$x_1 = x_2 = x_3 = x_4 = 1, x_5 = .84, x_6 = 0, z = 116.06$

(b) Add the constraint $x_2 \le x_6$

TORA optimum solution:

$x_1 = x_2 = x_3 = x_4 = x_6 = 1, x_5 = .03, z = 113.68$

(c) Let $S_i$ be the unused funds at the end of year $i$ and change the right-hand sides of constraints 2, 3, and 4 to $70 + S_1$, $35 + S_2$, and $20 + S_3$, respectively.

TORA optimum solution:

$x_1 = x_2 = x_3 = x_4 = x_5 = 1, \quad x_6 = .71$

$z = \$127.72$ (thousand)

The solution is interpreted as follows:

| $i$ | $S_i$ | $S_i - S_{i-1}$ | Decision |
|---|---|---|---|
| 1 | 4.96 | — | — |
| 2 | 7.62 | +2.66 | Don't borrow from yr 1 |
| 3 | 4.62 | −3.00 | Borrow \$3 from year 2 |
| 4 | 0 | −4.62 | Borrow \$4.62 from yr 2 |

The effect of availing excess money for use in later years is that the first five projected are completed and 71% of project 6 is undertaken.

The total revenue increases from \$116,060 to 127,720.

(d) The slack $S_i$ in period $i$ is treated as an unrestricted variable.

TORA optimum solution: $z = \$131.30$

$S_1 = 2.3, \ S_2 = .4, \ S_3 = -5, \ S_4 = -6.1$

This means that additional funds are needed in years 3 and 4.

Increase in return $= 131.30 - 116.06 = \$15.24$

Ignoring the time value of money, the amount borrowed $5 + 6.1 - (2.3 + .4) = \$8.4$. Thus,

$$\text{rate of return} = \frac{15.24 - 8.4}{8.4} \cong 81\%$$

continued...

**2**

$x_i$ = dollar investment in project $i$, $i = 1, 2, 3, 4$

$y_j$ = dollar investment in bank in year $j$, $j = 1, 2, 3, 4, 5$

Maximize $z = y_5$

Subject to

$$x_1 + x_2 + x_4 + y_1 \le 10{,}000$$
$$.5x_1 + .6x_2 - x_3 + .4x_4 + 1.065y_1 - y_2 = 0$$
$$.3x_1 + .2x_2 + .8x_3 + .6x_4 + 1.065y_2 - y_3 = 0$$
$$1.8x_1 + 1.5x_2 + 1.9x_3 + 1.8x_4 + 1.065y_3 - y_4 = 0$$
$$1.2x_1 + 1.3x_2 + .8x_3 + .95x_4 + 1.065y_4 - y_5 = 0$$

All variables $\ge 0$

TORA optimal solution:

$x_1 = 0, \ x_2 = \$10{,}000, \ x_3 = \$6000, \ x_4 = 0$

$y_1 = 0, \ y_2 = 0, \ y_3 = \$6800, \ y_4 = \$33{,}642$

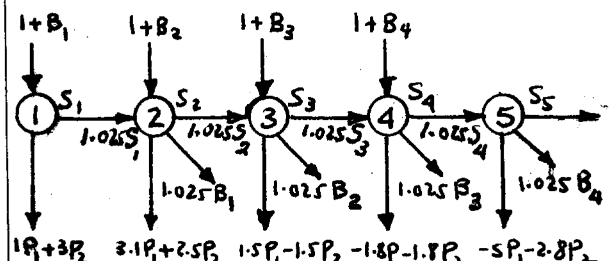
$z = \$53{,}628.73$ at the start of year 5

**3**

$P_i$ = fraction undertaken of project $i$, $i = 1, 2$

$B_j$ = million dollars borrowed in quarter $j$, $j = 1, 2, 3, 4$

$S_j$ = surplus million dollars at the start of quarter $j$, $j = 1, 2, 3, 4, 5$




(a) Maximize $z = S_5$

subject to

$$P_1 + 3P_2 + S_1 - B_1 = 1$$
$$3.1P_1 + 2.5P_2 - 1.02S_1 + S_2 + 1.025B_1 - B_2 = 1$$
$$1.5P_1 - 1.5P_2 - 1.02S_2 + S_3 + 1.025B_2 - B_3 = 1$$
$$-1.8P_1 - 1.8P_2 - 1.02S_3 + S_4 + 1.025B_3 - B_4 = 1$$
$$-5P_1 - 2.8P_2 - 1.02S_4 + S_5 + 1.025B_4 = 1$$
$$0 \le P_1 \le 1, \quad 0 \le P_2 \le 1$$
$$0 \le B_j \le 1, \quad j = 1, 2, 3, 4$$

<u>Optimum solution</u>:

$P_1 = .7113 \qquad P_2 = 0$

$z = 5.8366$ million dollars

$B_1 = 0$, $B_2 = .9104$ million dollars

$B_3 = 1$ million dollars, $B_4 = 0$

(b) $B_1 = 0$, $S_1 = .2887$ million \$

$B_2 = .9104$, $S_2 = 0$

$B_3 = 1$, $S_3 = 0$

$B_4 = 0$, $S_4 = 1.2553$

The solution shows that $B_i \cdot S_i = 0$, meaning that you can't borrow and also end up with surplus in any quarter. The result makes sense because the cost of borrowing (2.5%) is higher then the return on surplus funds (2%)

**4**

Assume that the investment program ends at the start of year 11. Thus, the 6-year bond option can be exercised in years 1, 2, 3, 4, and 5 only. Similarly, the 9-year bond can be used in years 1 and 2 only. Hence, from year 6 on, the only option available is insured savings at 7.5%.

Let

$I_i$ = insured savings investments in year $i$, $i = 1, 2, \dots, 10$

$G_i$ = 6-year bond investment in year $i$, $i = 1, 2, \dots, 5$

$M_i$ = 9-year bond investment in year $i$, $i = 1, 2$

The objective is to maximize total accumulation at the end of year 10; that is,

$$\text{maximize } z = 1.075I_{10} + 1.079G_5 + 1.085M_2$$

The constraints represent the balance equation for each year's cash flow.

$$I_1 + .98G_1 + 1.02M_1 = 2$$
$$I_2 + .98G_2 + 1.02M_2 = 2 + 1.075I_1 + .079G_1 + .085M_1$$
$$I_3 + .98G_3 = 2.5 + 1.075I_2 + .079(G_1 + G_2) + .085(M_1 + M_2)$$
$$I_4 + .98G_4 = 2.5 + 1.075I_3 + .079(G_1 + G_2 + G_3) + .085(M_1 + M_2)$$
$$I_5 + .98G_5 = 3 + 1.075I_4 + .079(G_1 + G_2 + G_3 + G_4) + .085(M_1 + M_2)$$
$$I_6 = 3.5 + 1.075I_5 + .079(G_1 + G_2 + G_3 + G_4 + G_5) + .085(M_1 + M_2)$$

continued...

$$I_7 = 3.5 + 1.075\, I_6 + 1.079\, G_1 + .079\,(G_2 + G_3 + G_4 + G_5) + .085\,(M_1 + M_2)$$

$$I_8 = 4 + 1.075\, I_7 + 1.079\, G_2 + .079\,(G_3 + G_4 + G_5) + .085\,(M_1 + M_2)$$

$$I_9 = 4 + 1.075\, I_8 + 1.079\, G_3 + .079\,(G_4 + G_5) + .085\,(M_1 + M_2)$$

$$I_{10} = 5 + 1.075\, I_9 + 1.079\, G_4 + .079\, G_5 + 1.085\, M_1 + .085\, M_2$$

all variables $\geq 0$

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 26a-14
Final iteration No: 14
Objective value (max) = 46.8500

| Variable | Value | Obj Coeff | Obj Val Contrib |
|---|---|---|---|
| x1 I1 | 0.0000 | 0.0000 | 0.0000 |
| x2 I2 | 0.0000 | 0.0000 | 0.0000 |
| x3 I3 | 0.0000 | 0.0000 | 0.0000 |
| x4 I4 | 0.0000 | 0.0000 | 0.0000 |
| x5 I5 | 0.0000 | 0.0000 | 0.0000 |
| x6 I6 | 4.6331 | 0.0000 | 0.0000 |
| x7 I7 | 9.6137 | 0.0000 | 0.0000 |
| x8 I8 | 15.4678 | 0.0000 | 0.0000 |
| x9 I9 | 24.6663 | 0.0000 | 0.0000 |
| x10 I10 | 37.5201 | 1.0750 | 40.3341 |
| x11 G1 | 0.0000 | 0.0000 | 0.0000 |
| x12 G2 | 0.0000 | 0.0000 | 0.0000 |
| x13 G3 | 2.9053 | 0.0000 | 0.0000 |
| x14 G4 | 3.1395 | 0.0000 | 0.0000 |
| x15 G5 | 3.9028 | 1.0790 | 4.2111 |
| x16 M1 | 1.9608 | 0.0000 | 0.0000 |
| x17 M2 | 2.1242 | 1.0850 | 2.3047 |

| Constraint | RHS | Slack(-)/Surplus(+) |
|---|---|---|
| 1 (=) | 2.0000 | 0.0000 |
| 2 (=) | 2.0000 | 0.0000 |
| 3 (=) | 2.5000 | 0.0000 |
| 4 (=) | 2.5000 | 0.0000 |
| 5 (=) | 3.0000 | 0.0000 |
| 6 (=) | 3.5000 | 0.0000 |
| 7 (=) | 3.5000 | 0.0000 |
| 8 (=) | 4.0000 | 0.0000 |
| 9 (=) | 4.0000 | 0.0000 |
| 10 (=) | 5.0000 | 0.0000 |

| Year | Recommendation |
|---|---|
| 1 | Invest all in 9-yr bond |
| 2 | Invest all in 9-yr. bond |
| 3 | Invest all in 6-yr bond |
| 4 | Invest all in 6-yr bond |
| 5 | Invest all in 6-yr bond |
| 7 | Invest all in insured savings |
| 8 | Invest all in insured savings |
| 9 | Invest all in insured savings |
| 10 | Invest all in insured savings |

**5**

$x_{iA}$ = amount invested in year $i$, plan A (1000\$)

$x_{iB}$ = amount invested in year $i$, plan B (1000\$)

Maximize $z = 3x_{2B} + 1.7x_{3A}$

subject to

$$x_{1A} + x_{1B} \leq 100$$

$$-1.7x_{1A} + x_{2A} + x_{2B} = 0$$

$$-3x_{1B} - 1.7x_{2A} + x_{3A} = 0$$

$$x_{iA}, x_{iB} \geq 0 \text{ for } i = 1, 2, 3$$

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 2.6a-15
Final iteration No: 4
Objective value (max) = 510.0000
==> ALTERNATIVE solution detected at x2

| Variable | Value | Obj Coeff | Obj Val Contrib |
|---|---|---|---|
| x1 x1A | 100.0000 | 0.0000 | 0.0000 |
| x2 x1B | 0.0000 | 0.0000 | 0.0000 |
| x3 x2A | 0.0000 | 0.0000 | 0.0000 |
| x4 x2B | 170.0000 | 3.0000 | 510.0000 |
| x5 x3A | 0.0000 | 1.7000 | 0.0000 |

| Constraint | RHS | Slack(-)/Surplus(+) |
|---|---|---|
| 1 (<) | 100.0000 | 0.0000- |
| 2 (<) | 0.0000 | 0.0000- |
| 3 (<) | 0.0000 | 0.0000- |

Optimum solution: Invest \$100,000 in A in yr 1 and \$170,000 in B in yr 2.
Alternative optimum: Invest \$100,000 in B in yr 1 and \$300,000 in A in yr 3.

**6**

$x_i$ = dollars allocated to choice $i$, $i = 1, 2, 3, 4$

$y$ = minimum return

$$\text{Maximize } z = \min \begin{cases} -3x_1 + 4x_2 - 7x_3 + 15x_4 \\ 5x_1 - 3x_2 + 9x_3 + 4x_4 \\ 3x_1 - 9x_2 + 10x_3 - 8x_4 \end{cases}$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 500$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The problem can be converted to a linear program as

continued...

Maximize $z = y$

subject to

$$-3x_1 + 4x_2 - 7x_3 + 15x_4 \geq y$$
$$5x_1 - 3x_2 + 9x_3 + 4x_4 \geq y$$
$$3x_1 - 9x_2 + 10x_3 - 8x_4 \geq y$$
$$x_1 + x_2 + x_3 + x_4 \leq 500$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$y$ unrestricted

*** OPTIMUM SOLUTION SUMMARY ***

Title:
Final iteration No: 5
Objective value (max) = 1175.0000

| Variable | Value | Obj Coeff | Obj Val Contrib |
|---|---|---|---|
| x1 | 0.0000 | 0.0000 | 0.0000 |
| x2 | 0.0000 | 0.0000 | 0.0000 |
| x3 | 287.5000 | 0.0000 | 0.0000 |
| x4 | 212.5000 | 0.0000 | 0.0000 |
| x5 y | 1175.0000 | 1.0000 | 1175.0000 |

| Constraint | RHS | Slack(-)/Surplus(+) |
|---|---|---|
| 1 (>) | 0.0000 | 0.0000+ |
| 2 (>) | 0.0000 | 2262.5000+ |
| 3 (>) | 0.0000 | 0.0000+ |
| 4 (<) | 500.0000 | 0.0000- |

Allocate \$287.50 to choice 3 and \$212.50 to choice 4. Return = \$1175.00

**7**

$$i = \begin{cases} 1, & \text{regular savings} \\ 2, & \text{3-month CD} \\ 3, & \text{6-month CD} \end{cases}$$

$x_{it}$ = Deposit in plan $i$ at start of month $t$

$$t = \begin{cases} 1, 2, \ldots, 12 & \text{if } i = 1 \\ 1, 2, \ldots, 10 & \text{if } i = 2 \\ 1, 2, \ldots, 7 & \text{if } i = 3 \end{cases}$$

$y_1$ = initial amount on hand to insure a feasible solution

$r_i$ = interest rate for plan $i = 1, 2, 3$

$$J_i = \begin{cases} 12, & i = 1 \\ 10, & i = 2 \\ 7, & i = 3 \end{cases}$$

continued...

$$P_i = \begin{cases} 1, & i = 1 \\ 3, & i = 2 \\ 6, & i = 3 \end{cases} \qquad d_t = \$\text{ demand for period } t$$

$$\text{Maximize } z = \sum_{\substack{t=1 \\ t-P_i>0}}^{12} \sum_{i=1}^{3} r_i \cdot x_{i,t-P_i} - y_1$$

s.t.

$$y_1 - x_{11} - x_{21} - x_{31} \geq d_1$$
$$1000 + \sum_{\substack{i=1 \\ t-P_i>0}}^{3} (1+r_i) x_{i,t-P_i} - \sum_{\substack{i=1 \\ t \leq J_i}}^{3} x_{it} \geq d_t, \; t = 2, \ldots, 12$$
$$x_{it}, \; y_1 \geq 0$$

Solution: (see file ampl2.3c-7.txt)

$y_1 = \$1200$, $z = -1136.29$

Interest amount $= 1200 - 1136.29 = \$63.71$

Deposits:

| $t$ | $x_{1t}$ | $x_{2t}$ | $x_{3t}$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | 200 | 0 |
| 3 | 286.48 | 313.53 | 0 |
| 4 | 0 | 587.43 | 0 |
| 5 | 314.37 | 289.30 | 0 |
| 6 | 0 | 734.69 | 0 |
| 7 | 0 | 98.20 | 0 |
| 8 | 0 | 294.60 | |
| 9 | 0 | 848.16 | |
| 10 | 0 | 0 | |
| 11 | 0 | | |
| 12 | 0 | | |

# Set 2.3d

**1**

$x_{W1}$ = # wrenches/wk using regular time
$x_{W2}$ = # wrenches/wk using overtime
$x_{W3}$ = # wrenches/wk using subcontracting
$x_{C1}$ = # chisels/wk using regular time
$x_{C2}$ = # chisels/wk using overtime
$x_{C3}$ = # chisels/wk using subcontracting

$$\text{Minimize } z = 2x_{W1} + 2.8x_{W2} + 3x_{W3} + 2.1x_{C1} + 3.2x_{C2} + 4.2x_{C3}$$

Subject to

$$x_{W1} \le 550, \quad x_{W2} \le 250$$

$$x_{C1} \le 620, \quad x_{C2} \le 280$$

$$\frac{x_{C1} + x_{C2} + x_{C3}}{x_{W1} + x_{W2} + x_{W3}} \ge 2$$

or

$$2x_{W1} + 2x_{W2} + 2x_{W3} - x_{C1} - x_{C2} - x_{C3} \le 0$$

$$x_{W1} + x_{W2} + x_{W3} \ge 1500$$

$$x_{C1} + x_{C2} + x_{C3} \ge 1200$$

all variables $\ge 0$

(a) Optimum from TORA:

$x_{W1} = 550$, $x_{W2} = 250$, $x_{W3} = 700$

$x_{C1} = 620$, $x_{C2} = 280$, $x_{C3} = 2100$

$z = \$14,918$

(b) Increasing marginal cost ensures that regular time capacity is used before that of overtime, and that overtime capacity is used before that of subcontracting. If the marginal cost function is not monotonically increasing, additional constraints are needed to ensure that the capacity restriction is satisfied.

continued...

**2**

$x_j$ = number of units produced of product $j$, $j = 1, 2, 3, 4$

Profit per unit:

Product 1 = $75 - 2\times10 - 3\times5 - 7\times4 = \$12$
Product 2 = $70 - 3\times10 - 2\times5 - 3\times4 = \$18$
Product 3 = $55 - 4\times10 - 1\times5 - 2\times4 = \$2$
Product 4 = $45 - 2\times10 - 2\times5 - 1\times4 = \$11$

$$\text{Maximize } z = 12x_1 + 18x_2 + 2x_3 + 11x_4$$

s.t.

$$2x_1 + 3x_2 + 4x_3 + 2x_4 \le 500$$
$$3x_1 + 2x_2 + x_3 + 2x_4 \le 380$$
$$7x_1 + 3x_2 + 2x_3 + x_4 \le 450$$
$$x_1, x_2, x_3, x_4 \ge 0$$

TORA Solution:

$x_1 = 0$, $x_2 = 133.33$, $x_3 = 0$, $x_4 = 50$

$z = \$2950$

**3**

$x_j$ = number of units of model $j$

$$\text{Maximize } z = 30x_1 + 20x_2 + 50x_3$$

Subject to

(1) $2x_1 + 3x_2 + 5x_3 \le 4000$
(2) $4x_1 + 2x_2 + 7x_3 \le 6000$
(3) $x_1 + .5x_2 + \frac{1}{3}x_3 \le 1500$
(4) $\frac{x_1}{3} = \frac{x_2}{2}$, or $2x_1 - 3x_2 = 0$
(5) $\frac{x_2}{2} = \frac{x_3}{5}$, or $5x_2 - 2x_3 = 0$

$x_1 \ge 200$, $x_2 \ge 200$, $x_3 \ge 150$

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 2.6a-12
Final iteration No: 4
Objective value (max) =41081.0820

| Variable | Value | Obj Coeff | Obj Val Contrib |
|---|---|---|---|
| x1 | 324.3243 | 30.0000 | 9729.7305 |
| x2 | 216.2162 | 20.0000 | 4324.3242 |
| x3 | 540.5405 | 50.0000 | 27027.0273 |

| Constraint | RHS | Slack(-)/Surplus(+) |
|---|---|---|
| 1 (<) | 4000.0000 | 0.0000- |
| 2 (<) | 6000.0000 | 486.4865- |
| 3 (<) | 1500.0000 | 887.3875- |
| 4 (=) | 0.0000 | 0.0000 |
| 5 (=) | 0.0000 | 0.0000 |
| LB-x1 | 200.0000 | 124.3243+ |
| LB-x2 | 200.0000 | 16.2162+ |
| LB-x3 | 150.0000 | 390.5405+ |

**4**

$x_{ij}$ = Nbr. cartons in month $i$ from supplier $j$

$I_i$ = End inventory in period $i$, $I_0 = 0$

$c_{ij}$ = Price per unit of $x_{ij}$

$h$ = Holding cost/unit/month

$C$ = Supplier capacity/month

$d_i$ = Demand for month $i$

$i = 1, 2, 3, \quad j = 1, 2$

$$\text{Minimize } z = \sum_{i=1}^{3}\sum_{j=1}^{2} c_{ij}\, x_{ij} + \frac{h}{2}\left(\sum_{i=1}^{3}\left(\sum_{j=1}^{2} x_{ij} + I_{i-1} + I_i\right)\right)$$

s.t. $x_{ij} \le C$, all $i$ and $j$

$$\sum_{j=1}^{2} x_{ij} + I_{i-1} - I_i = d_i, \text{ all } i$$

Optimum solution:

| $i$ | $x_{i1}$ | $x_{i2}$ | $I$ |
|---|---|---|---|
| 1 | 400 | 100 | 0 |
| 2 | 400 | 400 | 200 |
| 3 | 200 | 0 | 0 |

Total cost = \$167,450.

**5**

$x_i$ = Production amount in quarter $i$

$I_i$ = End inventory for quarter $i$

$$\text{Minimize } z = 20x_1 + 22x_2 + 24x_3 + 26x_4 + 3.5(I_1 + I_2 + I_3)$$

s.t.

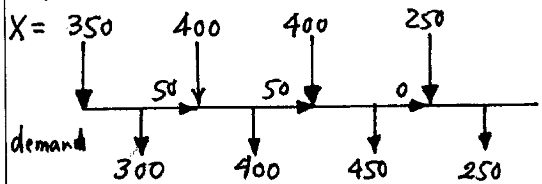
$x_1 = 300 + I_1$ $\qquad x_i \le 400,\ i = 1,2,3,4$

$I_1 + x_2 = 400 + I_2$ $\qquad I_i \le 100,\ i = 1,2,3$

$I_2 + x_3 = 450 + I_3$ $\qquad I_0 = I_4 = 0$

$I_3 + x_4 = 250$

Optimum solution:




Total cost = \$32,250

**6**

$x_{ij}$ = Qty of product $i$ in month $j$, $i = 1, 2$, $j = 1, 2, 3$

$I_{ij}$ = End inventory of product $i$ in month $j$

$$\text{Minimize } z = 30(x_{11} + x_{12} + x_{13}) + 28(x_{21} + x_{22} + x_{23}) + .9(I_{11} + I_{12} + I_{13}) + .75(I_{21} + I_{22} + I_{23})$$

s.t.

$$(x_{1j}/1.25) + x_{2j} \le \begin{cases} 3000, & j=1 \\ 3500, & j=2 \\ 3000, & j=3 \end{cases}$$

$$I_{1,j-1} + x_{1j} - I_{1j} = \begin{cases} 500, & j=1 \\ 5000, & j=2 \\ 750, & j=3 \end{cases}$$

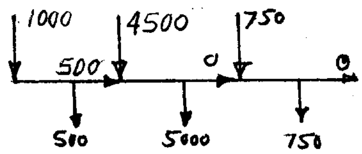
$$I_{2,j-1} + x_{2j} - I_{2j} = \begin{cases} 1000, & j=1 \\ 1200, & j=2 \\ 1200, & j=3 \end{cases}$$

$I_{i0} = 0,\ i = 1, 2$

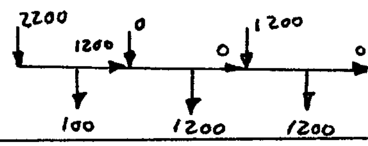
$x_{ij},\ I_{ij} \ge 0$

Optimum solution: Cost = \$284,050

Product 1:




Product 2:




**7**

$x_{ij}$ = Qty by operation $i$ in month $j$

$i = 1, 2, \quad j = 1, 2, 3$

$$\text{Minimize } z = .2\sum_{j=1}^{3} I_{1j} + .4\sum_{j=1}^{3} I_{2j} + 10x_{11} + 12x_{12} + 11x_{13} + 15x_{21} + 18x_{22} + 16x_{23}$$

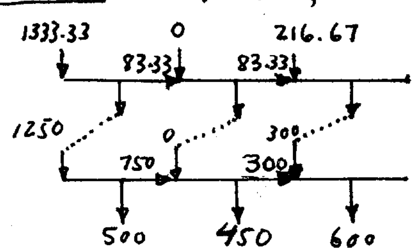
$.6x_{11} \le 800$, $.6x_{12} \le 700$, $.6x_{13} \le 550$

$.8x_{21} \le 1000$, $.8x_{22} \le 850$, $.8x_{23} \le 700$

$$\left.\begin{array}{l} x_{1j} + I_{1,j-1} = x_{2j} + I_{1j} \\ x_{2j} + I_{2,j-1} = I_{2j} + d_j \end{array}\right\} j = 1, 2, 3$$

$I_{i0} = 0,\ i = 1, 2$

Solution: Cost = \$39,720




$I_{ij}$ = Entering inv. of op. $i$ in month $j$

## Set 2.3d

**8**

$x_j$ = Units of product $j$, $j=1,2$

$y_i^-$ = Unused hours of machine $i$ ⎫
$y_i^+$ = Overtime hours of machine $i$ ⎭ $i=1,2$

Maximize $z = 110x_1 + 118x_2 - 100(y_1^+ + y_2^+)$

s.t.

$$\frac{x_1}{5} + \frac{x_2}{5} + y_1^- - y_1^+ = 8$$

$$\frac{x_1}{8} + \frac{x_2}{4} + y_2^- - y_2^+ = 8$$

$$y_1^+ \le 4, \quad y_2^+ \le 4$$

$$x_1, x_2, y_1^-, y_1^+, y_2^-, y_2^+ \ge 0$$

Solution:

Revenue = \$6,232

$x_1 = 56$, $y_1^+ = 4$ hrs

$x_2 = 4$, $y_2^+ = 0$

$y_1^-, y_2^- = 0$

**1**

$x_s$ = tons of strawberry / day
$x_g$ = tons of grapes / day
$x_a$ = tons of apples / day

$x_A$ = cans of drink A / day
$x_B$ = cans of drink B / day
$x_C$ = cans of drink C / day
(Each can holds one lb)

$x_{sA}$ = lb of strawberry used in drink A / day
$x_{sB}$ = lb of strawberry used in drink B / day
$x_{gA}$ = lb of grapes used in drink A / day
$x_{gB}$ = lb of grapes used in drink B / day
$x_{gC}$ = lb of grapes used in drink C / day
$x_{aB}$ = lb of apples used in drink B / day
$x_{aC}$ = lb of apples used in drink C / day

$$\text{Maximize } z = 1.15x_A + 1.25x_B + 1.2x_C - 200x_s - 100x_g - 90x_a$$

s.t.

$$x_s \le 200,\ x_g \le 100,\ x_a \le 150$$
$$x_{sA} + x_{sB} = 1500x_s$$
$$x_{gA} + x_{gB} + x_{gC} = 1200x_g$$
$$x_{aB} + x_{aC} = 1000x_a$$
$$x_A = x_{sA} + x_{gA}$$
$$x_B = x_{sB} + x_{gB} + x_{aB}$$
$$x_C = x_{gC} + x_{aC}$$
$$x_{sA} = x_{gA},$$
$$x_{sB} = x_{gB},\ x_{gB} = .5x_{aB}$$
$$3x_{gC} = 2x_{aC}$$

all variables $\ge 0$

Optimum solution:

$x_A = 90,000$ cans, $x_B = 300,000$ cans, $x_C = 0$

$x_{ij}$:

| $i$ \ $j$ | A | B | C |
|---|---|---|---|
| s | 45,000 | 75,000 | 0 |
| g | 45,000 | 75,000 | 0 |
| a | 0 | 150,000 | 0 |
| | 90,000 | 300,000 | 0 |

$x_s = 80$ tons, $x_g = 100$ tons, $x_a = 150$ tons

$z = \$439,000/\text{day}$

**2**

$x_s$ = lb of screws per package
$x_b$ = lb of bolts per package
$x_n$ = lb of nuts per package
$x_w$ = lb of washers per package

$$\text{Minimize } z = 1.1x_s + 1.5x_b + \frac{70}{80}x_n + \frac{20}{30}x_w$$

s.t.

$$Y = x_s + x_b + x_n + x_w$$
$$x_s \ge .1Y$$
$$x_b \ge .25Y,\ \frac{x_b}{50} \le x_w,\ \frac{x_b}{10} \le x_n$$
$$x_n \le .15Y$$
$$x_w \le .1Y$$
$$Y \ge 1$$

All variables are nonnegative

Optimum solution:

$Y = 1,\ x_s = .5,\ x_b = .25,\ x_n = .15,\ x_w = .1$

Cost = \$1.12

**3**

$x_{o,(A,B,C)}$ = lb of oats in cereals A, B, C
$x_{r,(A,C)}$ = lb of raisins in cereals A, C
$x_{c,(B,C)}$ = lb of coconuts in cereals B, C
$x_{a,(A,B,C)}$ = lb of almond in cereals A, B, C

$$Y_o = x_{oA} + x_{oB} + x_{oC}$$
$$Y_r = x_{rA} + x_{rC}$$
$$Y_c = x_{cB} + x_{cC}$$
$$Y_a = x_{aA} + x_{aB} + x_{aC}$$
$$W_A = x_{oA} + x_{rA} + x_{aA}$$
$$W_B = x_{oB} + x_{cB} + x_{aB}$$
$$W_C = x_{oC} + x_{rC} + x_{cC} + x_{aC}$$

$$\text{Maximize } z = \frac{1}{5}(2W_A + 2.5W_B + 3W_C) - \frac{1}{2000}(100Y_o + 120Y_r + 110Y_c + 200Y_n)$$

s.t.

$$W_A \le 500 \times 5 = 2500$$
$$W_B \le 600 \times 5 = 3000$$
$$W_C \le 500 \times 5 = 4000$$

continued...

# Set 2.3e

$Y_o \le 5 \times 2000 = 10{,}000$

$Y_r \le 2 \times 2000 = 4{,}000$

$Y_c \le 1 \times 2000 = 2{,}000$

$Y_a \le 1 \times 2000 = 2{,}000$

$X_{oA} = \frac{50}{5} X_{rA}$, $X_{oA} = \frac{50}{2} X_{aA}$

$X_{oB} = \frac{60}{2} X_{cB}$, $X_{oB} = \frac{60}{3} X_{aB}$

$X_{oC} = \frac{60}{3} X_{rC}$, $X_{oC} = \frac{60}{4} X_{cC}$, $X_{oC} = \frac{60}{2} X_{aC}$

all variables are nonnegative.

<u>Optimum solution</u>: $Z = \$5384.84/\text{day}$

$W_A = 2500$ lb or 500 boxes/day

$W_B = 3000$ lb or 600 boxes

$W_C = 5793.45$ lb or $\simeq 1158$ boxes

$X_o = 10{,}000$ lb or 5 tons/day

$X_r = 471.19$ lb or .236 ton

$X_c = 428.16$ lb or .214 ton

$X_a = 394.11$ lb or .197 ton

**4**

$$\left.\begin{array}{l} X_{Ai} = \text{bbl of gasoline A in fuel } i \\ X_{Bi} = \text{bbl of gasoline B in fuel } i \\ X_{Ci} = \text{bbl of gasoline C in fuel } i \\ X_{Di} = \text{bbl of gasoline D in fuel } i \end{array}\right\} i = 1, 2$$

$Y_A = X_{A1} + X_{A2}$

$Y_B = X_{B1} + X_{B2}$

$Y_C = X_{C1} + X_{C2}$

$Y_D = X_{D1} + X_{D2}$

$F_1 = X_{A1} + X_{B1} + X_{C1} + X_{D1}$

$F_2 = X_{A2} + X_{B2} + X_{C2} + X_{D2}$

Maximize $Z = 200F1 + 250F2 - (120Y_A + 90Y_B + 100Y_C + 150Y_D)$

continued…

s.t.

$X_{A1} = X_{B1}$, $X_{A1} = .5X_{C1}$, $X_{A1} = .25X_{D1}$

$X_{A2} = X_{B2}$, $X_{A2} = 2X_{C2}$, $X_{A2} = \frac{2}{3}X_{D2}$

$Y_A \le 1000$, $Y_B \le 1200$, $Y_C \le 900$, $Y_D \le 1500$

$F_1 \ge 200$, $F_2 \ge 400$

<u>Optimum solution</u>: $Z = \$495{,}416.67$

$Y_A = 958.33$ bbl/day

$Y_B = 958.33$ bbl/day

$Y_C = 516.67$ bbl/day

$Y_D = 1500$ bbl/day

$F1 = 200$ bbl/day

$F_2 = 3733.33$ bbl/day

**5**

$A$ = bbl of crude A/day

$B$ = bbl of crude B/day

$R$ = bbl of regular gasoline/day

$P$ = bbl of premium gasoline/day

$J$ = bbl of jet gasoline/day

Maximize $Z = 50(R - R^+) + 70(P - P^+) + 120(J - J^+) - (10R^- + 15P^- + 20J^-) - (2R^+ + 3P^+ + 4J^+) - (30A + 40B)$

s.t.

$A \le 2500$, $B \le 3000$

$R = .2A + .25B$, $R + R^- - R^+ = 500$

$P = .1A + .3B$, $P + P^- - P^+ = 700$

$J = .25A + .1B$, $J + J^- - J^+ = 400$

All variables $\ge 0$

<u>Optimum solution</u>:

$Z = \$21{,}852.94$

$A = 1176.47$ bbl/day

$B = 1058.82$ bbl/day

$R = 500$ bbl/day

$P = 435.29$ bbl/day

$J = 400$ bbl/day

**6**

NR = bbl/day of naphta used in regular
NP = bbl/day of naphta used in premium
NJ = bbl/day of naphta used in jet
LR = bbl/day of light used in regular
LP = bbl/day of light used in premium
LJ = bbl/day of light used in jet

Using the other notation in Problem 5,

Maximize $z = 50(R-R^+) + 70(P-P^+) + 120(J-J^+) - (10R^- + 15P^- + 20J^-) - (2R^+ + 3P^+ + 4J^+) - (30A + 40B)$

s.t.

$A \le 2500,\quad B \le 3000$

$R + R^- - R^+ = 500$

$P + P^- - P^+ = 700$

$J + J^- - J^+ = 400$

$.35A + .45B = NR + NP + NJ$

$.6A + .5B = LR + LP + LJ$

$R = NR + LR$

$P = NP + LP$

$J = NJ + LJ$

all variables are nonnegative

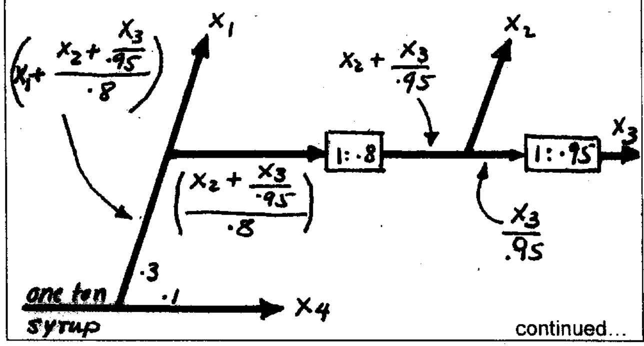
Optimum solution: $Z = \$71,473.68$

$A = 1684.21,\ B = 0$

$R = 500,\ P = 700,\ J = 400$

**7**

$x_1$ = tons of brown sugar per week
$x_2$ = tons of white sugar per week
$x_3$ = tons of powdered sugar per week
$x_4$ = tons of molasses per week




continued…

Maximize $Z = 150x_1 + 200x_2 + 230x_3 + 35x_4$

s.t.

$x_4 \le 4000 \times .1$

or $x_4 \le 400$

$x_1 + \left(\frac{x_2 + \frac{x_3}{.95}}{.8}\right) \le .3 \times 4000$

or $.76x_1 + .95x_2 + x_3 \le 912$

$x_1 \ge 25,\ x_2 \ge 25$

$x_3 \ge 25,\ x_4 \ge 0$

Optimum solution from TORA:

$x_1 = 25$ tons per week

$x_2 = 25$ tons per week

$x_3 = 869.25$ tons per week

$x_4 = 400$ tons per week

$Z = \$222,677.50$

**8**

A = bbl/hr of stock A
B = bbl/hr of stock B

$\left.\begin{array}{l} Y_{Ai} = \text{bbl/hr of A used in gasoline } i \\ Y_{Bi} = \text{bbl/hr of B used in gasoline } i \end{array}\right\} i = 1,2$

Maximize $Z = 7(Y_{A1} + Y_{B1}) + 10(Y_{A2} + Y_{B2})$

s.t.

$A = Y_{A1} + Y_{A2},\quad A \le 450$

$B = Y_{B1} + Y_{B2},\quad B \le 700$

$98Y_{A1} + 89Y_{B1} \ge 91(Y_{A1} + Y_{B1})$

$98Y_{A2} + 89Y_{B2} \ge 93(Y_{A2} + Y_{B2})$

$10Y_{A1} + 8Y_{B1} \le 12(Y_{A1} + Y_{B1})$

$10Y_{A2} + 8Y_{B2} \le 12(Y_{A2} + Y_{B2})$

all variables are nonnegative

Optimum solution:

$Z = \$10,675$

$A = 450$ bbl/hr

$B = 700$ bbl/hr

Gasoline 1 production $= Y_{A1} + Y_{B1} = 61.11 + 213.89 = 275$ bbl/hr

Gasoline 2 production $= Y_{A2} + Y_{B2} = 388.89 + 486.11 = 875$ bbl/hr

# Set 2.3e

**9**

$S$ = tons of steel scrap / day
$A$ = tons of alum. scrap / day
$C$ = tons of cast iron scrap / day
$Ab$ = tons of alum. briquettes / day
$Sb$ = tons silicon briquettes / day
$a$ = tons of alum. / day
$g$ = tons of graphite / day
$s$ = tons of silicon / day
$aI$ = tons of alum. in ingot I / day
$aII$ = tons of alum. in ingot II / day
$gI$ = tons of graphite in ingot I / day
$gII$ = tons of graphite in ingot II / day
$sI$ = tons of silicon in ingot I / day
$sII$ = tons of silicon in ingot II / day
$I_1$ = tons of ingot I / day
$I_2$ = tons of ingot II / day.

Minimize $z = 100S + 150A + 75C + 900Ab + 380Sb$

s.t. $S \le 1000,\ A \le 500,\ C \le 2500$

$$a = .1S + .95A + Ab$$
$$g = .05S + .01A + .15C$$
$$s = .04S + .02A + .08C + S_b$$
$$I_1 = aI + gI + sI$$
$$I_2 = aII + gII + sII$$
$$a_I + a_{II} \le a,\ sI + sII \le s,\ gI + gII \le g$$
$$.081 I_1 \le aI \le .108 I_1$$
$$.015 I_1 \le gI \le .03 I_1$$
$$.025 I_1 \le sI < \infty$$
$$.062 I_2 \le aII \le .089 I_2$$
$$.041 I_2 \le gII \le \infty$$
$$.028 I_2 \le sII \le .041 I_2$$
$$I_1 \ge 130,\quad I_2 \ge 250$$

Optimum solution:

$$z = \$117{,}435.65$$
$$S = 0,\ A = 38.2,\ C = 1489.41$$
$$Ab = Sb = 0$$
$$I_1 = 130,\quad I_2 = 250$$
$$a = 36.29,\ g = 223.79,\ s = 119.92$$

**10**

$x_{ij}$ = tons of ore $i$ allocated to alloy $k$
$w_k$ = tons of alloy $k$ produced

Maximize $z = 200\,w_A + 300\,w_B$
$\qquad - 30(x_{1A} + x_{1B})$
$\qquad - 40(x_{2A} + x_{2B})$
$\qquad - 50(x_{3A} + x_{3B})$

Subject to

Specification constraints:

$$.2x_{1A} + .1x_{2A} + .05x_{3A} \le .8\,w_A \quad (1)$$
$$.1x_{1A} + .2x_{2A} + .05x_{3A} \le .3\,w_A \quad (2)$$
$$.3x_{1A} + .3x_{2A} + .2x_{3A} \ge .5\,w_A \quad (3)$$
$$.1x_{1B} + .2x_{2B} + .05x_{3B} \ge .4\,w_B \quad (4)$$
$$.1x_{1B} + .2x_{2B} + .05x_{3B} \le .6\,w_B \quad (5)$$
$$.3x_{1B} + .3x_{2B} + .7x_{3B} \ge .3\,w_B \quad (6)$$
$$.3x_{1B} + .3x_{2B} + .2x_{3B} \le .7\,w_B \quad (7)$$

Ore constraints:

$$x_{1A} + x_{1B} \le 1000$$
$$x_{2A} + x_{2B} \le 2000$$
$$x_{3A} + x_{3B} \le 3000$$

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 26a-17
Final iteration No: 12
Objective value (max) =400000.0000

| Variable | Value | Obj Coeff | Obj Val Contrib |
|---|---|---|---|
| x1 wA | 1799.9999 | 200.0000 | 359999.9688 |
| x2 wB | 1000.0001 | 300.0000 | 300000.0312 |
| x3 x1A | 1000.0000 | -30.0000 | -30000.0000 |
| x4 x1B | 0.0000 | -30.0000 | -0.0000 |
| x5 x2A | 0.0000 | -40.0000 | -0.0000 |
| x6 x2B | 2000.0001 | -40.0000 | -80000.0078 |
| x7 x3A | 3000.0000 | -50.0000 | -150000.0000 |
| x8 x3B | 0.0000 | -50.0000 | -0.0000 |

| Constraint | RHS | Slack(-)/Surplus(+) |
|---|---|---|
| 1 (<) | 0.0000 | 1090.0000- |
| 2 (<) | 0.0000 | 290.0000- |
| 3 (>) | 0.0000 | 0.0000+ |
| 4 (>) | 0.0000 | 0.0000+ |
| 5 (<) | 0.0000 | 200.0000- |
| 6 (>) | 0.0000 | 300.0002+ |
| 7 (<) | 0.0000 | 100.0000- |
| 8 (<) | 1000.0000 | 0.0000- |
| 9 (<) | 2000.0000 | 0.0000- |
| 10 (<) | 3000.0000 | 0.0000- |

Solution:

Produce 1800 tons of alloy A and 1000 tons of alloy B.

**1**

$h$ = Regular pay hour

8-hr pay = $8h$

12-hr pay = $12h + \frac{4h}{2} = 14h$

$x_i$ = Nbr 8-hr buses starting in period $i$

$y_i$ = Nbr. of 12-hr buses starting in period $i$

Minimize $z = h\left(8\sum_{i=1}^{6} x_i + 14\sum_{i=1}^{6} y_i\right)$

s.t.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | 1 | 1 | | | | 1 | 1 | $\geq 4$ |
| 1 | 1 | | | | | 1 | 1 | | | | 1 | $\geq 8$ |
| | 1 | 1 | | | | 1 | 1 | 1 | | | | $\geq 10$ |
| | | 1 | 1 | | | | 1 | 1 | 1 | | | $\geq 7$ |
| | | | 1 | 1 | | | | 1 | 1 | 1 | | $\geq 12$ |
| | | | | 1 | 1 | | | | 1 | 1 | 1 | $\geq 4$ |

Solution: $z = 196h$

$x_1 = 4, x_2 = 4, x_4 = 2, x_5 = 4, x_3 = x_6 = 0$

$y_3 = 6, y_1 = y_2 = y_4 = y_5 = y_6 = 0$

For 8-hr only buses, solution is

$z = 208h$

$x_1 = x_2 = 4, x_3 = 6, x_4 = 1, x_5 = 11, x_6 = 0$

(8-hr + 12-hr) buses is cheaper.

**2**

$x_i$ = Nbr. of volunteers starting in hour $i$

Minimize $z = \sum_{i=1}^{14} x_i$

s.t.

(8:00) $x_1 \geq 4$

(9:00) $x_1 + x_2 \geq 4$

(10:00) $x_1 + x_2 + x_3 \geq 6$

(11:00) $x_2 + x_3 + x_4 \geq 6$

(12:00) $x_3 + x_4 + x_5 \geq 8$

(1:00) $x_4 + x_5 + x_6 \geq 8$

(2:00) $x_5 + x_6 + x_7 \geq 6$

(3:00) $x_6 + x_7 + x_8 \geq 6$

(4:00) $x_7 + x_8 + x_9 \geq 4$

(5:00) $x_8 + x_9 + x_{10} \geq 4$

(6:00) $x_9 + x_{10} + x_{11} \geq 6$

(7:00) $x_{10} + x_{11} + x_{12} \geq 6$

(8:00) $x_{11} + x_{12} + x_{13} \geq 8$

(9:00) $x_{12} + x_{13} \geq 8$

All $x_j \geq 0$

continued…

Solution: $z$ = 32 volunteers

$x_1 = 4, x_3 = 2, x_4 = 6, x_6 = 2, x_7 = 4, x_{10} = 6, x_{12} = 8$

all other $x_i = 0$

**3**

Same formulation as in Problem 2 with the added constraints $x_5 = 0, x_{11} = 0$

Optimum solution remains the same

**4**

$x_i$ = Nbr. of casuals starting on day $i$

($i = 1$: Monday, $i = 7$: Sunday)

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

s.t.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| M | 1 | | | 1 | 1 | 1 | 1 | $\geq 20$ |
| T | 1 | 1 | | | 1 | 1 | 1 | $\geq 14$ |
| W | 1 | 1 | 1 | | | 1 | 1 | $\geq 10$ |
| Th | 1 | 1 | 1 | 1 | | | 1 | $\geq 15$ |
| F | 1 | 1 | 1 | 1 | 1 | | | $\geq 18$ |
| Sat | | 1 | 1 | 1 | 1 | 1 | | $\geq 10$ |
| Sun | | | 1 | 1 | 1 | 1 | 1 | $\geq 12$ |

Solution: $z$ = 20 workers

$x_1 = 8, x_4 = 6, x_5 = 4, x_6 = 1, x_7 = 1$

**5**

$x_i$ = Nbr. Students starting at hour $i$

$i = 1$ (8:01), $i = 9$ (4:01), $x_5 = 0$

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + x_9$

s.t.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | |
|---|---|---|---|---|---|---|---|---|---|
| 8:01 | 1 | | | | | | | | $\geq 2$ |
| 9:01 | 1 | 1 | | | | | | | $\geq 2$ |
| 10:01 | 1 | 1 | 1 | | | | | | $\geq 3$ |
| 11:01 | | 1 | 1 | 1 | | | | | $\geq 4$ |
| 12:01 | | | 1 | 1 | | | | | $\geq 4$ |
| 1:01 | | | | 1 | 1 | | | | $\geq 3$ |
| 2:01 | | | | | 1 | 1 | | | $\geq 3$ |
| 3:01 | | | | | 1 | 1 | 1 | | $\geq 3$ |
| 4:01 | | | | | | 1 | 1 | 1 | $\geq 3$ |

Solution: $z$ = 9 students

$x_1 = 2, x_3 = 1, x_4 = 3, x_7 = 3$

# Set 2.3f

## 6

Let $x_i$ = Nbr. starting on day i and lasting for 7 days

$y_{ij}$= Nbr. starting shift on day i and *starting* their 2 days off on day $j$, $i \neq j$

Thus, of the $x_1$ workers who start on Monday, $y_{12}$ will take T and W off, $y_{13}$ will take W and Th off, and so on, as the following table shows.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| 1 | **start on Mon** | $y_{12}$ | $y_{12}+y_{13}$ | $y_{13}+y_{14}$ | $y_{14}+y_{15}$ | $y_{15}+y_{16}$ | $y_{16}$ |
| 2 | y27 | **Tue** | y23 | y23+y24 | y24+y25 | y25+y26 | y26+y27 |
| 3 | y31+y37 | y31 | **Wed** | y34 | y34+y35 | y35+y36 | y36+y37 |
| 4 | y41+y47 | y41+y42 | y42 | **Th** | y45 | y45+y46 | y46+y47 |
| 5 | y51+y57 | y51+y52 | y52+y53 | y53 | **Fri** | y56 | y56+y57 |
| 6 | y61+y67 | y61+y62 | y62+y63 | y63+y64 | y64 | **Sat** | y67 |
| 7 | y71 | y71+y72 | y72+y73 | y73+y74 | y74+y75 | y75 | **Su** |

**Minimize** z = $x_1+x_2+x_3+x_4+x_5+x_6+x_7$

**Each employee has 2 days off:** $x_i$ = sum{j in 1..7,j≠i}$y_{ij}$

**Mon (1) constraint:** $s$ -(y27 + y31+ y37 + y41+y47+ y51+ y57 + y61+y67 + y71)>= 12

**Tue (2) constraint:** $s$ -(y12 + y31 + y41+ y42 + y51+y52 + y61+y62 + y71+y72>= 18

**Wed (3) constraint:** $s$ -(y12+y13 + y23 + y42 + y52+y53 + y62+y63 + y72+y73>= 20

**Th (4) constraint:** $s$ -(y13+y14 + y23 +y24+ y24 + y53 + y63+y64 + y73+y74>= 28

**Fri (5) constraint:** $s$ -(y14+y15 + y24 +y25+ y34 +y35 + y45+ y64 + y74+y75>= 32

**Sat(6) constraint:** $s$ -(y15+y16+ y25 +y26+ y35 + y36 + y45+y46+ y56+y75>= 40

**Sun(7) constraint:** $s$ -( y16+ y26 +y27+ y36 + y37 +y46+y47+ y56+y57+y67)>= 40

continued

**Solution:** 42 employees

| Starting | | Nbr off | | | | | | |
|---|---|---|---|---|---|---|---|---|
| On | Nbr | M | Tu | Wed | Th | Fri | Sat | Sun |
| M | 16 | | 16 | 16 | | | | |
| Tu | 8 | | | | 8 | 8 | | |
| Wed | 8 | 8 | 8 | | | | | |
| Th | 0 | | | | | | | |
| Fri | 6 | | | 6 | 6 | | | |
| Sat | 2 | 2 | | | | | | 2 |
| Sun | 2 | | | | | 2 | 2 | |
| Nbr off | | 10 | 24 | 22 | 14 | 10 | 2 | 2 |
| Nbr at work | | 32 | 18 | 20 | 28 | 32 | 40 | 40 |
| Surplus above minimum | | 20 | 0 | 0 | 0 | 0 | 0 | 0 |

# Set 2.3g

**1**

| | Setting 1 | Setting 3 | Number produced | Surplus rolls |
|---|---|---|---|---|
| 5' | 0 | 2 | 200 | 50 |
| 7' | 1 | 0 | 200 | 0 |
| 9' | 1 | 1 | 300 | 0 |
| Loss/ft | 4 | 1 | | |
| No. rolls | 200 | 100 | | |

Trim loss area =
$L(200\times 4 + 100\times 1 + 50\times 5) = 1150L\ ft^2$

(b) 15' standard roll:

| | Setting 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 5' | 3 | 1 | 1 | 0 |
| 7' | 0 | 1 | 0 | 2 |
| 9' | 0 | 0 | 1 | 0 |
| trim loss per ft | 0 | 3 | 1 | 1 |

(c) $x_1 + x_2 + 2x_5 \geq 120$

New solution calls for decreasing the number of standard 20'-rolls by 30

(d) $x_1 + x_3 + 2x_6 \geq 240$

New solution calls for increasing the number of standard 20'-rolls by 50

**2**

$x_i$ = Space ($in^2$) allocated to cereal $i$

Maximize $z = 1.1x_1 + 1.3x_2 + 1.08x_3 + 1.25x_4 + 1.2x_5$

s.t.

$16x_1 + 24x_2 + 18x_3 + 22x_4 + 20x_5 \leq 5000$

$x_1 \leq 100,\ x_2 \leq 85,\ x_3 \leq 140,\ x_4 \leq 80,\ x_5 \leq 90$

$x_i \geq 0$ for all $i = 1, 2, \ldots, 5$

Solution:

$z = \$\,314/day$

$x_1 = 100,\ x_3 = 140,\ x_5 = 44$

$x_2 = x_4 = 0$

**3**

$x_i$ = Nbr. of ads for issue $i$, $i = 1,2,3,4$

Minimize $z = S_1^- + S_2^- + S_3^- + S_4^-$

s.t.

$(-30{,}000 + 60{,}000 + 30{,}000)x_1 + S_1^- - S_1^+ = .51\times 400{,}000$

$(80{,}000 + 30{,}000 - 45{,}000)x_2 + S_2^- - S_2^+ = .51\times 400{,}000$

$(40{,}000 + 10{,}000)x_3 + S_3^- - S_3^+ = .51\times 400{,}000$

$(90{,}000 - 25{,}000)x_4 + S_4^- - S_4^+ = .51\times 400{,}000$

$1500(x_1 + x_2 + x_3 + x_4) \leq 100{,}000$

$x_1, x_2, x_3, x_4 \geq 0$

Solution:

$x_1 = 3.4,\ x_2 = 3.14,\ x_3 = 4.08,\ x_4 = 3.14$

**4**

$x_{ij}$ = Units of part $j$ produced by department $i$, $i = 1,2,3$, $j = 1,2$

Maximize $z = \min\{x_{11} + x_{21},\ x_{12} + x_{22},\ x_{13} + x_{23}\}$

or

Maximize $z = y$

s.t.

$y \leq x_{11} + x_{21}$

$y \leq x_{12} + x_{22}$

$y \leq x_{13} + x_{23}$

$\frac{x_{11}}{8} + \frac{x_{12}}{5} + \frac{x_{13}}{10} \leq 100$

$\frac{x_{21}}{6} + \frac{x_{22}}{12} + \frac{x_{23}}{4} \leq 80$

all $x_{ij} \geq 0$

Solution:

Nbr. of assembly units $= y = 556.2 \simeq 557$

$x_{11} = 354.78,\ x_{12} = 0$

$x_{21} = 556.52,\ x_{22} = 201.74$

$x_{31} = 556.52,\ x_{32} = 0$

**5**

$x_i$ = tons of coal $i$, $i = 1,2,3$

Minimize $z = 30x_1 + 35x_2 + 33x_3$

s.t.

$2500x_1 + 1500x_2 + 1600x_3 \leq 2000(x_1 + x_2 + x_3)$

$x_1 \leq 30,\ x_2 \leq 30,\ x_3 \leq 30$

$x_1 + x_2 + x_3 \geq 50$

Solution: $z = \$1361.11$

$x_1 = 22.22$ tons, $x_2 = 0$, $x_3 = 27.78$ tons.

## 6

$t_i$ = Green time in secs for highway $i$, $i=1,2,3$

Maximize $z = 3\left(\frac{500}{3600}\right)t_1 + 4\left(\frac{600}{3600}\right)t_2 + 5\left(\frac{400}{3600}\right)t_3$

s.t.

$\left(\frac{500}{3600}\right)t_1 + \left(\frac{600}{3600}\right)t_2 + \left(\frac{400}{3600}\right)t_3 \le \frac{510}{3600}\left(2.2\times 60 - 3\times 10\right)$

$t_1 + t_2 + t_3 + 3\times 10 \le 2.2\times 60,\ t_1 \ge 25, t_2 \ge 25, t_3 \ge 25$

Solution: $z = \$58.04/\text{hr}$

$t_1 = 25,\ t_2 = 43.6,\ t_3 = 33.4$ sec

## 7

$y_i$ = observation $i$

Define straight line as

$\hat{y}_i = a + b,\ a, b$ unrestricted

Minimize $z = \sum_{i=1}^{10} |y_i - \hat{y}_i|$

$= \sum_{i=1}^{10} |y_i - ai - b|$

Let $d_i = |y_i - ai - b|$

Minimize $z = d_1 + d_2 + \cdots + d_{10}$

s.t.

$y_i - ai - b \le d_i$

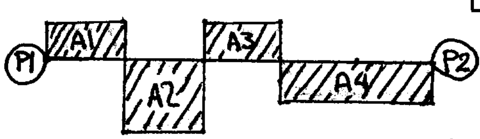
$y_i - ai - b \ge -d_i$

$a, b$, unrestricted

$d_i \ge 0$

Solution: $\hat{y}_i = 2.85714\,i + 6.42857$

## 8




$A1 = 2\times 1760\times 10\times 50 = 1760$ (thousand) $Yd^3$

$A2 = 3520,\ A3 = 1760,\ A4 = 3520$

Distances (center to center) in miles:

$$\begin{array}{c|cc} & A2 & A4 \\ \hline A1 & 2 & 7 \\ A3 & 2 & 3 \\ P1 & 3 & 8 \\ P2 & 7 & 2 \end{array}$$

continued...

Cost (\$) per cubic yd:

$$\begin{array}{cc|cc} & & (5) & (6) \\ & & A2 & A4 \\ \hline (1) & A1 & .2+2\times.15=.50 & .20+7\times.15=1.25 \\ (2) & A3 & .20+2\times.15=.50 & .20+3\times.15=.65 \\ (3) & P1 & 1.70+3\times.15=2.15 & 1.70+8\times.15=2.90 \\ (4) & P3 & 2.10+7\times.15=3.15 & 2.10+2\times.15=2.40 \end{array}$$

Using the code $A1 \equiv 1,\ A3 \equiv 2,\ P1 \equiv 3,\ P2 \equiv 4,\ A2 \equiv 5,\ A4 \equiv 6$, let

$x_{ij} = 10^3\, Yd^3$ from source $i$ to destination $j$

$i = 1,2,3,4,\ j = 5,6$

Minimize $z = 1000(.5x_{15} + 1.25x_{16} + .5x_{25} + .65x_{26} + 2.15x_{35} + 2.9x_{36} + 3.15x_{45} + 2.4x_{46})$

s.t.

$x_{15} + x_{16} \le 1760 \qquad x_{35} + x_{36} \le 20{,}000$

$x_{25} + x_{26} \le 1760 \qquad x_{45} + x_{46} \le 15{,}000$

$x_{15} + x_{25} + x_{35} + x_{45} \ge 3520$

$x_{16} + x_{26} + x_{36} + x_{46} \ge 3520$

Solution:

A1→A2: $x_{15} = 1760$ (1000 Cu Yd)

A1→A4: $x_{16} = 0$

A3→A2: $x_{25} = 0$

A3→A4: $x_{26} = 1760$

P1→A2: $x_{35} = 1760$

P1→A4: $x_{36} = 0$

P2→A2: $x_{45} = 0$

P2→A4: $x_{46} = 1760$

Cost = \$10,032,000

## 9

$x_{ij}$ = Blue regulars on front $i$ in defense line $j$, $i=$

$y_{ij}$ = Blue reserves on front $i$ in defense line $j$.

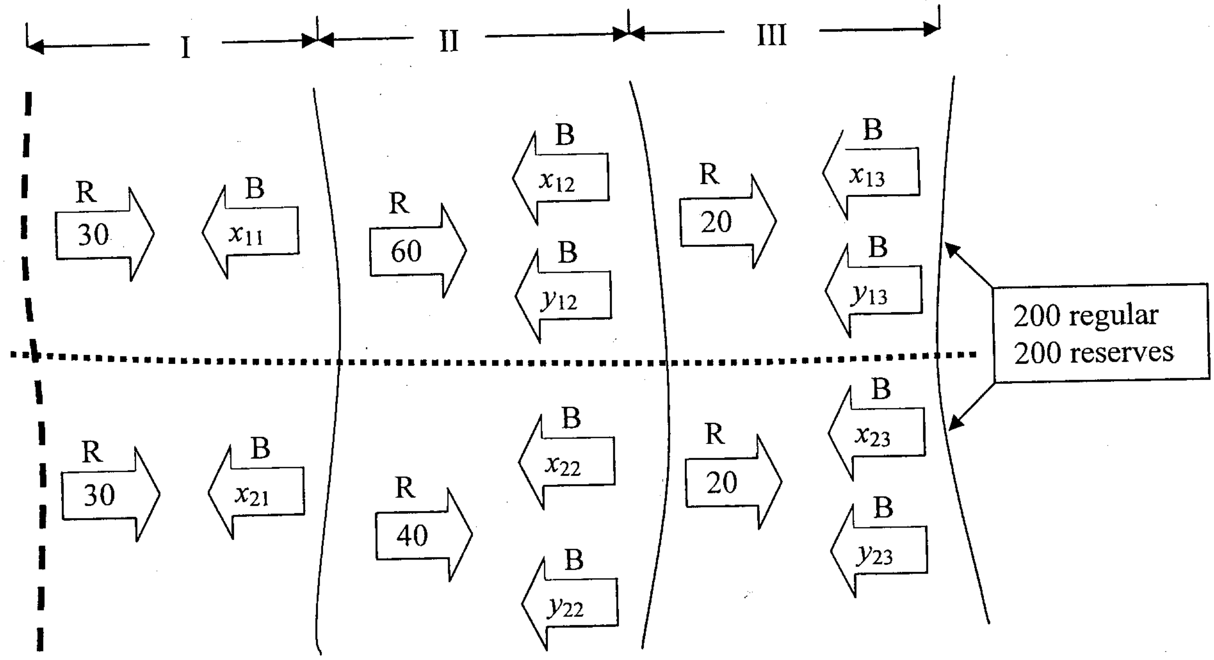
$t_{ij}$ = Delay days on front $i$ in defense line $j$.

Maximize $z = \min\{t_{11} + t_{12} + t_{13},\ t_{21} + t_{22} + t_{23}\}$

or

continued...

# Set 2.3g



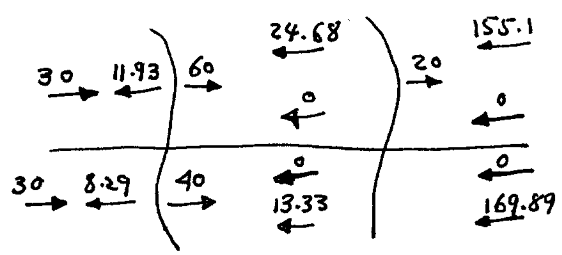

Maximize $z = T$

s.t.

$$T \le t_{11} + t_{12} + t_{13}$$
$$T \le t_{21} + t_{22} + t_{23}$$

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} \le 200$$
$$y_{12} + y_{13} + y_{22} + y_{23} \le 200$$
$$t_{11} = .5 + 8.8\,\frac{x_{11}}{30}$$
$$t_{12} = .75 + 7.9\,\frac{x_{12} + y_{12}}{60}$$
$$t_{13} = .55 + 10.2\,\frac{x_{13} + y_{13}}{20}$$
$$t_{21} = 1.1 + 10.5\,\frac{x_{21}}{30}$$
$$t_{22} = 1.3 + 8.1\,\frac{x_{22} + y_{22}}{40}$$
$$t_{23} = 1.5 + 9.2\,\frac{x_{23} + y_{23}}{20}$$

Solution: Battle duration = 87.65 days




**10**

$x_i$ = Efficiency of plant $i$

Minimize $z = .2(500)x_1 + .25(3000)x_2 + .15(6000)x_3 + .18(1000)x_4$

s.t.

$$500(1-x_1) \le .00085 \times 215{,}000$$
$$.94(500)(1-x_1) + 3000(1-x_2) \le .0009 \times 220{,}000$$
$$.94^2(500)(1-x_1) + .94(3000)(1-x_2) + 6000(1-x_3) \le .0008 \times 200{,}000$$
$$.94^3(500)(1-x_1) + .94^2(300)(1-x_2) + .94(6000)(1-x_3) + 1000(1-x_4) \le .0008 \times 210{,}000$$

$$0 \le x_1 \le .99$$
$$0 \le x_2 \le .99$$
$$0 \le x_3 \le .99$$
$$0 \le x_4 \le .99$$

Solution

Cost per ton = $1891.41

Plant 1 efficiency = .99

Plant 2 efficiency = .9661

Plant 3 efficiency = .99

Plant 4 efficiency = .9824

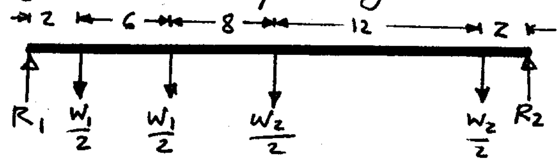
continued...

**11**

$W_i$ = Capacity of yoke $i$ (Kips)

$R_1$ = Reaction in Kips at left end

$R_2$ = Reaction in Kips at right end




Maximize $z = w_1 + w_2$

s.t.

$$R_1 + R_2 = w_1 + w_2$$

$$2\left(\frac{w_1}{2}\right) + 8\left(\frac{w_1}{2}\right) + 16\left(\frac{w_2}{2}\right) + 28\left(\frac{w_2}{2}\right) = 30R_2$$

$$R_1 \le 25, \quad R_2 \le 25$$

$$\frac{w_1}{2} \le 20, \quad \frac{w_2}{2} \le 20$$

Solution:

$w_1$ = 20.59 Kips

$w_2$ = 29.41 Kips

**12**

$x_{ij}$ = Nbr. of aircraft of type $i$ allocated to route $j$ ($i = 1,2,3,4$, $j = 1,2,3,4$)

$S_j$ = Nbr. of passengers not served on route $j$, $j = 1,2,3,4$

$$\begin{aligned} \text{Minimize } z = {} & 1000(3x_{11}) + 1100(2x_{12}) \\ & + 1200(2x_{13}) + 1500(x_{14}) \\ & + 800(4x_{21}) + 900(3x_{22}) \\ & + 1000(3x_{23}) + 1000(2x_{24}) \\ & + 600(5x_{31}) + 800(5x_{32}) \\ & + 800(4x_{33}) + 900(2x_{34}) \\ & + 40S_1 + 50S_2 + 45S_3 + 70S_4 \end{aligned}$$

Subject to

$$\sum_{j=1}^{4} x_{1j} \le 5, \quad \sum_{j=1}^{4} x_{2j} \le 8, \quad \sum_{j=1}^{4} x_{3j} \le 10$$

$$50(3x_{11}) + 30(4x_{21}) + 20(5x_{31}) + S_1 = 1000$$

$$50(2x_{12}) + 30(3x_{22}) + 20(5x_{32}) + S_2 = 2000$$

$$50(2x_{13}) + 30(3x_{23}) + 20(4x_{33}) + S_3 = 900$$

$$50(x_{14}) + 30(2x_{24}) + 20(2x_{34}) + S_4 = 1200$$

All $x_{ij}$ and $S_j \ge 0$

continued…

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 26a-16
Final iteration No: 16
Objective value (min) =221900.0000
==> ALTERNATIVE solution detected at x13

| Variable | Value | Obj Coeff | Obj Val Contrib |
|---|---|---|---|
| x1 x11 | 5.0000 | 3000.0000 | 14999.9990 |
| x2 x12 | 0.0000 | 2200.0000 | 0.0000 |
| x3 x13 | 0.0000 | 2400.0000 | 0.0000 |
| x4 x14 | 0.0000 | 1500.0000 | 0.0000 |
| x5 x21 | 0.0000 | 3200.0000 | 0.0000 |
| x6 x22 | 0.0000 | 2700.0000 | 0.0000 |
| x7 x23 | 0.0000 | 3000.0000 | 0.0000 |
| x8 x24 | 8.0000 | 2000.0000 | 15999.9990 |
| x9 x31 | 2.5000 | 3000.0000 | 7500.0015 |
| x10 x32 | 7.5000 | 4000.0000 | 29999.9980 |
| x11 x33 | 0.0000 | 3200.0000 | 0.0000 |
| x12 x34 | 0.0000 | 1800.0000 | 0.0000 |
| x13 s1 | 0.0000 | 40.0000 | 0.0000 |
| x14 s2 | 1250.0000 | 50.0000 | 62500.0000 |
| x15 s3 | 899.9998 | 45.0000 | 40499.9922 |
| x16 s4 | 720.0001 | 70.0000 | 50400.0078 |

| Constraint | RHS | Slack(-)/Surplus(+) |
|---|---|---|
| 1 (<) | 5.0000 | 0.0000- |
| 2 (<) | 8.0000 | 0.0000- |
| 3 (<) | 10.0000 | 0.0000- |
| 4 (=) | 1000.0000 | 0.0000 |
| 5 (=) | 2000.0000 | 0.0000 |
| 6 (=) | 900.0000 | 0.0000 |
| 7 (=) | 1200.0000 | 0.0000 |

Solution:

| Aircraft Type | Route | Nbr. aircraft |
|---|---|---|
| 1 | 1 | 5 |
| 2 | 4 | 8 |
| 3 | 1 | 2.5 |
| 3 | 2 | 7.5 |

Fractional solution must be rounded.

Cost = \$221,900

# CHAPTER 3

# The Simplex Method and Sensitivity Analysis

**1**

$(x_1, x_2) = (3, 1)$

$M1:\ S_1 = 24 - (6\times3 + 4\times1) = 2$ tons/day

$M2:\ S_2 = 6 - (1\times3 + 2\times1) = 1$ ton/day

**2**

$$S_1 = x_1 + x_2 - 800 = 500 + 600 - 800 = 300 \text{ lb}$$

**3**

$10x_1 - 3x_2 \geq -5 \equiv -10x_1 + 3x_2 \leq 5$

Thus, $-10x_1 + 3x_2 + S_1 = 5$ ①

Also, $10x_1 - 3x_2 \geq -5 \equiv 10x_1 - 3x_2 - S_2 = -5$

Thus, $-10x_1 + 3x_2 + S_2 = 5$ ②

① and ② are the same

**4**

$x_{ij}$ = number of units of product $i$ manufactured on machine $j$

LP model

Maximize $z = 10(x_{11} + x_{12}) + 15(x_{21} + x_{22})$

subject to

$$\begin{aligned} |(x_{11} + x_{21}) - (x_{12} + x_{22})| &\leq 5 \\ x_{11} + x_{21} &\leq 200 \\ x_{12} + x_{22} &\leq 250 \\ x_{ij} \geq 0 \text{ for all } i \,\&\, j \end{aligned}$$

Equation form:

$|(x_{11} + x_{21}) - (x_{12} + x_{22})| \leq 5$

to

$$\begin{aligned} x_{11} + x_{21} - x_{12} - x_{22} &\leq 5 \\ x_{11} + x_{21} - x_{12} - x_{22} &\geq -5 \end{aligned}$$

Maximize $z = 10x_{11} + 10x_{12} + 15x_{21} + 15x_{22}$

Subject to

$$\begin{aligned} x_{11} + x_{21} - x_{12} - x_{22} + S_1 &= 5 \\ -x_{11} - x_{21} + x_{12} + x_{22} + S_2 &= 5 \\ x_{11} + x_{21} + S_3 &= 200 \\ x_{12} + x_{22} + S_4 &= 250 \end{aligned}$$

$x_{ij} \geq 0$ for all $i$ and $j$

$S_i \geq 0$ for all $i$

continued...

**5**

$y = \max\{|x_1 - x_2 + 3x_3|, |-x_1 + 3x_2 - x_3|\}$

Hence

$|x_1 - x_2 + 3x_3| \leq y$

$|-x_1 + 3x_2 - x_3| \leq y$

LP model:

minimize $z = y$

Subject to

$$\begin{aligned} x_1 - x_2 + 3x_3 &\leq y \\ x_1 - x_2 + 3x_3 &\geq -y \\ -x_1 + 3x_2 - x_3 &\leq y \\ -x_1 + 3x_2 - x_3 &\geq -y \\ x_1, x_2, x_3, y &\geq 0 \end{aligned}$$

Equation form:

Minimize $z = y$

Subject to

$$\begin{aligned} -y + x_1 - x_2 + 3x_3 + S_1 &= 0 \\ -y - x_1 + x_2 - 3x_3 + S_2 &= 0 \\ -y - x_1 + 3x_2 - x_3 + S_3 &= 0 \\ -y + x_1 - 3x_2 + x_3 + S_4 &= 0 \end{aligned}$$

$x_1, x_2, x_3, y, S_1, S_2, S_3, S_4 \geq 0$

**6**

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \Leftrightarrow \begin{cases} \sum_{j=1}^{n} a_{ij}x_j \leq b_i & ① \\ \sum_{j=1}^{n} a_{ij}x_j \geq b_i & ② \end{cases}$$

From ②, for $i = 1, 2, \ldots, m$, we have

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \Leftrightarrow \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right) \geq \sum_{i=1}^{m} b_i \Leftrightarrow \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right)x_j \geq \sum_{i=1}^{m} b_i$$

Thus, ① and ② are equivalent to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i,\quad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right)x_j \geq \sum_{i=1}^{m} b_i$$

**1**

$x_1$ = Nbr. $\frac{1}{4}$-lb / day

$x_2$ = Nbr. cheeseburgers/day

Maximize $z = .2x_1 + .15x_2 - .25x_3^+$

s.t.

$$.25x_1 + .2x_2 + x_3^- - x_3^+ = 200$$

$$x_1 + x_2 \le 900$$

Solution: $z = \$173.35$

$x_1 = 900,\ x_2 = 0,\ x_3^+ = 25$ lb

**2**

(a) $x_j$ = #units of product $j$ per day, $j = 1, 2$

$x_3^+$ = unused minutes of machine time / day

$x_3^-$ = machine overtime per day in minutes

Maximize $z = 6x_1 + 7.5x_2 - .5x_3^-$

Subject to

$$10x_1 + 12x_2 + x_3^+ - x_3^- = 2500$$

$$150 \le x_1 \le 200$$

$$x_2 \le 45$$

$$x_1, x_2 \ge 0$$

$$x_3^+, x_3^- \ge 0$$

TORA optimum solution:

$x_1$ = 200 units/day

$x_2$ = 45 units/day

$x_3^-$ = overtime minutes = 40 minutes/day

$z = \$1517.50$

(b) Overtime at \$1.50/min yields $x_3^- = 0$, which means no overtime is needed

**3**

$x_j$ = # of units of products 1, 2, and 3

Maximize $z = 2x_1 + 5x_2 + 3x_3 - 15x_4^+ - 10x_5^+$

Subject to

$$2x_1 + x_2 + 2x_3 + x_4^- - x_4^+ = 80$$

$$x_1 + x_2 + 2x_3 + x_5^- - x_5^+ = 65$$

all variables $\ge 0$

Solution: $z = \$325$

$x_2 = 65$ units, $x_4^- = 15$

All other variables = 0

continued…

**4**

Formulation 1:

Maximize $z = -2x_1 + 3x_2^+ - 3x_2^- - 2x_3^+ + 2x_3^-$

Subject to

$$4x_1 - x_2^+ + x_2^- - 5x_3^+ + 5x_3^- = 10$$

$$2x_1 + 3x_2^+ - 3x_2^- + 2x_3^+ - 2x_3^- = 12$$

all variables $\ge 0$

Optimum solution:

$x_1 = 0$

$\left.\begin{matrix} x_2^+ = 6.15 \\ x_2^- = 0 \end{matrix}\right\} \Rightarrow x_2 = 6.15$

$\left.\begin{matrix} x_3^+ = 0 \\ x_3^- = 3.23 \end{matrix}\right\} \Rightarrow x_3 = -3.23$

$z = 24.92$

Formulation 2:

Maximize $z = -2x_1 + 3x_2^+ - 2x_3^+ - w$

Subject to

$$4x_1 - x_2^+ - 5x_3^+ + 6w = 10$$

$$2x_1 + 3x_2^+ + 2x_3^+ - 5w = 12$$

all variables $\ge 0$

Optimum solution:

$x_1 = 0$

$\left.\begin{matrix} x_2^+ = 9.38 \\ w = 3.23 \end{matrix}\right\} \Rightarrow x_2 = 9.38 - 3.23 = 6.15$

$\left.\begin{matrix} x_3^+ = 0 \\ w = 3.23 \end{matrix}\right\} \Rightarrow x_3 = 0 - 3.23 = -3.23$

$z = 24.92$

continued…

## Set 3.2a

**1**

(a)

Equation form:

Maximize $z = 2x_1 + 3x_2$

subject to

$$x_1 + 3x_2 + x_3 = 6$$
$$3x_1 + 2x_2 + x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(b) Basic $(x_1, x_2)$ (Point B):

$$x_1 + 3x_2 = 6$$
$$3x_1 + 2x_2 = 6$$

Solution: $(x_1, x_2) = (\frac{6}{7}, \frac{12}{7})$, $z = 6\frac{6}{7}$

Basic $(x_1, x_3)$ (Point E):

$$x_1 + x_3 = 6$$
$$3x_1 = 6$$

Solution: $(x_1, x_3) = (2, 4)$, $z = 4$

Basic $(x_1, x_4)$ (Point C):

$$x_1 = 6$$
$$3x_1 + x_4 = 6$$

Solution: $(x_1, x_4) = (6, -12)$
Unique but infeasible

Basic $(x_2, x_3)$ (Point A):

$$3x_2 + x_3 = 6$$
$$2x_2 = 6$$

Solution: $(x_2, x_3) = (3, -3)$
Unique but infeasible

Basic $(x_2, x_4)$ (Point D):

$$3x_2 = 6$$
$$2x_2 + x_4 = 6$$

Solution: $(x_2, x_4) = (2, 2)$, $Z = 6$

Basic $(x_3, x_4)$ (Point F):

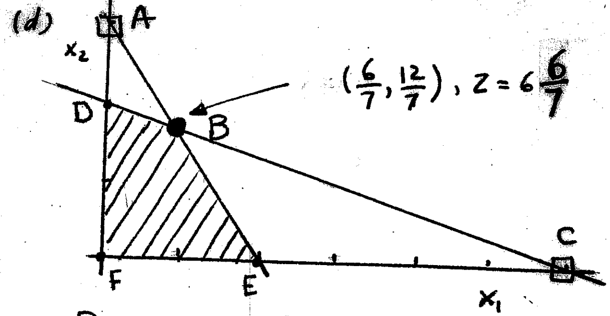
$$x_3 = 6$$
$$x_4 = 6$$

Solution: $(x_3, x_4) = (6, 6)$, $Z = 0$

(c) Optimum solution occurs at B:
$(x_1, x_2) = (\frac{6}{7}, \frac{12}{7})$ with $z = 6\frac{6}{7}$

continued...

(d)




(e) From the graph in (d), we have

A: $x_2 = 3$, $x_3 = -3$

C: $x_1 = 6$, $x_4 = -12$

**2**

(a) Maximize $z = 2x_1 - 4x_2 + 5x_3 - 6x_4$

subject to

$$x_1 + 4x_2 - 2x_3 + 8x_4 + x_5 = 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 + x_6 = 1$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

| Combination | Solution | Status | z |
|---|---|---|---|
| $x_1, x_2$ | 0, 1/2 | Feasible | -2 |
| $x_1, x_3$ | 8, 3 | Feasible | $\boxed{31}$ |
| $x_1, x_4$ | 0, 1/4 | Feasible | -3/2 |
| $x_1, x_5$ | -1, 3 | Infeasible | — |
| $x_1, x_6$ | 2, 3 | Feasible | 4 |
| $x_2, x_3$ | 1/2, 0 | Feasible | -2 |
| $x_2, x_4$ | 1/2, 0 | Feasible | -2 |
| $x_2, x_5$ | 1/2, 0 | Feasible | -2 |
| $x_2, x_6$ | 1/2, 0 | Feasible | -2 |
| $x_3, x_4$ | 0, 1/4 | Feasible | -3/2 |
| $x_3, x_5$ | 1/3, 8/3 | Feasible | 5/3 |
| $x_3, x_6$ | -1, 4 | Infeasible | — |
| $x_4, x_5$ | 1/4, 0 | Feasible | -3/2 |
| $x_4, x_6$ | 1/4, 0 | Feasible | -3/2 |
| $x_5, x_6$ | 2, 1 | Feasible | 0 |

Optimum Solution:

$x_1 = 8$, $x_2 = 0$, $x_3 = 3$, $x_4 = 0$

$Z = 31$

continued...

(b) Minimize $z = x_1 + 2x_2 - 3x_3 - 2x_4$

subject to

$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

| Combination | Solution | Status | z |
|---|---|---|---|
| $x_1, x_2$ | infinity of solutions | | — |
| $x_1, x_3$ | 4, 0 | Feasible | 4 |
| $x_1, x_4$ | 4, 0 | Feasible | 4 |
| $x_2, x_3$ | 2, 0 | Feasible | 4 |
| $x_2, x_4$ | 2, 0 | Feasible | 4 |
| $x_3, x_4$ | $-\frac{4}{7}, \frac{16}{7}$ | Infeasible | — |

Alternative optima:

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | z |
|---|---|---|---|---|
| 4 | 0 | 0 | 0 | 4 |
| 0 | 2 | 0 | 0 | 4 |

**3**

maximize $z = x_1 + x_2$

subject to

$$x_1 + 2x_2 + x_3 = 6$$
$$2x_1 + x_2 - x_4 = 16$$
$$x_1, x_2, x_3, x_4 \geq 0$$

| Combination | Solution | Status |
|---|---|---|
| $x_1, x_2$ | 26/3, −4/3 | Infeasible |
| $x_1, x_3$ | 8, −2 | Infeasible |
| $x_1, x_4$ | 6, −4 | Infeasible |
| $x_2, x_3$ | 16, −26 | Infeasible |
| $x_2, x_4$ | 3, −13 | Infeasible |
| $x_3, x_4$ | 6, −16 | Infeasible |

continued...

**4**

Maximize $z = 2x_1 + 3x_2^- - 3x_2^+ + 5x_3$

Subject to

$$-6x_1 + 7x_2^- - 7x_2^+ - 9x_3 - x_4 = 4$$
$$x_1 + x_2^- - x_2^+ + 4x_3 = 10$$
$$x_1, x_2^-, x_2^+, x_3, x_4 \geq 0$$

$(x_2, x_2)$:

$$7x_2^- - 7x_2^+ = 4$$
$$x_2^- - x_2^+ = 10$$

Since $(7x_2^- - 7x_2^+)$ and $(x_2^- - x_2^+)$ are dependent, it is impossible for $x_2^-$ and $x_2^+$ to be basic simultaneously. This means that at least $x_2^-$ and $x_2^+$ must be nonbasic at zero level; thus making it impossible for $x_2^-$ and $x_2^+$ to assume positive values simultaneously in any basic solution.

**5**

maximize $z = x_1 + 3x_2$

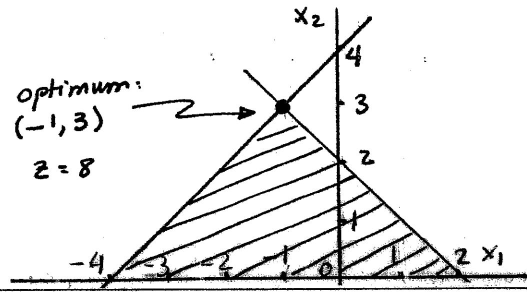
subject to

$$x_1 + x_2 + x_3 = 2$$
$$-x_1 + x_2 + x_4 = 4$$
$x_1$ unrestricted
$$x_2, x_3 \geq 0$$

| Combination | Solution | Status | z |
|---|---|---|---|
| $x_1, x_2$ | −1, 3 | Feasible | $\boxed{8}$ |
| $x_1, x_3$ | −4, 6 | Feasible | −4 |
| $x_1, x_4$ | 2, 6 | Feasible | 2 |
| $x_2, x_3$ | 4, −2 | Infeasible | — |
| $x_2, x_4$ | 2, 2 | Feasible | 6 |
| $x_3, x_4$ | 2, 4 | Feasible | 0 |

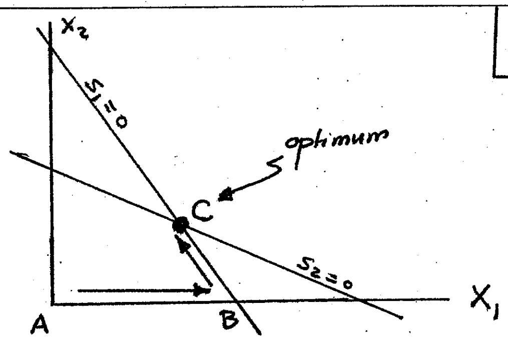
Optimum: $x_1 = -1$, $x_2 = 3$, $z = 8$

(c)



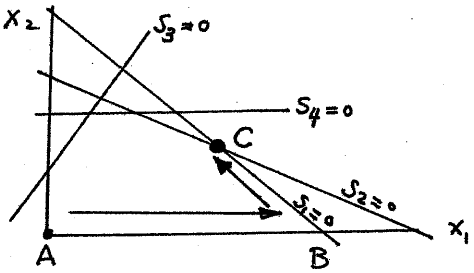
# Set 3.3a

**1**




| Extreme point | Basic | Nonbasic |
|---|---|---|
| A | $s_1, s_2$ | $x_1, x_2$ |
| B | $x_1, s_2$ | $x_2, s_1$ |
| C | $x_1, x_2$ | $s_1, s_2$ |

**2**




| Extreme point | Basic | Nonbasic |
|---|---|---|
| A | $s_1, s_2, s_3, s_4$ | $x_1, x_2$ |
| B | $x_1, s_2, s_3, s_4$ | $s_1, x_2$ |
| C | $x_1, x_2, s_3, s_4$ | $s_1, s_2$ |

**3**

(a) (A, B) adjacent, hence can be on a simplex path. Remaining pairs cannot be on a simplex path because they are not adjacent.

(b) (i) Yes, because connects adjacent extreme points

(ii) No, because C and I are not adjacent.

(iii) No, because the path returns to a previous extreme point.

**4**

| Extreme Point | Basic | Nonbasic |
|---|---|---|
| A | $s_1, s_2, s_3, s_4$ | $x_1, x_2, x_3$ |
| B | $s_1, x_1, s_3, s_4$ | $s_2, x_2, s_3$ |
| C | $x_2, s_2, s_3, s_4$ | $s_1, x_1, x_3$ |
| D | $s_1, s_2, x_3, s_4$ | $x_1, x_2, s_3$ |
| E | $x_1, x_2, s_3, s_4$ | $s_1, s_2, x_3$ |
| F | $x_2, s_2, x_3, s_4$ | $x_1, s_1, s_3$ |
| G | $s_1, x_1, x_3, s_4$ | $s_2, x_2, s_3$ |
| H | $s_1, x_1, x_2, x_3$ | $s_2, s_3, s_4$ |
| I | $x_1, x_2, x_3, s_3$ | $s_1, s_2, s_4$ |
| J | $x_1, s_2, x_2, x_3$ | $s_1, s_3, s_4$ |

**5**

(a) $x_3$ enters at value 1

$z = 0 + 3 \times 1 = 3$

(b) $x_1$ enters at value 1

$z = 0 + 5 \times 1 = 5$

(c) $x_2$ enters at value 1

$z = 0 + 7 \times 1 = 7$

(d) Tie broken arbitrarily between $x_1$, $x_2$, and $x_3$. Entering value = 1

$z = 0 + 1 \times 1 = 1$

**1**

| Basic | z | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | Sol |
|---|---|---|---|---|---|---|---|---|
| z | 1 | -5 | -4 | 0 | 0 | 0 | 0 | 0 |
| $s_1$ | 0 | 6 | 4 | 1 | 0 | 0 | 0 | 24 |
| $s_2$ | 0 | 1 | 2 | 0 | 1 | 0 | 0 | 6 |
| $s_3$ | 0 | -1 | 1 | 0 | 0 | 1 | 0 | 1 |
| $s_4$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 2 |
| z | 1 | 0 | 6 | 0 | 5 | 0 | 0 | 30 |
| $s_1$ | 0 | 0 | -8 | 1 | -6 | 0 | 0 | -12 |
| $x_1$ | 0 | 1 | 2 | 0 | 1 | 0 | 0 | 6 |
| $s_3$ | 0 | 0 | 3 | 0 | 1 | 1 | 0 | 7 |
| $s_4$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 2 |

**2**

(a)

| Basic | x1 | x2 | x3 | x4 | sx5 | sx6 | sx7 | Solution |
|---|---|---|---|---|---|---|---|---|
| z | -2.00 | -1.00 | 3.00 | -5.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1)sx5 | 1.00 | 2.00 | 2.00 | 4.00 | 1.00 | 0.00 | 0.00 | 40.00 |
| 2)sx6 | 2.00 | -1.00 | 1.00 | 2.00 | 0.00 | 1.00 | 0.00 | 8.00 |
| 3)sx7 | 4.00 | -2.00 | 1.00 | -1.00 | 0.00 | 0.00 | 1.00 | 10.00 |
| z | 3.00 | -3.50 | 5.50 | 0.00 | 0.00 | 2.50 | 0.00 | 20.00 |
| 1)sx5 | -3.00 | 4.00 | 0.00 | 0.00 | 1.00 | -2.00 | 0.00 | 24.00 |
| 2)x4 | 1.00 | -0.50 | 0.50 | 1.00 | 0.00 | 0.50 | 0.00 | 4.00 |
| 3)sx7 | 5.00 | -2.50 | 1.50 | 0.00 | 0.00 | 0.50 | 1.00 | 14.00 |
| z | 0.38 | 0.00 | 5.50 | 0.00 | 0.88 | 0.75 | 0.00 | 41.00 |
| 1)x2 | -0.75 | 1.00 | 0.00 | 0.00 | 0.25 | -0.50 | 0.00 | 6.00 |
| 2)x4 | 0.62 | 0.00 | 0.50 | 1.00 | 0.12 | 0.25 | 0.00 | 7.00 |
| 3)sx7 | 3.12 | 0.00 | 1.50 | 0.00 | 0.62 | -0.75 | 1.00 | 29.00 |

(b)

| Basic | x1 | x2 | x3 | x4 | sx5 | sx6 | sx7 | Solution |
|---|---|---|---|---|---|---|---|---|
| z | -8.00 | -6.00 | -3.00 | 2.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1)sx5 | 1.00 | 2.00 | 2.00 | 4.00 | 1.00 | 0.00 | 0.00 | 40.00 |
| 2)sx6 | 2.00 | -1.00 | 1.00 | 2.00 | 0.00 | 1.00 | 0.00 | 8.00 |
| 3)sx7 | 4.00 | -2.00 | 1.00 | -1.00 | 0.00 | 0.00 | 1.00 | 10.00 |
| z | 0.00 | -10.00 | -1.00 | 0.00 | 0.00 | 0.00 | 2.00 | 20.00 |
| 1)sx5 | 0.00 | 2.50 | 1.75 | 4.25 | 1.00 | 0.00 | -0.25 | 37.00 |
| 2)sx6 | 0.00 | 0.00 | 0.50 | 2.50 | 0.00 | 1.00 | -0.50 | 3.00 |
| 3)x1 | 1.00 | -0.50 | 0.25 | -0.25 | 0.00 | 0.00 | 0.25 | 2.50 |
| z | 0.00 | 0.00 | 6.00 | 17.00 | 4.00 | 0.00 | 1.00 | 170.00 |
| 1)x2 | 0.00 | 1.00 | 0.70 | 1.70 | 0.40 | 0.00 | -0.10 | 15.00 |
| 2)sx6 | 0.00 | 0.00 | 0.50 | 2.50 | 0.00 | 1.00 | -0.50 | 3.00 |
| 3)x1 | 1.00 | 0.00 | 0.60 | 0.60 | 0.20 | 0.00 | 0.20 | 10.00 |

(c)

| Basic | x1 | x2 | x3 | x4 | sx5 | sx6 | sx7 | Solution |
|---|---|---|---|---|---|---|---|---|
| z | -3.00 | 1.00 | -3.00 | -4.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1)sx5 | 1.00 | 2.00 | 2.00 | 4.00 | 1.00 | 0.00 | 0.00 | 40.00 |
| 2)sx6 | 2.00 | -1.00 | 1.00 | 2.00 | 0.00 | 1.00 | 0.00 | 8.00 |
| 3)sx7 | 4.00 | -2.00 | 1.00 | -1.00 | 0.00 | 0.00 | 1.00 | 10.00 |
| z | 1.00 | -1.00 | -1.00 | 0.00 | 0.00 | 2.00 | 0.00 | 16.00 |
| 1)sx5 | -3.00 | 4.00 | 0.00 | 0.00 | 1.00 | -2.00 | 0.00 | 24.00 |
| 2)x4 | 1.00 | -0.50 | 0.50 | 1.00 | 0.00 | 0.50 | 0.00 | 4.00 |
| 3)sx7 | 5.00 | -2.50 | 1.50 | 0.00 | 0.00 | 0.50 | 1.00 | 14.00 |
| z | 0.25 | 0.00 | -1.00 | 0.00 | 0.25 | 1.50 | 0.00 | 22.00 |
| 1)x2 | -0.75 | 1.00 | 0.00 | 0.00 | 0.25 | -0.50 | 0.00 | 6.00 |
| 2)x4 | 0.62 | 0.00 | 0.50 | 1.00 | 0.12 | 0.25 | 0.00 | 7.00 |
| 3)sx7 | 3.12 | 0.00 | 1.50 | 0.00 | 0.62 | -0.75 | 1.00 | 29.00 |
| z | 1.50 | 0.00 | 0.00 | 2.00 | 0.50 | 2.00 | 0.00 | 36.00 |
| 1)x2 | -0.75 | 1.00 | 0.00 | 0.00 | 0.25 | -0.50 | 0.00 | 6.00 |
| 2)x3 | 1.25 | 0.00 | 1.00 | 2.00 | 0.25 | 0.50 | 0.00 | 14.00 |
| 3)sx7 | 1.25 | 0.00 | 0.00 | -3.00 | 0.25 | -1.50 | 1.00 | 8.00 |

continued...

(d)

| Basic | x1 | x2 | x3 | x4 | sx5 | sx6 | sx7 | Solution |
|---|---|---|---|---|---|---|---|---|
| z | -5.00 | 4.00 | -6.00 | 8.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1)sx5 | 1.00 | 2.00 | 2.00 | 4.00 | 1.00 | 0.00 | 0.00 | 40.00 |
| 2)sx6 | 2.00 | -1.00 | 1.00 | 2.00 | 0.00 | 1.00 | 0.00 | 8.00 |
| 3)sx7 | 4.00 | -2.00 | 1.00 | -1.00 | 0.00 | 0.00 | 1.00 | 10.00 |
| z | -13.00 | 8.00 | -10.00 | 0.00 | 0.00 | -4.00 | 0.00 | -32.00 |
| 1)sx5 | -3.00 | 4.00 | 0.00 | 0.00 | 1.00 | -2.00 | 0.00 | 24.00 |
| 2)x4 | 1.00 | -0.50 | 0.50 | 1.00 | 0.00 | 0.50 | 0.00 | 4.00 |
| 3)sx7 | 5.00 | -2.50 | 1.50 | 0.00 | 0.00 | 0.50 | 1.00 | 14.00 |
| z | -7.00 | 0.00 | -10.00 | 0.00 | -2.00 | 0.00 | 0.00 | -80.00 |
| 1)x2 | -0.75 | 1.00 | 0.00 | 0.00 | 0.25 | -0.50 | 0.00 | 6.00 |
| 2)x4 | 0.62 | 0.00 | 0.50 | 1.00 | 0.12 | 0.25 | 0.00 | 7.00 |
| 3)sx7 | 3.12 | 0.00 | 1.50 | 0.00 | 0.62 | -0.75 | 1.00 | 29.00 |

**3**

| Basic | Ratios | | | |
|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
| $x_5$ | 4/1 | 4/2 | -- | 4/5 |
| $x_6$ | 8/5 | -- | -- | 8/6 |
| $x_7$ | 3/2 | 3/3 | -- | 3/3 |
| $x_8$ | -- | -- | 0/1 | -- |
| Value | 1.5 | 1 | 0 | 0.8 |
| Leaving var | $x_7$ | $x_7$ | $x_8$ | $x_5$ |

**4**

(a) Nonbasic $x_1$ will improve solution.

| Basic | $x_1$-ratios |
|---|---|
| $x_2$ | 4/5 $\Rightarrow x_2$ leaves, $x_1 = \frac{4}{5}$ |
| $x_3$ | 8/6 |
| $x_4$ | 3/3 |

$x_1 = \frac{4}{5} = .8$, $x_3 = 8 - 6 \times .8 = 3.6$, $x_4 = 3 - 3 \times .8 = .6$

$x_2 = 0$, $z = .8 \times 1 = .8$

(b) $x_1$ remains nonbasic at zero. Current solution, $x_2 = 4$, $x_3 = 8$, $x_4 = 3$, $z = 0$ is optimum

**5**

Basic solutions consist of one variable each. Thus,

$x_1 = 90/1 = 90$, $z = 5 \times 90 = 450$

$x_2 = 90/3 = 30$, $z = -6 \times 30 = -180$

$x_3 = 90/5 = 18$, $z = 3 \times 18 = 54$

$x_4 = 90/6 = 15$, $z = -5 \times 15 = -75$

$x_5 = 90/3 = 30$, $z = 12 \times 30 = 360$

Optimum solution:

$x_1 = 90$, $x_2 = x_3 = x_4 = x_5 = 0$, $z = 450$

**6**

(a) Basic: $(x_8, x_3, x_1) = (12, 6, 0)$, $z = 620$

Nonbasic: $(x_2, x_4, x_5, x_6, x_7) = (0, 0, 0, 0, 0)$

(b) $x_2$, $x_5$, $x_6$ will improve solution.

$x_2$ enters: $x_2 = \min(\frac{12}{3}, \frac{6}{1}, -) = 4$. Thus, $x_8$ leaves, $\Delta z = 4 \times 5 = 20$

continued...

# Set 3.3b

$x_5$ enters: $x_5 = \min(—, \frac{6}{1}, \frac{0}{6}) = 0$. Thus, $\Delta z = 1 \times 0 = 0$ ($x_1$ leaves)

$x_6$ enters: $x_6 = \min(—, —, —)$. Thus, no leaving variable and $x_6$ can be increased to $\infty$. $\Delta z = +\infty$

(c) $x_4$ can improve solution.

$x_4$ enters: $x_4 = \min(—, \frac{6}{3}, —) = 2$. Thus, $x_3$ leaves. $\Delta z = -4 \times 2 = -8$

(d) As shown in (b), $x_5$ cannot change $z$ because it enters the solution a level zero. $x_7$ cannot change $z$ either because its objective equation coefficient $= 0$. $\Delta z = 0 \times \min(\frac{12}{5}, \frac{6}{3}, —) = 0$

**7**

(a) Maximize $z = 3x_1 + 6x_2$:
$x_2$ is the first entering variable. Resulting path is $A \to G \to F \to E$.

(b) Maximize $z = 4x_1 + x_2$:
Entering variable $x_1 = \begin{pmatrix}\text{min intercept with}\\ x_1\text{-axis}\end{pmatrix}$

$x_1 = \min(2, 3, 5) = 2$ at $B$

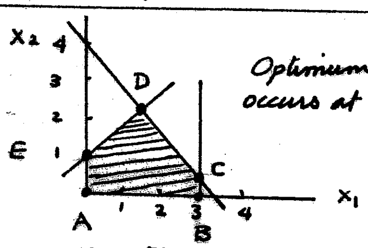
$\Delta z = 4 \times 2 = 8$

(c) Maximize $z = x_1 + 4x_2$:
Entering variable $x_2 = \begin{pmatrix}\text{min intercept}\\ \text{with } x_2\text{-axis}\end{pmatrix}$

$x_2 = \min(1, 2, 4) = 1$

$\Delta z = 4 \times 1 = 4$

**8**




Optimum occurs at $D$

(a) $x_1$ will enter first and the iterations will follow the path $A \to B \to C \to D$

(b) $x_2$ enters first and the iterations will follow the path $A \to E \to D$

(c) The most-negative criterion requires more iterations (4 vs. 3). This criterion is only a heuristic, and although it does not guarantee the smallest number of

continued...

iterations, computational experience demonstrates that, on the average, the most-negative criterion is more efficient.

(d) Iterations are identical, with the exception of the objective row, which should appear with an opposite sign

**9**

Optimum tableau:

| Basic | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | |
|---|---|---|---|---|---|---|---|
| $z$ | 0 | 0 | $\frac{3}{4}$ | $\frac{1}{2}$ | 0 | 0 | 21 |
| $x_1$ | 1 | 0 | ¼ | -1/2 | 0 | 0 | 3 |
| $x_2$ | 0 | 1 | -1/8 | ¾ | 0 | 0 | 3/2 |
| $s_3$ | 0 | 0 | 3/8 | -5/4 | 1 | 0 | 5/2 |
| $s_4$ | 0 | 0 | 1/8 | -3/4 | 0 | 1 | 1/2 |

If $s_1$ enters, its value $= \min\left\{\frac{3}{1/4}, —, \frac{5/2}{3/8}, \frac{1/2}{1/8}\right\} = 4$
New $z = 21 - 3/4 \times 4 = 18$

If $s_2$ enters, its value $= \min\left\{—, \frac{3/2}{3/4}, —, —\right\} = 2$
New $z = 21 - 1/2 \times 2 = 20$. The second best $z$ is associated with $s_2$ entering the basic solution

**10**

Not easily extendable because the third best solution may not be an adjacent corner point of the current optimum point.

**11**

$x_1$ = number of purses per day
$x_2$ = number of bags per day
$x_3$ = number of backpacks per day

Maximize $z = 24x_1 + 22x_2 + 45x_3$

subject to

$$2x_1 + x_2 + 3x_3 \le 42$$
$$2x_1 + x_2 + 2x_3 \le 40$$
$$x_1 + .5x_2 + x_3 \le 45$$
$$x_1, x_2, x_3 \ge 0$$

TORA's optimum solution:

$x_1 = 0$, $x_2 = 36$, $x_3 = 2$, $z = \$882$

Status of resources:

| Resource | Slack | Status |
|---|---|---|
| Leather | 0 | scarce |
| Sewing | 0 | scarce |
| Finishing | 25 | abundant |

**12**

From TORA Iterations module, click $\boxed{\text{All Iterations}}$, then go to the optimal iteration and click any of the associated nonbasic variables (X4, SX6, SX7, SX8). Now, click $\boxed{\text{Next Iteration}}$ to produce the new iteration in which the selected variable becomes basic. The associated value of Z will deteriorate.

**13**

To determine the next-best solution, follow the procedure in Problem 1. First, let X4 enter the basic solution and record the associated value of Z. Next, click $\boxed{\text{View/Modify Input Data}}$ and re-solve the problem to produce the same optimum tableau that was used before X4 was entered into the basic solution. Now, enter SX6 into the basic solution and record the associated value of Z. Repeat the procedure for SX7 and SX8. You will get the following results:

| Entering variable | Z |
|---|---|
| X4 | 2.63 |
| SX6 | 1.00 |
| SX7 | 6.40 |
| SX8 | 1.90 |

The next-best solution is associated with entering SX7 into the basic solution. Associated values of the variables are

$x_1 = 1.6$

$x_2 = 0$

$x_3 = 1.6$

$x_4 = 0$

$Z = 6.40$

# Set 3.4a

**1**

| Iteration | Basic | $x_1$ | $x_2$ | $x_3$ | $R_1$ | $R_2$ | $x_4$ | Solution |
|---|---|---|---|---|---|---|---|---|
| 0 (starting) | $z$ | $-4+7M$ | $-1+4M$ | $-M$ | 0 | 0 | 0 | $9M$ |
| $x_1$ enters | $R_1$ | 3 | 1 | 0 | 1 | 0 | 0 | 3 |
| $R_1$ leaves | $R_2$ | 4 | 3 | $-1$ | 0 | 1 | 0 | 6 |
| | $x_4$ | 1 | 2 | 0 | 0 | 0 | 1 | 4 |
| 1 | $z$ | 0 | $\frac{1+5M}{3}$ | $-M$ | $\frac{4-7M}{3}$ | 0 | 0 | $4+2M$ |
| $x_2$ enters | $x_1$ | 1 | 1/3 | 0 | 1/3 | 0 | 0 | 1 |
| $R_2$ leaves | $R_2$ | 0 | 5/3 | $-1$ | $-4/3$ | 1 | 0 | 2 |
| | $x_4$ | 0 | 5/3 | 0 | $-1/3$ | 0 | 1 | 3 |
| 2 | $z$ | 0 | 0 | 1/5 | $8/5-M$ | $-1/5-M$ | 0 | 18/5 |
| $x_3$ enters | $x_1$ | 1 | 0 | 1/5 | 3/5 | $-1/5$ | 0 | 3/5 |
| $x_4$ leaves | $x_2$ | 0 | 1 | $-3/5$ | $-4/5$ | 3/5 | 0 | 6/5 |
| | $x_4$ | 0 | 0 | 1 | 1 | $-1$ | 1 | 1 |
| 3 | $z$ | 0 | 0 | 0 | $7/5-M$ | $-M$ | $-1/5$ | 17/5 |
| (optimum) | $x_1$ | 1 | 0 | 0 | 2/5 | 0 | $-1/5$ | 2/5 |
| | $x_2$ | 0 | 1 | 0 | $-1/5$ | 0 | 3/5 | 9/5 |
| | $x_3$ | 0 | 0 | 1 | 1 | $-1$ | 1 | 1 |

**2**

$M=1$:

Optimum solution: $x_1=0$, $x_2=2$, $XR4=1$
$z=3$

Solution is infeasible because $XR4$ is positive. The reason $M=1$ produces an infeasible solution is that it does not play the role of a penalty relative to the objective coefficients of the real variables, $x_1$ and $x_2$. Using $M=1$ makes $XR4$ more attractive than $x_1$ from the standpoint of minimization.

$M=10$:

Optimum solution: $x_1=.4$, $x_2=1.8$, $z=3.4$

The solution is feasible because it does not include artificials at positive level. $M=10$ is relatively much larger than the objective coefficients of $x_1$ and $x_2$, and hence properly plays the role of a penalty.

$M=1000$:

It produces the optimum solution as with $M=10$. The conclusion is that it suffices to select $M$ reasonably larger than the objective coefficients of the real variables. Actually, $M=1000$ is an "overkill" in this case, and selecting such huge values could result in adverse roundoff error.

**3**

(a) Minimize $z = 4x_1 + x_2 + M(R_1+R_2+R_3)$

subject to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - S_2 + R_2 = 6$$
$$x_1 + 2x_2 - S_3 + R_3 = 4$$
$$x_1, x_2, S_2, S_3, R_1, R_2, R_3 \geq 0$$

| Basic | $x_1$ | $x_2$ | $S_2$ | $S_3$ | $R_1$ | $R_2$ | $R_3$ | |
|---|---|---|---|---|---|---|---|---|
| $z$ | $-4$ | $-1$ | | | $-M$ | $-M$ | $-M$ | 0 |
| $R_1$ | 3 | 1 | | | 1 | | | 3 |
| $R_2$ | 4 | 3 | $-1$ | | | 1 | | 6 |
| $R_3$ | 1 | 2 | | $-1$ | | | 1 | 4 |
| $z$ | $-4+8M$ | $-1+6M$ | $-M$ | $-M$ | 0 | 0 | 0 | $10M$ |
| $R_1$ | 3 | 1 | | | 1 | | | 3 |
| $R_2$ | 4 | 3 | $-1$ | | | 1 | | 6 |
| $R_3$ | 1 | 2 | | $-1$ | | | 1 | 4 |

(b) Minimize $z = 4x_1 + x_2 + MR_1$

subject to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 + S_2 = 6$$
$$x_1 + 2x_2 + S_3 = 4$$

| Basic | $x_1$ | $x_2$ | $R_1$ | $S_2$ | $S_3$ | |
|---|---|---|---|---|---|---|
| $z$ | $-4$ | $-1$ | $-M$ | | | 0 |
| $R_1$ | 3 | 1 | 1 | | | 3 |
| $S_2$ | 4 | 3 | | 1 | | 6 |
| $R_3$ | 1 | 2 | | | 1 | 4 |
| $z$ | $-4+3M$ | $-1+M$ | 0 | 0 | 0 | $3M$ |
| $R_1$ | 3 | 1 | 1 | | | 3 |
| $S_2$ | 4 | 3 | | 1 | | 6 |
| $R_3$ | 1 | 2 | | | 1 | 4 |

(c) Minimize $z = 4x_1 + x_2 + M(R_1+R_2)$

subject to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 + R_2 = 6$$
$$x_1 + 2x_3 + S_3 = 4$$

| Basic | $x_1$ | $x_2$ | $R_1$ | $R_2$ | $S_3$ | |
|---|---|---|---|---|---|---|
| $z$ | $-4$ | $-1$ | $-M$ | $-M$ | 0 | 0 |
| $R_1$ | 3 | 1 | 1 | | | 3 |
| $R_2$ | 4 | 3 | | 1 | | 6 |
| $S_3$ | 1 | 2 | | | 1 | 4 |
| $z$ | $-4+7M$ | $-1+4M$ | 0 | 0 | 0 | $9M$ |
| $R_1$ | 3 | 1 | 1 | | | 3 |
| $R_2$ | 4 | 3 | | 1 | | 6 |
| $S_3$ | 1 | 2 | | | 1 | 4 |

continued...

(d) Maximize $z = 4x_1 + x_2 - M(R_1 + R_2)$

subject to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - S_2 + R_2 = 6$$
$$x_1 + 2x_2 + S_3 = 4$$

| Basic | $x_1$ | $x_2$ | $S_2$ | $R_1$ | $R_2$ | $S_3$ | |
|---|---|---|---|---|---|---|---|
| z | -4 | -1 | 0 | M | M | 0 | 0 |
| $R_1$ | 3 | 1 | | 1 | | | 3 |
| $R_2$ | 4 | 3 | -1 | | 1 | | 6 |
| $S_3$ | 1 | 2 | | | | 1 | 4 |
| z | -4-7M | -1-4M | M | 0 | 0 | 0 | -9M |
| $R_1$ | 3 | 1 | | 1 | | | 3 |
| $R_2$ | 4 | 3 | -1 | | 1 | | 6 |
| $S_3$ | 1 | 2 | | | | 1 | 4 |

## 4

(a) Maximize $z = 5x_1 + 6x_2 - MR_1$

subject to

$$-2x_1 + 3x_2 + R_1 = 3 \quad (1)$$
$$x_1 + 2x_2 + S_3 = 5 \quad (3)$$
$$6x_1 + 7x_2 + S_4 = 3 \quad (4)$$

$$z - (5-2M)x_1 - (6+3M)x_2 = -3M$$

(b) Maximize $z = 2x_1 - 7x_2 - M(R_1 + R_2 + R_5)$

subject to

$$-2x_1 + 3x_2 + R_1 = 3 \quad (1)$$
$$4x_1 + 5x_2 - S_2 + R_2 = 10 \quad (2)$$
$$6x_1 + 7x_2 + S_4 = 3 \quad (4)$$
$$4x_1 + 8x_2 - S_5 + R_5 = 5 \quad (5)$$

$$z - (2+6M)x_1 - (-7+16M)x_2 + MS_2 + MS_5 = -18M$$

(c) Minimize $z = 3x_1 + 6x_2 + MR_5$

subject to

$$x_1 + 2x_2 + S_1 = 5 \quad (3)$$
$$6x_1 + 7x_2 + S_2 = 3 \quad (4)$$
$$4x_1 + 8x_2 - S_5 + R_5 = 5 \quad (5)$$

$$z - (3-4M)x_1 - (6-8M)x_2 - MS_5 = 5M$$

(d) Minimize $z = 4x_1 + 6x_2 + M(R_1 + R_2 + R_5)$

subject to

$$-2x_1 + 3x_2 + R_1 = 3 \quad (1)$$
$$4x_1 + 5x_2 - S_2 + R_2 = 10 \quad (2)$$
$$4x_1 + 8x_2 - S_5 + R_5 = 5 \quad (5)$$

$$z - (4-6M) - (6-16M)x_2 - MS_2 - MS_5 = 18M$$

(e) Minimize $z = 3x_1 + 2x_2 + M(R_1 + R_5)$

subject to

$$-2x_1 + 3x_2 + R_1 = 3 \quad (1)$$
$$4x_1 + 8x_2 - S_5 + R_5 = 5 \quad (5)$$

$$z - (3-2M)x_1 - (2-11M)x_2 - MS_5 = 8M$$

continued...

## 5

(a)

| | Basic | $x_1$ | $x_2$ | $x_3$ | $S_2$ | $R_1$ | $R_2$ | |
|---|---|---|---|---|---|---|---|---|
| 0 | z | -2-3M | -3+4M | 5-2M | M | 0 | 0 | -17M |
| | $R_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| | $R_2$ | $\boxed{2}$ | -5 | 1 | -1 | 0 | 1 | 10 |
| I | z | 0 | $-8-\frac{7M}{2}$ | $6-\frac{M}{2}$ | $-1-\frac{M}{2}$ | 0 | $1+\frac{3M}{2}$ | 10-2M |
| | $R_1$ | 0 | $\boxed{7/2}$ | 1/2 | 1/2 | 1 | -1/2 | 2 |
| | $x_1$ | 1 | -5/2 | 1/2 | -1/2 | 0 | 1/2 | 5 |
| II | z | 0 | 0 | 50/7 | 1/7 | 16/7+M | -1/7+M | $\frac{102}{7}$ |
| | $x_2$ | 0 | 1 | 1/7 | 1/7 | 2/7 | -1/7 | 4/7 |
| | $x_1$ | 1 | 0 | 6/7 | -1/7 | 5/7 | 1/7 | 45/7 |

(b)

| | Basic | $x_1$ | $x_2$ | $x_3$ | $S_2$ | $R_1$ | $R_2$ | Sol'n |
|---|---|---|---|---|---|---|---|---|
| 0 | z | -2+3M | -3-4M | 5+2M | -M | 0 | 0 | 17M |
| | $R_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| | $R_2$ | 2 | -5 | 1 | -1 | 0 | 1 | 10 |
| I | z | 0 | -8+7M/2 | 6+M/2 | -1+M/2 | 0 | 1-3M/2 | 10+2M |
| | $R_1$ | 0 | 7/2 | 1/2 | 1/2 | 1 | -1/2 | 2 |
| | $x_1$ | 1 | -5/2 | 1/2 | -1/2 | 0 | 1/2 | 5 |
| II | z | 0 | 0 | 50/7 | 1/7 | 16/7-M | -1/7-M | $\frac{102}{7}$ |
| | $x_2$ | 0 | 1 | 1/7 | 1/7 | 2/7 | -1/7 | 4/7 |
| | $x_1$ | 1 | 0 | 6/7 | -1/7 | 5/7 | 1/7 | 45/7 |
| III | z | 0 | -50 | 0 | -7 | -12-M | 7-M | -14 |
| | $x_3$ | 0 | 7 | 1 | 1 | 2 | -1 | 4 |
| | $x_1$ | 1 | -6 | 0 | -1 | -1 | 1 | 3 |

continued...

(c)

| | Basic | $x_1$ | $x_2$ | $x_3$ | $S_2$ | $R_1$ | $R_2$ | Sol'n |
|---|---|---|---|---|---|---|---|---|
| | $z$ | $-1$ | $-2$ | $-1$ | 0 | $M$ | $M$ | — |
| 0 | $R_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| | $R_2$ | 2 | $-5$ | 1 | $-1$ | 0 | 1 | 10 |
| | $z$ | $-1-3M$ | $-2+4M$ | $-1-2M$ | $M$ | 0 | 0 | $-17M$ |
| I | $R_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| | $R_2$ | 2 | $-5$ | 1 | $-1$ | 0 | 1 | 10 |
| | $z$ | 0 | $-9/2-7M/2$ | $-1/2-M/2$ | $-1/2-M/2$ | 0 | $1/2+3M/2$ | $5-2M$ |
| II | $R_1$ | 0 | $7/2$ | $1/2$ | $1/2$ | 1 | $-1/2$ | 2 |
| | $x_1$ | 1 | $-5/2$ | $1/2$ | $-1/2$ | 0 | $1/2$ | 5 |
| | $z$ | 0 | 0 | $1/7$ | $1/7$ | $9/7+M$ | $-1/7+M$ | $\frac{53}{7}$ |
| III | $x_2$ | 0 | 1 | $1/7$ | $1/7$ | $2/7$ | $-1/7$ | $4/7$ |
| | $x_1$ | 1 | 0 | $6/7$ | $-1/7$ | $5/7$ | $1/7$ | $\frac{45}{7}$ |

(d)

| | Basic | $x_1$ | $x_2$ | $x_3$ | $S_2$ | $R_1$ | $R_2$ | Sol'n |
|---|---|---|---|---|---|---|---|---|
| | $z$ | $-4$ | 8 | $-3$ | 0 | $-M$ | $-M$ | 0 |
| 0 | $R_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| | $R_2$ | 2 | $-5$ | 1 | $-1$ | 0 | 1 | 10 |
| | $z$ | $-4+3M$ | $8-4M$ | $-3+2M$ | $-M$ | 0 | 0 | $17M$ |
| I | $R_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| | $R_2$ | 2 | $-5$ | 1 | $-1$ | 0 | 1 | 10 |
| | $z$ | 0 | $-2+7M/2$ | $-1+M/2$ | $-2+M/2$ | 0 | $2-3M/2$ | $20+2M$ |
| II | $R_1$ | 0 | $7/2$ | $1/2$ | $1/2$ | 1 | $-1/2$ | 2 |
| | $x_1$ | 1 | $-5/2$ | $1/2$ | $-1/2$ | 0 | $1/2$ | 5 |
| | $z$ | 0 | 0 | $-5/7$ | $-12/7$ | $4/7-M$ | $12/7-M$ | $\frac{148}{7}$ |
| III | $x_2$ | 0 | 1 | $1/7$ | $1/7$ | $2/7$ | $-1/7$ | $4/7$ |
| | $x_1$ | 1 | 0 | $6/7$ | $-1/7$ | $-5/7$ | $1/7$ | $45/7$ |

**6**

In the first iteration, we must substitute out the starting solution variables, $x_3$ and $x_4$, in the $z$-equation, exactly as we do with the artificial variables

| | Basic | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Solution |
|---|---|---|---|---|---|---|
| | $z$ | $-2$ | $-4$ | $-4$ | 3 | — |
| 0 | $x_3$ | 1 | 1 | 1 | 0 | 4 |
| | $x_4$ | 1 | 4 | 0 | 1 | 8 |
| | $z$ | $-1$ | $-12$ | 0 | 0 | $-8$ |
| I | $x_3$ | 1 | 1 | 1 | 0 | 4 |
| | $x_4$ | 1 | $\boxed{4}$ | 0 | 1 | 8 |
| | $z$ | 2 | 0 | 0 | 3 | 16 |
| II | $x_3$ | $3/4$ | 0 | 1 | $-1/4$ | 2 |
| | $x_2$ | $1/4$ | 1 | 0 | $1/4$ | 2 |

**7**

After adding surplus $S_1$ and $S_2$, substitute out $x_3$ in the $z$-equation

| | Basic | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $x_3$ | $x_4$ | Solution |
|---|---|---|---|---|---|---|---|---|
| | $z$ | $-3$ | $-2$ | 0 | 0 | $-3$ | 0 | — |
| 0 | $x_3$ | 1 | 4 | $-1$ | 0 | 1 | 0 | 7 |
| | $x_4$ | 2 | 1 | 0 | $-1$ | 0 | 1 | 10 |
| | $z$ | 0 | 10 | $-3$ | 0 | 0 | 0 | 21 |
| I | $x_3$ | 1 | 4 | $-1$ | 0 | 1 | 0 | 7 |
| | $x_4$ | 2 | 1 | 0 | $-1$ | 0 | 1 | 10 |
| | $z$ | $-5/2$ | 0 | $-1/2$ | 0 | $-5/2$ | 0 | $7/2$ |
| II | $x_2$ | $1/4$ | 1 | $-1/4$ | 0 | $1/4$ | 0 | $7/4$ |
| | $x_4$ | $7/4$ | 0 | $1/4$ | $-1$ | $-1/4$ | 1 | $33/4$ |

**8**

Both $x_3$ and $R$ (the starting solution variables) must be substituted out in the $z$-equation

| | Basic | $x_1$ | $x_2$ | $x_3$ | $R$ | Solution |
|---|---|---|---|---|---|---|
| | $z$ | $-1$ | $-5$ | $-3$ | $M$ | — |
| 0 | $x_3$ | 1 | 2 | 1 | 0 | 3 |
| | $R$ | 2 | $-1$ | 0 | 1 | 4 |
| | $z$ | $2-2M$ | $1+M$ | 0 | 0 | $9-4M$ |
| I | $x_3$ | 1 | 2 | 1 | 0 | 3 |
| | $R$ | $\boxed{2}$ | $-1$ | 0 | 1 | 4 |
| | $z$ | 0 | 2 | 0 | $-1+M$ | 5 |
| II | $x_3$ | 0 | $5/2$ | 1 | $-1/2$ | 1 |
| | $x_1$ | 1 | $-1/2$ | 0 | $1/2$ | 2 |

**9**

Maximize $z = 2x_1 + 5x_2 - MR_1$

subject to

$$3x_1 + 2x_2 - S_1 + R_1 = 6$$
$$2x_1 + x_2 + S_2 = 2$$
$$x_1, x_2, S_1, R_1, S_2 \geq 0$$

| Basic | $x_1$ | $x_2$ | $S_1$ | $R_1$ | $S_2$ | |
|---|---|---|---|---|---|---|
| z | -2 | -5 | 0 | M | 0 | - |
| $R_1$ | 3 | 2 | -1 | 1 | 0 | 6 |
| $S_2$ | 2 | 1 | 0 | 0 | 1 | 2 |
| z | -2-3M | -5-2M | M | 0 | 0 | -6M |
| $R_1$ | 3 | 2 | -1 | 1 | 0 | 6 |
| $S_2$ | 2 | 1 | 0 | 0 | 1 | 2 |
| z | 0 | -4-M/2 | M | 0 | 1+3M/2 | -2+3M |
| $R_1$ | 0 | 1/2 | -1 | 1 | -3/2 | 3 |
| $x_1$ | 1 | 1/2 | 0 | 0 | 1/2 | 1 |
| z | 8+M | 0 | M | 0 | 5+2M | 10-2M |
| $R_1$ | -1 | 0 | -1 | 1 | -2 | 2 |
| $x_2$ | 2 | 1 | 0 | 0 | 1 | 2 |

The z-row shows that the solution is optimal (all nonbasic coefficients in the z-row are $\geq 0$). However, the solution is infeasible because the artificial variable $R_1$ assumes a positive value. Having a positive value for the artificial variable $R_1$ is the same as regarding the constraint $3x_1 + 2x_2 \geq 6$ as $3x_1 + 2x_2 \leq 6$, which violates the constraints of the original model.

**1**

In Phase I, we always minimize the sum of the artificial variables because the sum represents a measure of infeasibility in the problem

**2**

(a) Minimize $r = R_1$

(b) Minimize $r = R_1 + R_2 + R_5$

(c) Minimize $r = R_5$

(d) Minimize $r = R_1 + R_2 + R_5$

(e) Minimize $r = R_1 + R_5$

**3**

(a) Phase I:

| Basic | $x_1$ | $x_2$ | $x_3$ | $s_2$ | $R_1$ | $R_2$ | Sol |
|---|---|---|---|---|---|---|---|
| $r$ | 0 | 0 | 0 | 0 | -1 | -1 | 0 |
| $R_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| $R_2$ | 2 | -5 | 1 | -1 | 0 | 1 | 10 |
| $r$ | 3 | -4 | 2 | -1 | 0 | 0 | 17 |
| $R_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 7 |
| $R_2$ | 2 | -5 | 1 | -1 | 0 | 1 | 10 |
| $r$ | 0 | 7/2 | 1/2 | 1/2 | 0 | -3/2 | 2 |
| $R_1$ | 0 | 7/2 | 1/2 | 1/2 | 1 | -1/2 | 2 |
| $x_1$ | 1 | -5/2 | 1/2 | -1/2 | 0 | 1/2 | 5 |
| $r$ | 0 | 0 | 0 | 0 | -1 | -1 | 0 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 2/7 | -1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 5/7 | 1/7 | 45/7 |

Phase II

| Basic | $x_1$ | $x_2$ | $x_3$ | $s$ | Sol |
|---|---|---|---|---|---|
| $z$ | -2 | -3 | 5 | 0 | 0 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 45/7 |
| $z$ | 0 | 0 | 50/7 | 1/7 | 102/7 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 45/7 |

(b) Phase I is the same as in (a)

Phase II

| Basic | $x_1$ | $x_2$ | $x_3$ | $s_2$ | Sol |
|---|---|---|---|---|---|
| $z$ | -2 | -3 | 5 | 0 | 0 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 45/7 |
| $z$ | 0 | 0 | 50/7 | 1/7 | 102/7 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 45/7 |
| $z$ | 0 | -50 | 0 | -7 | -14 |
| $x_3$ | 0 | 7 | 1 | 1 | 4 |
| $x_1$ | 1 | -6 | 0 | -1 | 3 |

continued...

(c) Phase I is the same as in (a)

Phase II:

| Basic | $x_1$ | $x_2$ | $x_3$ | $s_2$ | Sol |
|---|---|---|---|---|---|
| $z$ | -1 | -2 | -1 | 0 | 0 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 45/7 |
| $z$ | 0 | 0 | 1/7 | 1/7 | 53/7 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 45/7 |

(d) Phase I is the same as in (a)

Phase II:

| Basic | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Sol |
|---|---|---|---|---|---|
| $z$ | -4 | 8 | -3 | 0 | 0 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 45/7 |
| $z$ | 0 | 0 | -5/7 | -12/7 | 21 1/7 |
| $x_2$ | 0 | 1 | 1/7 | 1/7 | 4/7 |
| $x_1$ | 1 | 0 | 6/7 | -1/7 | 45/7 |

**4**

Minimize $r = R_1$

subject to

$$3x_1 + 2x_2 - S_1 + R_1 = 6$$
$$2x_1 + x_2 + S_2 = 2$$
$$x_1,\ x_2,\ S_1,\ R_1,\ S_2 \geq 0$$

Solution of Phase I by TORA yields $r = 2$, which indicates that the problem has no feasible space

**5**

Minimize $Z = R_2$

subject to

$$2x_1 + x_2 + x_3 + S_1 = 2$$
$$3x_1 + 4x_2 + 2x_3 - S_2 + R_2 = 8$$
$$x_1,\ x_2,\ x_3,\ S_1,\ S_2,\ R_2 \geq 0$$

Phase I Optimal solution:

| Basic | $x_1$ | $x_2$ | $x_3$ | $S_2$ | $S_1$ | $R_2$ | Sol |
|---|---|---|---|---|---|---|---|
| $r$ | -5 | 0 | -2 | -1 | -4 | 0 | 0 |
| $x_2$ | 2 | 1 | 1 | 0 | 1 | 0 | 2 |
| $R_2$ | -5 | 0 | -2 | -1 | -4 | 1 | 0 |

$R_2 = 0$ is basic in the Phase I Solution

continued...

(b)

Phase I (continued): R2 leaves, x1 enters (also x3, s2, and s1 are candidates for the entering variable).

| | x1 | x2 | x3 | s2 | s1 | R2 | Sol. |
|---|---|---|---|---|---|---|---|
| r | -5 | 0 | -2 | -1 | -4 | 0 | 0 |
| x2 | 2 | 1 | 1 | 0 | 1 | 0 | 2 |
| R2 | -5 | 0 | -2 | -1 | -4 | 1 | 0 |
| r | 0 | 0 | 0 | 0 | 0 | -1 | |
| x2 | 0 | 1 | 1/5 | -2/5 | -3/5 | 2/5 | 2 |
| x1 | 1 | 0 | 2/5 | 1/5 | 4/5 | -1/5 | 0 |

Drop R2-column.

Phase II:

| | x1 | x2 | x3 | s2 | s1 | Sol. |
|---|---|---|---|---|---|---|
| z | -2 | -2 | -4 | 0 | 0 | 0 |
| x2 | 0 | 1 | 1/5 | -2/5 | -3/5 | 2 |
| x1 | 1 | 0 | 2/5 | 1/5 | 4/5 | 0 |
| z | 0 | 0 | -14/5 | -2/5 | 2/5 | 4 |
| x2 | 0 | 1 | 1/5 | -2/5 | -3/5 | 2 |
| x1 | 1 | 0 | 2/5 | 1/5 | 4/5 | 0 |
| z | 7 | 0 | 0 | 1 | 6 | 4 |
| x2 | -1/2 | 1 | 0 | -1/2 | -1 | 2 |
| x3 | 5/2 | 0 | 1 | 1/2 | 2 | 0 |

Optimum solution:

$x_1 = 0, x_2 = 2, x_3 = 0, z = 4$

Phase I:

| | x1 | x2 | x3 | R1 | R2 | R3 | Sol |
|---|---|---|---|---|---|---|---|
| r | -10 | 0 | -4 | -8 | 0 | 0 | 0 |
| x2 | 2 | 1 | 1 | 1 | 0 | 0 | 2 |
| R2 | -5 | 0 | -2 | -3 | 1 | 0 | 0 |
| R3 | -5 | 0 | -2 | -4 | 0 | 1 | 0 |
| r | 0 | 0 | 1 | -2 | -2 | 0 | 0 |
| x2 | 0 | 1 | 1/5 | -1/5 | 2/5 | 0 | 2 |
| x1 | 1 | 0 | 2/5 | 3/5 | -1/5 | 0 | 0 |
| R3 | 0 | 0 | 0 | -1 | -1 | 1 | 0 |

Remove R1- and R2 columns, which gives

| | x1 | x2 | x3 | R3 | Sol |
|---|---|---|---|---|---|
| r | 0 | 0 | 1 | 0 | 0 |
| x2 | 0 | 1 | 1/5 | 0 | 2 |
| x1 | 1 | 0 | 2/5 | 0 | 0 |
| R3 | 0 | 0 | 0 | 1 | 0 |

The R3-row is R3 = 0, which is redundant. Hence the R3-row and R3-column can be dropped from the tableau with no consequences.

Phase II:

| | x1 | x2 | x3 | Sol |
|---|---|---|---|---|
| z | -3 | -2 | -3 | 0 |
| x2 | 0 | 1 | 1/5 | 2 |
| x1 | 1 | 0 | 2/5 | 0 |
| z | 0 | 0 | -7/5 | 4 |
| x2 | 0 | 1 | 1/5 | 2 |
| x1 | 1 | 0 | 2/5 | 0 |
| z | 7/2 | 0 | 0 | 4 |
| x2 | -1/2 | 1 | 0 | 2 |
| x1 | 5/2 | 0 | 1 | 0 |

Optimum solution:

$x_1 = 0, x_2 = 2, x_3 = 0, z = 4$

**7**

If $x_1$, $x_3$, $x_4$, or $x_5$ assume a positive value, the value of the objective function at the end of Phase I must necessarily become positive. This follows because these variables have nonzero z-row coefficients in the optimal Phase I tableau. A positive objective value at the end of Phase I means that Phase I solution is infeasible. Since Phase II uses the same constraints as in Phase I, it follows that Phase II must have $x_1 = x_3 = x_4 = x_5 = 0$ as well.

Phase II:

| Basic | $x_2$ | $R$ | Sol'n |
|---|---|---|---|
| $z$ | -2 | 0 | 0 |
| $x_2$ | 1 | 0 | 2 |
| $R$ | 0 | 1 | 0 |
| $z$ | 0 | 0 | 4 |
| $x_2$ | 1 | 0 | 2 |
| $R$ | 0 | 1 | 0 |

Optimum solution:

$x_1 = 0 \quad x_2 = 2 \quad x_3 = x_4 = x_5 = 0$

$z = 4$

**8**

$$\begin{aligned} -5x_1 + 6x_2 - 2x_3 + x_4 \qquad\qquad &= -5 \\ x_1 - 3x_2 - 5x_3 \qquad + x_5 \qquad &= -8 \\ 2x_1 + 5x_2 - 4x_3 \qquad\qquad + x_6 &= 9 \end{aligned}$$

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $R$ | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | -1 | |
| -5 | 6 | -2 | 1 | 0 | 0 | -1 | -5 |
| 1 | -3 | -5 | 0 | 1 | 0 | $\boxed{-1}$ | -8 |
| 2 | 5 | -4 | 0 | 0 | 1 | 0 | 9 |
| -1 | 3 | 5 | 0 | -1 | 0 | 0 | 8 |
| -6 | 9 | 3 | 1 | -1 | 0 | 0 | 3 |
| -1 | 3 | 5 | 0 | -1 | 0 | 1 | 8 |
| 2 | 5 | -4 | 0 | 0 | 1 | 0 | 9 |

Phase I problem:

minimize $r = R$

subject to

$$\begin{aligned} -6x_1 + 9x_2 + 3x_3 + x_4 - x_5 \qquad &= 3 \\ -x_1 + 3x_2 + 5x_3 \qquad - x_5 + R &= 8 \\ 2x_1 + 5x_2 - 4x_3 \qquad\qquad + x_6 &= 9 \end{aligned}$$

all variables $\geq 0$

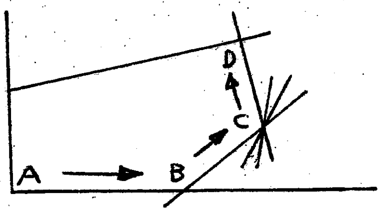
The logic of the procedure is as follows:

In the R-column, enter -1 for any constraint with negative RHS and 0 for all other constraints.

Next, use the R-column as a pivot column and select the pivot element as the one corresponding to the most negative RHS. This procedure will always require one artificial variable regardless of the number of constraints.

**1**

(a)

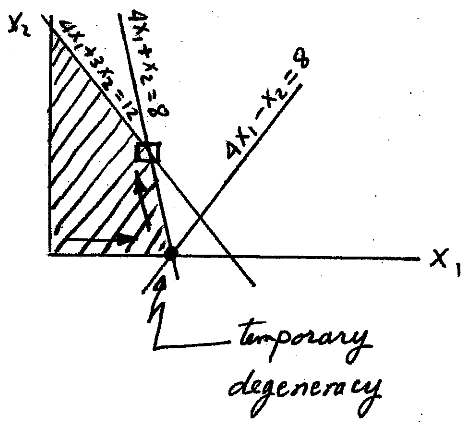



(b) $A: 1$, $B: 1$, $C: \binom{3}{2} = 3$, $D: 1$

**2**

(a) From TORA, iterations 2 and 3 are degenerate. Degeneracy is removed in iteration 4.

(b)




**3**

(a) Four iterations

(b) Three iterations: In iteration 2, degeneracy is removed because basic sX5 = 0 corresponds to a negative constraint coefficient in the entering variable column (X2).

(c) In part (a), solution encounters 2 degenerate basic solution at the same corner point. In part (b), only one basic solution was encountered.

## 1

| Iteration | Basic | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | Solution |
|---|---|---|---|---|---|---|---|---|
| | $z$ | $-1$ | $-2$ | $-3$ | 0 | 0 | 0 | 0 |
| 0 | $s_1$ | 1 | 2 | $\boxed{3}$ | 1 | 0 | 0 | 10 |
| | $s_2$ | 1 | 1 | 0 | 0 | 1 | 0 | 5 |
| | $s_3$ | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| | $z$ | 0 | $\boxed{0}$ | 0 | 1 | 0 | 0 | 10 |
| I | $x_3$ | $1/3$ | $2/3$ | 1 | $1/3$ | 0 | 0 | $10/3$ |
| | $s_2$ | 1 | $\boxed{1}$ | 0 | 0 | 1 | 0 | 5 |
| | $s_3$ | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| | $z$ | $\boxed{0}$ | 0 | 0 | 1 | 0 | 0 | 10 |
| II | $x_3$ | $-1/3$ | 0 | 1 | $1/3$ | $-2/3$ | 0 | 0 |
| | $x_2$ | 1 | 1 | 0 | 0 | 1 | 0 | 5 |
| | $s_3$ | $\boxed{1}$ | 0 | 0 | 0 | 0 | 1 | 1 |
| | $z$ | 0 | 0 | 0 | 1 | $\boxed{0}$ | 0 | 10 |
| III | $x_3$ | 0 | 0 | 1 | $1/3$ | $-2/3$ | $1/3$ | $1/3$ |
| | $x_2$ | 0 | 1 | 0 | 0 | $\boxed{1}$ | $-1$ | 4 |
| | $x_1$ | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| | $z$ | 0 | 0 | 0 | 1 | 0 | 0 | 10 |
| IV | $x_3$ | 0 | $2/3$ | 1 | $1/3$ | 0 | $-1/3$ | 3 |
| | $s_2$ | 0 | 1 | 0 | 0 | 1 | $-1$ | 4 |
| | $x_1$ | 1 | 0 | 0 | 0 | 0 | 1 | 1 |

Three alternative basic optima:

$$(x_1, x_2, x_3) = \begin{cases} (0, 0, 10/3) \\ (0, 5, 0) \\ (1, 4, 1/3) \end{cases}$$

The associated <u>nonbasic</u> alternative optima are

$$\tilde{x}_1 = \lambda_3$$

$$\tilde{x}_2 = 5\lambda_2 + 4\lambda_3$$

$$\tilde{x}_3 = 10/3\,\lambda_1 + 1/3\,\lambda_3$$

where

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$0 \le \lambda_i \le 1,\ i = 1, 2, 3$$

## 2

| Basic | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | |
|---|---|---|---|---|---|---|
| $z$ | $-2$ | 1 | 3 | 0 | 0 | 0 |
| $s_1$ | 1 | $-1$ | $\boxed{5}$ | 1 | 0 | 10 |
| $s_2$ | 2 | $-1$ | 3 | 0 | 1 | 40 |
| $z$ | $-7/5$ | $2/5$ | 0 | $3/5$ | 0 | 6 |
| $x_3$ | $\boxed{1/5}$ | $-1/5$ | 1 | $1/5$ | 0 | 2 |
| $s_2$ | $7/5$ | $-2/5$ | 0 | $-3/5$ | 1 | 34 |
| $z$ | 0 | $-1$ | 7 | 2 | 0 | 20 |
| $x_1$ | 1 | $-1$ | 5 | 1 | 0 | 10 |
| $s_2$ | 0 | $\boxed{1}$ | $-7$ | $-2$ | 1 | 20 |
| $z$ | 0 | 0 | $\boxed{0}$ | $\boxed{0}$ | 1 | 40 |
| $x_1$ | 1 | 0 | $-2$ | $-1$ | 0 | 30 |
| $x_2$ | 0 | 1 | $-7$ | $-2$ | 1 | 20 |

$x_3$ and $s_1$ can yield alternative optima. However, because all their constraint coefficients are negative (in general, $\le 0$), none can yield an alternative (corner point) basic solution.


## 3

| Iteration | Basic | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | Soln |
|---|---|---|---|---|---|---|---|---|
| | $z$ | $-3$ | $-1$ | 0 | 0 | 0 | 0 | 0 |
| 0 | $s_1$ | 1 | 2 | 0 | 1 | 0 | 0 | 5 |
| | $s_2$ | $\boxed{1}$ | 1 | $-1$ | 0 | 1 | 0 | 2 |
| | $s_3$ | 7 | 3 | $-5$ | 0 | 0 | 1 | 20 |
| | $z$ | 0 | 2 | $-3$ | 0 | 3 | 0 | 6 |
| I | $s_1$ | 0 | 1 | $\boxed{1}$ | 1 | $-1$ | 0 | 3 |
| | $x_1$ | 1 | 1 | $-1$ | 0 | 1 | 0 | 2 |
| | $s_3$ | 0 | $-4$ | 2 | 0 | $-7$ | 1 | 6 |
| | $z$ | 0 | 5 | 0 | 3 | $\boxed{0}$ | 0 | 15 |
| II | $x_3$ | 0 | 1 | 1 | 1 | $-1$ | 0 | 3 |
| | $x_1$ | 1 | 2 | 0 | 1 | 0 | 0 | 5 |
| | $s_3$ | 0 | $-6$ | 0 | $-2$ | $-5$ | 1 | 0 |

The optimum solution is degenerate because $s_3$ is basic and equal to zero. Also, it has alternative <u>nonbasic</u> solutions because $s_2$ has a zero coefficient in the z-row and all its constraint coefficients are $\le 0$.

**1**

| Basic | $x_1$ | $x_2$ | $s_1$ | $s_2$ | |
|---|---|---|---|---|---|
| $z$ | $-2$ | $-1$ | $0$ | $0$ | $0$ |
| $s_1$ | $1$ | $-1$ | $1$ | $0$ | $10$ |
| $s_2$ | $2$ | $0$ | $0$ | $1$ | $40$ |
| $z$ | $0$ | $-3$ | $2$ | $0$ | $20$ |
| $x_1$ | $1$ | $-1$ | $1$ | $0$ | $10$ |
| $s_2$ | $0$ | $2$ | $-2$ | $1$ | $20$ |
| $z$ | $0$ | $0$ | $-1$ | $3/2$ | $50$ |
| $x_1$ | $1$ | $0$ | $0$ | $1/2$ | $20$ |
| $x_2$ | $0$ | $1$ | $-1$ | $1/2$ | $10$ |

unbounded $\longrightarrow$ ↑

**2**

(a)

| $x_2$ |
|---|
| $-10$ |
| $-3$ |
| $0$ |
| $-1$ |

$\Rightarrow$ Solution space unbounded in the direction of $x_2$

(b) Objective value is unbounded because each unit increase in $x_2$ increases $z$ by 10

**3**

If, at any iteration, all the constraint coefficients of a variable are $\leq 0$, then the solution space is unbounded in the direction of that variable.

A more "foolproof" way of accomplishing this task is to solve a sequence of LPs in which the objective function is

Maximize $z = x_j$, $j = 1, 2, \dots, n$

subject to the constraints of the problem. For the unbounded variables, $z = \infty$.

**1**

$x_1$ = number of units of T1
$x_2$ = number of units of T2
$x_3$ = number of units of T3

Constraints:

$$3x_1 + 5x_2 + 6x_3 \le 1000$$
$$5x_1 + 3x_2 + 4x_3 \le 1200$$
$$x_1 + x_2 + x_3 \ge 500$$
$$x_1, x_2, x_3 \ge 0$$

We can use Phase I to see whether the problem has a feasible solution; that is,

minimize $r = R_3$

subject to

$$3x_1 + 5x_2 + 6x_3 + S_1 = 1000$$
$$5x_1 + 3x_2 + 4x_3 + S_2 = 1200$$
$$x_1 + x_2 + x_3 - S_3 + R_3 = 500$$
$$x_1, x_2, x_3, S_1, S_2, S_3, R_3 \ge 0$$

Optimum solution from TORA:

$R_3 = r = 225$ units

This is interpreted as a deficiency of 225 units. The most that can be produced is $500 - 225 = 275$ units

**2**

| Basic | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $R_1$ | Sol'n |
|---|---|---|---|---|---|---|---|
| $z$ | $-3-3M$ | $-2-4M$ | $-3-2M$ | $M$ | $0$ | $0$ | $-8M$ |
| $S_1$ | 2 | $\boxed{1}$ | 1 | 0 | 1 | 0 | 2 |
| $R_1$ | 3 | 4 | 2 | $-1$ | 0 | 1 | 8 |
| $z$ | $-1+5M$ | 0 | $-1+2M$ | $M$ | $2+4M$ | 0 | 4 |
| $x_2$ | 2 | 1 | 1 | 0 | 1 | 0 | 2 |
| $R_1$ | $-5$ | 0 | $-2$ | $-1$ | $-4$ | 1 | 0 |

Because $R_1 = 0$ in the optimal tableau, the problem has a feasible solution. The optimum solution is

$x_1 = 0$, $x_2 = 2$, $z = 4$

Note that in the first iteration, $R_1$ could have been used as the leaving variable, in which case it would not be basic in the optimum iteration.

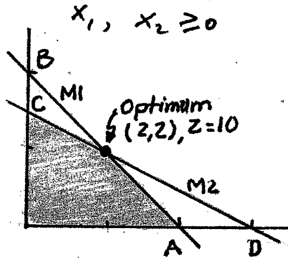
**1**

$x_1$ = Nbr. units of product A
$x_2$ = Nbr. units of product B

Maximize $z = 2x_1 + 3x_2$

s.t.

$2x_1 + 2x_2 \le 8$ (M1)
$3x_1 + 6x_2 \le 18$ (M2)
$x_1, x_2 \ge 0$




| | M1 | M2 | Z |
|---|---|---|---|
| A=(4,0) | | 12 | 8 |
| B=(0,4) | | 24 | 12 |
| C=(0,3) | 6 | | 9 |
| D=(6,0) | 12 | | 12 |

(a) M1 at C $= 2(0) + 2(3) = 6$
M1 at D $= 2(6) + 2(0) = 12$
z at C $= 2(0) + 3(3) = 9$
z at D $= 2(6) + 3(0) = 12$

Dual price $= \frac{12-9}{12-6} = \$.50$/unit

Allowable range $= (6 \le M1 \le 12)$

M2 at A $= 3(4) + 6(0) = 12$
M2 at B $= 3(0) + 6(4) = 24$
z at A $= 2(4) + 3(0) = 8$
z at B $= 2(0) + 3(4) = 12$

Dual price $= \frac{12-8}{24-12} = \$.33$/unit

Range: $12 \le M2 \le 24$

(b) Dual price = \$.50/unit valid in the range $6 \le M1 \le 12$
Increase in revenue $= .5 \times 4 = \$2.00$
Increase in cost $= .3 \times 4 = \$1.20$
Cost < Revenue — purchase recommended

(c) Dual price = \$.33/unit valid in the range $12 \le M2 \le 24$
Purchase price/unit < \$.33

(d) Dual price = \$.33/unit valid in the range $12 \le M2 \le 24$. M2 is increased from 18 to 23 units
Increase in revenue
$= 5 \times .33 = \$1.65$

New optimum revenue $= 10 + 1.65 = \$11.65$

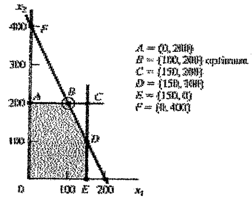
**2**

$x_1$ = daily number of type 1 hat
$x_2$ = daily number of type 2 hat

Maximize $z = 8x_1 + 5x_2$

$2x_1 + x_2 \le 400$
$x_1 \le 150$
$x_2 \le 200$
$x_1, x_2 \ge 0$

(a) Optimum occurs at B:
$x_1 = 100$ type 1 hats
$x_2 = 200$ type 2 hats
$z = \$1800$




(b) $A = (0, 200)$, $C = (150, 200)$

| | capacity | z |
|---|---|---|
| A | $2 \times 0 + 1 \times 200 = 200$ | $8 \times 0 + 5 \times 200 = 1000$ |
| C | $2 \times 150 + 1 \times 200 = 500$ | $8 \times 150 + 5 \times 200 = 2200$ |

worth/capacity unit $= \frac{2200 - 1000}{500 - 200}$
$=$ ~~\$4 per type 2 hat~~

Range: (200, 500)

(c) Dual price = 0 in the range $(100, \infty)$
Thus, change from $x_1 \le 150$ to $x_1 \le 120$ has no effect on optimum z.

(d) Let d = demand limit for type 2 hat

| | d | z |
|---|---|---|
| D(150,100) | 100 | $8(150) + 5(100) = \$1700$ |
| F(0,400) | 400 | $8(0) + 5(400) = \$2000$ |

Dual price $= \frac{2000-1700}{400-100} = \$1.00$

Range (100, 400)

Maximum increase in demand limit for type 2 hat $= 400 - 200 = 200$ hats

## Set 3.6b

**1**

(a) $\frac{3}{6} \le \frac{c_A}{c_B} \le \frac{2}{2}$, or

$.5 \le \frac{c_A}{c_B} \le 1$ or $1 \le \frac{c_B}{c_A} \le 2$

(b) Maximize $z = 2x_A + 3x_B$

$c_B = 3$: $3 \times .5 \le c_A \le 3 \times 1$

$1.5 \le c_A \le 3$

$c_A = 2$: $2 \times .5 \le c_B \le 2 \times 2$

$1 \le c_B \le 4$

(c) $\frac{c_A}{c_B} = \frac{5}{4} = 1.25$, which falls outside the range $.5 \le \frac{c_A}{c_B} \le 1$. Optimum solution changes and must be computed anew. New solution: $x_A = 4$, $x_B = 0$, $z = \$20$.

(d) Case 1: $z = 5x_A + 3x_B$

$c_A = 5$ falls outside the range $(1.5, 3)$, hence the optimum changes. New Optimum is $x_A = 4$, $x_B = 0$, $z = \$20$.

Case 2: $z = 2x_A + 4x_B$

$c_B = 4$ falls in the range $(1, 4)$, hence optimum is unchanged at $x_A = x_B = 2$,

$z = 2(2) + 4(2) = \$12$

**2**

(a) $\frac{1}{2} \le \frac{c_1}{c_2} \le \frac{6}{4}$, or

$.5 \le \frac{c_1}{c_2} \le 1.5$ or $\frac{2}{3} \le \frac{c_2}{c_1} \le 2$

(b) Given $c_1 = 5$, then

$5\left(\frac{2}{3}\right) \le c_2 \le 5(2)$, or $\frac{10}{3} \le c_2 \le 10$

(c) $\frac{c_1}{c_2} = \frac{5}{3} = 1.67$, which falls outside the range $.5 \le \frac{c_1}{c_2} \le 1.5$. Hence the solution changes

**3**

(a) $\frac{0}{1} \le \frac{c_1}{c_2} \le \frac{2}{1}$, or

$0 \le \frac{c_1}{c_2} \le 2$

(b) $\frac{c_1}{c_2} = 1$, which falls in the range $0 \le \frac{c_1}{c_2} \le 2$. Hence, the solution is unchanged.

**1**

Feasibility conditions:

$x_2 = 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2$
$x_3 = 230 + \frac{1}{2}D_2$
$x_6 = 20 - 2D_1 + D_2 + D_3$

(a) $D_1 = 438 - 430 = 8$ min
$D_2 = 500 - 460 = 40$
$D_3 = 410 - 420 = -10$

$x_2 = 100 + \frac{1}{2}(8) - \frac{1}{4}(40) = 94 > 0$
$x_3 = 230 + \frac{1}{2}(40) = 250 > 0$
$x_6 = 20 - 2(8) + 40 - 10 = 34 > 0$

Dual prices:
Resource 1 = \$1/min, $-200 \le D_1 \le 10$
2 = \$2/min, $-20 \le D_2 \le 400$
3 = \$0/min, $-20 \le D_3 < \infty$

New profit $= 1350 + D_1 + 2D_2 + 0D_3$
$= 1350 + 8 + 2 \times 40 = \$1438$

(b) $D_1 = 460 - 430 = 30$ min
$D_2 = 440 - 460 = -20$
$D_3 = 380 - 420 = -40$
$x_2 = 100 + \frac{1}{2}(30) - \frac{1}{4}(-20) = 120 > 0$
$x_3 = 230 + \frac{1}{2}(-20) = 220 > 0$
$x_6 = 20 - 2(30) - 20 - 40 = -100 < 0$

**2**

(a) Overtime cost $\frac{50}{60} = \$.83/\text{min}$
Revenue (dual price) for operation 1 is \$1/min.
Cost < Revenue $\Rightarrow$ advantageous

(b) Dual price for operation 2 = \$2/min valid in the range $-20 \le D_2 \le 400$
$D_2 = 120$ minutes
Revenue increase $= 120 \times 2 = \$240$
Cost increase $= 2(\$55) = \$110$
Revenue > cost $\Rightarrow$ accept.

(c) No, resource 3 is already abundant. This is the reason its dual price = 0

(d) Dual price for operation 1 is \$1/min, valid in the range $-200 \le D_1 \le 10$

continued...

$D_1 = 440 - 430 = 10$ min
Cost $= \frac{10}{60} \times 40 = \$6.67$
New revenue $= 1350 + 1 \times 10 = \$1360$
Net revenue $= 1360 - \$6.67 = \$1353.33$

(e) Dual price = \$2/min, $-20 \le D_2 \le 400$
$D_2 = -$ min
Decrease in cost $= \frac{15}{60} \times \$30 = \$7.50$
Lost revenue $= 15 \times \$2.00 = \$30.00$
Lost revenue > Decrease in cost
Not recommended.

**3**

$x_j$ = units of product $i = 1, 2, 3$

Maximize $z = 20x_1 + 50x_2 + 35x_3$

s.t.
$-.5x_1 + .5x_2 + .5x_3 \le 0$
$x_1 \le 75$
$2x_1 + 4x_2 + 3x_3 \le 240$
$x_1, x_2, x_3 \ge 0$

(a) Solution: $z = \$2800$
$x_1 = x_2 = 40, \; x_3 = 0$

| | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | |
|---|---|---|---|---|---|---|---|
| $z$ | 0 | 0 | 10/3 | 20/3 | 0 | 35/3 | 2800 |
| $x_2$ | 0 | 0 | 5/6 | 2/3 | 0 | 1/6 | 40 |
| $S_2$ | 1 | 0 | 1/6 | 4/3 | 1 | -1/6 | 35 |
| $x_1$ | 0 | 1 | -1/6 | -4/3 | 0 | 1/6 | 40 |

(b) $z + 10/3\,x_3 + 20/3\,S_1 + 0\,S_2 + 35/3\,S_3 = 2800$
Dual price for raw material = \$35/3 /lb

$$\left.\begin{aligned} x_2 &= 40 + D_3/6 \\ S_2 &= 35 - D_3/6 \\ x_1 &= 40 + D_3/6 \end{aligned}\right\} \Rightarrow -240 \le D_3 \le 210$$

$D_3 = 120$ lb falls in the range $(-240, 210)$

New Solution:
$x_1 = 40 + \frac{120}{6} = 60$ units
$x_2 = 40 + \frac{120}{6} = 60$ units
$x_3 = 0$

New revenue $= 2800 + (35/3)(120)$
$= \$4200$

continued...

(c) Dual price = 0, $-35 \le D_2 < \infty$

$\pm 10\%$ of 75 = $\pm 7.5$ or

Change has no effect on the solution

**4**

$x_j$ = units of product $j$, $j = 1, 2, 3$,

Maximize $z = 4.5x_1 + 5x_2 + 4x_3$

s.t.

$10x_1 + 5x_2 + 6x_3 \le 600$

$6x_1 + 8x_2 + 9x_3 \le 600$

$8x_1 + 10x_2 + 12x_3 \le 600$

$x_1, x_2, x_3 \ge 0$

(a) Solution: $z = \$325$

$x_1 = 50, x_2 = 20, x_3 = 0$

(b) Optimum tableau

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | |
|---|---|---|---|---|---|---|---|
| z | 0 | 0 | 2 | .083 | 0 | .458 | 325 |
| $x_1$ | 1 | 0 | 0 | .167 | 0 | -.083 | 50 |
| $s_2$ | 0 | 0 | -.6 | .067 | 1 | -.833 | 140 |
| $x_2$ | 0 | 1 | 1.2 | -.133 | 0 | .167 | 20 |

$z + 2x_3 + .083s_1 + 0s_2 + .458s_3 = 325$

Dual prices:

Process 1: \$.083/min

2: \$0/min

3: \$.458/min

Process3 > Process 1

(c) Process 1: $60 \times .083 = \$4.98$

2: 0

3: $60 \times .458 = \$27.48$

**5**

$x_1$ = Nbr. of practical courses

$x_2$ = Nbr. of humanistic courses

Maximize $z = 1500x_1 + 1000x_2$

$x_1 + x_2 + s_1 = 30$ (1)

$x_1 - s_2 = 10$ (2)

$x_2 - s_3 = 10$ (3)

$x_1, x_2, s_1, s_2, s_3 \ge 0$

(a) Solution:

$z = \$40{,}000$

$x_1 = 20$ courses

$x_2 = 10$ Courses

continued...

(b) From Tora,

$z + 1500s_1 + 0s_2 + 500s_3 = 40{,}000$

$s_1$ is a slack, $s_2$ and $s_3$ are surplus

Dual prices:

constraint 1: \$1500/course

constraint 2: \$0/min limit course

constraint 3: -\$500/min limit course

Dual price for constraint 1 equals the revenue per practical course. Hence, an additional course must necessarily be of the practical type.

(c) From TORA,

$$\left.\begin{array}{l} s_2 = 10 + D_1 \ge 0 \\ x_1 = 20 + D_1 \ge 0 \end{array}\right\} -10 \le D_1 < \infty$$

$x_2 = 10$

Thus, the dual price of \$1500 for constraint 1 is valid for any number of courses $\ge 30 - 10 = 20$.

(d) Dual price = -\$500. To determine the range where it applies, we have from TORA

$$\left.\begin{array}{l} s_1 = 10 - D_3 \ge 0 \\ x_1 = 20 - D_3 \ge 0 \\ x_2 = 10 + D_3 \ge 0 \end{array}\right\} -10 \le D_3 \le 10$$

A unit increase in lower limit on humanistic course offering (i.e. from 10 to 11) decreases revenue by \$500

**6**

$x_1$ = Radio minutes

$x_2$ = TV minutes

$x_3$ = Newspaper ads

Maximize $z = x_1 + 50x_2 + 5x_3$

s.t.

$15x_1 + 300x_2 + 50x_3 \le 10000$ (1)

$x_3 \ge 5$ (2)

$x_1 \le 400$ (3)

$-x_1 + 2x_2 \le 0$ (4)

$x_1, x_2, x_3 \ge 0$

Solution: $z = 1561.36$

$x_1 = 59.09$ min, $x_2 = 29.55$ min, $x_3 = 5$ ads

continued...

(b) $S_1, S_3, S_4$ = Slacks associated with constraints 1, 3, and 4
$S_2$ = Surplus associated with constraint 2

From TORA's optimum tableau:

$Z + 2.879 S_2 + .158 S_1 + 0 S_2 + 1.364 S_3 = 1561.36$

$59.091 + .006D_1 - .303D_2 \quad - .909D_4 \geq 0$

$5 \quad + D_2$

$340.909 - .006D_1 + .303D_2 + D_3 + .909D_4 \geq 0$

$29.545 + .003D_1 - .152D_2 \quad + .045D_4 \geq 0$

| Constraint | Dual Price | RHS Range† |
|---|---|---|
| 1 | .158 | (250, 66250) |
| 2 | −2.879* | (0, 2000) |
| 3 | 0 | (59.09, ∞) |
| 4 | 1.3636 | (−375, 65) |

* Negative because $S_2$ is a surplus variable

† These results are taken from TORA output. They differ from those computed from the given $D_i$ - conditions because of roundoff error

Conclusions:

1. Increasing the lower limit on the number of newspaper ads is not advantageous because the associated dual price is negative (= −2.879)
2. Increasing the upper limit on radio minutes is not warranted because its dual price is zero (the current limit is already abundant).

(c) Dual price = .158/budget \$ valid in the range $250 \leq \$ \leq 66250$.

50% budget increase = \$5000, or budget will be increased to 15,000.

Increase in Z = .158 × 5000 = 790

**7**

(a) $x_1$ = Nbr. shirts / week
$x_2$ = Nbr. blouses / week

Maximize $z = 8x_1 + 12x_2$

s.t.

$20x_1 + 60x_2 \leq 25 \times 60 \times 40 = 60{,}000$

$70x_1 + 60x_2 \leq 35 \times 60 \times 40 = 84{,}000$

$12x_1 + 4x_2 \leq 5 \times 60 \times 40 = 12{,}000$

$x_1, x_2 \geq 0$

continued...

Solution: $z = \$13920$ / week

$x_1 = 480$ shirts, $x_2 = 840$ blouses

(b) Let $s_1$, $s_2$, and $s_3$ be the slack variables associated with the cutting, sewing, and packaging constraints. From the optimum TORA tableau, we have

$z + .12S_1 + .08S_2 + 0S_3 = 13920$

| Dept. | Worth/hr (Dual price) | |
|---|---|---|
| Cutting | \$.12/ | = \$7.20/hr |
| Sewing | \$.08/min | = \$4.80/hr |
| Packaging | \$0/hr | |

(c) Breakeven wages are \$7.20/hr for cutting and \$4.80 for sewing

**8**

(a) $x_1$ = units of solution A
$x_2$ = units of solution B

Maximize $z = 8x_1 + 10x_2$

s.t.

$.5x_1 + .5x_2 \leq 150 \quad (1)$

$.6x_1 + .4x_2 \leq 145 \quad (2)$

$30 \leq x_1 \leq 150 \quad (3)$

$40 \leq x_2 \leq 200 \quad (4)$

Solution: $z = \$2800$

$x_1 = 100$ units, $x_2 = 200$ units

(b) Define

$S_1, S_2, S_3, S_4$ = slacks in constraints 1, 2, 3, 4

$S_5, S_6$ = surplus variables associated with the lower bounds of constraints 3 and 4.

From TORA's optimum tableau:

$z + 16S_1 + 0S_2 + 0S_3 + 2S_4 + 0S_5 + 0S_6 = 2800$

Conditions:

$S_5 = 70 + 2D_1 - D_4 - D_5 \geq 0$

$S_2 = 5 - 1.2D_1 + D_2 + .2D_4 \geq 0$

$S_3 = 50 - 2D_1 + D_3 + D_4 \geq 0$

$x_1 = 100 + 2D_1 - D_4 \quad \geq 0$

$x_2 = 200 + D_4 \quad \geq 0$

$S_4 = 160 + D_4 - D_6 \quad \geq 0$

continued...

## Set 3.6c

| Constraint | Dual price | RHS-range |
|---|---|---|
| 1 | 16 | (115, 154.17) |
| 2 | 0 | (140, ∞) |
| 3 (upper) | 0 | (100, ∞) |
| 3 (lower) | 0 | (−∞, 100) |
| 4 (upper) | 2 | (175, 270) |
| 4 (lower) | 0 | (−∞, 200) |

Increase in raw material 1 and in the upper bound on solution B is advantageous because their dual prices (16 and 2) are positive.

(c) Increase in revenue/unit = \$16
Increase in cost/unit = \$20
Not recommended!

(d) Dual price for raw material 2 is zero because it is abundant. No increase is warrented.

**9**

$x_1$ = Nbr. DiGi-1
$x_2$ = Nbr. DiGi-2
$S_i$ = Idle minutes for station $i$, $i = 1, 2, 3$

The objective is to minimize $S_1 + S_2 + S_3$. To express the objective function in terms of $x_1$ and $x_2$, consider

$$6x_1 + 4x_2 + S_1 = .9 \times 480 = 432$$
$$5x_1 + 4x_2 + S_2 = .86 \times 480 = 412.8$$
$$4x_1 + 6x_2 + S_3 = .88 \times 480 = 422.4$$

Thus, $S_1 + S_2 + S_3 = 1267.2 - 15x_1 - 14x_2$

(a)
Maximize $z = 15x_1 + 14x_2$
s.t.
$$6x_1 + 4x_2 + S_1 = 432$$
$$5x_1 + 4x_2 + S_2 = 412.8$$
$$4x_1 + 6x_2 + S_3 = 422.4$$
$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

$z$ represents the total used time in the three stations in minutes.

Solution: $z = 1241.28$ minutes
$x_1 = 45.12$ units, $x_2 = 40.32$ units

Utilization $= \frac{1241.28}{1267.20} \times 100 = 97.95\%$

continued...

(b) From TORA,

$$z + 1.7S_1 + 0S_2 + 1.2S_3 = 1241.28$$

Conditions:

$$x_1 = .3D_1 - .2D_3 + 45.12 \geq 0$$
$$S_2 = -.7D_1 + D_2 - .2D_3 + 25.92 \geq 0$$
$$x_2 = -.2D_1 + .3D_3 + 40.32 \geq 0$$

| Station | Dual Price | RHS Range |
|---|---|---|
| 1 | 1.7 | 281.6, 469.03 |
| 2 | 0 | 386.88, ∞ |
| 3 | 1.2 | 288, 552 |

1% decrease in maintenance time is equivalent to $D_1 = D_2 = D_3 = 4.8$ minutes. This is equivalent to having

| Station | Daily minutes |
|---|---|
| 1 | 436.8 |
| 2 | 417.6 |
| 3 | 427.2 |

All three daily minutes fall within the allowable ranges. Thus

| Station | Increase in utilized time/day |
|---|---|
| 1 | 4.8 × 1.7 = 8.16 minutes |
| 2 | 4.8 × 0 = 0 |
| 3 | 4.8 × 1.2 = 5.76 |

(c) $D_1 = .9(600 - 480) = 108$ min
$D_2 = .86(600 - 480) = 103.2$
$D_3 = .88(600 - 480) = 105.6$

From the conditions in (b)

$$x_1 = .3 \times 108 - .2 \times 105.6 + 45.12 = 56.4$$
$$S_2 = -.7 \times 108 + 103.2 - .2 \times 105.6 + 25.92 = 32.4$$
$$x_2 = -.2 \times 108 + .3 \times 105.6 + 40.32 = 50.4$$

Solution is feasible. Hence dual prices remain applicable and the net utilization is increased by $1.7 \times 108 + 0 \times 103.2 + 1.2 \times 105.6 = 310.32$ minutes. Because station 2 has zero dual price, its capacity need not be increased. The associated cost thus equals $1.5(600 - 480) + 0 + 1.5(600 - 480) = \$360$.

The proposal can be improved by recommending that station 2 time remain unchanged.

**10**

$x_1$ = Nbr. purses / day
$x_2$ = Nbr. bags / day
$x_3$ = Nbr. backpacks/day

Maximize $z = 24x_1 + 22x_2 + 45x_3$
s.t.

$$2x_1 + x_2 + 3x_3 \le 42$$
$$2x_1 + x_2 + 2x_3 \le 40$$
$$x_1 + .5x_2 + x_3 \le 45$$
$$x_1, x_2, x_3 \ge 0$$

Solution: $z = \$882$, $x_1 = 0, x_2 = 2, x_3 = 36$

Letting $s_1, s_2, s_3$ be the slacks in constraints 1, 2, and 3, we get

$$z + 20x_1 + s_1 + 21s_2 + 0s_3 = 882$$

Conditions:

$$x_3 = 2 + D_1 - D_2 \ge 0$$
$$x_2 = 36 - 2D_1 + 3D_2 \ge 0$$
$$s_3 = 25 - .5D_2 + D_3 \ge 0$$

| Resource | Dual price | RHS Ranges |
|---|---|---|
| Leather | 1 | (40, 60) |
| Sewing | 21 | (28, 42) |
| Finishing | 0 | (20, ∞) |

(a) Available leather = 45 ft² falls in the RHS range. Solution remains feasible. $D_1 = 45 - 42 = 3$. New solution:

$x_1 = 0$
$x_2 = 36 - 2 \times 3 = 30$
$x_3 = 2 + 3 = 5$
$z = 882 + 1 \times D_1 = 882 + 1 \times 3 = \$885$

(b) Available leather = 41 ft² falls in the RHS range and the solution remains feasible. $D_1 = 41 - 42 = -1$

$x_2 = 36 - (2 \times -1) = 38$
$x_3 = 2 - 1 = 1$
$z = 882 + (1 \times -1) = \$881$

(c) Sewing hours = 38 falls within the RHS range. $D_2 = 38 - 40 = -2$. Dual price = 21

$x_2 = 36 + 3 \times -2 = 30$
$x_3 = 2 - (-2) = 4$
$z = 882 + (21 \times -2) = \$840$

continued....

(d) Sewing hours = 46 hours falls outside the RHS range. Thus, the current optimum basic solution is infeasible. To obtain the new solution, either solve the problem anew or use the algorithms in chapter 4.

(e) Finishing hours = 15, which falls outside the RHS range. Hence, resolve the problem.

(f) Sewing hours = 50, which falls in the RHS range. $D_3 = 50 - 45 = 5$. Solution remains unchanged because dual price is zero and $D_3$ does not appear in the expression for $x_2$ or $x_3$.

(g) Dual price = \$21/hr, which is higher than the cost of an additional worker per hour. Hiring is recommended.

**11**

$x_1$ = Nbr. model 1 units
$x_2$ = Nbr. model 2 units

Maximize $z = 3x_1 + 4x_2$
s.t.

$$2x_1 + 3x_2 \le 1200$$
$$2x_1 + x_2 \le 1000$$
$$4x_2 \le 800$$
$$x_1, x_2 \ge 0$$

Solution: $z = \$1750$
$x_1 = 450$, $x_2 = 100$

(a) $S_1 = 0 \Rightarrow$ Resistors scarce
$S_2 = 0 \Rightarrow$ Capacitors scarce
$S_3 = 400 \Rightarrow$ chips abundant

(b) $z + \frac{5}{4}S_1 + \frac{1}{4}S_2 = 1750$

| Resource | Dual price |
|---|---|
| Resistors | \$1.25/resistor |
| Capacitors | \$.25/capacitor |
| Chips | \$0/chip |

(c) Conditions:

$$x_1 = 450 - \tfrac{1}{4}D_1 + \tfrac{3}{4}D_2 \ge 0$$
$$S_3 = 400 - 2D_1 + 2D_2 + D_3 \ge 0$$
$$x_2 = 100 + \tfrac{1}{2}D_1 - \tfrac{1}{2}D_2 \ge 0$$

Feasibility ranges:

$$\left.\begin{array}{l} 450 - .25D_1 \ge 0 \\ 400 - 2D_1 \ge 0 \\ 100 + .5D_1 \ge 0 \end{array}\right\} \Rightarrow -200 \le D_1 \le 200$$

continued...

$$\left.\begin{array}{l} 450 + .75D_2 \ge 0 \\ 400 + 2D_2 \ge 0 \\ 100 - .5D_2 \ge 0 \end{array}\right\} \Rightarrow -200 \le D_2 \le 200$$

$400 + D_3 \ge 0 \Rightarrow -400 \le D_3 < \infty$

(d) $D_1 = 1300 - 1200 = 100$ in the allowable range $-200 \le D_1 \le 200$.

$\Delta Z = 100 \times 1.25 = \$125$

$x_1 = 450 - .25 \times 100 = 425$

$x_2 = 100 + .5 \times 100 = 150$

New $Z = 1750 + \Delta Z = \$1875$

(e) $D_3 = 350 - 800 = -450$, which falls outside allowable range $-400 \le D_3$. Thus, basic solution and dual price change and the problem must be solved anew.

(f) $-200 \le D_2 \le 200$, dual price $= .25$. Thus,

$-200 \times .25 \le \Delta Z \le 200 \times .5$

$-50 \le \Delta Z \le 50$

$\$1700 \le Z \le \$1800$

$450 - .75 \times 200 \le x_1 \le 450 + .75 \times 200$

$100 - \frac{1}{2}(-200) \le x_2 \le 100 - \frac{1}{2}(+200)$

(g) Cost of purchasing 500 additional resistors $= 500 \times .40 = \$200$

$D_1 = 500$ resistors

Dual price of \$1.25 is valid in $-200 \le D_1 \le 200$. Thus, for the first 200 resistors alone, HiDec will get an additional revenue of $200 \times 1.25 = \$250$, which is more than the cost of all 500 resistors. Accept.

**12**

From Example 3.6-2, we have for the TOYCO model

$-200 \le D_1 \le 10$

$-20 \le D_2 \le 400$

$-20 \le D_3 < \infty$

(a) $D_1 = 8$, $D_2 = 40$, $D_3 = -10$

All $D_i$, $i = 1, 2, 3$ fall within the feasibility ranges. Thus

continued...

$r_1 = \frac{8}{10}$, $r_2 = \frac{40}{400}$, $r_3 = \frac{-10}{-20}$

$r_1 + r_2 + r_3 = .8 + .1 + .5 = 1.4 > 1$

Hence, no conclusion can be made about the feasibility of the new RHS (438, 500, 410). Problem 1(a) shows that these new values do pro-duce a feasible solution.

(b) $D_1 = 30$, $D_2 = -20$, $D_3 = -40$.

Because $D_1$ and $D_3$ fall outside the given feasibility ranges, the 100% rule cannot be applied in this case

**13**

(a) From TORA,

$x_1 = 2 + \frac{2}{3}D_1 - \frac{1}{3}D_2 \ge 0$

$x_2 = 2 - \frac{1}{3}D_1 + \frac{2}{3}D_2 \ge 0$

Feasibility ranges:

$-3 \le D_1 \le 6$

$-3 \le D_2 \le 6$

(b) $D_1 = D_2 = \Delta > 0$. Thus

$$\left.\begin{array}{l} x_1 = 2 + \Delta/3 > 0 \\ x_2 = 2 + \Delta/3 > 0 \end{array}\right\} \text{for all } \Delta > 0$$

100% rule for $0 < \Delta \le 3$:

$r_1 = r_2 = \frac{\Delta}{6} \le \frac{3}{6} \Rightarrow r_1 + r_2 < 1$, which confirms feasibility for $0 < D < 3$

100% rule for $3 < \Delta \le 6$:

$r_1 = r_2 = \frac{\Delta}{6} \Rightarrow \frac{3}{6} < r_1, r_2 \le \frac{6}{6}$

$r_1 + r_2 > 1 \Rightarrow$ cannot confirm feasibility.

100% rule for $\Delta > 6$:

$\Delta$ is outside $-3 \le D_1, D_2 \le 6$. Thus, the rule is not applicable.

**1**

From Section 3.6.3, we have the following optimality conditions for the TOYCO model:

$x_1$: $4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 \geq 0$

$x_4$: $1 + \frac{1}{2}d_2 \geq 0$

$x_5$: $2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 \geq 0$

(i) $Z = 2x_1 + x_2 + 4x_3$

$d_1 = 2-3 = -1$, $d_2 = 1-2 = -1$, $d_3 = 4-5 = -1$

$x_1$: $4 - \frac{1}{4}(-1) + \frac{3}{2}(-1) - (-1) = 3.75 > 0$

$x_4$: $1 + \frac{1}{2}(-1) = .5 > 0$

$x_5$: $2 - \frac{1}{4}(-1) + \frac{1}{2}(-1) = 1.75 > 0$

Conclusion: Solution is unchanged

(ii) $Z = 3x_1 + 6x_2 + x_3$

$d_1 = 3-3 = 0$, $d_2 = 6-2 = 4$, $d_3 = 1-5 = -4$

$x_1$: $4 - \frac{1}{4}(4) + \frac{3}{2}(4) - (0) = -3 < 0$

Conclusion: solution changes

(iii) $Z = 8x_1 + 3x_2 + 9x_3$

$d_1 = 8-3 = 5$, $d_2 = 3-2 = 1$, $d_3 = 9-5 = 4$

$x_1$: $4 - \frac{1}{4}(1) + \frac{3}{2}(4) - (5) = 4.75 > 0$

$x_4$: $1 + \frac{1}{2}(1) = 1.5 > 0$

$x_5$: $2 - \frac{1}{4}(1) + \frac{1}{2}(4) = 3.75 > 0$

Conclusion: Solution is unchanged

**2**

$x_1$ = Nbr. cans of A1

$x_2$ = Nbr. cans of A2

$x_3$ = Nbr. cans of BK

Maximize $Z = 80x_1 + 70x_2 + 60x_3$

s.t.

$x_1 + x_2 + x_3 \leq 500 \quad \Leftarrow S_1$

$x_1 \geq 100 \quad \Leftarrow S_2$

$4x_1 - 2x_2 - 2x_3 \leq 0 \quad \Leftarrow S_3$

$x_1, x_2, x_3 \geq 0$

TORA optimum tableau:

| Basic | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $S_2$ | $s_3$ | Solution |
|---|---|---|---|---|---|---|---|
| $z$ | 0 | 0 | 10 | 73.33 | 0 | 1.67 | 36666.67 |
| $x_2$ | 0 | 1 | 1 | .67 | 0 | -.17 | 333.33 |
| $x_1$ | 1 | 0 | 0 | .33 | 0 | .17 | 166.67 |
| $S_2$ | 0 | 0 | 0 | .33 | 1 | .17 | 66.67 |

continued...

(a) $Z = \$366.67$

$x_1 = 166.67$, $x_2 = 333.33$, $x_3 = 0$

(b) Reduced cost for $x_3$ = 10 cents. Price should be increased by more than 10 cents/can

(c) $d_1 = d_2 = d_3 = -5$ cents

From the optimum tableau, reduced costs:

$x_3$: $10 + d_2 - d_3 = 10 - 5 - (-5) = 10 > 0$

$S_1$: $73.33 + .67d_2 + .33d_1$
$= 73.33 + .67(-5) + .33(-5) = 68.33 > 0$

$S_3$: $1.67 - .17d_2 + .17d_1 = 1.67 - .17(-5) + .17(-5)$
$= 1.67 > 0$

Conclusion: Solution is unchanged.

**3**

(a) Available carpenter hours in a 10-day period $= 4 \times 10 \times 8 = 320$

$x_1$ = Nbr. chairs assembled in 10 days

$x_2$ = Nbr. tables assembled in 10 days

Maximize $Z = 50x_1 + 135x_2$

s.t.

$.5x_1 + 2x_2 \leq 320$

$4 \leq \frac{x_1}{x_2} \leq 6 \Rightarrow \begin{cases} x_1 - 4x_2 \geq 0 \\ x_1 - 6x_2 \leq 0 \end{cases}$

$x_1, x_2 \geq 0$

Solution: $Z = \$27{,}840$, $x_1 = 384$, $x_2 = 64$

(b) Optimum tableau:

| | $x_1$ | $x_2$ | $s_1$ | $S_2$ | $s_3$ | solution |
|---|---|---|---|---|---|---|
| $z$ | 0 | 0 | 87 | 0 | 6.5 | 27840 |
| $x_2$ | 0 | 1 | .2 | 0 | -.1 | 64 |
| $x_1$ | 1 | 0 | 1.2 | 0 | .4 | 384 |
| $S_2$ | 0 | 0 | .4 | 1 | .8 | 128 |

Optimality conditions:

$s_1$: $87 + 1.2d_1 + .2d_2 \geq 0$

$S_3$: $6.5 + .4d_1 - .1d_2 \geq 0$

For $d_1 = -5$, $d_2 = -13.5$:

$S_1$: $87 + 1.2(-5) + .2(-13.5) = 78.3 > 0$

$S_3$: $6.5 + .4(-5) - .1(-13.5) = 5.85 > 0$

Solution remains the same

(c) $d_1 = 25 - 50 = -25$, $d_2 = 120 - 135 = -15$

$S_1$: $87 + 1.2(-25) + .2(-15) = 58.5 > 0$

$S_3$: $6.5 + .4(-25) - .1(-15) = -2 < 0$

Solution changes

# Set 3.6d

**4**

(a) $x_1$ = Amt. of personal loan (\$)
$x_2$ = Amt. of car loan (\$)

Maximize $z = .14(x_1 - .03x_1) + .12(x_2 - .02x_2) - .03x_1 - .02x_2$
$= .1058x_1 + .0976x_2$

s.t.

$x_1 + x_2 \le 200{,}000$

$\frac{x_2}{x_1} \ge 2$ or $2x_1 - x_2 \le 0$

$x_1, x_2 \ge 0$

Solution: $z = \$20{,}067$
$x_1 = \$66{,}667$, $x_2 = \$133{,}333$

Rate of return $= \frac{20{,}067}{200{,}000} \times 100 = 10.03\%$

(b) Optimum tableau:

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | Solution |
|---|---|---|---|---|---|
| z | 0 | 0 | .1003 | .0027 | 20066.67 |
| $x_2$ | 0 | 1 | .6667 | -.3333 | 133333.33 |
| $x_1$ | 1 | 0 | .3333 | .3333 | 66666.67 |

Optimality conditions:
$s_1$: $.1003 + .333d_1 + .6667d_2 \ge 0$
$s_2$: $.0027 + .3333d_1 - .3333d_2 \ge 0$

New $x_1$-objective coefficient $= .14(1-.04) - .04$
$= .0944$
New $x_2$-objective coefficient $= .12(1-.03) - .03$
$= .0864$

$d_1 = .0944 - .1058 = -.0114$
$d_2 = .0864 - .0976 = -.0112$

$s_1$: $.1003 + .3333(-.0114) + .6667(-.0112)$
$= .08907 > 0$

$s_2$: $.0027 + .3333(-.0114) - .3333(-.0112)$
$= .00267 > 0$

Solution does not change

**5**

(a) $x_i$ = Nbr of units of motor $i$, $i = 1,2,3,4$

Maximize $z = 60x_1 + 40x_2 + 25x_3 + 30x_4$

s.t. $8x_1 + 5x_2 + 4x_3 + 6x_4 \le 8000$
$x_1 \le 500$, $x_2 \le 500$, $x_3 \le 800$, $x_4 \le 750$
$x_1, x_2, x_3, x_4 \ge 0$

Solution: $z = \$59{,}375$, $x_1 = 500$, $x_2 = 500$, $x_3 = 375$
$x_4 = 0$

continued...

(b) Optimality conditions (from TORA):
$x_4$: $7.5 + 1.5d_3 - d_4 \ge 0$
$s_1$: $6.25 + .25d_3 \ge 0$
$s_2$: $10 - 2d_3 + d_1 \ge 0$
$s_3$: $8.75 - 1.25d_3 + d_2 \ge 0$

From $s_3$, $8.75 + d_2 \ge 0 \Rightarrow -8.75 \le d_2 < \infty$

Thus, price of type 2 motor can be reduced by at most \$8.75 without causing a solution change.

(c) $d_1 = -15$, $d_2 = -10$, $d_3 = -6.25$, $d_4 = -7.5$
Solution remains the same because
$x_4$: $7.5 + 1.5(-6.25) - (-7.5) = 5.625 > 0$
$s_1$: $6.25 + .25(-6.25) = 4.6875 > 0$
$s_2$: $10 - 2(-6.25) + (-15) = 7.5 > 0$
$s_3$: $8.75 - 1.25(-6.25) + (-10) = 6.5625 > 0$

(d) Reduced cost for $x_4 = 7.5$. Increase price of type 4 motor by more than \$7.50.

**6**

(a) $x_1$ = Cases of juice/day
$x_2$ = Cases of sauce/day
$x_3$ = Cases of paste/day

Maximize $z = 21x_1 + 9x_2 + 12x_3$

s.t.
$(1 \times 24)x_1 + (\frac{1}{2} \times 24)x_2 + (\frac{3}{4} \times 24)x_3 \le 60{,}000$
$x_1 \le 2000$, $x_2 \le 5000$, $x_3 \le 6000$
$x_1, x_2, x_3 \ge 0$

Solution: $z = \$51{,}000$
$x_1 = 2000$, $x_2 = 1000$, $x_3 = 0$

(b) From TORA, optimality conditions given $d_2$:
$x_3$: $1.5 + 1.5d_2 \ge 0 \Rightarrow d_2 \ge -1$
$s_1$: $.75 + .083d_2 \ge 0 \Rightarrow d_2 \ge -9$
$s_2$: $3 - 2d_2 \ge 0 \Rightarrow d_2 \le 1.5$

Thus, $-1 \le d_2 \le 1.5$, or
$9 - 1 \le$ price/case of sauce $\le 9 + 1.5$

Solution mix remains the same if the price per case of sauce remains between \$8 and \$10.50.

**7**

(a) $x_1$ = Nbr. regular cabinets / day
$x_2$ = Nbr. deluxe cabinets / day

Maximize $z = 100x_1 + 140x_2$

s.t.

$$.5x_1 + x_2 \le 180$$
$$x_1 \le 200$$
$$x_2 \le 150$$
$$x_1, x_2 \ge 0$$

Solution: $z = \$31,200$
$x_1 = 200$ regular
$x_2 = 80$ deluxe

(b) From TORA, optimality conditions:

$s_1: 140 + d_2 \ge 0$
$s_2: 30 + d_1 - .5d_2 \ge 0$

$d_1 = 80 - 100 = -20$
$d_2 = 80 - 140 = -60$

$s_1: 140 + (-60) = 80 > 0$
$s_2: 30 + (-20) - .5(-60) = 40 > 0$

Solution remains the same

**8**

(a) For the original TOYCO model, TORA gives (also see Section 3.6.3)

$$-\infty < d_1 \le 4, \; -2 \le d_2 \le 8, \; -8/3 \le d_3 < \infty$$

(ii) Original $z = 3x_1 + 2x_2 + 5x_3$
New $z = 3x_1 + 6x_2 + x_3$

| $i$ | $d_i$ | $u_i$ | $v_i$ | $r_i$ |
|---|---|---|---|---|
| 1 | 0 | | 4 | $0/4 = 0$ |
| 2 | 4 | | 8 | $4/8 = 1/2$ |
| 3 | $-4$ | $-8/3$ | | $-4/-\frac{8}{3} = 3/2$ |

$$r_1 + r_2 + r_3 = 0 + 1/2 + 3/2 = 2 > 1$$

The 100% rule is nonconclusive in this case. The solution in Problem 1 (ii) shows that the solution will change

(iii) Original $z = 3x_1 + 2x_2 + 5x_3$
New $z = 8x_1 + 3x_2 + 9x_3$

| $i$ | $d_i$ | $u_i$ | $v_i$ | $r_i$ |
|---|---|---|---|---|
| 1 | 5 | | 4 | $5/4$ |
| 2 | 1 | | 8 | $1/8$ |
| 3 | 4 | | $\infty$ | $4/\infty = 0$ |

$$r_1 + r_2 + r_3 = \frac{5}{4} + \frac{1}{8} = \frac{11}{8} > 1$$

continued…

The 100% rule is nonconclusive. Yet Problem 1(iii) shows that the solution remains unchanged.

The two cases demonstrate that the 100% rule is too weak to be effective in decision making, and that it is more reliable to utilize the simultaneous optimality conditions given in Section 3.6.3.

(b) $-30 \le d_1 < \infty, \; -140 \le d_2 \le 60$

New $Z = 80x_1 + 80x_2$
Original $z = 100x_1 + 140x_2$

| $i$ | $d_i$ | $u_i$ | $v_i$ | $r_i$ |
|---|---|---|---|---|
| 1 | $-20$ | $-30$ | $\infty$ | $-20/-30 = 2/3$ |
| 2 | $-60$ | $-140$ | 60 | $-60/-140 = 3/7$ |

$$r_1 + r_2 = 2/3 + 3/7 = \frac{23}{21} > 1$$

The 100% rule is nonconclusive. Yet, Problem 7(b) shows that the solution remains unchanged.

# Set 3.6e

**1**

See file solver3.6e-1.xls in ch3Files

Dual prices for years 1, 2, 3, and 4 are 0, 0, 0, 2.89. Thus, for year 4, one (thousand) additional dollars increases Z by \$2.89 thousand. It is worthwhile to increase the funding for year 4.

**2**

See file tora3.6e-2.txt

| Constraint | Dual Price | Range |
|---|---|---|
| 1 | 5.36 | $(0, \infty)$ |
| 2 | −3.73 | $(-\infty, 6000)$ |
| 3 | −1.13 | $(-\infty, 6800)$ |
| 4 | −1.07 | $(-\infty, 33642)$ |
| 5 | −1.00 | $(-\infty, 53628.73)$ |

(a) Constraint 1: $x_1 + x_2 + x_4 + y_1 \le 10{,}000$

Dual price = \$5.36/invested \$

Rate of return = 536%

(b) Constraint 2: \$1000 spend on pleasure

$.5x_1 + .6x_2 - x_3 + .4x_4 + 1.065y_1 - y_2 = 1000$

Dual price = −3.73/pleasure \$

Range = $(-\infty, 6000)$

Spending \$1000 at end of year 1 reduces total return by \$3730.

**3**

See file tora3.6e-3.txt in ch3Files

| Quarter | Dual price | Range |
|---|---|---|
| 1 | 1.2488 | .6647, 2.5806 |
| 2 | 1.2443 | .6580, 2.6122 |
| 3 | 1.1945 | −.2646, 1.1245 |
| 4 | 1.0200 | −.2553, $\infty$ |
| 5 | 1.0000 | −4.8366, $\infty$ |

(a) An additional \$ available at the start of quarter 1 is worth \$1.24888 at the end of 4 quarters. Similarly, an additional dollar at the start of periods 2, 3, and 4 is worth \$1.2443, \$1.1945, and \$1.02, respectively. The dual price for quarter 4 (= \$1.02) shows that all we can do with the money then is to invest it at 2% for the quarter.

We can use the dual price to determine the rate of return for each quarter — namely,

<u>quarter 1</u>:

$1.2488 = 1.2243(1+i_1) \Rightarrow i_1 = .02$

<u>quarter 2</u>:

$1.2243 = 1.1945(1+i_2) \Rightarrow i_2 = .025$

<u>quarter 3</u>:

$1.1945 = 1.02(1+i_3) \Rightarrow i_3 = .171$

<u>quarter 4</u>:

$1.02 = 1.0(1+i_4) \Rightarrow i_4 = .02$

(b) The dual price associated with the upper bound on $B_3$ (UB-X10) is \$.149. It represents the networth per dollar borrowed in period 3. Also, an extra dollar in period 3 is worth \$1.1945 at the end of the horizon. However, if that dollar is borrowed, it must be repaid as \$1.025 in the next quarter. The repayment is equivalent to forgoing making 2% in interest. Thus, the networth of borrowing in period 3 is

$1.1945 - 1.025 \times 1.02 = .149$

This result is consistent with the dual price for the upper bound on $B_3$

continued...

**4**

| Constraint | Current RHS | Min RHS | Max RHS | Dual Price |
|---|---|---|---|---|
| 1 (=) | 2.0000 | 0.0000 | infinity | 2.1756 |
| 2 (=) | 2.0000 | -0.1667 | infinity | 2.0173 |
| 3 (=) | 2.5000 | -0.3472 | infinity | 1.8647 |
| 4 (=) | 2.5000 | -0.5767 | infinity | 1.7296 |
| 5 (=) | 3.0000 | -0.8248 | infinity | 1.6044 |
| 6 (=) | 3.5000 | -1.1331 | infinity | 1.4356 |
| 7 (=) | 3.5000 | -6.1137 | infinity | 1.3355 |
| 8 (=) | 4.0000 | -11.4678 | infinity | 1.2423 |
| 9 (=) | 4.0000 | -20.6663 | infinity | 1.1556 |
| 10 (=) | 5.0000 | -32.5201 | infinity | 1.0750 |

The dual price provides the worth per additional \$ at the end of year 10.

Annual rate of return:

Period 1: $2.1756 = 2.0173(1+i_1) \Rightarrow i_1 = .0785$

Period 2: $2.0173 = 1.8647(1+i_2) \Rightarrow i_2 = .0818$

Period 3: $1.8647 = 1.7296(1+i_3) \Rightarrow i_3 = .0781$

Period 4: $1.7296 = 1.6044(1+i_4) \Rightarrow i_4 = .0780$

etc...

**5**

See file tora3.6e-5.txt in ch3Files

The dual price for constraint 1

$$x_{1A} + x_{1B} \le 100{,}000$$

is \$5.10. Thus, each invested \$ is worth \$5.10 at the end of the investment horizon. Range $(0, \infty)$

**6**

Dual price for the constraint

$$x_1 + x_2 + x_3 + x_4 \le 500$$

is \$2.35 per \$ invested, range $(0, \infty)$

The gambler should bet the largest amount possible.

**7**

See file tora3.6e-7.txt in ch3Files

For, $x_{W1} + x_{W2} + x_{W3} \ge 1500$, the dual price is \$11.4, range $(800, \infty)$

One extra wrench automatically requires the production of two chisels, thus leading to the following changes:

Cost of one wrench using subcont. = \$3.00

Cost of 2 chisels using subcont. = 2x \$4.20

total = \$11.40

$x_{W1} \le 550$, dual price = -\$1, range $(-\infty, 1250)$. If regular time capacity for wrenches is increased by 1 unit, one less wrench will be produced by subcontractor, which saves \$3 - \$2 = \$1.

Similar interpretations can be given for the remaining dual prices

**8**

See file tora3.6e-8.txt in ch3Files

| Machine | Capacity | Dual price | Range |
|---|---|---|---|
| 1 | 500 | 2 | (253.33, 570) |
| 2 | 380 | 12 | (333.33, 750) |

The company should pay less than \$2/hr for machine 1 and less than \$12/hr for machine 2.

**9**

See file tora3.6e-9.txt in ch3Files

(a) Constraint $2x_1 + 3x_2 + 5x_3 \le 4000$ corresponds to raw material A. Its dual price is \$10.27/lb. For a purchase price of \$12/lb, acquisition of additional raw material A is not recommended.

(b) Constraint $4x_1 + 2x_2 + 7x_3 \le 6000$ is associated with raw material B. Its dual price is \$0/lb. Resource B is already abundant. Thus, no additional purchase is recommended.

**10**

(a) See file tora3.6e-10.txt

| Constraint | Dual price |
|---|---|
| 1 | 0 |
| 2 | 0 |
| 3 | -400 |
| 4 | -750 |
| 5 | 0 |
| 6 | 0 |
| 7 | 0 |

Constraints 3 and 4 have negative dual price. These correspond respectively to the third specification for alloy A and the first specification for alloy B. Changes in these specifications affects profit adversely

(b) For the ore constraints, the dual prices are \$90, \$110, and \$30 per additional ton of ores 1, 2, and 3, respectively. These are the maximum prices the company should pay.

# CHAPTER 5

## Transportation Model and its Variants

# Set 5.1a

**1**

(a) False
(b) True
(c) True

**2**

(a) $\sum a_i = 25$, $\sum b_j = 31$
Add a dummy source whose supply amount is $31-25=6$ units

(b) $\sum a_i = 74$, $\sum b_j = 65$
add a dummy destination whose demand amount is $74-65=9$ units

**3**

Denver will be 150 cars short. Similarly, Miami will be 50 cars short of satisfying its demand

**4**

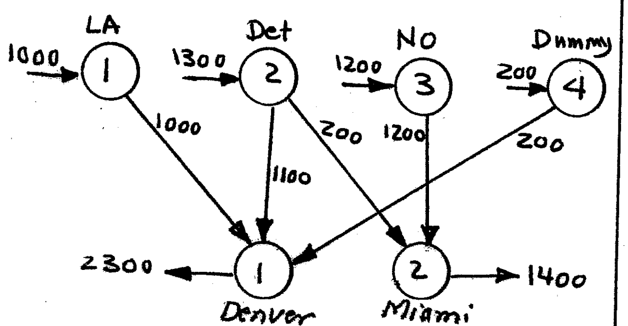
assign a very high cost M to the route from Detroit to Dummy

**5**

| | | Den 1 | Miami 2 | |
|---|---|---|---|---|
| LA | 1 | (80) 1000 | (M) | 1000 |
| Det | 2 | (100) 1100 | (108) 200 | 1300 |
| NO | 3 | (102) | (68) 1200 | 1200 |
| Dummy | 4 | (200) 200 | (300) | 200 |
| | | 2300 | 1400 | |

use M = \$1000 in TORA

Optimum solution from TORA



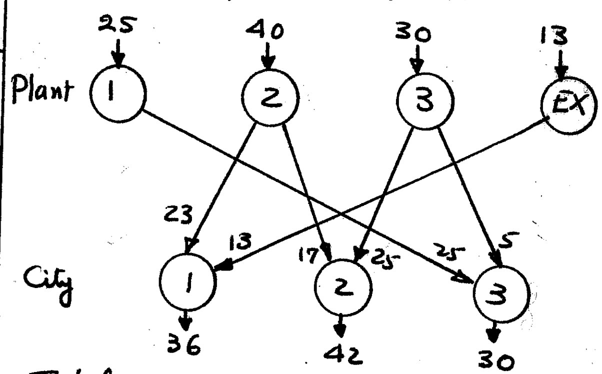

Denver is 200 cars short, Cost = \$33,200

**6**

(a)

| | City 1 | 2 | 3 | |
|---|---|---|---|---|
| Plant 1 | (600) | (700) | (400) 25 | 25 |
| Plant 2 | (320) 23 | (300) 17 | (350) | 40 |
| Plant 3 | (500) | (480) 25 | (450) 5 | 30 |
| Excess plant 4 | (1000) 13 | (1000) | (M) | 13 |
| | 36 | 42 | 30 | |

(b) M = \$10,000 in TORA




Total cost = \$49,710

(c) City 1 excess cost = $13 \times 1000$
= \$13,000

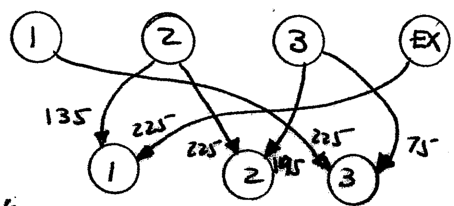
**7**

Assume units in 100,000 kWh

(a)

| | city 1 | 2 | 3 | |
|---|---|---|---|---|
| Plant 1 | (60) | (70) | (40) 225 | 225 |
| 2 | (32) 135 | (30) 225 | (35) | 360 |
| 3 | (50) | (48) 195 | (45) 75 | 270 |
| EX | (100) 225 | (100) | (M) | 225 |
| | 360 | 420 | 300 | |

(b)




(c) City 1 excess cost = \$22,500

Optimum cost = \$55,305

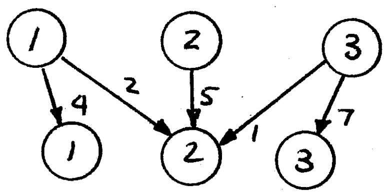
**8**

Unit transportation cost in thousand \$ per million gallons $= \left(\left(\frac{10¢}{1000} \times 10^6 \times \text{mileage}\right) \times \frac{1}{100}\right) \times \frac{1}{1000}$

$= \frac{\text{mileage}}{10}$

Distribution Area

| | 1 | 2 | 3 | |
|---|---|---|---|---|
| Ref. 1 | 4 (12) | 2 (18) | (M) | 6 |
| 2 | (30) | 5 (10) | (8) | 5 |
| 3 | (20) | 1 (25) | 7 (12) | 8 |
| | 4 | 8 | 7 | |




Total cost = \$243,000

**9**

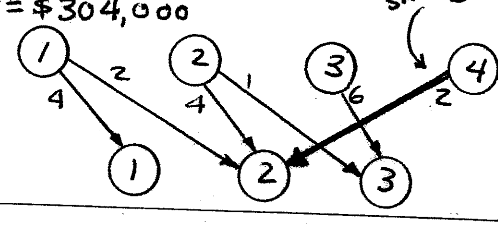
Unit cost in thousand \$ from Dummy source to distribution areas 2 or 3

$= \frac{5¢}{100} \times \frac{10^6}{10^3} = 50$ thousand \$/million gal

Distribution area

| | 1 | 2 | 3 | |
|---|---|---|---|---|
| Ref. 1 | 4 (12) | 2 (18) | (M) | 6 |
| 2 | (30) | 4 (10) | 1 (8) | 5 |
| 3 | (20) | (25) | 6 (12) | 6 |
| Dummy | (M) | 2 (50) | (50) | 2 |
| | 4 | 8 | 7 | |

Cost = \$304,000




**10**

Unit costs in thousand \$ per million gallons:

from refinery 1 to Dummy

$= \frac{\$1.50}{100} \times \frac{10^6}{10^3} = 15$

from refinery 2 to Dummy

$= \frac{\$2.20}{100} \times \frac{10^6}{10^3} = 22$

| | 1 | 2 | 3 | Dummy | |
|---|---|---|---|---|---|
| Ref. 1 | 4 (12) | 2 (18) | (M) | (15) | 6 |
| 2 | (30) | 5 (10) | (8) | (22) | 5 |
| 3 | (20) | 1 (25) | 4 (12) | 3 (0) | 8 |
| | 4 | 8 | 4 | 3 | |

Refinery 3 diverts 3 million gallons for use within.

Total cost = \$207,000

**11**

(a) Total supply = 150+200+250 = 600 crates

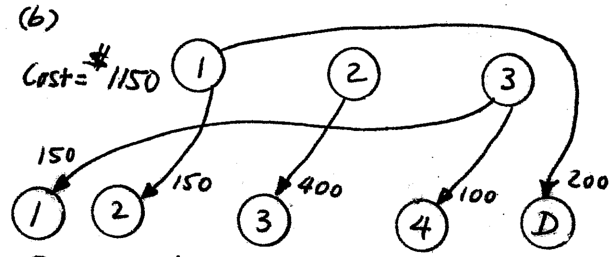
Total demand = 150+150+400+100 = 800 crates

$\left(\begin{array}{l}\text{Potential overtime supply}\\ \text{by each of orchards 1 \& 2}\end{array}\right) = 800 - 600 = 200$ crates

| | 1 | 2 | 3 | 4 | Dummy | |
|---|---|---|---|---|---|---|
| Orch 1 | (1) | 150 (2) | (3) | (2) | 200 (0) | 150+200 |
| 2 | (2) | (4) | 400 (1) | (2) | (0) | 200+200 |
| 3 | 150 (1) | (3) | (5) | 100 (3) | (M) | 250 |
| | 150 | 150 | 400 | 100 | 200 | |

(b)

Cost = \$1150




Problem has alternative optima.

(c) Orchad 1 = 0 overtime crates

Orchad 2 = 200 overtime crates

# Set 5.1a

**12**

Supply/demand quantities are expressed in truckloads, determined by dividing the number of cars by 18 and rounding the result up, if necessary. For example, supply amount at center 1 is $\frac{400}{18} = 22.22$ or 23 truckloads. Expressing unit transportation costs in \$1000 per truck load, we get

| | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| 1 | 2.5 (6) | 3.75 | 5 | 3.5 (9) | .875 (8) | 23 |
| 2 | 1.25 | 1.75 (3) | 1.5 (9) | 1.625 | 2 | 12 |
| 3 | 1 | 2.25 (9) | 2.5 | 3.75 | 3.25 | 9 |
| | 6 | 12 | 9 | 9 | 8 | |

(b) alternative solution exists

Cost = \$92,500

**13**

| | | N.O. M2 | N.O. M1 | Det M4 | Det M2 | Det M1 | L.A. M4 | L.A. M3 |
|---|---|---|---|---|---|---|---|---|
| Denver M1 | 630 | | 130 (102) | | | 500 (100) | | |
| Denver M2 | 480 | 102 | | | 450 (100) | | | |
| Denver M3 | 400 | | | | | | | 400 (80) |
| Denver M4 | 480 | | | 180 (100) | | | 300 (80) | |
| Denver M1/2 | 120 | 102 | 50 (102) | | 70 (100) | 100 | | |
| Denver M3/4 | 220 | | | 100 (100) | | | 80 | 120 (80) |
| Miami M1 | 540 | | 540 (68) | | | 108 | | |
| Miami M2 | 475 | 400 (68) | | | 75 (108) | | | |
| Miami M3 | 180 | | | | | | | 180 (215) |
| Miami M4 | 95 | | | 95 (108) | | | 215 | |
| Miami M1/3 | 80 | | 80 (68) | | | 108 | | 215 |
| Miami M2/4 | 30 | 68 | | 25 (108) | 5 (108) | | 215 | |
| | | 400 | 800 | 400 | 600 | 500 | 300 | 700 |

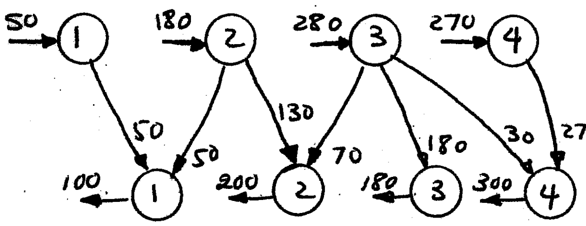
Optimum solution:

LA-Denver M4 = 300 cars
Det.-Denver M1 = 500 cars
Det.-Denver M2 = 450 cars
Det.-Denver M1/M2 = 70 cars
Det.-Miami M2 = 75 cars
Det.-Miami M2/4 = 5 cars
Det.-Denver M4 = 180 cars
Det.-Denver M3/4 = 100 cars
Det.-Miami M4 = 95 cars
Det.-Miami M2/4 = 25 cars
N.O.-Denver M1 = 130 cars
N.O.-Denver M1/2 = 50 cars
N.O.-Miami M1 = 540 cars
N.O.-Miami M1/3 = 80 cars
N.O.-Miami M2 = 400 cars

Total cost = \$343,620

## 1

| | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| 1 | 40 (50) | 40.4 | 40.7 | 41.4 | 50 |
| 2 | 42 (50) | 40 (130) | 40.3 | 41 | 180 |
| 3 | 44 | 42 (70) | 40 (180) | 40.7 (30) | 280 |
| 4 | 46 | 44 | 42 | 40 (270) | 270 |
| | 100 | 200 | 180 | 300 | |




Cost = $31,461

## 2

Least-cost starting solution.
(Problem has alternative optima.)

| | Mon | Tue | Wed | Thu | Fri | Sat | Sun | Disposal | |
|---|---|---|---|---|---|---|---|---|---|
| New | 12 (24) | 12 (12) | 12 (2) | 12 | 12 | 12 | 12 | 0 (86) | 124 |
| Mon | | 6 | 6 | 3 (6) | 1 (18) | 1 | 1 | 0 | 24 |
| Tue | | | 6 (12) | 6 | 3 | 1 | 1 | 0 | 12 |
| Wed | | | | 6 (14) | 6 | 3 | 1 | 0 | 14 |
| Thu | | | | | 6 | 6 | 3 (20) | 0 | 20 |
| Fri | | | | | | 6 (14) | 6 | 0 (4) | 18 |
| Sat | | | | | | | 6 (2) | 0 (12) | 14 |
| Sun | | | | | | | | 0 (22) | 22 |
| | 24 | 12 | 14 | 20 | 18 | 14 | 22 | 124 | |

The given optimum solution is interpreted as summarized below.

Total cost = $804

continued…

| Day | New | Sharpening Service: Overnite | 2-day | 3-day | Disposal |
|---|---|---|---|---|---|
| Mon | 24 | 0 | 6 | 18 | 0 |
| Tue | 12 | 12 | 0 | 0 | 0 |
| Wed | 2 | 14 | 0 | 0 | 0 |
| Thu | 0 | 0 | 20 | 0 | 0 |
| Fri | 0 | 14 | 0 | 0 | 4 |
| Sat | 0 | 2 | 0 | 0 | 12 |
| Sun | 0 | 0 | 0 | 0 | 22 |

## 3

| | Mon | Tue | Wed | Thu | Fri | Sat | Sun | Disposal | |
|---|---|---|---|---|---|---|---|---|---|
| New | 12 (24) | 12 | 12 (8) | 12 | 12 | 12 | 12 | 0 (92) | 124 |
| Mon | | 6 | 6.5 | 3 (12) | 3.5 | 4 | 4.5 | 0 | 24 |
| Tue | | | 6 (6) | 6.5 | 3 (6) | 3.5 | 4 | 0 | 12 |
| Wed | | | | 6 (8) | 6.5 | 3 (6) | 3.5 | 0 | 14 |
| Thu | | | | | 6 (12) | 6.5 | 3 (8) | 0 | 20 |
| Fri | | | | | | 6 (8) | 6.5 | 0 (10) | 18 |
| Sat | | | | | | | 6 (14) | 0 | 14 |
| Sun | | | | | | | | 0 (22) | 22 |
| | 24 | 12 | 14 | 20 | 18 | 14 | 22 | 124 | |

| Day | New | Sharpening Service: Overnight | 2-day | Disposal |
|---|---|---|---|---|
| Mon | 24 | 12 | 12 | 0 |
| Tue | 0 | 6 | 6 | 0 |
| Wed | 8 | 8 | 6 | 0 |
| Thu | 0 | 12 | 8 | 0 |
| Fri | 0 | 8 | 0 | 10 |
| Sat | 0 | 14 | 0 | 0 |
| Sun | 0 | 0 | 0 | 22 |

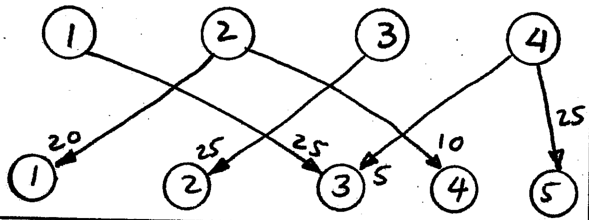
Total cost = $840
alternative solution exists

## Set 5.2a

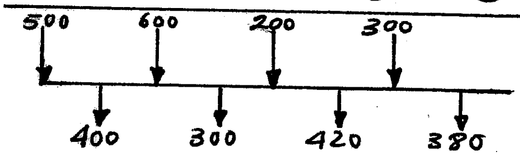
### 4

| Machine \ Task | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| 1 | 10 | 2 | 3 (25) | 15 | 9 | 25 |
| 2 | 5 (20) | 10 | 15 | 2 (10) | 4 | 30 |
| 3 | 15 | 5 (20) | 14 | 7 | 15 | 20 |
| 4 | 20 | 15 | 13 (5) | M | 8 (25) | 30 |
| | 20 | 20 | 30 | 10 | 25 | |

Total cost = $560




### 5



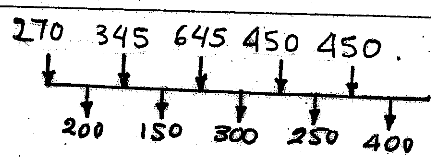

| | | | | |
|---|---|---|---|---|
| c: | $100 | $140 | $120 | $150 |
| h: | $3 | $3 | $3 | $3 |

| | 1 | 2 | 3 | 4 | Surplus | |
|---|---|---|---|---|---|---|
| 1 | 100 (400) | 103 (100) | 106 | M | 0 | 500 |
| 2 | M | 140 (200) | 143 (220) | 146 (180) | 0 | 600 |
| 3 | M | M | 120 (200) | 123 | 0 | 200 |
| 4 | M | M | M | 150 (200) | 0 (100) | 300 |
| | 400 | 300 | 420 | 380 | 100 | |

Cost = $190,040, alternative solution exists

| Period | Capacity | Amt Prod. | Delivery |
|---|---|---|---|
| 1 | 500 | 500 | 400 for 1<br>100 for 2 |
| 2 | 600 | 600 | 200 for 2<br>220 for 3<br>180 for 4 |
| 3 | 200 | 200 | 200 for 3 |
| 4 | 300 | 200 | 200 for 4 |

### 6




| | 1 | 2 | 3 | 4 | 5 | Surplus | |
|---|---|---|---|---|---|---|---|
| $R_1$ | 100 (180) | 104 | 108 | 112 | 116 | 0 | 180 |
| $O_1$ | 150 (20) | 154 | 158 | 162 | 166 | 0 (70) | 90 |
| $R_2$ | | 96 (150) | 100 (80) | 104 | 108 | 0 | 230 |
| $O_2$ | | 144 | 148 | 152 | 156 | 0 (115) | 115 |
| $R_3$ | | | 116 (220) | 120 | 124 (50) | 0 (160) | 430 |
| $O_3$ | | | 174 | 178 | 182 | 0 (215) | 215 |
| $R_4$ | | | | 102 (250) | 106 (50) | 0 | 300 |
| $O_4$ | | | | 153 | 157 | 0 (150) | 150 |
| $R_5$ | | | | | 106 (300) | 0 | 300 |
| $O_5$ | | | | | 159 | 0 (150) | 150 |
| | 200 | 150 | 300 | 250 | 400 | 860 | |

Cost = $137,720
alternative solution exists.

| Period | Production schedule |
|---|---|
| 1 | Regular – 180 engines<br>Overtime – 20 engines |
| 2 | Regular: 230 engines |
| 3 | Regular 270 engines |
| 4 | Regular 300 engines |
| 5 | Regular 300 engines |

**7**

| | 1 | 2 | 3 | 4 | 5 | 6 | Disposal | |
|---|---|---|---|---|---|---|---|---|
| New | 200 (200) | 210 (180) | 220.5 (140) | 231.53 | 243.1 | 255.26 | 0 (878) | 1398 |
| 1 | | 120 | 121.5 (12) | 35 (188) | 36.5 | 38 | 0 | 200 |
| 2 | | | 120 (148) | 121.5 | 35 (32) | 36.5 | 0 | 180 |
| 3 | | | | 120 (10) | 121.5 | 35 (290) | 0 | 300 |
| 4 | | | | | 120 (198) | 121.5 | 0 | 198 |
| 5 | | | | | | 120 | 0 (230) | 230 |
| 6 | | | | | | | 0 (290) | 290 |
| | 200 | 180 | 300 | 198 | 230 | 290 | 1398 | |

Cost = \$170,698

alternative solution exists

| Month | New | Overhaul 1-day | Overhaul 3-day | Disposal |
|---|---|---|---|---|
| 1 | 200 | 12 | 188 | 0 |
| 2 | 180 | 148 | 32 | 0 |
| 3 | 140 | 10 | 290 | 0 |
| 4 | 0 | 198 | 0 | 0 |
| 5 | 0 | 0 | 0 | 230 |
| 6 | 0 | 0 | 0 | 290 |

**8**

(a) Use negative cost values

| Loc \ Bidder | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| 1 | −520 | M | −650 **10** | −180 | **10** |
| 2 | −210 | −510 **20** | M | −430 | **20** |
| 3 | −570 | −495 | −240 | −710 **30** | **30** |
| Dummy | 0 **30** | 0 **10** | 0 **20** | 0 | **60** |
| | **30** | **30** | **30** | **30** | |

(b) Bidder 1 = 0 acre

Bidder 2 = 20 acres (location 1)

Bidder 3 = 10 acres (location 2)

Bidder 4 = 30 acres (location 3)

# Set 5.3a

(a)

Northwest:

Cost = \$42

| | | | |
|---|---|---|---|
| 5 (0) | 1 (2) | (1) | 6 |
| (2) | 4 (1) | 3 (5) | 7 |
| (2) | (4) | 7 (3) | 7 |
| 5 | 5 | 10 | |

Least-cost:

Cost = \$37

| | | | |
|---|---|---|---|
| 5 (0) | (2) | 1 (1) | 6 |
| (2) | 5 (1) | 2 (5) | 7 |
| (2) | (4) | 7 (3) | 7 |
| 5 | 5 | 10 | |

Vogel:

Cost = \$37

| | | | | Penalty | Penalty |
|---|---|---|---|---|---|
| 5 (0) | (2) | 1 (1) | 6 | 1 | 1 |
| (2) | 5 (1) | (5) | 7 | 1 | 4 |
| (2) | (4) | 7 (3) | 7 | 1 | 1 |
| 5 | 5 | 10 | | | |

| | | | | |
|---|---|---|---|---|
| Penalty | 2 | 1 | 2 | ← Step 1 |
| Penalty | – | 1 | 2 | ← Step 2 |

(b)

Northwest:

Cost = \$94

| | | | |
|---|---|---|---|
| 7 (1) | (2) | (6) | 7 |
| 3 (0) | 9 (4) | (2) | 12 |
| (3) | 1 (1) | 10 (5) | 11 |
| 10 | 10 | 10 | |

Least-Cost:

Cost = \$61

| | | | |
|---|---|---|---|
| (1) | (2) | 7 (6) | 7 |
| 10 (0) | (4) | 2 (2) | 12 |
| (3) | 10 (1) | 1 (5) | 11 |
| 10 | 10 | 10 | |

VAM:

Cost = \$40

| | | | Penalties | | |
|---|---|---|---|---|---|
| 7 (1) | (2) | (6) | 1 | 1 | 1 |
| 2 (0) | (4) | 10 (2) | 2 | 4 | – |
| 1 (3) | 10 (1) | (5) | 2 | 2 | 2 |

| | | | | |
|---|---|---|---|---|
| Penalties | 1 | 1 | 3 | ← Step 1 |
| | 2 | 1 | – | ← Step 2 |
| | 2 | 1 | – | ← Step 3 |

(c)

Northwest:

Cost = \$104

| | | | |
|---|---|---|---|
| 9 (5) | 3 (1) | (8) | 12 |
| (2) | 7 (4) | 7 (0) | 14 |
| (3) | (6) | 4 (7) | 4 |
| 9 | 10 | 11 | |

Least-Cost

Cost = \$38

| | | | |
|---|---|---|---|
| 2 (5) | 10 (1) | (8) | 12 |
| 3 (2) | (4) | 11 (0) | 14 |
| 4 (3) | (6) | (7) | 4 |
| 9 | 10 | 11 | |

VAM:

Cost = \$38

| | | | | Penalties | |
|---|---|---|---|---|---|
| 2 (5) | 10 (1) | (8) | 12 | 4 | 4 |
| 3 (2) | (4) | 11 (0) | 14 | 2 | 2 |
| 4 (3) | (6) | (7) | 4 | 3 | 3 |
| 9 | 10 | 11 | | | |

| | | | | |
|---|---|---|---|---|
| Penalties | 1 | 3 | 7 | ← Step 1 |
| | 1 | 3 | – | ← Step 2 |

continued…

continued…

(i)

| u \ v | 0 | 2 | 6 | |
|---|---|---|---|---|
| 0 | c=0: 5 | c=2: 1 − | c=1: 5 + | 6 |
| −1 | c=2: −3 | c=1: 4 + | c=5: 5 − | 9 |
| −3 | c=2: −5 | c=4: −5 | c=3: 5 | 5 |
| | 5 | 5 | 10 | |

| u \ v | 0 | −3 | 1 | |
|---|---|---|---|---|
| 0 | c=0: 5 − | c=2: −5 | c=1: 1 + | 6 |
| 4 | c=2: 2 + | c=1: 5 | c=5: 4 − | 9 |
| 7 | c=2: 0 | c=4: −5 | c=3: 5 | 5 |
| | 5 | 5 | 10 | |

| u \ v | 0 | −1 | 1 | |
|---|---|---|---|---|
| 0 | c=0: 1 | c=2: −3 | c=1: 5 | 6 |
| 2 | c=2: 4 | c=1: 5 | c=5: −2 | 9 |
| 2 | c=2: 0 | c=4: −3 | c=3: 5 | 5 |
| | 5 | 5 | 10 | |

cost = \$33

alternative solution exists

(ii)

| u \ v | 0 | 4 | 2 | |
|---|---|---|---|---|
| 0 | c=0: 7 | c=4: 1 | c=2: 0 | 8 |
| −1 | c=2: −3 | c=3: 5 | c=4: −3 | 5 |
| −2 | c=1: −3 | c=2: 0 | c=0: 6 | 6 |
| | 7 | 6 | 6 | |

Problem has alternative optima. Cost = \$19

Note: If $x_{23}$ were selected as the zero in place of $x_{32}$, solution would require one more iteration.

continued...

(iii)

| u \ v | M | M−3 | M−5 | |
|---|---|---|---|---|
| 0 | c=M: 4 − | c=3: M−6 + | c=5: M−10 | 4 |
| 7−M | c=7: 1 + | c=4: 6 − | c=9: −7 | 7 |
| 11−M | c=1: | c=8: 0 | c=6: 19 | 19 |
| | 5 | 6 | 19 | |

| u \ v | 6 | 3 | 1 | |
|---|---|---|---|---|
| 0 | c=M: 6−M | c=3: 4 | c=5: −4 | 4 |
| 1 | c=7: 5 − | c=4: 2 + | c=9: −7 | 7 |
| 5 | c=1: 10 + | c=8: 0 − | c=6: 19 | 19 |
| | 5 | 6 | 19 | |

| u \ v | 6 | 3 | 11 | |
|---|---|---|---|---|
| 0 | c=M: 6−M | c=3: 4 − | c=5: 6 + | 4 |
| 1 | c=7: 5 − | c=4: 2 + | c=9: 3 | 7 |
| −5 | c=1: 0 + | c=8: −10 | c=6: 19 − | 19 |
| | 5 | 6 | 19 | |

| u \ v | 0 | −3 | 5 | |
|---|---|---|---|---|
| 0 | c=M: −M | c=3: −6 | c=5: 4 | 4 |
| 7 | c=7: 1 − | c=4: 6 | c=9: 3 + | 7 |
| 1 | c=1: 4 + | c=8: −10 | c=6: 15 − | 19 |
| | 5 | 6 | 19 | |

| u \ v | 0 | 0 | 5 | |
|---|---|---|---|---|
| | c=M: −M | c=3: −3 | c=5: 4 | 4 |
| | c=7: −3 | c=4: 6 | c=9: 1 | 7 |
| | c=1: 5 | c=8: −7 | c=6: 14 | 19 |
| | 5 | 6 | 19 | |

Cost = \$142

continued...

## Set 5.3b

(c)

| Method | Nbr. of iterations (i) | (ii) | (iii) |
|---|---|---|---|
| NW | 3 | 4 | 5 |
| Least cost | 2 | 2 | 2 |
| Vogel | 2 | 1 | 1 |

Least-cost starting solution: **2**

| u \ v | 2 | 1 | 2 | |
|---|---|---|---|---|
| | [5] −3 | (10) [1] | [7] −5 | 10 |
| | (70) [6] − | [4] 1 + | (10) [6] | 80 |
| | (5) [3] + | (10) [2] − | [5] −2 | 15 |
| | [5] −3 | [3] −2 | (40) [2] | 40 |
| | 75 | 20 | 50 | |

| u \ v | 3 | 1 | 3 | |
|---|---|---|---|---|
| 0 | [5] −2 | (10) [1] | [7] −4 | 10 |
| 3 | (60) [6] | (10) [4] | (10) [6] | 80 |
| 0 | (15) [3] | [2] −1 | [5] −2 | 15 |
| −1 | [5] −3 | [3] −3 | (40) [2] | 40 |
| | 75 | 20 | 50 | |

Destination 3 will be 40 units short. Optimum cost = $595

Least-cost starting solution: **3**

| u \ v | 2 | 1 | 1 | |
|---|---|---|---|---|
| 0 | [5] −3 | 10 [1] | [7] −6 | 10 |
| 4 | 30 [6] | [4] 1 | 50 [6] | 80 |
| 1 | 5 [3] | 10 [2] | [5] −3 | 15 |
| −2 | 40 [0] | [0] −1 | [M] −1−M | 40 |
| | 75 | 20 | 50 | |

| u \ v | 3 | 1 | 3 | |
|---|---|---|---|---|
| 0 | [5] −2 | 10 [1] | [7] −4 | 10 |
| 3 | 20 [6] | 10 [4] | 50 [6] | 80 |
| 0 | 15 [3] | [2] −1 | [5] −2 | 15 |
| −3 | 40 [0] | [0] −2 | [M] −M | 40 |
| | 75 | 20 | 50 | |

Total cost = $515. Dest. 1 is 40 units short.

Vogel method: **4**

| | | | | |
|---|---|---|---|---|
| [1] | [2] | [1] | [5] | 0 |
| [3] | [4] | [5] | [M] | 1 |
| [2] | [3] | [3] | 20 [3] | 1 |
| 1 | 1 | 2 | 2 | |

| | | | | |
|---|---|---|---|---|
| [1] | [2] | 20 [1] | | 0 |
| [3] | [4] | [5] | | 1 |
| [2] | [3] | [3] | | 1 |
| 1 | 1 | 2 | | |

| | | | | |
|---|---|---|---|---|
| | | | | |
| [3] | [4] | [5] | | 1 |
| [2] | [3] | 0 [3] | | 1 |
| 1 | 1 | 2 | | |

| | | | | |
|---|---|---|---|---|
| | | | | |
| 20 [3] | 20 [4] | | | 1 |
| 10 [2] | [3] | | | 1 |
| 1 | 1 | | | |

| u \ v | 0 | 1 | 1 | 1 | |
|---|---|---|---|---|---|
| 0 | $-1$ ${}^{1}$ | $-1$ ${}^{2}$ | **20** ${}^{1}$ | $-2$ ${}^{3}$ | **20** |
| 3 | **20** ${}^{3}$ | **20** ${}^{4}$ | $-1$ ${}^{5}$ | $4-M$ ${}^{M}$ | **40** |
| 2 | **10** ${}^{2}$ | 0 ${}^{3}$ | **0** ${}^{3}$ | **20** ${}^{3}$ | **30** |
| | **30** | **20** | **20** | **20** | |

Cost = \$240 — alternative solution exists

**5**

| u \ v | 2 | 5 | 10 | |
|---|---|---|---|---|
| −2 | 15 $c_{11}$ | $c_{12}$ | $c_{13}$ | 15 |
| 3 | 5 $c_{21}$ | 25 $c_{22}$ | $c_{23}$ | 30 |
| 5 | $c_{31}$ | 5 $c_{32}$ | 80 $c_{33}$ | 85 |
| | 20 | 30 | 80 | |

(a) $c_{ij} = u_i + v_j$ for basic $x_{ij}$

Thus,

$c_{11} = 2-2 = 0$

$c_{21} = 3+2 = 5$

$c_{22} = 3+5 = 8$

$c_{32} = 5+5 = 10$

$c_{33} = 5+10 = 15$

Cost $= 15\times 0 + 5\times 5 + 25\times 8 + 5\times 10 + 80\times 15 = \$1475$

(b) $u_i + v_j - c_{ij} \le 0$ for nonbasic $x_{ij}$

$-2+5-c_{12} \le 0 \implies c_{12} \ge 3$

$-2+10-c_{13} \le 0 \implies c_{13} \ge 8$

$3+10-c_{23} \le 0 \implies c_{23} \ge 13$

$5+2-c_{31} \le 0 \implies c_{31} \ge 7$

continued...

Problems 6 and 7 on next page

continued...

**6**

(a) For basic $x_{ij}$, $c_{ij} = u_i + v_j$.

| u \ v | 2 | 2 | 5 | |
|---|---|---|---|---|
| 1 | $c_{11}=3$ (10) | $1+2\theta$ | $1+3\theta$ | 10 |
| -1 | $2+\theta$ | $c_{22}=1$ (20) | $c_{23}=4$ (20) | 40 |
| | 10 | 20 | 20 | |

$$\text{Cost} = 3\times 10 + 1\times 20 + 4\times 20 = \$130$$

(b) For nonbasic $x_{ij}$: $u_i + v_j - c_{ij} \le 0$ to satisfy optimality. Hence

$$2+1-(1+2\theta) \le 0 \Rightarrow \theta \ge 1$$

$$5+1-(1+3\theta) \le 0 \Rightarrow \theta \ge 5/3$$

$$2-1-(2+\theta) \le 0 \Rightarrow \theta \ge -1$$

Take $\theta = \frac{5}{3}$ to yield $x_{13} = 0$ as the zero basic variable.

**7**

| | $x_{11}$ | $x_{12}$ | $x_{13}$ | $x_{21}$ | $x_{22}$ | $x_{23}$ | |
|---|---|---|---|---|---|---|---|
| min Z= | 1 | 1 | 2 | 6 | 5 | 1 | |
| s.t. | | | | | | | |
| | 1 | 1 | 1 | | | | $\ge 5$ |
| | | | | 1 | 1 | 1 | $\ge 6$ |
| | 1 | | | 1 | | | $\ge 2$ |
| | | 1 | | | 1 | | $\ge 7$ |
| | | | 1 | | | 1 | $\ge 1$ |

$x_{ij} \ge 0$ for all $i$ and $j$

Optimum LP solution using TORA:

$$Z = 15,\ x_{11} = 2,\ x_{12} = 7,\ x_{23} = 6$$

If we replace the first two constraints with <u>equations</u>, we get the optimum solution:

$$Z = 27,\ x_{11} = 2,\ x_{12} = 3,$$
$$x_{22} = 4,\ x_{23} = 2$$

The new solution is worse!

**1**

| | $u_1$ | $u_2$ | $u_3$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | |
|---|---|---|---|---|---|---|---|---|
| max | 15 | 25 | 10 | 5 | 15 | 15 | 15 | |
| s.t. | | | | | | | | |
| | 1 | | | 1 | | | | $\le 10$ |
| | 1 | | | | 1 | | | $\le 2$ |
| | 1 | | | | | 1 | | $\le 20$ |
| | 1 | | | | | | 1 | $\le 11$ |
| | | 1 | | 1 | | | | $\le 12$ |
| | | 1 | | | 1 | | | $\le 7$ |
| | | 1 | | | | 1 | | $\le 9$ |
| | | 1 | | | | | 1 | $\le 20$ |
| | | | 1 | 1 | | | | $\le 4$ |
| | | | 1 | | 1 | | | $\le 14$ |
| | | | 1 | | | 1 | | $\le 16$ |
| | | | 1 | | | | 1 | $\le 18$ |

From Table 5-25:

$u_1 = 0,\ u_2 = 5,\ u_3 = 7$

$v_1 = -3,\ v_2 = 2,\ v_3 = 4,\ v_4 = 11$

$$\text{Optimum } w = 15\times 0 + 25\times 5 + 10\times 7 + 5\times -3 + 15\times 2 + 15\times 4 + 15\times 11 = \$435$$

**2**

$$\text{minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n$$

Next, consider

$$z' = \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij} + K)\, x_{ij}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + K\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + K\sum_{i=1}^{m} a_i$$

continued...

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + K, \quad K \text{ is a constant}$$

$$= z + K$$

This result shows that optimization using $z$ and $z'$ yield the same optimum values of $x_{ij}$.

To show why the dual values associated with a given primal basic solution are not unique, note that, for any constant $K$,

$$\begin{pmatrix}\text{Dual} \\ \text{Values}\end{pmatrix} = \begin{pmatrix}\text{Original basic} \\ \text{obj. coefficients} \\ + \\ K\end{pmatrix} \times \text{Inverse}$$

This means that even though the optimal primal solution is unique for all $K$, there are infinity of dual values, each corresponding to a given value of $K$.

The conclusion is that an arbitrary value assigned to one of the dual variables (e.g., $u_1 = 0$) implies a specific value for the constant $K$.

## Set 5.4a

**1**

(a-i)

| | | | | | Row min |
|---|---|---|---|---|---|
| 3 | 8 | 2 | 10 | 3 | 2 |
| 8 | 7 | 2 | 9 | 7 | 2 |
| 6 | 4 | 2 | 7 | 5 | 2 |
| 8 | 4 | 2 | 3 | 5 | 2 |
| 9 | 10 | 6 | 9 | 10 | 6 |

| | | | | |
|---|---|---|---|---|
| 1 | 6 | 0 | 8 | 1 |
| 6 | 5 | 0 | 7 | 5 |
| 4 | 2 | 0 | 5 | 3 |
| 6 | 2 | 0 | 1 | 3 |
| 3 | 4 | 0 | 3 | 4 |
| Col min → 1 | 2 | 0 | 1 | 1 |

Assignment:
1-5
2-3
3-2
4-4
5-1
Cost = $21

| | | | | |
|---|---|---|---|---|
| 0 | 4 | 2 | 7 | $\boxed{0}$ |
| 3 | 1 | $\boxed{0}$ | 4 | 2 |
| 3 | $\boxed{0}$ | 2 | 4 | 2 |
| 5 | 0 | 2 | $\boxed{0}$ | 2 |
| $\boxed{0}$ | 0 | 0 | 0 | 1 |

(a-ii)

| | | | | | |
|---|---|---|---|---|---|
| 3 | 9 | 2 | 2 | 7 | 2 |
| 6 | 1 | 5 | 6 | 6 | 1 |
| 9 | 4 | 7 | 10 | 3 | 3 |
| 2 | 5 | 4 | 2 | 1 | 1 |
| 9 | 6 | 2 | 4 | 6 | 2 |

| | | | | |
|---|---|---|---|---|
| 1 | 7 | 0 | 1 | 5 |
| 5 | 0 | 4 | 5 | 5 |
| 6 | 1 | 4 | 7 | 0 |
| 1 | 4 | 3 | 1 | 0 |
| 7 | 4 | 0 | 2 | 4 |
| Col min 1 | 0 | 0 | 1 | 0 |

continued…

| | | | | |
|---|---|---|---|---|
| $\boxed{0}$ | 7 | 0 | 0 | 5 |
| 4 | $\boxed{0}$ | 4 | 4 | 5 |
| 5 | 1 | 4 | 6 | $\boxed{0}$ |
| 0 | 4 | 3 | $\boxed{0}$ | 0 |
| 6 | 4 | $\boxed{0}$ | 1 | 4 |

Optimum:
1-1
2-2
3-5
4-4
5-3
Cost = $11

**2**

| | | | |
|---|---|---|---|
| 5 | 5 | M | 2 |
| 7 | 4 | 2 | 3 |
| 9 | 3 | 5 | M |
| 7 | 2 | 6 | 7 |

| | | | |
|---|---|---|---|
| 3 | 3 | M-2 | 0 |
| 5 | 2 | 0 | 1 |
| 6 | 0 | 2 | M-3 |
| 5 | 0 | 4 | 5 |

(All entries are divided by 10 for convenience)

| | | | |
|---|---|---|---|
| 0 | 3 | M-2 | 0 |
| 2 | 2 | 0 | 1 |
| 3 | 0 | 2 | M-3 |
| 2 | 0 | 4 | 5 |

| | | | |
|---|---|---|---|
| 0 | 5 | M-2 | $\boxed{0}$ |
| 2 | 4 | $\boxed{0}$ | 1 |
| 1 | $\boxed{0}$ | 0 | M-5 |
| $\boxed{0}$ | 0 | 4 | 5 |

Optimum: 1-4, 2-3, 3-2, 4-1
Cost = $140

**3**

| Worker \ Job | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 50 | 50 | M | 20 | 0 |
| 2 | 70 | 40 | 20 | 30 | 0 |
| 3 | 90 | 30 | 50 | M | 0 |
| 4 | 70 | 20 | 60 | 70 | 0 |
| 5 | 60 | 45 | 30 | 80 | 0 |

Job 5 is dummy

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 30 | M-20 | 0 | 0 |
| 2 | 20 | 20 | 0 | 10 | 0 |
| 3 | 40 | 10 | 30 | M-20 | 0 |
| 4 | 20 | 0 | 40 | 50 | 0 |
| 5 | 10 | 25 | 10 | 60 | 0 |

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 30 | M-20 | $\boxed{0}$ | 10 |
| 2 | 20 | 20 | $\boxed{0}$ | 10 | 10 |
| 3 | 30 | 0 | 20 | M-30 | $\boxed{0}$ |
| 4 | 20 | $\boxed{0}$ | 40 | 50 | 10 |
| 5 | $\boxed{0}$ | 15 | 0 | 50 | 0 |

Optimum:
1-4
2-3
3-5
4-2
5-1

Worker 3 is assigned to dummy job 5. Thus, worker 5 must replace worker 3.

**4**

Add a "dummy" operator with zero assignment cost to each job (including the fifth). The optimal solution will show the replacement by indicating which of the current jobs (1 thru 4) is assigned to the dummy operator. If the dummy operator is assigned to the new job, then the new job must assume lower priority to the current four jobs.

(all assignment cost are divided by 10 for convenience.)

| Operator \ Job | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 5 | 5 | M | 2 | 2 |
| 2 | 7 | 4 | 2 | 3 | 1 |
| 3 | 9 | 3 | 5 | M | 2 |
| 4 | 7 | 2 | 6 | 7 | 8 |
| 5 | 0 | 0 | 0 | 0 | 0 |

← Dummy

| | | | | |
|---|---|---|---|---|
| 3 | 3 | M-3 | 0 | 0 |
| 6 | 3 | 1 | 2 | 0 |
| 7 | 1 | 3 | M-2 | 0 |
| 5 | 0 | 4 | 5 | 0 |
| 0 | 0 | 0 | 0 | 0 |

| | | | | |
|---|---|---|---|---|
| 2 | 2 | M-4 | $\boxed{0}$ | 0 |
| 5 | 2 | $\boxed{0}$ | 2 | 0 |
| 6 | 0 | 2 | M-2 | $\boxed{0}$ |
| 5 | $\boxed{0}$ | 4 | 6 | 7 |
| $\boxed{0}$ | 0 | 0 | 1 | 0 |

Optimum:
1 – 4
2 – 3
3 – 5
4 – 2
5 – 1

Since dummy operator is assigned to job 1, new job 5 has higher priority over job 1.

**5**

Define the following two sets:

Set 1: (DA,3), (DA,10), (DA,17), (DA,25)

continued...

Set 2: (AT,7), (AT,12), (AT,21), (AT,28).

The idea is to match one element from Set 1 with another element from Set 2. The matching automatically decides the date and location for the purchase of each ticket. For example, consider the following assignment:

(DA,3) – (AT,21)
(DA,10) – (AT,7)
(DA,17) – (AT,28)
(DA,25) – (AT,12)

This assignment can be interpreted as follows:

| | | |
|---|---|---|
| Ticket 1: | June 3 | DA → AT |
| | June 21 | AT → DA |
| Ticket 2: | June 7 | AT → DA |
| | June 10 | DA → AT |
| Ticket 3: | June 17 | DA → AT |
| | June 28 | AT → DA |
| Ticket 4: | June 12 | AT → DA |
| | June 25 | DA → AT |

The complete assignment model is given below

| | A,7 | A,12 | A,21 | A,28 |
|---|---|---|---|---|
| D,3 | 400 | 300 | 300 | 280 |
| D,10 | 300 | 400 | 300 | 300 |
| D,17 | 300 | 300 | 400 | 300 |
| D,25 | 300 | 300 | 300 | 400 |

Optimum:

(D,3) – (A,28)   (A,21) – (D,25)
(A,7) – (D,10)   (A,12) – (D,17)

Problem has alternative optima.

**6**

Distance matrix in meters:

$$\begin{array}{cc|cccc|}
 & & \multicolumn{4}{c}{\text{candidate areas}} \\
 & & a & b & c & d \\
 & 1 & 50 & 50 & 95 & 45 \\
\text{existing} & 2 & 30 & 30 & 55 & 65 \\
\text{centers} & 3 & 70 & 50 & 25 & 55 \\
 & 4 & 100 & 60 & 55 & 25
\end{array}$$

A measure of the optimal assignment of new centers to candidate locations must reflect both distance and frequency of trips; that is

$$\begin{array}{cc}
 & \begin{array}{cccc} & \text{existing} & & \\ 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} \text{I} \\ \text{II} \\ \text{new III} \\ \text{IV} \end{array} & \begin{bmatrix} 10 & 7 & 0 & 11 \\ 2 & 1 & 8 & 4 \\ 4 & 9 & 6 & 0 \\ 3 & 5 & 2 & 7 \end{bmatrix}
\end{array}
\begin{array}{c}
\begin{array}{cccc} & \text{candidate} & & \\ a & b & c & d \end{array} \\
\begin{bmatrix} 50 & 50 & 95 & 45 \\ 30 & 30 & 55 & 65 \\ 70 & 50 & 25 & 55 \\ 100 & 60 & 55 & 25 \end{bmatrix}
\end{array}$$

$$\begin{array}{ccc}
 & & \begin{array}{cccc} a & b & c & d \end{array} \\
 & \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} &
\begin{bmatrix} 1810 & 1370 & 1940 & 1180 \\ 1090 & 770 & 665 & 695 \\ 890 & 770 & 1025 & 1095 \\ 1140 & 820 & 995 & 745 \end{bmatrix}
\end{array}$$

New

TORA optimum assignment:

I - d

II - c

III - a

IV - b

**7**

The ranking of the projects by the different teams can use the following numeric score

1 : Highest preference

1 : lowest preference

A tie in preference between two or more projects is indicated by assigning the projects the same score. For example, the scores

| Project | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Score | 9 | 9 | 8 | 7 | 3 | 5 | 4 | 1 | 2 | 6 |

indicate that project 8 is the most preferred and projects 1 and 2 tie for the least preferred status.

For the development of the model, we use the following numeric designations for the projects

| Project nbr. | Project name |
|---|---|
| 1 | Boeing-F15 |
| 2 | Boeing-F18 |
| 3 | Boeing Simulation |
| 4 | Cargil |
| 5 | Cobb-Vantress |
| 6 | ConAgra |
| 7 | Cooper |
| 8 | DaySpring (layout) |
| 9 | DaySpring (Materials) |
| 10 | JB Hunt |
| 11 | Raytheon |
| 12 | Tyson South |
| 13 | Tyson East |
| 14 | WAL-MART |
| 15 | Yellow |

continued…

The following is a typical summary of preference scores submitted by the 11 teams:

| Project | Team 1* | 2 | 3 | 4 | 5 | 6* | 7* | 8 | 9* | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1⊗ | — | 1 | 2 | 2 | 1 | — | — | 1 | — | 2 | 15 |
| 2⊗ | — | 1 | 3 | 1 | 2 | — | — | 1 | — | 10 | 12 |
| 3 | 1 | 2 | 5 | 3 | 2 | 13 | 5 | 1 | 4 | 15 | 1 |
| 4 | 2 | 3 | 6 | 4 | 10 | 5 | 14 | 2 | 1 | 4 | 14 |
| 5 | 3 | 5 | 4 | 5 | 9 | 4 | 12 | 3 | 3 | 13 | 13 |
| 6 | 3 | 4 | 2 | 5 | 9 | 8 | 12 | 1 | 2 | 1 | 13 |
| 7 | 4 | 6 | 1 | 12 | 8 | 9 | 10 | 2 | 5 | 2 | 5 |
| 8 | 5 | 6 | 7 | 14 | 7 | 9 | 10 | 4 | 6 | 3 | 15 |
| 9 | 7 | 8 | 9 | 14 | 7 | 1 | 1 | 15 | 1 | 15 | 1 |
| 10 | 7 | 9 | 12 | 15 | 6 | 3 | 9 | 5 | 4 | 7 | 5 |
| 11⊗ | — | 9 | 13 | 6 | 5 | — | — | 7 | — | 6 | 7 |
| 12 | 13 | 10 | 14 | 7 | 4 | 2 | 8 | 9 | 15 | 4 | 9 |
| 13 | 14 | 11 | 1 | 8 | 3 | 13 | 7 | 8 | 1 | 8 | 9 |
| 14 | 15 | 12 | 5 | 9 | 1 | 14 | 7 | 6 | 2 | 9 | 10 |
| 15 | 15 | 13 | 7 | 10 | 2 | 15 | 6 | 1 | 3 | 1 | 11 |

* Team does not meet citizenship requirements

⊗ project requiring US citizenship

The problem is modeled as an assignment model. Entries — are replaced by M, a very large value. The model is unbalanced. Thus, 4 artificial teams must be added to balance the model. In the end four projects will not be assigned.

TORA solution:

| Project | Team | Score |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 4 | 1 |
| 3 | 11 | 1 |
| 4 | 1 | 1 |
| 5 | None | — |
| 6 | 8 | 1 |
| 7 | 3 | 1 |
| 8 | None | — |
| 9 | 7 | 1 |
| 10 | None | — |
| 11 | None | — |
| 12 | 6 | 2 |
| 13 | 10 | 1 |
| 14 | 5 | 1 |
| 15 | 10 | 1 |
| | Total score | 13 |

continued....

$$\text{Average score} = \frac{13}{11} = 1.18$$

The average score is close to 1, meaning that all preferences are well met.

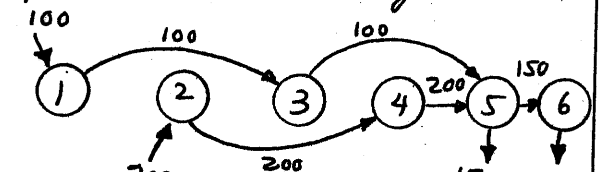
# Set 5.5a

## 1

| | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|
| 1 | 1 (100) | 4 | M | M | 100 |
| 2 | 3 | 2 (200) | M | M | 200 |
| 3 | 0 (200) | 1 | 6 (100) | M | B |
| 4 | 3 | 0 (100) | 5 (200) | 8 | B |
| 5 | M | M | 0 (150) | 1 (150) | B |
| | B | B | 150+B | 150 | |

Let B = 300 units

Optimum solution using TORA:



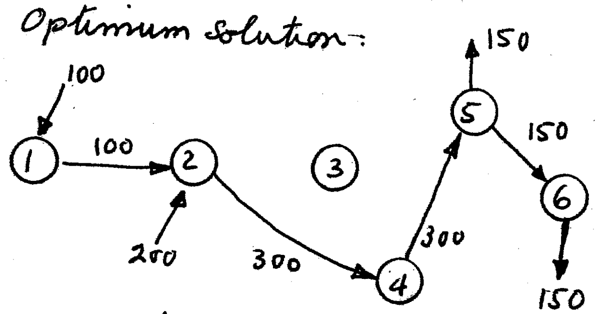

Cost = $2250

## 2

B = 300 units

| | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|
| 1 | 1 (100) | 5 | 4 | M | M | 100 |
| 2 | 0 (200) | 3 (0) | 2 (300) | M | M | 200+B |
| 3 | M | 0 (300) | 1 | 6 (0) | M | B |
| 4 | M | 3 | 0 | 5 (300) | 8 | B |
| 5 | M | M | M | 0 (150) | 1 (150) | B |
| | B | B | B | 150+B | 150 | |

Optimum solution:

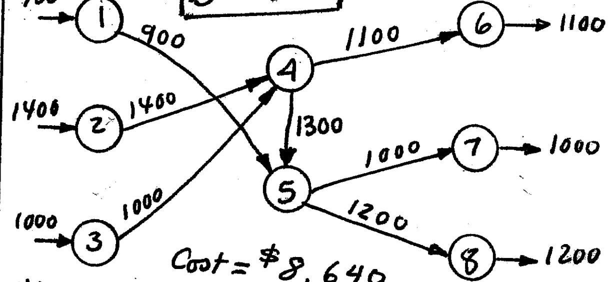



Cost = $2,350

## 3

| | 4 | 5 | 6 | 7 | 8 | |
|---|---|---|---|---|---|---|
| 1 | 1 | .3 (900) | M | M | M | 900 |
| 2 | .8 (1400) | 4.3 | M | M | M | 1400 |
| 3 | 2 (1000) | 4.6 | M | M | M | 1000 |
| 4 | 0 (900) | .5 (1300) | .2 (1100) | 4.5 | 6 | B |
| 5 | M | 0 (1100) | 3 | 2.1 (1000) | 1.9 (1200) | B |
| | B | B | 1100 | 1000 | 1200 | |

$\boxed{B = 3300}$

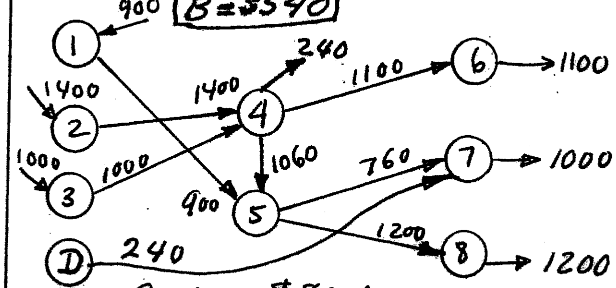



Cost = $8,640

(b)

| | 4 | 5 | 6 | 7 | 8 | |
|---|---|---|---|---|---|---|
| 1 | 1 | .3 (900) | M | M | M | 900 |
| 2 | .8 (1400) | 4.3 | M | M | M | 1400 |
| 3 | 2 (1000) | 4.6 | M | M | M | 1000 |
| 4 | 0 (1380) | .5 (1060) | .2 (1100) | 4.5 | 6 | B |
| 5 | M | 0 (1580) | 3 | 2.1 (760) | 1.9 (1200) | B |
| Dummy | 0 | 0 | 0 | 0 (240) | 0 | 240 |
| | 240+B | B | 1100 | 1000 | 1200 | |

$\boxed{B = 3540}$




Cost = $8,016

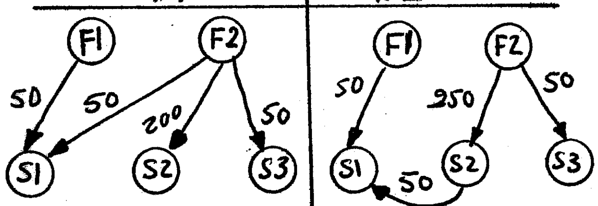
**4**

| | F1 | F2 | S1 | S2 | S3 | Dum | |
|---|---|---|---|---|---|---|---|
| F1 | 0 (500) | 6 | 7 (50) | 8 | 9 | 0 (150) | 200 + B |
| F2 | 6 | 0 (500) | 5 (50) | 4 (200) | 3 (50) | 0 | 300 + B |
| S1 | 7 | 2 | 0 (500) | 5 | 1 | 0 | B |
| S2 | 1 | 5 | 1 | 0 (500) | 4 | 0 | B |
| S3 | 8 | 9 | 7 | 6 | 0 (500) | 0 | B |
| | B | B | 100 + B | 200 + B | 50 + B | 150 | |

$\boxed{B = 500}$

Alternative optima using TORA:

#1 #2

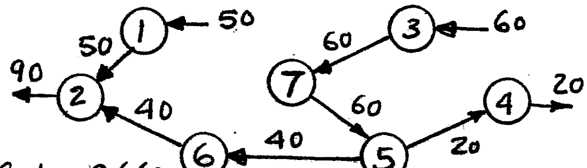



Cost = \$1,550

**5**

assume that units of supply and demand are in thousand gallons. $\boxed{B = 110}$

| | 2 | 4 | 5 | 6 | 7 | |
|---|---|---|---|---|---|---|
| 1 | 20 (50) | M | M | M | 3 | 50 |
| 3 | M | 30 | M | M | 9 (60) | 60 |
| 5 | M | 2 (20) | 0 (50) | 4 (40) | 10 | B |
| 6 | 8 (40) | M | 4 | 0 (70) | M | B |
| 7 | 40 | M | 10 (60) | M | 0 (50) | B |
| | 90 | 20 | B | B | B | |




Cost = 2,660,000 gallon miles

**6**

| | 2 | 3 | 4 | 5 | 6 | 7 | |
|---|---|---|---|---|---|---|---|
| 1 | 5 | 3 (1) | M | M | M | M | 1 |
| 2 | 0 (1) | 4 | 1 | 7 | M | M | 0+1 |
| 3 | 6 | 0 | 5 | 1 | 2 (1) | M | 0+1 |
| 4 | M | M | 0 (1) | 9 | M | 4 | 0+1 |
| 5 | M | M | 2 | 0 (1) | 5 | 8 | 0+1 |
| 6 | M | 3 | M | 7 | 0 | 3 (1) | 0+1 |
| | 0+1 | 0+1 | 0+1 | 0+1 | 0+1 | 1 | |

Optimum route using TORA:

$1 \rightarrow 3 \rightarrow 6 \rightarrow 7$

Distance $= 3 + 2 + 3 = 8$

**7**

minimize z.

| | $x_{13}$ | $x_{14}$ | $x_{23}$ | $x_{24}$ | $x_{34}$ | $x_{35}$ | $x_{36}$ | $x_{46}$ | $x_{47}$ | $x_{56}$ | $x_{67}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| z= | 3 | 4 | 2 | 5 | 7 | 8 | 6 | 4 | 9 | 5 | 3 | |
| (1) | 1 | 1 | | | | | | | | | | = 1000 |
| (2) | | | 1 | 1 | | | | | | | | = 1200 |
| (3) | -1 | | -1 | | 1 | 1 | 1 | | | | | = 0 |
| (4) | | -1 | | -1 | -1 | | | 1 | 1 | | | = 0 |
| (5) | | | | | | -1 | | | | 1 | | = -800 |
| (6) | | | | | | | -1 | -1 | | -1 | 1 | = -900 |
| (7) | | | | | | | | | -1 | | -1 | = -500 |

Each node yields a constraint. The special characteristics of the model show that each column has exactly +1 and -1, with the remainder of the elements equal to zero.

**8**

$x_{ij}$ = number of laborers hired at the start of period $i$ and terminated at the start of period $j$.

Define nodes 1, 2, 3, 4, and 5 to correspond to the five months of the horizon. Node 6 is added to allow defining the variables $x_{i6}$ that terminate at the end of the five-month planning horizon. The associated LP is defined below.

| | $x_{12}$ | $x_{13}$ | $x_{14}$ | $x_{15}$ | $x_{16}$ | $x_{23}$ | $x_{24}$ | $x_{25}$ | $x_{26}$ | $x_{34}$ | $x_{35}$ | $x_{36}$ | $x_{45}$ | $x_{46}$ | $x_{56}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 100 | 130 | 180 | 220 | 250 | 100 | 130 | 180 | 220 | 100 | 130 | 180 | 100 | 130 | 100 | min |
| (1) | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | $\geq 100$ |
| (2) | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | $\geq 120$ |
| (3) | | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | | | | $\geq 80$ |
| (4) | | | | 1 | 1 | | | 1 | 1 | | 1 | 1 | 1 | 1 | | $\geq 170$ |
| (5) | | | | | 1 | | | | 1 | | | 1 | | 1 | 1 | $\geq 50$ |

Let $S_1, S_2, S_3, S_4$, and $S_5$ be the surplus variables associated with constraints 1, 2, 3, 4, and 5, respectively. The LP after adding the surplus variables thus appears as

| $x_{12}$ | $x_{13}$ | $x_{14}$ | $x_{15}$ | $x_{16}$ | $x_{23}$ | $x_{24}$ | $x_{25}$ | $x_{26}$ | $x_{34}$ | $x_{35}$ | $x_{36}$ | $x_{45}$ | $x_{46}$ | $x_{56}$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 130 | 180 | 220 | 250 | 100 | 120 | 180 | 220 | 100 | 130 | 180 | 100 | 130 | 100 | | | | | | min |
| 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | −1 | | | | | 100 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | | −1 | | | | 120 |
| | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | −1 | | | 80 |
| | | | 1 | 1 | | | 1 | 1 | | 1 | 1 | 1 | 1 | | | | | −1 | | 170 |
| | | | | 1 | | | | 1 | | | 1 | | 1 | 1 | | | | | −1 | 50 |

Next, perform the following transformation:

1. Leave equation (1) unchanged.
2. Replace equation(2) with (2) − (1).
3. Replace equation (3) with (3) − (2).
4. Replace equation (4) with (4) − (3).
5. Replace equation (5) with (5) − (4)
6. Add a new equation that equals −(5).

These transformations lead to the following LP

| $x_{12}$ | $x_{13}$ | $x_{14}$ | $x_{15}$ | $x_{16}$ | $x_{23}$ | $x_{24}$ | $x_{25}$ | $x_{26}$ | $x_{34}$ | $x_{35}$ | $x_{36}$ | $x_{45}$ | $x_{46}$ | $x_{56}$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 130 | 180 | 220 | 250 | 100 | 130 | 180 | 220 | 100 | 130 | 180 | 100 | 130 | 100 | | | | | | min |
| 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | −1 | | | | | 100 |
| −1 | | | | | 1 | 1 | 1 | 1 | | | | | | | 1 | −1 | | | | 20 |
| | −1 | | | | −1 | | | | 1 | 1 | 1 | | | | | 1 | −1 | | | −40 |
| | | −1 | | | | −1 | | | −1 | | | 1 | 1 | | | | 1 | −1 | | 90 |
| | | | −1 | | | | −1 | | | −1 | | −1 | | 1 | | | | 1 | −1 | −120 |
| | | | | −1 | | | | −1 | | | −1 | | −1 | −1 | | | | | 1 | −50 |

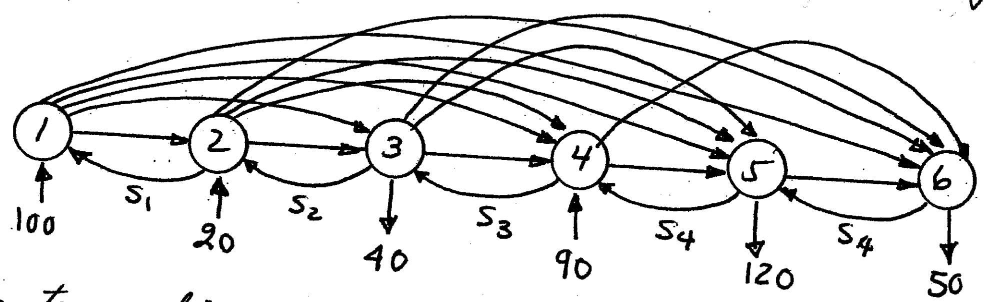
continued...

The last LP has the structure of a transshipment model (see Problem 7). Let

$S_1 = x_{21}$ $\quad S_3 = x_{43}$ $\quad S_5 = x_{65}$

$S_2 = x_{32}$ $\quad S_4 = x_{54}$

Then the LP above can be translated as a network as follows:

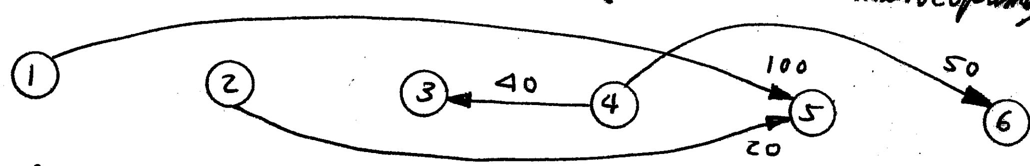



The transshipment model thus appears as

| | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 100 | 130 | 180 | 220 | 250 | 100 + B |
| 2 | 0 | 0 | 100 | 130 | 180 | 220 | 20 + B |
| 3 | M | 0 | 0 | 100 | 130 | 180 | B |
| 4 | M | M | 0 | 0 | 100 | 130 | 90 + B |
| 5 | M | M | M | 0 | 0 | 100 | B |
| 6 | M | M | M | M | 0 | 0 | B |
| | B | B | 40 + B | B | 120 + B | 50 + B | |

$B = 550$

The optimum solution from TORA is (Problem has alternative optima)




This solution can be interpreted as follows

1. Hire 100 laborers at the start of period 1 and terminate them at the start of period 5.
2. Hire 20 workers at the start of period 2 and terminate them at the start of period 5.
3. Hire 50 workers at the start of period 4 and terminate them at the start of period 6.

The solution satisfies the labor requirements exactly, except for period 3 where there is a surplus of 40 workers ($x_{43} = 40$).

# CHAPTER 11

# Deterministic Inventory Models

**1**

$y^* = \sqrt{\frac{2KD}{h}}$, $t_0 = \frac{y^*}{D}$, $TCU(y^*) = \sqrt{2KDh}$

a) $y^* = \sqrt{\frac{2\times100\times30}{.05}} = 346.4$ units

$t_0 = \frac{346.4}{30} = 11.55$ days

$TCU(y^*) = \frac{100\times30}{346.4} + \frac{.05\times346.4}{2} = \$17.32$

Policy: order 346.4 units whenever inventory drops to 207.2 units

Effective lead time = 6.91 days

b) $y^* = \sqrt{\frac{2\times50\times30}{.05}} \cong 245$ units

$t_0^* = \frac{245}{30} = 8.16$ days

$L_e = 5.51$ days

$TCU(y^*) = \frac{50\times30}{245} + \frac{.05\times245}{2} = \$12.25$

Policy: order 245 units whenever inventory drops to 165.15 units

c) $y^* = \sqrt{\frac{2\times100\times40}{.01}} = 894.4$ units

$t_0 = \frac{894.4}{40} = 22.36$ days

$L_e = 7.64$ days

$TCU(y^*) = \frac{100\times40}{894.4} + \frac{.01\times894.4}{2} = \$8.94$

Policy: Order 894.4 units whenever inventory drops to 305.57 units.

d) $y^* = \sqrt{\frac{2\times100\times20}{.04}} = 316.23$ units

$t_0^* = \frac{316.23}{20} = 15.81$ days

$L_e = 14.19$ days

$TCU(y^*) = \frac{100\times20}{316.23} + \frac{.04\times316.23}{2} = \$12.65$

Policy: Order 316.23 units whenever inventory drops to 283.8 units.

**2**

$D = 300$ lb/wk, $K = \$20$, $h = \$.03$/lb/day

(a) $TC/wk = \frac{KD}{y} + \frac{hy}{2}$

$= \frac{20\times300}{300} + \frac{7\times.03\times300}{2} = \$51.50$

(b) $y^* = \sqrt{\frac{2\times20\times300}{(.03\times7)}} = 239$ lb

$t_0^* = \frac{239}{300/7} = .8$ wk

$TC/wk = \sqrt{2\times20\times300\times.03\times7}$

$= \$50.20$

continued...

$L_e = 0$ days

Policy: Order 239 lb whenever inventory drops to zero level.

c) Cost difference $= 51.50 - 50.20$

$= \$1.30$

**3**

a) $h = \frac{.35}{7} = \$.05$/unit/day

$D = 50$ units/day, $K = \$20$

$y^* = \sqrt{\frac{2\times20\times50}{.05}} = 200$ units

$t_0 = \frac{200}{50} = 4$ days

$L = 7$ days, $L_e = 3$ days

$R = 3\times50 = 150$ units

Policy: Order 200 units whenever inventory drops to 150 units.

b) Optimum number of orders $= \frac{365}{4}$

$\cong 91$ orders

**4**

(a) Policy 1: $D = \frac{R}{L_e} = \frac{50}{10} = 5$ units/day

$Cost/day = \frac{KD}{y} + \frac{hy}{2}$

$= \frac{20\times5}{150} + \frac{.02\times150}{2} = \$2.17$

Policy 2: $D = \frac{75}{15} = 5$ units/day

$Cost/day = \frac{20\times5}{200} + \frac{.02\times200}{2} = \$2.50$

choose policy 1.

(b) $K = \$20$, $D = 5$ units/day

$h = \$.02$, $L = 22$ days

$y^* = \sqrt{\frac{2\times20\times5}{.02}} = 100$ units

$t_0 = \frac{100}{5} = 20$ days

$L_e = 22 - 20 = 2$ days

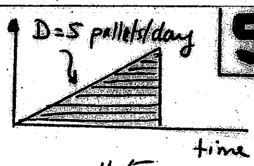
Reorder level $= 2\times5 = 10$ units

Order 100 units whenever the level drops to 10 units

$Cost/day = \frac{20\times5}{100} + \frac{.02\times100}{2} = \$2.00$

**5**

$D = 5$ units/day
$h = \$.10$/day
$K = \$100$

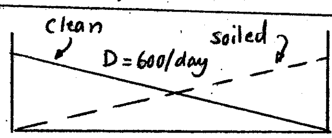



$$y^* = \sqrt{\frac{2\times 5\times 100}{.1}} = 100 \text{ pallets}$$

$$t_0 = \frac{100}{5} = 20 \text{ days}$$

Pick up 100 pallets every 20 days.

**6**




$$TC/day = \frac{K}{y/D} + \frac{h_1 y}{2} + \frac{h_2 y}{2} + .6D$$

$$= \frac{KD}{y} + (h_1 + h_2)\frac{y}{2} + .6D$$

$$y^* = \sqrt{\frac{2KD}{(h_1+h_2)}} = \sqrt{\frac{2\times 81\times 600}{(.01+.02)}} = 1800 \text{ towels}$$

$$t_0 = 1800/600 = 3 \text{ days}$$

$$Cost/day = \frac{81\times 600}{1800} + \frac{.03\times 1800}{2} = \$54$$

Optimal policy: Pick up soiled towels and deliver an equal batch of 1800 towels every 3 days

**7**

The basic assumption is that the employee will deposit sufficient funds in Europe to take advantage of the higher interest rate and periodically send lump sums to the US to take care of the obligations. This problem in the context of an application of the simple economic lot size formula with no shortages. The idea is that it may be more economical to hold funds longer in European banks to take advantage of their considerably higher interest rate. The cost of wiring funds from overseas (= \$50) may be regarded as the "setup" cost and the lost interest per dollar per year (=.065 - .015 = \$.05) can be treated as the "holding" cost. Using this information, the economic lot size formula will yield

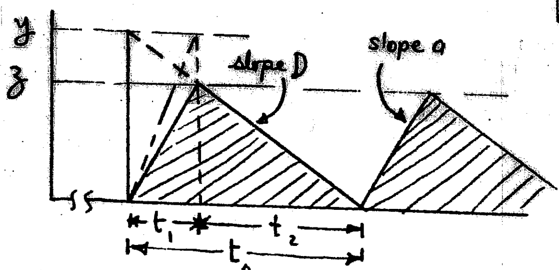
$$\text{Deposit amount} = \sqrt{\frac{2kD}{h}} = \sqrt{\frac{2\times 50\times 12000}{.05}} = \$4899$$

$$\text{Time between deposits} = t_0 = \frac{4899}{12000} = .408 \text{ year}$$

$$= 4.9 \text{ months}$$

**Optimal policy:** Send \$4899 ( ≈\$5000) every 4.9 ( ≈ 5) months to the US. The first installment occurs at the start of the year

**8**




a) From the geometry of the figure,

$$z = t_1(a-D) = \frac{y}{a}(a-D) = y\left(1-\frac{D}{a}\right)$$

b)
$$TCU(y) = \frac{K + (z/2)\,t_0 * h}{t_0}$$

$$= \frac{KD}{y} + \frac{h}{2}\left(1-\frac{D}{a}\right)y$$

(c) $\frac{\partial TCU(y)}{\partial y} = 0$ gives

$$-\frac{KD}{y^2} + \frac{h}{2}\left(1-\frac{D}{a}\right) = 0$$

$$y^* = \sqrt{\frac{2KD}{h\left(1-\frac{D}{a}\right)}}$$

(d)
$$\lim_{a\to\infty} \sqrt{\frac{2KD}{h\left(1-\frac{D}{a}\right)}} = \sqrt{\frac{2KD}{h}}$$

**9**

Alternative 1: Produce

$$y^* = \sqrt{\frac{2KD}{h\left(1-\frac{D}{a}\right)}}$$

$$= \sqrt{\frac{2\times 20\times \frac{26000}{365}}{.02\left(1-\frac{26000/365}{100}\right)}} = 703.7 \text{ units}$$

Total cost/day

$$= \frac{KD}{y^*} + \frac{h}{2}\left(1-\frac{D}{a}\right)y^*$$

$$= \frac{200\times \frac{2600}{365}}{703.7} + \frac{.02}{2}\left(1-\frac{26000}{100\times 365}\right)\times 703.7$$

$$= \$4.05 \text{ per day}$$

continued...

alternative 2: Buy

$$y^* = \sqrt{\frac{2KD}{h}}$$

$$= \sqrt{\frac{2 \times 15 \times \frac{26000}{365}}{.02}}$$

$$= 326.87 \text{ units}$$

Total cost/day

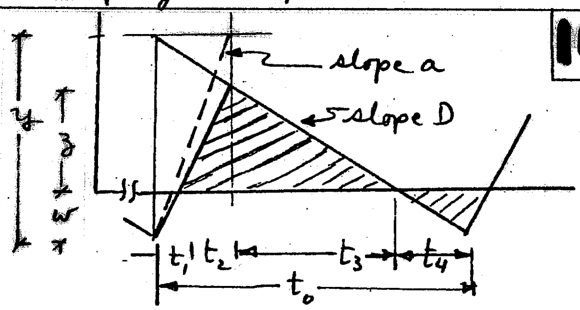
$$= \frac{KD}{y^*} + \frac{h}{2} y^*$$

$$= \frac{15 \times \frac{26000}{365}}{377.45} + \frac{.02}{2} \times 377.45$$

$$= \$6.54/\text{day}$$

The company should produce its own.

**10**




$$z = y\left(1 - \frac{D}{a}\right) - w$$

$$TCU(y,w) = \left[K + \frac{h\{y(1-\frac{D}{a}) - w\}^2 + pw^2}{2D(1-D/a)}\right] / t_0$$

$$= \frac{KD}{y} + \frac{h\{y(1-\frac{D}{a}) - w\}^2 + pw^2}{2y(1-D/a)}$$

Partial derivatives = 0 give

$$-\frac{KD}{y^2} + h\left(\frac{1}{2}\left(1-\frac{D}{a}\right) - \frac{w^2}{2y^2(1-D/a)}\right) - \frac{pw^2}{2y^2(1-\frac{D}{a})} = 0$$

$$h\left(\frac{w}{y(1-\frac{D}{a})} - 1\right) + \frac{pw}{y(1-D/a)} = 0$$

This gives,

$$y^* = \sqrt{\frac{2KD(p+h)}{ph(1-D/a)}}, \quad w^* = \sqrt{\frac{2KDh(1-\frac{D}{a})}{p(p+h)}}$$

**1**

EOQ before quantity dis-count = 1800 towels per Problem 6, Set 11.2a.

Total cost/day given batches of 1800 towels

$$= DC_1 + \frac{KD}{y} + \frac{h_1+h_2}{2}y$$

$$= 600\times.6 + \frac{81\times600}{1800} + \frac{.03\times1800}{2} = \$414$$

Total cost/day given batches of 2500 towels

$$= DC_2 + \frac{KD}{y} + \frac{(h_1+h_2)}{2}y$$

$$= 600\times.5 + \frac{81\times600}{2500} + \frac{.03\times2500}{2} = \$356.94$$

Take advantage of price discount.

**2**

$$y_m = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2\times100\times30}{.05}} = 346.41$$

$q = 500$ units

Because $y_m < q$, we need to compute $Q$.

$$TCU_1(y_m) = DC_1 + \frac{KD}{y_m} + \frac{hy_m}{2}$$

$$= 30\times10 + \frac{100\times30}{346.41} + \frac{.05\times346.41}{2}$$

$$= 317.32$$

The equation for computing $Q$ is

$$Q^2 + \left(\frac{2(8\times30-317.32)}{.05}\right)Q + \frac{2\times100\times30}{.05} = 0$$

or

$$Q^2 - 3092.82Q + 120000 = 0$$

This yields $Q = 3053.52$ units

Because $y_m < q < Q \Rightarrow y^* = q = 500$

$t_0 = \frac{500}{30} = 16.67$ days $\Rightarrow L_e = 4.33$

Order 500 units when inventory drop to 130.

**3**

$$y_m = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2\times50\times20}{.3}}$$

$= 81.65$ units

Because $q > y_m$, we need to compute $Q$.

$$TCU_1(y_m) = 20\times25 + \frac{50\times20}{81.65} + \frac{.3\times81.65}{2}$$

$= \$524.49$

$Q$-equation:

$$Q^2 + \left(\frac{2(22.5\times20-524.49)}{.3}\right)Q + \frac{2\times50\times20}{.3} = 0$$

$$Q^2 - 496.63Q + 6666.67 = 0$$

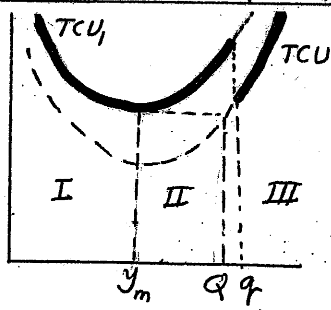
continued...

Thus, $Q = 482.83$

Because $y_m < q < Q \Rightarrow y^* = 150$

Order 150 units when inventory drops to 0

**4**




From the preceding figure, the dis-count is not advantageous if

$$TCU_1(y_m) \le TCU_2(q)$$

or

$$DC_1 + \frac{KD}{y_m} + \frac{hy_m}{2} \le DC_2 + \frac{KD}{q} + \frac{hq}{2}$$

or

$$20C_1 + \frac{50\times20}{81.65} + \frac{.3\times81.65}{2} \le 20C_2 + \frac{50\times20}{150} + \frac{.3\times150}{2}$$

Thus, the condition reduces to

$$C_1 - C_2 \le .23359$$

Let $d$ = discount factor $(<1)$.

Then $C_2 = (1-d)C_1$, $0 < d < 1$

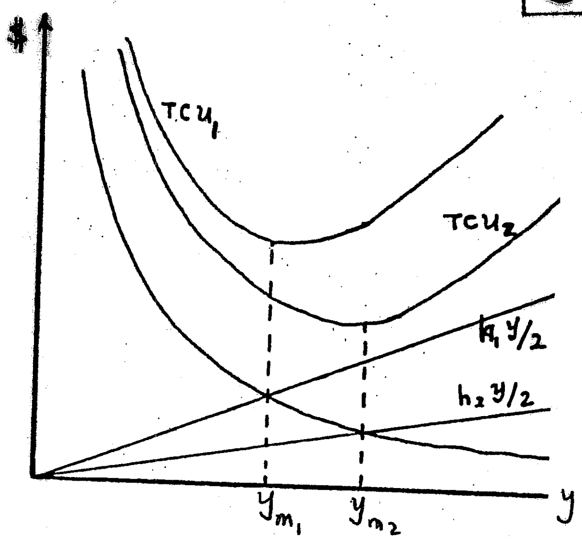
Given $C_1 = 25$, we have

$$25d \le .233588$$

or $$d \le .009344$$

Thus, no advantage if the % discount is $\le .9344\%$ $(\simeq 1\%)$

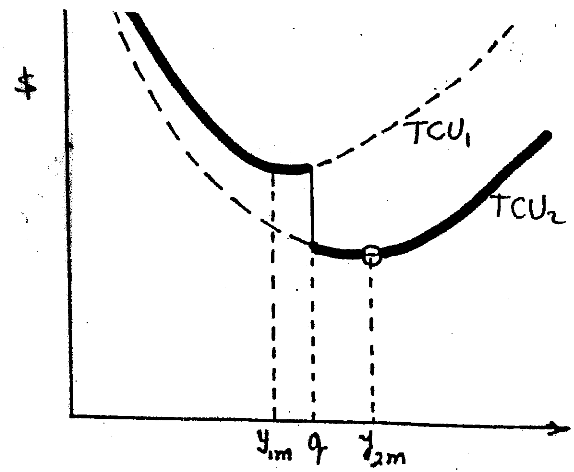
**5**




$$TCU_1(y) = \frac{KD}{y} + \frac{h_1 y}{2}$$

$$TCU_2(y) = \frac{KD}{y} + \frac{h_2 y}{2}$$

Case 1: $q < y_{2m}$




Solution:

$$y^* = y_{2m}$$

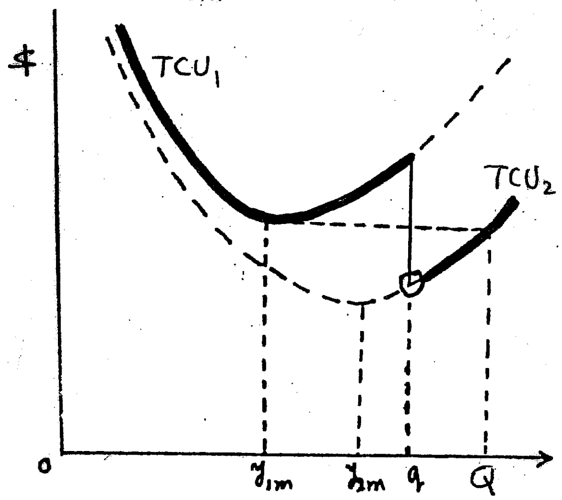
$$TCU(y^*) = TCU_2(y_{2m})$$

continued...

Case 2: $y_{2m} < q \le Q$

The value of $Q$ is determined from the equation:

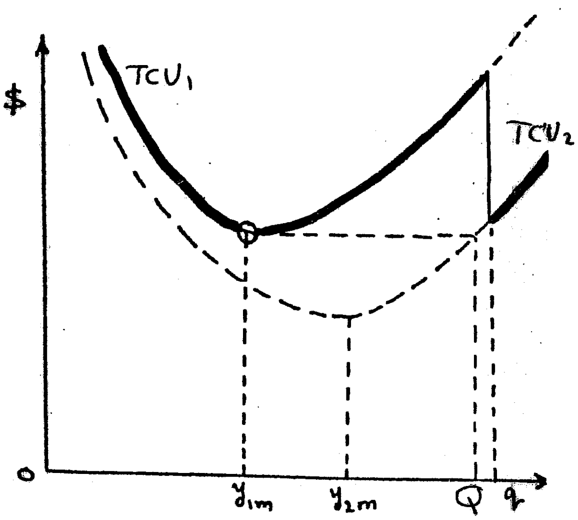
$$TCU_1(y_{1m}) = TCU_2(Q)$$




Solution: $y^* = q$

$$TCU(y^*) = TCU_2(q)$$

Case 3: $y_{2m} < Q < q$




Solution: $y^* = y_{1m}$, $TCU(y^*) = TCU_1(y_{1m})$

$$TCU(y^*) = \begin{cases} TCU_2(y_{2m}), & q < y_{2m} \\ TCU_2(q), & y_{2m} < q \le Q \\ TCU_1(y_{1m}), & y_{2m} < Q < q \end{cases}$$

1

See file ampl11.3c-1.txt.
AMPL model will not converge unless $K_iD_i/y_i$ is replaced with $K_iD_i/(y_i+\varepsilon)$, where $\varepsilon>0$ and very small.

SOLUTION:
Total cost = 568.11
$y_1 = 4.42$
$y_2 = 6.87$
$y_3 = 4.12$
$y_4 = 7.20$
$y_5 = 5.80$

2

See file ampl11.3c-2.txt.
New constraint:
$(1/2)(y_1 + y_2 + y_3) \leq 25$

SOLUTION:
Total cost = 10.42
$y_1 = 10.83$
$y_2 = 16.85$
$y_3 = 22.32$

3

See file ampl11.3c-3.txt.
New constraint:
Average inventory for item i = $y_i/2$.
$(1/2)(100y_1 + 55y_2 + 100y_3) \leq 1000$

SOLUTION:
Total cost = 14.31
y1 = 5.58
y2 = 7.90
y3 = 10.07

4

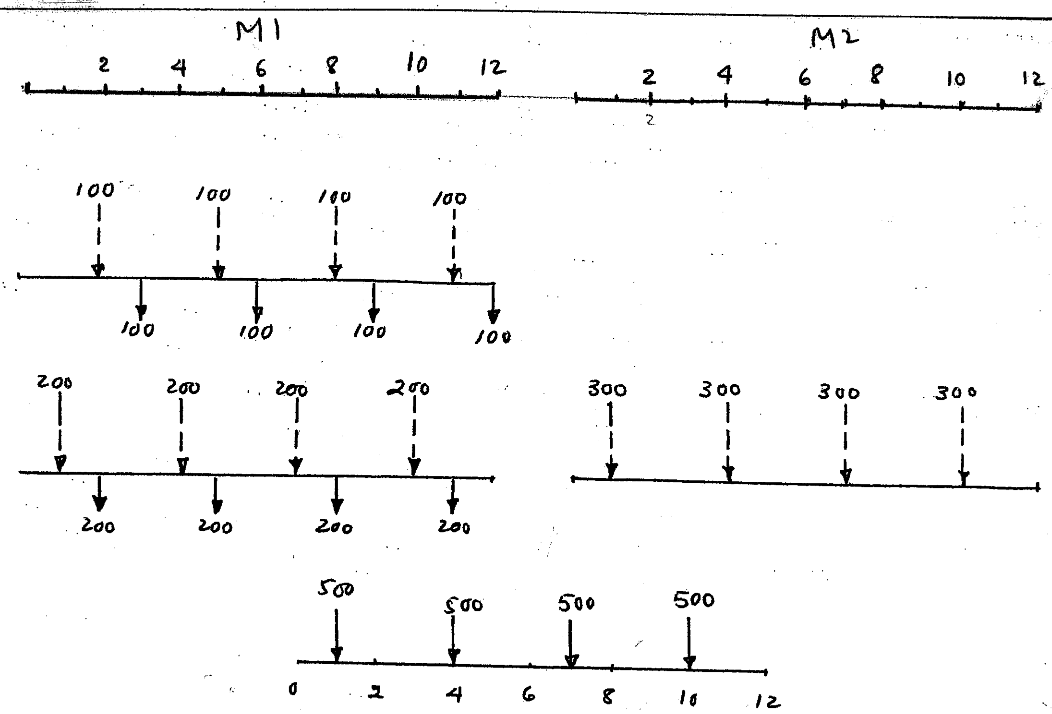
See file ampl11.3c-4.txt.
AMPL model will not converge unless $K_iD_i/y_i$ is replaced with $K_iD_i/(y_i+\varepsilon)$, where $\varepsilon>0$ and very small.

New constraint:
$365(10/y_1 + 20/y_2 + 5/y_3 + 10/y_4) \leq 150$

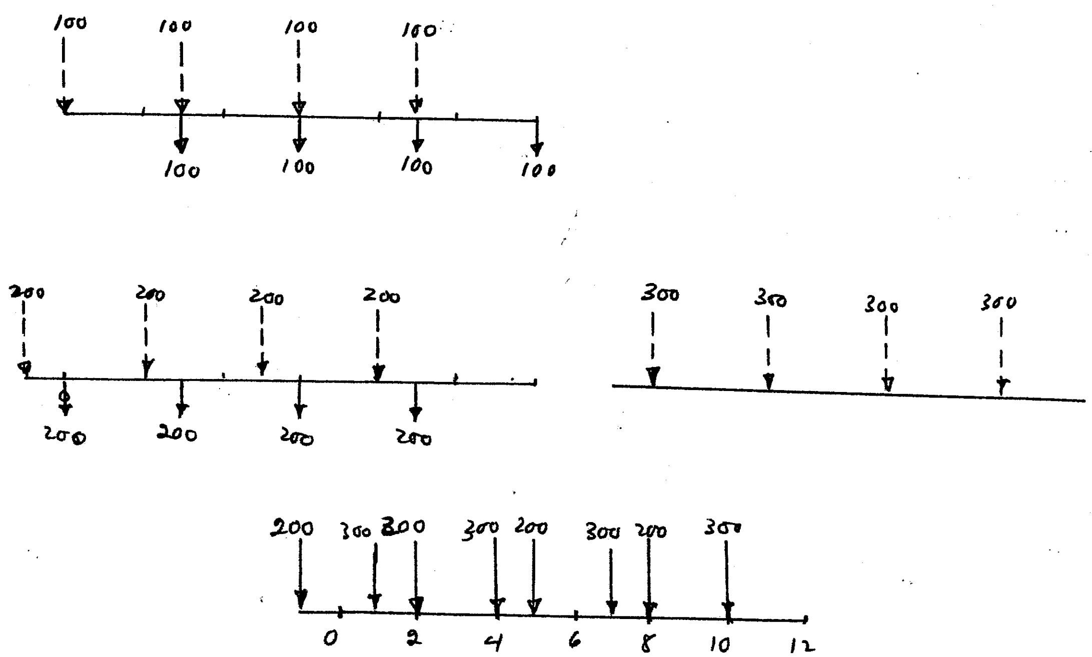
SOLUTION:
Total cost = 54.71
y1 = 155.30
y2 = 118.81
y3 = 74.36
y4 = 90.09

# Set 11.4a

(a)


M1
2 4 6 8 10 12
M2
2 4 6 8 10 12
100 100 100 100
100 100 100 100
200 200 200 200
200 200 200 200
300 300 300 300
500 500 500 500
0 2 4 6 8 10 12

(b)


100 100 100 100
100 100 100 100
200 200 200 200
200 200 200 200
300 300 300 300
200 300 300 300 200 300 200 300
0 2 4 6 8 10 12

**1**

| | 1 | 2 | 3 | 4 | Surplus | |
|---|---|---|---|---|---|---|
| $R_1$ | 90 (5) | 5.1 | 5.25 | 5.37 | 0 | 90 |
| $O_1$ | 10 (7.5) | 30 (7.6) | 10 (7.75) | 7.87 | 0 | 50 |
| $R_2$ | | 100 (3) | 3.15 | 3.27 | 0 | 100 |
| $O_2$ | | 60 (4.5) | 4.65 | 4.77 | 0 | 60 |
| $R_3$ | | | 120 (4) | 4.12 | 0 | 120 |
| $O_3$ | | | 80 (6) | 6.12 | 0 | 80 |
| $R_4$ | | | | 110 (1) | 0 | 110 |
| $O_4$ | | | | 50 (1.5) | 20 (0) | 70 |
| | 100 | 190 | 210 | 160 | 20 | |

**2**

(a)

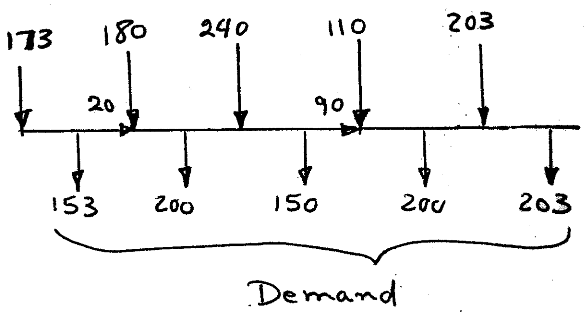
| | 1 | 2 | 3 | 4 | Surplus | |
|---|---|---|---|---|---|---|
| I | 11 (1) | 1.3 | 1.65 | 1.85 | 0 | 11 |
| II | 2 | 1 (2.3) | 3 (2.65) | 2.85 | 0 | 4 |
| III | 5 | 5.3 | 5.65 | 5 5 (5.85) | 5 0 (0) | 10 |
| I | X | 3 (2) | 2.25 | 2.55 | 0 | 3 |
| II | X | 4 | 11 (4.35) | 1 (4.55) | 0 | 12 |
| III | X | 6 | 6.35 | 5 (6.55) | 10 5 (0) | 10 |
| I | X | X | 3 (2) | 2.2 | 0 | 3 |
| II | X | X | 5 | 8 (5.2) | 0 | 8 |
| III | X | X | 7 | 7.2 | 4 (0) | 4 |
| IV | X | X | 10 | 10.2 | 10 (0) | 10 |
| I | X | X | X | 3 (3) | 0 | 3 |
| II | X | X | X | 8 (4) | 0 | 8 |
| III | X | X | X | 4 (5) | 0 | 4 |
| IV | X | X | X | 7 | 10 (0) | 10 |
| | 11 | 4 | 17 | 29 | 39 | |

(b) Additional 10 units are produced as shown by the circled entries in period 4. The problem has alternative solutions.

**3**

| | | 1 | 2 | 3 | 4 | 5 | Surplus | |
|---|---|---|---|---|---|---|---|---|
| 1 | R | 100 (4) | 4.5 | 5 | 5.5 | 6 | 0 | 100 |
| | O | 50 (6) | 6.5 | 7 | 7.5 | 8 | 0 | 50 |
| | S | 3 (7) | 20 (7.5) | 8 | 8.5 | 9 | 7 (0) | 30 |
| 2 | R | X | 40 (4) | 4.5 | 5 | 6 | 0 | 40 |
| | O | X | 60 (6) | 6.5 | 7 | 7.5 | 0 | 60 |
| | S | X | 80 (7) | 7.5 | 8 | 8.5 | 0 | 80 |
| 3 | R | X | X | 90 (4) | 4.5 | 5 | 0 | 90 |
| | O | X | X | 60 (6) | 20 (6.5) | 7 | 0 | 80 |
| | S | X | X | 7 | 70 (7.5) | 8 | 0 | 70 |
| 4 | R | X | X | X | 60 (4) | 4.5 | 0 | 60 |
| | O | X | X | X | 50 (6) | 6.5 | 0 | 50 |
| | S | X | X | X | 7 | 7.5 | 20 (0) | 20 |
| 5 | R | X | X | X | X | 70 (4) | 0 | 70 |
| | O | X | X | X | X | 50 (6) | 0 | 50 |
| | S | X | X | X | X | 83 (7) | 17 (0) | 100 |
| | | 153 | 200 | 150 | 200 | 203 | 44 | |

Solution summary

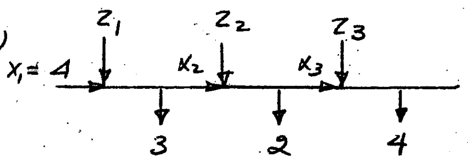


**1**

(a) No, because inventory should not be held needlessly at the end of planning horizon

(b)

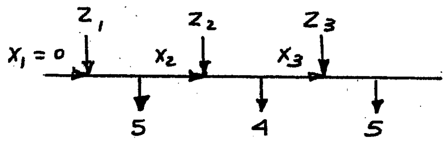
(i)




$0 \le z_1 \le 5$, $1 \le z_2 \le 5$, $0 \le z_3 \le 4$

$x_1 = 4$, $1 \le x_2 \le 6$, $0 \le x_3 \le 4$

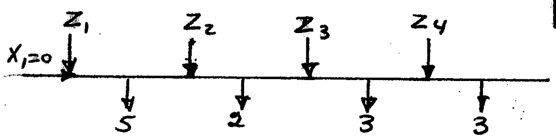
(ii)




$5 \le z_1 \le 14$, $0 \le z_2 \le 9$, $0 \le z_3 \le 5$

$x_1 = 0$, $0 \le x_2 \le 9$, $0 \le x_3 \le 5$

**2**




Stage 1: $f_1(x_2) = \min_{z_1 = D_1 + x_2} \{K_1 + C_1(z_1) + h_1 x_2\}$

where $C_i(z_i) = \begin{cases} 1z_i, & 0 \le z_i \le 6 \\ 2z_i, & z_i \ge 7 \end{cases}$ $i = 1, 2, \dots, 4$

| $x_2 \backslash z_1$ | $K_1 = 5$, $h_1 = 1$ | | | | | | | | | Opt. Sol. | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $f_1$ | $z_1$ |
| 0 | 10 | | | | | | | | | 10 | 5 |
| 1 | | 12 | | | | | | | | 12 | 6 |
| 2 | | | 15 | | | | | | | 15 | 7 |
| 3 | | | | 18 | | | | | | 18 | 8 |
| 4 | | | | | 21 | | | | | 21 | 9 |
| 5 | | | | | | 24 | | | | 24 | 10 |
| 6 | | | | | | | 27 | | | 27 | 11 |
| 7 | | | | | | | | 30 | | 30 | 12 |
| 8 | | | | | | | | | 33 | 33 | 13 |

Stage 2:

$$f_2(x_3) = \min_{0 \le z_2 \le D_2 + x_3} \{K_2 + C_2(z_2) + h_2 x_3 + f_1(x_3 + D_2 - z_2)\}$$

$0 \le z_2 \le 8$, $0 \le x_3 \le 6$, $D_2 = 2$

| $x_3 \backslash z_2$ | $K_2 = 7$, $h_2 = 1$ | | | | | | | | | Opt. Sol. | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $f_2$ | $z_2$ |
| 0 | 15 | 20 | 19 | | | | | | | 15 | 0 |
| 1 | 19 | 24 | 22 | 21 | | | | | | 19 | 0 |
| 2 | 23 | 28 | 26 | 24 | 23 | | | | | 23 | 0,4 |
| 3 | 27 | 32 | 30 | 28 | 26 | 25 | | | | 25 | 5 |
| 4 | 31 | 36 | 34 | 32 | 30 | 28 | 27 | | | 27 | 6 |
| 5 | 35 | 40 | 38 | 36 | 34 | 32 | 30 | 30 | | 30 | 6 |
| 6 | 39 | 44 | 42 | 40 | 38 | 36 | 34 | 33 | 33 | 33 | 7,8 |

Stage 3: $0 \le z_3 \le 6$, $0 \le x_4 \le 3$, $D_3 = 3$

| $x_4 \backslash z_3$ | $K_3 = 9$, $h_3 = 1$ | | | | | | | Opt. Sol. | |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $f_3$ | $z_3$ |
| 0 | 25 | 33 | 30 | 27 | | | | 25 | 0 |
| 1 | 28 | 36 | 35 | 32 | 29 | | | 28 | 0 |
| 2 | 32 | 39 | 38 | 37 | 34 | 31 | | 31 | 5 |
| 3 | 36 | 43 | 41 | 40 | 39 | 36 | 33 | 33 | 6 |

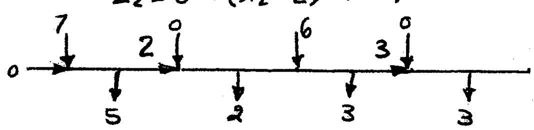
Stage 4: $0 \le z_4 \le 3$, $x_5 = 0$, $D_4 = 3$

| $x_5 \backslash z_4$ | $K_4 = 7$, $h_4 = 1$ | | | | Opt. Sol. | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | $f_4$ | $z_4$ |
| 0 | 33 | 39 | 37 | 35 | 33 | 6 |

Solution:

$(x_5 = 0) \rightarrow z_4 = 0 \rightarrow (x_4 = 3) \rightarrow z_3 = 6 \rightarrow (x_3 = 0) \rightarrow$

$z_2 = 0 \rightarrow (x_2 = 2) \rightarrow z_1 = 7$




Total cost = \$33

continued...

**3**

$$f_1(x_2) = \min_{0 \le z_1 \le D_1 + x_2}\left\{C_1(z_1) + K_1 + h_1\left(\frac{x_1 + z_1 + x_2}{2}\right)\right\}$$

$$= \min_{0 \le z_1 \le D_1 + x_2}\left\{K_1 + c_1(z_1) + h_1\left(x_2 + \frac{D_1}{2}\right)\right\}$$

$$f_i(x_{i+1}) = \min_{0 \le z_i \le D_i + x_{i+1}}\left\{K_i + C_i(z_i) + h_i\left(x_{i+1} + \frac{D_i}{2}\right) + f_{i-1}(x_{i+1} + D_i - z_i)\right\}$$

**4**

$$f_n(x_n) = \min_{z_n + x_n = D_n}\left\{K_n + C_n(z_n)\right\}$$

$$f_i(x_i) = \min_{D_i \le x_i + z_i \le D_1 + \cdots + D_n}\left\{K_i + C_i(z_i) + h_i(x_i + z_i - D_i) + f_{i+1}(x_i + z_i - D_i)\right\}$$

Stage 3: $D_3 = 4$, $0 \le x_3 \le 4$

| $x_3$ | $z_3 = 0$ | 1 | 2 | 3 | 4 | Opt. Sol. $f_3$ | $z_3$ |
|---|---|---|---|---|---|---|---|
| 0 | | | | | 56 | 56 | 4 |
| 1 | | | | 36 | | 36 | 3 |
| 2 | | | 26 | | | 26 | 2 |
| 3 | | 16 | | | | 16 | 1 |
| 4 | 0 | | | | | 0 | 0 |

Stage 2: $D_2 = 2$

| $x_2$ | $z_2 = 0$ | 1 | 2 | 3 | 4 | 5 | 6 | Opt. Sol. $f_2$ | $z_2$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | | 83 | 76 | 89 | 102 | 109 | 76 | 3 |
| 1 | | 73 | 66 | 69 | 82 | 89 | | 66 | 2 |
| 2 | 56 | 56 | 59 | 62 | 69 | | | 56 | 0, 1 |
| 3 | 39 | 49 | 52 | 49 | | | | 34 | 0 |
| 4 | 32 | 42 | 39 | | | | | 32 | 0 |
| 5 | 25 | 29 | | | | | | 25 | 0 |
| 6 | 12 | | | | | | | 12 | 0 |

continued...

Stage 1: $D_1 = 3$

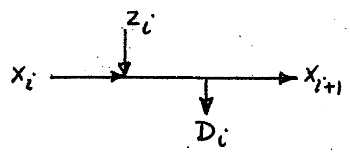
| $x_1$ | $z_1 = 2$ | 3 | 4 | 5 | 6 | 7 | 8 | Opt. Sol. $f_1$ | $z_1$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 99 | 100 | 111 | 115 | 129 | 193 | 151 | 99 | 2 |

Solution:

$(x_1 = 1) \rightarrow z_1 = 2 \rightarrow (x_2 = 0) \rightarrow z_2 = 3 \rightarrow$
$(x_3 = 1) \rightarrow z_3 = 3$

Cost = \$99

**5**




$$\text{Average inventory} = \frac{x_i + z_i + x_{i+1}}{2} = \frac{x_i + z_i + x_i + z_i - D_i}{2} = x_i + z_i - \frac{D_i}{2}$$

Replace $h_i(x_i + z_i - D_i)$ with $h_i\left(x_i + z_i - \frac{D_i}{2}\right)$ in the backward formulation of problem 4.

# Set 11.4d

1

Period 1:

| | A | B | C | D | E | F | G | H | | O | P | Q | R | S | T | U |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | |
| 2 | Number of periods, N= | | 4 | | Current period= | | 1 | | | | | | | | | |
| 3 | Period | 1 | 2 | 3 | 4 | | | | | | | | | | | |
| 4 | c(1 to 4)= | 2 | 2 | 2 | 2 | | | | | Optimum Solution Summary | | | | | | |
| 5 | K(1 to 4)= | 98 | 114 | 185 | 70 | | | | | | | | | | | |
| 6 | h(1 to 4)= | 1 | 1 | 1 | 1 | | | | | x | f | z | x | f | z | |
| 7 | D(1 to 4)= | 0 | 22 | 90 | 67 | | | | Current | | period 1 | | | | | |
| 8 | Are D values correct? | | | | | | | | optimum | 0 | 0 | 0 | | | | |
| 9 | Period 0 | | z1= | 0 | 22 | 112 | 179 | | Period 1 | 22 | 164 | 22 | | | | |
| 10 | f0 | | C1(z1)= | 0 | 142 | 322 | 456 | | f1 | z1 | 112 | 434 | 112 | | | |
| 11 | | x2= | 0 | 0 | 111111 | 111111 | 111111 | | 0 | 0 | 179 | 635 | 179 | | | |
| 12 | | x2= | 22 | 111111 | 164 | 111111 | 111111 | | 164 | 22 | | | | | | |
| 13 | | x2= | 112 | 111111 | 111111 | 434 | 111111 | | 434 | 112 | | | | | | |
| 14 | | x2= | 179 | 111111 | 111111 | 111111 | 635 | | 635 | 179 | | | | | | |

Period 2:

| | A | B | C | D | E | F | G | H | | O | P | Q | R | S | T | U |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | |
| 2 | Number of periods, N= | | 4 | | Current period= | | 2 | | | | | | | | | |
| 3 | Period | 1 | 2 | 3 | 4 | | | | | | | | | | | |
| 4 | c(1 to 4)= | 2 | 2 | 2 | 2 | | | | | Optimum Solution Summary | | | | | | |
| 5 | K(1 to 4)= | 98 | 114 | 185 | 70 | | | | | | | | | | | |
| 6 | h(1 to 4)= | 1 | 1 | 1 | 1 | | | | | x | f | z | x | f | z | |
| 7 | D(1 to 4)= | 0 | 22 | 90 | 67 | | | | Current | | period 1 | | | period 2 | | |
| 8 | Are D values correct? | | | | | | | | optimum | 0 | 0 | 0 | 0 | 158 | 22 | |
| 9 | Period 1 | | z2= | 0 | 22 | 112 | 179 | | Period 2 | 22 | 164 | 22 | 90 | 428 | 112 | |
| 10 | f1 | | C2(z2)= | 0 | 158 | 338 | 472 | | f2 | z2 | 112 | 434 | 112 | 157 | 629 | 179 |
| 11 | 0 | x3= | 0 | 164 | 158 | 111111 | 111111 | | 158 | 22 | 179 | 635 | 179 | | | |
| 12 | 164 | x3= | 90 | 524 | 111111 | 428 | 111111 | | 428 | 112 | | | | | | |
| 13 | 434 | x3= | 157 | 792 | 111111 | 111111 | 629 | | 629 | 179 | | | | | | |
| 14 | 635 | | | | | | | | | | | | | | | |

Period 3:

| | A | B | C | D | E | F | G | H | | O | P | Q | R | S | T | U |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | |
| 2 | Number of periods, N= | | 4 | | Current period= | | 3 | | | | | | | | | |
| 3 | Period | 1 | 2 | 3 | 4 | | | | | | | | | | | |
| 4 | c(1 to 4)= | 2 | 2 | 2 | 2 | | | | | Optimum Solution Summary | | | | | | |
| 5 | K(1 to 4)= | 98 | 114 | 185 | 70 | | | | | | | | | | | |
| 6 | h(1 to 4)= | 1 | 1 | 1 | 1 | | | | | x | f | z | x | f | z | |
| 7 | D(1 to 4)= | 0 | 22 | 90 | 67 | | | | Current | | period 1 | | | period 2 | | |
| 8 | Are D values correct? | | | | | | | | optimum | 0 | 0 | 0 | 0 | 158 | 22 | |
| 9 | Period 2 | | z3= | 0 | 90 | 157 | | | Period 3 | 22 | 164 | 22 | 90 | 428 | 112 | |
| 10 | f2 | | C3(z3)= | 0 | 365 | 499 | | | f3 | z3 | 112 | 434 | 112 | 157 | 629 | 179 |
| 11 | 158 | x4= | 0 | 428 | 523 | 111111 | | | 428 | 0 | 179 | 635 | 179 | | | |
| 12 | 428 | x4= | 67 | 696 | 111111 | 724 | | | 696 | 0 | | period 3 | | | | |
| 13 | 629 | | | | | | | | | | 0 | 428 | 0 | | | |
| 14 | | | | | | | | | | | 67 | 696 | 0 | | | |

Period 4:

V13 = 67

| | A | B | C | D | E | F | G | H | | O | P | Q | R | S | T | U |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | |
| 2 | Number of periods, N= | | 4 | | Current period= | | 4 | | | | | | | | | |
| 3 | Period | 1 | 2 | 3 | 4 | | | | | | | | | | | |
| 4 | c(1 to 4)= | 2 | 2 | 2 | 2 | | | | | Optimum Solution Summary | | | | | | |
| 5 | K(1 to 4)= | 98 | 114 | 185 | 70 | | | | | | | | | | | |
| 6 | h(1 to 4)= | 1 | 1 | 1 | 1 | | | | | x | f | z | x | f | z | |
| 7 | D(1 to 4)= | 0 | 22 | 90 | 67 | | | | Current | | period 1 | | | period 2 | | |
| 8 | Are D values correct? | | | | | | | | optimum | 0 | 0 | 0 | 0 | 158 | 22 | |
| 9 | Period 3 | | z4= | 0 | 67 | | | | Period 4 | 22 | 164 | 22 | 90 | 428 | 112 | |
| 10 | f3 | | C4(z4)= | 0 | 204 | | | | f4 | z4 | 112 | 434 | 112 | 157 | 629 | 179 |
| 11 | 428 | x5= | 0 | 696 | 632 | | | | 632 | 67 | 179 | 635 | 179 | | | |
| 12 | 696 | | | | | | | | | | | period 3 | | | period 4 | |
| 13 | | | | | | | | | | | 0 | 428 | 0 | 0 | 632 | 67 |
| 14 | | | | | | | | | | | 67 | 696 | 0 | | | |

Optimum solution:

| $z_4$ | $x_4$ | $z_3$ | $x_3$ | $z_2$ | $x_2$ | $z_1$ |
|---|---|---|---|---|---|---|
| 67 | 0 | 0 | 90 | 112 | 0 | 0 |

Cost = \$632

2

Stage 1: $D_1 = 150,\ x_1 = 50$

| $x_2$ | $z_1=100$ | 200 | 220 | 260 | 330 | 420 | 550 | 730 | 870 | 920 | Opt. Sol. $f_1$ | $z_1^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 700 | | | | | | | | | | 700 | 100 |
| 100 | | 1400 | | | | | | | | | 1400 | 200 |
| 120 | | | 1540 | | | | | | | | 1540 | 220 |
| 160 | | | | 1820 | | | | | | | 1820 | 260 |
| 230 | | | | | 2310 | | | | | | 2110 | 330 |
| 320 | | | | | | 2940 | | | | | 2940 | 420 |
| 450 | | | | | | | 3850 | | | | 3850 | 550 |
| 630 | | | | | | | | 5110 | | | 5110 | 730 |
| 770 | | | | | | | | | 6090 | | 6090 | 870 |
| 820 | | | | | | | | | | 6940 | 6940 | 920 |

Stage 2: $D_2 = 100$

| $x_3$ | $z_2=0$ | 100 | 120 | 160 | 230 | 320 | 450 | 630 | 770 | 820 | Opt. Sol. $f_2$ | $z_2^*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1400 | 1400 | | | | | | | | | 1400 | 100 |
| 20 | 1560 | | 1540 | | | | | | | | 1540 | 120 |
| 60 | 1880 | | | 1820 | | | | | | | 1820 | 160 |
| 130 | 2440 | | | | 2310 | | | | | | 2310 | 230 |
| 220 | 3160 | | | | | 2940 | | | | | 2940 | 320 |
| 350 | 4200 | | | | | | 3850 | | | | 3850 | 450 |
| 530 | 5690 | | | | | | | 5110 | | | 5110 | 630 |
| 670 | 6760 | | | | | | | | 6090 | | 6090 | 770 |
| 720 | 7160 | | | | | | | | | 6940 | 6940 | 820 |

Stage 3: $D = 20$

| $x_4$ | $z_3=0$ | 20 | 60 | 130 | 220 | 350 | 530 | 670 | 720 | Opt. Sol. $f_3$ | $z_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1540 | 1580 | | | | | | | | 1540 | 0 |
| 90 | 1900 | | 1820 | | | | | | | 1820 | 60 |
| 110 | 2530 | | | 2240 | | | | | | 2240 | 130 |
| 200 | 3340 | | | | 2780 | | | | | 2780 | 220 |
| 330 | 4510 | | | | | 3560 | | | | 3560 | 350 |
| 510 | 6130 | | | | | | 4640 | | | 4640 | 530 |
| 650 | 7390 | | | | | | | 5480 | | 5480 | 670 |
| 700 | 7840 | | | | | | | | 5780 | 5780 | 720 |

Stage 4: $D_4 = 40$

| $x_5$ | $z_4=0$ | 40 | 110 | 200 | 330 | 510 | 650 | 750 | Opt. Sol. $f_4$ | $z_4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1820 | 1900 | | | | | | | 1820 | 0 |
| 70 | 2310 | | 2250 | | | | | | 2250 | 110 |
| 160 | 2940 | | | 2700 | | | | | 2700 | 200 |
| 240 | 3850 | | | | 3350 | | | | 3350 | 330 |
| 470 | 5110 | | | | | 4250 | | | 4250 | 510 |
| 610 | 6090 | | | | | | 4950 | | 4950 | 650 |
| 660 | 6440 | | | | | | | 5200 | 5200 | 700 |

Stage 5: $D_5 = 70$

| $x_6$ | $z_5=0$ | 70 | 160 | 290 | 470 | 610 | 660 | Opt. Sol. $f_5$ | $z_5$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 2250 | 2440 | | | | | | 2250 | 0 |
| 90 | 2880 | | 3160 | | | | | 2880 | 0 |
| 220 | 3790 | | | 4200 | | | | 3790 | 0 |
| 400 | 5050 | | | | 5640 | | | 5050 | 0 |
| 540 | 6030 | | | | | 6761 | | 6030 | 0 |
| 590 | 6380 | | | | | | 7160 | 6380 | 0 |

continued...

Stage 6: $D_6 = 90$

| $x_7$ | $z_6 = 0$ | 90 | 220 | 400 | 540 | 590 | Opt. Sol. $f_6$ | $z_6$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 2880 | 3110 | | | | | 2880 | 0 |
| 130 | 4180 | | 4600 | | | | 4180 | 0 |
| 310 | 5980 | | | 6580 | | | 5980 | 0 |
| 450 | 7380 | | | | 8120 | | 7380 | 0 |
| 500 | 7880 | | | | | 8670 | 7880 | 0 |

Stage 7: $D_7 = 130$

| $x_8$ | $z_7 = 0$ | 130 | 310 | 450 | 500 | Opt. Sol. $f_7$ | $z_7$ |
|---|---|---|---|---|---|---|---|
| 0 | 4180 | 3700 | | | | 3700 | 130 |
| 180 | 6160 | | 4600 | | | 4600 | 310 |
| 320 | 7700 | | | 5300 | | 5300 | 450 |
| 370 | 8250 | | | | 5550 | 5550 | 500 |

Stage 8: $D_8 = 180$

| $x_9$ | $z_8 = 0$ | 180 | 320 | 370 | Opt. Sol. $f_8$ | $z_8$ |
|---|---|---|---|---|---|---|
| 0 | 4600 | 4720 | | | 4600 | 0 |
| 140 | 5860 | | 5840 | | 5840 | 320 |
| 190 | 6310 | | | 6240 | 6240 | 370 |

Stage 9: $D_9 = 140$

| | $z_9 = 0$ | 140 | 190 | Opt. Sol. $f_9$ | $z_9$ |
|---|---|---|---|---|---|
| 0 | 5840 | 5180 | | 5180 | 140 |
| 50 | 6340 | | 5380 | 5380 | 190 |

Stage 10: $D_{10} = 50$

| $x_{11}$ | $z_{10} = 0$ | 50 | Opt. Sol. $f_{10}$ | $z_{10}$ |
|---|---|---|---|---|
| 0 | 5380 | 5780 | 5380 | 0 |

Solution:

| Period | Order Amount |
|---|---|
| 1 | 100 |
| 2 | 120 |
| 3 | 0 |
| 4 | 200 |
| 5 | 0 |
| 6 | 0 |
| 7 | 310 |
| 8 | 0 |
| 9 | 190 |
| 10 | 0 |

Minimum cost = \$5380

Period 1:

| | A | B | C | D | E | F | G | H | I | N | O | P | Q | R | S | T | U | V |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| 2 | | Number of periods, N= | | | 5 | Current period= | | | 1 | <<Go | | | Optimum Solution Summary | | | | | |
| 3 | I | Period | 1 | 2 | 3 | 4 | 5 | | | | | | | | | | | |
| 4 | N | c(1 to 5) = | 10 | 10 | 10 | 10 | 10 | | | | | | | | | | | |
| 5 | P | K(1 to 5) = | 80 | 70 | 60 | 80 | 60 | | | | | | | | | | | |
| 6 | U | h(1 to 5) = | 1 | 1 | 1 | 1 | 1 | | | | | | x | f | z | x | f | z |
| 7 | T | D(1 to 5) = | 50 | 70 | 100 | 30 | 60 | | | | Current | | | period 1 | | | | |
| 8 | | Are z1-values correct? | | | yes | yes | yes | yes | yes | | optimum | | 0 | 580 | 50 | | | |
| 9 | | Period 0 | | z1= | 50 | 120 | 220 | 250 | 310 | | Period 1 | | 70 | 1350 | 120 | | | |
| 10 | | f0 | C1(z1)= | | 580 | 1280 | 2280 | 2580 | 3180 | | f1 | z1 | 170 | 2450 | 220 | | | |
| 11 | S | | x2= | 0 | 580 | 111111 | 111111 | 111111 | 111111 | | 580 | 50 | 200 | 2780 | 250 | | | |
| 12 | T | | x2= | 70 | 111111 | 1350 | 111111 | 111111 | 111111 | | 1350 | 120 | 260 | 3440 | 310 | | | |
| 13 | A | | x2= | 170 | 111111 | 111111 | 2450 | 111111 | 111111 | | 2450 | 220 | | | | | | |
| 14 | G | | x2= | 200 | 111111 | 111111 | 111111 | 2780 | 111111 | | 2780 | 250 | | | | | | |
| 15 | E | | x2= | 260 | 111111 | 111111 | 111111 | 111111 | 3440 | | 3440 | 310 | | | | | | |

Period 2:

| | A | B | C | D | E | F | G | H | I | N | O | P | Q | R | S | T | U | V |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| 2 | | Number of periods, N= | | | 5 | Current period= | | | 2 | <<Go | | | Optimum Solution Summary | | | | | |
| 3 | I | Period | 1 | 2 | 3 | 4 | 5 | | | | | | | | | | | |
| 4 | N | c(1 to 5) = | 10 | 10 | 10 | 10 | 10 | | | | | | | | | | | |
| 5 | P | K(1 to 5) = | 80 | 70 | 60 | 80 | 60 | | | | | | | | | | | |
| 6 | U | h(1 to 5) = | 1 | 1 | 1 | 1 | 1 | | | | | | x | f | z | x | f | z |
| 7 | T | D(1 to 5) = | 50 | 70 | 100 | 30 | 60 | | | | Current | | | period 1 | | | | |
| 8 | | Are z2-values correct? | | | yes | yes | yes | yes | yes | | optimum | | 0 | 580 | 50 | | | |
| 9 | | Period 1 | | z2= | 0 | 70 | 170 | 200 | 260 | | Period 2 | | 70 | 1350 | 120 | | | |
| 10 | | f1 | C2(z2)= | | 0 | 770 | 1770 | 2070 | 2670 | | f2 | z2 | 170 | 2450 | 220 | | | |
| 11 | S | 580 | x3= | 0 | 1350 | 1350 | 111111 | 111111 | 111111 | | 1350 | 0 | 200 | 2780 | 250 | | | |
| 12 | T | 1350 | x3= | 100 | 2550 | 111111 | 2450 | 111111 | 111111 | | 2450 | 170 | 260 | 3440 | 310 | | | |
| 13 | A | 2450 | x3= | 130 | 2910 | 111111 | 111111 | 2780 | 111111 | | 2780 | 200 | | period 2 | | | | |
| 14 | G | 2780 | x3= | 190 | 3630 | 111111 | 111111 | 111111 | 3440 | | 3440 | 260 | 0 | 1350 | 0 | | | |
| 15 | E | 3440 | | | | | | | | | | | 100 | 2450 | 170 | | | |
| 16 | | | | | | | | | | | | | 130 | 2780 | 200 | | | |
| 17 | C | | | | | | | | | | | | 190 | 3440 | 260 | | | |
| 18 | O | | | | | | | | | | | | | | | | | |

Period 3:

| | A | B | C | D | E | F | G | H | I | N | O | P | Q | R | S | T | U | V |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| 2 | | Number of periods, N= | | | 5 | Current period= | | | 3 | <<Go | | | Optimum Solution Summary | | | | | |
| 3 | I | Period | 1 | 2 | 3 | 4 | 5 | | | | | | | | | | | |
| 4 | N | c(1 to 5) = | 10 | 10 | 10 | 10 | 10 | | | | | | | | | | | |
| 5 | P | K(1 to 5) = | 80 | 70 | 60 | 80 | 60 | | | | | | | | | | | |
| 6 | U | h(1 to 5) = | 1 | 1 | 1 | 1 | 1 | | | | | | x | f | z | x | f | z |
| 7 | T | D(1 to 5) = | 50 | 70 | 100 | 30 | 60 | | | | Current | | | period 1 | | | period 3 | |
| 8 | | Are z3-values correct? | | | yes | yes | yes | yes | | | optimum | | 0 | 580 | 50 | 0 | 2410 | 100 |
| 9 | | Period 2 | | z3= | 0 | 100 | 130 | 190 | | | Period 3 | | 70 | 1350 | 120 | 30 | 2740 | 130 |
| 10 | | f2 | C3(z3)= | | 0 | 1060 | 1360 | 1960 | | | f3 | z3 | 170 | 2450 | 220 | 90 | 3400 | 190 |
| 11 | S | 1350 | x4= | 0 | 2450 | 2410 | 111111 | 111111 | | | 2410 | 100 | 200 | 2780 | 250 | | | |
| 12 | T | 2450 | x4= | 30 | 2810 | 111111 | 2740 | 111111 | | | 2740 | 130 | 260 | 3440 | 310 | | | |
| 13 | A | 2780 | x4= | 90 | 3530 | 111111 | 111111 | 3400 | | | 3400 | 190 | | period 2 | | | | |
| 14 | G | 3440 | | | | | | | | | | | 0 | 1350 | 0 | | | |
| 15 | E | | | | | | | | | | | | 100 | 2450 | 170 | | | |
| 16 | | | | | | | | | | | | | 130 | 2780 | 200 | | | |
| 17 | C | | | | | | | | | | | | 190 | 3440 | 260 | | | |

Period 4:

| | A | B | C | D | E | F | G | H | I | N | O | P | Q | R | S | T | U | V |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| 2 | | Number of periods, N= | | | 5 | Current period= | | | 4 | <<Go | | | Optimum Solution Summary | | | | | |
| 3 | I | Period | 1 | 2 | 3 | 4 | 5 | | | | | | | | | | | |
| 4 | N | c(1 to 5) = | 10 | 10 | 10 | 10 | 10 | | | | | | | | | | | |
| 5 | P | K(1 to 5) = | 80 | 70 | 60 | 80 | 60 | | | | | | | | | | | |
| 6 | U | h(1 to 5) = | 1 | 1 | 1 | 1 | 1 | | | | | | x | f | z | x | f | z |
| 7 | T | D(1 to 5) = | 50 | 70 | 100 | 30 | 60 | | | | Current | | | period 1 | | | period 3 | |
| 8 | | Are z4-values correct? | | | yes | yes | yes | | | | optimum | | 0 | 580 | 50 | 0 | 2410 | 100 |
| 9 | | Period 3 | | z4= | 0 | 30 | 90 | | | | Period 4 | | 70 | 1350 | 120 | 30 | 2740 | 130 |
| 10 | | f3 | C4(z4)= | | 0 | 380 | 980 | | | | f4 | z4 | 170 | 2450 | 220 | 90 | 3400 | 190 |
| 11 | S | 2410 | x5= | 0 | 2740 | 2790 | 111111 | | | | 2740 | 0 | 200 | 2780 | 250 | | period 4 | |
| 12 | T | 2740 | x5= | 60 | 3460 | 111111 | 3450 | | | | 3450 | 90 | 260 | 3440 | 310 | 0 | 2740 | 0 |
| 13 | A | 3400 | | | | | | | | | | | | period 2 | | 60 | 3450 | 90 |
| 14 | G | | | | | | | | | | | | 0 | 1350 | 0 | | | |
| 15 | E | | | | | | | | | | | | 100 | 2450 | 170 | | | |
| 16 | | | | | | | | | | | | | 130 | 2780 | 200 | | | |
| 17 | C | | | | | | | | | | | | 190 | 3440 | 260 | | | |
| 18 | O | | | | | | | | | | | | | | | | | |

continued...

# Set 11.4d

## Period 5:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| Number of periods, N= | | | 5 | Current period= | | | 5 | <<Ma | | | Optimum Solution Summary | | | | | |
| Period | 1 | 2 | 3 | 4 | 5 | | | | | | | | | | | |
| c(1 to 5)= | 10 | 10 | 10 | 10 | 10 | | | | | | | | | | | |
| K(1 to 5)= | 80 | 70 | 60 | 80 | 60 | | | | | | | | | | | |
| h(1 to 5)= | 1 | 1 | 1 | 1 | 1 | | | | | | x | f | z | x | f | z |
| D(1 to 5)= | 50 | 70 | 100 | 30 | 60 | | | | Current | | | period 1 | | | period 3 | |
| Are z-values correct? | | | yes | yes | | | | | optimum | | 0 | 580 | 50 | 0 | 2410 | 100 |
| Period 4 | | z5= | 0 | 60 | | | | | Period 5 | | 70 | 1350 | 120 | 30 | 2740 | 130 |
| 14 | | C5(z5)= | 0 | 660 | | | | | f5 | z5 | 170 | 2450 | 220 | 90 | 3400 | 190 |
| 2740 | x6= | 0 | 3450 | 3400 | | | | | 3400 | 60 | 200 | 2780 | 250 | | period 4 | |
| 3450 | | | | | | | | | | | 260 | 3440 | 310 | 0 | 2740 | 0 |
| | | | | | | | | | | | | period 2 | | 60 | 3450 | 90 |
| | | | | | | | | | | | 0 | 1350 | 0 | | period 5 | |
| | | | | | | | | | | | 100 | 2450 | 170 | 0 | 3400 | 60 |
| | | | | | | | | | | | 130 | 2780 | 200 | | | |
| | | | | | | | | | | | 190 | 3440 | 260 | | | |

Summary of optimum solution:

z = 120   0   130   0   60

x = 70   0   30   0   0

D = 50   70   100   30   60

i = 1   2   3   4   5

Cost = $3400

4

## Period 1:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| Number of periods, N= | | | 6 | Current period= | | 1 | <<Ma | | | | Optimum Solution Summary | | | | | |
| Period | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | | | | | |
| c(1 to 6)= | 2 | 2 | 2 | 2 | 2 | 2 | | | | | | | | | | |
| K(1 to 6)= | 20 | 17 | 10 | 18 | 5 | 50 | | | | | | | | | | |
| h(1 to 6)= | 1 | 1 | 1 | 3 | 1 | 1 | | | | | x | f | z | x | f | z |
| D(1 to 6)= | 10 | 15 | 7 | 20 | 13 | 25 | | | Current | | | period 1 | | | | |
| Are z1-values correct? | | | yes | yes | yes | yes | yes | yes | optimum | | 0 | 40 | 10 | | | |
| Period 0 | | z1= | 10 | 25 | 32 | 52 | 65 | 90 | Period 1 | | 15 | 85 | 25 | | | |
| 10 | | C1(z1)= | 40 | 70 | 84 | 124 | 150 | 200 | f1 | z1 | 22 | 106 | 32 | | | |
| | x2= | 0 | 40 | 1111111 | 1111111 | 1111111 | 1111111 | 1111111 | 40 | 10 | 42 | 166 | 52 | | | |
| | x2= | 15 | 1111111 | 85 | 1111111 | 1111111 | 1111111 | 1111111 | 85 | 25 | 55 | 205 | 65 | | | |
| | x2= | 22 | 1111111 | 111111 | 106 | 1111111 | 1111111 | 1111111 | 106 | 32 | 80 | 280 | 90 | | | |
| | x2= | 42 | 1111111 | 111111 | 1111111 | 166 | 1111111 | 1111111 | 166 | 52 | | | | | | |
| | x2= | 55 | 1111111 | 111111 | 1111111 | 1111111 | 205 | 1111111 | 205 | 65 | | | | | | |
| | x2= | 80 | 1111111 | 111111 | 1111111 | 1111111 | 1111111 | 280 | 280 | 90 | | | | | | |

## Period 2

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | |
| periods, N= | | 6 | Current period= | | | 2 | <<Ma | | | Optimum Solution Summary | | | | | |
| 1 | 2 | 3 | 4 | 5 | 6 | | | | | | | | | | |
| 2 | 2 | 2 | 2 | 2 | 2 | | | | | | | | | | |
| 20 | 17 | 10 | 18 | 5 | 50 | | | | | | | | | | |
| 1 | 1 | 1 | 3 | 1 | 1 | | | | | x | f | z | x | f | z |
| 10 | 15 | 7 | 20 | 13 | 25 | | | Current | | | period 1 | | | | |
| s correct? | | yes | yes | yes | yes | yes | yes | optimum | | 0 | 40 | 10 | | | |
| | z2= | 0 | 15 | 22 | 42 | 55 | 80 | Period 2 | | 15 | 85 | 25 | | | |
| | C2(z2)= | 0 | 47 | 61 | 101 | 127 | 177 | f2 | z2 | 22 | 106 | 32 | | | |
| x3= | 0 | 85 | 87 | 1111111 | 1111111 | 1111111 | 1111111 | 85 | 0 | 42 | 166 | 52 | | | |
| x3= | 7 | 113 | 111111 | 108 | 1111111 | 1111111 | 1111111 | 108 | 22 | 55 | 205 | 65 | | | |
| x3= | 27 | 193 | 111111 | 1111111 | 168 | 1111111 | 1111111 | 168 | 42 | 80 | 280 | 90 | | | |
| x3= | 40 | 245 | 111111 | 1111111 | 1111111 | 207 | 1111111 | 207 | 55 | | period 2 | | | | |
| x3= | 65 | 345 | 111111 | 1111111 | 1111111 | 1111111 | 282 | 282 | 80 | 0 | 85 | 0 | | | |
| | | | | | | | | | | 7 | 108 | 22 | | | |
| | | | | | | | | | | 27 | 168 | 42 | | | |
| | | | | | | | | | | 40 | 207 | 55 | | | |
| | | | | | | | | | | 65 | 282 | 80 | | | |

continued...

## Period 3:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| Number of periods, N= | | | 6 | Current period= | | 3 | <<Ma | | | | Optimum Solution Summary | | | | | |
| Period | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | | | | | |
| c(1 to 6)= | 2 | 2 | 2 | 2 | 2 | 2 | | | | | | | | | | |
| K(1 to 6)= | 20 | 17 | 10 | 18 | 5 | 50 | | | | | | | | | | |
| h(1 to 6)= | 1 | 1 | 1 | 3 | 1 | 1 | | | | | x | f | z | x | f | z |
| D(1 to 6)= | 10 | 15 | 7 | 20 | 13 | 25 | | | Current | | | period 1 | | | period 3 | |
| Are z-values correct? | | | yes | yes | yes | yes | yes | | optimum | | 0 | 40 | 10 | 0 | 108 | 0 |
| Period 2 | | z3= | 0 | 7 | 27 | 40 | 65 | | Period 3 | | 15 | 85 | 25 | 20 | 169 | 27 |
| 12 | | C3(z3)= | 0 | 24 | 64 | 90 | 140 | | f3 | z3 | 22 | 106 | 32 | 33 | 208 | 40 |
| 85 | x4= | 0 | 108 | 109 | 1111111 | 1111111 | 1111111 | | 108 | 0 | 42 | 166 | 52 | 58 | 283 | 65 |
| 108 | x4= | 20 | 188 | 1111111 | 169 | 1111111 | 1111111 | | 169 | 27 | 55 | 205 | 65 | | | |
| 169 | x4= | 33 | 240 | 1111111 | 1111111 | 208 | 1111111 | | 208 | 40 | 80 | 280 | 90 | | | |
| 207 | x4= | 58 | 340 | 1111111 | 1111111 | 1111111 | 283 | | 283 | 65 | | period 2 | | | | |
| 282 | | | | | | | | | | | 0 | 85 | 0 | | | |
| | | | | | | | | | | | 7 | 108 | 22 | | | |
| | | | | | | | | | | | 27 | 168 | 42 | | | |
| | | | | | | | | | | | 40 | 207 | 55 | | | |
| | | | | | | | | | | | 65 | 282 | 80 | | | |

## Period 4:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| Number of periods, N= | | | 6 | Current period= | | 4 | <<Ma | | | | Optimum Solution Summary | | | | | |
| Period | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | | | | | |
| c(1 to 6)= | 2 | 2 | 2 | 2 | 2 | 2 | | | | | | | | | | |
| K(1 to 6)= | 20 | 17 | 10 | 18 | 5 | 50 | | | | | | | | | | |
| h(1 to 6)= | 1 | 1 | 1 | 3 | 1 | 1 | | | | | x | f | z | x | f | z |
| D(1 to 6)= | 10 | 15 | 7 | 20 | 13 | 25 | | | Current | | | period 1 | | | period 3 | |
| Are z-values correct? | | | yes | yes | yes | yes | | | optimum | | 0 | 40 | 10 | 0 | 108 | 0 |
| Period 3 | | z4= | 0 | 20 | 33 | 58 | | | Period 4 | | 15 | 85 | 25 | 20 | 169 | 27 |
| 13 | | C4(z4)= | 0 | 58 | 84 | 134 | | | f4 | z4 | 22 | 106 | 32 | 33 | 208 | 40 |
| 108 | x5= | 0 | 169 | 166 | 1111111 | 1111111 | | | 166 | 20 | 42 | 166 | 52 | 58 | 283 | 65 |
| 169 | x5= | 13 | 247 | 1111111 | 231 | 1111111 | | | 231 | 33 | 55 | 205 | 65 | | period 4 | |
| 208 | x5= | 38 | 397 | 1111111 | 1111111 | 356 | | | 356 | 58 | 80 | 280 | 90 | 0 | 166 | 20 |
| 283 | | | | | | | | | | | | period 2 | | 13 | 231 | 33 |
| | | | | | | | | | | | 0 | 85 | 0 | 38 | 356 | 58 |
| | | | | | | | | | | | 7 | 108 | 22 | | | |
| | | | | | | | | | | | 27 | 168 | 42 | | | |
| | | | | | | | | | | | 40 | 207 | 55 | | | |
| | | | | | | | | | | | 65 | 282 | 80 | | | |

## Period 5:

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| Number of periods, N= | | | 6 | Current period= | | 5 | <<Ma | | | | Optimum Solution Summary | | | | | |
| Period | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | | | | | |
| c(1 to 6)= | 2 | 2 | 2 | 2 | 2 | 2 | | | | | | | | | | |
| K(1 to 6)= | 20 | 17 | 10 | 18 | 5 | 50 | | | | | | | | | | |
| h(1 to 6)= | 1 | 1 | 1 | 3 | 1 | 1 | | | | | x | f | z | x | f | z |
| D(1 to 6)= | 10 | 15 | 7 | 20 | 13 | 25 | | | Current | | | period 1 | | | period 3 | |
| Are z-values correct? | | | yes | yes | yes | | | | optimum | | 0 | 40 | 10 | 0 | 108 | 0 |
| Period 4 | | z5= | 0 | 13 | 38 | | | | Period 5 | | 15 | 85 | 25 | 20 | 169 | 27 |
| 14 | | C5(z5)= | 0 | 31 | 81 | | | | f5 | z5 | 22 | 106 | 32 | 33 | 208 | 40 |
| 166 | x6= | 0 | 231 | 197 | 1111111 | | | | 197 | 13 | 42 | 166 | 52 | 58 | 283 | 65 |
| 231 | x6= | 25 | 381 | 1111111 | 272 | | | | 272 | 38 | 55 | 205 | 65 | | period 4 | |
| 356 | | | | | | | | | | | 80 | 280 | 90 | 0 | 166 | 20 |
| | | | | | | | | | | | | period 2 | | 13 | 231 | 33 |
| | | | | | | | | | | | 0 | 85 | 0 | 38 | 356 | 58 |
| | | | | | | | | | | | 7 | 108 | 22 | | period 5 | |
| | | | | | | | | | | | 27 | 168 | 42 | 0 | 197 | 13 |
| | | | | | | | | | | | 40 | 207 | 55 | 25 | 272 | 38 |
| | | | | | | | | | | | 65 | 282 | 80 | | | |

## Period 6

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wagner-Whitin (Forward) Dynamic Programming Inventory Model | | | | | | | | | | | | | | | | |
| Number of periods, N= | | | 6 | Current period= | | 6 | <<Ma | | | | Optimum Solution Summary | | | | | |
| Period | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | | | | | |
| c(1 to 6)= | 2 | 2 | 2 | 2 | 2 | 2 | | | | | | | | | | |
| K(1 to 6)= | 20 | 17 | 10 | 18 | 5 | 50 | | | | | | | | | | |
| h(1 to 6)= | 1 | 1 | 1 | 3 | 1 | 1 | | | | | x | f | z | x | f | z |
| D(1 to 6)= | 10 | 15 | 7 | 20 | 13 | 25 | | | Current | | | period 1 | | | period 3 | |
| Are z-values correct? | | | yes | yes | | | | | optimum | | 0 | 40 | 10 | 0 | 108 | 0 |
| Period 5 | | z6= | 0 | 25 | | | | | Period 6 | | 15 | 85 | 25 | 20 | 169 | 27 |
| 15 | | C6(z6)= | 0 | 100 | | | | | f6 | z6 | 22 | 106 | 32 | 33 | 208 | 40 |
| 197 | x7= | 0 | 272 | 297 | | | | | 272 | 0 | 42 | 166 | 52 | 58 | 283 | 65 |
| 272 | | | | | | | | | | | 55 | 205 | 65 | | period 4 | |
| | | | | | | | | | | | 80 | 280 | 90 | 0 | 166 | 20 |
| | | | | | | | | | | | | period 2 | | 13 | 231 | 33 |
| | | | | | | | | | | | 0 | 85 | 0 | 38 | 356 | 58 |
| | | | | | | | | | | | 7 | 108 | 22 | | period 5 | |
| | | | | | | | | | | | 27 | 168 | 42 | 0 | 197 | 13 |
| | | | | | | | | | | | 40 | 207 | 55 | 25 | 272 | 38 |
| | | | | | | | | | | | 65 | 282 | 80 | | period 6 | |
| | | | | | | | | | | | | | | 0 | 272 | 0 |

Optimum: Cost = $272

$z_1 = 10$, $z_2 = 22$, $z_3 = 0$

$z_4 = 20$, $z_5 = 38$, $z_6 = 0$

**1**

$i=1,\ K_1=250$:

| Period, t | $D_t$ | TC(1,t) | TCU(1,t) |
|---|---|---|---|
| 1 | 60 | 250 | 250/1 = 250 |
| 2 | 70 | 250+1×70 = 320 | 320/2 = 160* |
| 3 | 80 | 320+2×80 = 480 | 480/3 = 160* |
| 4 | 90 | 480+3×90 = 750 | 750/4 = 187.50 |

Produce 60+70+80 = 210 for 1, 2, and 3

$i=4,\ K_4=300$

| Period, t | $D_t$ | TC(4,t) | TCU(4,t) |
|---|---|---|---|
| 4 | 90 | 300 | 300/1 = 300 |
| 5 | 85 | 300+85 = 385 | 385/2 = 192.5 |
| 6 | 80 | 385+2×80 = 545 | 545/3 = 181.67 |
| 7 | 75 | 545+3×75 = 770 | 770/4 = 192.5 |

Produce 90+85+80 = 255 for 4, 5, and 6

$i=7,\ K_7=\$250$:

| Period t | $D_t$ | TC(7,t) | TCU(7,t) |
|---|---|---|---|
| 7 | 75 | 250 | 250/1 = 250 |
| 8 | 70 | 250+70 = 320 | 320/2 = 160 |
| 9 | 65 | 320+2×65 = 450 | 450/3 = 150 |
| 10 | 60 | 450+3×60 = 630 | 630/4 = 157.50 |

Produce 75+70+65 = 210 for 7, 8, and 9

$i=10,\ K_{10}=\$250$:

| Period t | $D_t$ | TC(10,t) | TCU(10,t) |
|---|---|---|---|
| 10 | 60 | 250 | 250/1 = 250 |
| 11 | 55 | 250+1×55 = 305 | 305/2 = 152.50 |
| 12 | 50 | 305+2×50 = 405 | 405/3 = 135 |

Produce 60+55+50 = 165 for 10, 11, and 12

**2**

$i=1,\ K=200$:

| t | $D_t$ | TC(1,t) | TCU(1,t) |
|---|---|---|---|
| 1 | 100 | 200 | 200/1 = 200 |
| 2 | 120 | 200+144 = 344 | 344/2 = 172 |
| 3 | 50 | 344+2×1.2×50 = 464 | 464/3 = 154.67 |
| 4 | 70 | 464+3×1.2×70 = 716 | 716/4 = 179 |

$i=4,\ K=\$200$:

| t | $D_t$ | TC(4,t) | TCU(4,t) |
|---|---|---|---|
| 4 | 70 | 200 | 200/1 = 200 |
| 5 | 90 | 200+1.2×90 = 308 | 308/2 = 154 |
| 6 | 105 | 308+2×1.2×105 = 560 | 560/3 = 186.67 |

$i=6,\ K=\$200$:

| t | $D_t$ | TC(6,t) | TCU(6,t) |
|---|---|---|---|
| 6 | 105 | 200 | 200/1 = 200 |
| 7 | 115 | 200+1.2×115 = 338 | 338/2 = 169 |
| 8 | 95 | 338+2×1.2×95 = 566 | 566/3 = 188.67 |

$i=8,\ K=\$200$:

| t | $D_t$ | TC(8,t) | TCU(8,t) |
|---|---|---|---|
| 8 | 95 | 200 | 200/1 = 200 |
| 9 | 80 | 200+1.2×80 = 296 | 296/2 = 148 |
| 10 | 85 | 296+2×1.2×85 = 500 | 500/3 = 166.67 |

$i=10,\ K=\$200$:

| t | $D_t$ | TC(10,t) | TCU(10,t) |
|---|---|---|---|
| 10 | 85 | 200 | 200/1 = 200 |
| 11 | 100 | 200+1.2×100 = 320 | 320/2 = 160 |
| 12 | 110 | 320+2×1.2×110 = 584 | 548/3 = 194.67 |

Schedule:

| Produce | For periods |
|---|---|
| 270 | 1, 2, and 3 |
| 160 | 4, and 5 |
| 220 | 6 and 7 |
| 175 | 8 and 9 |
| 185 | 10 and 11 |
| 110 | 12 |

Continued...

# Chapter 15

# Queuing Systems

**1**

(a) Efficiency = $100 - 29 = 71\%$

(b) For average waiting time $\le 3$ minutes, at least 5 cashiers are needed

For efficiency $\ge 90\%$, the associated idleness percentage is $\le 10\%$. The corresponding number of cashiers is at most 2.

Conclusion:

The two conditions cannot be satisfied simultaneously. At least one of the two conditions must be relaxed.

**2**

$C_A = \$18$ per hour

$C_B = \$25$ per hour

Length of queue A = 4 jobs

Length of queue B = $.7 \times 4 = 2.8$ jobs

Cost of A = $\$18 + 4 \times \$10 = \$58$ per hour

Cost of B = $\$25 + 2.8 \times \$10 = \$53$ per hour

Decision:

Select Model B.

**1**

| Situation | Customer | Server |
|---|---|---|
| a | Plane | Runway |
| b | Passenger | Taxi |
| c | Machinist | Clerk at tool crib |
| d | Letter | Clerk |
| e | Student | Registrar's office |
| f | Cases | Judge |
| g | Shopper | Cashier |
| h | Car | Parking space |

**2**

| Situation | Calling Source | Customers arrival |
|---|---|---|
| a | $\infty$ | Individual |
| b | $\infty$ | Individual |
| c | $\infty$ | Individual |
| d | $\infty$ | Bulk |
| e | $\infty$ | Individual |
| f | $\infty$ | Individual |
| g | $\infty$ | Individual |
| h | $\infty$ | Individual |

| Situation | Interarrival time | Service time |
|---|---|---|
| a | Probabilistic | Time to clear runway |
| b | Probabilistic | Ride time |
| c | Probabilistic | Time to receive tool |
| d | Deterministic | Time to process letter |
| e | Probabilistic | Time to process registr$^n$ |
| f | Probabilistic | Trial time |
| g | Probabilistic | check-out time |
| h | Probabilistic | Parking time. |

| Situation | Queue Capacity | Queue Discipline |
|---|---|---|
| a | $\infty$ | FIFO |
| b | $\infty$ | FIFO |
| c | $\infty$ | FIFO |
| d | $\infty$ | Random |
| e | $\infty$ | FIFO |
| f | $\infty$ | FIFO |
| g | $\infty$ | FIFO |
| h | 0 | None |

**3**

| # | Queueing situation | Customers |
|---|---|---|
| 1 | Arrival of orders | Orders |
| 2 | Processing (single machine) | Rush orders |
| 3 | Processing (single machine) | Regular jobs |
| 4 | Processing (Prod. line) | Rush jobs |
| 5 | Processing (Prod. line) | Regular jobs |
| 6 | Receipt of completed jobs | Completed orders |
| 7 | Tool crib | Tools |
| 8 | Machine breakdown | machines |

| # | Servers | Discipline | Service time | Queue length | Source |
|---|---|---|---|---|---|
| 1 | Foreman | Priority | Sorting time | $\infty$ | $\infty$ |
| 2 | machine | FIFO | Prod. time | $\infty$ | $\infty$ |
| 3 | machine | FIFO | Prod. time | $\infty$ | $\infty$ |
| 4 | Prod. line | FIFO | Prod. time | $\infty$ | $\infty$ |
| 5 | Prod. line | FIFO | Prod. time | $\infty$ | $\infty$ |
| 6 | Shipping facilities | FIFO | Loading time | finite | $\infty$ |
| 7 | Tool crib | Priority | Exchange time | finite | finite |
| 8 | Repair persons | Priority | Repair time | finite | finite |

**4**

(a) T. (b) T. (c) T.

**5**

(a) None.
(b) None.
(c) None.
(d) None.
(e) Jockey or balk
(f) None
(g) Jockey
(h) None

**1**

(a) Av. interarrival time (in time units)

$$= \frac{1}{\text{arrival rate } \lambda \text{ (in customers/unit time)}}$$

(b) Let $\bar{I}$ = av. interarrival time

(i) $\lambda = \frac{60}{10} = 6$ arrivals/hr

$\bar{I} = 10$ minutes $= \frac{1}{6}$ hour

(ii) $\lambda = \frac{60}{3} = 20$ arrivals/hr

$\bar{I} = \frac{6}{2} = 3$ minutes $= \frac{1}{20}$ hr

(iii) $\lambda = \frac{10}{30} \times 60 = 20$ arrivals/hr

$\bar{I} = \frac{30}{10} = 3$ minutes $= \frac{1}{20}$ hour

(iv) $\lambda = 1/.5 = 2$ arrivals/hour

$\bar{I} = .5$ hour

(c) Let $\bar{S}$ = av. service time

(i) $\mu = \frac{60}{12} = 5$ services/hour

$\bar{S} = 12$ minutes $= .2$ hour

(ii) $\mu = \frac{60}{7.5} = 8$ services/hr

$\bar{S} = 7.5$ min $= .125$ hr

(iii) $\mu = \frac{5}{30} \times 60 = 10$ services/hr

$\bar{S} = \frac{30}{5} = 6$ min $= 1/10$ hr

(iv) $\mu = \frac{1}{.3} = 3.33$ services/hr

$\bar{S} = .3$ hour

**2**

(a) $\lambda_{hour} = .2$ failures/hr

$\lambda_{week} = .2 \times 24 \times 7 = 33.6$ failures/wk

(b) P{at least one failure in 2 hours}

= P{time betn. failures $\le 2$}

$= P\{t \le 2\} = 1 - e^{-.2 \times 2} \cong .33$

(c) $P\{t > 3 \text{ hrs}\} = 1 - P\{t \le 3\} = e^{-.2 \times 3} \cong .55$

(d) $P\{t \le 1 \text{ hour}\} = 1 - e^{-.2 \times 1} = .18$

**3**

$\lambda = \frac{1}{.05} = 20$ arrivals/hr

(a) $f(t) = \lambda e^{-\lambda t}$

$= 20 e^{-20t}, \quad t > 0$

(b) $P\{t > \frac{15}{60}\} = P\{t > .25\}$

$= e^{-20 \times .25}$

$= .00674$

(c) $P\{t \le \frac{3}{60}\} = P\{t \le .05\}$

$= 1 - e^{-20 \times .05} = .632$

$P\{t > \frac{5}{60}\} = e^{-\frac{20 \times 5}{60}} = .189$

(d) $t = 45 - 10 = 35$ minutes

Av. # of arrivals in 35 min.

$= 20 \times \frac{35}{60} = 11.67$ arrivals

**4**

$\lambda = \frac{1}{6}$ arrivals/hr

$P\{t \ge 1\} = e^{-1/6 \times 1} = .846$

$P\{t \le .5\} = 1 - e^{-1/6 \times .5}$

$= 1 - e^{-1/12} = .08$

**5**

(a) $\lambda = \frac{60}{10} = 6$ arrivals/hr

(b) $P\{t \ge \frac{15}{60}\} = e^{-6 \times \frac{15}{60}} = .223$

(c) $P\{t \le \frac{20}{60}\} = 1 - e^{-6 \times \frac{20}{60}} = .865$

**6**

(a) $P\{t \le \frac{2}{60}\} = 1 - e^{-35(2/60)}$

$= .6886$

(b) $P\{\frac{2}{60} \le t \le \frac{3}{60}\}$

$= P\{t \le 3/60\} - P\{t \le \frac{2}{60}\}$

$= (1 - e^{-35 \times 3/60}) - (1 - e^{-35 \times 2/60})$

$= e^{-70/60} - e^{-105/60} = .1376$

(c) $P\{t \ge 3/60\} = e^{-35(3/60)}$

$= .1738$

**7**

$\lambda = \frac{60}{1.5} = 40$ arrivals/hr

| Jim's Payoff | $-2¢$ | $+2¢$ |
|---|---|---|
| Prob. | $P\{t \geq 1\}$ | $P\{t \leq 1\}$ |

$P\{t \geq 1\} = e^{-40(1/60)} = .5134$

$P\{t \leq 1\} = 1 - .5134 = .4866$

Jim's exp. payoff/arriving customer

$= -2 \times .5134 + 2 \times .4866$

$= -.0536$ cent

Jim's exp. payoff/8 hours

$= -.0536(8\lambda)$

$= -.0536 \times 8 \times 40$

$\cong -17.15$ cent

Conclusion: Jim will pay Ann an average of 17 cents every 8 hrs

**8**

| Jim's payoff | 2 | 0 | -2 |
|---|---|---|---|
| Probability | $P\{t \leq 1\}$ | $P\{1 \leq t \leq 1.5\}$ | $P\{t \geq 1.5\}$ |

$P\{t \leq 1\} = .4866$

$P\{t \geq 1.5\} = e^{-40(1.5/60)}$

$= .3679$

| 2 | 0 | -2 |
|---|---|---|
| .4866 | .1455 | .3679 |

Jim's expected payoff/8 hours

$= [2 \times .4866 + 0 \times .1455 - 2 \times .3679] \times 40 \times 8$

$\cong 76$ cents

**9**

| $2¢$ | $3¢$ | $-5¢$ | $-6¢$ |
|---|---|---|---|
| $t \leq 1$ | $1 \leq t \leq 1.5$ | $1.5 \leq t \leq 2$ | $t \geq 2$ |

$\lambda = 40$ arrivals/hr

$P\{t \leq 1\} = 1 - e^{-40/60} = .4866$

$P\{1 \leq t \leq 1.5\} = e^{-40(1/60)} - e^{-40(1.5/60)}$

$= .1455$

$P\{1.5 \leq t \leq 2\} = e^{-40(1.5/60)} - e^{-40(2/60)}$

$= .1043$

$P\{t \geq 2\} = e^{-40(2/60)} = .2636$

Jim's exp. payoff/8 hours

$= 8 \times 40\,(2 \times .4866 + 3 \times .1455 - 5 \times .1043 - 6 \times .2636)$

$\simeq -222$ cents

Jim pays Ann an average of \$2.22/8 hours.

**10**

(a) $\lambda = \frac{60}{6} = 10$ customers/hr

$P\{t \leq 4 \text{ min}\} = 1 - e^{-10(4/60)}$

$= .4866$

(b)

$$\%\text{discount} = \begin{cases} 10\%, & \text{if } t \leq 4 \\ 6\%, & \text{if } 4 < t \leq 5 \\ 2\%, & \text{if } t > 5 \end{cases}$$

$P\{t \leq 4\} = .4866$

$P\{4 < t \leq 5\} = e^{-10(4/60)} - e^{-10(5/60)}$

$= .0788$

$P\{t > 5\} = e^{-10(5/60)} =$

$= .4346$

Expected % discount

$= 10 \times .4866 + 6 \times .0788 + 2 \times .4346$

$= 6.208\%$

**11**

$$\lambda = \frac{365 \times 24}{9000} = .973 \text{ failure/yr}$$

$$P\{t \le 1\} = 1 - e^{-.973 \times 1}$$
$$= .622$$

**12**

Lack-of-memory property applies.

(a) The waiting time for the green bus is exponential with mean 10 minutes:

$$f(t) = .1 \, e^{-.1t}, \quad t \ge 0$$

(b) The waiting time for the red bus is exponential with mean 7 minutes:

$$f(t) = \frac{1}{7} e^{-t/7}, \quad t \ge 0$$

**13**

$$E\{t\} = \int_0^\infty t \lambda e^{-\lambda t} dt$$
$$= -\int_0^\infty t \, d e^{-\lambda t}$$
$$= -\left(t e^{-\lambda t} - \int_0^\infty e^{-\lambda t} dt\right)$$
$$= -\left(t e^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t}\right)\Big]_0^\infty$$
$$= 1/\lambda$$

$$E\{t^2\} = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$
$$= -\int_0^\infty t^2 d e^{-\lambda t}$$
$$= -\left[t^2 e^{-\lambda t} - \int_0^\infty 2t \, e^{-\lambda t} dt\right]$$

continued…

$$= -\left[t^2 e^{-\lambda t} - \frac{2}{\lambda}\int_0^\infty t \lambda e^{-\lambda t} dt\right]_0^\infty$$
$$= +\frac{2}{\lambda^2}$$

$$Var\{t\} = E\{t^2\} - E^2\{t\}$$
$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$
$$= \frac{1}{\lambda^2}$$

**1**

TORA input $= (5, 0, 0, \infty, \infty)$

$p_{n\geq 5}(t=1hr) = 1 - [p_0(1) + \cdots + p_4(1)]$

$= 1 - e^{-5}\left(1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right)$

$= 1 - .44049 = .55951$

**2**

$\lambda = 1$ trip/month

(a) $\lambda t = 3$: TORA input $= (3, 0, 0, \infty, \infty)$

$p_0(3) = \frac{(1\times 3)^0 e^{-1\times 3}}{0!} = .049787$

(b) $\lambda t = 12$: TORA input $= (12, 0, 0, \infty, \infty)$

$p_{n\leq 8}(t=12) = p_0(12) + \cdots + p_8(12)$

$= \frac{12^0 e^{-12}}{0!} + \frac{12^1 e^{-12}}{1!} + \cdots + \frac{12^8 e^{-12}}{8!}$

$= .15503$

(c) $p_0(1) = \frac{1^0 e^{-1}}{0!} = e^{-1} = .3679$

TORA input $= (1, 0, 0, \infty, \infty)$

**3**

$\lambda = 2$ arrivals/minute

(a) $\lambda t = 2\times 5 = 10$ arrivals

(b) $\lambda t = 2\times .5 = 1$

TORA input $= (1, 0, 0, \infty, \infty)$

$p_0(t=.5) = e^{-2\times .5} = .3679$

(c) $1 - p_0(t=.5) = 1 - .3679 = .6321$

(d) $\lambda t = 2\times 3 = 6$ arrivals

TORA input $= (6, 0, 0, \infty, \infty)$

$p_0(t=3) = \frac{(2\times 3)^0 e^{-2\times 3}}{0!} = .00248$

**4**

$\lambda = 1/5 = .2$ arrival/min

(a) $p_2(t=7) = \frac{(.2\times 7)^2 e^{-.2\times 7}}{2!} = .24167$

TORA input $= (1.4, 0, 0, \infty, \infty)$

(b) $p_1(t=5) = \frac{(.2\times 5)^1 e^{-.2\times 5}}{1!} = .36788$

**5**

$\lambda = 25$ books per day

(a) $\lambda t = 25\times 30 = 750$ books $= 7.5$ shelves

(b) 10 bookcases $= 10\times 5\times 100 = 5000$ books

$p_{n>5000}(t=30) = 1 - [p_0(30) + \cdots + p_{5000}(30)]$

$\simeq 0$

**6**

(a) $\lambda_{green} = .1$ stop/min, $\lambda_{red} = 1/7$ stop/min

$\lambda$ combined $= .1 + \frac{1}{7} = .24286$ stop/min

$p_2(5) = \frac{(.24286\times 5)^2 e^{-.24286\times 5}}{2!} = .219$

The two buses could be 2 G, 2R or 1G and 1R.

(b) $P\{t \leq 2\} = 1 - e^{-.243\times 2} = .3849$

**7**

$E\{n|t\} = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

$= \lambda t\, e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$

$= \lambda t\, e^{-\lambda t} e^{+\lambda t} = \lambda t$

$E\{n^2|t\} = \sum_{n=0}^{\infty} n^2 \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

$= \sum_{n=1}^{\infty} n^2 \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

$= \lambda t\, e^{-\lambda t} \sum_{n=1}^{\infty} \frac{n(\lambda t)^{n-1}}{(n-1)!}$

$= \lambda t\, e^{-\lambda t} \frac{\partial}{\partial \lambda t}\left(\lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}\right)$

$= \lambda t\, e^{-\lambda t} \frac{d}{\partial \lambda t}\left(\lambda t\, e^{\lambda t}\right)$

$= \lambda t\, e^{-\lambda t}\left(\lambda t\, e^{\lambda t} + e^{\lambda t}\right)$

$= (\lambda t)^2 + \lambda t$

Thus,

$var\{n|t\} = (\lambda t)^2 + \lambda t - (\lambda t)^2$

$= \lambda t$

**8**

$$p_0'(t) = -\lambda p_0(t) \quad (1)$$

$$p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \quad (2)$$

From (1)

$$d p_0(t) = -\lambda p_0(t) dt$$

which yields

$$p_0(t) = A e^{-\lambda t}$$

Because $p_0(0) = 1 \Rightarrow A = 1$, $p_0(t) = e^{-\lambda t}$

For $n = 1$:

$$p_1'(t) = -\lambda p_1(t) + \lambda p_0(t)$$

$$= -\lambda p_1(t) + \lambda e^{-\lambda t}$$

or

$$p_1'(t) + \lambda p_1(t) = \lambda e^{-\lambda t}$$

This yields the solution:

$$p_1(t) = e^{-\int \lambda dt}\left\{\int \lambda e^{-\lambda t} e^{-\int \lambda dt} dt + C\right\}$$

$$= \lambda t e^{-\lambda t} + C$$

Because $p_1(0) = 0$, $C = 0$, and

$$p_1(t) = \frac{\lambda t e^{-\lambda t}}{1!}$$

Induction proof:

Given

$$p_i(t) = \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

then

$$p_{i+1}'(t) + \lambda p_{i+1}(t) = \lambda \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

The solution is

$$p_{i+1}(t) = e^{-\int \lambda dt}\left\{\int \frac{\lambda (\lambda t)^i e^{-\lambda t} e^{\int \lambda dt}}{i!} dt + C\right\}$$

$$= \frac{e^{-\lambda t} (\lambda t)^{i+1}}{(i+1)!} + C$$

Because $p_{i+1}(0) = 0$, $C = 0$, and

$$p_{i+1}(t) = \frac{e^{-\lambda t} (\lambda t)^{i+1}}{(i+1)!}$$

continued...

**2**

$\mu = 3$ dozens/day, $N=18$

TORA input data $=(0, \mu t, 1, 18, 18)$

(a) $\mu = 3\times3 = 9$

$p_0(t=3) = .00532$ (from TORA)

(b) $\mu t = 3\times2 = 6$

$\sum_{n=0}^{18} n\, p_n(2) = 11.955$

(c) This part can be solved using Poisson or exponential distributions:

Poisson: $\mu t = 3\times1 = 3$

Probability $= p_0(1)+p_1(1)+\dots+p_{17}(1)$

$= .9502$ (from TORA)

Exponential: mean $= 1/3$ day

P{purchasing at least one dozen in 1 day}

$= P\{\text{time between purchases} \le 1\}$

$= 1-e^{-3\times1} = .9502$

(d) Exponential: $P\{t\le .5\} = 1-e^{-3\times.5} = .7769$

Poisson: $p_0(.5)+p_1(.5)+\dots+p_{17}(.5) = .7769$

(e) $p_0(1) = 0$ $(\mu t = 3\times1 = 3)$

**3**

$N=40$, $\mu = 10$ calls/hr

TORA input $(0, \mu t, 1, 40, 40)$

(a) $p_{n>0}(t=4) = 1-p_0(4)$

$= 1-.521 = .479$

(b) $E\{n|t=4\} = \sum_{n=0}^{40} n\, p_n(4) \simeq 2.5$ blocks $\cong 25$ tickets

**4**

$N=48$, $\mu = \frac{4\times10}{8} = 5$ cans/hr

$\mu t = 5\times4 = 20$ cans

$p_0(4) \cong .000005$ (from TORA)

$N=48$, $\mu t = 5\times8 = 40$, $p_0(8) = .11958$

**5**

$\mu = 1/1 = 1$ withdrawl/week

$N=5$, $\mu t = 4$

$p_0(4) = .37116$

**6**

$N=80$ items, $\mu = 5$ items/day

(a) $\mu t = 5\times2 = 10$ items

$p_{70}(2) = .1251$

(b) $\mu t = 5\times4 = 20$ items

$p_0(4) = .00001$

(c) $\mu t = 5\times4 = 20$ items

$E\{n|4\text{days}\} = \sum_{n=0}^{80} n\, p_n(4) \simeq 60$ items

Av. # of withdrawls $= 80-60$

$= 20$ items

**7**

$\mu = 1/1 = 1$ breakdown/day

$N=10$, $\mu t = 1\times2 = 2$

From TORA, $p_0(2) = .00005$

**8**

(a) $N=25$, $\mu = 3$/day

$t = 6$ days, $\mu t = 18$

Av. stock remaining after 6 days

$= E\{n|t=6\} = 7.11$

Av. order size $= 25-7.11$

$\simeq 18$ items

(b) $t=4$, $\mu t = 3\times4 = 12$

$p_0(4) = .00069$

(c) $t=6$, $\mu t = 3\times6 = 18$

$p_{n\le14}(6) = p_0(6)+\dots+p_{14}(6) = .9696$

**9**

P{time betw. departures $>T$}

$= P\{\text{no departures during } T\}$

$= P\{N \text{ left after time } T\}$

$= p_N(T)$

$P\{t>T\} = p_N(T) = \frac{(\mu T)^0 e^{-\mu T}}{0!}$

$= e^{-\mu T}$

**10**

$p_N'(t) = -\mu\, p_N(t)$ (1)

$p_n'(t) = -\mu\, p_n(t) + \mu\, p_{n+1}(t),\quad 0 \le n < N$ (2)

From (1), we get

$$p_N(t) = C e^{-\mu t}$$

Given $p_N(0) = 1$, then $c = 1$ and

$$p_N(t) = e^{-\mu t}$$

Next, consider (2) for $n = N-1$

$$p_{N-1}'(t) = -\mu\, p_{N-1}(t) + \mu\, p_N(t)$$
$$= -\mu\, p_{N-1}(t) + \mu\, e^{-\mu t}$$

Thus,

$$p_{N-1}(t) = e^{-\int \mu t}\left\{\int \mu e^{-\mu t} e^{\int \mu dt} dt + C\right\}$$
$$= e^{-\mu t}\mu t + C$$

Because $p_{N-1}(0) = 0$, $C = 0$ and $p_{N-1}(t) = (\mu t)e^{-\mu t}$

<u>Induction proof</u>:

Given $p_{n+1}(t) = \dfrac{(\mu t)^{N-n-1} e^{-\mu t}}{(N-n-1)!}$, then

$$p_n'(t) = -\mu\, p_n(t) + \frac{\mu\, (\mu t)^{N-n-1} e^{-\mu t}}{(N-n-1)!}$$

Solution gives

$$p_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}$$

**1**

(a) $P\{0 \text{ counter open}\} = p_0 = \frac{1}{55}$

$P\{1 \text{ counter open}\} = p_1 + p_2 + p_3$

$= \frac{1}{55}(2 + \quad + 8) = \frac{14}{55}$

$P\{2 \text{ counters open}\} = p_4 + p_5 + p_6$

$= \frac{1}{55}(8+8+8) = \frac{24}{55}$

$P\{3 \text{ counters open}\} = p_7 + p_8 + \cdots$

$= 1 - (p_0 + \cdots + p_6)$

$= 1 - (\frac{1}{55} + \frac{14}{55} + \frac{24}{55}) = \frac{16}{55}$

(b) Av. # busy counters

$= 0 \times \frac{1}{55} + 1 \times \frac{14}{55} + 2 \times \frac{24}{55} + 3 \times \frac{16}{55}$

$= 2$ counters

(c) Av. # idle counters $= 3 - 2 = 1$

**2**

$\lambda = 1/5 = .2$ arrival/min

$= 12$ arrivals/hr

(a)
$$\mu_n = \begin{cases} 5 \text{ customers/hr}, & n = 0,1,2 \\ 10 \text{ customers/hr}, & n = 3,4 \\ 15 \text{ customers/hr}, & n = 5,6 \\ 20 \text{ customers/hr}, & n \geq 7 \end{cases}$$

$p_1 = \frac{12}{5} p_0 = 2.4 p_0$

$p_2 = (\frac{12}{5})^2 p_0 = 5.76 p_0$

$p_3 = (\frac{12}{5})^2 (\frac{12}{10}) p_0 = 6.912 p_0$

$p_4 = (\frac{12}{5})^2 (\frac{12}{10})^2 p_0 = 8.2944 p_0$

$p_5 = (\frac{12}{5})^2 * (\frac{12}{10})^2 (\frac{12}{15}) p_0 = 6.63552 p_0$

$p_6 = (\frac{12}{5})^2 (\frac{12}{10})^2 (\frac{12}{15})^2 p_0 = 5.308416 p_0$

$p_{n\geq7} = (\frac{12}{5})^2 (\frac{12}{10})^2 (\frac{12}{15})^2 (\frac{12}{20})^{n-6} p_0 = 5.308416 (.6)^{n-6} p_0$

From $\sum_{n=0}^{\infty} p_n = 1$, we get $p_0 = .002587$

$p_1 = .05421$, $p_2 = .13010$, $p_3 = .15612$

$p_4 = .18735$, $p_5 = .14988$, $p_6 = .1199$

$p_{n\geq7} = .1199(.6)^{n-6}$

(b) $p_{n\geq7} = 1 - (p_0 + p_1 + \cdots + p_6) = .8$

Continued…

(c) $P\{0 \text{ counter}\} \Rightarrow p_0 = .002587$

$P\{1 \text{ counter}\} = p_1 + p_2 = .18431$

$P\{2 \text{ counters}\} = p_3 + p_4 = .34347$

$P\{3 \text{ counters}\} = p_5 + p_6 = .26978$

$P\{4 \text{ counters}\} = p_7 + p_8 + \cdots = .199853$

Av. # idle counters

$= 4 - (1 \times .18431 + 2 \times .34347 + 3 \times .26978$

$+ 4 \times .199853) \cong 1.52$

**3**

$$\mu_n = \begin{cases} 5n, & n = 1,2 \\ 15, & n = 3,4,\ldots \end{cases}$$

$p_1 = (\frac{10}{5}) p_0 = 2 p_0$

$p_2 = (\frac{10}{5})(\frac{10}{10}) p_0 = 2 p_0$

$p_{n\geq3} = (\frac{10}{5})(\frac{10}{10})(\frac{10}{15})^{n-2} p_0 = 2(\frac{2}{3})^{n-2} p_0$

Thus,

$p_0 + 2p_0 + 2p_0 + [2(\frac{2}{3}) + 2(\frac{2}{3})^2 + \cdots] p_0 = 1$

which gives $p_0 = .1111$

(a) Prob that 3 counters are in use

$= p_{n\geq3} = 1 - (p_0 + p_1 + p_2)$

$= 1 - (.1111 + .2222 + .2222)$

$= .4445$

(b) $p_{n\leq2} = p_0 + p_1 + p_2 = .5555$

**4**

$$\lambda_n = \begin{cases} 12 \text{ cars/hr}, & n = 0,1,\ldots,10 \\ 0 & n \geq 11 \end{cases}$$

$\mu_n = 60/6 = 10$ cars/hr

$p_n = (\frac{12}{10})^n p_0, \quad n = 1,2,\ldots,10$

$= 0, \quad n \geq 11$

$p_0(1 + 1.2 + 1.2^2 + \cdots + 1.2^{10}) = p_0 \frac{1 - 1.2^{11}}{1 - 1.2}$

Thus, $p_0 = .0311$

Continued…

4 continued

(a) $p_{10} = \left(\frac{12}{10}\right)^{10} p_0 = .19259$

(b) $p_{n\geq 1} = 1 - p_0 = 1 - .0311 = .9689$

(c) Av. length of the lane

$= 0p_0 + 1p_1 + \cdots + 10p_{10}$

$= 1\times.03732 + 2\times.04479 + 3\times.05375$
$+ 4\times.0645 + 5\times.0774 + 6\times.09288$
$+ 7\times.11145 + 8\times.13374 + 9\times.16049$
$+ 10\times.19259 = 6.71071$

5

$\lambda_n = 6$ arrivals/hr, $n = 0, 1, \ldots, 8$

$= 5$ arrivals/hr, $n = 9, 10, \cdots 11, 12$

$\mu_n = n/.5 = 2n$/hr, $n = 1,2,3,4$

$= 10$/hr, $n \geq 5$

| $n$ | $p_n$ |
|---|---|
| 0 | 0.049526 |
| 1 | 0.148578 |
| 2 | 0.222866 |
| 3 | 0.222866 |
| 4 | 0.16715 |
| 5 | 0.10029 |
| 6 | 0.060174 |
| 7 | 0.036104 |
| 8 | 0.021663 |
| 9 | 0.010831 |
| 10 | 0.005416 |
| 11 | 0.002708 |
| 12 | 0.001354 |

$p_1 = \frac{6}{2} p_0 = 3p_0$

$p_2 = \frac{6}{2}\cdot\frac{6}{4} p_0 = 4.5p_0$

$p_3 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot p_0 = 4.5p_0$

$p_4 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8} p_0 = 3.375p_0$

$p_5 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\cdot\frac{6}{10} p_0 = 2.025p_0$

$p_6 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\cdot\frac{6}{10}\cdot\frac{6}{10} p_0 = 1.215p_0$

$p_7 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\cdot\frac{6}{10}\cdot\frac{6}{10}\cdot\frac{6}{10} p_0 = .729p_0$

$p_8 = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\left(\frac{6}{10}\right)^4 p_0 = .4374p_0$

$p_{n\geq 9} = \frac{6}{2}\cdot\frac{6}{4}\cdot\frac{6}{6}\cdot\frac{6}{8}\left(\frac{6}{10}\right)^4\left(\frac{5}{10}\right)^{n-8} p_0 = .4374(.5)^{n-8} p_0$

From $\sum_{n=0}^{12} p_n = 1$, we get $p_0 = .0495$

(a) $p_{12} = .4374\times.5^4\times.0495 = .00135$

(b) $p_{n\geq 5} = 1 - (p_0 + p_1 + \cdots + p_4) = .2385$

(c) Av. # busy tables $= 0p_0 + 1p_1 + 2p_2 + 3p_3$
$+ 4p_4 + 5p_{n\geq 5} = 2.9768$

5 continued

(d) $1p_6 + 2p_7 + \cdots + 7p_{12}$

$= 1\times.0602 + 2\times.0361 + 3\times.0217$
$+ 4\times.0108 + 5\times.0054 +$
$6\times.0027 + 7\times.00135$

$= .2935$ pair

6

$\lambda = 4$ customers/hr

$\lambda_n = \begin{cases} 4, & n = 0, 1, \ldots, 4 \\ 0, & n \geq 5 \end{cases}$

$\mu_n = \frac{60}{15} = 4$ customers/hr

(a) $p_1 = \frac{4}{4} p_0$

$p_2 = \left(\frac{4}{4}\right)^2 p_0$

$p_3 = \left(\frac{4}{4}\right)^3 p_0$

$p_4 = \left(\frac{4}{4}\right)^4 p_0$

$p_0 + p_1 + \cdots + p_4 = 1 \implies p_0 = 1/5$

$p_0 = p_1 = p_2 = p_3 = p_4 = 1/5$

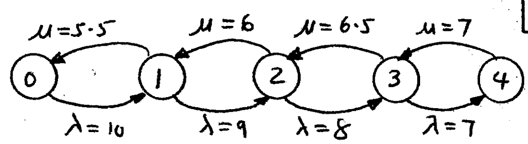
(b) expected # in shop $=$

$0p_0 + 1p_1 + 2p_2 + 3p_3 + 4p_4$

$= \frac{1}{5}(1+2+3+4) = 2$

(c) $p_4 = .2$

7




(a) $5.5p_1 = 10p_0$

$10p_0 + 6p_2 = (5.5+9)p_1$

$9p_1 + 6.5p_3 = (6+8)p_2$

$8p_2 + 7p_4 = (6.5+7)p_3$

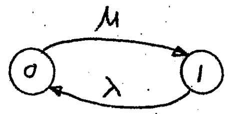
(b) $p_1 = 1.82p_0$, $p_2 = 2.727p_0$

$p_3 = 3.3566p_0$, $p_4 = 3.3566p_0$

$p_0 + p_1 + \cdots + p_4 = 1 \implies p_0 = .08882$

$p_1 = .1614$, $p_2 = .2422$, $p_3 = .2981$, $p_4 =$

**8**




(a) $\mu p_1 = \lambda p_0$

$p_1 = \frac{\lambda}{\mu} p_0$

(b) $p_0 + \frac{\lambda}{\mu} p_0 = 1$

$p_0 = \frac{1}{1+\rho}$ , $\rho = \lambda/\mu$

$p_1 = \frac{\rho}{1+\rho}$

(c) $L_s = 0p_0 + 1p_1 = \frac{\rho}{1+\rho}$

(d) $\lambda_{eff} = \lambda p_0 = \frac{\lambda}{1+\rho}$

(e) $W_q = \frac{L_s}{\lambda_{eff}} - \frac{1}{\mu}$

$= \frac{\rho/(1+\rho)}{\lambda/(1+\rho)} - \frac{1}{\mu} = 0$

**9**

$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} =$

$\lambda_{n-1}\left(\frac{\lambda_0}{\mu_1} \cdot \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-2}}{\mu_{n-1}}\right) +$

$\mu_{n+1}\left(\frac{\lambda_0}{\mu_1} \cdot \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_n}{\mu_{n+1}}\right)$

$= \mu_n\left(\frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-1}}{\mu_n}\right) +$

$\lambda_n\left(\frac{\lambda_0}{\mu_1} \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{n-1}}{\mu_n}\right)$

$= \mu_n p_n + \lambda_n p_n$

$= (\mu_n + \lambda_n) p_n$

**1**

(a) $L_q = \sum_{n=6}^{8}(n-5)p_n$

$= 1p_6 + 2p_7 + 3p_8$

$= 1 \times .05847 + 2 \times .03508 + 3 \times .02105$

$= .19177$

(b) $W_q = \dfrac{L_q}{\lambda_{eff}}$

$= \dfrac{.1917}{5.8737} = .03265$ hour

$W_s = W_q + \dfrac{1}{\mu}$

$= .03264 + \dfrac{1}{2} = .53265$ hour

(c) $\lambda_{lost} = \lambda p_8$

$= 6 \times .02105 = .1263$ car/hr

Number lost/8 hrs $= .1263 \times 8 = 1.01$ cars

(d) Average number of empty spaces

$= c - (L_s - L_q)$

$= c - \sum_{n=0}^{8} np_n + \sum_{n=c+1}^{8}(n-c)p_n$

$= \left(c\sum_{n=0}^{8} p_n - c\sum_{n=c+1}^{8} p_n\right) - \left(\sum_{n=0}^{8} np_n - \sum_{n=c+1}^{8} np_n\right)$

$= c\sum_{n=0}^{c} p_n - \sum_{n=0}^{c} np_n$

$= \sum_{n=0}^{c-1}(c-n)p_n$

**2**

(a)

$\lambda_n = 6$ cars/hr, $n = 0, 1, \ldots, 6$

$\mu_n = \begin{cases} \left(\frac{4}{3}\right)n, & n = 1, 2, \ldots, 6 \\ 8, & n = 7, 8, 9, 10 \end{cases}$

$p_n = \left(\frac{6}{4/3}\right)^n \frac{1}{n!} p_0,\ n = 0, 1, \ldots, 6$

continued...

2 continued

$p_n = \dfrac{\left(\frac{6}{4/3}\right)^n}{6!\ 6^{n-6}} p_0,\ n = 7, 8, 9, 10$

$p_0\left(1 + \frac{9/2}{1!} + \frac{(9/2)^2}{2!} + \frac{(9/2)^3}{3!} + \frac{(9/2)^4}{4!} + \frac{(9/2)^5}{5!} + \frac{(9/2)^6}{6!} + \frac{(9/2)^7}{6!\,6} + \frac{(9/2)^8}{6!\,6^2} + \frac{(9/2)^9}{6!\,6^3} + \frac{(9/2)^{10}}{6!\,6^4}\right) = 1$

Thus, $p_0 = .0004$

| $n$ | $p_n$ | $n$ | $p_n$ |
|---|---|---|---|
| 1 | .00304 | 6 | .10027 |
| 2 | .01141 | 7 | .12534 |
| 3 | .02852 | 8 | .15667 |
| 4 | .05348 | 9 | .19584 |
| 5 | .08022 | 10 | .24480 |

(b) $\lambda_{eff} = \lambda(1 - p_{10}) = 10(1 - .2448)$

$= 7.552$ cars/hr

(c) $L_s = 0p_0 + 1p_1 + 2p_2 + \cdots + 10p_{10}$

$= 7.6941$ cars

(d) $W_s = \dfrac{L_s}{\lambda_{eff}} = \dfrac{7.6941}{7.552} = 1.0155$ cars

$W_q = 1.0155 - \dfrac{1}{4/3} = .2655$

(e) $L_q = \lambda_{eff} W_q$

$= .2655 \times 7.552$

$= 2.005$ cars

Average number of occupied spaces $= L_s - L_q$

$= 7.6941 - 2.005$

$= 5.6891$ spaces

## 1

(a) $\%\text{ utilization} = 100(1-p_0)$

$$= 100\frac{\lambda}{\mu}$$

$$= 100\left(\frac{4}{6}\right) = 66.67\%$$

(b) $p_{n\geq 1} = 1 - p_0 = \frac{\lambda}{\mu} = \frac{4}{6} = .6667$

(c) $p_{n\leq 7} = p_0 + p_1 + \cdots + p_7$

$$= 1 - \left(\frac{\lambda}{\mu}\right)^8 = 1 - \left(\frac{4}{6}\right)^8 = .961$$

(d) $p_0 + p_1 + \cdots + p_K \geq .99$

From Figure 17-6, $K = 11$

Also, we can determine $K$ from

$$1 - \rho^{K+1} \geq .99$$

$$(K+1) \geq \frac{\ln .01}{\ln(4/6)} = 11.$$

or $K \geq 11.350 - 1 = 10.358$

Thus, $K \geq 11$

Note that the desired number of parking spaces is almost doubled (from 5 to 11) to accommodate the increase in the acceptance percentage from 90% to 99%.

## 2

$\lambda = 1/5 = .2$ job/day

$\mu = 1/4 = .25$ job/day

From the TORA output on the next column,

(a) $p_0 = .2$

(b) Av. income/month = $\$50\mu t$

$$= 50 \times .25 \times 30$$

$$= \$375$$

(c) Av. number of jobs awaiting completion $= L_q = 3.2$ jobs

cost $= 3.2 \times \$40 = \$128$

Continued...

| | | | |
|---|---|---|---|
| Lambda = | 0.20000 | Mu = | 0.25000 |
| Lambda eff = | 0.20000 | Rho/c = | 0.80000 |
| Ls = | 4.00000 | Lq = | 3.20000 |
| Ws = | 20.00000 | Wq = | 16.00000 |

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.20000 | 0.20000 | 23 | 0.00118 | 0.99528 |
| 1 | 0.16000 | 0.36000 | 24 | 0.00094 | 0.99622 |
| 2 | 0.12800 | 0.48800 | 25 | 0.00076 | 0.99698 |
| 3 | 0.10240 | 0.59040 | 26 | 0.00060 | 0.99758 |
| 4 | 0.08192 | 0.67232 | 27 | 0.00048 | 0.99807 |
| 5 | 0.06554 | 0.73786 | 28 | 0.00039 | 0.99845 |
| 6 | 0.05243 | 0.79028 | 29 | 0.00031 | 0.99876 |
| 7 | 0.04194 | 0.83223 | 30 | 0.00025 | 0.99901 |
| 8 | 0.03355 | 0.86578 | 31 | 0.00020 | 0.99921 |
| 9 | 0.02684 | 0.89263 | 32 | 0.00016 | 0.99937 |
| 10 | 0.02147 | 0.91410 | 33 | 0.00013 | 0.99949 |
| 11 | 0.01718 | 0.93128 | 34 | 0.00010 | 0.99959 |
| 12 | 0.01374 | 0.94502 | 35 | 0.00008 | 0.99968 |
| 13 | 0.01100 | 0.95602 | 36 | 0.00006 | 0.99974 |
| 14 | 0.00880 | 0.96482 | 37 | 0.00005 | 0.99979 |
| 15 | 0.00704 | 0.97185 | 38 | 0.00004 | 0.99983 |
| 16 | 0.00563 | 0.97748 | 39 | 0.00003 | 0.99987 |
| 17 | 0.00450 | 0.98199 | 40 | 0.00003 | 0.99989 |
| 18 | 0.00360 | 0.98559 | 41 | 0.00002 | 0.99991 |
| 19 | 0.00288 | 0.98847 | 42 | 0.00002 | 0.99993 |
| 20 | 0.00231 | 0.99078 | 43 | 0.00001 | 0.99995 |
| 21 | 0.00184 | 0.99262 | 44 | 0.00001 | 0.99996 |
| 22 | 0.00148 | 0.99410 | | | |

## 3

$\lambda = 1/4 = .25$ case/wk

$\mu = 1/1.5 = .66667$ case/wk

**M/M/c/GD/N/K Queueing Model**

| Input Data | | | |
|---|---|---|---|
| $\lambda$ = | 0.25 | | 0.66667 |
| c = | 1 | [redacted] | |
| Sys. Lim., N= | infinity | Source limit, K = | infinity |
| **Output Results** | | | |
| $\lambda_{eff}$ = | 0.2500 | | 0.3750 |
| Ls = | 0.6000 | Lq = | 0.2250 |
| Ws = | 2.4000 | Wq = | 0.9000 |

| n | Pn | CPn | 1-CPn |
|---|---|---|---|
| 0 | 0.625002 | 0.625002 | 0.374998 |
| 1 | 0.234375 | 0.859376 | 0.140624 |
| 2 | 0.087890 | 0.947266 | 0.052734 |
| 3 | 0.032959 | 0.980225 | 0.019775 |
| 4 | 0.012359 | 0.992584 | 0.007416 |
| 5 | 0.004635 | 0.997219 | 0.002781 |
| 6 | 0.001738 | 0.998957 | 0.001043 |
| 7 | 0.000652 | 0.999609 | 0.000391 |
| 8 | 0.000244 | 0.999853 | 0.000147 |
| 9 | 0.000092 | 0.999945 | 0.000055 |
| 10 | 0.000034 | 0.999979 | 0.000021 |
| 11 | 0.000013 | 0.999992 | 0.000008 |
| 12 | 0.000005 | 0.999997 | 0.000003 |
| 13 | 0.000002 | 0.999999 | 0.000001 |
| 14 | 0.000001 | 1.000000 | 0.000000 |

(a) $L_q = .225$ case

(b) $1 - p_0 = 1 - .625 = .375$ or $37.5\%$

(c) $W_s = 2.4$ weeks

## 4

Present situation:

$\lambda = 90$ cars/hr

$\mu = \frac{3600}{38} = 94.7368$ cars/hr

New situation:

$\lambda = 90$ cars per hour

$\mu = \frac{3600}{30} = 120$ cars per hour

Continued...

Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 90.00000 | 94.73680 | 90.00000 | 0.05000 | 19.00017 | 18.05017 | 0.21111 | 0.20056 |
| 2 | 1 | 90.00000 | 120.00000 | 90.00000 | 0.25000 | 3.00000 | 2.25000 | 0.03333 | 0.02500 |

$L_s$ (present) = 19 cars

% of idle time (new) = $p_0$(new) × 100

= 100 × .25 = 25%

The device can be justified based on the number of waiting customers, $L_s$, in the present system, but not on the basis of % idle time in the new one.

**5**

Scenario 1-- (M/M/1):(GD/infinity/infinity)

Lambda = 0.40000 Mu = 0.66667
Lambda eff = 0.40000 Rho/c = 0.60000

Ls = 1.49998 Lq = 0.89998
Ws = 3.74995 Wq = 2.24996

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.40000 | 0.40000 | 11 | 0.00145 | 0.99782 |
| 1 | 0.24000 | 0.64000 | 12 | 0.00087 | 0.99869 |
| 2 | 0.14400 | 0.78400 | 13 | 0.00052 | 0.99922 |
| 3 | 0.08640 | 0.87040 | 14 | 0.00031 | 0.99953 |
| 4 | 0.05184 | 0.92224 | 15 | 0.00019 | 0.99972 |
| 5 | 0.03110 | 0.95335 | 16 | 0.00011 | 0.99983 |
| 6 | 0.01866 | 0.97201 | 17 | 0.00007 | 0.99990 |
| 7 | 0.01120 | 0.98320 | 18 | 0.00004 | 0.99994 |
| 8 | 0.00672 | 0.98992 | 19 | 0.00002 | 0.99996 |
| 9 | 0.00403 | 0.99395 | 20 | 0.00001 | 0.99998 |
| 10 | 0.00242 | 0.99637 | | | |

(a) $p_0 = .4$

(b) $L_q = .9$ car

(c) $W_q = 2.25$ minutes

(d) $P_{n \geq 11} = 1 - CP_{10} = 1 - .99637 = .0036$

**6**

Scenario 1-- (M/M/1):(GD/infinity/infinity)

Lambda = 10.00000 Mu = 12.00000
Lambda eff = 10.00000 Rho/c = 0.83333

Ls = 5.00000 Lq = 4.16667
Ws = 0.50000 Wq = 0.41667

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.16667 | 0.16667 | 27 | 0.00121 | 0.99393 |
| 1 | 0.13889 | 0.30556 | 28 | 0.00101 | 0.99494 |
| 2 | 0.11574 | 0.42130 | 29 | 0.00084 | 0.99579 |
| 3 | 0.09645 | 0.51775 | 30 | 0.00070 | 0.99649 |
| 4 | 0.08038 | 0.59812 | 31 | 0.00059 | 0.99707 |
| 5 | 0.06698 | 0.66510 | 32 | 0.00049 | 0.99756 |
| 6 | 0.05582 | 0.72092 | 33 | 0.00041 | 0.99797 |
| 7 | 0.04651 | 0.76743 | 34 | 0.00034 | 0.99831 |
| 8 | 0.03876 | 0.80619 | 35 | 0.00028 | 0.99859 |
| 9 | 0.03230 | 0.83849 | 36 | 0.00024 | 0.99882 |
| 10 | 0.02692 | 0.86541 | 37 | 0.00020 | 0.99902 |
| 11 | 0.02243 | 0.88784 | 38 | 0.00016 | 0.99918 |
| 12 | 0.01869 | 0.90654 | 39 | 0.00014 | 0.99932 |
| 13 | 0.01558 | 0.92211 | 40 | 0.00011 | 0.99943 |
| 14 | 0.01298 | 0.93509 | 41 | 0.00009 | 0.99953 |
| 15 | 0.01082 | 0.94591 | 42 | 0.00008 | 0.99961 |
| 16 | 0.00901 | 0.95493 | 43 | 0.00007 | 0.99967 |
| 17 | 0.00751 | 0.96244 | 44 | 0.00005 | 0.99973 |
| 18 | 0.00626 | 0.96870 | 45 | 0.00005 | 0.99977 |
| 19 | 0.00522 | 0.97392 | 46 | 0.00004 | 0.99981 |
| 20 | 0.00435 | 0.97826 | 47 | 0.00003 | 0.99984 |
| 21 | 0.00362 | 0.98189 | 48 | 0.00003 | 0.99987 |
| 22 | 0.00302 | 0.98491 | 49 | 0.00002 | 0.99989 |
| 23 | 0.00252 | 0.98742 | 50 | 0.00002 | 0.99991 |
| 24 | 0.00210 | 0.98952 | 51 | 0.00002 | 0.99992 |
| 25 | 0.00175 | 0.99126 | 52 | 0.00001 | 0.99994 |
| 26 | 0.00146 | 0.99272 | 53 | 0.00001 | 0.99995 |

(a) $p_0 + p_1 + p_2 = .4213$

continued...

(b) $1 - CP_2 = 1 - .4213 = .5787$

(c) $W_q = .417$ hour

(d) Let N = spaces (including car being served)

$CP_{N-1} \geq .9$

Because $CP_{11} = .88784$ and $CP_{12} = .90654$,

$N-1 \geq 12 \Rightarrow N \geq 13$.

**7**

In general, $L_s < L_q + 1$. The reason is that $p_0 > 0$, usually. Consider

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n$$

$$= \sum_{n=1}^{\infty} n p_n - \sum_{n=1}^{\infty} p_n$$

$$= L_s - (1 - p_0)$$

The closer $p_0$ is to zero, the more likely $L_s \cong L_q + 1$ will hold.

**8**

Consider

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n$$

$$= \sum_{n=1}^{\infty} (n-1)(1-\rho)\rho^n$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho}\left(\sum_{n=1}^{\infty} \rho^{n-1}\right)$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n$$

$$= (1-\rho)\rho^2 \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= \rho^2 (1-\rho) \frac{1}{(1-\rho)^2}$$

$$= \frac{\rho^2}{1-\rho}$$

**9**

(a) $P\{j \text{ in queue} \mid j \geq 1\}$

$$= p\{n \text{ in system} \mid n \geq 2\}$$

$$= \frac{p_n}{\sum_{j=2}^{\infty} p_j}$$

Thus,

$$\text{expected number} = \sum_{n=2}^{\infty}(n-1)\frac{p_n}{\sum_{j=2}^{\infty} p_j}$$

$$= \frac{\sum_{n=2}^{\infty} np_n - \sum_{n=2}^{\infty} p_n}{\sum_{n=2}^{\infty} p_n}$$

$$= \frac{\sum_{n=1}^{\infty} np_n - p_1}{\sum_{n=2}^{\infty} p_n} - 1$$

$$= \frac{\frac{\rho}{1-\rho} - \rho(1-\rho)}{1-[(1-\rho)+\rho(1-\rho)]} - 1$$

$$= \frac{1}{1-\rho}$$

(b) Exp. number in queue given the system is not empty

$$= \sum_{n=1}^{\infty}(n-1)\left(\frac{p_n}{\sum_{j=1}^{\infty} p_j}\right)$$

$$= \frac{\sum_{n=1}^{\infty} np_n - \sum_{n=1}^{\infty} p_n}{\sum_{j=1}^{\infty} p_j}$$

$$= \frac{\left(\frac{\rho}{1-\rho}\right) - \rho}{\rho}$$

$$= \frac{\rho}{1-\rho}$$

Continued...

Thus,

Exp. waiting time in queue for those who must wait

$$= \frac{\rho/(1-\rho)}{\lambda}$$

$$= \frac{1}{\mu-\lambda}$$

**1**

$w(\tau) = (\mu-\lambda)e^{-(\mu-\lambda)\tau}, \quad \tau > 0$

$\lambda = 1/4 = .25/\text{wk}$
$\mu = 1/1.5 = .667/\text{wk}$ $\Big\}\ (\mu-\lambda) = .417$

$\rho = \lambda/\mu = \frac{1.5}{4} = .375$

$w(\tau) = .417e^{-.417\tau}, \quad \tau > 0$

$P\{\tau > 1\} = e^{-.417\times 1} = .659$

**2**

(a) Standard deviation $= \frac{1}{\mu-\lambda} = \frac{1}{6-4} = .5$

(b) $w(\tau) = (\mu-\lambda)e^{-(\mu-\lambda)\tau}, \quad \tau > 0$

$P\left\{\frac{1}{2(\mu-\lambda)} \le \tau \le \frac{3}{2(\mu-\lambda)}\right\}$
$= (1-e^{-1.5}) - (1-e^{-.5})$
$= e^{-.5} - e^{-1.5}$
$= .3834$

**3**

$W_s \le 10$ minutes, $\lambda = 4/\text{hr}$

$\frac{1}{(\mu-\lambda)} \le \frac{10}{60}$ hr

or $\mu - \lambda \ge 6$

or $\mu \ge 6 + \lambda = 10/\text{hr}$

**4**

$P\{\tau > \frac{10}{60}\} \le .1$, or

$e^{-\frac{1}{6}(\mu-4)} \le .1$

$\mu - 4 \ge 13.8$

$\mu \ge 17.8/\text{hr}$

**5**

$P\{\tau > 5\} = e^{-(\mu-\lambda)t} = e^{-.267\times 5} = .2636$

where $\lambda = .4/\text{min}$, $\mu = .667/\text{min}$

Exp. # customers in a 12-hr day
$= \lambda \times 12 \times 60 = .4 \times 12 \times 60 = 288$ cust.

Exp. cost $= 288 \times .2636 \times .5 = \$37.95$

**6**

Let

$w_{n+1}(t|n)$ = conditional pdf for waiting in queue given there are $n$ customers ahead
= $n$-fold convolution of the exponential pdf

$$= \frac{\mu(\mu t)^{n-1}e^{-\mu t}}{(n-1)!}$$

$w(t)$ = absolute pdf of waiting time in queue

$g(t,n)$ = joint pdf of $t$ and $n$

$= w_{n+1}(t|n)\,p_n$

$$= \frac{\mu(\mu t)^{n-1}e^{-\mu t}}{(n-1)!}\rho^n(1-\rho)$$

(a) For $t > 0$

$$w(t) = \sum_{n=1}^{\infty} g(t,n)$$

$$= \frac{\mu\rho e^{-\mu t}(1-\rho)}{e^{-\mu\rho t}}\sum_{n=1}^{\infty}\frac{(\mu\rho t)^{n-1}e^{-\mu\rho t}}{(n-1)!}$$

$$= \mu\rho(1-\rho)e^{-\mu(1-\rho)t}, \quad t > 0$$

For $t = 0$, $w(0) = p_0 = (1-\rho)$

$$w(t) = \begin{cases} 1-\rho, & t = 0 \\ \mu\rho(1-\rho)e^{-\mu(1-\rho)t}, & t > 0 \end{cases}$$

(b) $W_q = E\{t\}$

$$= \int_0^{\infty} t\,w(t)\,dt$$

$$= 0\,w(0) + \int_{0^+}^{\infty} t\,w(t)\,dt$$

$$= \int_{0^+}^{\infty} \mu\rho t(1-\rho)e^{-\mu(1-\rho)t}\,dt$$

$$= \frac{\rho}{\mu(1-\rho)}$$

**1**

(a) $p_0 = .3654$

(b) $W_q = .207$ hour

(c) Average number of empty spaces $= 4 - L_q$

$= 4 - .788$

$= 3.212$ spaces

(d) $p_5 = .04812$

(e) $W_s \le 10$ minutes

Title: 17.6d-1
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 4.00000 | 6.00000 | 3.80752 | 0.36541 | 1.42256 | 0.78797 | 0.37362 | 0.20695 |
| 2 | 1 | 4.00000 | 7.00000 | 3.89179 | 0.44403 | 1.11691 | 0.56094 | 0.28699 | 0.14413 |
| 3 | 1 | 4.00000 | 8.00000 | 3.93651 | 0.50794 | 0.90476 | 0.41270 | 0.22984 | 0.10484 |
| 4 | 1 | 4.00000 | 9.00000 | 3.96116 | 0.55987 | 0.75340 | 0.31327 | 0.19020 | 0.07908 |
| 5 | 1 | 4.00000 | 10.00000 | 3.97532 | 0.60247 | 0.64199 | 0.24446 | 0.16149 | 0.06149 |

| $\mu$ (cars/hr) | $W_s$ (hrs) | $W_s$ (min) |
|---|---|---|
| 6 | .3736 | 22.4 |
| 7 | .287 | 17.16 |
| 8 | .23 | 13.80 |
| 9 | .19 | 11.40 |
| 10 | .16 | 9.60 |

Desired service rate = 10 cars/hr
Thus, the service time must be reduced from $\frac{60}{6} = 10$ minutes to $\frac{60}{10} = 6$ minutes, a 40% reduction

**2**

$m$ = number of parking spaces
An arriving car will not find a space if there are $m+1$ cars in the system. Thus, find $m$ such that $p_{m+1} \le .01$

TORA input $= (4, 6, 1, m+1, \infty)$

| $m$ | $N = m+1$ | $p_N$ |
|---|---|---|
| 4 | 5 | .04812 |
| 5 | 6 | .0311 |
| 6 | 7 | .0203 |
| 7 | 8 | .01335 |
| 8 | 9 | .009 |

Select the number of parking spaces $m \ge 8$

Continued...

**3**

$m$ = number of seats.
The $N = m+1$, and

$\lambda_{eff} = \lambda P_N = 5 P_N$ customers/hr

TORA input $= (6, 5, 1, N, \infty)$

Title: 17.6d-3
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 6.00000 | 5.00000 | 3.62637 | 0.27473 | 1.12088 | 0.39560 | 0.30909 | 0.10909 |
| 2 | 1 | 6.00000 | 5.00000 | 4.06855 | 0.18629 | 1.72578 | 0.91207 | 0.42418 | 0.22418 |
| 3 | 1 | 6.00000 | 5.00000 | 4.32810 | 0.13438 | 2.35949 | 1.49387 | 0.54516 | 0.34516 |
| 4 | 1 | 6.00000 | 5.00000 | 4.49647 | 0.10071 | 3.02117 | 2.12188 | 0.67190 | 0.47190 |
| 5 | 1 | 6.00000 | 5.00000 | 4.61288 | 0.07742 | 3.70984 | 2.78726 | 0.80423 | 0.60423 |

| $m$ | $N = m+1$ | $\lambda_{eff}$ (customers/hr) |
|---|---|---|
| 1 | 2 | 3.63 |
| 2 | 3 | 4.07 |

Use two seats or less

**4**

$\lambda = 10$ generators per hour
$\mu = \frac{60}{15} = 4$ generators per hour
$N = 7 + 1 = 8$

Title: 17.6d-4
Scenario 1-- (M/M/1):(GD/8/Infinity)

Lambda = 10.00000 Mu = 4.00000
Lambda eff = 3.99843 Rho/c = 2.50000

Ls = 7.33569 Lq = 6.33609
Ws = 1.83464 Wq = 1.58464

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.00039 | 0.00039 | 5 | 0.03841 | 0.06375 |
| 1 | 0.00098 | 0.00138 | 6 | 0.09603 | 0.15978 |
| 2 | 0.00246 | 0.00383 | 7 | 0.24006 | 0.39984 |
| 3 | 0.00615 | 0.00998 | 8 | 0.60016 | 1.00000 |
| 4 | 0.01536 | 0.02534 | | | |

(a) $p_8 \cong .6$

(b) $L_q = 6.34$ generators

(c) Let $C$ = belt capacity. Thus, $N = C + 1$. The assembly department is kept in operation so long as at least one empty space remains on the belt; that is,

$$P\{\text{empty space on belt}\} = p_0 + p_1 + \cdots + p_C$$

$$= \frac{1-\rho}{1-\rho^{C+2}} \sum_{n=0}^{C} \rho^n$$

$$= \frac{1-\rho}{1-\rho^{C+2}} \cdot \frac{1-\rho^{C+1}}{1-\rho}$$

$$= \frac{1-\rho^{C+1}}{1-\rho^{C+2}}$$

Continued...

$$\lim_{c\to\infty} \frac{1-\rho^{c+1}}{1-\rho^{c+2}} = \lim_{c\to\infty} \frac{-(c+1)\rho^{c}}{-(c+2)\rho^{c+1}}$$

$$= \lim_{c\to\infty} \frac{c+1}{(c+2)\rho}$$

$$= \lim_{c\to\infty} \left(\frac{1+1/c}{1+2/c}\right)\frac{1}{\rho}$$

$$= \frac{1}{\rho}$$

In the present example, $\rho = 10/4$ and $1/\rho = .4$. Thus,

$$\lim_{c\to\infty} (p_0+p_1+\cdots+p_c) = 1/\rho = .4$$

This result means that regardless of how large the belt is, the probability of finding an empty space cannot exceed .4. Thus, achieving a 95% utilization for the assembly dept. is impossible.

The result makes sense because the arrival rate $\lambda$ (=10/hr) is $2\frac{1}{2}$ times larger than the service rate (= 4). The only way we can accomplish the desired result is to reduce $\lambda$ and/or increase $\mu$.

**5**

(a) $p_{50} \simeq .00002$

(b) $P\{\text{wish is not fulfilled}\}$

$= P\{48 \text{ or more in resturant}\}$

$= p_{48}+p_{49}+p_{50}$

$= 1-(p_0+p_1+\cdots+p_{47})$

$= 1-.99993$

$= .00007$

continued...

Title: 17.6d-5
Scenario 1-- (M/M/1):(GD/50/infinity)

TORA input = $(10, 12, 1, 50, \infty)$

| | | | |
|---|---|---|---|
| Lambda = | 10.00000 | Mu = | 12.00000 |
| Lambda eff = | 9.99982 | Rho/c = | 0.83333 |
| Ls = | 4.99533 | Lq = | 4.16201 |
| Ws = | 0.49954 | Wq = | 0.41621 |

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.16668 | 0.16668 | 26 | 0.00146 | 0.99281 |
| 1 | 0.13890 | 0.30558 | 27 | 0.00121 | 0.99402 |
| 2 | 0.11575 | 0.42133 | 28 | 0.00101 | 0.99504 |
| 3 | 0.09646 | 0.51779 | 29 | 0.00084 | 0.99588 |
| 4 | 0.08038 | 0.59818 | 30 | 0.00070 | 0.99658 |
| 5 | 0.06699 | 0.66516 | 31 | 0.00059 | 0.99717 |
| 6 | 0.05582 | 0.72098 | 32 | 0.00049 | 0.99765 |
| 7 | 0.04652 | 0.76750 | 33 | 0.00041 | 0.99806 |
| 8 | 0.03876 | 0.80627 | 34 | 0.00034 | 0.99840 |
| 9 | 0.03230 | 0.83857 | 35 | 0.00028 | 0.99868 |
| 10 | 0.02692 | 0.86549 | 36 | 0.00024 | 0.99892 |
| 11 | 0.02243 | 0.88792 | 37 | 0.00020 | 0.99911 |
| 12 | 0.01869 | 0.90662 | 38 | 0.00016 | 0.99928 |
| 13 | 0.01558 | 0.92220 | 39 | 0.00014 | 0.99941 |
| 14 | 0.01298 | 0.93518 | 40 | 0.00011 | 0.99952 |
| 15 | 0.01082 | 0.94600 | 41 | 0.00009 | 0.99962 |
| 16 | 0.00902 | 0.95501 | 42 | 0.00008 | 0.99970 |
| 17 | 0.00751 | 0.96253 | 43 | 0.00007 | 0.99976 |
| 18 | 0.00626 | 0.96879 | 44 | 0.00005 | 0.99982 |
| 19 | 0.00522 | 0.97401 | 45 | 0.00005 | 0.99986 |
| 20 | 0.00435 | 0.97835 | 46 | 0.00004 | 0.99990 |
| 21 | 0.00362 | 0.98198 | 47 | 0.00003 | 0.99993 |
| 22 | 0.00302 | 0.98500 | 48 | 0.00003 | 0.99996 |
| 23 | 0.00252 | 0.98751 | 49 | 0.00002 | 0.99998 |
| 24 | 0.00210 | 0.98961 | 50 | 0.00002 | 1.00000 |
| 25 | 0.00175 | 0.99136 | | | |

**6**

TORA input = $(20, 7.5, 1, 15, \infty)$

Title: 17.6d-6
Scenario 1-- (M/M/1):(GD/15/infinity)

| | | | |
|---|---|---|---|
| Lambda = | 20.00000 | Mu = | 7.50000 |
| Lambda eff = | 7.50000 | Rho/c = | 2.66667 |
| Ls = | 14.40000 | Lq = | 13.40000 |
| Ws = | 1.92000 | Wq = | 1.78667 |

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.00000 | 0.00000 | 8 | 0.00065 | 0.00104 |
| 1 | 0.00000 | 0.00000 | 9 | 0.00174 | 0.00278 |
| 2 | 0.00000 | 0.00000 | 10 | 0.00463 | 0.00742 |
| 3 | 0.00000 | 0.00001 | 11 | 0.01236 | 0.01978 |
| 4 | 0.00001 | 0.00002 | 12 | 0.03296 | 0.05273 |
| 5 | 0.00003 | 0.00005 | 13 | 0.08789 | 0.14062 |
| 6 | 0.00009 | 0.00015 | 14 | 0.23438 | 0.37500 |
| 7 | 0.00024 | 0.00039 | 15 | 0.62500 | 1.00000 |

(a) $p_0 \cong 0.$

(b) $p_{n\le 14} = p_0+\cdots+p_{14} = .375$

(c) $W_s = 1.92$ hours

**7**

(a) $p_{n\le 4} = p_0+p_1+\cdots+p_4$

$= .962$

(b) $\lambda_{lost} = \lambda p_5$

$= 5 \times .038 = .19$ cust./hr

(c) $L_s = 0\times.399+1\times.249+2\times.156$

$+ 3\times.097+4\times.061$

$+ 5\times.038$

$= 1.286$

continued...

(d) $W_q = W_s - \frac{1}{\mu}$

$\lambda_{eff} = 5(1 - .038) = 4.81$ cust/hr

$W_s = \frac{L_s}{\lambda_{eff}}$

$= \frac{1.286}{4.81}$

$= .2675$ hour

$W_q = .2675 - 1/8$

$= .1424$ hour

**8**

$$p_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}}$$

$$\lim_{\rho \to 1} p_n = \lim_{\rho \to 1} \frac{\rho^n - \rho^{n+1}}{1-\rho^{N+1}}$$

$$= \lim_{\rho \to 1} \frac{n\rho^{n-1} - (n+1)\rho^n}{-(N+1)\rho^N}$$

$$= \frac{1}{N+1}$$

Thus,

$$L_s = \sum_{n=0}^{N} n p_n$$

$$= \frac{1}{N+1} \sum_{n=0}^{N} n$$

$$= \frac{N(N+1)}{2(N+1)} = \frac{N}{2}$$

**9**

$$W_s = W_q + \frac{1}{\mu}$$

$$\lambda_{eff} W_s = \lambda_{eff} W_q + \frac{\lambda_{eff}}{\mu}$$

Thus,

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

or

$$\lambda_{eff} = \mu(L_s - L_q)$$

**1**

TORA input = $(8, 5, 2, \infty, \infty)$

Title: 17.6e-1
Scenario 1– (M/M/2):(GD/infinity/infinity)

Lambda = 8.00000 Mu = 5.00000
Lambda eff = 8.00000 Rho/c = 0.80000

Ls = 4.44444 Lq = 2.84444
Ws = 0.55556 Wq = 0.35556

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.11111 | 0.11111 | 23 | 0.00131 | 0.99475 |
| 1 | 0.17778 | 0.28889 | 24 | 0.00105 | 0.99580 |
| 2 | 0.14222 | 0.43111 | 25 | 0.00084 | 0.99664 |
| 3 | 0.11378 | 0.54489 | 26 | 0.00067 | 0.99731 |
| 4 | 0.09102 | 0.63591 | 27 | 0.00054 | 0.99785 |
| 5 | 0.07282 | 0.70873 | 28 | 0.00043 | 0.99828 |
| 6 | 0.05825 | 0.76698 | 29 | 0.00034 | 0.99862 |
| 7 | 0.04660 | 0.81359 | 30 | 0.00028 | 0.99890 |
| 8 | 0.03728 | 0.85087 | 31 | 0.00022 | 0.99912 |
| 9 | 0.02983 | 0.88070 | 32 | 0.00018 | 0.99930 |
| 10 | 0.02386 | 0.90456 | 33 | 0.00014 | 0.99944 |
| 11 | 0.01909 | 0.92365 | 34 | 0.00011 | 0.99955 |
| 12 | 0.01527 | 0.93892 | 35 | 0.00009 | 0.99964 |
| 13 | 0.01222 | 0.95113 | 36 | 0.00007 | 0.99971 |
| 14 | 0.00977 | 0.96091 | 37 | 0.00006 | 0.99977 |
| 15 | 0.00782 | 0.96873 | 38 | 0.00005 | 0.99982 |
| 16 | 0.00625 | 0.97498 | 39 | 0.00004 | 0.99985 |
| 17 | 0.00500 | 0.97998 | 40 | 0.00003 | 0.99988 |
| 18 | 0.00400 | 0.98399 | 41 | 0.00002 | 0.99991 |
| 19 | 0.00320 | 0.98719 | 42 | 0.00002 | 0.99992 |
| 20 | 0.00256 | 0.98975 | 43 | 0.00002 | 0.99994 |
| 21 | 0.00205 | 0.99180 | 44 | 0.00001 | 0.99995 |
| 22 | 0.00164 | 0.99344 | | | |

TORA input = $(16, 5, 4, \infty, \infty)$

Title: 17.6e-1
Scenario 2– (M/M/4):(GD/infinity/infinity)

Lambda = 16.00000 Mu = 5.00000
Lambda eff = 16.00000 Rho/c = 0.80000

Ls = 5.58573 Lq = 2.38573
Ws = 0.34911 Wq = 0.14911

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.02730 | 0.02730 | 24 | 0.00138 | 0.99450 |
| 1 | 0.08737 | 0.11467 | 25 | 0.00110 | 0.99560 |
| 2 | 0.13979 | 0.25446 | 26 | 0.00088 | 0.99648 |
| 3 | 0.14911 | 0.40357 | 27 | 0.00070 | 0.99718 |
| 4 | 0.11929 | 0.52285 | 28 | 0.00056 | 0.99775 |
| 5 | 0.09543 | 0.61828 | 29 | 0.00045 | 0.99820 |
| 6 | 0.07634 | 0.69463 | 30 | 0.00036 | 0.99856 |
| 7 | 0.06107 | 0.75570 | 31 | 0.00029 | 0.99885 |
| 8 | 0.04886 | 0.80456 | 32 | 0.00023 | 0.99908 |
| 9 | 0.03909 | 0.84365 | 33 | 0.00018 | 0.99926 |
| 10 | 0.03127 | 0.87492 | 34 | 0.00015 | 0.99941 |
| 11 | 0.02502 | 0.89994 | 35 | 0.00012 | 0.99953 |
| 12 | 0.02001 | 0.91995 | 36 | 0.00009 | 0.99962 |
| 13 | 0.01601 | 0.93596 | 37 | 0.00008 | 0.99970 |
| 14 | 0.01281 | 0.94877 | 38 | 0.00006 | 0.99976 |
| 15 | 0.01025 | 0.95901 | 39 | 0.00005 | 0.99981 |
| 16 | 0.00820 | 0.96721 | 40 | 0.00004 | 0.99985 |
| 17 | 0.00656 | 0.97377 | 41 | 0.00003 | 0.99988 |
| 18 | 0.00525 | 0.97901 | 42 | 0.00002 | 0.99990 |
| 19 | 0.00420 | 0.98321 | 43 | 0.00002 | 0.99992 |
| 20 | 0.00336 | 0.98657 | 44 | 0.00002 | 0.99994 |
| 21 | 0.00269 | 0.98926 | 45 | 0.00001 | 0.99995 |
| 22 | 0.00215 | 0.99140 | 46 | 0.00001 | 0.99996 |
| 23 | 0.00172 | 0.99312 | | | |

(a) $\underline{C=2:}$

$P\{\text{all servers are busy}\} = (P_{n\geq 2})^2$
$= (1-.29)^2$
$= .504$

$\underline{C=4:}$

$P\{\text{all servers are busy}\} = 1 - P_{n\leq 3}$
$= 1 - .404$
$= .596$

C = 4 yields a higher probability that all servers are busy.

continued...

(b)

Title: 17.6e-1
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 16.00000 | 5.00000 | 16.00000 | 0.02730 | 5.58573 | 2.38573 | 0.34911 | 0.14911 |
| 2 | 5 | 16.00000 | 5.00000 | 16.00000 | 0.03715 | 3.71299 | 0.51299 | 0.23206 | 0.03206 |
| 3 | 6 | 16.00000 | 5.00000 | 16.00000 | 0.03977 | 3.34526 | 0.14526 | 0.20908 | 0.00908 |

For $C=5$, $W_q = .032$ hour $\cong$ 2 min
$C=4$, $W_q = .149$ hour $\simeq$ 9 min
Select $C \geq 5$

**2**

$\underline{C=2}$: $\lambda = 8$ calls/hr
$\mu = \frac{60}{14.5} = 4.1379$ calls/hr

$\underline{C=4}$: $\lambda = 16$ calls/hr
$\mu = 4.1379$ calls per hour

utilization $= \lambda/\mu c = .967$

Title: 6e-2
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 8.00000 | 4.13790 | 8.00000 | 0.01695 | 29.49805 | 27.56471 | 3.68726 | 3.44559 |
| 2 | 4 | 16.00000 | 4.13790 | 16.00000 | 0.00332 | 30.75857 | 26.89187 | 1.92241 | 1.68074 |

$$W_q = \begin{cases} 3.446 \text{ hours for } C=2 \\ 1.681 \text{ hours for } C=4 \end{cases}$$

Consolidation reduces the waiting time by more than 51%.

**3**

(a) $\lambda = \frac{60}{5} = 12$ per hour
$\mu = 10$ per hour
$c > \frac{\lambda}{\mu} = 1.2 \Rightarrow c \geq 2$

(b) $\lambda = \frac{60}{2} = 30$ per hour
$\mu = \frac{60}{6} = 10$ per hour
$c > \frac{\lambda}{\mu} = \frac{30}{10} = 3 \Rightarrow c \geq 4$

(c) $\lambda = 30$ per hour, $\mu = 40$ per hr.
$c > \frac{30}{40} = .75 \Rightarrow c \geq 1$

**4**

$\lambda = 45$ customers/hr
$\mu = \frac{60}{5} = 12$ customers/hr
$c > \frac{45}{12}$ or $c \geq 4$
Desired $W_q \leq 30$ seconds $= .0083$ hr

Title: 6e-4
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 45.00000 | 12.00000 | 45.00000 | 0.00656 | 16.72545 | 12.97545 | 0.37168 | 0.28834 |
| 2 | 5 | 45.00000 | 12.00000 | 45.00000 | 0.01868 | 5.13537 | 1.38537 | 0.11412 | 0.03079 |
| 3 | 6 | 45.00000 | 12.00000 | 45.00000 | 0.02208 | 4.12903 | 0.37903 | 0.09176 | 0.00842 |
| 4 | 7 | 45.00000 | 12.00000 | 45.00000 | 0.02309 | 3.86873 | 0.11873 | 0.08597 | 0.00264 |

Select $C \geq 7$.

**5**

TORA input: (20, 12, 3, ∞, ∞)

Title: 17.6e-5
Scenario 1-- (M/M/3):(GD/infinity/infinity)

| | | | |
|---|---|---|---|
| Lambda = | 20.00000 | Mu = | 12.00000 |
| Lambda eff = | 20.00000 | Rho/c = | 0.55556 |
| Ls = | 2.04137 | Lq = | 0.37470 |
| Ws = | 0.10207 | Wq = | 0.01874 |

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.17266 | 0.17266 | 10 | 0.00218 | 0.99728 |
| 1 | 0.28777 | 0.46043 | 11 | 0.00121 | 0.99849 |
| 2 | 0.23981 | 0.70024 | 12 | 0.00067 | 0.99916 |
| 3 | 0.13323 | 0.83347 | 13 | 0.00037 | 0.99953 |
| 4 | 0.07401 | 0.90748 | 14 | 0.00021 | 0.99974 |
| 5 | 0.04112 | 0.94860 | 15 | 0.00012 | 0.99986 |
| 6 | 0.02284 | 0.97144 | 16 | 0.00006 | 0.99992 |
| 7 | 0.01269 | 0.98414 | 17 | 0.00004 | 0.99996 |
| 8 | 0.00705 | 0.99119 | 18 | 0.00002 | 0.99998 |
| 9 | 0.00392 | 0.99510 | 19 | 0.00001 | 0.99999 |

$m$ = size of waiting room.

$p_0 + p_1 + \cdots + p_{m+2} \geq .999 \Rightarrow m \geq 10$

**6**

$C = 2$, $\lambda_{windows} = .8 \times \frac{60}{3} = 16$ /hr

$\mu = \frac{60}{5} = 12$ per hour

Title: 6e-6
Scenario 1-- (M/M/2):(GD/infinity/infinity)

| | | | |
|---|---|---|---|
| Lambda = | 16.00000 | Mu = | 12.00000 |
| Lambda eff = | 16.00000 | Rho/c = | 0.66667 |
| Ls = | 2.40000 | Lq = | 1.06667 |
| Ws = | 0.15000 | Wq = | 0.06667 |

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.20000 | 0.20000 | 14 | 0.00137 | 0.99726 |
| 1 | 0.26667 | 0.46667 | 15 | 0.00091 | 0.99817 |
| 2 | 0.17778 | 0.64444 | 16 | 0.00061 | 0.99878 |
| 3 | 0.11852 | 0.76296 | 17 | 0.00041 | 0.99919 |
| 4 | 0.07901 | 0.84198 | 18 | 0.00027 | 0.99946 |
| 5 | 0.05267 | 0.89465 | 19 | 0.00018 | 0.99964 |
| 6 | 0.03512 | 0.92977 | 20 | 0.00012 | 0.99976 |
| 7 | 0.02341 | 0.95318 | 21 | 0.00008 | 0.99984 |
| 8 | 0.01561 | 0.96879 | 22 | 0.00005 | 0.99989 |
| 9 | 0.01040 | 0.97919 | 23 | 0.00004 | 0.99993 |
| 10 | 0.00694 | 0.98613 | 24 | 0.00002 | 0.99995 |
| 11 | 0.00462 | 0.99075 | 25 | 0.00002 | 0.99997 |
| 12 | 0.00308 | 0.99383 | 26 | 0.00001 | 0.99998 |
| 13 | 0.00206 | 0.99589 | | | |

(a) $P_{n \geq 2} = 1 - (p_0 + p_1)$
$= 1 - .46667$
$= .5333$

(b) $p_0 = .2$

(c) $L_q = 1.067$

(d) NO, because $\lambda > \mu$. The minimum number of windows should $\geq \frac{\lambda}{\mu} = \frac{16}{12} = 1.33$

Number of windows $\geq 2$

**7**

$\lambda = 25 \times \frac{60}{15} = 100$ jobs/hour

$\mu = \frac{60}{2} = 30$ jobs/hour, $C = 4$

Title: 6e-7
Scenario 1-- (M/M/4):(GD/infinity/infinity)

| | | | |
|---|---|---|---|
| Lambda = | 100.00000 | Mu = | 30.00000 |
| Lambda eff = | 100.00000 | Rho/c = | 0.83333 |
| Ls = | 6.62194 | Lq = | 3.28861 |
| Ws = | 0.06622 | Wq = | 0.03289 |

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.02131 | 0.02131 | 28 | 0.00138 | 0.99311 |
| 1 | 0.07103 | 0.09234 | 29 | 0.00115 | 0.99425 |
| 2 | 0.11839 | 0.21073 | 30 | 0.00096 | 0.99521 |
| 3 | 0.13154 | 0.34228 | 31 | 0.00080 | 0.99601 |
| 4 | 0.10962 | 0.45190 | 32 | 0.00066 | 0.99668 |
| 5 | 0.09135 | 0.54325 | 33 | 0.00055 | 0.99723 |
| 6 | 0.07613 | 0.61937 | 34 | 0.00046 | 0.99769 |
| 7 | 0.06344 | 0.68281 | 35 | 0.00038 | 0.99808 |
| 8 | 0.05286 | 0.73568 | 36 | 0.00032 | 0.99840 |
| 9 | 0.04405 | 0.77973 | 37 | 0.00027 | 0.99866 |
| 10 | 0.03671 | 0.81644 | 38 | 0.00022 | 0.99889 |
| 11 | 0.03059 | 0.84703 | 39 | 0.00019 | 0.99907 |
| 12 | 0.02549 | 0.87253 | 40 | 0.00015 | 0.99923 |
| 13 | 0.02125 | 0.89377 | 41 | 0.00013 | 0.99936 |
| 14 | 0.01770 | 0.91148 | 42 | 0.00011 | 0.99946 |
| 15 | 0.01475 | 0.92623 | 43 | 0.00009 | 0.99955 |
| 16 | 0.01229 | 0.93853 | 44 | 0.00007 | 0.99963 |
| 17 | 0.01025 | 0.94877 | 45 | 0.00006 | 0.99969 |
| 18 | 0.00854 | 0.95731 | 46 | 0.00005 | 0.99974 |
| 19 | 0.00711 | 0.96443 | 47 | 0.00004 | 0.99978 |
| 20 | 0.00593 | 0.97035 | 48 | 0.00004 | 0.99982 |
| 21 | 0.00494 | 0.97530 | 49 | 0.00003 | 0.99985 |
| 22 | 0.00412 | 0.97941 | 50 | 0.00002 | 0.99988 |
| 23 | 0.00343 | 0.98284 | 51 | 0.00002 | 0.99990 |
| 24 | 0.00286 | 0.98570 | 52 | 0.00002 | 0.99991 |
| 25 | 0.00238 | 0.98809 | 53 | 0.00001 | 0.99993 |
| 26 | 0.00199 | 0.99007 | 54 | 0.00001 | 0.99994 |
| 27 | 0.00165 | 0.99173 | 55 | 0.00001 | 0.99995 |

(a) $P_{n \geq 4} = 1 - CP_3$
$= 1 - .34228 = .65772$

(b) $W_s = .06622$ hour

(c) $L_q = 3.29$ jobs

(d) $p_0 = .021 \Rightarrow 2.1\%$ idleness

(e) Av. # of idle computers $= 4 - (L_s - L_q)$
$= 4 - (6.62 - 3.29) = .67$

**8**

$\lambda = 15 + 10 + 20 = 45$ customers/hour

$\mu = \frac{60}{6} = 10$ customers/hour

$C > 45/10 = 4.5 \Rightarrow C \geq 5$

Title: 6e-8
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 45.00000 | 10.00000 | 45.00000 | 0.00496 | 11.36244 | 6.86244 | 0.25250 | 0.15250 |
| 2 | 6 | 45.00000 | 10.00000 | 45.00000 | 0.00914 | 5.76496 | 1.26496 | 0.12811 | 0.02811 |
| 3 | 7 | 45.00000 | 10.00000 | 45.00000 | 0.01046 | 4.89100 | 0.39100 | 0.10869 | 0.00869 |

(a) $W_s \leq 15/60 = .25$ hour $\Rightarrow C \geq 6$

(b) % idle $= \frac{C - (L_s - L_q)}{C} \times 100$

| C | $L_s$ | $L_q$ | $C-(L_s-L_q)$ | % idle |
|---|---|---|---|---|
| 5 | 11.362 | 6.862 | .5 | 10% |
| 6 | 5.765 | 1.265 | 1.5 | 25% |

Select $C = 5$

(c)

| C | 5 | 6 | 7 |
|---|---|---|---|
| $p_0$ | .00496 | .00914 | .01046 |

Select $C \leq 6$

**9**

1. Limited space inside a bank or a grocery store
2. Multiple queues appear to offer more courteous service.

**10**

For $c$ parallel servers:

$$L_q = \frac{\rho}{c-\rho}, \text{ provided } \frac{\rho}{c} \to 1$$

Thus,

$$W_{q_c} = \frac{1}{\lambda_c}\frac{\rho}{c-\rho} = \frac{1}{(c\mu-\lambda_c)}$$

For a single server

$$W_{q_1} = \frac{\lambda_1}{\mu(\mu-\lambda_1)}$$

Because $\lambda_c = c\lambda_1$, we have

$$\frac{W_{q_c}}{W_{q_1}} = \left(\frac{\frac{1}{c(\mu-\lambda_1)}}{\frac{\lambda_1}{\mu(\mu-\lambda_1)}}\right) = \frac{1}{c\left(\frac{\lambda_1}{\mu}\right)}$$

$$= \frac{1}{c\left(\frac{\lambda c/\mu}{c}\right)}$$

$$= \frac{1}{c\,(\rho/c)}$$

$$\lim_{\frac{\rho}{c}\to 1} \frac{W_{q_c}}{W_{q_1}} = \frac{1}{c}$$

**11**

Determination of $p_0$ involves the finite series sum

$$\sum_{n=c}^{\infty}\left(\frac{\rho}{c}\right)^{n-c} = \sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu c}\right)^{j}$$

The series will diverge if $\lambda \geq \mu c$. The condition requires that customers be serviced at a rate faster than the rate at which they arrive at the facility. Else, the queue will build up to infinity.

**12**

$$L_q = \sum_{n=c}^{\infty}(n-c)p_n$$

$$= \sum_{n=c}^{\infty} n p_n - c\sum_{n=c}^{\infty} p_n + \sum_{n=0}^{c-1} n p_n - \sum_{n=0}^{c-1} n p_n + c\sum_{n=0}^{c-1} p_n - c\sum_{n=0}^{c-1} p_n$$

$$= \sum_{n=0}^{\infty} n p_n - c\sum_{n=0}^{\infty} p_n + \sum_{n=0}^{c-1}(c-n)p_n$$

$$= L_s - c + (\text{number of idle servers})$$

$$= L_s - \bar{c}$$

Now, by definition

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

It follows that $\bar{c} = \frac{\lambda_{eff}}{\mu}$

**13**

$$p_n = \begin{cases} \frac{\lambda^n}{n!\,\mu^n}p_0, & n \leq c \\ \frac{\lambda^n}{c!\,c^{n-c}\mu^n}p_0, & n \geq c \end{cases}$$

for $c = 1$,

$$p_n = \begin{cases} \frac{\lambda}{\mu}p_0 & n = 1 \\ \left(\frac{\lambda}{\mu}\right)^n p_0 & n \geq 1 \end{cases}$$

Thus,

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, \quad n = 1, 2, \dots$$

**14**

$$L_q = p_0\frac{1}{c!}\sum_{n=c+1}^{\infty}(n-c)\frac{(\lambda/\mu)^n}{c^{n-c}}$$

$$= p_0\frac{(\lambda/\mu)^c}{c!}\sum_{n=c+1}^{\infty}(n-c)\left(\frac{\lambda}{\mu c}\right)^{n-c}$$

$$= p_0\frac{(\lambda/\mu)^c}{c!}\sum_{j=1}^{\infty} j\left(\frac{\lambda}{\mu c}\right)^{j}$$

$$= p_0\frac{(\lambda/\mu)^c}{c!}\frac{\lambda}{\mu c}\frac{d}{d\left(\frac{\lambda}{\mu c}\right)}\sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu c}\right)^{j}$$

$$= p_0\frac{(\lambda/\mu)^c}{c!}\left\{\frac{\lambda/\mu c}{(1-\lambda/\mu c)^2}\right\}$$

$$= p_c\frac{\rho/c}{(1-\rho/c)^2} = \frac{\rho}{(c-\rho)^2}p_c$$

**15**

(a) $P\{\text{a customer is waiting}\}$

$$= P\{\text{at least } c+1 \text{ in system}\}$$

$$= \sum_{n=c+1}^{\infty} p_n$$

$$= \sum_{n=c}^{\infty} p_n - p_c$$

$$= p_0 \frac{\rho^c}{c!} \frac{1}{1-\frac{\rho}{c}} - p_c$$

$$= p_c \left\{ \frac{1}{1-\frac{\rho}{c}} - 1 \right\}$$

$$= p_c \left( \frac{\rho}{c-\rho} \right)$$

(b) Expected number in queue given the queue is not empty

$$= \sum_{i=c+1}^{\infty} (i-c) \frac{p_i}{\sum_{j=c+1}^{\infty} p_j}$$

$$= \frac{L_q}{\sum_{j=c+1}^{\infty} p_j} = \frac{L_q}{p_c\left(\frac{\rho}{c-\rho}\right)}$$

Now, $L_q = \dfrac{p_0}{c!} \displaystyle\sum_{n=c+1}^{\infty} (n-c) \frac{\rho^n}{c^{n-c}}$

$$= p_0 \frac{\rho^c}{c!} \sum_{j=1}^{\infty} j \left(\frac{\rho}{c}\right)^j$$

$$= p_0 \frac{\rho^c}{c!} \left( \frac{\rho/c}{(1-\rho/c)^2} \right), \quad \frac{\rho}{c} < 1$$

$$= p_c \left\{ \frac{c\rho}{(c-\rho)^2} \right\}, \quad \frac{\rho}{c} < 1$$

Substitution for $L_q$ yield the desired result.

(c) Exp. waiting time for those who must wait = Exp. waiting time given there are $c$ in the system.

$$= \frac{1}{\lambda} \sum_{i=c+1}^{\infty} (i-c) \frac{p_i}{\sum_{n=0}^{\infty} p_n}$$

$$= \frac{L_q/\lambda}{p_c/(1-\rho/c)} = \frac{1}{\mu(c-\rho)}$$

**16**

First convert the $c$-channel case into an equivalent single channel. Let the customer just arriving be the $j^{th}$ in queue. Because there are $c$ channels in parallel, the service time, $t$, of each of the other $j-1$ customers and the (one) customer in service are determined as follows: Let $t_1, t_2, \ldots, t_c$ be the actual service times in the $c$ channels. Then,

$$P\{t > T\} = P\{\min_{1 \le i \le c} t_i > T\}$$

$$= \left(e^{-\mu T}\right)^c = e^{-\mu c T}$$

This is true because if $\min_i > T$, then every $t_i$ must be $> T$.

Now,

$$F_t(T) = 1 - P\{t > T\}$$

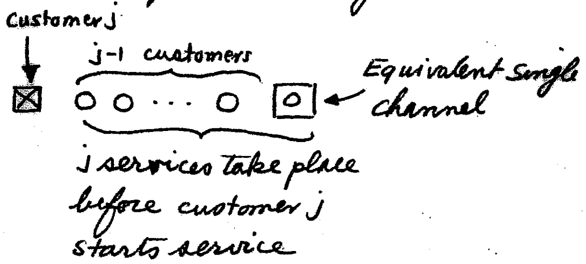
$$= 1 - e^{-\mu c T}, \quad T > 0$$

Thus,

$$f(T) = \frac{\partial F_t(T)}{\partial T} = \mu c e^{-\mu c T}, \quad T > 0$$

which is exponential with mean $\frac{1}{\mu c}$.

The $c$ channels can be converted into an equivalent single channel as




Before customer $j$ starts service, $j$ other customers each with a service time $T$ must be processed first.

Continued…

The assumption here is that all $c$ channels are busy. If there are any idle servers, arriving customer $j$ will have zero waiting time in queue and the special case is treated separately.

Let $\tau$ be the waiting time in queue given there are $j$ other customer yet to be serviced. Then

$$\tau = T_1' + T_2 + \cdots + T_j$$

where $T_1', T_2, \ldots, T_j$ are exponential with mean $1/\mu c$. $T_1'$ represents the remaining service time for the customer already in service. The lack of memory property indicate that $T_1'$ is also exponential with mean $1/\mu c$. Thus,

$$W_q(\tau|j) = \frac{\mu c(\mu c\tau)^{j-1} e^{-\mu c\tau}}{(j-1)!}, \quad \tau > 0$$

Let $W_q(\tau)$ be the absolute pdf, then

$$W_q(\tau) = \sum_{j=1}^{\infty} W_q(\tau|j)\, q_j$$

where

$$q_j = \begin{cases} \sum_{k=0}^{c-1} p_k, & j = 0 \\ p_{c+j-1}, & j > 0 \end{cases}$$

Hence, for $\tau > 0$

$$W_q(\tau) = \sum_{j=1}^{\infty} \frac{\mu c(\mu c\tau)^{j-1} e^{-\mu c\tau}}{(j-1)!} \frac{\rho^{c+j-1}}{c!\, c^{j-1}} p_0$$

$$= \frac{\rho^c \mu c\, e^{-\mu c\tau}}{c!} p_0 \sum_{j=0}^{\infty} \frac{(\rho\mu c\tau/c)^j}{j!}$$

$$= \frac{\rho^c \mu c\, e^{-\mu c\tau}}{c!} p_0\, e^{\lambda\tau}$$

$$= \frac{\rho^c \mu\, e^{-\mu(c-\rho)\tau}}{(c-1)!} p_0$$

continued...

For $\tau = 0$, the corresponding probability is $\sum_{k=0}^{c-1} p_k$, or

$$1 - \sum_{k=c}^{\infty} p_k = 1 - \sum_{j=0}^{\infty} p_{c+j}$$

$$= 1 - \sum_{j=0}^{\infty} \frac{\rho^{c+j}}{c!\, c^j} p_0$$

$$= 1 - \frac{\rho^c}{c!} \left( \frac{p_0}{1-\frac{\rho}{c}} \right)$$

$$= 1 - \left\{ \frac{\rho^c p_0}{(c-1)!(c-\rho)} \right.$$

Hence,

$$W_q(\tau) = \begin{cases} 1 - \dfrac{\rho^c p_0}{(c-1)!(c-\rho)}, & \tau = 0 \\ \dfrac{\mu\rho^c e^{-\mu(c-\rho)\tau}}{(c-1)!} p_0, & \tau > 0 \end{cases}$$

**17**

$$P\{\tau > y\} = \int_y^{\infty} w_q(\tau)\, d\tau$$

$$= \frac{c\mu\rho^c p_0}{c!} \int_y^{\infty} e^{-(c\mu-\lambda)\tau} d\tau$$

$$= \frac{\rho^c c\mu}{c!(c\mu-\lambda)} e^{-(c\mu-\lambda)y} p_0$$

$$= \frac{\rho^c p_0}{c!\,(1-\frac{\rho}{c})} e^{-(c\mu-\lambda)y}$$

$$= P\{\tau > 0\}\, e^{-(c\mu-\lambda)y}$$

where $P\{\tau > 0\} = 1 - P\{\tau = 0\}$

**18**

From Problem 16, the waiting time in the system is computed as

$$T = T_1' + T_2 + \cdots + T_j + t_j$$

where

$t_j =$ actual service time for customer $j$.

$t_j$ is exponential with mean $1/\mu$

Thus, $T$ is the convolution of the waiting time in queue and the actual service time of customer $j$. This means that $w(T)$ is the convolution of $w_q(\tau)$ and $g(t)$; that is,

$$w(T) = w_q(\tau) * g(t)$$

where

$$g(t) = \mu e^{-\mu t}, \quad t > 0$$

$$w(T) = w_q(0)g(T) + \int_{0+}^{T} w_q(\tau)g(T-\tau)\,d\tau$$

$$= \left(1 - \frac{\rho^c p_0}{(c-1)!(c-\rho)}\right)\mu e^{-\mu T} + p_0\int_{0+}^{T}\frac{\mu\rho^c e^{-\mu(c-\rho)\tau}}{(c-1)!}\mu e^{-\mu(T-\tau)}\,d\tau$$

$$= \left(1 - \frac{\rho^c p_0}{(c-1)!(c-\rho)}\right)\mu e^{-\mu T} + \frac{\mu\rho^c e^{-\mu T}}{(c-1)!(c-1-\rho)}p_0\left\{1 - e^{-\mu(c-1-\rho)T}\right\}$$

$$= \mu e^{-\mu T} - \frac{\rho^c p_0 \mu e^{-\mu T}}{(c-1)!(c-1-\rho)}\frac{(c-\rho-1)}{(c-\rho)} + \frac{\mu\rho^c e^{-\mu T}p_0}{(c-1)!(c-1-\rho)} - \frac{\mu\rho^c e^{-\mu T}e^{-\mu(c-1-\rho)T}}{(c-1)!(c-1-\rho)}$$

Continued...

$$= \mu e^{-\mu T} + \frac{\rho^c p_0 \mu e^{-\mu T}}{(c-1)!(c-1-\rho)}\left\{\frac{1}{c-\rho} - e^{-\mu(c-1-\rho)T}\right\}$$

$$T > 0$$

# Set 15.6f

**1**

(a) $c-(L_s-L_q)=4-(4.24-1.54)$

$= 1.3$ cabs

(b) $p_q = .04468$

(c) Title: 6f-1
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 16.00000 | 5.00000 | 15.42815 | 0.03121 | 4.23984 | 1.15421 | 0.27481 | 0.07481 |
| 2 | 4 | 16.00000 | 5.00000 | 15.25869 | 0.03236 | 4.02634 | 0.97460 | 0.26387 | 0.06387 |
| 3 | 4 | 16.00000 | 5.00000 | 15.02834 | 0.03393 | 3.78470 | 0.77903 | 0.25184 | 0.05184 |
| 4 | 4 | 16.00000 | 5.00000 | 14.70690 | 0.03613 | 3.51216 | 0.57078 | 0.23881 | 0.03881 |
| 5 | 4 | 16.00000 | 5.00000 | 14.24151 | 0.03931 | 3.20550 | 0.35719 | 0.22508 | 0.02508 |

$m$ = length of waiting list

$N = m+4$

| $m$ | $N$ | $W_q$(hr) | $W_q$(min) |
|---|---|---|---|
| 6 | 10 | .075 | 4.5 |
| 5 | 9 | .064 | 3.83 |
| 4 | 8 | .052 | 3.12 |
| 3 | 7 | .039 | 2.33 |
| 2 | 6 | .025 | 1.5 |

Select $m \le 3$

**2**

$c=2$, $\lambda = 20/hr$, $N=5$

$\mu = 60/6 = 10/hr$

(a) $p_5 = .1818$ or $18.18\%$

(b) $p_1 = .1818$ or $18.18\%$

(c) % utilization $= 100\left(\frac{L_s-L_q}{c}\right)$

$= \frac{2.727-1.091}{2} \times 100$

$= 81.8\%$

(d) Probability $= p_2+p_3+p_4 = .54546$

(e) $p_N \le .1$

| $N$ | 5 | ... | 8 | 9 | 10 |
|---|---|---|---|---|---|
| $p_N$ | .1818 | | .1176 | .1053 | .0952 |

$N \ge 10$ spaces (including the pumps)

continued...

(f) $p_0 \le .05$

| $N$ | 5 | ... | 8 | 9 | 10 |
|---|---|---|---|---|---|
| $p_N$ | .0909 | | .0588 | .0526 | .0476 |

$N \ge 10$

**3**

$\lambda = 60/10 = 6/hr$

$\mu = 60/30 = 2/hr$, $N=18$

(a) # idle mechanics

$= c-(L_s-L_q)$

$= 3-(9.54-6.71) = .17$

(b) $p_{18} = .0559$

$\lambda_{lost} = .0559 \times 6 = .3354$ job/hr

# lost jobs in 10 hrs $= 3.354$ jobs

(c) $p_{n\le 17} = p_0+p_1+\cdots+p_{17}$

$= .9441$

(d) $p_{n\le 2} = p_0+p_1+p_2 = .10559$

(e) $L_q = 6.7081$ mower

(f) $\frac{L_s-L_q}{c} = \frac{9.54-6.71}{3} = .944$

**4**

$N=40$, $c=30$, $\lambda = 20/hr$

$\mu = 60/60 = 1/hr$

(a) $p_{40} = .00014$

(b) $p_{30}+p_{31}+\cdots+p_{39} = p_{n\le 39} - p_{n\le 29}$

$= .99986 - .97533$

$= .02453$

(c) $p_{29} = .01248$

(d) $L_s - L_q = 20.043 - .046 \approx 20$ spaces

(e) $L_q = .046$

continued...

(f) If there are 30 cars or more in the lot, the student will not make it to class. Thus,

$P\{\text{not finding a parking space}\}$

$$= p_{30} + p_{31} + \cdots + p_{40} = 1 - P_{n \le 29}$$

$$= 1 - .97533 = .02467$$

No. of students who cannot park during an 8-hr period $= 20 \times .02467 \times 8$

$$\cong 4 \text{ students}$$

**5**

$$1 = p_0 \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \sum_{n=c}^{N} \left(\frac{\rho}{c}\right)^{n-c} \right\}$$

$$= p_0 \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \frac{1 - (\rho/c)^{N-c+1}}{(1 - \rho/c)} \right\}$$

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \left( \frac{1 - (\rho/c)^{N-c+1}}{1 - \rho/c} \right) \right\}$$

**6**

$$\bar{c} = L_s - L_q$$

$$= \lambda_{eff} (W_s - W_q)$$

$$= \lambda_{eff} \left(\frac{1}{\mu}\right)$$

**7**

$$1 = \frac{p_0}{c!} \sum_{n=c}^{N} \frac{\rho^n}{c^{n-c}} + p_0 \sum_{n=0}^{c-1} \frac{\rho^n}{n!}$$

$$= \frac{p_0 \rho^c}{c!} \sum_{n=0}^{N-c} \left(\frac{\rho}{c}\right)^n + p_0 \sum_{n=0}^{c-1} \frac{\rho^n}{n!}$$

$$= \frac{p_0 \rho^c}{c!} (N-c+1) + p_0 \sum_{n=0}^{c-1} \frac{\rho^n}{n!}$$

Thus,

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} (N-c+1) \right\}^{-1}$$

$$L_q = \sum_{n=c}^{N} (n-c) p_n$$

$$= \sum_{j=0}^{N-c} j \, p_{j+c}$$

$$= \frac{\rho}{c!} \frac{\rho}{c} \sum_{j=0}^{N-c} j \left(\frac{\rho}{c}\right)^{j-1} p_0$$

continued...

$$= \frac{\rho^c}{c!} \sum_{j=0}^{N-c} j \, p_0 \quad \left(\text{because } \frac{\rho}{c} = 1\right)$$

$$= \frac{\rho^c}{c!} \frac{(N-c)(N-c+1)}{2} p_0$$

$$= \frac{\rho^c (N-c)(N-c+1)}{2c!} p_0$$

**8**

$$\lambda_n = \begin{cases} \lambda, & n = 0, 1, 2, \ldots, c-1 \\ 0, & n = c \end{cases}$$

$$\mu_n = n\mu, \quad n = 0, 1, \ldots, c$$

Thus,

$$p_n = \frac{\rho^n}{n!} p_0, \quad n = 0, 1, 2, \ldots, c$$

$$\sum_{n=0}^{c} p_n = \sum_{n=0}^{c} \frac{\rho^n}{n!} p_0 = 1$$

$$p_0 = \left\{ \sum_{n=0}^{c} \frac{\rho^n}{n!} \right\}^{-1}$$

## Set 15.6g

**1**

(a) $p_0 = 0$

(b) $p_{n\geq 10} = 1 - p_{n\leq 9} = 1$

(c) $p_{n\leq 40} - p_{n\leq 29} = .7771 - .13787$
$= .63923$

(d) $L_s = 36$

Net annual equity
$= \$1000 \times 36 \{.1(1-.3) + .9(1+.15)\}$
$= \$39,780$

**2**

$\lambda = \frac{100}{8} = 12.5/hr$

$\mu = \frac{60}{30} = 2/hr$

(a) $L_s = 6.25 \simeq 7$ seats

(b) $p_{n\geq 8} = 1 - (p_0 + p_1 + \cdots + p_7)$
$= 1 - .7089 = .2911$

(c) $p_0 = .00193$

**3**

$\rho = .1$

| c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|
| 2 | 1.00000 | 10.00000 | 1.00000 | 0.90476 | 0.10025 | 0.00025 | 0.10025 | 0.00025 |
| 4 | 1.00000 | 10.00000 | 1.00000 | 0.90484 | 0.10000 | 0.00000 | 0.10000 | 0.00000 |
| 10 | 1.00000 | 10.00000 | 1.00000 | 0.90484 | 0.10000 | 0.00000 | 0.10000 | 0.00000 |
| 20 | 1.00000 | 10.00000 | 1.00000 | 0.90484 | 0.10000 | 0.00000 | 0.10000 | 0.00000 |
| 50 | 1.00000 | 10.00000 | 1.00000 | 0.90484 | 0.10000 | 0.00000 | 0.10000 | 0.00000 |
| 9999 | 1.00000 | 10.00000 | 1.00000 | 0.90484 | 0.10000 | 0.00000 | 0.10000 | 0.00000 |

**4**

$\rho = 9$

| $c$ | $\lambda$ | $\mu$ | $\lambda_{eff}$ | $p_0$ | $L_s$ | $L_q$ | $W_s$ | $W_q$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 9.00000 | 1.00000 | 9.00000 | 0.00007 | 15.01858 | 6.01858 | 1.66873 | 0.66873 |
| 15 | 9.00000 | 1.00000 | 9.00000 | 0.00012 | 9.07235 | 0.07235 | 1.00804 | 0.00804 |
| 25 | 9.00000 | 1.00000 | 9.00000 | 0.00012 | 9.00000 | 0.00000 | 1.00000 | 0.00000 |
| 50 | 9.00000 | 1.00000 | 9.00000 | 0.00012 | 9.00000 | 0.00000 | 1.00000 | 0.00000 |
| 9999 | 9.00000 | 1.00000 | 9.00000 | 0.00012 | 9.00000 | 0.00000 | 1.00000 | 0.00000 |

1. For very small $\rho$, $(M/M/\infty):(GD/\infty/\infty)$ provides reliable estimates for $(M/M/c):(GD/\infty/\infty)$.
2. For large $\rho$, $(M/M/\infty)$ gives reliable estimates only if $c$ is large

**1**

(a) $R=1$: $\lambda_{eff} = \lambda(22-L_s)$

$= .5(22 - 12.004)$

$= 4.998$

$R=4$: $\lambda_{eff} = .5(22-2.1) = 9.95$

(b) No. of idle repair persons

$= 4-(L_s - L_q)$

$= 4-(2.1-.11) = 2.01$

(c) $p_0 = .10779$

(d) $R=3$:

$P\{2 \text{ or } 3 \text{ are idle}\} = p_0 + p_1$

$= .34492$

Title: 6h-1
Scenario 3-- (M/M/3):(GD/22/22)

Lambda = 0.50000 Mu = 5.00000
Lambda eff = 9.76696 Rho/c = 0.03333

Ls = 2.46596 Lq = 0.51257
Ws = 0.25248 Wq = 0.05248

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.10779 | 0.10779 | 8 | 0.00953 | 0.99244 |
| 1 | 0.23713 | 0.34492 | 9 | 0.00445 | 0.99689 |
| 2 | 0.24899 | 0.59390 | 10 | 0.00193 | 0.99881 |
| 3 | 0.16599 | 0.75990 | 11 | 0.00077 | 0.99959 |
| 4 | 0.10513 | 0.86502 | 12 | 0.00028 | 0.99987 |
| 5 | 0.06308 | 0.92810 | 13 | 0.00009 | 0.99996 |
| 6 | 0.03574 | 0.96384 | 14 | 0.00003 | 0.99999 |
| 7 | 0.01906 | 0.98291 | | | |

**2**

Productivity of repair persons

$$= \frac{\text{Av. \# busy repair persons}}{R}$$

$$= \frac{L_s - L_q}{R}$$

| R | Repair prod. | Shop prod. |
|---|---|---|
| 1 | 100% | 45.44% |
| 2 | 88.2% | 80.15% |
| 3 | 65.1% | 88.7% |
| 4 | 49.7% | 90.45% |

$R=2$ yield 80.15% shop productivity and also maintain repair productivity at 88.2%

**3**

Increasing R, in effect, increases the number of machines that remain operative, and hence the chance of additional breakdowns. Stated differently, if all machines remain broken, there will be no new calls for repair service, and $\lambda_{eff} = 0$

**4**

$\lambda = \frac{60}{45} = 1.33$ machines/hr

$\mu = \frac{60}{8} = 7.5$ machines/hr

$R=1$, $K=5$

Title: 6h-4
Scenario 1-- (M/M/1):(GD/5/5)

Lambda = 1.33333 Mu = 7.50000
Lambda eff = 4.99939 Rho/c = 0.17778

Ls = 1.25045 Lq = 0.58386
Ws = 0.25012 Wq = 0.11679

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.33341 | 0.33341 | 3 | 0.11240 | 0.95293 |
| 1 | 0.29637 | 0.62978 | 4 | 0.03996 | 0.99290 |
| 2 | 0.21075 | 0.84053 | 5 | 0.00710 | 1.00000 |

(a) $L_s = 1.25$ machines

(b) $p_0 = .33341$

(c) $W_s = .25$ hour

**5**

$\lambda = 60/45 = 1.33/hr$

$\mu = 60/20 = 3/hr$

$R=4$, $K=4$

Title: 6h-5
Scenario 1-- (M/M/4):(GD/4/4)

Lambda = 1.33333 Mu = 3.00000
Lambda eff = 3.69230 Rho/c = 0.11111

Ls = 1.23077 Lq = 0.00000
Ws = 0.33333 Wq = 0.00000

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.22972 | 0.22972 | 3 | 0.08067 | 0.99104 |
| 1 | 0.40839 | 0.63811 | 4 | 0.00896 | 1.00000 |
| 2 | 0.27226 | 0.91037 | | | |

(a) $L_s = 1.23$ worker

(b) $p_0 = .22922$

**6**

$\lambda = \frac{60}{30} = 2$ calls/hr/baby

$\mu = \frac{60}{120} = .5/\text{hr}$

$R = 5, \quad K = 5$

Title: 6h-6
Scenario 1-- (M/M/5):(GD/5/5)

| | | | |
|---|---|---|---|
| Lambda = 2.00000 | | Mu = 0.50000 | |
| Lambda eff = 2.00000 | | Rho/c = 0.80000 | |
| Ls = 4.00000 | | Lq = 0.00000 | |
| Ws = 2.00000 | | Wq = 0.00000 | |

| n | Probability, pn | Cumulative, Pn | n | Probability, pn | Cumulative, Pn |
|---|---|---|---|---|---|
| 0 | 0.00032 | 0.00032 | 3 | 0.20480 | 0.26272 |
| 1 | 0.00640 | 0.00672 | 4 | 0.40960 | 0.67232 |
| 2 | 0.05120 | 0.05792 | 5 | 0.32768 | 1.00000 |

(a) No. "awake" babies

$= 5 - L_s = 5 - 4 = 1$ baby

(b) $p_5 = .32768$

(c) $p_{n\le 2} = p_0 + p_1 + p_2 = .05792$

**7**

$$p_n = \begin{cases} \frac{K\lambda}{\mu}\frac{(K-1)\lambda}{2\mu}\cdots\frac{(K-n)\lambda}{n\mu}p_0, & 0\le n\le R \\ \frac{K\lambda}{\mu}\frac{(K-1)\lambda}{2\mu}\cdots\frac{(K-R)\lambda}{R\mu}\cdots\frac{K-n}{R\mu}p_0, & R\le n\le K \end{cases}$$

Thus,

$$p_n = \begin{cases} \frac{K(K-1)\cdots(K-n)}{1\times 2\times\cdots\times n}\left(\frac{\lambda}{\mu}\right)^n p_0, & 0\le n\le R \\ \frac{C_n^k\, n!}{R!\,R^{n-R}}\left(\frac{\lambda}{\mu}\right)^n p_0, & R\le n\le K \end{cases}$$

$$= \begin{cases} C_n^K \rho^n p_0, & 0\le n\le R \\ C_n^k \frac{n!\,\rho^n}{R!\,R^{n-R}} p_0, & R\le n\le K \end{cases}$$

**8**

$\bar{R} = L_s - L_q$

$= \lambda_{eff}(W_s - W_q)$

$= \lambda_{eff}\left(\frac{1}{\mu}\right)$

hence $\lambda_{eff} = \mu\bar{R}$

**9**

$$p_n = \begin{cases} C_n^k \rho^n n!\, p_0, & n = 0, 1 \\ C_n^k n!\, \rho^n p_0, & n = 1, 2, \dots, K \end{cases}$$

$$= \frac{K!}{(K-n)!}\rho^n p_0, \quad n = 0, 1, 2, \dots, K$$

$$L_s = \sum_{n=0}^{K} n p_n = p_0 K! \sum_{n=0}^{K} \frac{n\rho^n}{(K-n)!}$$

$$= K - \left(\frac{1-p_0}{\rho}\right)$$

**1**

$$\% \text{idle} = \frac{1-(L_s-L_q)}{1} \times 100$$
$$= [1-(L_s-L_q)] \times 100$$
$$= (1-1.333+.667) \times 100$$
$$= 33.3\%$$

**2**

(a) $E\{t\} = 14$ min

$Var\{t\} = \frac{(20-8)^2}{12} = 12 \text{ min}^2$

$\lambda = 4/hr = .0667/min$

$L_s = 7.867$ cars

$W_s = 118$ min $= 1.967$ hours

$L_q = 6.933$ cars

$W_q = 104$ min $= 1.733$ hours

(b) $E\{t\} = 12$ min

$Var\{t\} = 9 \text{ min}^2$

$\lambda = .0667$ /min

$L_s = 2.5$ cars

$W_s = 37.5$ min $= .625$ hour

$L_q = 1.7$ cars

$W_q = 25.5$ min $= .425$ hr

(c) $E\{t\} = 4\times.2 + 8\times.6 + 15\times.2 = 8.6$ min

$Var\{t\} = (4-8.6)^2(.2) + (8-8.6)^2(.6) + (15-8.6)^2(.2) = 12.64 \text{ min}^2$

$L_s = 1.0244$ cars

$W_s = 15.3657$ min $= .256$ hr

$L_q = .451$ car

$W_q = 6.765$ min $= .113$ hr

**3**

$\lambda = .3$ job/day

Service time distribution:

$f(t) = .5, \quad 2 \le t \le 4$ days

$E\{t\} = 3$ days

$Var\{t\} = \frac{4}{12} = .333 \text{ days}^2$

(a) $L_q = 4.2$ homes

(b) $W_s = 17$ days

(c) $E\{t\} = 1.5$, $Var\{t\} = \frac{1}{12} = .0833$

$L_q = .191$ home

$W_s = 2.14$ days

**4**

$\lambda = \frac{30}{8\times60} = .0625$ prescr./min

$E\{t\} = 12+3 = 15$ min

$Var\{t\} = 9 + \frac{(4-2)^2}{12} = 9.333 \text{ min}^2$

(a) $p_0 = .0625$

(b) $L_q = 7.3$ prescriptions

(c) $W_s = 132.17$ min $= 2.2$ hours

**5**

$\lambda = 1/45$ /min $= .0222$ /min

$E\{t\} = 28+4.5 = 32.5$ min

$Var\{t\} = \frac{(6-3)^2}{12} = .75$

(a) $L_q = .9395$ item

(b) $p_0 = .278$

(c) $W_s = 74.78$ min

**6**

$$L_s = \lambda E\{t\} + \frac{\lambda^2(E^2\{t\} + Var\{t\})}{2(1-\lambda E\{t\})}$$
$$= \lambda E\{t\} + \frac{(\lambda E\{t\})^2}{2(1-\lambda E\{t\})}$$
$$= \rho + \frac{\rho^2}{2(1-\rho)}$$

**7**

$$L_s = \frac{m\lambda}{\mu} + \frac{\lambda^2\left(\frac{m^2}{\mu^2} + \frac{n}{\mu^2}\right)}{2\left(1 - \frac{m\lambda}{\mu}\right)}$$

$$= m\rho + \frac{m^2\rho^2 + m\rho^2}{2(1-m\rho)}$$

$$= m\rho + \frac{m(m+1)\rho^2}{2(1-m\rho)}$$

**8**

$$E\{t\} = \frac{1}{\mu},\ Var\{t\} = \frac{1}{\mu^2}$$

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2\left(\frac{1}{\mu^2} + \frac{1}{\mu^2}\right)}{2(1-\lambda/\mu)}$$

$$= \rho + \frac{\rho^2}{1-\rho}$$

$$= \frac{\rho}{1-\rho}$$

**9**

(a) Because each server receives every $c^{th}$ customer and the interarrival time at the channel is exponential with mean $1/\lambda$, the interarrival time at each server is the convolution of c exponential distributions each with mean $\frac{1}{\mu}$. This means that the interarrival time is gamma with mean $c/\lambda$ and variance $c/\lambda^2$.

(b) The interarrival time at the $i^{th}$ server is exponential with mean $\frac{1}{\alpha_i \lambda}$. This means that the arrivals at server $i$ is Poisson with mean $\alpha_i \lambda$, $i=1,2,\ldots,c$

**1**

(a) $\mu_2 = \dfrac{24}{\left(\frac{1000}{36}\right)\frac{1}{60}} = 5.184$ jobs/day

$\mu_3 = \dfrac{24}{\left(\frac{1000}{50}\right)\times\frac{1}{60}} = 7.2$ jobs/day

$\mu_4 = \dfrac{24}{\left(\frac{1000}{66}\right)\times\frac{1}{60}} = 9.5$ jobs/day

(b) $ETC_i = 24\, C_{1i} + 80\, Lq_i$

| $i$ | $\lambda_i$ | $\mu_i$ | $Lq_i$ | $C_{1i}$ | $ETC_i$ |
|---|---|---|---|---|---|
| 1 | 4 | 4.32 | 11.57 | \$15 | \$1285.60 |
| 2 | 4 | 5.18 | 2.62 | 20 | 689.60 |
| 3 | 4 | 7.20 | .69 | 24 | 631.20 |
| 4 | 4 | 9.50 | .31 | 27 | 672.80 |

Select model 3.

**2**

$\lambda = 3/hr$

$\mu_1 = 5/hr, \quad C_1 = \$15$

$\mu_2 = 8/hr, \quad C_2 = \$20$

Cost/Broken machine = \$50/hr

(M/M/1):(GD/10/10):

$\lambda = 3,\ \mu = 5 \Rightarrow L_{s_1} = 8.33$

(M/M/1):(GD/10/10):

$\lambda = 3,\ \mu = 8 \Rightarrow L_{s_2} = 7.33$

$TC_1 = 50L_{s_1} + 15 = 50\times 8.33 + 15$
$= \$431.50/hr$

$TC_2 = 50L_{s_2} + 20 = 50\times 7.33 + 20$
$= \$386.50/hr$

Hire second repair person.

**3**

$\lambda = 10/hr = .167/min$

Scanner A:

Service time distribution:

$f_A(t) = \dfrac{1}{\left(\frac{35}{10}\right)-\left(\frac{25}{10}\right)} = 1,\ 2.5 \le t \le 3.5$

Continued...

$E_A\{t\} = 3$ min

$Var_A\{t\} = \frac{1}{12}$ min$^2$

Scanner B:

$f_B(t) = \dfrac{1}{\frac{35}{15} - \frac{25}{15}} = 1.5,\quad 5/3 \le t \le 7/3$

$E_B\{t\} = 2$ min

$Var_B\{t\} = \dfrac{(2/3)^2}{12} = 1/27$ min$^2$

From Excel file PKFormula.xls,

$L_{s_A} = .755$ customer

$L_{s_B} = .419$ customer

$TC_A = .2L_{s_A} + C_A$
$= \left(.2\times .755 + \dfrac{\$10}{10\times 60}\right)\times 60 = \$10.06/hr$

$TC_B = .2L_{sB} + C_B$
$= \left(.2\times .419 + \dfrac{\$15}{10\times 60}\right)\times 60 = \$6.53/hr$

Select scanner B

**4**

(a)

$\mu$ = number of filled orders/hr

$\lambda$ = number of requested orders/hr

$C_1$ = cost/unit increase in production rate

$C_2$ = cost of waiting/unit waiting time/cust.

$TC(\mu)$ = Total cost/unit waiting time given $\mu$

$= C_1\mu + C_2 L_s$

$= C_1\mu + C_2 \dfrac{\lambda}{\mu - \lambda}$

(b)

$\dfrac{\partial TC(\mu)}{\partial \mu} = C_1 - C_2 \dfrac{\lambda}{(\mu-\lambda)^2} = 0$

$\mu = \lambda + \sqrt{\dfrac{C_2}{C_1}\lambda}$

(c) $\lambda = 3,\ C_1 = .1\times 500 = \$50,\ C_2 = \$100$

$\mu = 3 + \sqrt{\dfrac{100}{50}\times 3} = 5.45$ orders/hr

Optimum production rate

$= 500\times 5.45 \simeq 2725$ pieces/hr

**5**

$\lambda = 80$ jobs/wk

$C_1 = \$250/\text{wk}$ $\quad C_2 = \$500/\text{job}/\text{wk}$

$$\mu = \lambda + \sqrt{\frac{C_2}{C_1}\lambda}$$

$$= 80 + \sqrt{\frac{500}{250} \times 80} = 92.65 \text{ jobs/wk}$$

**6**

$\lambda = 25$ groups/hr

Model A: $\mu_A = 26/\text{hr}$, $N = 20$

Operating cost $C_A = \$12000$ /month

From TORA: $p_{20} = .03128$

$L_q = 7.65$ groups

Cost/hr = operating cost/hr + waiting cost/hr + cost of lost customers/hr

$$= \frac{C_A}{30 \times 10} + 10L_q + \lambda p_N \times 15$$

$$= \frac{12000}{30 \times 10} + 10 \times 7.65 + 25 \times .03128 \times 15$$

$$= \$128.23/\text{hr}$$

Model B: $\mu_B = 29/\text{hr}$, $N = 30$

$C_B = \$16000$/month

From TORA: $p_{30} = .0016$

$L_q = 5.07$ groups

$$\text{Cost/hr} = \frac{\$16000}{30 \times 10} + 10 \times 5.07 + 25 \times .0016 \times 15$$

$$= \$104.63$$

Select model B

**7**

Let

$C_3$ = cost/unit time/additional capacity unit.

The cost model in Problem 6 is modified by adding the term $C_3 N$ to the cost equation.

**8**

$p_0$ is the probability of running out of stock. Thus,

Cost of lost sales per hour $= C_1 \lambda p_0$

$E\{\text{cost}\}$/unit time

$= E\{\text{lost sales cost}\}$/unit time $+ E\{\text{holding cost}\}$/unit time

$$= C_1 \lambda p_0 + C_2 L_s$$

For $(M/M/1):(GD/\infty/\infty)$

$$p_0 = (1-\rho)$$

$$L_s = \frac{\rho}{1-\rho}$$

Thus,

$$E\{\text{cost}\}/\text{unit time} = C_1\lambda(1-\rho) + C_2\frac{\rho}{1-\rho}$$

$$\frac{\partial E\{\text{cost}\}}{\partial \rho} = -C_1\lambda + \frac{C_2}{(1-\rho)^2} = 0$$

Thus,

$$\rho = 1 \pm \sqrt{\frac{C_1 \lambda}{C_2}}$$

Under steady state, $\rho$ must be less than 1. Thus,

$$\rho = 1 - \sqrt{\frac{C_1\lambda}{C_2}}$$

The solution requires $\sqrt{\frac{C_1\lambda}{C_2}} < 1$ in order for $\rho$ not to assume a negative value. Note that $\rho = \frac{\lambda}{\mu}$, where $\lambda$ is a constant. This means that $\mu$ is the actual optimization variable.

**1**

$C_1 = \$20$, $C_2 = \$45$,
$\lambda = 17.5/\text{hr}$, $\mu = 10/\text{hr}$

Title: 17.9b-1 (M/M/c)(GD/infinity/infinity)
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 17.50000 | 10.00000 | 17.50000 | 0.06667 | 7.46667 | 5.71667 | 0.42667 | 0.32667 |
| 2 | 3 | 17.50000 | 10.00000 | 17.50000 | 0.15564 | 2.21712 | 0.46712 | 0.12669 | 0.02669 |
| 3 | 4 | 17.50000 | 10.00000 | 17.50000 | 0.17038 | 1.84206 | 0.09206 | 0.10526 | 0.00526 |
| 4 | 5 | 17.50000 | 10.00000 | 17.50000 | 0.17314 | 1.76962 | 0.01962 | 0.10112 | 0.00112 |

$ETC(c) = 20c + 45L_s$

| $c$ | $L_s(c)$ | $ETC(c)$ |
|---|---|---|
| 2 | 7.467 | $20\times2+45\times7.467=\$376.02$ |
| → 3 | 2.217 | $20\times3+45\times2.217=\$159.77$ |
| 4 | 1.842 | $20\times4+45\times1.842=\$162.89$ |
| 5 | 1.770 | $20\times5+45\times1.770=\$179.65$ |

Use three clerks

**2**

$\text{Cost/hr} = C_1 L_s + C_2 c$

$C_1 = \$30$, $C_2 = \$18$

$(M/M/c):(GD/10/10)$: $\lambda = 1/20 = 0.05/\text{hr}$
$\mu = 1/3 = 0.333/\text{hr}$

Title: 9b-2
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0.05000 | 0.33300 | 0.41603 | 0.21439 | 1.67942 | 0.43010 | 4.03683 | 1.03383 |
| 2 | 3 | 0.05000 | 0.33300 | 0.43187 | 0.24269 | 1.36246 | 0.06554 | 3.15476 | 0.15175 |

$(\text{Cost/hr for } c=2) = 30\times1.68+18\times2 = \$86.40$
$(\text{Cost/hr for } c=3) = 30\times1.36+18\times3 = \$94.80$

(a) No, because the cost is higher

(b) Schedule loss/breakdown $= C_1 W_s$

$c=2$: $W_s = 4.037$ hours
schedule loss $= 30\times4.037 = \$121.11$
$c=3$: $W_s = 3.155$ hours
Schedule loss $= 30\times3.155 = \$94.65$

**3**

The problem is similar to the machine repair model. The executives are the "machines" and the WATS line is the "server"

arrival rate/executive = 2 calls/day

service rate $= \frac{480}{6}$
$= 80$ calls/day

Continued...

TORA input:

$R=1$: $(2, 80, 1, 100, 100)$

$R=2$: $(2, 80, 2, 100, 100)$

Title: 9b-3
Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2.00000 | 80.00000 | 80.00000 | 0.00000 | 59.99881 | 58.99881 | 0.74999 | 0.73749 |
| 2 | 2 | 2.00000 | 80.00000 | 159.28000 | 0.00200 | 20.35920 | 18.36820 | 0.12782 | 0.11532 |

(a) No WATS:

Cost/month = (2 calls/8 hrs/exec) × (100 exec) × (6 min/call) × (50 ¢/min) × (200 hrs/month)
$= \$15{,}000$/month

One WATS Line: $L_q = 59$

Cost/month = cost of WATS line + $C_1 L_q$
$= \$2000/\text{month} + 59\left(\frac{1¢}{100}\times60\times200\right)$
$= \$9080$

Savings $= 15{,}000 - 9080$
$= \$5920$/month

(b) Two WATS lines: $L_q = 18.4$

Cost/month $= 2\times2000 + 18.4\left(\frac{1¢}{100}\times200\times60\right)$
$= \$6200$

Additional savings
$= 9080 - 6200 = \$2880$

Lease a second WATS line

**4**

Rate of breakdown/machine, $\lambda$

$= \frac{57.8}{8 \times 20} = .36125/hr$

$\mu = \frac{60}{6} = 10/hr$

TORA model: (M/M/3):(GD/20/20)

$W_s$ = lost time per breakdown

$\lambda$ = number of breakdowns/hr/mach

lost time per mach/hr $= \lambda W_s$

From TORA, $W_s = .10118$ hr

Lost revenue/machine/hr

$= 25 \times (.36125 \times .10118) \times \$2$

$= \$1.83$

Lost revenue for all machines

$= 20 \times 1.83 = \$36.60$

Cost of 3 repairpersons/hr

$= 3 \times 20 = \$60.$

**5**

$TC(c) = cC_1 + C_2 L_s(c)$

$TC(c-1) = (c-1)C_1 + C_2 L_s(c-1)$

$TC(c+1) = (c+1)C_1 + C_2 L_s(c+1)$

$TC(c-1) - TC(c)$

$= -C_1 + C_2\{L_s(c-1) - L_s(c)\}$

$TC(c+1) - TC(c)$

$= C_1 - C_2\{L_s(c) - L_s(c+1)\}$

At a minimum point, we must have

$TC(c-1) \geq TC(c)$

$TC(c+1) \geq TC(c)$

Thus,

$L_s(c-1) - L_s(c) \geq \frac{C_1}{C_2}$

$L_s(c) - L_s(c+1) \leq \frac{C_1}{C_2}$

continued...

or

$L_s(c) - L_s(c+1) \leq \frac{C_1}{C_2} \leq L_s(c-1) - L_s(c)$

$\frac{C_1}{C_2} = \frac{12}{50} = .24$

| $c$ | $L_s(c)$ | $L_s(c) - L_s(c+1)$ |
|---|---|---|
| 2 | 7.467 | – |
| 3 | 2.217 | 5.25 |
| 4 | 1.842 | .375 |
| 5 | 1.764 | .078 |

$\leftarrow \frac{C_1}{C_2} = .24$

$c^* = 4$

**1**

$\lambda = 1/7 = .1428$ breakdown/hr

$\mu = .25$ repair per hour

TORA model: $(M/M/R):(GD/10/10)$

Comparative Analysis

| Scenario | c | Lambda | Mu | L'da eff | p0 | Ls | Lq | Ws | Wq |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.14280 | 0.25000 | 0.25000 | 0.00001 | 8.24932 | 7.24934 | 32.99772 | 28.99772 |
| 2 | 1 | 0.14280 | 0.25000 | 0.25000 | 0.00001 | 8.24932 | 7.24934 | 32.99772 | 28.99772 |
| 3 | 3 | 0.14280 | 0.25000 | 0.71071 | 0.00540 | 5.02302 | 2.18017 | 7.06758 | 3.06758 |
| 4 | 4 | 0.14280 | 0.25000 | 0.83618 | 0.00899 | 4.14443 | 0.79972 | 4.95641 | 0.95641 |
| 5 | 5 | 0.14280 | 0.25000 | 0.88773 | 0.01043 | 3.78339 | 0.23247 | 4.26187 | 0.26187 |
| 6 | 6 | 0.14280 | 0.25000 | 0.90407 | 0.01081 | 3.66899 | 0.05272 | 4.05831 | 0.05831 |
| 7 | 7 | 0.14280 | 0.25000 | 0.90807 | 0.01089 | 3.64096 | 0.00867 | 4.00955 | 0.00955 |
| 8 | 8 | 0.14280 | 0.25000 | 0.90878 | 0.01091 | 3.63602 | 0.00091 | 4.00100 | 0.00100 |

(a) From TORA's output

$L_s < 4 \Rightarrow R \geq 5$

(b) From TORA's output

$W_q < 1 \Rightarrow R \geq 4$

**2**

$C_1 = \$12$

| c | $L_s$ |
|---|---|
| 2 | 7.467 |
| 3 | 2.217 |
| 4 | 1.842 |

$$2.217 - 1.842 \leq \frac{12}{C_2} \leq 7.467 - 2.217$$

$$.375 \leq \frac{12}{C_2} \leq 5.25$$

or

$$\$2.29 \leq C_2 \leq \$32$$

# Chapter 16

# Simulation Modeling

# Set 16.1a

## 1

| R1 | R2 | X | Y | (X-1)^2+(Y-2)^2 | 1=in, 0=out |
|---|---|---|---|---|---|
| 0.0589 | 0.6733 | -3.411 | 3.733 | 22.46021 | 1 |
| 0.4799 | 0.9486 | 0.799 | 6.486 | 20.164597 | 1 |
| 0.6139 | 0.5933 | 2.139 | 2.933 | 2.16781 | 1 |
| 0.9341 | 0.1782 | 5.341 | -1.218 | 29.199805 | 0 |
| 0.3473 | 0.5644 | -0.527 | 2.644 | 2.746465 | 1 |
| 0.3529 | 0.3646 | -0.471 | 0.646 | 3.997157 | 1 |
| 0.7676 | 0.8931 | 3.676 | 5.931 | 22.613737 | 1 |
| 0.3919 | 0.7876 | -0.081 | 4.876 | 9.439937 | 1 |
| 0.5199 | 0.6358 | 1.199 | 3.358 | 1.883765 | 1 |
| 0.7472 | 0.8954 | 3.472 | 5.954 | 21.7449 | 1 |
| | | | | Total= | 9 |
| | | | | Area estimate= | 90 |

Exact area = 78.54 cm2. Estimate from Figure 18-2 = 78.56 cm2 for a sample size of n=30,000. Current estimate = 90 cm2, which is unreliable because the sample size is too small.

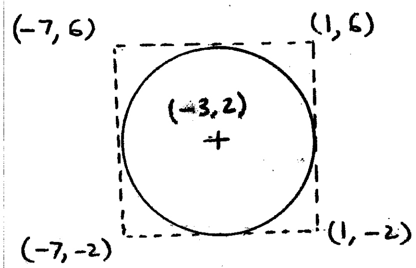
## 2

(a) $X = -7 + 8R_1$

$Y = -2 + 8R_2$

$f(x) = 1/8, \quad -7 \le x \le 1$

$f(y) = 1/8, \quad -2 \le x \le 6$




(b)

**Monte Carlo Estimation of the Area of a Circle**

| Input data | | | |
|---|---|---|---|
| Nbr. Replications, N = | 10 | | |
| Sample size, n = | 100,000 | Steps = | 1 |
| Radius, r = | 4 | | |
| Center, cx = | -3 | | |
| Center, cy = | 2 | | |

**Output results**

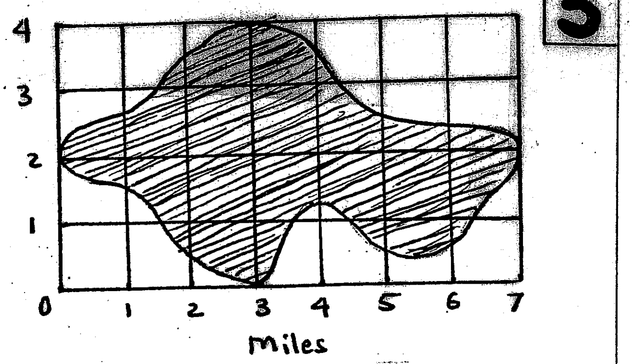
Exact area = 50.265

Press to Execute Monte Carlo

Monte Carlo Calculations:

| | n=100000 |
|---|---|
| Replication 1 | 50.223 |
| Replication 2 | 50.378 |
| Replication 3 | 50.113 |
| Replication 4 | 50.260 |
| Replication 5 | 50.244 |
| Replication 6 | 50.330 |
| Replication 7 | 50.327 |
| Replication 8 | 50.252 |
| Replication 9 | 50.236 |
| Replication 10 | 50.467 |
| | |
| Mean = | 50.283 |
| Std. Deviation = | 0.099 |
| | |
| 95% lower conf. limit = | 50.212 |
| 95% upper conf. limit = | 50.354 |

## 3




| $R_1$ | $R_2$ | X | Y | in? |
|---|---|---|---|---|
| .0589 | .6733 | .4123 | 2.6932 | No |
| .4799 | .9486 | 3.3593 | 3.7944 | Yes |
| .6139 | .5933 | 4.2973 | 2.3732 | Yes |
| .9341 | .1782 | 6.5387 | .7128 | NO |
| .3473 | .5644 | 2.4311 | 2.2576 | Yes |
| .3529 | .3646 | 2.4703 | 1.4584 | Yes |
| .7676 | .8931 | 5.3732 | 3.5724 | No |
| .3919 | .7876 | 2.7433 | 3.1504 | Yes |
| .5199 | .6358 | 3.6393 | 2.5432 | No |
| .7472 | .8954 | 5.2304 | 3.5816 | No |

# points in = 5

Area estimate $= \frac{5}{10} \times (4 \times 7) = 14$ miles$^2$

## 4

(a) $P\{H\} = .5 \qquad P\{T\} = .5$

If $0 \le R \le .5$, Jim gets \$10

$.5 < R \le 1$, Jan gets \$10

(b)

| R | Jan's pay | R | Jan's pay |
|---|---|---|---|
| .0589 | -10 | .3529 | -10 |
| .6733 | 10 | .3646 | -10 |
| .4799 | -10 | .7676 | 10 |
| .9486 | 10 | .8931 | 10 |
| .6139 | 10 | .3919 | -10 |
| .5933 | 10 | .7876 | 10 |
| .9341 | 10 | .5199 | 10 |
| .1782 | -10 | .6358 | 10 |
| .3473 | -10 | .7472 | 10 |
| .5644 | 10 | .8954 | 10 |
| | $\bar{X}_1$ = \$2 | | $\bar{X}_2$ = \$4 |

continued…

| $R$ | Jan's pay | $R$ | Jan's pay | $R$ | Jan's pay |
|---|---|---|---|---|---|
| .5861 | 10 | .3455 | -10 | .7900 | 10 |
| .1281 | -10 | .4871 | -10 | .7698 | 10 |
| .2867 | -10 | .8111 | 10 | .2871 | -10 |
| .8216 | 10 | .8912 | 10 | .9534 | 10 |
| .3866 | -10 | .4291 | -10 | .1394 | -10 |
| .7125 | 10 | .2302 | -10 | .9025 | 10 |
| .2108 | -10 | .5423 | 10 | .1605 | -10 |
| .3575 | -10 | .4208 | -10 | .3567 | -10 |
| .2926 | -10 | .6975 | 10 | .3070 | -10 |
| .8261 | 10 | .5954 | 10 | .5513 | 10 |
| $\bar{X}_3 = -\$2$ | | $\bar{X}_4 = \$0$ | | $\bar{X}_5 = \$0$ | |

(b) Av. Jan's pay based on 5 repls.

$= 2+4-2+0+0$

$= \$.8$

$$S = \sqrt{\frac{(2-.8)^2+(4-.8)^2+(-2-.8)^2+2(0-.8)^2}{5-1}}$$

$$= \sqrt{\frac{80.8}{4}} = 2.28$$

Confidence interval:

$$.8 - \frac{2.28}{\sqrt{5}}\, t_{.025,4} \le \mu \le .8 + \frac{2.28}{\sqrt{5}}\, t_{.025,4}$$

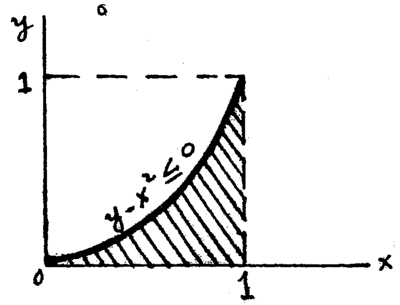
Given $t_{.025,4} = 2.776$, the 95% confidence interval is

$$-2.03 \le \mu \le 3.63$$

(c) Theoretical Jan's payoff = \$0.

## 5

Estimate $\int_0^1 x^2 dx$




Continued...

(a) Let x=R1 and y=R2.
Experiment: If R2<R1^2, count point "in".
Estimate of integral = (1x1)(Points "in")/5

(b)

| | R1 | R2 | 1=in, 0=out |
|---|---|---|---|
| Rep 1 | 0.0589 | 0.6733 | 0 |
| | 0.4799 | 0.9486 | 0 |
| | 0.6139 | 0.5933 | 0 |
| | 0.9341 | 0.1782 | 1 |
| | 0.3473 | 0.5644 | 0 |
| | Integral estimate = | | 0.2 |
| Rep 2 | 0.3529 | 0.3646 | 0 |
| | 0.7676 | 0.8931 | 0 |
| | 0.3919 | 0.7876 | 0 |
| | 0.5199 | 0.6358 | 0 |
| | 0.7472 | 0.8954 | 0 |
| | Integral estimate = | | 0 |
| Rep 3 | 0.5869 | 0.1281 | 1 |
| | 0.2867 | 0.8216 | 0 |
| | 0.8261 | 0.3866 | 1 |
| | 0.7125 | 0.2108 | 1 |
| | 0.3575 | 0.2926 | 0 |
| | Integral estimate = | | 0.6 |
| Rep 4 | 0.3455 | 0.4871 | 0 |
| | 0.8111 | 0.8912 | 0 |
| | 0.4291 | 0.2302 | 0 |
| | 0.5954 | 0.5423 | 0 |
| | 0.4208 | 0.6975 | 0 |
| | Integral estimate = | | 0 |
| overall integral estimate = | | | 0.2 |
| Std. Deviation = | | | 0.244949 |
| 95% lower confidence limit = | | | -0.189714 |
| 95% upper confidence limit = | | | 0.5485706 |
| Exact integral value = | | | 0.3333 |

The given estimate is not "good" when compared with the exact value because sample size (n = 5) is too small.

## 6

$7 \equiv (6,1),(5,2),(4,3),(3,4),(2,5),(1,6)$

$11 \equiv (6,5),(5,6)$

Monte Carlo experiment:

| $R$ | outcome |
|---|---|
| $0 \le R \le 1/6$ | 1 |
| $1/6 < R \le 1/3$ | 2 |
| $1/3 < R \le 1/2$ | 3 |
| $1/2 < R \le 2/3$ | 4 |
| $2/3 < R \le 5/6$ | 5 |
| $5/6 < R \le 1$ | 6 |
| $0 \le R \le .167$ | 1 |
| $.167 < R \le .333$ | 2 |
| $.333 < R \le .5$ | 3 |
| $.5 < R \le .667$ | 4 |
| $.667 < R \le .833$ | 5 |
| $.833 < R \le 1$ | 6 |

Continued...

| $R_1$ | $R_2$ | Sum | Payoff |
|---|---|---|---|
| .0589 | .6733 | $1+5=\boxed{6}$ point | |
| .4799 | .9486 | $3+6=9$ | |
| .6139 | .5933 | $4+4=8$ | |
| .9341 | .1782 | $6+2=8$ | |
| .3473 | .5644 | $3+4=7 \rightarrow$ | $-\$10$ |
| .3529 | .3646 | $3+3=\boxed{6}$ point | |
| .7676 | .8931 | $5+6=11$ | |
| .3919 | .7876 | $3+5=8$ | |
| .5199 | .6358 | $4+4=8$ | |
| .7472 | .8954 | $5+6=11$ | |
| .5869 | .1281 | $4+1=5$ | |
| .2867 | .8216 | $2+5=7 \rightarrow$ | $-\$10$ |
| .8261 | .3866 | $5+3=\boxed{8}$ point | |
| .7125 | .2108 | $5+2=7 \rightarrow$ | $-\$10$ |
| .3575 | .2926 | $3+2=\boxed{5}$ point | |
| .3455 | .4871 | $3+3=6$ | |
| .8111 | .8912 | $5+6=11$ | |
| .4291 | .2302 | $3+2=5 \rightarrow$ | $\$10$ |
| .5954 | .5423 | $4+4=\boxed{8}$ point | |
| .4208 | .6975 | $3+5=8 \rightarrow$ | $\$10$ |

**7**

Lead time:

$0 \le R \le .5$, $L = 1$ day

$.5 < R \le 1$, $L = 2$ days

Demand/day:

$0 \le R \le .2$, $d = 0$ unit

$.2 < R \le .9$, $d = 1$ unit

$.9 < R \le 1.$ $d = 2$ units

Let $p(d, L)$ be the joint pdf of demand and lead time. The procedure calls for constructing a frequency table of demand and lead time. The maximum demand during lead time is $2 \times 2 = 4$ units, so that the demand $d = 0, 1, 2, 3, 4$. We will use the random numbers in Table 16-1 in the following manner: First use a random number to generate a lead time. If $L=1$ day, use one

continued...

7 continued

random number to generate the demand in that day. If $L=2$ days, use two random numbers to generate the demands for the two days. For example, $R = .058962$ yields $L=1$. Next, $R = .6733$ gives $d = 1$. Thus, we update the frequency table by increasing the frequency of the entry $(d=1, L=1)$ by one. The frequency table using the first two columns of $R$ in Table 16-1 is

| L \ d | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 1 | I | ~~IIII~~ II | II | 0 | 0 |
| 2 | II | 0 | ~~IIII~~ II | IIII | 0 |

| L \ d | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 1 | 1 | 7 | 2 | 0 | 0 |
| 2 | 2 | 0 | 7 | 4 | 0 |

Total $n = 23$

Relative frequency table:

| L \ d | 0 | 1 | 2 | 3 | 4 | P(L) |
|---|---|---|---|---|---|---|
| 1 | 1/23 | 7/23 | 2/23 | 0 | 0 | $\frac{10}{23}$ |
| 2 | 2/23 | 0 | 7/23 | 4/23 | 0 | $\frac{13}{23}$ |
| p(d) | 3/23 | 7/23 | 9/23 | 4/23 | 0 | |

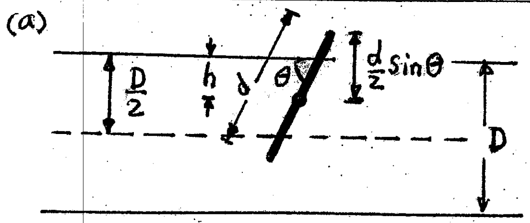
Notice that

$$p(d) = \sum_L p(d, L)$$

$$p(L) = \sum_d p(d, L)$$

**8**

(a)




From graph, needle will touch line or cross it is

$$h \le \frac{d}{2}\sin\theta$$

(b) Generate $h = R_1 \times D/2$

$$\theta = \pi \times R_2$$

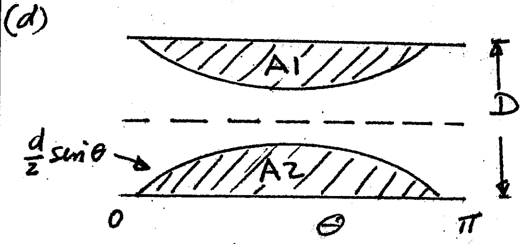
If $h \le \frac{d}{2}\sin\theta$, needle touches. Else it doesn't.

Probability estimate $= \dfrac{\#\text{ touches}}{\text{Sample size}}$

(c)

| A | B | C | D | E |
|---|---|---|---|---|
| | D= | 20 | d= | 10 |
| | (RAND()*$C$1)*0.5 | RAND()*PI() | $E$1*0.5*SIN(C4) | IF(B4<=D4,1,0) |
| | h | theta | d*sin(theta)/2 | 1=touch, 0=else |
| Rep 1 | 8.396953573 | 1.3165558 | 4.839272983 | 0 |
| | 7.107859045 | 2.9048959 | 1.172463622 | 0 |
| | 0.27542965 | 0.8440783 | 3.736795168 | 1 |
| | 1.267504547 | 2.8354706 | 1.506816139 | 1 |
| | 9.237262421 | 0.7436482 | 3.38488765 | 0 |
| | 2.495379696 | 2.9719552 | 0.844125326 | 0 |
| | 4.253169953 | 2.8396976 | 1.486650397 | 0 |
| | 8.516662244 | 1.4161445 | 4.940326141 | 0 |
| | 4.224254495 | 0.7887632 | 3.547410981 | 0 |
| | 3.690266876 | 3.0811599 | 0.301979787 | 0 |
| | | Estimate of probability= | | 0.2 |
| Rep 2 | 0.712918949 | 1.5238102 | 4.994481772 | 1 |
| | 9.381794079 | 2.5979258 | 2.586388239 | 0 |
| | 1.360072144 | 2.0189288 | 4.506289193 | 1 |
| | 8.477675064 | 1.9724771 | 4.60202594 | 0 |
| | 0.99443686 | 1.300734 | 4.81877136 | 1 |
| | 5.170438974 | 1.4568612 | 4.967582038 | 0 |
| | 5.056822846 | 1.6844549 | 4.967739087 | 0 |
| | 5.864264693 | 0.0683356 | 0.341412027 | 0 |
| | 6.87137267 | 2.6283793 | 2.454895584 | 0 |
| | 1.092023022 | 2.6522347 | 2.350296303 | 1 |
| | | Estimate of probability= | | 0.4 |
| Rep3 | 9.712756211 | 1.694489 | 4.961799031 | 0 |
| | 6.686447356 | 1.2243834 | 4.702983326 | 0 |
| | 6.436673778 | 2.4581589 | 3.157296664 | 0 |
| | 1.324134345 | 2.2441568 | 3.908652279 | 1 |
| | 1.775706228 | 2.255079 | 3.874363448 | 1 |
| | 0.090587765 | 2.7080167 | 2.100592855 | 1 |
| | 4.979938633 | 2.5138689 | 2.936520016 | 0 |
| | 8.678634219 | 2.7348178 | 1.978247037 | 0 |
| | 2.179672677 | 1.8339609 | 4.827857959 | 1 |
| | 9.640572895 | 1.2431615 | 4.734030551 | 0 |
| | | Estimate of probability= | | 0.4 |
| Rep 4 | 8.227016322 | 2.6999829 | 2.136976805 | 0 |
| | 8.757368267 | 2.1537385 | 4.174233356 | 0 |
| | 4.203914479 | 0.1860064 | 0.92467824 | 0 |
| | 6.098369885 | 2.1672345 | 4.13670754 | 0 |
| | 4.960185836 | 0.7841548 | 3.531135292 | 0 |
| | 3.899078191 | 1.8047989 | 4.863730557 | 1 |
| | 5.840727605 | 0.727722 | 3.325852126 | 0 |
| | 6.645324046 | 0.498725 | 2.391531067 | 0 |
| | 5.361422671 | 0.89898 | 3.91346242 | 0 |
| | 3.223016816 | 1.6715052 | 4.974665749 | 1 |
| | | Estimate of probability= | | 0.2 |
| | | | Mean value = | 0.3 |
| | | | Std. Deviation = | 0.1155 |
| | | | 95% LCL = | 0.1163 |
| | | | 95%UCL = | 0.4837 |

(d)




$$\text{Exact probability} = \frac{A_1 + A_2}{\pi D}$$

$$= \frac{2\int_0^{\pi}\frac{d}{2}\sin\theta\, d\theta}{\pi}$$

$$= \frac{2d}{\pi D}$$

(e) From (c),

$$\tilde{p} = .3$$

Thus,

$$\frac{2d}{\pi D} = .3$$

or $$\pi \simeq \frac{2d}{.3D}$$

$$\simeq \frac{2 \times 10}{.3 \times 20}$$

$$\simeq 3.33$$

## Set 16.2a

**1**

(a) Discrete

(b) Continuous

(c) Discrete

**2**

In discrete simulation, there are two main events: arrivals and departures. An arrival event may experience delay before starting service. When service has been completed, customer leaves the facility.

The description of the discrete simulation situation by arrival and departure events is the reason discrete simulation is associated with queues.

**1**

Events:

$A_1$ = rush job arrives
$A_2$ = regular job arrives
$D_1$ = rush job departs
$D_2$ = regular job departs

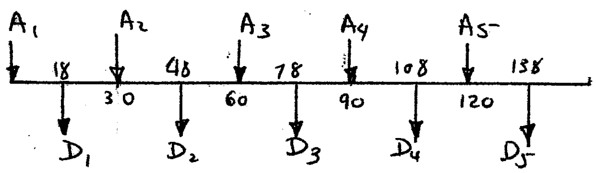
**2**

$A_0$ = job arrives at carousel
$A_1$ = job arrives at station 1
$A_2$ = job arrives at station 2
$A_3$ = job arrives at station 3
$D_1$ = job departs station 1
$D_2$ = job departs station 2
$D_3$ = job departs station 3

**3**

$A_1$ = car enters lane 1
$A_2$ = car enters lane 2
$A_3$ = car goes elsewhere
$D_1$ = car departs lane 1
$D_2$ = car departs lane 2.

**4**



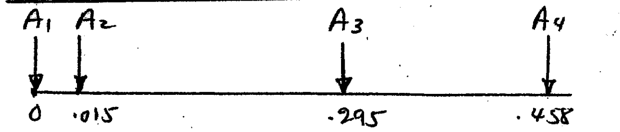
# Set 16.3b

**1**

$t = -\frac{1}{\lambda}\ln(1-R)$

$\lambda = 4$ customers/hr

| Customer | R | t(hrs) | Arrival time |
|---|---|---|---|
| 1 | — | — | 0 |
| 2 | .0589 | .015 | 0+.015 = .015 |
| 3 | .6733 | .280 | .015+.28 = .295 |
| 4 | .4799 | .163 | .295+.163 = .458 |




**2**

$f(t) = \frac{1}{b-a}, \quad a \le t \le b$

$F(t) = \int_0^t \frac{1}{b-a}dx = \frac{t-a}{b-a}, \quad a \le t \le b$

$R = \frac{t-a}{b-a}$

$t = a + (b-a)R$

**3**

$f_1(t_1) = .5e^{-.5t}, \quad \lambda = 1/2$ arrival/hr

$f_2(t) = \frac{1}{.9}, \quad 1.1 < t < 2$

$R = .0589, \ a_1 = -2\ln(1-.0589) = .12$ hr

$R = .6733, \ d_1 = 1.1 + .9 \times .6733 = 1.71$ hrs

$R = .4799, \ a_2 = -2\ln(1-.4799) = 1.31$ hrs

$R = .9486, \ a_3 = -2\ln(1-.9486) = 5.94$ hrs

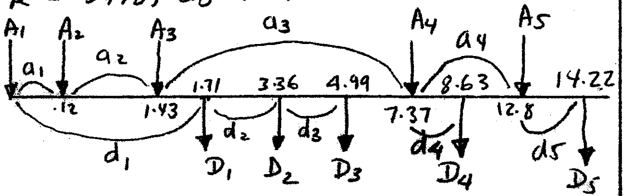
$R = .6139, \ d_2 = 1.1 + .9 \times .6139 = 1.65$ hrs

$R = .5933, \ d_3 = 1.1 + .9 \times .5933 = 1.63$ hrs

$R = .9341, \ a_4 = -2\ln(1-.9341) = 5.44$ hrs

$R = .1782, \ d_4 = 1.1 + .9 \times .1782 = 1.26$ hrs

$R = .3473, \ d_5 = 1.1 + .9 \times .3473 = 1.41$ hrs




**4**

(a) $0 \le R < .2, \quad d = 0$

$.2 \le R < .5, \quad d = 1$

$.5 \le R < .9, \quad d = 2$

$.9 \le R \le 1., \quad d = 3$

(b)

| Day | R | Demand d | Stock level |
|---|---|---|---|
| 0 | — | — | 5 |
| 1 | .0589 | 0 | 5 |
| 2 | .6733 | 2 | 3 |
| 3 | .4799 | 1 | 2 |

Replenish stock on day 3

**5**

Repair/.2, Package/.8:

$0 \le R < .2$, go to Repair

$.2 \le R \le 1.$, go to Package

Package/.8, Repair/.2:

$0 \le R < .8$, go to Package

$.8 \le R \le 1$, go to Repair

Example: $R = .1$ leads to Repair in the first case and to Package in the second case

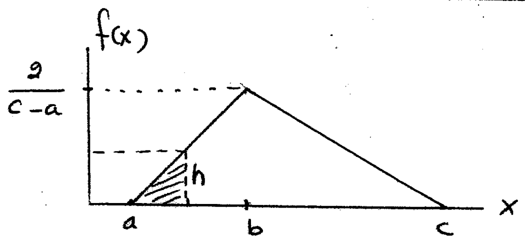
**6**

$0 \le R < .5$ : H

$.5 \le R \le 1.$ : T

| n | R | outcome | Payoff |
|---|---|---|---|
| 1 | .0589 | H | \$2 |
| 1 | .6733 | T | 0 |
| 2 | .4799 | H | $2^2 = \$4$ |

**7**




continued…

**7 continued**

(a)
$$F(x)=\begin{cases}\dfrac{(x-a)^2}{(b-a)(c-a)}, & a\le x\le b\\[2ex] 1-\dfrac{(c-x)^2}{(c-b)(c-a)}, & b\le x\le c\end{cases}$$

For $R=\dfrac{(x-a)^2}{(b-a)(c-a)}$,

$$x=a+\sqrt{R(b-a)(c-a)},\quad 0\le R\le \frac{b-a}{c-a}$$

For $R=1-\dfrac{(c-x)^2}{(c-b)(c-a)}$,

$$x=c-\sqrt{(c-b)(c-a)(1-R)},\quad \frac{b-a}{c-a}\le R\le 1$$

(b) $a=1,\ b=3,\ c=7$

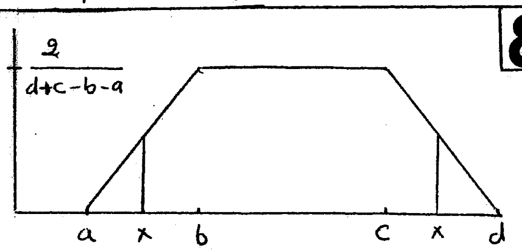
$$\frac{b-a}{c-a}=\frac{3-1}{7-1}=.333$$

Thus,

$$x=\begin{cases}1+\sqrt{(3-1)(7-1)R}\\ \quad =1+\sqrt{12R}, & 0\le R\le .333\\[2ex] 7-\sqrt{(7-3)(7-1)(1-R)}\\ \quad =7-\sqrt{24(1-R)}, & .333\le R\le 1\end{cases}$$

| $R$ | $x$ |
|---|---|
| .0589 | 1.84 |
| .6733 | 4.20 |
| .4799 | 3.47 |
| .9486 | 5.89 |
| .6139 | 3.96 |

**8**




(a)
$$F(x)=\begin{cases}\dfrac{(x-a)^2}{(b-a)(d+c-b-a)} & a\le x\le b\\[2ex] \dfrac{1}{(b-a)(d+c-b-a)}+\dfrac{2(x-a)}{(d+c-b-a)} & b\le x\le c\\[2ex] 1-\dfrac{(d-x)^2}{(d-c)(d+c-b-a)} & c\le x\le d\end{cases}$$

Continued...

$R=\dfrac{(x-a)^2}{(b-a)(d+c-b-a)}$ gives

$$x=a+\sqrt{(b-a)(d+c-b-a)R},\quad 0\le R\le \frac{b-a}{(d+c-b-a)}$$

$R=\dfrac{1}{(b-a)(d+c-b-a)}+\dfrac{2(x-b)}{(d+c-b-a)}$ gives

$$x=\frac{1}{2}\left(R-\frac{1}{(b-a)(d+c-b-a)}\right)(d+c-b-a),$$
$$\frac{b-a}{d+c-b-a}\le R\le 1-\frac{d-c}{(d+c-b-a)}$$

$$R=1-\frac{(d-x)^2}{(d-c)(d+c-b-a)}$$

$$x=d-\sqrt{(d-c)(d+c-b-a)(1-R)},$$
$$1-\frac{d-c}{(d+c-b-a)}\le R\le 1$$

(b) $a=1,\ b=2,\ c=4,\ d=6$

$$1+\sqrt{(2-1)(6+4-2-1)R}=1+\sqrt{7R},\quad 0\le R\le .143$$

$$2+\frac{6+4-2-1}{2}\left(R-\frac{1}{(2-1)(6+4-2-1)}\right)$$
$$=2+3.5(R-.143),$$
$$.143\le R\le .714$$

$$6-\sqrt{(6-4)(6+4-2-1)(1-R)}$$
$$=6-\sqrt{14(1-R)}$$
$$.714\le R\le 1$$

| $R$ | $x$ |
|---|---|
| .0589 | 1.64 |
| .6733 | 3.86 |
| .4799 | 3.18 |
| .9486 | 5.15 |
| .6139 | 3.65 |

**9**

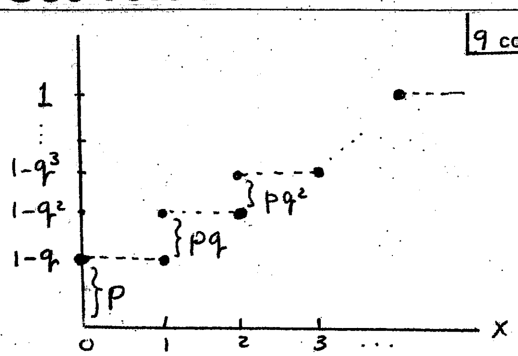
$f(x)=pq^x,\quad x=0,1,2,\ldots$

$(p+q)=1$

$$F(x)=p\sum_{t=0}^{x}q^t$$
$$=1-q^{x+1},\quad x=0,1,2,\ldots$$

Continued...

9 continued




Sampling procedure:

if $0 \le R \le p$, then $x = 0$.

For $p < R \le 1$, we have

$$1-q^n \le R \le 1-q^{n+1}$$

or

$$n \le \frac{\ln(1-R)}{\ln q} \le n+1$$

Thus, for $p \le R \le 1$, compute

$$x = \left[\frac{\ln(1-R)}{\ln q}\right]$$

where $[a]$ is the largest integer less than or equal to $a$.

For $p = .6$, $q = .4$, we have

| $R$ | $\frac{\ln(1-R)}{\ln q}$ | $x$ |
|---|---|---|
| .0589 | — | 0 |
| .6733 | 1.22 | 1 |
| .4799 | — | 0 |
| .9486 | 3.24 | 3 |
| .6139 | 1.03 | 1 |

**10**

$$f(x) = \alpha\beta^{-\alpha} x^{\alpha-1} e^{-(x/\beta)^\alpha}, \quad x > 0$$

$$= \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha}, \quad x > 0$$

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}, \quad x > 0$$

Thus,

$$R = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}$$

or

$$x = \beta\left[-\ln(1-R)\right]^{1/\alpha}$$

**1**

$y = -\frac{1}{5}\ln\{(.0589 \times .6733 \times .4799 \times .9486\}$

$= .803$ hour

**2**

$\lambda = 5$ events/hr, $t = 1$

$e^{-5\times 1} = e^{-5} = .00673$

| $i$ | $R_1R_2 \cdots R_i$ | |
|---|---|---|
| 1 | .0589 | |
| 2 | $.0589 \times .6733$ | $= .0397$ |
| 3 | $.0397 \times .4799$ | $= .0190$ |
| 4 | $.0190 \times .9486$ | $= .0181$ |
| 5 | $.0181 \times .6139$ | $= .0111$ |
| 6 | $.0111 \times .5933$ | $= .00656$ |
| 7 | $.00656 \times .9341$ | $= .00614$ |

Hence $n = 6$

**3**

$\mu = 8$, $\sigma = 1$, $N(8,1)$

Convolution method:

$x = R_1 + R_2 + \cdots + R_{12} = 6.1094$

$y = 8 + 1(6.1094 - 6) = 8.1094$

Box-Miller method:

$x = \sqrt{-2\ln R_1}\cos(2\pi R_2)$

$= \sqrt{-2\ln .0589}\cos(2\pi \times .6733)$

$\cong -1.103$

$y = 8 + 1(-1.103) = 6.897$

**4**

$\lambda = 6$/day $\quad m = 5$

$y = -\frac{1}{6}\ln(.0589 \times .6733 \times .4799 \times .9486 \times .6139) = .751$ hour

**5**

$N(27,3)$: $\mu = 27$, $\sigma = 3$

Given $R_1$ and $R_2$, we have

$x_1 = \sqrt{-2\ln R_1}\cos(2\pi R_2)$

$x_2 = \sqrt{-2\ln R_1}\sin(2\pi R_2)$

$y_1 = \mu + \sigma x_1$

$y_2 = \mu + \sigma x_2$

Continued...

| | J | K | L | M | N | O |
|---|---|---|---|---|---|---|
| | Mean = 27 | | Std. Dev = 3 | | | |
| | | | | | | |
| | R1 | R2 | x1 | x2 | y1 | y2 |
| 5 | 0.0589 | 0.6733 | -1.1030306 | -2.108827 | 23.69091 | 20.67352 |
| 6 | 0.4799 | 0.9486 | 1.149111 | -0.384576 | 30.44733 | 25.84627 |
| 7 | 0.6139 | 0.5933 | -0.8229152 | -0.546495 | 24.53125 | 25.36051 |
| | | | | mean y= | 25.09163 | |
| | | | | Sy | 3.197533 | |

Formulas:

L5= SQRT(-2*LN(J5))*COS(2*PI()*K5)
M4= SQRT(-2*LN(J5))*SIN(2*PI()*K5)
N4= $K$1+L4*$M$1
O4= $K$1+M4*$M$1

**6**

$x_i = 10 + (20-10)R_i$

$= 10 + 10R_i$, $i = 1, 2, 3, 4$

$t = x_1 + x_2 + x_3 + x_4$

$= 40 + 10(R_1 + R_2 + R_3 + R_4)$

**7**

| | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $t$ (sec) | $\Sigma t$ |
|---|---|---|---|---|---|---|
| 1 | .0589 | .6733 | .4799 | .9486 | 61.61 | 61.60 |
| 2 | .6139 | .5933 | .9341 | .1782 | 63.20 | 124.81 |
| 3 | .3473 | .7676 | .8931 | .3919 | 64.00 | 188.81 |
| 4 | .7876 | .5199 | .6358 | .7472 | 66.91 | 255.72 |
| 5 | .8954 | .5869 | .1281 | .2867 | 58.94 | 314.69 |

The number of mice that exit the maze in 300 seconds is 4

**8**

Let $x_1, x_2, \ldots, x_r$ be $r$ successive random deviates obtained from the geometric distribution as given in Problem 9, Set 18.3b. Then

$x_i = \left[\frac{\ln R_i}{\ln(1-p)}\right]$, $i = 1, 2, \ldots, r$

Because the negative binomial is the convolution of $r$ independent geometric random variables, it follows that a random negative binomial sample can be determined as

$$x = \sum_{i=1}^{r}\left[\frac{\ln R_i}{\ln(1-p)}\right]$$

Note that $[a]$ represents the largest integer $\le a$

**1**

Step 1: $R = .6139$
$X = .6139$

Step 2: $R = .5933$

Step 3: $\dfrac{f(.6139)}{g(.6139)} = .948 > .5933$
Reject X

Step 1: $R = .9341$, $X = .9341$

Step 2: $R = .1782$

Step 3: $\dfrac{f(.9341)}{g(.9341)} = \dfrac{.3693}{1.5} = .246 > .1782$
Reject X

Step 1: $R = .3473$, $X = .3473$

Step 2: $R = .5644$

Step 3: $\dfrac{f(.3473)}{g(.3473)} = .9067 > .5644$
Reject X

Step 1: $R = .3529$, $X = .3529$

Step 2: $R = .3646$

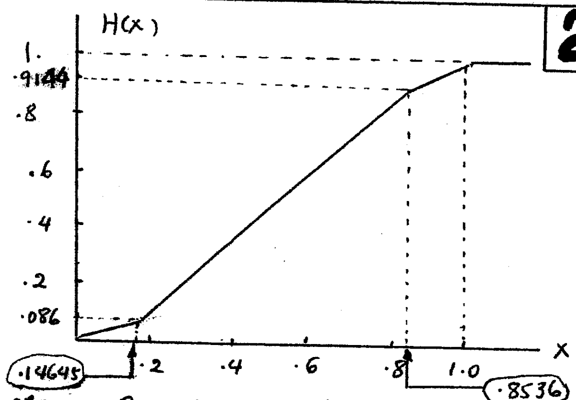
Step 3: $\dfrac{f(.3529)}{g(.3529)} = .913 > .3646$
Reject X

Step 1: $R = .7676$, $X = .7676$

Step 2: $R = .8931$

Step 3: $\dfrac{f(.7676)}{g(.7676)} = .7135 < .8931$
accept $X = .7676$

**2**




Step 1: $R = .4799$, $X = .4831$

Step 2: $R = .9486$

Step 3: $\dfrac{f(.4831)}{g(.4831)} = .9988 > .9486$
Reject X

Step 1: $R = .6139$, $X = .5974$

Step 2: $R = .5933$

continued...

2 continued

Step 3: $\dfrac{f(.5974)}{g(.5974)} = .962 > .5933$
reject X

Step 1: $R = .9341$ $X = .8804$

Step 2: $R = .1782$

Step 3: $\dfrac{f(.8804)}{g(.8804)} = .842 > .1782$
Reject X

Step 1: $R = .3529$, $X = .375$

Step 2: $R = .3646$

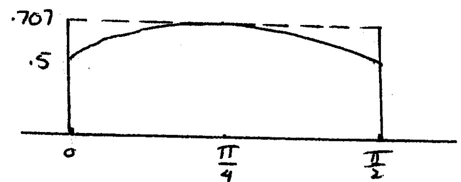
Step 3: $\dfrac{f(.375)}{g(.375)} = .937 > .3646$
Reject X

Step 1: $R = .7676$, $X = .7286$

Step 2: $R = .8931$

Step 3: $\dfrac{f(.7286)}{g(.7286)} = \dfrac{1.186}{1.5} = .791 < .8931$
accept X

**3**




$f(x) = \dfrac{\sin(x) + \cos(x)}{2}$ $\quad 0 \le x \le \frac{\pi}{2}$

$\max f(x) = .707$ at $x = \frac{\pi}{4}$

$g(x) = .707$ $\quad 0 \le x \le \pi/2$

$h(x) = \dfrac{g(x)}{\text{area under } g(x)}$

$= \dfrac{.707}{.707 \times \frac{\pi}{2}} = .637$ $\quad 0 \le x \le \frac{\pi}{2}$

**4**

$\int_{12}^{20} \frac{K_1}{t}\,dt = K_1 \ln \frac{20}{12} = 1$

Thus, $K_1 = 1.96$

$\int_{18}^{22} \frac{K_2}{t^2}\,dt = K_2\left(\frac{1}{18} - \frac{1}{22}\right) = 1$

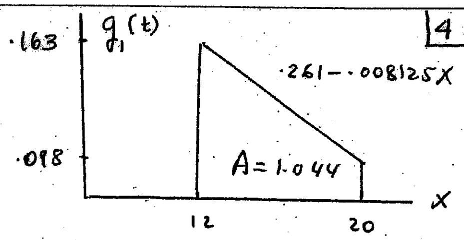
Thus, $K_2 = 99$

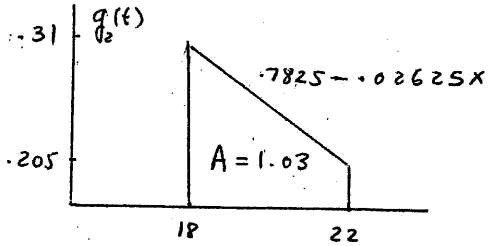
$f_1(t) = \dfrac{1.96}{t}$, $\quad 12 \le t \le 20$

$f_2(t) = \dfrac{99}{t^2}$, $\quad 18 \le t \le 22$

continued...

4 continued




$$h_1(t) = \frac{.261 - .008125t}{1.044}$$

$$= .25 - .007783t$$

$$H_1(t) = .025x - .00778\frac{x^2}{2}\Big|_{12}^{t}$$

$$= .25t - .003892t^2 - 2.44$$

$$h_2(t) = \frac{.7825 - .02625t}{1.03}$$

$$= .76 - .0255t$$

$$H_2(t) = .76t - .01275t^2 - 9.55$$

Sample computations from $H_2(t)$:

Step 1: $R_1 = .0589$

$$.76t - .01275t^2 - 9.55 = .0589$$

$$t^2 - 59.6t + 753.64 = 0$$

$$t = \frac{59.6 \pm \sqrt{(-59.6)^2 - 4 \times 753.64}}{2}$$

$$= 18.2$$

Step 2: $R = .6733$

Step 3: $$\frac{f_2(18.21)}{g_2(18.21)} = \frac{\left(\frac{99}{18.21^2}\right)}{.7825 - .02625 \times 18.21}$$

$$= .98 > .6733$$

Reject $t$.

continued...

# Set 16.4a

1

**Multiplicative Congruential Method**

| Input data | |
|---|---|
| b = | 17 |
| c = | 111 |
| u0 = | 7 |
| m = | 103 |
| How many numbers? | 50 |
| Output results | |
| Press to Generate Sequence | |

Generated random numbers:

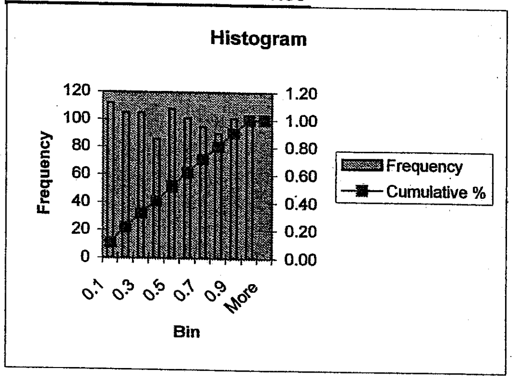
| | |
|---|---|
| 1 | 0.23301 |
| 2 | 0.03883 |
| 3 | 0.73786 |
| 4 | 0.62136 |
| 5 | 0.64078 |
| 6 | 0.97087 |
| 7 | 0.58252 |
| 8 | 0.98058 |
| 9 | 0.74757 |
| 10 | 0.78641 |
| 11 | 0.44660 |
| 12 | 0.66990 |
| 13 | 0.46602 |
| 14 | 0.00000 |
| 15 | 0.07767 |
| 16 | 0.39806 |
| 17 | 0.84466 |
| 18 | 0.43689 |
| 19 | 0.50485 |
| 20 | 0.66019 |
| 21 | 0.30097 |
| 22 | 0.19417 |
| 23 | 0.37864 |
| 24 | 0.51456 |
| 25 | 0.82524 |
| 26 | 0.10680 |
| 27 | 0.89320 |
| 28 | 0.26214 |
| 29 | 0.53398 |
| 30 | 0.15534 |
| 31 | 0.71845 |
| 32 | 0.29126 |
| 33 | 0.02913 |
| 34 | 0.57282 |
| 35 | 0.81553 |
| 36 | 0.94175 |
| 37 | 0.08738 |
| 38 | 0.56311 |
| 39 | 0.65049 |
| 40 | 0.13592 |
| 41 | 0.38835 |
| 42 | 0.67961 |
| 43 | 0.63107 |
| 44 | 0.80583 |
| 45 | 0.77670 |
| 46 | 0.28155 |
| 47 | 0.86408 |
| 48 | 0.76699 |
| 49 | 0.11650 |
| 50 | 0.05825 |

2

| R=RAND() | Bin |
|---|---|
| 0.813455 | 0.1 |
| 0.21757 | 0.2 |
| 0.937991 | 0.3 |
| 0.840823 | 0.4 |
| 0.19536 | 0.5 |
| 0.681599 | 0.6 |
| 0.829291 | 0.7 |
| 0.377723 | 0.8 |
| 0.149187 | 0.9 |
| 0.965781 | 1 |
| 0.808752 | |
| 0.957601 | |
| 0.502469 | |
| 0.620944 | |
| 0.992405 | |
| 0.97218 | |
| 0.051905 | |
| 0.144368 | |
| 0.129308 | |
| 0.676603 | |
| 0.140868 | |
| 0.486705 | |
| 0.12415 | |
| 0.821802 | |
| 0.954853 | |
| 0.301267 | |
| 0.827929 | |
| 0.917179 | |
| 0.07369 | |
| 0.462159 | |
| 0.333902 | |
| 0.390604 | |
| 0.723163 | |
| 0.041401 | |
| 0.805603 | |
| 0.556012 | |

| *Bin* | *Frequency* | *umulative %* |
|---|---|---|
| 0.1 | 112 | 0.11 |
| 0.2 | 105 | 0.22 |
| 0.3 | 105 | 0.32 |
| 0.4 | 86 | 0.41 |
| 0.5 | 108 | 0.52 |
| 0.6 | 101 | 0.62 |
| 0.7 | 95 | 0.71 |
| 0.8 | 90 | 0.80 |
| 0.9 | 101 | 0.90 |
| 1 | 97 | 1.00 |
| More | 0 | 1.00 |

Sample size = 1000



1

$C = 2$ barbers

$f_1(t) = .1e^{-.1t}, \quad t > 0$

$f_2(t) = \frac{1}{15}, \quad 15 \le t \le 30$

$t_1 = -12 \ln R$

$t_2 = 15 + 15R$

$A_1$ at $T = 0$:

$T(A_2) = 0 + (-10 \ln .0589) = 28.3$

$T(D_2) = 0 + (15 + 15 \times .6733) = 25.1$

Barber 1 busy

$D_2$ at $T = 25.1$:

Barber 1 idle

$A_2$ at $T = 28.3$:

$T(A_3) = 28.3 - 10 \ln .4799 = 35.6$

$T(D_2) = 28.3 + (15 + 15 \times .9486) = 57.5$

Barber 1 busy $\boxed{A_3 \ D_2}$

$A_3$ at $T = 35.6$:

$T(A_4) = 35.6 - 10 \ln .6139 = 40.5$

$T(D_3) = 35.6 + (15 + 15 \times .5933) = 59.5$

Barber 2 busy $\boxed{A_4 \ D_2 \ D_3}$

$A_4$ at $T = 40.5$:

$T(A_5) = 40.5 - 10 \ln .9341 = 41.2$

$A_4$ waits in queue

$\boxed{A_5 \ D_2 \ D_3}$ $\quad \boxed{A_4}$ ← queue

$A_5$ at $T = 41.2$:

$T(A_6) = 41.2 - 10 \ln .1782 = 58.4$

$A_5$ waits in queue

$\boxed{D_2 \ A_6 \ D_3}$ $\quad \boxed{A_4 \ A_5}$ ← queue

continued...

$D_2$ at $T = 57.5$:

Barber 1 idle

Take $A_4$ out of queue

$T(D_4) = 57.5 + 15 + 15 \times .3473 = 77.7$

Barber 1 busy

$\boxed{A_6 \ D_3 \ D_4}$ $\quad \boxed{A_5}$ ← queue

$A_6$ at $T = 58.4$:

$T(A_7) = 58.4 - 10 \ln .5644 = 64.1$

Put $A_6$ in queue $\boxed{D_3 \ A_7 \ D_4}$

$D_3$ at $T = 59.5$: $\quad \boxed{A_5 \ A_6}$ ← queue

Barber 2 idle

Take $A_5$ out of queue

$T(D_5) = 59.5 + 15 + 15 \times .3529 = 79.8$

Barber 2 busy

$\boxed{A_7 \ D_4 \ D_5}$ $\quad \boxed{A_6}$ ← queue

$A_7$ at $T = 64.1$:

$T(A_8) = 64.1 - 10 \ln .3646 = 74.2$

Put $A_7$ in queue

$\boxed{A_8 \ D_4 \ D_5}$ $\quad \boxed{A_6 \ A_7}$ ← queue

$A_8$ at $T = 74.2$:

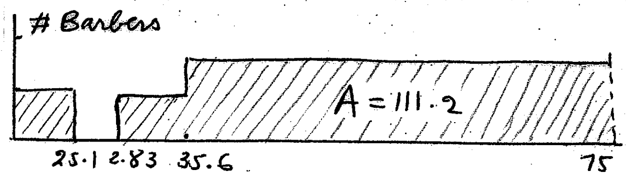
$T(A_9) = 74.2 + (-10 \ln .7676)$

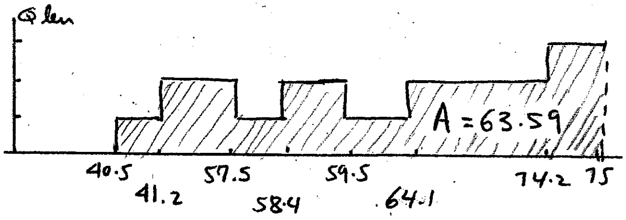
$= 76.8$

Place $A_8$ in queue.

$\boxed{A_9 \ D_4 \ D_5}$ $\quad \boxed{A_6 \ A_7 \ A_8}$ ← queue

continued...




Av. facility utilization $= \frac{111.2}{75}$

$= 1.48$ barbers

Av. queue length $= \frac{63.59}{75} = .8$ customer

Av. waiting time in queue $= \frac{63.59}{8}$

$= 7.95$ min

Av. waiting time for those who must wait $= \frac{63.59}{5} = 12.72$ min

**2**

(a) Observation.

(b) Time.

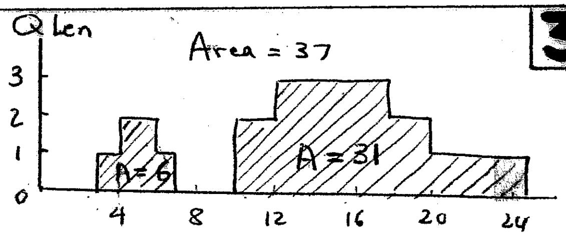
(c) Observation.

(d) Observation.

(e) Observation.

(f) Time.

**3**

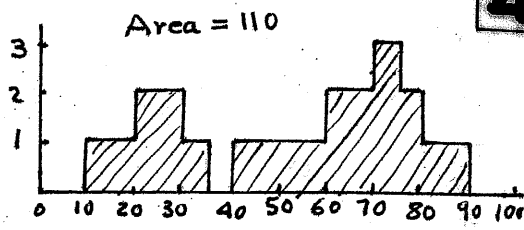



(a) $\bar{Q} = \frac{37}{25} = 1.48$ customers

(b) Number of waiting customers = 5

$\bar{W} = \frac{37}{5} = 7.4$ hours

**4**




(a) Average utilization

$= \frac{110}{100} = 1.1$ barber

(b) Average idle time

$= \frac{10 + (40-35) + (100-90)}{3}$

$= \frac{25}{3}$

$= 8.33$ minutes

**1**

**Simulation of a Single-Server Queueing Model**

| | | | | | | |
|---|---|---|---|---|---|---|
| Nbr of arrivals = | 10 | | | | | |
| Enter x in column A to select interarrival pdf: | | | | | | |
| | Constant = | | | | | |
| x | Exponential: | $\lambda$ = | 0.0667 | | | |
| | Uniform: | a = | 8 | b = | 9 | |
| | Triangular: | a = | | b = | | c = |
| Enter x in column A to select service time pdf: | | | | | | |
| | Constant = | | | | | |
| | Exponential: | $\mu$ = | | | | |
| x | Uniform: | a = | 10 | b = | 15 | |
| | Triangular: | a = | | b = | | c = |

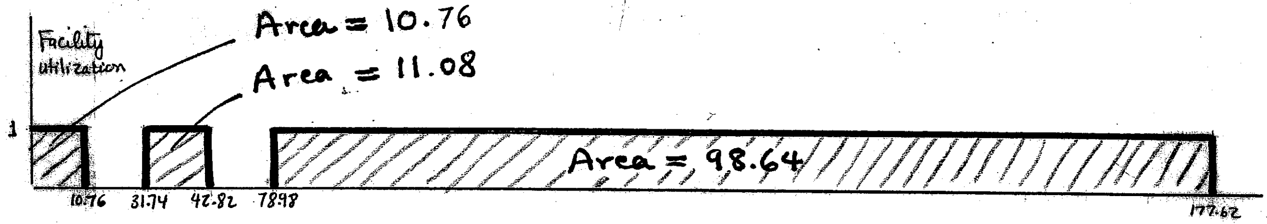
**Output Summary**

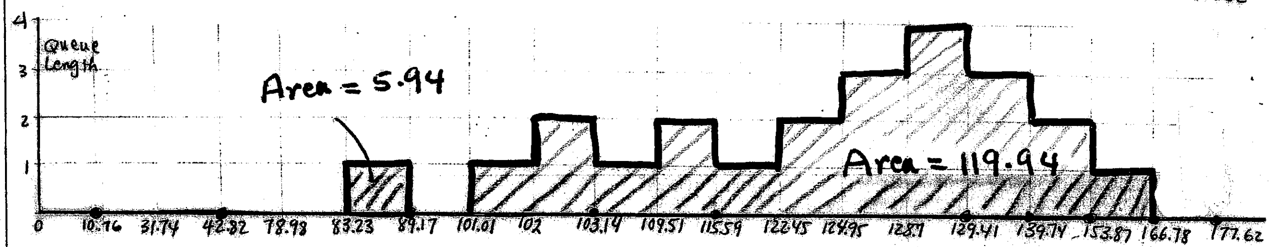
| | |
|---|---|
| Av. facility utilization = | 0.68 |
| Percent idleness (%) = | 32.17 |
| Maximum queue length= | 4 |
| Av. queue length, Lq = | 0.71 |
| Av. nbr in system, Ls = | 1.39 |
| Av. queue time, Wq = | 12.58 |
| Av. system time, Ws = | 24.63 |
| Sum(ServiceTime) = | 120.47 |
| Sum(Wq) = | 125.85 |
| Sum(Ws) = | 246.32 |

Press F9 to trigger a new simulation run.

**Simulation Calculations**

| Nbr | InterArvlTime | ServiceTime | ArrvlTime | DepartTime | Wq | Ws |
|---|---|---|---|---|---|---|
| 1 | 31.74 | 10.76 | 0.00 | 10.76 | 0.00 | 10.76 |
| 2 | 47.24 | 11.07 | 31.74 | 42.82 | 0.00 | 11.07 |
| 3 | 4.25 | 10.19 | 78.98 | 89.17 | 0.00 | 10.19 |
| 4 | 17.78 | 13.96 | 83.23 | 103.14 | 5.94 | 19.91 |
| 5 | 0.99 | 12.45 | 101.01 | 115.59 | 2.13 | 14.58 |
| 6 | 7.51 | 13.82 | 102.00 | 129.41 | 13.59 | 27.41 |
| 7 | 12.94 | 10.33 | 109.51 | 139.74 | 19.90 | 30.23 |
| 8 | 2.51 | 14.13 | 122.45 | 153.87 | 17.29 | 31.42 |
| 9 | 3.74 | 12.90 | 124.95 | 166.78 | 28.92 | 41.82 |
| 10 | 9.02 | 10.84 | 128.70 | 177.62 | 38.08 | 48.92 |




From the graph:

$$\sum \text{service times} = 10.76 + 11.08 + 98.64 = 120.48$$

$$\sum \text{queue waiting times} = 5.94 + 119.94 = 125.88$$

(The small difference between these answers and the simulation output is because of roundoff error.)

$$\text{Av. facility utilization} = \frac{120.48}{177.62} = .6783$$

$$\text{Av. queue length} = \frac{125.88}{177.62} = .7087$$

$$\text{Av. waiting time in queue} = \frac{125.88}{10} = 12.588$$

$$\text{Av. waiting time in system} = \frac{120.48 + 125.88}{10} = 24.636$$

# Set 16.5b

**2**

| | | | | |
|---|---|---|---|---|
| Nbr of arrivals = | 500 | <<Maximum 5 | | |
| Enter x in column A to select interarri | | | | |
| | Constant = | | | |
| x | Exponential: | $\lambda$ = | 4 | |
| | Uniform: | a = | | b = |
| | Triangular: | a = | | b = |
| Enter x in column A to select service t | | | | |
| | Constant = | | | |
| x | Exponential: | $\mu$ = | 6 | |
| | Uniform: | a = | | b = |
| | Triangular: | a = | | b = |
| Output Summar | | | | |

① 

| | |
|---|---|
| Av. facility utilization = | 0.66 |
| Percent idleness (%) = | 33.84 |
| Maximum queue length= | 0 |
| Av. queue length, Lq = | 1.42 |
| Av. nbr in system, Ls = | 2.08 |
| Av. queue time, Wq = | 0.37 |
| Av. system time, Ws = | 0.54 |

②

| | |
|---|---|
| Av. facility utilization = | 0.61 |
| Percent idleness (%) = | 38.65 |
| Maximum queue length= | 0 |
| Av. queue length, Lq = | 0.91 |
| Av. nbr in system, Ls = | 1.52 |
| Av. queue time, Wq = | 0.24 |
| Av. system time, Ws = | 0.40 |

③

| | |
|---|---|
| Av. facility utilization = | 0.65 |
| Percent idleness (%) = | 35.11 |
| Maximum queue length= | 0 |
| Av. queue length, Lq = | 0.91 |
| Av. nbr in system, Ls = | 1.56 |
| Av. queue time, Wq = | 0.22 |
| Av. system time, Ws = | 0.38 |

④

| | |
|---|---|
| Av. facility utilization = | 0.68 |
| Percent idleness (%) = | 31.70 |
| Maximum queue length= | 0 |
| Av. queue length, Lq = | 1.35 |
| Av. nbr in system, Ls = | 2.03 |
| Av. queue time, Wq = | 0.32 |
| Av. system time, Ws = | 0.48 |

⑤

| | |
|---|---|
| Av. facility utilization = | 0.60 |
| Percent idleness (%) = | 39.83 |
| Maximum queue length= | 0 |
| Av. queue length, Lq = | 1.14 |
| Av. nbr in system, Ls = | 1.74 |
| Av. queue time, Wq = | 0.30 |
| Av. system time, Ws = | 0.46 |

continued...

Summary:

| | utiliz | Lq | Ls | Wq | Ws |
|---|---|---|---|---|---|
| mean | .64 | 1.146 | 1.786 | .29 | .452 |
| Std. Dev. | .0339 | .2388 | .2598 | .0608 | .0642 |

95% confidence limits:

$t_{4,.025} = 2.776$

$UCL = \bar{x} + \frac{2.776S}{\sqrt{n}} = \bar{x} + 1.245$

$LCL = \bar{x} - 1.245$

| | utiliz | Lq | Ls | Wq | Ws |
|---|---|---|---|---|---|
| LCL | .598 | .850 | 1.464 | .215 | .372 |
| UCL | .682 | 1.442 | 2.108 | .365 | .531 |

Poisson queue output:

Scenario 1-- (M/M/1):(GD/infinity/infinity)

| | | | |
|---|---|---|---|
| Lambda = | 4.00000 | Mu = | 6.00000 |
| Lambda eff = | 4.00000 | Rho/c = | 0.66667 |
| Ls = | 2.00000 | Lq = | 1.33333 |
| Ws = | 0.50000 | Wq = | 0.33333 |

**3**

| | | | | | | |
|---|---|---|---|---|---|---|
| s = | 200 | <<Maximum 500 | | | | |
| umn A to select interarrival pdf: | | | | | | |
| = | 11.5 | | | | | |
| ial: | $\lambda$ = | | | | | |
| | a = | | b = | | | |
| r: | a = | | b = | | c = | |
| umn A to select service time pdf: | | | | | | |
| = | | | | | | |
| ial: | $\mu$ = | | | | | |
| | a = | | b = | | | |
| r: | a = | 9 | b = | 9.5 | c = | 15 |

①

| | |
|---|---|
| Av. facility utilization = | 0.96 |
| Percent idleness (%) = | 4.20 |
| Maximum queue length= | 2 |
| Av. queue length, Lq = | 0.12 |
| Av. nbr in system, Ls = | 1.08 |
| Av. queue time, Wq = | 1.36 |
| Av. system time, Ws = | 12.38 |

continued...

②

| | |
|---|---|
| Av. facility utilization = | 0.96 |
| Percent idleness (%) = | 3.85 |
| Maximum queue length= | 2 |
| Av. queue length, Lq = | 0.12 |
| Av. nbr in system, Ls = | 1.08 |
| Av. queue time, Wq = | 1.33 |
| Av. system time, Ws = | 12.39 |

③

| | |
|---|---|
| Av. facility utilization = | 0.97 |
| Percent idleness (%) = | 2.98 |
| Maximum queue length= | 2 |
| Av. queue length, Lq = | 0.19 |
| Av. nbr in system, Ls = | 1.16 |
| Av. queue time, Wq = | 2.14 |
| Av. system time, Ws = | 13.33 |

④

| | |
|---|---|
| Av. facility utilization = | 0.96 |
| Percent idleness (%) = | 3.58 |
| Maximum queue length= | 2 |
| Av. queue length, Lq = | 0.16 |
| Av. nbr in system, Ls = | 1.13 |
| Av. queue time, Wq = | 1.88 |
| Av. system time, Ws = | 12.97 |

⑤

| | |
|---|---|
| Av. facility utilization = | 0.97 |
| Percent idleness (%) = | 3.39 |
| Maximum queue length= | 2 |
| Av. queue length, Lq = | 0.17 |
| Av. nbr in system, Ls = | 1.14 |
| Av. queue time, Wq = | 2.00 |
| Av. system time, Ws = | 13.12 |

utilization:

$$\text{mean} = \frac{.96+.96+.97+.96+.97}{5}$$

$$= .964$$

$$\text{St. dev.} = .0311$$

**1**

$W_1 = \frac{14}{3} = 4.67$ (time units)

$W_2 = \frac{10}{4} = 2.5$

$W_3 = \frac{11}{3} = 3.67$

$W_4 = \frac{6}{3} = 2$

$W_5 = \frac{15}{4} = 3.75$

$\bar{W} = \frac{4.67+2.5+3.67+2+3.75}{5}$

$= 3.32$ time units

**2**

Dis-card observations during the transient period $(0, 100)$

$W_1 = \frac{12+30+10+14+16}{5} = 16.4$ time units

$W_2 = \frac{15+17+20+22}{4} = 18.5$

$W_3 = \frac{10+20+30+15+25+31}{6} = 21.83$

$W_4 = \frac{15+17+20+14+13}{5} = 15.8$

$W_5 = \frac{25+30+15}{3} = 23.33$

$\bar{W} = 19.17 \qquad S = 3.3$

Confidence interval

$\bar{W} \pm t_{.025,4} \frac{S}{\sqrt{n}}$

$= 19.17 \pm 2.776 \frac{3.3}{\sqrt{5}}$

or

$15.07 \le \mu \le 23.27$

**3**

| Batch | $a_i$ | $b_i$ | $y_i$ |
|---|---|---|---|
| 1 | 6 | 7 | .869 |
| 2 | 10 | 7 | 1.369 |
| 3 | 6 | 9 | .584 |
| | $\bar{a} = 7.33$ | $\bar{b} = 7.67$ | $\bar{y} = .941$ |
| | | | $S_y = .397$ |

continued...

3 continued

$y_i = \frac{3 \times 7.33}{7.67} - \frac{(3-1)(3 \times 7.33 - a_i)}{3 \times 7.67 - b_i}$

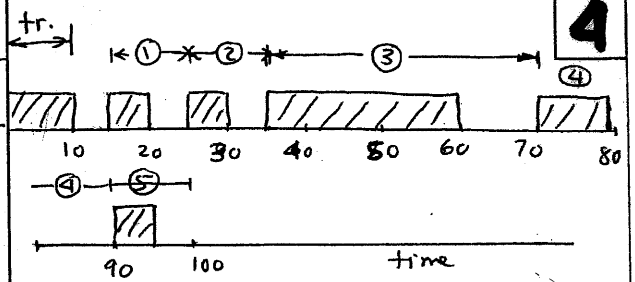
$= 2.867 - \frac{43.98 - 2a_i}{23.01 - b_i}$

95% confidence interval:

$.941 - 2.776 \frac{.397}{\sqrt{3}} \le \mu \le .941 + 2.776 \frac{.397}{\sqrt{3}}$

$.305 \le \mu \le 1.577$

**4**




(a) Start points are 15, 25, 35, 70, 90

(b)

| Batch | $a_i$ | $b_i$ | $y_i$ |
|---|---|---|---|
| 1 | 5 | 10 | .54 |
| 2 | 5 | 10 | .54 |
| 3 | 25 | 35 | .94 |
| 4 | 10 | 20 | .45 |
| 5 | 5 | 10 | .54 |
| | $\bar{a} = 10$ | 17 | $\bar{y} = .602$ |
| | | | $S_y = .193$ |

$y_i = \frac{5 \times 10}{17} - \frac{4(5 \times 10 - a_i)}{5 \times 17 - b_i}$

$= 2.94 - \frac{200 - 4a_i}{85 - b_i}$

$.602 - 2.776 \frac{.193}{\sqrt{5}} \le \mu \le .602 + 2.776 \frac{.193}{\sqrt{5}}$

or $.36 \le \mu \le .84$

(c) $t = \frac{90}{5} = 18$

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $A$ | 8 | 13 | 14 | 10 | 5 |
| $u_i$ | .44 | .72 | .78 | .56 | .28 |

Mean = .556, Std. Dev. = .2042