Gravitational Potential FEM Solution

Artur Dworkin

January 2024

$$\frac{d^2\phi}{dx^2} = 4\pi G\rho(x)$$

$$\Omega = [0;3]$$

$$\phi(0) = 5$$

$$\phi(3) = 7$$

$$\rho(x) = \begin{cases} 0 \text{ for } x \in [0,1] \\ 1 \text{ for } x \in (1,2] \\ 0 \text{ for } x \in (2,3] \end{cases}$$

Assume $G = 1$

$$V = \{v(x) : v(0) = 0 \wedge v(3) = 0\}$$

$$w \in V$$

$$\phi = w + \tilde{\phi}$$

$$\phi(0) = \tilde{\phi}(0) = 5$$

$$\phi(3) = \tilde{\phi}(3) = 7$$

$$\tilde{\phi} = \frac{2}{3}x + 5$$

$$\int\limits_0^3 v\phi'' dx = 4\pi G \int\limits_0^3 v\rho dx$$

$$[v\phi']_0^3 - \int\limits_0^3 v'\phi' dx = 4\pi G \int\limits_0^3 v\rho dx$$

$$v(3)\phi'(3) = 0$$

$$v(0)\phi'(0) = 0$$

$$-\int_0^3 v'\phi' dx = 4\pi G \int_0^3 v\rho dx$$

$$-\int_0^3 v'\phi' dx = 4\pi G \int_1^2 v dx$$

$$B(\phi, v) = -\int_0^3 v'\phi' dx$$

$$L(v) = 4\pi G \int_1^2 v dx$$

$$B(\phi, v) = B(w + \tilde{\phi}, v) = B(w, v) + B(\tilde{\phi}, v)$$
$$B(w, v) = L(v) - B(\tilde{\phi}, v)$$

Lets choose N points on whole Ω with boundaries. Since Dirichlet conditions have been specified on both sides of the interval, the last and first test functions can be omitted. We define the test functions $e_n(x)$ as follows:

$$\text{Let h} = \frac{3}{N-1} \text{ and } n \in \{1, 2, ..., (N-2)\}$$
$$e_n(x) = \begin{cases} \frac{x}{h} - n + 1 \text{ for } x \in [h(n-1), hn] \\ \frac{-x}{h} + n + 1 \text{ for } x \in [hn, h(n+1)] \end{cases}$$

$$\begin{bmatrix} B(e_1, e_1) & B(e_2, e_1) & \dots & B(e_{N-2}, e_1) \\ B(e_1, e_2) & B(e_2, e_2) & \dots & B(e_{N-2}, e_2) \\ \vdots & \vdots & \ddots & \vdots \\ B(e_1, e_{N-2}) & B(e_2, e_{N-2}) & \dots & B(e_{N-2}, e_{N-2}) \end{bmatrix} \cdot \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-2} \end{bmatrix} = \begin{bmatrix} L(e_1) - B(\tilde{\phi}, e_1) \\ L(e_2) - B(\tilde{\phi}, e_2) \\ \vdots \\ L(e_{N-2}) - B(\tilde{\phi}, e_{N-2}) \end{bmatrix}$$

The ϕ function we are looking for is:

$$\phi(x) = w + \tilde{\phi} = \phi_1 \cdot e_1 + \phi_2 \cdot e_2 + \ldots + \phi_{N-2} \cdot e_{N-2} + \frac{2}{3}x + 5$$