Multivariate
Statistics
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Exercise E1.

Let $\mathbb{X}, \mathbb{Y}$ be two real-valued stochastic variables with finite variance. Prove the following relation:

$$Cor(\mathbb{X}, \mathbb{Y}) = 1 \iff V(\mathbb{X}) > 0 \; and \; \exists \; constants \; a, b \; with \; b \neq 0: \mathbb{X} = a + b\mathbb{Y} \text{ with probability 1.}$$

Interpretation¿If two variables have correlation 1, they essentially contribute with the same information, since one can get one directly from the other.

Hint: Start by showing that a variable with variance 0 is constant with probability 1.

Solution:

We start by showing that a variable with variance 0 is constant with probability 1.

Assume that $V(\mathbb{Z}) = 0$ and put $c = E(\mathbb{Z}), p_0 = P(|\mathbb{Z} - c| > 0)$. Note that

$$p_0 = P\left(\bigcap_{n=1}^{\infty} |\mathbb{Z} - c| < \frac{1}{n}\right) = \lim_{n\to\infty} P\left(|\mathbb{Z} - c| < \frac{1}{n}\right)$$

WE do a counterposed proof: Suppose that $p_0 < 0$, and choose an N such that $q_N = P\left(|\mathbb{Z} - c| < \frac{1}{N}\right) < 1$. Then

$$0 = V(\mathbb{Z}) = E((\mathbb{Z} - c)^2) \geq E\left(\frac{1}{N^2} 1_{\left\{|\mathbb{Z}-c| \geq \frac{1}{N}\right\}}\right) = \frac{1}{N^2}(1 - q_N) > 0$$

Contradiction. Thus, $P(\mathbb{Z} = c) = 1. \square$

Now the actual proof:

"⇐" : We start by noting that

$$V(\mathbb{Y}) = V\left(\frac{1}{b}\mathbb{X} - \frac{a}{b}\right) = V\left(\frac{1}{b}\mathbb{X}\right) = b^{-2}V(\mathbb{X}) > 0.$$

Denote the variances of $\mathbb{X}, \mathbb{Y}$ by $\sigma_x^2,\ \sigma_y^2$, respectively, so that $\sigma_x^2 = b^2\sigma_y^2$. Now,

$$Cov(\mathbb{X}, \mathbb{Y}) = Cov(a + b\mathbb{Y}, \mathbb{Y}) = bCov(\mathbb{Y}, \mathbb{Y}) = b\sigma_y^2$$

Thus,

$$Cor(\mathbb{X},\mathbb{Y}) = \frac{Cov(\mathbb{X},\mathbb{Y})}{\sqrt{\sigma_x^2\sigma_y^2}} = \frac{b\sigma_y^2}{\sqrt{b^2\sigma_y^2\sigma_y^2}} = 1.$$

"⇒": First observe that $V(\mathbb{X}) > 0$ and $V(\mathbb{Y}) > 0$ simply from the fact that $Cor(\mathbb{X},\mathbb{Y})$ is well defined.

Denote the covariance between $\mathbb{X}$ and $\mathbb{Y}$ by σ_{xy}. Now,

$$1 = cor(\mathbb{X},\mathbb{Y}) = \frac{\sigma_{xy}}{\sigma_x\sigma_y},$$

So that

$$\sigma_{xy} = \sigma_x\sigma_y.$$

Now, define $\mathbb{Z} = \mathbb{X} - \frac{\sigma_x}{\sigma_y}\mathbb{Y}$, and note that

$$V(\mathbb{Z}) = V\left(\mathbb{X} - \frac{\sigma_x}{\sigma_y}\mathbb{Y}\right) = Cov\left(\mathbb{X} - \frac{\sigma_x}{\sigma_y}\mathbb{Y}, \mathbb{X} - \frac{\sigma_x}{\sigma_y}\mathbb{Y}\right)$$

$$= Cov(\mathbb{X},\mathbb{X}) + \frac{\sigma_x^2}{\sigma_y^2}Cov(\mathbb{Y},\mathbb{Y}) - 2\frac{\sigma_x}{\sigma_y}Cov(\mathbb{X},\mathbb{Y})$$

$$= \sigma_x^2 + \frac{\sigma_x^2}{\sigma_y^2}\sigma_y^2 - 2\frac{\sigma_x}{\sigma_y}\sigma_{xy}$$

$$= \sigma_x^2 + \sigma_x^2 - 2\frac{\sigma_x}{\sigma_y}\sigma_x\sigma_y = 0$$

$\mathbb{Z}$ is therefore constant with probability 1. Take $a = E(\mathbb{Z}), b = \frac{\sigma_x}{\sigma_y} \neq 0$. Then with probability 1,

$$\mathbb{X} = \mathbb{X} - \frac{\sigma_x}{\sigma_y}\mathbb{Y} + \frac{\sigma_x}{\sigma_y}\mathbb{Y} = \mathbb{Z} + \frac{\sigma_x}{\sigma_y}\mathbb{Y} = a + b\mathbb{Y}. \square$$