Exercise 7.5

Consider the independent random variables

$$\boldsymbol{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \sim N(\boldsymbol{0}, \boldsymbol{I}) = N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

and

$$\boldsymbol{G} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \sim N\left(\boldsymbol{0}, diag\left(\frac{4}{9}, \frac{7}{9}, \frac{4}{9}\right)\right) = N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{9} & 0 & 0 \\ 0 & \frac{7}{9} & 0 \\ 0 & 0 & \frac{4}{9} \end{bmatrix}\right)$$

Define $\boldsymbol{X}$ as

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

or shortly

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{F} + \boldsymbol{G}$$

This is the factor model with *observation* $\boldsymbol{X}$, *factor loadings* $\boldsymbol{A}$, *common factors* $\boldsymbol{F}$, and *unique factors* $\boldsymbol{G}$, cf. textbook p. 402-403.

1. What are the 3 communalities?
2. What is the distribution of $\boldsymbol{X}$ (mean and dispersion)?
3. What are the squared multiple correlations $\rho^2_{1|23}$, $\rho^2_{2|13}$ and $\rho^2_{3|12}$ between an $\boldsymbol{X}$-coordinate and the two other $\boldsymbol{X}$-coordinates?
4. Show that

$$\left(D(\boldsymbol{X})\right)^{-1} = \frac{1}{71}\begin{bmatrix} 80 & -27 & 3 \\ -27 & 81 & -9 \\ 3 & -9 & 72 \end{bmatrix}$$

5. Determine the distribution of

$$\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{X} \end{bmatrix} = (F_1, F_2, X_1, X_2, X_3)^T$$

6. Determine the conditional distribution of $(\boldsymbol{F}|\boldsymbol{X} = \boldsymbol{x})$
7. How may we predict $\boldsymbol{F}$ if we have observed $\boldsymbol{X}$?
8. What are the uncertainties on the predicted values?

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