Solution Exercise 6.2

We assume that the observations from each of the two bed groups are independent and normally distributed

$$\boldsymbol{X}_1 \sim N_3(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}) \quad \text{and} \quad \boldsymbol{X}_2 \sim N_3(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$$

1. We use theorem 5.12 and Mahalanobis' distance (section 5.1.3) becomes

$$D^2 = (\widehat{\boldsymbol{\mu}}_1 - \widehat{\boldsymbol{\mu}}_2)^T \widehat{\boldsymbol{\Sigma}}^{-1} (\widehat{\boldsymbol{\mu}}_1 - \widehat{\boldsymbol{\mu}}_2) = \begin{bmatrix} 75.9 - 78.2 \\ 45.3 - 39.5 \\ 41.8 - 44.1 \end{bmatrix}^T \begin{bmatrix} -0.88 \\ 1.98 \\ -1.39 \end{bmatrix} = [-2.3 \quad 5.8 \quad -2.3] \begin{bmatrix} -0.88 \\ 1.98 \\ -1.39 \end{bmatrix} = 16.705$$

The test statistic for the hypothesis $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ against the alternative $\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$ becomes:

$$Z = \frac{493+73-3-1}{3(493+73-2)} \times \frac{493 \times 73}{493+73} \times 16.705$$

Since $F(3{,}562)_{0.9995} \cong 6.0$, the hypothesis is strongly rejected. Therefore we may assume that it is possible to use the 3 variables to discriminate between the two bed groups.

2. We now investigate the marginal effect of neglecting two of the variables (bands 1 and 3). We already have Mahalanobis' distance based on all three bands ($D_3^2 = 16.705$). Mahalanobis' distance based only on band 4 becomes

$$D_1^2 = (-2.3) 16.6^{-1} (-2.3) = 0.32$$

The test statistic thus becomes (theorem 5.13)

$$Z = \frac{493+73-3-1}{2} \times \frac{493 \times 73(16.705 - 0.32)}{(493+73)(493+73-2) + 493 \times 73 \times 0.32} = 501.5$$

Since $F(2{,}562)_{0.9995} \cong 7.9$, the hypothesis that bands 1 and 3 are non-informative with respect to discriminating between the two bed groups is strongly rejected.

3. We have that the priors are equal and since nothing is given about losses we also assume that non-zero losses are equal and obtain that c in the Bayes solution is equal to 1 and hence $\log c = 0$. Therefore the decision function becomes (section 5.1.2 + example 5.9)

$$d(\boldsymbol{x}) = \boldsymbol{x}' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) - \frac{1}{2} \boldsymbol{\mu}_1' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_1 + \frac{1}{2} \boldsymbol{\mu}_2' \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_2$$

$$= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} -0.88 \\ 1.98 \\ -1.39 \end{bmatrix} - \frac{1}{2} 310.4 + \frac{1}{2} 396.5$$

$$= -0.88 x_1 + 1.98 x_2 - 1.39 x_3 + 43.05$$

The classification rule thus becomes:

If $d(\boldsymbol{x}) \geq 0$, we classify $\boldsymbol{x}$ as bed group 10, if $d(\boldsymbol{x}) < 0$, then as bed group 13.

4. We are still assuming equal non-zero losses. The estimated prior probabilities become (theorem 5.1)

$$\hat{p}_1 = \frac{493}{493+73} = 0.8710 \quad \text{and} \quad \hat{p}_2 = \frac{73}{493+73} = 0.1290$$

and we have

$$\log c = log\frac{\hat{p}_2}{\hat{p}_1} = log\frac{0.1290}{0.8710} = log\frac{73}{493} = -1.91$$

The decision function becomes

$$d_2(\boldsymbol{x}) = d(\boldsymbol{x}) + 1.91 = -0.88x_1 + 1.98x_2 - 1.39x_3 + 44.96$$

and the classification rule

If $d_2(\boldsymbol{x}) \geq 0$, we classify $\boldsymbol{x}$ as bed group 10, if $d_2(\boldsymbol{x}) < 0$, then as bed group 13.

5. The observation

$$\boldsymbol{x} = \begin{bmatrix} 74 \\ 43 \\ 40 \end{bmatrix}$$

inserted in d gives

$$d(\boldsymbol{x}) = -0.88 \times 74 + 1.98 \times 43 - 1.39 \times 40 + 43.05 = 7.47 > 0$$

so the observation is classified as bed group 10. See also Example 5.9

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