Exercises
Multivariate Statistics

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2022 Autumn, v. 0.12.1

Chapter 1

Exercise 1.1

Consider the matrix

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}$$

a) For which values of ρ is this a valid dispersion matrix? (Hint: Both dispersion and correlation matrices must be positive semi-definite)

Solution

$\boldsymbol{\rho}$ must be positive semidefinite, i.e. it must have non-negative eigenvalues. We have

$$\det \begin{pmatrix} 1-\lambda & \rho & \rho \\ \rho & 1-\lambda & \rho \\ \rho & \rho & 1-\lambda \end{pmatrix} = (1-\lambda)^3 + 2\rho^3 - 3\rho^2(1-\lambda)$$

Substituting $x = (1-\lambda)$ we get $x^3 - 3\rho^2 x + 2\rho^3 = 0$. The roots of which are ρ, ρ, -2ρ. This can be trivially solved in e.g. Maple, or we can guess a root ρ and find the remaining by reducing:

$$\frac{x^3 - 3\rho^2 x + 2\rho^3}{x-\rho} = x^2 + \rho x - 2\rho^2$$

The eigenvalues then become:

$$1-\lambda=\begin{cases}\rho\\ \rho\\ -2\rho\end{cases} \implies \lambda=\begin{cases}1-\rho\\ 1-\rho\\ 1+2\rho\end{cases}$$

these are all non-negative iff $\underline{\underline{-\frac{1}{2}\leq\rho\leq 1}}$

Assume that $\boldsymbol{\rho}$ is a valid dispersion matrix. Let furthermore the 3-dimensional random variable $\boldsymbol{X}$ be normally distributed

$$X\sim \mathrm{N}(0,\boldsymbol{\rho})$$

b) Determine the axes in the contour ellipsoid for the probability density function of $\boldsymbol{X}$

Solution

This is equivalent to finding the eigenvectors. It can easily be solved with Maple - we show how to do it by hand below. We found the eigenvalues in the previous question and will start with $\lambda_1=1+2\rho$. We subtract it from the diagonal and must then solve:

$$\begin{bmatrix}-2\rho & \rho & \rho\\ \rho & -2\rho & \rho\\ \rho & \rho & -2\rho\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

thus the first eigenvector is of the form

$$s\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}$$

and the normed version is

$$v_1=\begin{bmatrix}\sqrt{3}/3\\ \sqrt{3}/3\\ \sqrt{3}/3\end{bmatrix}$$

We then consider $\lambda_2=\lambda_3=1-\rho$ and get

$$\begin{bmatrix}\rho & \rho & \rho\\ \rho & \rho & \rho\\ \rho & \rho & \rho\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

or

$$x_1 + x_2 + x_3 = 0$$

This spans a plane and we may choose

$$v_2 = s \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$$

Finally, v_3 must satisfy

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

i.e. $x_3 = -2x_1$ and $x_2 = x_1$, which gives

$$v_3 = s \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{6}/6 \\ -\sqrt{6}/3 \end{bmatrix}$$

Exercise 1.2 EXAM 2013 - Problem 2

We consider a random variable

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

with mean and dispersion matrix respectively equal to

$$\mu = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{\Sigma} = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}$$

Furthermore we consider the random variables

$$U = X + Y + Z$$

$$V = 2X - Y - Z$$

a) [org. 2.1] The mean value of the two-dimensional random variable $\begin{bmatrix} U \\ V \end{bmatrix}$ is?

Solution

We use theorem 1.3 and 1.2.

$$\mathrm{E}(U) = \mathrm{E}(X+Y+Z) = \mathrm{E}(X) + \mathrm{E}(Y) + \mathrm{E}(Z) = 1+1+1 = 3$$

$$\mathrm{E}(V) = \mathrm{E}(2X-Y-Z) = \mathrm{E}(2X) - \mathrm{E}(Y) - \mathrm{E}(Z) = 2-1-1 = 0$$

Collecting the terms: $\mathrm{E}\left(\begin{bmatrix} U \\ V \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

b) [org. 2.2] The variance of U is?

Solution

We use theorem 1.8.

$$\begin{aligned}
\mathrm{D}(U) =& \mathrm{D}(X+Y+Z) = \mathrm{C}(X+Y+Z, X+Y+Z) \\
=& \mathrm{C}(X,X) + \mathrm{C}(X,Y) + \mathrm{C}(X,Z) + \\
& \mathrm{C}(Y,X) + \mathrm{C}(Y,Y) + \mathrm{C}(Y,Z) + \\
& \mathrm{C}(Z,X) + \mathrm{C}(Z,Y) + \mathrm{C}(Z,Z) \\
=& 1 + \rho + \rho + \\
& \rho + 1 + \rho + \\
& \rho + \rho + 1 \\
=& 3 + 6\rho
\end{aligned}$$

Note that $\mathrm{C}(X,X) = \mathrm{D}(X)$

c) [org. 2.3] The variance of V is?

Solution

We use theorem 1.8 and 1.6

$$
\begin{aligned}
D(V) =& D(2X - Y - Z) = C(2X - Y - Z, 2X - Y - Z) \\
=& C(2X, 2X) + C(2X, -Y) + C(2X, -Z) + \\
& C(-Y, 2X) + C(-Y, -Y) + C(-Y, -Z) + \\
& C(-Z, 2X) + C(-Z, -Y) + C(-Z, -Z) \\
=& 4 - 2\rho - 2\rho \\
& -2\rho + 1 + \rho \\
& -2\rho + \rho + 1 \\
=& 6 - 6\rho
\end{aligned}
$$

Note that $C(-X, -X) = -1 \cdot C(X, X) \cdot (-1) = D(X)$ as the minuses cancels each other.

d) [org. 2.4] The covariance between U and V is?

Solution

We use theorem 1.8 and define our original variables as $W = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$, then

$$C(U, V) = C(AW, BW)$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & -1 & -1 \end{bmatrix}$$

thus

$$
\begin{aligned}
C(U, V) = C(AW, BW) &= A\,C(W, W)B^T \\
&= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \\
&= 0
\end{aligned}
$$

Note that $C(W, W) = D(W)$ which was given in the exercise.

e) [org. 2.5] The conditional mean $\mathrm{E}\left(\begin{bmatrix} X \\ Y \end{bmatrix} \middle| Z = z\right)$ is equal to?

Solution

We use theorem 1.27

$$\begin{aligned}
\mathrm{E}\left(\begin{bmatrix} X \\ Y \end{bmatrix} \middle| Z = z\right) &= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2 \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \rho \\ \rho \end{bmatrix} \left[1\right]^{-1}(z-1) \\
&= \begin{bmatrix} 1 + \rho(z-1) \\ 1 + \rho(z-1) \end{bmatrix}
\end{aligned}$$

Exercise 1.3

The goal of this exercise, is to further familiarise you with calculations of statistical moments on multivariate data, as well as the interpretation of partial correlations.

We consider a three-dimensional random variable

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

with dispersion matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}$$

We now consider the linear combination $Y + aZ$. We want to estimate a such that the quantity

$$\mathrm{Corr}(X, Y + aZ)^2$$

is maximised.

a) What is the variance of X, the variance of $Y + aZ$ and the covariance $\mathrm{Cov}(X, Y + aZ)$?

Solution

We use theorem 1.8 and remark 1.9.

$$\begin{aligned}
\mathrm{V}(X) &= 1 \\
\mathrm{V}(Y + aZ) &= \mathrm{V}(Y) + \mathrm{V}(aZ) + 2\,\mathrm{Cov}(Y, aZ) \\
&= \mathrm{V}(Y) + a^2\,\mathrm{V}(Z) + 2a\,\mathrm{Cov}(Y, Z) \\
&= 1 + a^2 + 2a\rho \\
\mathrm{Cov}(X, Y + aZ) &= \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, aZ) \\
&= \mathrm{Cov}(X, Y) + a\,\mathrm{Cov}(X, Z) = \\
&= \rho + a\rho^2
\end{aligned}$$

b) Using the results above, write up the expression for $\mathrm{Corr}(X, Y + aZ)^2$.

Solution

We use the formula for correlation found in section 1.1.3, and insert the values found above in a).

$$\begin{aligned}
\mathrm{Corr}(X, Y + aZ)^2 &= \left(\frac{\mathrm{Cov}(X, Y + aZ)}{\sqrt{\mathrm{V(X)}\,\mathrm{V}(Y + aZ)}}\right)^2 \\
&= \frac{\mathrm{Cov}(X, Y + aZ)^2}{\mathrm{V(X)}\,\mathrm{V}(Y + aZ)} \\
&= \frac{(\rho + a\rho^2)^2}{1 + a^2 + 2a\rho}
\end{aligned}$$

c) Determine a such that $\mathrm{Corr}(X, Y + aZ)^2$ is maximised (Hint: You may consider using a symbolic solver, but it is possible by hand).

|||| Solution

We had that

$$\mathrm{Corr}(X, Y + aZ)^2 = \frac{(\rho + a\rho^2)^2}{1 + a^2 + 2a\rho}$$

which we can expand to

$$\rho^2 \frac{1 + a^2\rho^2 + 2a\rho}{1 + a^2 + 2a\rho}$$

To find the maximum as a function of a we take the derivative of the numerator with respect to a

$$\rho^2 \left[(2a\rho^2 2\rho)(1 + a^2 + 2a\rho) - (1 + a^2\rho^2 + 2a\rho)(2a + 2\rho) \right] = 2\rho^2(\rho^2 - 1)a(a\rho + 1)$$

The zeros for this expression is $a = 0$ for the maximum and $a = -\frac{1}{\rho}$ for the minimum.

The answer is thus $a = 0$!

d) Using the value found for a, determine the maximum of the squared correlation.

|||| Solution

We insert in the equation found in b)

$$\mathrm{Corr}(X, Y + aZ)^2 = \frac{(\rho + a\rho^2)^2}{1 + a^2 + 2a\rho} = \frac{(\rho + 0\rho^2)^2}{1 + 0^2 + 2 \cdot 0\rho} = \rho^2$$

e) Compare the result in d) with the squared multiple correlation between X and $(Y, Z)^T$

|||| Solution

We use theorem 1.42

$$\rho^2_{X|YZ} = 1 - \frac{\det \mathbf{\Sigma}}{\det \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}} = 1 - \frac{1 + \rho^4 - 2\rho^2}{1 - \rho^2} = \frac{-\rho^4 + \rho^2}{1 - \rho^2} = \rho^2$$

The same result as in d)!

f) Find the partial correlation between X and Z given Y, and between X and Y given Z.

|||| Solution

We use the equation on the bottom of page 34.

$$\rho_{XZ|Y} = \frac{\rho^2 - \rho\rho}{\sqrt{(1-\rho^2)(1-\rho^2)}} = 0$$

$$\rho_{XY|Z} = \frac{\rho - \rho^2\rho}{\sqrt{(1-\rho^4)(1-\rho^2)}} = \frac{\rho}{\sqrt{1+\rho^2}}$$

g) Comment on the results. Can you relate the value you found for a, with the partial correlation values?

|||| Solution

The result in c) tells us that when we want to predict X based on Y and Z we should only use Y. Z will not contribute any further. This is in accordance with the result in f) namely that $\rho_{XZ|Y} = 0$, i.e. given Y then X and Z are uncorrelated, i.e. Z contains no additional information on X if we know Y.