Exercises
Multivariate Statistics

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Chapter 1

Exercise 1.1

Consider the matrix

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}$$

a) For which values of ρ is this a valid dispersion matrix? (Hint: Both dispersion and correlation matrices must be positive semi-definite)

Assume that $\boldsymbol{\rho}$ is a valid dispersion matrix. Let furthermore the 3-dimensional random variable $\boldsymbol{X}$ be normally distributed

$$X \sim \mathrm{N}(0, \boldsymbol{\rho})$$

b) Determine the axes in the contour ellipsoid for the probability density function of $\boldsymbol{X}$

Exercise 1.2 EXAM 2013 - Problem 2

We consider a random variable

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

with mean and dispersion matrix respectively equal to

$$\mu = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{\Sigma} = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}$$

Furthermore we consider the random variables

$$U = X + Y + Z$$

$$V = 2X - Y - Z$$

a) [org. 2.1] The mean value of the two-dimensional random variable $\begin{bmatrix} U \\ V \end{bmatrix}$ is?

b) [org. 2.2] The variance of U is?

c) [org. 2.3] The variance of V is?

d) [org. 2.4] The covariance between U and V is?

e) [org. 2.5] The conditional mean $\mathrm{E}\left(\begin{bmatrix} X \\ Y \end{bmatrix} \middle| Z = z\right)$ is equal to?

Exercise 1.3

The goal of this exercise, is to further familiarise you with calculations of statistical moments on multivariate data, as well as the interpretation of partial correlations.

We consider a three-dimensional random variable

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

with dispersion matrix

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}$$

We now consider the linear combination $Y + aZ$. We want to estimate a such that the quantity

$$\text{Corr}(X, Y + aZ)^2$$

is maximised.

a) What is the variance of X, the variance of $Y + aZ$ and the covariance $\text{Cov}(X, Y + aZ)$?

b) Using the results above, write up the expression for $\text{Corr}(X, Y + aZ)^2$.

c) Determine a such that $\text{Corr}(X, Y + aZ)^2$ is maximised (Hint: You may consider using a symbolic solver, but it is possible by hand).

d) Using the value found for a, determine the maximum of the squared correlation.

e) Compare the result in d) with the squared multiple correlation between X and $(Y, Z)^T$

f) Find the partial correlation between X and Z given Y, and between X and Y given Z.

g) Comment on the results. Can you relate the value you found for a, with the partial correlation values?