Solution to exercise 3.1

In the following we use ' to denote the transpose matrix!

1. We have the following model
$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$$

or

$$\boldsymbol{Y} = \boldsymbol{x\theta} + \boldsymbol{\varepsilon}$$

It is stated in the text that $\boldsymbol{\varepsilon}$ represents random errors so we assume that the components are independent with the same variance σ^2, i.e. $D(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{I}$ **.W**e find

$$\boldsymbol{x}'\boldsymbol{x} = \begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 10 \\ 10 & 6 \end{bmatrix}$$

$$\boldsymbol{x}'\boldsymbol{Y} = \begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ -2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 17 \\ 9 \end{bmatrix}$$

Since

$$\begin{bmatrix} 18 & 10 \\ 10 & 6 \end{bmatrix}^{-1} = \frac{1}{8} \begin{bmatrix} 6 & -10 \\ -10 & 18 \end{bmatrix}$$

we get

$$\widehat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = -\frac{1}{16} \begin{bmatrix} 6 & -10 \\ -10 & 18 \end{bmatrix} \begin{bmatrix} 17 \\ 9 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix}$$

Furthermore

$$D(\widehat{\boldsymbol{\theta}}) = \sigma^2 (\boldsymbol{x}'\boldsymbol{x})^{-1} = \sigma^2 \frac{1}{8} \begin{bmatrix} 6 & -10 \\ -10 & 18 \end{bmatrix}$$

The vector of predicted values becomes

$$\widehat{\boldsymbol{Y}} = \boldsymbol{x}\widehat{\boldsymbol{\theta}} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -2 \end{bmatrix}$$

The vector of residuals is thus

$$\boldsymbol{Y} - \widehat{\boldsymbol{Y}} = \boldsymbol{Y} - \boldsymbol{x}\widehat{\boldsymbol{\theta}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix}$$

and

$$\left\|\boldsymbol{Y} - \boldsymbol{x}\widehat{\boldsymbol{\theta}}\right\|^2 = \left(\boldsymbol{Y} - \boldsymbol{x}\widehat{\boldsymbol{\theta}}\right)'\left(\boldsymbol{Y} - \boldsymbol{x}\widehat{\boldsymbol{\theta}}\right) = \frac{1}{4} + \frac{1}{16} + \frac{1}{16} + 0 = \frac{3}{8}$$

and we have

$$\hat{\sigma}^2 = \frac{1}{4-2}\frac{3}{8} = \frac{3}{16}$$

$$\widehat{D}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} \widehat{V}(\hat{a}) & \widehat{Cov}(\hat{a}, \hat{b}) \\ \widehat{Cov}(\hat{a}, \hat{b}) & \widehat{V}(\hat{b}) \end{bmatrix} = \hat{\sigma}^2 \frac{1}{8} \begin{bmatrix} 6 & -10 \\ -10 & 18 \end{bmatrix} = \begin{bmatrix} \frac{9}{64} & -\frac{15}{64} \\ -\frac{15}{64} & \frac{27}{64} \end{bmatrix}$$

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