

02409 Multivariate Statistics

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(1-3) 60%

Clustering 4 groups

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Groups

28

16

1

Factor 1 [41%]

Factor 3 [19%]

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Agenda

- CCA – recap
- Discrimination
 - The elements of discrimination and classification
 - Bayes and MINIMAX solutions
 - The Normal Case
 - Linear Discriminant Analysis
 - Mahalanobis' distance

Canonical variables and correlations

We consider a random variable

$$\mathbf{Z} \sim N_{p+q}(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where $p \leq q$ and \mathbf{Z} and the parameters have been partitioned as follows:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{X} \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_y \\ \mu_x \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{bmatrix}.$$

Definition 6.11

Consider \mathbf{Z} as above. Then the first *pair of canonical variables* is the pair of linear combinations

$$V_1 = \mathbf{a}_1^T \mathbf{Y} \text{ and } W_1 = \mathbf{b}_1^T \mathbf{X}$$

each having variance 1 that maximize the correlation $\rho(\mathbf{a}^T \mathbf{Y}, \mathbf{b}^T \mathbf{X})$ for all (\mathbf{a}, \mathbf{b}) . The maximum correlation ϱ_1 is the *first canonical correlation*. For $r \leq p$ we define the *r'th pair of canonical variables* as the pair of linear combinations

$$V_r = \mathbf{a}_r^T \mathbf{Y} \text{ and } W_r = \mathbf{b}_r^T \mathbf{X},$$

which each has the variance 1, which are uncorrelated with the previous $r - 1$ pairs of canonical variables, and which maximizes the correlation $\rho(\mathbf{a}^T \mathbf{Y}, \mathbf{b}^T \mathbf{X})$ under those constraints. The maximum correlation ϱ_r is the *r'th canonical correlation*.

CCA – salesdata

- Sales Performance:
 - Sales Growth
 - Sales Profitability
 - New Account Sales
- Test Scores as a Measure of Intelligence
 - Creativity
 - Mechanical Reasoning
 - Abstract Reasoning
 - Mathematics

Obs	growth	profit	new	create	mech	abs	math
1	93.0	96.0	97.8	9	12	9	20
2	88.8	91.8	96.8	7	10	10	15
3	95.0	100.3	99.0	8	12	9	26
4	101.3	103.8	106.8	13	14	12	29
5	102.0	107.8	103.0	10	15	12	32
6	95.8	97.5	99.3	10	14	11	21
7	95.5	99.5	99.0	9	12	9	25
8	110.8	122.0	115.3	18	20	15	51
9	102.8	108.3	103.8	10	17	13	31
10	106.8	120.5	102.0	14	18	11	39

CCA – salesdata

To what extent is sales performance correlated with measures of intelligence?

We will use canonical correlation analysis to investigate that.

CCA – salesdata

We take

$$Y = (\textit{growth}, \textit{profit}, \textit{new})$$
$$X = (\textit{create}, \textit{mech}, \textit{abs}, \textit{mat})$$

```
salesdata<-read.csv2("Data/Salesdata.csv")
names(salesdata)
[1] "growth" "profit" "new"      "create" "mech"    "abs"     "math"

Y<-salesdata[,1:3]
X<-salesdata[,4:7]
```

CCA – salesdata

Constructing the matrices E_1, E_2 :

```
Sigmayy<-var(Y)
Sigmaxx<-var(X)
Sigmayx<-cov(Y,X)
Sigmaxy<-t(Sigmayx)
```

$$E_1 = \Sigma_{yy}^{-1} \Sigma_{yx} \Sigma_{xx}^{-1} \Sigma_{xy}$$
$$E_2 = \Sigma_{xx}^{-1} \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx}$$

E_1 : 3 × 3 matrix

```
E1<-solve(Sigmayy)%*%Sigmayx%%solve(Sigmaxx)%*%Sigmaxy
E2<-solve(Sigmaxx)%*%Sigmaxy%%solve(Sigmayy)%*%Sigmayx
E1
```

	growth	profit	new
growth	0.5241851	0.1481294	0.33088173
profit	0.1273726	0.8098589	-0.05362605
new	0.4729584	0.1452355	0.57317616

E2

	create	mech	abs	math
create	0.3936913	0.04077489	0.21007893	0.3414371
mech	-0.1095057	0.20397352	-0.09896763	0.6466981
abs	0.4429526	-0.08171082	0.58337701	0.1264595
math	0.1160912	0.19365385	0.02750066	0.7261784

E_2 : 4 × 4 matrix

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Extracting the canonincal correlations:

We find the canonical correlation coefficients as the eigenvalues for E_1 :

```
(my.cor<-sqrt(eigen(E1)$values))  
[1] 0.9944827 0.8781065 0.3836057
```

Thus,

$$\hat{q} = (0.9944827, 0.8781065, 0.3836057)$$

Note that

```
sqrt(round(eigen(E2)$values,digits=6))  
[1] 0.9944828 0.8781065 0.3836053 0.0000000
```

E_2 is singular, but the first three eigenvalues are equal to the eigenvalues of E_1 .

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Finding the canonical vectors:

We find the canonical vectors as the eigenvectors for E_1 and E_2 (the first 3):

```
A<-eigen(E1)$vectors
B<-eigen(E2)$vectors[,1:3]
```

Checking that the signs are right, so that we have positive correlations:

```
round(cov(as.matrix(Y)%*%A,as.matrix(X)%*%B),digits=4)
      [,1]      [,2]      [,3]
[1,] 72.9489  0.0000  0.0000
[2,]  0.0000 -4.0158  0.0000
[3,]  0.0000  0.0000  1.7533
```

Correcting (arbitrarily choosing A) :

```
A[,2]<--A[,2]
```

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Standardizing the canonical vectors:

We must have $A^T \Sigma_{yy} A = I$, $B^T \Sigma_{xx} B = I$. Current values:

```
round(t(A)%%Sigmayy%%A,digits=4)
```

```
      [,1] [,2] [,3]  
[1,] 95.6643 0.0000 0.0000  
[2,]  0.0000 6.8621 0.0000  
[3,]  0.0000 0.0000 3.3337
```

```
round(t(B)%%Sigmaxx%%B,digits=4)
```

```
      [,1] [,2] [,3]  
[1,] 56.2462 0.0000 0.0000  
[2,]  0.0000 3.0479 0.0000  
[3,]  0.0000 0.0000 6.2665
```

Diagonal matrix but not the identity. Reason: eigenvectors are given as **unit vectors**. We standardize:

```
A<-A%%diag(1/sqrt(diag(t(A)%%Sigmayy%%A)))  
B<-B%%diag(1/sqrt(diag(t(B)%%Sigmaxx%%B)))
```

- Because we multiply with a diagonal matrix, we don't change the direction of the vectors, but only the length. Compare with the approach in the book page [383](#).

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The canonical vectors:

A

	[,1]	[,2]	[,3]
[1,]	0.06237788	-0.1740703	-0.3771529
[2,]	0.02092564	0.2421641	0.1035150
[3,]	0.07825817	-0.2382940	0.3834151

B

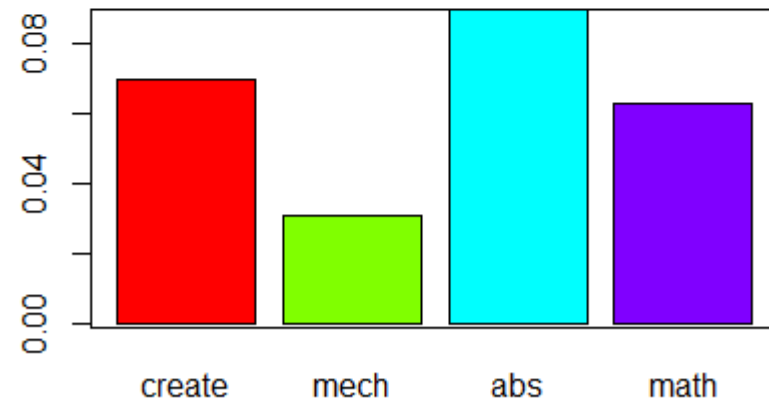
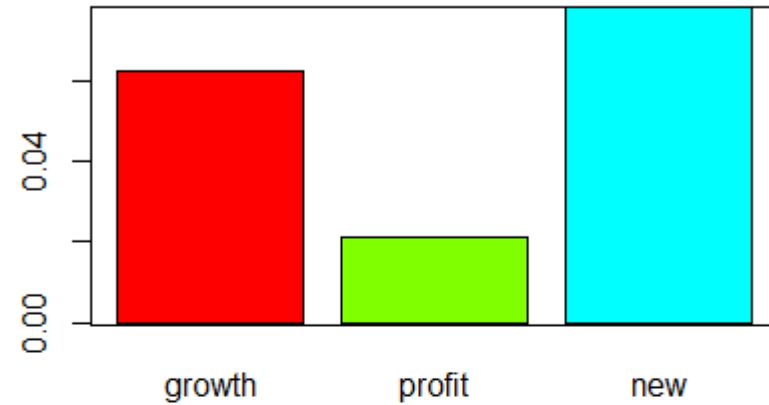
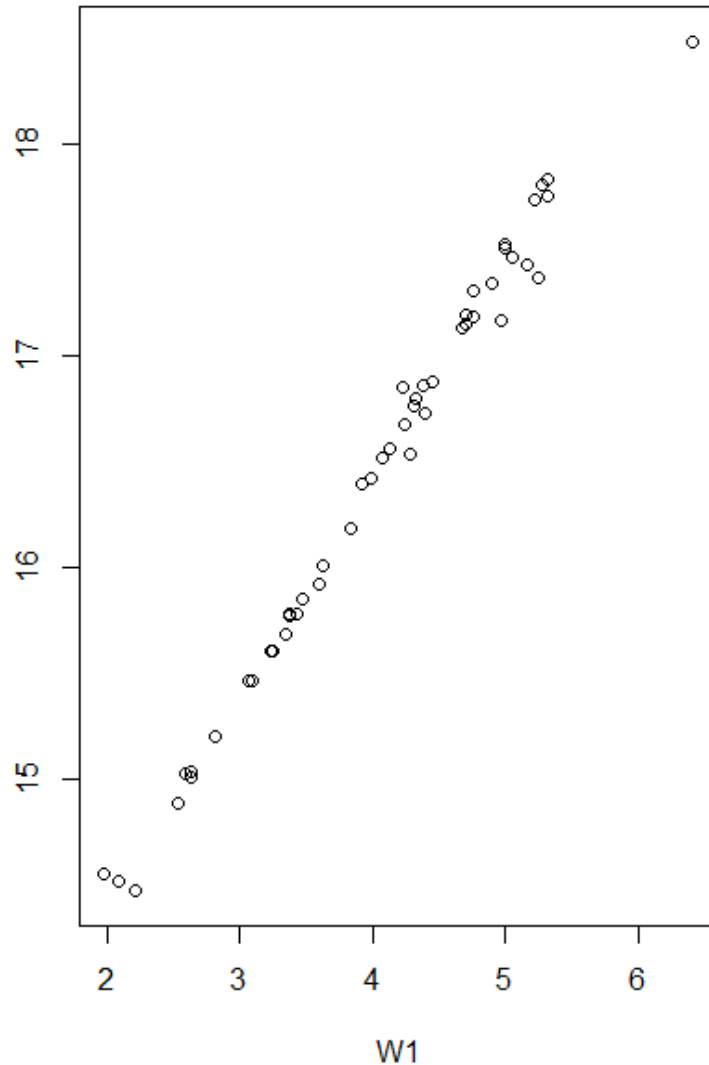
	[,1]	[,2]	[,3]
[1,]	0.06974814	-0.19239132	0.24655659
[2,]	0.03073830	0.20157438	-0.14189528
[3,]	0.08956418	-0.49576326	-0.28022405
[4,]	0.06282997	0.06831607	0.01133259

The linear combinations of sales performance and intelligence measures that correlates the most:

$$V_1 = 0.06 * growth + 0.02 * profit + 0.08 * new$$

$$W_1 = 0.07 * create + 0.03 * mech + 0.09 * abs + 0.06 * math$$

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- New account sales and sales growth correlate with measures of intelligence, lesser so for sales profitability.
- In particular Abstract reasoning, Creativity and to some extent Mathematics correlate with sales performance, less so for mechanical thinking.

CCA – salesdata

Testing: Assume that (Y, X) are normally distributed.

$$V \begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix}, \quad \text{Cor} \begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} I_3 & \begin{pmatrix} \varrho_1 & & \\ & \varrho_2 & \\ & & \varrho_3 \end{pmatrix} \\ * & I_3 \end{pmatrix}$$

$\text{rank}(\Sigma_{yx}) \leq p$: Testing for no canonical correlation corresponds to testing that $V \begin{pmatrix} Y \\ X \end{pmatrix}$ is a diagonal matrix.

- Test for no canonical correlation:

$$H_0: \Sigma = V \begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} \Sigma_{vv} & \\ & \Sigma_{ww} \end{pmatrix} \quad \text{vs.} \quad H: \Sigma = V \begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix}$$

Test statistic the p canonical correlations are 0:

$$Q_p = LR^{2/n} = \frac{\det(\hat{\Sigma})}{\det(\hat{\Sigma}_{yy})\det(\hat{\Sigma}_{xx})}$$

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$$V(AX) = AV(X)A^T$$

$$Q = \frac{\det(\hat{\Sigma})}{\det(\hat{\Sigma}_{yy})\det(\hat{\Sigma}_{xx})}$$

Note that

$$V \begin{pmatrix} Y - \Sigma_{yx}\Sigma_{xx}^{-1}X \\ X \end{pmatrix} = \begin{pmatrix} I & -\Sigma_{yx}\Sigma_{xx}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Sigma_{xx}^{-1}\Sigma_{xy} & I \end{pmatrix} = \begin{pmatrix} \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy} & 0 \\ 0 & \Sigma_{xx} \end{pmatrix} = \begin{pmatrix} \Sigma_{y|x} & 0 \\ 0 & \Sigma_{xx} \end{pmatrix}$$

and that

$$\det \begin{pmatrix} I & -\Sigma_{yx}\Sigma_{xx}^{-1} \\ 0 & I \end{pmatrix} = 1$$

Thus,

$$\det(\Sigma) = \det(\Sigma_{y|x})\det(\Sigma_{xx})$$

and

$$\begin{aligned} Q_p &= \frac{\det(\hat{\Sigma})}{\det(\hat{\Sigma}_{yy})\det(\hat{\Sigma}_{xx})} = \frac{\det(\hat{\Sigma}_{y|x})\det(\hat{\Sigma}_{xx})}{\det(\hat{\Sigma}_{yy})\det(\hat{\Sigma}_{xx})} = \frac{\det(\hat{\Sigma}_{y|x})}{\det(\hat{\Sigma}_{yy})} = \det(\hat{\Sigma}_{yy}^{-1}\hat{\Sigma}_{y|x}) = \det(I - \hat{\Sigma}_{yy}^{-1}\hat{\Sigma}_{yx}\hat{\Sigma}_{xx}^{-1}\hat{\Sigma}_{xy}) = \det(I - \hat{E}_1) \\ &= \det \left(I - \begin{pmatrix} \hat{\varrho}_1^2 & & \\ & \ddots & \\ & & \hat{\varrho}_p^2 \end{pmatrix} \right) = \prod_{i=1}^p (1 - \hat{\varrho}_i^2) \end{aligned}$$

follows.

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$$Q_0 = \frac{\det(\hat{\Sigma})}{\det(\hat{\Sigma}_{vv})\det(\hat{\Sigma}_{ww})} = \prod_{i=1}^p (1 - \hat{\varrho}_i^2)$$

Note that $\text{rank}(\Sigma_{yx}) = \text{rank}(E_1) = \# \text{ nonzero canonical correlations}$. From the estimation procedure it is immediate that the test statistic for

$$H_0: \Sigma = V \begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} \Sigma_{vv} & \\ & \Sigma_{ww} \end{pmatrix} \quad \text{vs.} \quad H_r: \Sigma = V \begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix} \text{ with } \text{rank}(\Sigma_{yx}) = r$$

is maximizing $\det(I - \hat{E}_1)$ under the assumption that $\text{rank}(E_1) = r$, ie.

$$\prod_{i=p-r+1}^p (1 - \hat{\varrho}_i^2)$$

Thus, from simple division, the test statistic for that r canonical correlations are equal to 0:

$$H_r: \Sigma = V \begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix} \text{ with } \text{rank}(\Sigma_{yx}) = p - r \quad \text{vs} \quad H: \Sigma = V \begin{pmatrix} Y \\ X \end{pmatrix} = \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix}$$

is

$$Q_r = \prod_{i=1}^r (1 - \hat{\varrho}_i^2)$$

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Testing

$$H_0: \Sigma = \begin{pmatrix} \Sigma_{yy} & \\ & \Sigma_{xx} \end{pmatrix} \text{ vs. } H: \Sigma = \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix}$$

dim(Σ_{yy}) = p
dim(Σ_{xx}) = q

is

$$Q_0 = \frac{\det(\hat{\Sigma})}{\det(\hat{\Sigma}_{yy})\det(\hat{\Sigma}_{xx})}$$

Under H_0 , Q_0 is Wilks distributed, $\wedge (p, n - 1 - q, q)$.

df from unbiased estimation of variances

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Heuristics for the df for Q_r :

Let the columns of W^\perp span the orthogonal complement of W . W^\perp will have $q - p$ columns.

Taking $Y' = V_{p-r+1;\dots;p}$ (r variables), $X' = (W_{p-r+1;\dots;p}; W^\perp)$ ($q-p+r$ variables),

then

$$\text{cor}(Y', X') = \begin{pmatrix} \varrho_{p-r+1} & & 0 & & \\ & \ddots & & & \\ & & \varrho_p & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

Then H_r vs. H corresponds to testing that $V \begin{pmatrix} Y' \\ X' \end{pmatrix}$ is a diagonal matrix, with blocks of dimensions $r \times r, (q - p + r) \times (q - p + r)$.

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Hypothesis	$Q_{1:3}$	Value	Distribution
$H_3: \varrho_1 = \varrho_2 = \varrho_3 = 0$	$(1 - \hat{\varrho}_1^2)(1 - \hat{\varrho}_2^2)(1 - \hat{\varrho}_3^2)$	0.8528	$\Lambda(3, n - 5, 4)$
$H_2: \varrho_2 = \varrho_3 = 0$	$(1 - \hat{\varrho}_2^2)(1 - \hat{\varrho}_3^2)$	0.1952	$\Lambda(2, n - 5, 3)$
$H_1: \varrho_3 = 0$	$(1 - \hat{\varrho}_3^2)$	0.0021	$\Lambda(1, n - 5, 2)$

$$\Lambda(r, n - 1 - q, q - p + r), r = 1, \dots, 3:$$

Wilks' distribution. Super complicated, and not tabled in base R. Wilks' distributions have **small values critical** for the test.

- Therefore, we transform ourselves to lesser complicated distributions.
- Or use the approximative Barlett's test (lecture E).

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Λ	f_1	f_2	$F(f_2, f_1)$	F Exact or approximative?
$\Lambda(3, n-5, 4)$	12	$v(n-5) - 5$	$\frac{1 - \Lambda^{1/v} f_1}{\Lambda^{1/v} f_2}$	Approximative
$\Lambda(2, n-5, 3)$	6	$2(n-5) - 2$	$\frac{1 - \Lambda^{1/2} f_1}{\Lambda^{1/2} f_2}$	Exact
$\Lambda(1, n-5, 2)$	2	$n-5$	$\frac{1 - \Lambda f_1}{\Lambda f_2}$	Exact

Conversion from $\Lambda(r, n-1-q, q-p+r)$:

$$f_1 = r(q-p+r),$$

$$\text{If } r = 1 \text{ take } v = 1; \text{ if } r > 1 \text{ take } v = \sqrt{\frac{f_1^2 - 4}{r^2 + (q-p+r)^2 - 5}}$$

$$f_2 = v(n-1-q) - \frac{f_1}{2} + 1$$

$$r = 2: v = 2. \quad r = 3: v = \sqrt{7}.$$

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- **Barlett's test:**
- In general,

$$-\left(n - \frac{p + q + 3}{2}\right) \log(Q_i) \sim \chi^2_{r(q-p+r)}$$

The same df as f_1 on the previous slide. Note that if $p = q$, the degrees of freedom is equal to r^2 ;

CCA – salesdata

Hypothesis	$Q_{3:1}$	Value	f_1	f_2	$F(f_1, f_2)$	p_F	$\chi^2_{2,6,12}$	p_{chisq}
$H_3: \varrho_1 = \varrho_2 = \varrho_3 = 0$	$(1 - \hat{\varrho}_1^2)(1 - \hat{\varrho}_2^2)(1 - \hat{\varrho}_3^2)$	0.0021	12	108.0588	87.39	< 0.001	276.43	< 0.001
$H_2: \varrho_2 = \varrho_3 = 0$	$(1 - \hat{\varrho}_2^2)(1 - \hat{\varrho}_3^2)$	0.1952	6	88	17.53	< 0.001	73.51	< 0.001
$H_1: \varrho_3 = 0$	$(1 - \hat{\varrho}_3^2)$	0.8528	2	45	3.88	0.03	7.16	0.03

- Data do not support deletion of any canonical correlations ($p=0.03$).

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- Same values using `p.asym`:

```
p.asym(my.cor,n,p,q, tstat="Wilks")
Wilks' Lambda, using F-approximation (Rao's F):
```

	stat	approx	df1	df2	p.value
1 to 3:	0.002148472	87.391525	12	114.0588	0.000000e+00
2 to 3:	0.195241267	18.526265	6	88.0000	8.248957e-14
3 to 3:	0.852846693	3.882233	2	45.0000	2.783536e-02

Exactly the same values, but without the Chisquare approximation.

CCA – salesdata

Correlation between Y, X, V, W :

	growth	profit	new	create	mech	abs	math	V1	V2	V3	W1	W2	W3
growth	1.00	0.93	0.88	0.57	0.71	0.67	0.93	0.98	0.00	-0.20	0.97	0.00	-0.08
profit	0.93	1.00	0.84	0.54	0.75	0.47	0.94	0.95	0.32	0.01	0.94	0.28	0.00
new	0.88	0.84	1.00	0.70	0.64	0.64	0.85	0.95	-0.19	0.24	0.95	-0.16	0.09
create	0.57	0.54	0.70	1.00	0.59	0.15	0.41	0.63	-0.19	0.25	0.64	-0.22	0.65
mech	0.71	0.75	0.64	0.59	1.00	0.39	0.57	0.72	0.21	-0.03	0.72	0.24	-0.07
abs	0.67	0.47	0.64	0.15	0.39	1.00	0.57	0.64	-0.44	-0.22	0.65	-0.50	-0.57
math	0.93	0.94	0.85	0.41	0.57	0.57	1.00	0.94	0.17	-0.04	0.94	0.20	-0.09
V1	0.98	0.95	0.95	0.63	0.72	0.64	0.94	1.00	0.00	0.00	0.99	0.00	0.00
V2	0.00	0.32	-0.19	-0.19	0.21	-0.44	0.17	0.00	1.00	0.00	0.00	0.88	0.00
V3	-0.20	0.01	0.24	0.25	-0.03	-0.22	-0.04	0.00	0.00	1.00	0.00	0.00	0.38
W1	0.97	0.94	0.95	0.64	0.72	0.65	0.94	0.99	0.00	0.00	1.00	0.00	0.00
W2	0.00	0.28	-0.16	-0.22	0.24	-0.50	0.20	0.00	0.88	0.00	0.00	1.00	0.00
W3	-0.08	0.00	0.09	0.65	-0.07	-0.57	-0.09	0.00	0.00	0.38	0.00	0.00	1.00

Canonocal Correlation analysis

What are the strengths and weaknesses of CCA?

- **Best:**

Questions concerning the number and nature of mutually independent relations between two sets of variables. For the sales data: 3 dimensions.

- **Mediocre:**

Questions concerning the degree of overlap or redundancy between two sets of variables.

- **Not Very Well:**

Questions concerning the similarity between two within-set correlation or covariance matrices.

Discriminant Analysis

- So far we have studied relations between multivariate data through the variance matrix.
- We have separated parts of the data, and studied correlations between subsets.
- But what if the correlation comes from a (n unknown) substructure in the data, dividing the study population into discrete groups, each having the same distribution of the data?
- Discriminant analysis is about modeling and uncovering groupings within the data.

Discrimination and Classification

1. The elements of discrimination and classification
2. Bayes and MINIMAX solutions
3. The Normal Case:
 1. Linear Discriminant Analysis
 2. Quadratic Discriminant Analysis
4. Mahalanobis' distance
5. Estimation of classification errors in LANDSAT data
6. Canonical Discriminant Analysis

Discriminant Analysis

- Multivariate data in a population with two subgroups π_1, π_2 :

$$X_j = \begin{pmatrix} X_{j1} \\ \vdots \\ X_{jp} \end{pmatrix}, j = 1, \dots, n.$$

Density for X_j :

$$f_1(x), \text{ if } X_j \in \pi_1;$$

$$f_2(x), \text{ if } X_j \in \pi_2.$$

Decision functions

- We seek a function to determine, based on observational data, if a subject belongs to π_1 or π_2 :

$$d: \mathbb{R}^p \rightarrow \{\pi_1, \pi_2\}$$

- Since d can only attain two values, it will necessarily subdivide \mathbb{R}^p into two disjoint subsets R_1, R_2 :

$$d(x) = \begin{cases} \pi_1 & \text{if } x \in R_1 \\ \pi_2 & \text{if } x \in R_2 \end{cases}$$

- If we choose R_1 , we choose d !

Loss

- We shall assume that it is not without consequences if we make a mistake. If X really belongs to π_1 , we suffer a **loss** $L(\pi_1, \pi_2)$ if we classify X_j to π_2 via the decision function.

From		Classify as	
		π_1	π_2
Nature	π_1	0	$L(\pi_1, \pi_2) = L_{12}$
	π_2	$L(\pi_2, \pi_1) = L_{21}$	0

- A good decision function **minimizes the loss**.

A Bayesian framework

- Let us suppose that we have an idea about the distribution of the subpopulations π_1, π_2 . Maybe we have seen other observations from the same subpopulations.
- Let us therefore assume that, prior to any observations if the values of X_j , there is a probability distribution Π on the subclasses in the population, such that for all individuals it holds that

$$P(\Pi = \pi_1) = g(\pi_1) = 1 - P(\Pi = \pi_2) = 1 - g(\pi_2)$$

- In the following, we will write p_i for $g(\pi_i)$, $i = 1, 2$.

Posterior Probability

We define the *posterior probability*

$$k(\pi_i|x) := \frac{p_i f_i(x)}{p_1 f_1(x) + p_2 f_2(x)}, i = 1, 2$$

- In contrast to the a priori distribution, the posterior distribution k holds the probability of the classes AFTER observation of the data X_j .

Expected loss

- The class $\Pi(j)|X_j = x$ now has a distribution.
- Then the loss, $L(\Pi(i), d(X_j))|X_j = x$ has a distribution.
- Let us calculate the average loss given $X_j = x$:

$$\begin{aligned} & E\left(L\left(\Pi(j), d(X_j)\right) \mid X_j = x\right) \\ &= E\left(L\left(\Pi(j), d(x)\right) \mid X_j = x\right) \\ &= \begin{cases} L(\pi_2, \pi_1)k(\pi_2|x) & \text{if } x \in R_1 \\ L(\pi_1, \pi_2)k(\pi_1|x) & \text{if } x \in R_2 \end{cases} \end{aligned}$$

The Bayes solution

$$L(\pi_2, \pi_1)k(\pi_2|x) \quad \text{if } x \in R_1$$

$$L(\pi_1, \pi_2)k(\pi_1|x) \quad \text{if } x \in R_2$$

- **Loss minimization:** Choose R_1 to be the the area where the top line loss is less than the bottom line loss:

$$\begin{aligned} R_1 &= \{x \in \mathbb{R}^p \mid L(\pi_1, \pi_2)k(\pi_2|x) \leq L(\pi_2, \pi_1)k(\pi_1|x)\} \\ &= \left\{x \in \mathbb{R}^p \mid \frac{L_{12}k(\pi_1|x)}{L_{21}k(\pi_2|x)} \geq 1\right\} \\ &= \left\{x \in \mathbb{R}^p \mid \frac{L_{12}p_1f_1(x)}{L_{21}p_2f_2(x)} \geq 1\right\} \\ &= \left\{x \in \mathbb{R}^p \mid \frac{f_1(x)}{f_2(x)} \geq \frac{L_{21}p_2}{L_{12}p_1}\right\} \end{aligned}$$

The Bayes solution

||| Theorem 5.1

The *Bayes solution* to the classification problem is given by the region

$$R_1 = \left\{ x \mid \frac{f_1(x)}{f_2(x)} \geq \frac{L_{21}p_2}{L_{12}p_1} \right\} .$$

Risk

- The **loss** is what we face if we make a mis-classification – the CONSEQUENCE of a misclassification.
- But if the probability of making a misclassification is small, then the consequences will likely not occur.
- Let us focus on **risk**: *probability \times consequence*.
- Instead of minimizing the loss, we could minimize the risk that we run;
- We can do so by selecting the region R_1 so that the risks in R_1 and R_2 are the same:

Risk

- If $\Pi(j) = \pi_1$ (ie. Individual j belongs to π_1), the expected risk is

$$R = \textit{consequence} \times \textit{probability} = L_{12}P(X \in R_2)$$

If $\Pi(j) = \pi_2$ (ie. Individual j belongs to π_2), the expected risk is

$$R = \textit{consequence} \times \textit{probability} = L_{21}P(X \in R_1)$$

Minimax solution

||| Theorem 5.2

The *minimax solution* for the classification problem is given by the region

$$R_1 = \left\{ x \mid \frac{f_1(x)}{f_2(x)} \geq c \right\} .$$

where c is determined by

$$L_{12}P \left\{ \frac{f_1(x)}{f_2(x)} < c \mid \pi_1 \right\} = L_{21}P \left\{ \frac{f_1(x)}{f_2(x)} \geq c \mid \pi_2 \right\} .$$

- It is seen that the last line exactly expresses that risk = *consequence* \times $P(\text{misclassification})$ is independent of the true state Π .

The Normal Distribution

Assume that individuals in π_1, π_2 have normally distributed data, $N_p(\mu_1, \Sigma), N_p(\mu_2, \Sigma)$, respectively, with Σ invertible.

Introduce the inner product $\langle, \rangle_{\Sigma^{-1}}$ and the corresponding norm $\| \cdot \|_{\Sigma^{-1}}$ on \mathbb{R}^p by

$$\langle x, y \rangle_{\Sigma^{-1}} = x^T \Sigma^{-1} y$$

$$f_1(x) = \frac{1}{(2\pi)^{p/2} \sqrt{\det(\Sigma)}} \exp(-)$$

The Normal Distribution

Densities:

$$f_i(x) = \frac{1}{(2\pi)^{p/2} \sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2} \|x - \mu_i\|_{\Sigma^{-1}}^2\right), i = 1, 2.$$

Thus:

$$\frac{f_1(x)}{f_2(x)} = \exp\left(-\frac{1}{2} (\|x - \mu_1\|_{\Sigma^{-1}}^2 - \|x - \mu_2\|_{\Sigma^{-1}}^2)\right)$$

The Normal Distribution

$$\begin{aligned}\frac{f_1(x)}{f_2(x)} \geq c &\Leftrightarrow -\|x - \mu_1\|_{\Sigma^{-1}}^2 + \|x - \mu_2\|_{\Sigma^{-1}}^2 \geq 2 \log(c) \\ &\Leftrightarrow 2 \langle x, \mu_1 - \mu_2 \rangle_{\Sigma^{-1}} - \|\mu_1\|_{\Sigma^{-1}}^2 + \|\mu_2\|_{\Sigma^{-1}}^2 \geq 2 \log(c)\end{aligned}$$

||| Theorem 5.4

Let $\pi_1 \sim N(\mu_1, \Sigma)$ and $\pi_2 \sim N(\mu_2, \Sigma)$. Then we have

$$\begin{aligned}\frac{f_1(x)}{f_2(x)} \geq c &\Leftrightarrow x^T \Sigma^{-1} (\mu_1 - \mu_2) - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_2^T \Sigma^{-1} \mu_2 \geq \log c \\ &\Leftrightarrow \left[x^T \Sigma^{-1} \mu_1 - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 \right] - \left[x^T \Sigma^{-1} \mu_2 - \frac{1}{2} \mu_2^T \Sigma^{-1} \mu_2 \right] \geq \log c.\end{aligned}$$

The Normal Distribution

Note that the decision function is a *separating hyperplane*:

The hyperplane satisfies

$$2 \langle x, \mu_1 - \mu_2 \rangle_{\Sigma^{-1}} - \|\mu_1\|_{\Sigma^{-1}}^2 + \|\mu_2\|_{\Sigma^{-1}}^2 = 2 \log(c),$$

ie. it has the form $x = \tilde{c}v + \beta$, with $v = \Sigma^{-1}(\mu_1 - \mu_2)$, and $\beta \perp v$ with β varying. The hyperplane has dimension $p - 1$.

Linear Discriminant Functions

- Bayes: $R_1 = \left\{ x \in \mathbb{R}^p \mid \frac{f_1(x)p_1}{f_2(x)p_2} \geq \frac{L_{21}}{L_{12}} \right\}$
- Minimax: $R_1 = \left\{ x \in \mathbb{R}^p \mid \frac{f_1(x)}{f_2(x)} \geq c \right\}$

Linear discriminant functions:

Bayes:
$$\langle x, \mu_i \rangle_{\Sigma^{-1}} - \frac{1}{2} \|\mu_i\|_{\Sigma^{-1}}^2 + \log(p_i), \quad i = 1, 2;$$

Minimax:
$$\langle x, \mu_i \rangle_{\Sigma^{-1}} - \frac{1}{2} \|\mu_i\|_{\Sigma^{-1}}^2, \quad i = 1, 2.$$

Linear Discriminator

Linear discriminator between π_1 and π_2 :

$$\langle x, \mu_1 - \mu_2 \rangle_{\Sigma^{-1}} - \frac{1}{2} \|\mu_1\|_{\Sigma^{-1}}^2 + \frac{1}{2} \|\mu_2\|_{\Sigma^{-1}}^2 + \log(c)$$

Separating
hyperplane

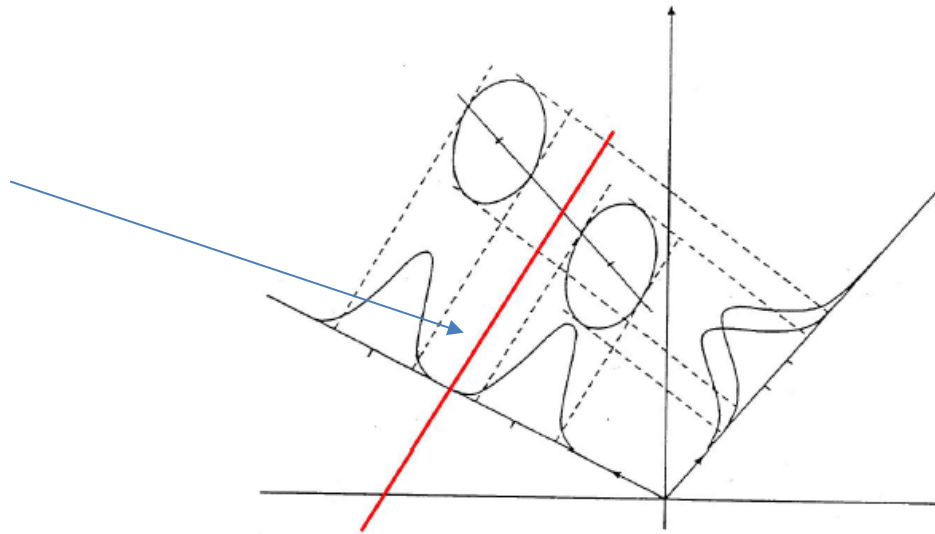


Figure 5.4 – Classification example

Linear Discriminator

Linear discriminator between π_1 and π_2 :

$$\langle x, \mu_1 - \mu_2 \rangle_{\Sigma^{-1}} - \frac{1}{2} \|\mu_1\|_{\Sigma^{-1}}^2 + \frac{1}{2} \|\mu_2\|_{\Sigma^{-1}}^2 - \log(c)$$

Separating
hyperplane

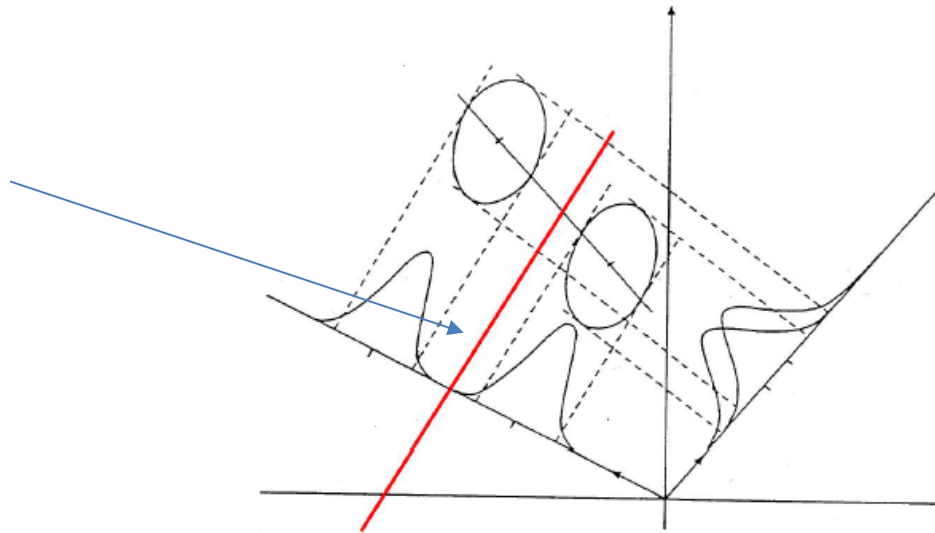


Figure 5.4 – Classification example

Selecting the Optimal Discriminator

- Take

$$\delta = \Sigma^{-1}(\mu_1 - \mu_2)$$

||| Theorem 5.6

The vector δ has the property that it maximizes the function

$$g(d) = \frac{[E_1(X^T d) - E_2(X^T d)]^2}{V(X^T d)} = \frac{[(\mu_1 - \mu_2)^T d]^2}{d^T \Sigma d}$$

Distribution of the Linear Discriminator

||| Theorem 5.8

We consider the random variable defined by the linear discriminator (omitting the term $-\log c$), i.e.

$$Z = X^T \Sigma^{-1} (\mu_1 - \mu_2) - \frac{1}{2} \mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_2^T \Sigma^{-1} \mu_2 .$$

Then

$$Z \sim \begin{cases} \text{N} \left(+\frac{1}{2} \|\mu_1 - \mu_2\|_{\Sigma^{-1}}^2, \|\mu_1 - \mu_2\|_{\Sigma^{-1}}^2 \right) & \text{if } \pi_1 \text{ is true} \\ \text{N} \left(-\frac{1}{2} \|\mu_1 - \mu_2\|_{\Sigma^{-1}}^2, \|\mu_1 - \mu_2\|_{\Sigma^{-1}}^2 \right) & \text{if } \pi_2 \text{ is true} \end{cases} .$$

Example

Suppose that

$$\pi_1 \leftrightarrow N(\mu_1, \Sigma), \quad \pi_2 \leftrightarrow N(\mu_2, \Sigma)$$

with

$$\mu_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mu_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \Sigma^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Losses:

From	Classify as	
	π_1	π_2
Nature	π_1	0
	π_2	$L_{12} = 2$
		$L_{21} = 1$
		0

We seek the minimax solution.

Example

$$\mu_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mu_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \Sigma^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, L_{12} = 2, L_{21} = 1$$

Note that

$$\|\mu_1 - \mu_2\|_{\Sigma^{-1}}^2 = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 13$$

Must find c so that

$$2P\left(\frac{f_1(x)}{f_2(x)} < c \middle| \pi_1\right) = P\left(\frac{f_1(x)}{f_2(x)} \geq c \middle| \pi_2\right)$$

Example

$$\|\mu_1 - \mu_2\|_{\Sigma^{-1}}^2 = 13$$

Theorem 5.8:

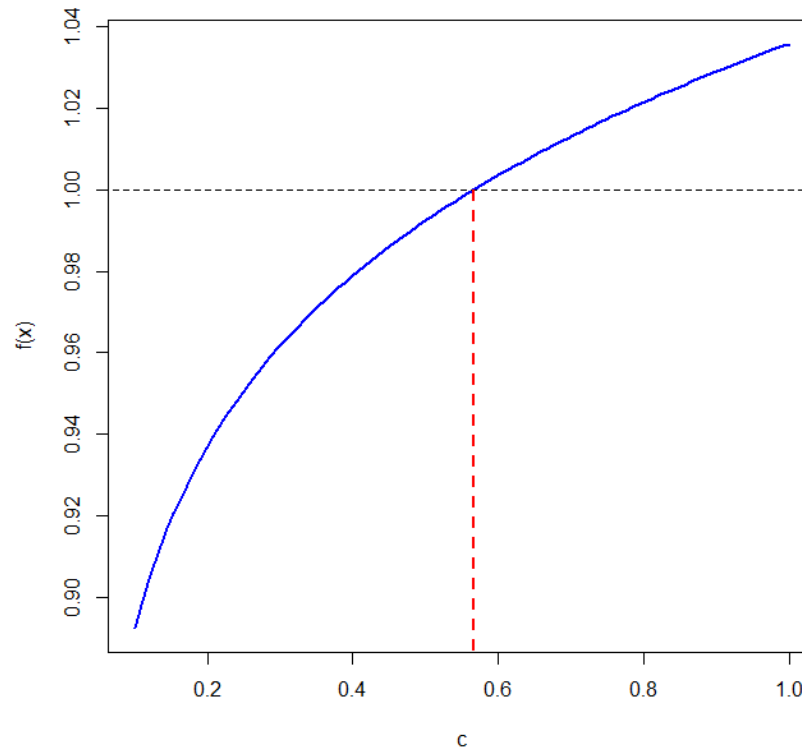
$$\begin{aligned} 2P\left(\frac{f_1(x)}{f_2(x)} < c \middle| \pi_1\right) &= P\left(\frac{f_1(x)}{f_2(x)} \geq c \middle| \pi_2\right) \Leftrightarrow \\ 2P(Z < \log(c) | \pi_1) &= P(Z \geq \log(c) | \pi_2) \Leftrightarrow \\ 2P\left(N\left(\frac{1}{2} \cdot 13, 13\right) < \log(c)\right) &= P\left(N\left(-\frac{1}{2} \cdot 13, 13\right) \geq \log(c)\right) \Leftrightarrow \\ 2P\left(N(0,1) < \frac{\log(c) - 6.5}{\sqrt{13}}\right) &= P\left(N(0,1) \geq \frac{\log(c) + 6.5}{\sqrt{13}}\right) \Leftrightarrow \\ 2\Phi\left(\frac{\log(c) - 6.5}{\sqrt{13}}\right) &= 1 - \Phi\left(\frac{\log(c) + 6.5}{\sqrt{13}}\right) \Leftrightarrow \\ 2\Phi\left(\frac{\log(c) - 6.5}{\sqrt{13}}\right) + \Phi\left(\frac{\log(c) + 6.5}{\sqrt{13}}\right) &= 1 \end{aligned}$$

with Φ the standard normal distribution function.

Example

$$2\Phi\left(\frac{\log(c) - 6.5}{\sqrt{13}}\right) + \Phi\left(\frac{\log(c) + 6.5}{\sqrt{13}}\right) = 1$$

$$f(c) = 2\Phi\left(\frac{\log(c) - 6.5}{\sqrt{13}}\right) + \Phi\left(\frac{\log(c) + 6.5}{\sqrt{13}}\right)$$



**We read off that
 $c = 0.5666$**

Example

- Misclassification probabilities:

$$\text{If } \pi_1: \Phi \left(\frac{(\log(0.5666) - 6.5)}{\sqrt{13}} \right) = 0.025$$

$$\text{If } \pi_2: 1 - \Phi \left(\frac{(\log(0.5666) + 6.5)}{\sqrt{13}} \right) = 0.05$$

Linear discriminator:

$$\begin{aligned} & \langle x, \mu_1 - \mu_2 \rangle_{\Sigma^{-1}} - \frac{1}{2} \|\mu_1\|_{\Sigma^{-1}}^2 + \frac{1}{2} \|\mu_2\|_{\Sigma^{-1}}^2 - \log(c) \\ & = 5x_1 - 2x_2 - 8.93, \end{aligned}$$

Separates at 0.

Example

Linear discriminator:

$$5x_1 - 2x_2 - 8.93,$$

Separates at 0.

Suppose that we observe $X_j = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$. The linear discriminator is

$$45 - 0 - 8.93 = 36.93 > 0.$$

Participant j is therefore classified to π_1 .

Discrimination With Unknown Parameters

Assume that individuals in π_1, π_2 have normally distributed data, $N_p(\mu_1, \Sigma), N_p(\mu_2, \Sigma)$, respectively, with Σ invertible. This time, with unknown parameters.

- We use the estimated values

$$\hat{\mu}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} X_{1i} = \bar{X}_1, \hat{\Sigma}_1 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (X_{1i} - \bar{X}_1)(X_{1i} - \bar{X}_1)^T$$

$$\hat{\mu}_2 = \frac{1}{n_2} \sum_{i=1}^{n_2} X_{2i} = \bar{X}_2, \hat{\Sigma}_2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (X_{2i} - \bar{X}_2)(X_{2i} - \bar{X}_2)^T$$

$$\hat{\Sigma} = \frac{1}{n_1 + n_2 - 2} \left((n_1 - 1)\hat{\Sigma}_1 + (n_2 - 1)\hat{\Sigma}_2 \right)$$

Discrimination With Unknown Parameters

- Estimated decision rule (Theorem 5.4):

$$x^T \hat{\Sigma}^{-1}(\hat{\mu}_1 - \hat{\mu}_2) - \frac{1}{2} \|\hat{\mu}_1\|_{\hat{\Sigma}^{-1}} + \frac{1}{2} \|\hat{\mu}_2\|_{\hat{\Sigma}^{-1}}$$

- For large sample sizes we can use the distribution of Z from Theorem 5.8 to estimate the separating hyperplane.

Discrimination With Unknown Parameters

- Theorem 5.8 utilizes the so-called *Mahalanobis distance*

$$\Delta_{\Sigma^{-1}}^2(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{\Sigma^{-1}}^2$$

We define the *empirical Mahalanobis distance*

$$D_{\hat{\Sigma}^{-1}}^2(\hat{\mu}_1, \hat{\mu}_2) = \|\hat{\mu}_1 - \hat{\mu}_2\|_{\hat{\Sigma}^{-1}}^2$$

D^2 is linked in distribution to Hotellings T^2 , in that

$$D = \frac{n_1 + n_2}{n_1 n_2} T^2$$

Testing the Optimal Discriminator

- The optimal discriminator is

$$\hat{\delta} = \hat{\Sigma}^{-1}(\hat{\mu}_1 - \hat{\mu}_2)$$

found from optimizing

$$\hat{g}(d) = \frac{((\hat{\mu}_1 - \hat{\mu}_2)^T d)^2}{\|d\|_{\hat{\Sigma}^{-1}}^2}$$

Optimized value:

$$\hat{g}(\hat{\delta}) = \frac{\left((\hat{\mu}_1 - \hat{\mu}_2)^T \hat{\Sigma}^{-1}(\hat{\mu}_1 - \hat{\mu}_2)\right)^2}{(\hat{\mu}_1 - \hat{\mu}_2)^T \hat{\Sigma}^{-1}(\hat{\mu}_1 - \hat{\mu}_2)} = (\hat{\mu}_1 - \hat{\mu}_2)^T \hat{\Sigma}^{-1}(\hat{\mu}_1 - \hat{\mu}_2) = D^2$$

Testing the Optimal Discriminator

Define for any d_0

$$D_0^2 = \hat{g}(d_0) = \frac{((\hat{\mu}_1 - \hat{\mu}_2)^T d_0)^2}{\|d_0\|_{\hat{\Sigma}^{-1}}^2}$$

||| Theorem 5.13

The statistic

$$Z = \frac{n_1 + n_2 - p - 1}{p - 1} \cdot \frac{n_1 n_2 (D^2 - D_0^2)}{(n_1 + n_2)(n_1 + n_2 - 2) + n_1 n_2 D_0^2}$$

may be used in testing the hypothesis that the linear projection determined by d_0 is the best discriminator against all alternatives. Z is $F(p - 1, n_1 + n_2 - p - 1)$ -distributed under the hypothesis and large values of Z are critical, i.e., the critical region is

$$C = \{x_{11}, \dots, x_{1n_1}, x_{21}, \dots, x_{2n_2} \mid z > F(p - 1, n_1 + n_2 - p - 1)_{1-\alpha}\}$$

if we use the significance level α . Here z is the observed value of Z .

Exercises

- 6.1: General introduction to Linear Discriminant Analysis
- 6.2: Test of means, test for further information, classification
- 6.3: Test of means, classification