

Solution to Exercise 7.5

The model is presented in section 6.3, but the solution only requires theory from chapter 1.

1. The communalities are (p. 404)

$$h_1^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{5}{9}$$

$$h_2^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{2}{9}$$

$$h_3^2 = \left(\frac{-1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9}$$

2. Since $\mathbf{X} = \mathbf{AF} + \mathbf{G}$ we according to remark 1.10 have

$$E(\mathbf{X}) = \mathbf{AE}(F) + E(\mathbf{G}) = \mathbf{A}\mathbf{0} + \mathbf{0} = \mathbf{0}$$

$$D(\mathbf{X}) = \Sigma = \mathbf{AD}(F)\mathbf{A}^T + D(\mathbf{G}) = \mathbf{AA}^T + \text{diag}(\frac{4}{9}, \frac{7}{9}, \frac{4}{9})$$

$$= \frac{1}{9} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} \frac{4}{9} & 0 & 0 \\ 0 & \frac{7}{9} & 0 \\ 0 & 0 & \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{2}{9} & \frac{1}{9} \\ 0 & \frac{1}{9} & \frac{5}{9} \end{bmatrix} + \begin{bmatrix} \frac{4}{9} & 0 & 0 \\ 0 & \frac{7}{9} & 0 \\ 0 & 0 & \frac{4}{9} \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 0 \\ 1/3 & 1 & 1/9 \\ 0 & 1/9 & 1 \end{bmatrix}$$

Furthermore \mathbf{X} is normally distributed so

$$\mathbf{X} \sim N(\mathbf{0}, \Sigma)$$

3.. We use the formula from theorem 1.42

$$D\left(\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\right) = \Sigma = \begin{bmatrix} 1 & [1/3 & 0] \\ [1/3] & \begin{bmatrix} 1 & 1/9 \\ 1/9 & 1 \end{bmatrix} \end{bmatrix}$$

Then

$$1 - \rho_{1|2,3}^2 = \frac{\det \Sigma}{1 \cdot \det \begin{bmatrix} 1 & 1/9 \\ 1/9 & 1 \end{bmatrix}} = \frac{71/81}{1 \cdot 80/81} = \frac{71}{80}$$

$$\rho_{1|2,3}^2 = \frac{9}{80} \cong 0.1125$$

We rearrange and obtain

$$D\left(\begin{bmatrix} X_2 \\ X_1 \\ X_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1/3 & 1/9 \\ 1/3 & 1 & 0 \\ 1/9 & 0 & 1 \end{bmatrix}$$

and

$$1 - \rho_{2|1,3}^2 = \frac{71/81}{1 \cdot 1} = \frac{71}{81}$$

$$\rho_{2|1,3}^2 = \frac{10}{81} \cong 0.1235$$

Similarly

$$1 - \rho_{3|1,2}^2 = \frac{71/81}{1 \cdot \det \begin{bmatrix} 1 & 1/3 \\ 1/3 & 1 \end{bmatrix}} = \frac{71/81}{8/9} = \frac{71}{72}$$

$$\rho_{3|1,2}^2 = \frac{1}{72} \cong 0.0139$$

4. Can be verified by direct computation.

5. We define $\Delta = \text{diag} \left(\frac{4}{9}, \frac{7}{9}, \frac{4}{9} \right) = D(\mathbf{G})$. Since we have

$$\begin{bmatrix} \mathbf{F} \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{A}\mathbf{F} + \mathbf{G} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$$

we get – following remark 1.10 – that the distribution is normal with

$$E \begin{bmatrix} \mathbf{F} \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$D \begin{bmatrix} \mathbf{F} \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \Delta \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}^T \\ \mathbf{0}^T & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A} & \Delta \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}^T \\ \mathbf{0}^T & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{A}\mathbf{A}^T + \Delta \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{A}^T \\ \mathbf{A} & \Sigma \end{bmatrix}$$

This is a somewhat cumbersome way to obtain the almost obvious result that

$$C(\mathbf{F}, \mathbf{AF} + \mathbf{G}) = C(\mathbf{F}, \mathbf{AF}) + C(\mathbf{F}, \mathbf{G}) = \mathbf{IA}^T + \mathbf{0} = \mathbf{A}^T$$

so that

$$D \begin{bmatrix} \mathbf{F} \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} D(\mathbf{F}) & C(\mathbf{F}, \mathbf{X}) \\ C(\mathbf{X}, \mathbf{F}) & D(\mathbf{X}) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{A}^T \\ \mathbf{A} & \Sigma \end{bmatrix}$$

6.-8. We have (theorem 1.27) that $(\mathbf{F}|\mathbf{X})$ is normally distributed with

$$E(\mathbf{F}|\mathbf{X} = \mathbf{x}) = \mathbf{0} + \mathbf{A}^T \Sigma^{-1} \mathbf{x}$$

$$D(\mathbf{F}|\mathbf{X} = \mathbf{x}) = \mathbf{I} - \mathbf{A}^T \Sigma^{-1} \mathbf{A}$$

Furthermore we have

$$\mathbf{A}^T \Sigma^{-1} = \frac{1}{71 \cdot 3} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 80 & -27 & 3 \\ -27 & 81 & -9 \\ 3 & -9 & 72 \end{bmatrix} = \frac{1}{213} \begin{bmatrix} 130 & 36 & -75 \\ 59 & 36 & 138 \end{bmatrix}$$

and the predictor is

$$E(\mathbf{F}|X = \mathbf{x}) \cong \begin{bmatrix} 0.6103 & 0.1690 & -0.3521 \\ 0.2770 & 0.1690 & 0.6479 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since

$$\mathbf{A}^T \boldsymbol{\Sigma}^{-1} \mathbf{A} = \frac{1}{3 \cdot 213} \begin{bmatrix} 130 & 36 & -75 \\ 59 & 36 & 138 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{639} \begin{bmatrix} 371 & 16 \\ 16 & 371 \end{bmatrix}$$

we get the dispersion matrix

$$D(\mathbf{F}|X = \mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{639} \begin{bmatrix} 371 & 16 \\ 16 & 371 \end{bmatrix} = \frac{1}{639} \begin{bmatrix} 268 & -16 \\ -16 & 268 \end{bmatrix} \cong \begin{bmatrix} 0.4194 & -0.0250 \\ -0.0250 & 0.4194 \end{bmatrix}$$

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