SIAM Review **Vol. 57, Issue 1 (March 2015)**

Book Reviews

BOOK REVIEWS

This issue marks a changing of the guard. After fifteen years as editor of book reviews, Bob O'Malley is stepping down. I am sure you will all agree with me that Bob has done a tremendous job, and I will admit that I feel some trepidation as I attempt to step into his shoes. At the same time I am excited for the opportunity to serve SIAM in this capacity. I have put together an editorial board to help me with the task; I thank Michele Benzi, Krešo Josić, Hinke Osinga, Nick Trefethen, and Thaleia Zariphopoulou for agreeing to work with me.

The book reviews in this issue were assembled by Bob; soon it will be my turn. The featured review by Andreas Albrecht takes a detailed look at Max Tegmark's quest for the ultimate nature of physical reality. He recommends the book and even suggests that it would be of value to readers who are not technically oriented. He declares himself unready to jump on Tegmark's bandwagon but finds that the book was well worth reading nevertheless. Each physicist defines boundaries for himself (or herself) and walks a line between being too conservative and being a crackpot. Albrecht writes, "Reading this book has caused me to stop and reflect on my own boundaries and the ideas that create structure for my own research."

As usual we have reviews on a wide variety of topics. The book *Introduction to Global Optimization Exploiting Space-Filling Curves* by Sergeyev, Strongin, and Lera, reviewed by Sergiy Butenko, particularly caught my eye. I would never have thought that space-filling curves could be practical tools for global optimization. I can see that I am going to learn a lot in this new job.

The issue concludes with two reviews by Bob O'Malley himself, one on a collection of recollections about the brilliant ex-mathematician Alexander Grothendieck, the other on a different sort of collection: views of mathematicians on the subject of creativity.

David S. Watkins
Section Editor
siam.book.review@gmail.com

SIAM REVIEW
Vol. 57, No. 1, pp. 153–164

Book Reviews

Edited by David S. Watkins

Featured Review: Our Mathematical Universe: My Quest for the Ultimate Nature of Reality. *By Max Tegmark.* Alfred A. Knopf, New York, 2014. $30.00. 432 pp., hardcover. ISBN 978-0-307-59980-3.

"What is reality?" "The proton mass is 938.3 MeV." *Our Mathematical Universe* is a remarkable work in which practical facts and philosophical questions exist (perhaps even happily) side by side.

Our Mathematical Universe is many things. For one, it is a lively and passionate telling of the personal story of a prominent cosmologist. Given the astonishing transformation the field of cosmology (the study of the history of the cosmos) has seen during Tegmark's career, his significant contributions to these developments, and his energetic and open way of writing (with a fun dose of the earlier history of the field thrown in), this book is highly worthwhile and will be loved by many. But Tegmark seeks to accomplish much more. A great deal of this work is devoted to exploring the nature of physics itself, and especially its relationship to mathematics. In this sphere, Tegmark transcends mere exploration and engages in the strong advocacy of a particular set of views.

In many circles, physics is seen as the most rigorous science. While some fields slip quickly into hand-waving arguments, physics has "fundamental equations," with "fundamental constants," and many of the fundamental insights they represent appear to form the bedrock of the rest of science, engineering, and life itself. When other fields of research appear less rigorous, this often can be due to the enormous difficulty inherent in computing the behavior of the large complex systems that describe molecules, materials, living things, and other parts of our world based on the quantitative description of their component parts in terms of fundamental physics.

However, some areas of physics are devoted to looking beyond the established fundamental equations in the hopes of finding something better: more fundamental, more beautiful, or perhaps more complete. Sometimes this search is driven simply by curiosity, but, as is often the case in the field of cosmology, the tension that arises when our fundamental picture seems inadequate to address the questions we care about can drive the search for an enhanced understanding and reveal a very different face of physics. While other fields of science that emerge in one way or another from the underlying physics can at least be guided by their emergence, the questions of what steps and vision might guide us to a deeper understanding of physics itself is extremely subjective. Ideas of what forms a good path forward are diverse, and sometimes the field can even be split about what constitutes progress. When this aspect of physics is viewed up close, our reputation for solidity and rigor can seem difficult to reconcile with the chaotic processes by which we search for a deeper understanding.

Still, our reputation for rigor does not come from nowhere, and this reputation is highly valued by even the most adventurous explorer on the path to deeper insights.

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *Market St.,* 6^{th} *Floor, Philadelphia, PA* 19104-2688.

Our goal, after all, is to ultimately find laws of physics that are even more solid and rigorous than those we have discovered so far. Most who venture into these uncharted areas of physics reconcile the chaotic nature of the process with the wish for rigor by setting boundaries of one sort or another. Of course, there is often disagreement about where these boundaries should lie. Some feel that quantum mechanics must remain part of any deeper picture, while space and time itself could well become something that only emerges from a very different fundamental picture (perhaps loops of string) when the physical world is viewed from a limited everyday perspective. Others take the opposite view, holding onto spacetime "all the way down," but readily giving up quantum mechanics for a vision of a deeper theory. Then, generally, everyone has some particular tolerance of how wild or poorly-formed they will let the conversation become before stepping away from it, declaring the conversation too ill-defined or unlikely to get anywhere. The debate about which boundaries are appropriate is itself often passionate and can add to the impression of chaos at the frontier. Still, these boundaries do often have a positive role in keeping research moving in a positive direction.

Among all the colleagues I have known in my career, two stand out as being least limited by such boundaries: John Wheeler (who Tegmark and I and many others regard as a heroic figure) and Max Tegmark. Wheeler (inventor of the term "black hole" and known for his major contributions to gravitational and nuclear physics) was not uniformly this way, and the story of his discomfort in the 1950s with his student Hugh Everett's (then) revolutionary ideas about quantum physics is a fascinating part of the history of physics. (The story of Wheeler and Everett is not specifically discussed in this book, although it is hinted at, but it is covered in some of the recommended reading.) By the time I started my first postdoctoral position, with an office across the hall from Wheeler's (at the University of Texas at Austin), he was expressing bold and difficult-to-pin-down ideas about the connections between physics and information. I admit I was a bit wary of falling under his spell lest I became so distracted by vaguely posed ideas such as "it from bit" that I would never write another physics paper of any substance. But by the time I met Tegmark I had relaxed my boundaries considerably, and the numerous adventurous discussions we have had about all aspects of physics have been a very special part of my research life.

Generally, to a given physicist, colleagues with boundaries much more open than their own seem at least a little bit lost and unproductive, while colleagues with much tighter boundaries seem unreasonably conservative and narrow. An interesting thread that runs throughout this book tracks Tegmark's path as he navigates his own boundaries and those of his colleagues. An email quoted verbatim from an (unnamed) senior colleague and journal editor warns of dire consequences for Tegmark's future if he does not separate his "crackpot" ideas from his "serious research" (and perhaps abandon the former entirely). Tegmark also reflects on the experiences of others (including Everett and Einstein) whose best work pushed beyond the comfort zones of most of their colleagues. Happily, things have worked out just fine for Tegmark, and his work on both "serious" and "crackpot" projects has continued to thrive. To be fair, there are many others who perhaps should have more carefully heeded such warnings, and it is not yet clear whether Tegmark's work that so concerned the editor will eventually be understood as a significant contribution.

The ultimate focus of this book is Tegmark's "Mathematical Universe Hypothesis" (MUH) and the "Level IV Multiverse" that he argues follows from the MUH. The MUH itself is simply stated as follows: "Our external physical reality is a mathematical structure." (For Tegmark, using the word "is" as opposed to "is represented by" in this definition is crucial.) From there Tegmark argues for a kind of "mathematical democracy," where any mathematical structure is just as real as any other (even if we only experience one of them as "our reality").

Tegmark takes many gradual steps to reach these grand ideas. He carefully illustrates the different ways mathematics has often led the way in the development of physics, from the Everett many worlds, to the Friedmann cosmology, to inflationary cosmology (and the idea of "eternal inflation") and more recently the string landscape. He takes his time to explain each of these developments (and many others) in a relaxed and accessible manner. He works earnestly to define his terms, including "reality" and "mathematical structure" (and, of course, defines the Level I, II, and III Multiverses). In making his larger leaps, Tegmark takes to heart the words of Nobel Prize–winning physicist Steven Weinberg, who noted that often in physics "our mistake is not that we take our theories too seriously, but that we do not take them seriously enough."

To illustrate, from the point of view of the Level IV Multiverse there are presumably all possible realities: Some with spacetime "all the way down" but with emergent quantum physics, and some with the reverse. Some where neither spacetime nor quantum physics is fundamental, and others where both are. There is no absolute right answer as to which of these is true: They all are in the multiverse, although we still can ask which of them is true for "our particular reality."

Although mathematics occupies a central role in Tegmark's story, his narrative takes the informal style of a particularly easy-going physicist. For example, the three dimensions of our space are presented by asking, "how many pencils can you arrange so they are all perpendicular to each other?" (Presumably these pencils have bits of shavings clinging to them from the sharpener and well-chewed erasers.) The book is thoughtfully and clearly organized, with the thirteen chapters grouped into three large sections covering cosmology, elementary particles (especially quantum physics), and the Level IV Multiverse. Each chapter ends with a conveniently presented outline of the main points covered within, and the preface offers a diagram suggesting how readers with different backgrounds might approach reading the book.

Much of the first two sections of the book covers established physics (though often carefully designed to ultimately drive home more bold and controversial points). The reader will learn the basic distances and sizes corresponding to planets, galaxies, and so on, culminating in the question, "what is space?" Classic topics such as the cosmic microwave background and galaxy clustering and basic facts from atomic, nuclear, and particle physics are presented. Even the established topics are treated in a very relaxed and personal manner, such as Tegmark's reflections on the experience of doing research (including "the joys of being scooped" and that sign mistake that almost ruined a talk). Also, Tegmark shares some lovely personal impressions of some of our colleagues. Little attempt is made to provide a textbook-level systematic treatment, which I imagine will make this book more accessible and fun for many readers (an extensive "further reading" list is also presented, with more emphasis on comprehensiveness than on guiding the reader to the next few books to read on these subjects).

As the narrative moves toward the main, and more philosophical, points, the reading becomes a more challenging experience. I think any reader will, as I did, find many questions that are not addressed or at least not addressed in a satisfactory manner; this is perhaps not surprising for topics like "what is time?" and "what is consciousness?" But Tegmark just keeps on making the case for his passionately held vision. As I finish writing this review while at a conference in Germany, the latter parts of the book conjure up the image of being on a road trip with Tegmark, blasting down the autobahn at 150 mph, with Tegmark fully confident he knows where he is going, and that you will like it when you get there. Meanwhile, as the passenger, I keep wondering why he passed by this or that interesting place without stopping for a closer look, and how he is going to manage to avoid various obstacles that loom on the horizon.

One of my big questions has been how Tegmark's ways of thinking might change the way I go about my physics research, should I be brought on board. It is hard to see how anything would change. He suggests that if we find physical phenomena that do not allow a mathematical description, we will have falsified his MUH, but how should we know whether such a situation is just a temporary setback due to our own lack of creativity? In any case, I am still very curious about how physics works in "my particular reality," and still hopeful that my colleagues and I can extract insights from how physics has developed so far to guide us to a deeper understanding. I'm not sure how saying that this is just one of many possible realities will significantly change this. As Tegmark acknowledges, the idea of drawing our reality from an ensemble and making statistical predictions about what we will discover is not sufficiently well developed to yield clear results. Maybe if I were sold on Tegmark's approach I would invest some of my time in developing a technical understanding of the Level IV Multiverse and the search for ourselves within it, but here I run up against clashes with my own research (or boundaries, in the language I used above), in which I have come to regard probabilities and ensembles in a manner incompatible with Tegmark's program.

In the end, though, this book is the better for not convincing me (and no doubt many other readers) to jump on his bandwagon. As Tegmark readily admits, his ideas are not advanced enough for people to take sides. Instead, I have gained something much more. Reading this book has caused me to stop and reflect on my own boundaries and the ideas that create structure for my own research. Now I want to go back out on the autobahn in my own car, so I can stop at some of the places Tegmark whizzed by and look as long as I like. And perhaps I will end up on a different highway altogether. I believe every reader will finish this book with new questions in mind, questions that may well change how they approach their own work and how they see the world.

I should add that although I address this review to SIAM members who probably all have some kind of technical profession, I feel this book would be readily accessible to a much wider range of readers, in fact to anyone curious about how science works and where it is headed. For good measure, Tegmark throws in a cheerful dose of informality. He pauses here and there for entertaining notions such as the dominance of napkins (instead of envelopes) used for performing "back of the envelope calculations" or to offer a list of cool questions whose answer is "42." Also, his frank and warmhearted account of the ups and downs of starting a career in physics will resonate with and offer support to individuals considering or starting out on such a path themselves.

The relationship between mathematics and the physical world is a truly wonderful thing. It is something most definitely worthy of a pause to celebrate and reflect upon. And those of us whose work exploits this relationship can especially hope that insights into this relationship can guide and enhance our work. Tegmark is to be congratulated for boldly and very accessibly sharing his radical and very thoughtful reflections with all of us. The reader is treated to a clear, personal, and passionate account of exciting advances in physics, especially in cosmology, over the last several decades, and of the joys of being part of these developments. The reader is also exposed to bold and controversial ideas about the relationships between physics, mathematics, and reality. The subject matter is way too subjective, and the book leaves too many questions unanswered, to expect that a majority of readers will become convinced of all of Tegmark's bold positions, but I expect most readers will get a great deal out of these adventures, as I did, because of the way he stimulated and challenged my own thinking on these deep topics. While I feel lucky that I don't need to think about all of this to write interesting physics papers, I have emerged from reading this book more aware and more curious about how my beliefs about these deeper questions shape my research and my perceptions about the world around me.

ANDREAS ALBRECHT
University of California, Davis

Stability of Functional Equations in Random Normed Spaces. *By Yeol Je Cho, Themistocles M. Rassias, and Reza Saadati.* Springer, New York, 2013. $109.00. xx+246 pp., hardcover. ISBN 978-1-4614-8476-9.

Several books on stability of functional equations have been published in the last few years, but this one by Yeol Je Cho, Themistocles M. Rassias, and Reza Saadati is quite exceptional, because it is the first to consider the issue of such stability in random normed spaces. Moreover, it focuses only on random spaces.

The book is Volume 86 of the series *Springer Optimization and its Applications*, but there are actually no comments in the book on connections between the stability of functional equations and the optimization issues. Certainly, some connections exist and experienced mathematicians notice them easily, but for those less experienced it can be quite difficult. The preface, which introduces the main subjects of the book, gives only a brief account of the history of functional equations and their stability, with some references to suitable monographs and surveys.

At present, it is commonly held that investigations of the stability of functional equations began with a question raised by S.M. Ulam in 1940 and a partial answer to it published by D.H. Hyers [2] in 1941. We quote that question below, but we are not actually sure what motivated Ulam to pose it, though some indications can be deduced from the remarks in [4]. For instance, maybe the motivation was the observation that a natural phenomenon is often subject to disturbances, which means that it should be described by inequalities rather than by equations. Therefore, it is important to know when, why, and to what extent we can replace those inequalities by suitable (or corresponding) equations.

The question of Ulam has been quoted in numerous papers on the subject, in various forms, but mainly as follows (cf. [2]):

Let G_1 be a group and (G_2, d) a metric group. Given $\epsilon > 0$, does there exist $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$d(f(xy), f(x)f(y)) < \delta, \qquad x, y \in G_1,$$

then a homomorphism $T : G_1 \to G_2$ exists with

$$d(f(x), T(x)) < \epsilon, \qquad x \in G_1?$$

Below we present the answer of Hyers [2].

Let X and Y be Banach spaces and $\varepsilon > 0$. Then for every $g : X \to Y$ with

(1) $$\|g(x + y) - g(x) - g(y)\| \le \epsilon, \qquad x, y \in X,$$

there exists a unique function $f : X \to Y$ such that

$$\|g(x) - f(x)\| \le \epsilon, \qquad x \in X,$$

and

(2) $$f(x + y) = f(x) + f(y), \qquad x, y \in X.$$

The latter result says that *the Cauchy functional equation* (2) *is Hyers–Ulam stable (or has the Hyers–Ulam stability) in the class of functions mapping X into Y* (i.e., in Y^X).

Let us now mention that we are aware of an earlier result of this type, due to

Gy. Pólya and G. Szegö, published in 1925, which reads as follows:

For every real sequence $(a_n)_{n\in\mathbb{N}}$ *with* $\sup_{n,m\in\mathbb{N}} |a_{n+m} - a_n - a_m| \le 1$, *there is a real number* ω *such that* $\sup_{n\in\mathbb{N}} |a_n - \omega n| \le 1$. *Moreover,* $\omega = \lim_{n\to\infty} a_n/n$.

Based on those results we could state a (very rough) formal general definition: the equation $\mathbb{T}\phi = \mathbb{S}\phi$ is stable provided every function ψ that solves the equation approximately (in some sense, say $d_1(\mathbb{S}\psi, \mathbb{T}\psi) \le \epsilon$, where d_1 measures a distance in a suitable family of functions and ϵ is a positive real number) is close in some sense to an exact solution ϕ of the equation (e.g., $d_2(\psi, \phi) \le \kappa\epsilon$ with some real constant $\kappa > 0$, where again d_2 measures a distance in a suitable family of functions). Clearly, this approach can be applied not only to functional equations (in one or many variables), but also to difference, differential, and integral equations and other mathematical objects (see [1]).

There have been various attempts to generalize that definition, which mainly take two directions. In the first, the constant ϵ is replaced by a function $\epsilon(x_1, \dots, x_n)$, with n a fixed positive integer. In the second, new ways of measuring the distances between $\mathbb{S}\psi$ and $\mathbb{T}\psi$ and between ψ and ϕ are applied. In the outcomes concerning the Cauchy functional equation considered in the book, the norm in (1) is replaced by various random norms; clearly, in such cases the function $\epsilon(x, y)$ must take values in the space of all distribution functions, i.e., in the space of all functions $F : \mathbb{R}\cup\{-\infty, +\infty\} \to [0, 1]$, which are left-continuous and nondecreasing on $\mathbb{R}$ and such that $F(0) = 0$, $F(+\infty) = 1$. Similar approaches have been applied to other functional equations.

The authors present various, mainly new results obtained under spatial t-norm and arbitrary t-norms. Next, they describe outcomes obtained in non-Archimedean random spaces and in Banach random algebras. A more detailed guide to the topics covered in the monograph is given in the contents.

Readers should be careful when applying or citing the results presented in the book, because it seems that there are several gaps. Maybe this is not the place to list them all, but, for instance, the assumption of completeness of the RN space (Y, μ, T_M) seems to be missing in the theorems contained in Chapter 3. Certainly, this assumption has been used in the proofs, but it is not specified clearly in the theorems.

Below we present several examples of functional equations that are considered in the book (we hope that this might be helpful information for potential readers of the book):

$$f(x+y) = f(x) + f(y) \quad \text{(Cauchy additive)},$$

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{(quadratic)},$$

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \quad \text{(cubic)},$$

$$3f(x+3y) + f(3x-y) = 15f(x+y) + 15f(x-y) + 80f(x) \quad \text{(cubic)},$$

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y) \quad \text{(quartic)},$$

$$\sum_{i=1}^{m} f\Big(mx_i + \sum_{j=1, j\neq i}^{m} x_j\Big) + f\Big(\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} mx_i\Big) \quad \text{(additive in } m\text{-variables)},$$

$$11f(x+2y) + 11f(x-2y) = 44f(x+y) + 44f(x-y) + 12f(3y) - 48f(2y) + 60f(y) - 66f(x) \quad \text{(ACQ} - \text{additive-cubic-quartic)},$$

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y)$$
$$\text{(AQCQ} - \text{additive-quadratic-cubic-quartic)}.$$

There are also results for several other, even more complicated functional equations.

The names given to the equations in the book are in general derived from the names of monomials and polynomials that are examples of their solutions. For example, the quadratic real function $f(x) \equiv x^2$ is a solution to the quadratic functional equation and the function $f(x) \equiv x + x^3 + x^4$ fulfills the ACQ-equation. However, quite often it is known (and is stated in the book) that all solutions to some of those equations must be of similar forms, under suitable assumptions. For instance, it is well known that every real solution f (without any assumption on its regularity) to the quadratic equation must be of the form $f(x) \equiv L(x,x)$ with some biadditive function L. We point out that the authors provide information on solutions of only some of the equations considered in the book. Some equations are presented without comment on their general solutions, and only the issue of their stability is considered.

Chapter 8, the last but one in the book, is devoted to the stability of random homomorphisms and random derivations in random Banach algebras, $*$-algebras, C^*-algebras (also non-Archimedean), and Lie C^*-algebras. The last chapter contains a few results on stability of several functional equations, systems of them, and functional inequalities in latticetic random φ-normed spaces (the definition of those spaces given there is not very precise). The authors also claim there that they provide stability outcomes in homogeneous probabilistic modular spaces, but I did not find them here, not even a definition of such spaces.

What is missing (for me) in the book? First of all there is no information on nor examples of various possible definitions of stability for functional equations; we refer the interested reader to [1, 3]. There are no comments, exercises, or remarks to help explain and compare the presented outcomes. The content is presented in a dry way, as a list of theorems. Fortunately, there are several examples which should be really helpful for less experienced readers.

Finally, I would like to formulate some general opinions. Despite some critical remarks, this is a good, well-composed, and self-contained monograph, written in mainly decent and clear English (though with numerous mistakes and missing words). The volume is focused on a reasonably narrow area with a very limited range of equations considered. The proofs are transparent.

I find the book interesting and very useful; it seems to be quite a satisfactory introduction to the particular field of research and it gives a decent account of some of the results that have been obtained so far, presents numerous new ones, and provides suitable references.

The book should interest any professional mathematician whose research is connected with functional equations, especially their stability in random spaces; I also can recommend it for graduate students interested in the subject. It could serve as a complete and independent introduction to the field of stability of functional equations in random spaces and as an excellent source of references for further study.

REFERENCES

[1] N. Brillouët-Belluot, J. Brzdęk, and K. Ciepliński, *On some recent developments in Ulam's type stability*, Abstr. Appl. Anal., 2012, article 716936.

[2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, 27 (1941), pp. 222–224.

[3] Z. Moszner, *On the stability of functional equations*, Aequationes Math., 77 (2009), pp. 33–88.

[4] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, London, 1960.

Janusz Brzdęk
Pedagogical University of Cracow

Introduction to Global Optimization Exploiting Space-Filling Curves. *By Yaroslav D. Sergeyev, Roman G. Strongin, and Daniela Lera.* Springer, New York, 2013, $49.99. x+125 pp., softcover. ISBN 978-1-4614-8042-6.

The dynamically evolving field of global optimization deals with some of the most challenging mathematical optimization problems, where one seeks to characterize and compute global optima of multiextremal functions over possibly nonconvex domains. Such problems are very common in scientific, engineering, and industrial applications. Despite their enormous complexity, significant progress has been made in all aspects of global optimization in the last two decades, as evidenced by numerous research monographs, textbooks, and surveys devoted to the topic; see, e.g., [2, 6, 1, 3, 5].

This work, which is a part of the *Springer Briefs in Optimization* series, provides a concise introduction to global optimization methods that take advantage of space-filling curves. The main idea behind these methods is quite natural and appealing: Use space-filling curves to convert a multidimensional optimization problem into an equivalent single-dimensional one, and then apply a univariate method to solve the resulting global optimization problem in one dimension. However, bringing this mathematically elegant idea to life is not an easy task, which could explain the relatively low level of activity in this research direction. The authors of this book are responsible for most of the notable contributions exploiting space-filling curves in a global optimization context. Strongin and Sergeyev published a major (over 700 pages) monograph [6] describing state-of-the-art developments (as of 2000) in detail. The book under review can be thought of as a gentle introduction to [6] that takes into account the advances made since 2000.

To somewhat "ease the pain" of dealing with a general global optimization problem, the authors restrict their attention to the problem of minimizing a Lipschitz-continuous function over an N-dimensional unit hypercube. The Lipschitz constant is not given and no smoothness assumptions are made, that is, the problem of interest is essentially a zero-order black-box optimization problem. With these assumptions, the problem is still widely applicable and extremely hard to solve. In fact, Nesterov [4] used the so-called "resisting strategy" to show the existence of a problem instance that requires at least $\lfloor L/2\epsilon \rfloor^N$ function evaluations in order to find an ϵ-approximate solution, where L is the Lipschitz constant (for the infinity norm). This observation prompted him to argue that, in general, optimization problems are unsolvable and one should focus on convex problems. Luckily, many practical problems can be solved much faster than prescribed by this pessimistic lower complexity bound, and the authors demonstrate how the Lipschitz assumption can be effectively used for this purpose.

The book consists of five chapters. The first chapter introduces the notion of space-filling curves and states the global optimization problem of interest mathematically. Having quoted Felix Klein ("Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions") in the epigraph, the authors proceed to define a space-filling (Peano) curve as "a curve that passes through every point of a multidimensional region." Remarkably, the discussion of the history behind this concept is complemented by figures containing the full versions of the original articles by Peano (1890, in French) and Hilbert (1891, in German). The first of these papers introduces space-filling curves for the first time, whereas the second one describes a geometric approach to constructing such curves.

The three chapters that follow constitute the core of this work. Chapter 2 describes algorithms and software designed to approximate Peano curves. In addition to numerous figures illustrating the ideas, a C code implementing the proposed algorithms is provided. Chapters 3 and 4 focus on development and acceleration, respectively, of global optimization algorithms utilizing the one-dimensional representation of multidimensional problems. The proposed methods take advantage of geometric properties of the univariate function (such as the Hölder condition) resulting from the Lipschitz assumption imposed on its multivariate counterpart, and the acceleration methods balance the local and global Lipschitz constant estimates in order to speed up the convergence. Proofs of convergence as well as results of numerical experiments are given. Finally, Chapter 5 provides a brief two-page conclusion, mainly suggesting pointers for further reading.

In summary, this book is an excellent introduction to global optimization methods based on space-filling curves. The very useful content and engaging style of presentation make it a pleasant, interesting, and valuable read, and I enthusiastically recommend it to anyone interested in optimization.

REFERENCES

[1] C. A. Floudas and C. E. Gounaris, *A review of recent advances in global optimization*, J. Global Optim., 45 (2009), pp. 3–38.

[2] R. Horst, P. M. Pardalos, and N. V. Thoai, *Introduction to Global Optimization*, 2nd ed., Kluwer Academic, Dordrecht, The Netherlands, 2000.

[3] M. Locatelli and F. Schoen, *Global Optimization: Theory, Algorithms, and Applications*, SIAM, Philadelphia, PA, 2013.

[4] Yu. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, Kluwer Academic, Dordrecht, The Netherlands, 2003.

[5] L. M. Rios and N. V. Sahinidis, *Derivative-free optimization: A review of algorithms and comparison of software implementations*, J. Global Optim., 56 (2013), pp. 1247–1293.

[6] R. G. Strongin and Ya. D. Sergeyev, *Global Optimization with Non-convex Constraints: Sequential and Parallel Algorithms*, Kluwer Academic, Dordrecht, The Netherlands, 2000.

Sergiy Butenko
Texas A&M University

Introduction to Finite and Spectral Element Methods Using MATLAB. Second Edition. *By C. Pozrikidis.* CRC Press, Boca Raton, FL, 2014. $132.95. xxvi+804 pp., hardcover. ISBN 978-1-4822-0915-0.

Nine years have passed since the first edition of this introduction, but the objective remains to provide a self-contained presentation of the finite and spectral element methods. The discussion explains the basic ideas behind these two popular methods and the details of a practical implementation. The author provides a library of MATLAB functions and the codes are described within the text. The revision for this second version is extensive—the number of pages has increased by more than 20%—and includes clarifications, detailed explanations, and new solved problems.

The book covers a large amount of material without requiring many prerequisites. After working with it as a self-study or as a textbook, the reader new to the field will be able to solve, with the provided software, some practical problems from beginning to end (generate the mesh in $\mathbb{R}^2$ or $\mathbb{R}^3$, assemble the system matrix, incorporate the boundary conditions, and solve the algebraic system of equations). The author chooses to ignore error analysis and convergence studies. One downside to this approach is that a novice reader could be challenged to verify their own implementation of a problem not covered in the textbook.

Even though the finite and spectral element methods have been covered in many other books, this particular combination of a text with its MATLAB library constitutes a valuable contribution. This introductory text should be accessible to a wide range of readers, but their understanding of why these methods work will remain limited.

Ulrich Hetmaniuk
University of Washington

Dynamics of Cancer: Mathematical Foundations of Oncology. *By Dominik Wodarz and Natalia L. Komorova.* World Scientific, Singapore, 2014. $148.00. xviii+514 pp., hardcover. ISBN 978-981-4566-36-0.

Mathematical and computational modeling approaches have been applied to every aspect of tumor growth from mutation acquisition and tumorigenesis to metastasis and therapeutic response. At universities across the country, courses are now being developed to teach interested students how to formulate, analyze, and simulate mathematical models of the complex dynamics that underlie tumor growth and treatment.

Dynamics of Cancer: Mathematical Foundations of Oncology is both timely and unique. To my knowledge there are no other books quite like this one currently on the market. As a textbook, it provides an instructional tool for teaching the mathematical underpinnings of tumor initiation, progression, and treatment. Existing mathematical models in oncology employ a number of mathematical methodologies including deterministic and stochastic approaches, discrete and/or hybrid models, and cell-based formulations. This text covers most of these mathematical strategies in a way that is accessible to students.

A remarkable feature of this book is that it truly written for two audiences: those with a solid mathematical background who expect to find detailed mathematical derivations, and those with less formal mathematical training who prefer to see a general outline of how theory has contributed to cancer research. In writing to these two distinct types of scholars, there is no lapse in the flow of the reading because the more advanced mathematics is self-contained within a grey background that can be skipped without loss of information or content. Many of the models presented are taken from the current literature, making the references provided another important resource for those interested in more detail or further exploration.

The material in this book encompasses a broad sampling of topics ranging from cancer genetics to immune interactions and treatment. A nice balance is maintained among mathematical exercises, computational exercises, and well-thought-out research projects that students will appreciate. This book could be used as a required text for an advanced undergraduate or entry-level graduate course in mathematical oncology. The authors even provide instructional tips and sample syllabi for using the text to teach a course aimed at students with either strong mathematical or strong biological backgrounds.

Natalia Komarova and Dominik Wodarz are world-renowned leaders in the field of mathematical oncology. I expect that this book will become a staple in the teaching of mathematical modeling of cancer dynamics.

TRACHETTE L. JACKSON
University of Michigan

Derivative Securities and Difference Methods. Second Edition. *By Y.-I. Zhu, X. Wu, I.-L. Chern, and Z. Sun*. Springer, New York, 2013. $149.00. xxii+647 pp., hardcover. ISBN 978-1-4614-7305-3.

Zhu, Wu, Chern, and Sun have recently published the second edition of their book *Derivative Securities and Difference Methods*, another in a growing number of books on numerical pricing of derivative financial instruments. As indicated by the title, it is mainly devoted to finite difference methods, and it is intended for researchers as well as graduate students. It is the most complete and useful book on the subject I have seen.

As one would expect, the book begins with several chapters containing basic facts about the most common derivative securities. Standard European and American options are discussed, as well as a large number of exotics including barrier, lookback, and discretely sampled Asian options. The authors explain precisely what the options do and how they are used in practice, although they do not delve into specific mechanics of option markets and other factors affecting prices such as commissions and taxation which can easily be found elsewhere. The differential equations the option prices satisfy are carefully derived, as are initial and boundary conditions. In contrast to most other books, this one provides proofs of uniqueness and shows how to convert pricing problems originally defined on infinite domains into problems defined on finite domains without requiring any artificial boundary conditions. The emphasis is on differential equations, and although stochastic processes and Itô's lemma and its generalizations are discussed, relatively little of the presentation involves probability. (Martingales are only mentioned once, in passing.) The explanations are clear and concise and provide the reader with sufficient background to analyze many of the other derivative securities that are constantly being developed. When they are available, analytic pricing formulas are also provided. The book also discusses many of the models and interest rate derivative securities that are amenable to pricing by finite difference methods, so the Vasicek, Cox–Ingersoll–Ross, Ho–Lee and Hull–White models are discussed, but the HJM (Heath–Jarrow–Morton) model is not. A three factor interest rate model is also provided.

The rest of the book is devoted to numerical methods for pricing the derivative securities discussed in the first part. Not only does the book contain the basic facts about solving parabolic differential equations, including convergence and stability, it also gives a basic introduction to interpolation, the solution of linear equations, quadrature, and pseudospectral methods. The authors also discuss inverse problems, which are rarely treated adequately in other financial mathematics texts. This part of the book is very heavy on proofs, and full details are provided. Many of the differences between this book and most of the others on the same topics are due to the fact that this one is clearly written by people in a math department. Among other things this means that when they say they have proved something, they actually have a proof. A large number of other books have vague derivations or just present algorithms. Although this leads to easier and more enjoyable reading, it doesn't provide the reader with any understanding of how to apply the methods to the more complicated problems that always arise. The authors do not skirt the complicating issues that most of the standard books avoid. In particular, they talk about the effect of discontinuities in the derivatives of terminal values of option prices, and how to deal with them. That is, they tell the reader how to use a "singularity separating method" and provide results of calculations where this method is used and where it isn't.

There are many other things I especially like about this book. What practitioners should particularly appreciate are the very detailed numerical experiments, which often include nine digits of results as well as execution times. Many derivatives are also priced using several different methods, so clear comparisons can be seen.

In addition, the book contains an excellent set of exercises and a very good set of references. Although it doesn't contain code or pseudocode as many elementary texts do, it does provide adequate information so the algorithms can easily be implemented by anyone with a reasonable numerical analysis background.

However, though the writing is clear and precise, if one were to compare this book with, say, one of Wilmott's, it's obvious which would be more pleasant to read. Proofs sometimes go on for several pages of correct, but occasionally mind-numbing manipulations. They may be necessary (this is, after all) numerical analysis, but they don't make for the most exciting reading

Also, one should note that although the book is more than 600 pages long, there are still, of course, many topics of current interest which are not covered. For example, there is no reference to jump diffusion models, and no specific stochastic volatility models are discussed. For obvious reasons, the important topics in financial mathematics and option pricing in particular change rather quickly, and one cannot expect any one text to cover them all.

Finally, it should be mentioned that this type of book cannot be viewed as a guide to everything one needs to know about derivative securities. In the last few years people have become aware of how investment banks and hedge funds make a lot of their money, and unfortunately it doesn't always involve long, delicate convergence proofs for differential equations. During the weekend I had originally decided to write my review of this excellent book, the Michael Lewis blockbuster *Flash Boys* was released. I commute into New York on a train with a large number of people who work at financial institutions (it is easy to distinguish them from the lawyers, and even easier to distinguish them from the people who work at nonprofit organizations). Not surprisingly, a fair number were reading Lewis' book. A week later several were reading Scott Patterson's *Dark Pools*, another fun book which gives an exciting account of how high-frequency trading practices affect option as well as futures, fixed income, and foreign currency markets (although to a lesser degree). I rarely find people reading books on differential equations, although it has happened.

Nevertheless, *Derivative Securities and Difference Methods* is a really good book that anyone studying or working in this field should own.

ANITA MAYO
Baruch College

Alexandre Grothendieck: A Mathematical Portrait. *Edited by Leila Schneps.* International Press of Boston, Somerville, MA, 2014. $85.00. viii+315 pp., hardcover, ISBN 978-1-57146-282-4.

Alexandre Grothendieck (1928–2014) had a tremendous impact on algebraic geometry, but essentially dropped out of mathematics around 1970, and his current whereabouts remain unknown. The current book is essentially the outcome of a conference held in Peyresq in 2008. The prominent expert chapter authors are Joe Diestal, Max Karoubi, Michel Raynaud, Steven

Kleiman, David Mumford, Carlos Simpson, Jacob Murre, Robin Hartshorne, Luc Illusie, Leila Schneps, Frans Oort, Pierre Cartier, and Yuri Manin. Aside from two authors, born in the 1960s, all the writers knew Grothendieck in his mathematical prime. The article by Cartier is the most biographical. All present their stories in an understandable, fairly nontechnical manner, providing a broad international outlook, emphasizing Grothendieck's interactions with Bourbaki and other French and American mathematicians of the time. After writing 10,000 pages, Grothendieck abandoned mathematics, leaving others much to contemplate his legacy!

This book will be valuable to all wanting to understand what it was all about.

ROBERT E. O'MALLEY, JR.
University of Washington

Mathematicians on Creativity. *Edited by P. Borwein, P. Liljedahl, and H. Zhai.* Mathematical Association of America, Washington, D.C., 2014. $30.00. xviii+199 pp., softcover, ISBN 978-0-88385-574-4.

This publication is a result of a survey of prominent mathematicians about the process of mathematical discovery conducted early in this century, supplemented by material taken from sources such as *Mathematical People* and popular biographies and articles. Survey responses are published from twenty-five or so mathematicians, including E. Bombieri, D. Donoho, G. Faltings, C. Fefferman, P. Huber, D. Kleitman, J. Marsden, G. Papanicolaou, J. Taylor, and G. Wahba. Predictably, they don't all quite agree. For example, C. Peskin wrote, "I do my best work while asleep," and A. Weinstein wrote, "Most of what I 'discover' in my dreams turns out to be nonsense." The extent of nonoriginal material might correspond to the A listing: Aitkin, d'Alembert, Andrews, Aristotle, Askey, and Atiyah. Browsing is recommended. Read your heroes' thoughts.

ROBERT E. O'MALLEY, JR.
University of Washington

SIAM Review Vol. 57, Issue 2 (June 2015)

Book Reviews

BOOK REVIEWS

I continue to benefit from the labors of my predecessor; that will begin to change with the next issue.

Our featured review by Edgar Knobloch discusses Timour Radko's *Double-Diffusive Convection.* The density of salt water depends upon both temperature and salt concentration. Both heat and salt diffuse, but at very different rates. In the oceans, conditions often arise in which these two contributors to buoyancy (or lack thereof) work against each other, leading to interesting and important phenomena such as salt fingers and thermohaline staircases. Global warming could have effects on these phenomena that we should strive to understand, given that we depend upon the oceans for our survival.

Speaking of survival, take a look at Hans Kaper's review of John Drake's *Climate Modeling for Scientists and Engineers.* For obvious reasons, a great deal of effort has gone into the development of climate models in recent years. This is an activity that occupies many SIAM members and will do so for the foreseeable future.

In addition to these, we have many other informative reviews, including books on iterative methods for linear systems, implicit function theorems for variational analysis, calculus without derivatives (nonsmooth functions), singular perturbations, calculus of variations, asymptotics, introductory differential equations, contract theory, and stochastic processes.

David S. Watkins
Section Editor
siam.book.review@gmail.com

SIAM REVIEW
Vol. 57, No. 2, pp. 295–318

Book Reviews

Edited by David S. Watkins

Featured Review: Double-Diffusive Convection. *By Timour Radko.* Cambridge University Press, Cambridge, UK, 2013. $120.00. xii+342 pp, hardcover. ISBN 978-0-521-88074-9.

The subject of *Double-Diffusive Convection*[1] by Timour Radko is of great importance in oceanography as well as in other areas of fluid mechanics. The book is written by an oceanographer from the point of view of oceanography and is an homage to the author's thesis advisor, the eminent oceanographer Melvin E. Stern (1929–2010).

What is double-diffusion and why write a book about it? The basic idea is simple and goes back to a thought experiment by Henry Stommel. In a paper entitled "An Oceanographic Curiosity: The Perpetual Salt Fountain" [1], Stommel, Arons, and Blanchard consider a thermally conducting vertical pipe, open at both ends, inserted into an ocean environment in which saltier but warmer water overlies fresher and colder water. The overall water density, which depends on both the salt content and temperature, is assumed to decrease upward, indicating so-called static stability. However, Stommel recognized that potential energy is stored in the unstable salt stratification and that it could be tapped by dissipation. He imagined displacing the liquid in the pipe upward. Since the pipe conducts heat the displaced fluid parcel will warm and so acquire upward buoyancy. As a result, it will continue to rise, carrying fresher water upward. The process continues until such time that the resulting dilution destroys the unstable salt stratification. Stern's great contribution was to realize that he did not need the pipe[2] for this to be the case. He reasoned correctly that the low diffusion rate of salt relative to that of heat would play the same role as the pipe, and noted that if a fluid parcel were displaced downward, it would cool, and since it retained its salt content, it would acquire negative buoyancy and hence continue to sink. The idea of *salt fingers* was thus born [2]. These take the form of descending parcels of salt-rich fluid interspersed with rising parcels of fresher fluid, a process maintained essentially indefinitely by the unstable salt stratification. The net effect is to (a) mix the upper regions of the ocean, an essential process involved in the oxygenation of the upper ocean, in the bringing of nutrients to the surface, and in increasing the rate of CO_2 absorption, and (b) gradually lower the center of mass of the liquid column. Thus, perhaps counterintuitively, the inclusion of dissipation leads to a vitally important instability. Instabilities of this type are called diffusive or secular instabilities and take place on a dissipative and therefore long timescale. Their effect is therefore seen in regions of the ocean that are stable with respect to dynamic overturning.

[1] The term *double-diffusive* has become accepted terminology in the field, although I well remember E. A. Spiegel fretting in the 1970s that only *doubly diffusive* is grammatically correct.

[2] I remember Judy Holyer saying this in a presentation at the Woods Hole Oceanographic Institution's Summer Program in Geophysical Fluid Dynamics in 1978, in an unintended pun greeted with great amusement by the audience, since Stern—who was in the audience—was almost never seen without his pipe.

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *Market St.,* 6^{th} *Floor, Philadelphia, PA* 19104-2688.

There is a related instability that takes place when cold fresh water overlies warm and salty water in such a way that density again decreases upward. In this case a fluid displaced downward is arrested by the higher salt content that surrounds it, and because it warms up its upward buoyancy is enhanced and so it overshoots its equilibrium condition on its way up. If the restoring force due to the salt concentration is strong enough to overcome viscous damping, growing oscillations result: the continuously stratified system is Hopf-unstable. This process, called diffusive convection in the book and overstable convection elsewhere, also leads to mixing and the gradual lowering of the center of mass of the water column.

The presence of these instabilities is readily confirmed by examining the linear stability properties of the conduction (no motion) state in a horizontal plane layer of depth h. The equations are

$$\mathrm{Pr}^{-1}\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \nabla^2\mathbf{u} + (Ra_T T - Ra_S S)\mathbf{g}/g, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

$$\partial_t T + (\mathbf{u}\cdot\nabla)T = \nabla^2 T, \tag{3}$$

$$\partial_t S + (\mathbf{u}\cdot\nabla)S = \tau\nabla^2 S. \tag{4}$$

Here the first equation is the Navier–Stokes equation for the velocity field $\mathbf{u} \equiv (u, v, w)$, nondimensionalized with respect to the vertical thermal diffusion time across the layer, h^2/κ_T, and assumed to be incompressible ($\nabla\cdot\mathbf{u} = 0$); p is the pressure. The flow is driven by buoyancy forces generated by temperature and salinity changes. The magnitude of these forces is measured by the thermal and solutal Rayleigh numbers $Ra_T = g\alpha\triangle\tilde{T}h^3/\nu\kappa_T$ and $Ra_S = g\beta\triangle\tilde{S}h^3/\nu\kappa_T$, where $\alpha \equiv -\frac{1}{\rho}\frac{\partial\rho}{\partial T} > 0$ is the thermal expansion coefficient and $\beta \equiv \frac{1}{\rho}\frac{\partial\rho}{\partial S} > 0$ is its solutal analogue. The kinematic viscosity ν and the thermal diffusivity κ_T are used to define the Prandtl number $\mathrm{Pr} \equiv \nu/\kappa_T$, a property of the fluid. The quantities $\triangle\tilde{T} > 0$ and $\triangle\tilde{S} > 0$ measure the imposed temperature and salinity drops across the layer. We have written $T = (\tilde{T} - \tilde{T}_r)/\triangle\tilde{T}$, $S = (\tilde{S} - \tilde{S}_r)/\triangle\tilde{S}$, where the $\tilde{}$ indicates dimensional quantities and the subscript r indicates reference values; $\mathbf{g}$ is the gravitational acceleration whose modulus is written as g. Both T and S are advected by the flow but also diffuse, with the salt diffusing more slowly than heat: $\tau \equiv \kappa_S/\kappa_T < 1$. Here κ_S is the solutal diffusivity. In other situations cross-diffusion effects may be important, and these are described by adding the term $\kappa_{ST}\nabla^2 S$ (Dufour effect) to the right side of (3) and $\kappa_{TS}\nabla^2 T$ (Soret effect) to the right side of (4). These effects are not discussed in the book.

These equations must be supplemented by appropriate boundary conditions. Stress-free conditions at top and bottom correspond to $u_z = v_z = w = 0$ on $z = 0, 1$. The conditions $T = S = 1$ on $z = 1$, $T = S = 0$ on $z = 0$ correspond to the finger regime; the conditions $T = S = 0$ on $z = 1$, $T = S = 1$ on $z = 0$ correspond to the diffusive regime. With periodic boundary conditions in the horizontal these boundary conditions are conducive to analytical progress. In two dimensions linearization around the conduction state $u = v = w = 0$, $T = S = z$ or $u = v = w = 0$, $T = S = 1 - z$ yields a simple cubic dispersion relation for the growth rate $a\sigma$ of the disturbance [2]:

$$\sigma^3 + (1 + \mathrm{Pr} + \tau)\sigma^2 + [\mathrm{Pr} + \tau + \mathrm{Pr}\,\tau \pm \mathrm{Pr}\,r_T \mp \mathrm{Pr}\,r_S]\sigma + \mathrm{Pr}(\tau \pm \tau r_T \mp r_S) = 0, \tag{5}$$

where the upper sign corresponds to the finger case and the lower to the diffusive case. Here $a \equiv k_c^2 + \pi^2$, $r_T \equiv Ra_T/Ra_0$, $r_S \equiv Ra_S/Ra_0$, $Ra_0 \equiv a^3/k_c^2$, and $k_c \equiv \pi/\sqrt{2}$

is the critical horizontal wavenumber for the onset of convection in a pure fluid with stress-free boundaries (because of the boundaries in the vertical, the critical vertical wavenumber is π). It follows that the onset of fingering occurs at $r_S = \tau(1 + r_T)$ as r_S increases and is steady ($\sigma = 0$), while the onset of the diffusive instability is oscillatory ($\sigma = i\omega$) and occurs at $r_T = \frac{1}{\mathrm{Pr}}(1 + \tau)(\mathrm{Pr} + \tau) + \left(\frac{\mathrm{Pr}+\tau}{1+\mathrm{Pr}}\right)r_S$, provided the stabilizing salt stratification is sufficiently strong: $r_S > \frac{\tau^2(1+\mathrm{Pr})}{\mathrm{Pr}(1-\tau)}$. Observe that this is possible only when $\tau < 1$, i.e., because heat diffuses faster than temperature.

Why write a whole book about these ideas? From the point of view of oceanography and the author the reason is the small scale mixing that results and the fact that this mixing can have a profound effect not only on the small scale of the fingers (typically on the order of a centimeter or less) but also on large scales. Conditions for the salt-finger instability are found in many locations, including the Caribbean and the outflow regions from the Mediterranean, where large regions of warmer and saltier water overlying a cooler and fresher body of water are found, largely an effect of higher evaporation at low latitudes. In contrast, conditions for the diffusive instability are found in the Arctic and Antarctic, where cooler fresher water generated by melting ice overlies warmer saltier water brought in by the thermohaline circulation from lower latitudes. The thermohaline "conveyor belt" is arguably one of the most important processes in the oceans, whereby warm salty water from lower latitudes is brought to high latitudes (e.g., via the Gulf Stream), where it cools and sinks to the bottom before returning along the ocean bottom and upwelling in the Southern Ocean. Doubly diffusive mixing reduces the efficiency of the buoyancy forces driving this circulation, and stronger evaporation arising from global warming enhances the mixing arising from the salt-finger instability and renders new regions susceptible to it. The diminution of the vigor of the Gulf Stream, or its possible shut-off by these developments, would have catastrophic consequences for European climate and the climate in Ireland and the United Kingdom in particular. Clearly, developing an understanding of these mixing processes is a very worthwhile undertaking.

The main reason for concern is that doubly diffusive mixing is much more efficient than is suggested by the qualitative linear stability arguments sketched above, since both the salt-finger instability and the diffusive instability tend to form layers. The resulting salt-finger staircases are a feature of large parts of the ocean (see Figure 1), and the same is true for layers generated in the diffusive regime (see Figure 2). The staircases consist of well-mixed (turbulent) regions of nearly constant temperature and salt concentration, separated by boundary layers through which large vertical fluxes can be transported, essentially by diffusive or other small scale processes, and may be coherent over lateral distances exceeding 100km. As a result the formation of salt-finger staircases and the associated diffusive layering is of fundamental importance in oceanography. Early ideas based on the presence of a collective instability of the salt-finger field (via the generation of internal gravity waves [3]) have been superseded by a flux instability favored by the author; indeed, the subject is all about fluxes, and the dependence of the flux ratio $\gamma \equiv \frac{\alpha F_T}{\beta F_S} \approx \frac{\overline{wT'}}{\overline{wS'}}$ on the density ratio $R_\rho \equiv \frac{\alpha \bar{T}_z}{\beta \bar{S}_z}$ is of paramount importance. Here F_T, F_S are the vertical heat and salinity fluxes, w is the vertical velocity, and T' and S' are the temperature and salinity fluctuations; $\bar{T}_z$ and $\bar{S}_z$ are the background temperature and salinity gradients in the vertical. The thermal expansion coefficient α and the corresponding salinity coefficient β are incorporated into the definitions of the fluctuations T' and S'. When γ decreases with R_ρ the system is unstable to layering; this is the case for $1 < R_\rho \lesssim 2$. These results are based on a combination of observations and numerical simulation, both in two

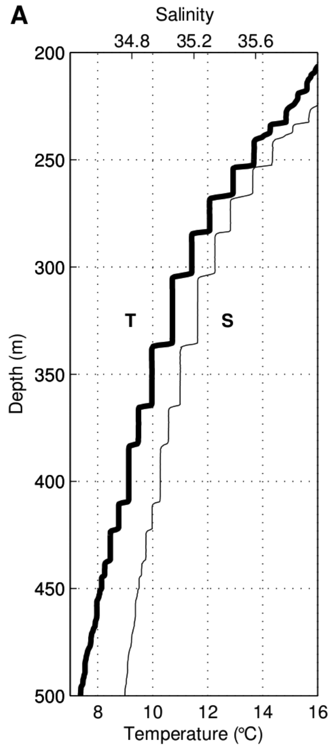

Fig. 1 *Typical profiles of potential temperature and salinity in the tropical Atlantic staircase taken during the SFTRE program. From* [6]. *Reprinted with permission from AAAS.*

dimensions and, more recently, in three dimensions. It is worth mentioning that no two-dimensional simulations of the salt-finger or diffusive regimes between horizontal boundaries with fixed temperature and salinity have ever identified layering. Only when the problem was reformulated in flux form with periodic boundary conditions in all three directions did simulations finally generate thermohaline layers [4, 5]. I consider these simulations to be a remarkable achievement and perhaps the single most important advance in the field since Stern's early work (see Figure 3).

The book presents the basic ideas behind double-diffusion in Chapter 1 and then proceeds to discuss linear stability (in Chapter 2) and, in considerable detail, the unbounded gradient model (in Chapter 3). The starting point for the analysis is the observation that boundary conditions at top and bottom are unlikely to be relevant under ocean conditions. The problem is therefore reformulated in terms of fluxes, leading to a characteristic finger scale $d \equiv (\kappa_T \nu / g\alpha|\bar{T}_z|)^{1/4}$, where $\bar{T}_z$ is the ambient thermal gradient in the vertical. With this length scale used to nondimensionalize lengths one obtains the following equations for the salt-finger regime:

$$\mathrm{Pr}^{-1}\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \nabla^2 \mathbf{u} + (T' - S')\mathbf{g}/g, \tag{6}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{7}$$

$$\partial_t T' + (\mathbf{u}\cdot\nabla)T' - w = \nabla^2 T', \tag{8}$$

$$\partial_t S' + (\mathbf{u}\cdot\nabla)S' - \frac{w}{R_\rho} = \tau \nabla^2 S', \tag{9}$$

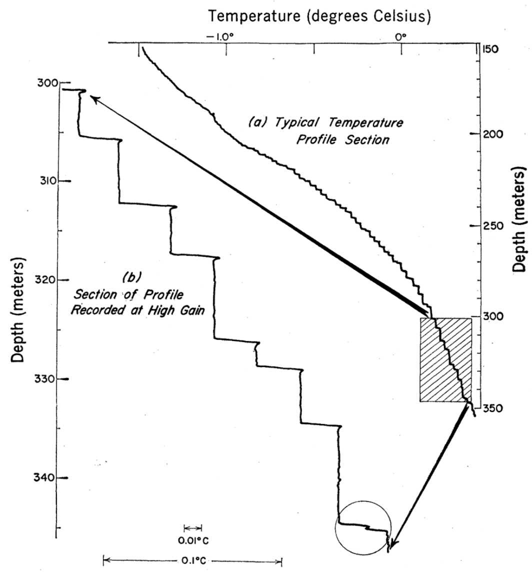

Fig. 2 *A sample temperature profile in the Arctic. From* [7]. *Reprinted with permission from AAAS.*

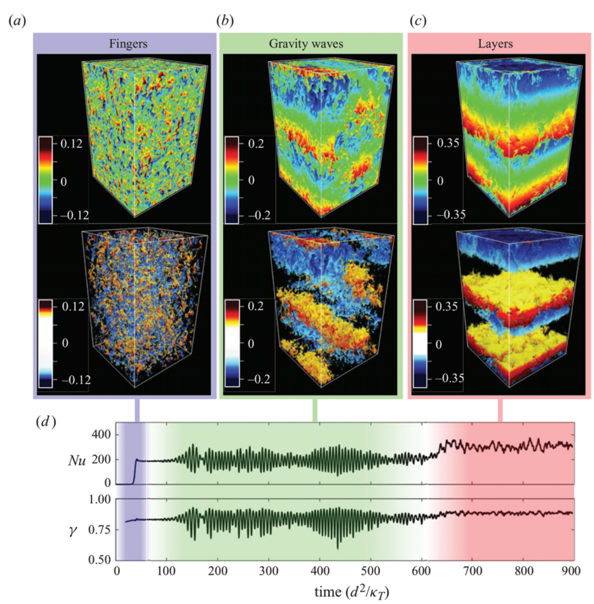

Fig. 3 *Simulated layer formation in a large triply periodic domain. Temperature perturbations are shown in color, with their respective color scales in each panel. The top panels are visualizations of the data-cube faces, while the lower ones are volume-rendered images. Snapshots are shown for three characteristic dynamical phases:* (a) *homogeneous fingering convection,* (b) *flow pattern dominated by gravity waves, and* (c) *formation of vigorously convecting layers separated by thin fingering interfaces.* (d) *Time series of the Nusselt number Nu (normalized convective flux) and of the turbulent flux ratio γ. From* [5].

where R_ρ is again the density ratio. In the diffusive regime it is customary to use $R_\rho^* \equiv 1/R_\rho$ as the relevant density ratio. This formulation has the advantage of both being natural and having one less parameter to specify the system, a consequence of the absence of the layer depth h from the formulation. Of course, for this formulation to be well-posed one must assume that the ambient gradients $\bar{T}_z$, $\bar{S}_z$, and hence R_ρ are somehow maintained. The basic issue is then to determine the resulting flux ratio γ. Incidentally, the same assumption is made about the Kepler shear in a magnetized accretion disk around a central object in the wake of the magnetorotational instability [8]. However, here the assumption that the Kepler flow is somehow immutable is likely to be less good since the magnetorotational instability is dynamic and not secular.

The above formulation is used throughout the remainder of the book, starting with the two-layer model (in Chapter 4) and the bounded layer model, described by (1)–(4), in Chapter 5. The major contribution of the book can be found in Chapter 6 on the collective instability of salt fingers, in Chapter 7 on thermohaline intrusions, and, in particular, in Chapter 8 on thermohaline staircases. Chapter 6 describes a mechanism favored by Stern that might be responsible for the formation of staircases, while Chapter 7 concentrates on horizontal intrusions driven by small lateral gradients of temperature and salt concentration that may be present in addition to the vertical stratification. These intrusions lead to finger-driven interleaving of water masses, and can extend horizontally over many kilometers; a full discussion of the beautiful experiments by Huppert and Turner on icebergs (or icicles) melting in a vertical salt gradient [9] would have complemented the chapter well. Chapter 8 is perhaps the best of the book. It surveys the remarkable observations of thermohaline staircases in different regions of the world and discusses a number of scenarios that may be responsible for their formation. Chapter 9 is very brief and devoted to demonstrating similarities at the linear level between the different scenarios for staircase generation in vertically and laterally driven systems. Chapters 10–14 are discourses on the influence of double-diffusion on transport and mixing in the oceans and in applications ranging from geology and astrophysics to metal casting and solar pond technology. The author is careful throughout to provide numerical estimates of the values of the physical parameters relevant to the oceans and other applications.

The emphasis of the book is not on mathematics, but on phenomena. It is a monograph, not a text, and succeeds very well at describing the essential doubly diffusive processes together with oceanographic observations and laboratory experiments. It presents the results of numerical simulations in both two and three dimensions of a variety of model systems, while rightly cautioning against overinterpretation of both laboratory experiments and the simulations. The former often employ dissolved salt-sugar mixtures instead of salt-heat (to control heat losses) at the cost of a rather different diffusivity ratio τ; the latter find it difficult to reach the parameter values relevant in many applications ($\tau \approx 10^{-2}$ for the salt-heat system). Because of its focus on an oceanographic audience, a SIAM reader might at times find the book frustrating. There are very few equations, and the theory that is presented does not go beyond linear stability analysis. Even that is incomplete since no boundary conditions are imposed—the author assumes that boundaries are unimportant and, in effect, assumes that the flow (u, v, w, T', S') is triply periodic. Throughout he is interested in maximal growth rates at the expense of instability thresholds such as those obtained below (1)–(4). As a result, no bifurcation analysis is performed and no weakly nonlinear studies of the evolution of the salt-finger and diffusive instabilities are presented. Likewise, no strongly nonlinear results such as those of Proctor [10] on the subcriticality of thermohaline convection in the limit $\tau \to 0$ are described. This is a shame since the subject has a great deal to offer an applied mathematician, and its importance is without doubt.

The author barely mentions the fact that diffusive convection has been used for many years as the prototypical system for a detailed study of the transition to chaos in a fluid flow. This is because the subcriticality of the flow reduces the Rayleigh numbers at which chaos sets in over, for example, ordinary Rayleigh–Bénard convection [11]. As a result the chaotic flow can be faithfully analyzed within a Lorenz-like fifth order Galerkin truncation [12, 13], and indeed direct numerical simulations of the governing equations (1)–(4) with the diffusive case boundary conditions (see above) reveal a complex route to chaos via cascades of period-doubling bifurcations [14, 15]. The result was the first demonstration of Shil'nikov dynamics [16] in a nominally infinite-dimensional fluid system; this led directly to the subsequent study of the Shil'nikov bifurcation by Glendinning and Sparrow [17], a worthy companion to related results on the Lorenz attractor which, sadly, is a poor model of two-dimensional Rayleigh–Bénard convection for which it was originally derived [18, 19].

These early studies focused on the properties of standing waves generated via the Hopf bifurcation of the conduction state. In fact, this bifurcation generates not only standing waves, but also traveling waves, a fact established via the development of a theory for the Hopf bifurcation with O(2) symmetry, the rotations and reflections of a circle, inherited from the imposition of periodic boundary conditions in the horizontal on a translation-invariant system [20]. The computation of the corresponding normal form coefficients predicts that standing waves are unstable to traveling waves, a result in accord with simulations [21] and, indeed, experiments [22]—on p. 83 the author states, incorrectly, that no experiments have ever been performed with imposed salt concentration at the upper and lower boundaries. Much of this type of analysis was motivated by the beautiful experiments of Paul Kolodner on binary fluid mixtures heated from below. This system is superior for experimental studies of doubly diffusive convection because it relies on the presence of a cross-diffusion effect to generate the conditions favorable for diffusive convection: the components of miscible two-component mixtures such as water-alcohol or normal He^3-He^4 mixtures tend to separate in an imposed thermal gradient, as in fact do salt-water mixtures [23]; in a mixture with a negative separation ratio heated from below the heavier component migrates toward the hotter bottom boundary while the lighter component migrates toward the upper colder boundary, setting up the same base state with cold fresh fluid overlying a hot salty liquid. Indeed, under appropriate conditions it is possible to transform the equations of binary fluid convection with the Soret and Dufour cross-diffusion terms to the equations describing doubly diffusive convection [24]. In small scale experiments such as those performed by Kolodner the temperatures at the top and bottom can be exquisitely controlled, unlike the "bucket" experiments favored by oceanographers, as can the alignment and smoothness of the top and bottom plates. As a result experimental results of exceptional quality are obtained, and these have been used to study the flows arising from the presence of the Hopf bifurcation—not only the traveling waves predicted by the theory, but also localized traveling waves [25, 26] and localized steady convection [27, 28], and the dispersive chaos states [29, 30] from which these are born. This continues to be a lively area of research [31, 32] among those of us interested in nonlinear dynamics, but sadly cross-diffusion is not discussed in the book and neither are the waves generated by overstability nor the reasons for the preference of traveling waves over standing waves (or other questions of planform selection by nonlinear terms). A thorough discussion of these notions may be found in another Cambridge University Press publication, the recent book *Magnetoconvection* by N. O. Weiss and M. R. E. Proctor [33]. Convection in an imposed magnetic field, or magnetoconvection, is also a double-diffusive system and it shares many of the properties exhibited by thermohaline convection (although apparently no staircases). In fact, for an imposed horizontal field the inclusion of magnetic buoyancy results in a system identical to doubly diffusive convection [34]. The book is also motivated by observations, mostly of the Sun, but adopts a different and more mathematically informed approach to understanding these observations, although direct numerical simulations still play an essential role.

While it is unlikely that these types of studies have any bearing on the oceanographic aims of the book, there are many opportunities for an applied mathematician to make a substantial impact on the field. The layer formation in the salt-finger regime mentioned above can be described by a negative (eddy) density diffusion coefficient, since the upper parts of the ocean become less dense as a result of the instability while the lower parts become denser. In addition, the layers are observed to coarsen over time and may only be stabilized by vertical heterogeneities in the temperature or salt stratification. All these suggest that the staircase formation phenomenon could be described by Cahn–Hilliard dynamics [35], and indeed that the coarsening process could be arrested, at least in some cases, by the presence of oscillatory spatial eigenvalues in a spatial dynamics formulation of the problem as occurs, for example, in the complex-valued Swift–Hohenberg equation [36]. The issues to overcome, and there are many, include the multiscale nature of the problem, the inclusion of turbulent processes both within the layers and in the interfaces between them, vertical heterogeneities, etc., but these are challenges that applied mathematicians relish. In fact, it is remarkable how little attention has been paid by applied mathematicians to layer formation in fluid flows, albeit with notable exceptions [37], and if this review is to have any value my hope would be that it will stimulate work in this area. The book under review is a perfect place to start, since it provides a comprehensive and up-to-date summary of in situ observations and their current interpretation by oceanographers.

From a more mathematical point of view I would have liked to have seen a mention of the so-called negative energy states, of which the spinning top is a classic example [38], and of dissipation-induced instabilities [39, 40], both very substantial fields in their own right. Of course, it is easy to be critical of any book of this type, but it is my view that the book would have been more effective had it started with an overview of what is known about the oceans, the thermohaline circulation, etc., if only to make it clear at the outset the importance of the interaction of heat and salt in the ocean, even if the circulation is driven by (nondiffusive) sinking of cold fresh fluid at high latitudes, and to introduce terminology such as the notion of a thermocline that is used throughout but not defined. Overall I found Chapter 8 on thermohaline staircases the most informative. The chapter includes, among other gems, the observation that relative to the 1985 Arctic Internal Wave Experiment, the present-day staircase region in the same ocean region is approximately 100m shallower and the height of the steps is twice as large ($\sim$ 3m average, with most steps in the 1–5m range). Other possible consequences of global warming and their possible mitigation via salt-finger-induced mixing [41] are mentioned throughout the book. Chapters 11–13 are perhaps the weakest since they are largely descriptions of processes in words without either the substance an applied mathematician would derive from the presence of equations or the relevant details that would make an nonexpert appreciate the subtleties associated with metalicity observations in stars (the term is never defined) or those arising from crystallization and solute rejection in magma chambers. I did, however, find the discussion of salt-fountain ocean farms, based on Stommel's "oceanographic curiosity," rather intriguing.

I also find it remarkable that the book does not mention the great advances that have been made in our understanding of mixing in the oceans using ideas from dynamical systems such as lobe theory [42, 43]—after all, the whole book is about mixing. Is this because these developments primarily concern two-dimensional mixing on isopycnal surfaces, mainly nondiffusive mixing, relying on diffusion only for the final, irreversible process? Or is it that the vast majority of oceanographers are simply unaware of these developments or too set in their ways to readily absorb new techniques? In this connection I found the author's lament that courses in "nonlinear instability—the heart and soul of double-diffusion—are notably absent in most oceanographic curricula" rather telling. If true, this is a sad reflection of today's compartmentalization and an added reason why the Summer Program in Geophysical Fluid Dynamics at the Woods Hole Oceanographic Institution is such an indispensable program for the future of oceanographic research.

In summary, this is a welcome addition to the oceanography literature, written in an engaging style ("double-diffusion is not for weaklings; it requires the patience of a saint and the intensity of a Rottweiler"). The book is nicely produced, as one would expect from a Cambridge University monograph, with remarkably few misprints. There is a nice set of color plates with the most important observational and numerical results, although most of the illustrations are in black and white. The list of references to oceanographic literature and, in particular, to oceanographic measurements is comprehensive, although clearly far from complete—the author estimates that some 250,000 papers have already been written on the topic of salt fingers, surely a testament to the importance of the subject!

Acknowledgment. This work was supported by the National Science Foundation under grant DMS-1317596.

REFERENCES

[1] H. Stommel, A. B. Arons, and D. Blanchard, *An oceanographic curiosity: The perpetual salt fountain*, Deep-Sea Res., 3 (1956), pp. 152–153.
[2] M. E. Stern, *The "salt-fountain" and thermohaline convection*, Tellus, 12 (1960), pp. 172–175.
[3] M. E. Stern, *Collective instability of salt fingers*, J. Fluid Mech., 35 (1969), pp. 209–218.
[4] T. Radko, *A mechanism for layer formation in a double-diffusive fluid*, J. Fluid Mech., 497 (2003), pp. 365–380.
[5] S. Stellmach, A. Traxler, P. Garaud, N. Brummell, and T. Radko, *Dynamics of fingering convection* II: *The formation of thermohaline staircases*, J. Fluid Mech., 677 (2011), pp. 554–571.
[6] R. W. Schmitt, J. R. Leadwell, E. T. Montgomery, K. L. Polzin, and J. M. Toole, *Enhanced diapycnal mixing by salt fingers in the thermocline of the tropical Atlantic*, Science, 308 (2005), pp. 685–688.
[7] V. T. Neal, S. Neshyba, and W. Denner, *Thermal stratification in the Arctic Ocean*, Science, 166 (1969), pp. 373–374.
[8] S. A. Balbus and J. F. Hawley, *Instability, turbulence, and enhanced transport in accretion disks*, Rev. Modern Phys., 70 (1998), pp. 1–53.
[9] H. E. Huppert and J. S. Turner, *Ice blocks melting into a salinity gradient*, J. Fluid Mech., 100 (1980), pp. 367–384.
[10] M. R. E. Proctor, *Steady subcritical thermohaline convection*, J. Fluid Mech., 105 (1981), pp. 507–521.
[11] H. E. Huppert and D. R. Moore, *Nonlinear double-diffusive convection*, J. Fluid Mech., 78 (1976), pp. 821–854.
[12] G. Veronis, *On finite amplitude instability in thermohaline convection*, J. Mar. Res., 23 (1965), pp. 1–17.

[13] L. N. Da Costa, E. Knobloch, and N. O. Weiss, *Oscillations in double-diffusive convection*, J. Fluid Mech., 109 (1981), pp. 25–43.

[14] D. R. Moore, J. Toomre, E. Knobloch, and N. O. Weiss, *Chaos in thermosolutal convection: Period doubling for partial differential equations*, Nature, 303 (1983), pp. 663–667.

[15] E. Knobloch, D. R. Moore, J. Toomre, and N. O. Weiss, *Transitions to chaos in two-dimensional double-diffusive convection*, J. Fluid Mech., 166 (1986), pp. 409–448.

[16] L. Shil'nikov, *A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium of saddle-focus type*, Math. USSR Sb., 10 (1970), pp. 91–102.

[17] P. Glendinning and C. Sparrow, *Local and global behavior near homoclinic orbits*, J. Statist. Phys., 35 (1984), pp. 645–696.

[18] E. N. Lorenz, *Deterministic nonperiodic flow*, J. Atmospheric Sci., 20 (1963), pp. 130–141.

[19] D. R. Moore and N. O. Weiss, *Two-dimensional Rayleigh-Bénard convection*, J. Fluid Mech., 58 (1973), pp. 289–312.

[20] J. D. Crawford and E. Knobloch, *Symmetry and symmetry-breaking bifurcations in fluid mechanics*, Ann. Rev. Fluid Mech., 23 (1991), pp. 341–387.

[21] E. Knobloch, A. Deane, J. Toomre, and D. R. Moore, *Doubly diffusive waves*, in Multiparameter Bifurcation Theory, M. Golubitsky and J. Guckenheimer, eds., Contemp. Math. 56, AMS, Providence, RI, 1986, pp. 203–216.

[22] A. A. Predtechensky, W. D. McCormick, J. B. Swift, A. G. Rossberg, and H. L. Swinney, *Traveling wave instability in sustained double-diffusive convection*, Phys. Fluids., 6 (1994), pp. 3923–3935.

[23] D. R. Caldwell, *Experimental studies of the onset of thermohaline convection*, J. Fluid Mech., 64 (1974), pp. 347–367.

[24] E. Knobloch, *Convection in binary fluids*, Phys. Fluids, 23 (1980), pp. 1918–1920.

[25] P. Kolodner, *Drifting pulses of traveling wave convection*, Phys. Rev. Lett., 66 (1991), pp. 1165–1168.

[26] W. Barten, M. Lücke, M. Kamps, and R. Schmitz, *Convection in binary mixtures.* II. *Localized traveling waves*, Phys. Rev. E, 51 (1995), pp. 5662–5680.

[27] P. Kolodner, *Coexisting traveling waves and steady rolls in binary-fluid convection*, Phys. Rev. E, 48 (1993), pp. R665–R668.

[28] O. Batiste, E. Knobloch, A. Alonso, and I. Mercader, *Spatially localized binary fluid convection*, J. Fluid Mech., 560 (2006), pp. 149–158.

[29] C. Bretherton and E. A. Spiegel, *Intermittency through modulational instability*, Phys. Lett. A, 96 (1983), pp. 152–156.

[30] P. Kolodner, J. A. Glazier, and H. Williams, *Dispersive chaos in one-dimensional traveling-wave convection*, Phys. Rev. Lett., 65 (1990), pp. 1579–1582.

[31] T. Watanabe, M. Iima, and Y. Nishiura, *Spontaneous formation of traveling localized structures and their asymptotic behaviours in binary fluid convection*, J. Fluid Mech., 712 (2012), pp. 219–243.

[32] I. Mercader, O. Batiste, A. Alonso, and E. Knobloch, *Travelling convectons in binary fluid convection*, J. Fluid Mech., 722 (2013), pp. 240–266.

[33] N. O. Weiss and M. R. E. Proctor, *Magnetoconvection*, Cambridge University Press, Cambridge, UK, 2014.

[34] D. W. Hughes and N. O. Weiss, *Double-diffusive convection with two stabilizing gradients: Strange consequences of magnetic buoyancy*, J. Fluid Mech., 301 (1995), pp. 383–406.

[35] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system.* I. *Interfacial free energy*, J. Chem. Phys., 28 (1958), pp. 258–267.

[36] L. Gelens and E. Knobloch, *Coarsening and frozen faceted structures in the supercritical complex Swift-Hohenberg equation*, Eur. Phys. J. D, 59 (2010), pp. 23–36.

[37] G. I. Barenblatt, M. Bertsch, R. Dal Passo, V. M. Prostokishin, and M. Ughi, *A mathematical model of turbulent heat and mass transfer in stably stratified shear flow*, J. Fluid Mech., 253 (1993), pp. 341–358.

[38] P. J. Morrison, *Hamiltonian description of the ideal fluid*, Rev. Modern Phys., 70 (1998), pp. 467–521.

[39] R. Krechetnikov and J. E. Marsden, *Dissipation-induced instabilities in finite dimensions*, Rev. Modern Phys., 79 (2007), pp. 519–553.

[40] R. Krechetnikov and J. E. Marsden, *Dissipation-induced instability phenomena in infinite-dimensional systems*, Arch. Rat. Mech. Anal., 194 (2009), pp. 611–668.

[41] G. C. Johnson and K. A. Kearney, *Ocean climate change fingerprints attenuated by salt-fingering?*, Geophys. Res. Lett., 36 (2009), L21603.

[42] M. Branicki and S. Wiggins, *Finite-time Lagrangian transport analysis: Stable and unstable manifolds of hyperbolic trajectories and finite-time Lyapunov exponents*, Nonlinear Process. Geophys., 17 (2010), pp. 1–36.

[43] M. J. Olascoaga and G. Haller, *Forecasting sudden changes in environmental contamination patterns*, Proc. Natl. Acad. Sci. USA, 109 (2012), pp. 4738–4743.

Edgar Knobloch
University of California, Berkeley

Iterative Methods for Linear Systems: Theory and Applications. *By Maxim A. Olshanskii and Eugene E. Tyrtyshnikov.* SIAM, Philadelphia, 2014. $85.00, xvi+247 pp., softcover. ISBN 978-1-611973-45-7.

With so many good books on iterative methods published in the 1990s and early 2000s, e.g., [1, 9, 11, 12, 14], it was surprising to see a new book on this subject, and I wondered what new material it might contain. The answer is that there is not a lot new on the basic iterative methods—conjugate gradients, GMRES, BiCG, QMR, etc., are the same as they were 10 years ago—but this book contains new material on preconditioners, Toeplitz and circulant matrices, and multigrid and domain decomposition methods, with a very nice arrangement of topics to tie the ideas together. The theoretical analysis is rigorous and makes use of a wide range of diverse techniques. The book would be difficult for someone who has little prior knowledge of iterative methods, but for those who are familiar with the basics and wish to learn more about analysis and applications, it is an excellent resource.

Chapter 1 deals with Krylov subspace methods and presents rigorous mathematical theory of convergence as well as optimality of approximations. Convergence estimates for nonnormal matrices are presented in terms of pseudospectra and the numerical range. I had hoped to see some of the more recent estimates such as those derived by Beckermann et al. (including one of the authors of this book) [2, 3, 4] and some discussion of the implications of Crouzeix's theorem or conjecture [7, 8] for the convergence of GMRES. Still, the theory that is presented is stated clearly and proved elegantly. It also might have been interesting to see some mention of the recently popularized IDR method [13, 10]. Algorithms are presented in a form that is easy to understand but not always best for computation. (For example, the unmodified Gram–Schmidt version of GMRES is given, with no discussion of how to update the QR factorization of the Hessenberg matrix.) The authors state that their book should be considered complementary to other books on iterative methods, and this is one of the areas in which other books should be consulted.

Chapter 2 deals with Toeplitz matrices and preconditioners, a topic that typically receives less attention in books on iterative methods. Simple and optimal circulant preconditioners are defined and analyzed, and the relation with the fast Fourier transform is explained. The topic of multilevel Toeplitz and circulant matrices is also addressed and leads naturally into the following chapters on multigrid and domain decomposition methods for problems arising from partial differential equations.

Chapter 3 gives a very nice introduction to multigrid methods, used either as stand-alone solvers or as preconditioners for conjugate gradient or other iterative methods. Relevant properties of finite element approximations are clearly delineated and then used to provide a clear and succinct proof of convergence (at a rate that is independent of the mesh size; i.e., with error reduction factor $1/2$ at each step) for a two-grid method applied to the model problem of Poisson's equation. This is a chapter that I would recommend to students with or without a background in multigrid methods. I often recommend *A Multigrid Tutorial* by Briggs et al. [6], but while that book provides a very friendly, easy-to-read introduction to how multigrid methods work, the lack of mathematical analysis can lead to confusion. This book makes the mathematics very clear. I was especially interested in the proof (p. 113) that the 2-norm of the iteration matrix for the model problem is less than or equal to $1/2$, as proofs that I had seen previously dealt with the A-norm of the iteration matrix. After the section on the two-grid method there is a nice explanation of how two-grid theory can be used to establish convergence of the multigrid V-cycle or W-cycle.

Chapter 4 deals with space decomposition methods, such as alternating Schwarz methods, domain decomposition methods, and the additive multigrid method viewed as a space decomposition algorithm. There is discussion of the BPX (Bramble–Pasciak–Xu) [5] preconditioner and also of hierarchical basis methods [15]. The unifying theme is decomposition of the problem into different parts (domains, frequencies, etc.), and this point of view yields a unifying analysis.

The final chapter in the book discusses applications and modifications of the general theory that are often required to develop algorithms that work in practice. Problems considered include singularly perturbed partial differential equations and certain problems in fluid mechanics. There is discussion of different multigrid smoothers that may be needed in order to obtain convergence rates that are independent of the mesh size. Examples are ILU smoothers and a number of others designed for specific types of problems.

Overall, I enjoyed this book very much. There are a number of typos and occasional changes in notation (e.g., sometimes the grid size is $1/N$ and sometimes it is $1/(N+1)$) and I wish the authors would avoid the expression $\mathrm{cond}(BA)$ for the ratio of the largest to smallest eigenvalue of the preconditioned matrix BA (or AB) in the conjugate gradient algorithm. This leads to confusion in Theorem 1.33 and the discussion afterwards, where the norm being used is not specified. Hopefully these small issues can be fixed if a second edition appears. This book demonstrates the blending that has taken place during the last 20 years or so of what were once considered somewhat separate and maybe even competing ideas—iterative methods, multigrid, and domain decomposition—into one cohesive whole. The mathematical theory is there, and there are helpful exercises at the end of each chapter to drive home the main points. It is a valuable new resource for the research community in iterative methods.

REFERENCES

[1] O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, New York, 1994.

[2] B. Beckermann, *Image numérique, GMRES et polynômes de Faber*, C.R. Acad. Sci. Paris, Ser. I, 340 (2005), pp. 855–860.

[3] B. Beckermann and M. Crouzeix, *Faber Polynomials of Matrices for Nonconvex Sets*, preprint, arXiv.1310.1356, to appear in JAEN J. of Approx.

[4] B. Beckermann, S. A. Goreinov, and E. E. Tyrtyshnikov, *Some remarks on the Elman estimate for GMRES*, SIAM J. Matrix Anal. Appl., 27 (2006), pp. 772–778.

[5] J. Bramble, J. Pasciak, and J. Xu, *Parallel multilevel preconditioners*, Math. Comp., 55 (1990), pp. 1–22.

[6] W. L. Briggs, V. E. Henson, and S. F. McCormick, *A Multigrid Tutorial*, SIAM, Philadelphia, 2000.

[7] M. Crouzeix, *Bounds for analytical functions of matrices*, Integral Equations Oper. Theory, 48 (2004), pp. 461–477.

[8] M. Crouzeix, *Numerical range and functional calculus in Hilbert space*, J. Funct. Anal., 244 (2007), pp. 668–690.

[9] A. Greenbaum, *Iterative Methods for Solving Linear Systems*, SIAM, Philadelphia, 1997.

[10] M. Gutknecht, *IDR explained*, Electron. Trans. Numer. Anal., 36 (2009), pp. 126–148.

[11] W. Hackbush, *Iterative Solution of Large Sparse Systems of Equations*, Springer, New York, Berlin, 1994.

[12] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd ed., SIAM, Philadelphia, 2003.

[13] P. Sonneveld and M. B. Van Gijzen, *IDR(s): A family of simple and fast algorithms for solving large nonsymmetric systems of linear equations*, SIAM J. Sci. Comput., 31 (2008), pp. 1035–1062.

[14] H. van der Vorst, *Iterative Krylov Methods for Large Linear Systems*, Cambridge University Press, Cambridge, UK, 2003.

[15] H. Yserentant, *Old and new convergence proofs for multigrid methods*, Acta Numer., 1993 (1993), pp. 285–326.

Anne Greenbaum
University of Washington

Climate Modeling for Scientists and Engineers. *By John B. Drake*. SIAM, Philadelphia, 2014. $69.00, viii+165 pp., softcover. ISBN 978-1-611973-53-2.

Earth's climate system has all the characteristics of a *complex system*. It has many components (atmosphere, oceans, cryosphere, biosphere, lithosphere), all components interact in complicated nonlinear ways, and there are many feedback loops that can be positive (reinforcing) or negative (inhibiting) depending on the state of the system. It is self-organizing and shows more emergent phenomena than we can handle. Unfortunately, controlled experiments are impossible: there is no Planet B. Since the beginning of the industrial revolution we have been engaged in an uncontrolled experiment, and the results don't look good. Climate change is a fact, and we had better gather our intellectual resources to see what might happen if we change our habits or if we do nothing.

Fortunately, climate models enable us to explore "what-if" scenarios through large-scale numerical simulations. In fact, the computer has become the laboratory of the climate scientist and so-called community climate models (CCMs) are the in-silico tools of the trade. The Intergovernmental Panel on Climate Change (IPCC) assesses the state of our knowledge of the climate system every four years or so on the basis of results obtained with a collection of "sanctioned" CCMs.

Climate models are *process models* that incorporate many of the processes which determine the dynamics of our climate system. But they cannot incorporate all the processes, either because we don't know enough about them or because they play out on scales that cannot be captured. As a consequence, projections generated by CCMs for the future are always subject to uncertainty. This makes climate modeling an interesting topic of research for the SIAM community.

CCMs are essentially algorithmic implementations of the laws of nature. These laws determine how the system evolves in time and are commonly formulated as (ordinary or partial) differential equations. The equations are discretized according to any of a number of approximation procedures and implemented in numerical algorithms designed for the particular computer architecture on which the simulations are to be performed. All this is the bread and butter of computational science.

The book under review was written from the perspective of a particular climate model, namely, the Community Climate System Model (CCSM), which was (and is still being) developed by a research community sponsored by the U.S. Department of Energy. The book gives an overview of what goes into the CCSM and how the algorithms are formulated and implemented. The focus is on the atmosphere and ocean, the two primary components of the climate system.

The book opens with a chapter on climate data and the basic circulation patterns of Earth's atmosphere and oceans. Chapter 2 is devoted to the formulation of the basic equations, the conservation equations of mass, momentum, and energy for a fluid on the surface of a rotating sphere. The Coriolis effect plays an important role and, depending on the scale of interest, various simplifications are possible (geostrophic wind approximation, hydrostatic approximation, shallow water approximation, etc.). The chapter ends with two examples: the hydrostatic baroclinic equations used in the atmospheric component of the Community Atmosphere Model and the continuity, momentum, and hydrostatic equations for the Parallel Ocean Program.

Chapter 3 is devoted to the discretization of the basic equations, in both the spatial domain and the time domain. The author describes in detail the control-volume method, the semi-Lagrangian transport method, and Galerkin spectral methods. In the course of the discussion, the reader learns about various solution methods (factorization, conjugate gradient, GMRES), concepts (consistency, stability, convergence, computer architectures), and special functions (spherical harmonics).

The short Chapter 4 presents two case studies: a paper by Barron and Washington [1] that looks into the distant past and analyzes climate conditions warmer than the present-day conditions, and a paper by Lawrence and Chase [2] that concerns the climate impact of global land-cover change. For each case, the author explains the methodology, highlights the conclusions, and raises some critical issues that are inherent in the use of climate models.

The final Chapter 5 addresses climate analysis. As the author notes, a significant problem is that the observational network is sparse, especially over the Earth's oceans. The author discusses several methods to approximate functions on a sphere, including spectral methods and empirical orthogonal functions, as well as statistical and data assimilation techniques.

A chapter called "Conclusion" summarizes likely future developments in climate modeling. The book ends with a bibliography and index. Supplementary material is available on the web at www.siam.org/books/mm19.

The book is a useful introduction to computational climate science and makes for a good topic course in computational science. The style is somewhat informal and exercises are sprinkled throughout the text. Of course, the book does not give the whole story. Our climate system consists of more than atmosphere and oceans. To be useful, a CCM must also account for snow and ice, forests, land cover, atmospheric chemistry, and a host of other processes. Some of these topics are discussed in the supplementary material.

The actual text could have benefited from some careful editing. Style, grammar, and punctuation are not always what they should be, and this reviewer spotted several errors (for example, "Gibb's phenomena" on p. 96; "Li's Principle Sums" on p. 91, taken from unpublished work [114], should most likely have been "Li's Principal Sums"; "Delaney triangulation" on p. 102; "Lorentz grid" on p. 108). Some additional figures would have made it easier for the reader to follow the arguments (for example, in the discussion of coordinates in section 2.2). The image on the cover, though colorful, is identified only as "Visualization of time dependent fields of the CCSM." It would have been nice to learn a few more details. Also, this reviewer noted several errors and missed items in the index.

Despite these few criticisms, this reviewer enjoyed reading the book and appreciates the author's efforts to make this information available to the computational science community. The book is a welcome complement to the textbook [3].

REFERENCES

[1] E. J. Barron and W. M. Washington, *The atmospheric circulation during warm geologic periods: Is the equator-to-pole surface temperature gradient the controlling factor?*, Geology, 10 (1982), pp. 633–636.

[2] P. J. Lawrence and Th. N. Chase, *Investigating the climate impacts of global land cover change in the community climate system model*, Int. J. Climatology, 30 (2010), pp. 2066–2087.

[3] H. Kaper and H. Engler, *Mathematics and Climate*, SIAM, Philadelphia, 2013.

Hans G. Kaper
Georgetown University

Implicit Functions and Solution Mappings: A View from Variational Analysis. Second Edition. *By A. L. Dontchev and R. T. Rockafellar*. Springer, New York, 2014. $71.99. xxviii+466 pp., hardcover. ISBN 978-1-4939-1036-6.

The inverse and implicit function theorems are two of the most important theorems in mathematical analysis and are basic tools in many applications, in particular, in theory and solution methods for various types of equations. A central issue is the question of whether the solution to an equation involving parameters can be considered (locally) as a function of these parameters, and, if so, what properties this function has. In the classical theory, differentiability assumptions are standard.

These days, however, many models of "generalized equation" type are of growing interest, like systems of inequalities, variational inequalities, nonsmooth equations, optimization problems and related systems of optimality conditions, equilibrium problems, and more. In order to derive variants of inverse and implicit function theorems on this level of generality, one has to deal with inclusions and set-valued solution mappings and to relax the standard concepts of differentiability. In modern variational analysis, many tools have been developed for handling set-valued mappings, and there are helpful extensions of continuity, Lipschitz continuity, differentiability, and regularity.

In the last two decades, several fundamental monographs have been published, devoted to many aspects of variational analysis, including the topics just mentioned. The first edition of the present book, published in 2009 in the series *Springer Monographs in Mathematics*, was one of them. This second edition now appears in the *Springer Series in Operations Research and Financial Engineering.* It is considerably enlarged compared with the first edition and has about 90 more pages. The authors' intent is to present "an updated and more complete picture of the field by including solutions of problems that have been solved since the first edition was published, and places old and new results in a broader perspective" (cited from the second edition's cover).

The authors' purpose is to provide a unified collection of ideas and results in the general context of implicit functions and solution mappings. In my opinion, these aims have been very successfully realized.

Even those who appreciated the first edition of the book will greatly benefit from the present update. A lot of new material has been added, some chapters and sections have been essentially reconstituted (and so the presentation has won), mistakes of the first edition have been corrected, a complete list of theorems has been added, and the index has doubled in size. Both authors are leading specialists in their field and have a deep knowledge of its classical and modern developments. Similar to the first edition, each chapter is concluded with bibliographical and historical commentaries; in the second edition, many interesting new commentaries have been added, and the authors present an insightful personal view on both the history and the present state of the art focused on the content of the book; for a more detailed bibliographical discussion from the historical viewpoint, see the monograph written by the second author with R. J.-B. Wets, *Variational Analysis* (Springer, 1998).

A big merit of the book is the systematic and well-ordered setup of the topic's rich material. For example, several basic ideas are followed throughout the whole book: to use the concept of (single-valued or multivalued) localization, to show the pathway from results in the classical framework to those in the modern framework of generalized equations, or to complement the study of different types of regularity by formulas for the Lipschitz or regularity modulus. Moreover, the authors' terminology is well reasoned and should become the standard in the literature on variational analysis, which often has a confusing variety of notation and names for the same objects. Furthermore, the authors frequently put a lot of effort into providing different views of the same result or notion, from the advanced textbook level to the high-end level of the discipline. This makes the book suitable not only for researchers, but also for teachers and students in advanced calculus courses.

In Chapter 1, the authors start with the classical implicit function paradigm, then they gradually relax the differentiability assumptions in various ways and end up in a Lipschitz continuity setting. Complementing this basic material, the authors have added a new section 1.8 [1.H] to the second edition, which covers the case of monotone functions which may be not differentiable. An interesting discovery by the authors added to the commentary to this chapter is apparently the first nonsmooth inverse function theorem by Hildebrand and Graves, published in 1927, about half a century before nonsmooth analysis officially entered the stage.

Chapter 2 is devoted to solution mappings for variational problems. In particular, Robinson's implicit function theorem for generalized equations is presented, and various types of variational inequalities are studied. Similar to the classical setting, again the question handled asks when a solution mapping can be localized to a function with some continuity properties. The presentation is self-contained and can be read by advanced students and others without special background. In Chapter 3, the authors turn to set-valued solution maps and study properties like metric regularity and set-valued Lipschitz properties.

While Chapters 2 and 3 follow the first edition with relatively minor but important changes, substantial additions start to appear in Chapter 4. The first is a new proof (cf. section 4.3 [4C]) of the coderivative criterion for metric regularity based on the equality of the quantities appearing in the derivative and coderivative criteria; a direct proof of this equality was recently given in a recent paper by the first author with H. Frankowska. Section 4.4 [4D] has been reconstituted and contains important results like a strict derivative condition for metric regularity and the inverse function theorems of Clarke and Kummer. Many applications to parametric optimization and variational problems are presented.

Chapter 5 deals with extensions of the Banach open mapping theorem; it includes fundamental theorems like those due to Robinson and Ursescu, Lyusternik, Graves, and Bartle and Graves. In the second edition, several recent results from the literature since 2009 are included, in particular, there are now five different proofs of the Lyusternik–Graves theorem as well as variations and extensions of this result. In particular, one proof of this basic theorem is based on the equivalence of linear openness with relaxed linear openness given in Theorem 5H.8. The connection of metric regularity with fixed points is highlighted in a new section 5.9 [5.I]. The existence of a calm local selection to the inverse of a function which is merely directionally differentiable is proved in the new section 5.11 [5.K]. Particularly interesting is the personal letter of R. G. Bartle, added to the commentary to that chapter; this letter gives an authentic insight into the history of the Bartle–Graves theorem.

In Chapter 6, the authors handle crucial applications to numerical analysis, like iterative procedures for generalized equations, metric regularity of Newton's iterations, inexact (generalized) Newton methods, and others. By comparison with the first edition, this chapter contains a large number of changes and additions. Of particular interest is the extension of the paradigm of the Lyusternik–Graves theorem to mappings defined by iterative methods, given in section 6.4 [6.D]. A new look at the path-following methods for variational inequalities together with error estimates are presented in section 6.7 [6G]. The role of metric regularity in optimal control problems is demonstrated in section 6.9 [6I].

In summary, the book represents the state of the art of the modern theory of inverse and implicit functions and provides an important source for studies of numerical methods and applications in this area. It can be warmly recommended to all specialists and advanced students working in optimization, analysis, numerical mathematics, and other mathematical fields, as well as to all those who apply variational analysis in engineering, physics, operations research, economics, finance, and more.

DIETHARD KLATTE
Universität Zürich

Calculus Without Derivatives. *By Jean-Paul Penot.* Springer, New York, 2013. $89.95. xx+524 pp., hardcover. ISBN 978-1-4614-4537-1.

Roz Chast has a wonderful cartoon entitled "Falling Off the Math Cliff" [*New Yorker*, March 6, 2006] showing a student at various stages of his mathematical education, visualized as scaling a rugged mountain peak and ending in a precipitous drop just as the student seems confident of his understanding. How true! It is at those moments, just before the fall, that I remember Jamie Sethian saying in one of his undergraduate lectures at Berkeley that mathematicians are junkies for those exhilarating flashes, lasting maybe two minutes, where they feel like they understand what is going on.

For most North American teenagers, calculus stands as a daunting, if not terrifying, barrier to the vast world of science and engineering, dry hills and dark canyons of exotic symbols littered with the corpses of pilgrims who didn't make it. It is known as "Introductory Analysis" for most other students throughout the world, but these foothills to the greater Analysis Range are pretty much the same everywhere: a highly refined canon of facts and computational techniques of integration and differentiation through which instructors shepherd their young charges with inspirational peeks at the promised land that lies beyond. It is hard to fully appreciate the amount of midnight oil spent by some of the most celebrated—and many of the uncelebrated—minds of recorded history to characterize, synthesize, and simplify the vast array of ideas, arguments, and computational tricks tossed about over centuries so that, over the course of one year, a student can leap past hundreds of years of intellectual development and bitter philosophical battles just to get to the beginning of the beginning of what we could call "university mathematics." I'm not sure whether, despite their disagreements, Leibniz and Newton (and Fermat!) would be unanimous in their horror or delight at what has become of their brainchildren, but modern mathematicians would certainly agree that integral and differential calculus today has succumbed to the exigencies of an audience with very broad horizons and strict limitations on time, interest, and patience. Not only are functions conflated with their values, but, worse, the study of limits is often crowded out by formal calculation and applications. These formal calculations are so effective that one can easily be lulled into the illusion that smooth functions are the only ones that are relevant, and that the story ends when you hit a kink.

Ask any beginning student about a tangent to the graph of $x \mapsto |x|$ at $x = 0$ and she will probably tell you that it doesn't exist or, at least, that it is not well-defined. After all, this function is not differentiable at the origin. We equip students to travel through gently rolling hills, so it is quite natural to avoid the rough patches. Until recently, most could ignore this part of the intellectual universe and be no worse off for it. Classical mechanics is predicated on the continuum and deterministic, one-to-one models—this is the legacy of integral and differential calculus. But then there are the laws of Man, contradictory, asymmetric, and political, and certainly not logical. Not only Man, but even Nature often resists the tools of differential calculus: think of quantum phenomena, breaking waves, and explosions. The usual calculus of differentiation is useless in these situations and collapses almost completely when you remove the basic assumption that directional derivatives must meet at a point. All of the facts that are more or less taken for granted—the gradient and derivative of a function are one and the same—can no longer be glossed over. An entirely new vocabulary is needed to handle these exceptional cases. For continuous convex functions on open subsets of Banach spaces with separable duals, the set of points where the Fréchet derivative exists is *dense* [1], so we can be comforted by the fact that a smooth handhold is never far away. Yet often, and particularly in optimization, the solutions to our problems lie *exactly* at the exceptional points where classical differential calculus breaks down, as the problem of minimizing the function $f(x) = |x|$ illustrates.

The title of Jean-Paul Penot's recently published collection of lecture notes in analysis, *Calculus Without Derivatives*, is provocative and evocative of Chast's math cliff. These 478 pages (plus references and index) encapsulate everything that your calculus teacher, following generations of educators, avoided on your first introduction to analysis. It is a bit like learning arithmetic all over again in a university course on real analysis: you see for the first time the bulk of the iceberg beneath the surface. Much of the content is, on mathematical timescales, recent, having its origins in the latter half of the 20th century. The book collects three different branches of analysis: differential calculus, convex analysis, and nonsmooth analysis. The last of these three topics is properly a generalization of the first two, which are evolving to become different species of the same genus called nonsmooth or variational analysis, depending on who you ask. A survey of facts and tools from topology and functional analysis supports these three themes.

Unlike the usual calculus texts, Penot does not shy away from precision. The book unfolds slowly and deliberately, starting with Zorn's lemma and convergence of nets in topological spaces. The latter is necessary for weak* topologies on dual Banach spaces, which is not only a natural setting for *subdifferentials* (generalized derivatives) and *Fenchel conjugates* of functions on normed spaces, but is necessary for characterizing the right notions of compactness of subsets of, and functions on, normed spaces, which generates new spaces (*weakly compactly generated spaces*, in particular) that arise in nonsmooth analysis. Set-valued mappings are treated early and without much ado. In a twist on the usual order, optimization and control theory are briefly mentioned as motivations for set-valued mappings, but without any prior development of variational problems or convex analysis—this comes later. Convex analysis of sets (topological properties and separation) is presented in considerable detail. The Hahn–Banach–Fenchel duality cycle, of which so much is made in [2], for instance, is nearly completed in this subsection, but Penot delays more than a passing mention of Fenchel duality until Chapter 3 where the analytical properties of convex functions (continuity, differentiability, calculus rules) are developed. Variational principles close out the introductory material, which are immediately put to use in a sort of proto-numerical analysis of variational methods including descent methods, error bounds, regularization and penalization methods, metric regularity, and the extension of Lipschitz continuity of set-valued mappings. This is not unreasonable, given the focus of the first chapter on topological and metric properties of sets. It is, however, a juxtaposition noticeable in its shift in tone and focus from the previous elements of the chapter, and it is one of the few places in the book where references to numerical methods can be found, other prominent occurrences being in Chapter 2 (Newton's method, of course, and the method of characteristics). This first quarter of the book concludes with the notion of well-posedness for variational problems.

The famous saying "standing on the shoulders of giants" attributed to Isaac Newton fittingly consecrates the second chapter, which is dedicated to a much more rigorous and general treatment of differential calculus than you would find in a calculus text. I can't help but wonder if Penot knew of the snarky context of Newton's lovely words penned in a letter to his rival, Robert Hooke, who was the physical antithesis of a giant. The central goal of the chapter is an analysis of the invertability of nonlinear maps culminating in the theorem of Lyusternik–Graves, the inverse mapping theorem, and their many applications. Applications to optimization and the calculus of variations, which have long been the driving force behind nonsmooth analysis, are treated in separate subsections. The normal and tangent cones, which figure so prominently in other books on variational/nonsmooth analysis and generalized differentiation, are introduced in the subsection presenting applications to optimization.

Convex functions serve as the vehicle transporting us away from classical differential calculus in Chapter 3. Here most of the remaining canon of convex analysis is developed: continuity, (generalized) differentiability, subdifferential calculus and the Legendre–Fenchel transformation. The final two subsections of the chapter concern the choice of spaces on which to formulate a given problem. Naturally, vector spaces with strictly convex norm are desirable, as are spaces on which continuous convex functions are differentiable (Fréchet or Hadamard) on dense subsets of open sets—the smooth handholds alluded to earlier.

Subsequent chapters deal with nonsmooth analysis outside of the context of convexity, which requires a closer attention to detail. Here the "four pillars of nonsmooth analysis" are presented in their fullest generality: normal cones, subdifferentials, tangent cones, and directional derivatives. (There are even six pillars if you count coderivatives and graphical derivatives.) The convex (Moreau–Rockafellar) subdifferential bifurcates into the Fréchet and the directional (or Dini–Hadamard, Bouligand, or contingent) subdifferential, which in finite-dimensional normed spaces are equivalent. Calculus rules for sums and compositions of functions are developed, and the relationships with geometrical counterparts, normal cones, and tangent cones, are explored. The last section of Chapter 4 contains a broad array of applications—the only direct nod to applications encountered in the latter third of the book. Chapter 5 is a survey of one of the most famous generalizations of the derivative due to Clarke, where the duality persists between normal and tangent cones and subdifferentials and directional derivatives. The cost of this clean picture is loss of information: the Clarke subdifferential is often too large for an inclusion of the form $0 \in \partial f(x)$ to be meaningful, the consequences of this lack of precision leading rather abruptly to a sound and fury signifying nothing. Chapters 6 and 7 are a study of limiting and graded subdifferentials, respectively and offer the most complete and informative theory of generalized differentiation, extending naturally to multivalued mappings. This is presented at an expert level and is, by the author's own admission, not easy going. It is, however, essential for seeing the wild frontiers of nonsmooth analysis.

There is by now an established list of topics and ideas attached to nonsmooth (or variational) analysis, and Penot's book does not deviate from this list. Other books from this field may differ in emphasis and depth on selected topics, but few are as broad and complete in their scope. What makes Penot's work stand out is his path through the material and the clean and scholarly presentation. It is well suited for individual study or a classroom, though classroom use would require a considerable commitment at the outset. As preparation for the rough road ahead of us in the coming decades, it might be worth the investment.

REFERENCES

[1] E. Asplund, *Fréchet differentiability of convex functions*, Acta Math., 121 (1968), pp. 31–47.

[2] J. Borwein and J. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counterexamples*, Cambridge University Press, Cambridge, UK, 2010.

Russell Luke
Universität Göttingen

Singular Perturbations: Introduction to System Order Reduction Methods with Applications. *By Elena Shchepakina, Vladimir Sobolev, and Michael P. Mortell.* Springer, London, 2014. $59.99. xiv+212 pp., softcover. Lecture Notes in Mathematics. Vol. 2114. ISBN 978-3-319-09569-1.

It's common knowledge, based on the Tikhonov–Levinson theory from the 1950s, that the asymptotic solution to initial value problems for singularly perturbed slow-fast vector systems of ordinary differential equations (under appropriate stability and smoothness hypotheses) consists of a thin initial layer of nonuniform convergence tending to an outer expansion. When the initial values lie on the appropriate slow initial manifold, the initial layer of rapid change is unnecessary. The impact of such reduced-order models on computations and asymptotic solutions is important in applications ranging from chemical kinetics to control theory.

Vladimir Sobolev and his Russian coworkers have been pursuing such studies for forty years, while his former student from Samara, Elana Shchepakina, has largely specialized in applications involving canards and the even rarer black swans. Michael Mortell studied mechanics and asymptotics at Caltech and has maintained those and related interests, even after serving as president of University College, Cork. The late professor Alexei Pokrovskii actively encouraged this collaboration, which has now resulted in this significant monograph.

The book's success lies in its detailed analysis of many specific examples. The material, necessarily a bit complicated, has been presented and illustrated very clearly, with lots of motivation from a variety of applied fields and with sophisticated extensions that would typically be overlooked. This is the place to learn about the asymptotics of enzyme kinetics, high gain control, combustion, conditionally stable manifolds, and canard cascades.

Robert E. O'Malley, Jr.
University of Washington

A First Course in the Calculus of Variations. *By Mark Kot.* American Mathematical Society, Providence, RI, 2014. $50.00. x+298 pp., softcover. ISBN 978-1-4704-1495-5.

I've long known that my colleague Mark Kot is a great scholar and dedicated teacher, as you'll learn by reading his new book on the calculus of variations. In a rich bibliography, Mark refers you to over twenty books on the calculus of variations and to quite a few on its history. The subject has been of great interest to him since his undergraduate days and he's since taught the subject many times, honing his lectures to the most perceptive approaches, clearing up old misunderstandings, and illustrating the necessary and sufficient conditions using specific illustrations and just-right examples. I know this is not an easy task, since I still recall my course on the subject from a distinguished mathematician who repeatedly got stuck putting together the needed sequence of clever arguments.

Mark demonstrates in substantial detail how the subject developed through the work of Euler, Lagrange, Legendre, Weierstrass, and Carathéodory, among others. Although optimization continues to develop in new directions and applications of variational calculus and optimal control arise frequently, most of the basics were available eighty years ago. The material will nonetheless remain of value to graduate students from many fields. It relies on good working skills in calculus, geometry, and mechanics. Following Mark's example, it's a good idea for instructors to have students write or present term papers on applied problems from the literature close to their own specialties in addition to figuring out some of the challenging exercises posed.

In summary, this is an outstanding and well-motivated book on a classical method of applied mathematics. The cover art combines figures that Euler used and some that Kot developed.

Robert E. O'Malley, Jr.
University of Washington

Asymptopia. *By Joel Spencer with Laura Florescu*. American Mathematical Society, Providence, RI, 2014. $30.00. xiv+183 pp., softcover. ISBN 978-1-4704-0904-3.

There are many good books on asymptotics with similar standard content, like Copson's *Asymptotic Expansions* (Cambridge, 1965). Spencer and his graduate student here provide quite a different introduction, based on insights gathered during Spencer's important life work which includes random graphs, prime numbers, combinatorics, theoretical computer science, and Ramsey theory. They show that doing asymptotics (rigorously!) is an art and suggest that *Asymptopia* is a world inspired by Paul Erdos.

They begin by using calculus arguments to obtain Stirling's approximation, which they call the most beautiful asymptotic formula, and follow with a chapter called "Big Oh, Little Oh, and All That." Ultimately, however, they consider binomial coefficients, unicyclic graphs, large deviations, algorithms, the iterated logarithm, and really big numbers. Despite very clear and well-motivated writing, readers will need some preliminary familiarity with probability and combinatorics, but will be convinced that asymptotics provides a powerful tool to do lots of significant mathematics.

Robert E. O'Malley, Jr.
University of Washington

Differential Equations: Theory, Technique, and Practice. Second Edition. *By Steven G. Krantz*. CRC Press, Boca Raton, FL, 2014. $99.95. xiv+541 pp., hardcover. ISBN 978-1-4822-4702-2.

George F. Simmons published the especially readable McGraw-Hill textbook *Differential Equations with Applications and Historical Notes* in 1972. A second edition (with John S. Robertson) appeared in 1991, and a third edition is scheduled for 2015. The first edition of this book under review had Simmons as the first author, while this edition is dedicated to Simmons "For the example that he set." Steven Krantz is, of course, the author of dozens of mathematics books and an expert on complex and harmonic analysis.

Its chapters are "What Is a Differential Equation?," "Second-Order Linear Equations," "Qualitative Properties and Theoretical Aspects," "Power-Series Solutions and Special Functions," "Numerical Methods, Fourier Series: Basic Concepts," "Partial Differential Equations and Boundary Value Problems," "Laplace Transforms," "The Calculus of Variations," "Systems of First-Order Equations," "The Nonlinear Theory," and "Dynamical Systems." Compared to most other textbooks for a course on ordinary differential equations, the coverage is extensive. Moreover, the writing is appealing, leading readers on in a sensibly motivated manner. Unusual topics include Picard iteration, the Sturm comparison theorem, the hypergeometric equation, and the Poincaré–Bendixson theorem. Interested students might want to see more on dynamical systems, however, and less about the tautochrone. The problems are nontrivial and traditional applications to physics and mechanics are plentiful, in addition to predator-prey models.

The historical notes and nuggets are very informative and cover most of the great contributors to differential equations and their applications. You might be surprised to read about Hermite, Green, Liapunov, Lienard, Poisson, Smale, Steinmetz, Volterra, and Wronski. Regarding Runge, Krantz curiously tells about a fistfight Szego had on a train (though it might have been Polya). Students will understand that those who came before were diverse and certainly not all boring theorem provers.

You will find this to be an outstanding text, especially for good students wanting a challenge.

ROBERT E. O'MALLEY, JR.
University of Washington

Contract Theory in Continuous-Time Models. *By J. Cvitanic and J. Zhang.* Springer, New York, 2013. $79.95. xii+255 pp., hardcover. ISBN: 978-3-642-14199-7.

Contract theory deals with contracts between a Principal and her Agent. The Principal can be an investor or a company, and the Agent the portfolio manager or the chief executive. The Principal must propose a contract to the Agent, which should satisfy two requirements: it should be acceptable to the Agent, and the Principal should get the most out of the Agent's actions. There are three classical versions of this problem. In the first one, called "Risk Sharing," both the Principal and the Agent have access to the same information. In the second, called "Hidden Action," the action of the Agent is not observable by the Principal. In the third, called "Hidden Type," the Principal (P) does not know all characteristics of the Agent (A) (for instance, P does not know A's risk aversion).

Of course, there is a lot of uncertainty in this kind of situation. The models are considered in continuous time, and the uncertainty is modeled by Brownian motion. Each one of the two contractants wants to maximize its utility, so there are in principle two intricate stochastic control problems associated with such a contract.

There are two main theoretical tools to deal with a stochastic control problem, namely, the theory of Hamilton–Jacobi–Bellman equations and the theory of the maximum principle. The first approach requires the state process to be Markovian and is well suited for the numerical computation of an approximate optimal control. The maximum principle leads to a characterization of an optimal strategy, which can lead to an explicit formula in some particular cases. The book takes this approach.

For deterministic control problems, Pontryagin's maximum principle, which gives necessary conditions (and in certain cases sufficient conditions) for a control to be optimal, has been known since the 1950s. The maximum principle is expressed in terms of the state and an adjoint state, the latter being in general the solution of an ODE with a final (rather than an initial) condition, i.e., it is a backward ODE. In the stochastic case, the problem is to get an adapted (i.e., depending at each time t upon the past of the driving Brownian motion and not upon its future increments) solution to a backward stochastic differential equation (BSDE). This problem motivated S. G. Peng at the end of the 1980s to study such equations; see [2]. Let us explain how this problem is solved (in the particular case of scalar solutions for simplicity). The solution of a stochastic differential equation (SDE) is a process $\{X_t,\ t \geq 0\}$ which solves

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

where $\{W_t,\ t \geq 0\}$ is a Brownian motion and the stochastic integral is interpreted in the Itô sense. If b and σ are Lipschitz continuous mappings, this SDE has a unique solution. Now let $T > 0$, let ξ be a square-integrable random variable which is a functional of $\{W_t,\ 0 \leq t \leq T\}$, and let $f(\omega, t, y, z)$ be jointly measurable, a functional of $\{W_s,\ 0 \leq s \leq t\}$ for each $t > 0$, and uniformly Lipschitz in y and z. Then there is a unique pair $\{(Y_t, Z_t),\ 0 \leq t \leq T\}$ of adapted processes giving a solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s.$$

Here we need the second process Z to make Y adapted (in the case $f \equiv 0$, Z is provided by Itô's martingale representation theorem). Note that ξ could be of the form $\xi = g(X_T)$ and f of the form $f(\omega, t, y, z) = h(X_t(\omega), y, z)$, where X is the solution of the previous SDE. Such a pairing of an SDE and a BSDE is called a forward-backward SDE (FBSDE) (in this case a "decoupled" FBSDE, since the coefficients of the forward SDE do not depend upon (Y, Z)). Finally, the coefficients b, σ, and h may depend in addition upon a control u. In such a case, with the above choice of ξ and f, we have

$$Y_0^u = \mathbb{E}\left[\int_0^T h(X_t^u, u_t, Y_t^u, Z_t^u)dt + g(X_T^u)\right],$$

where X^u solves the SDE

$$X_t^u = x + \int_0^t b(X_s^u, u_s)ds + \int_0^t \sigma(X_s^u, u_s)dW_s$$

and (Y^u, Z^u) solves the corresponding BSDE. We recognize that Y_0 has the general form of the cost functional in a stochastic control problem. Hence, it is natural to formulate a stochastic control in continuous time as the problem of controlling a BSDE. Since BSDEs will enter the scene for the maximum principle, there is no way to avoid them anyway.

The last part of the book under review consists of three chapters: Chapter 9 gives a clear and concise presentation of the theory of BSDEs, including the class of BSDEs whose coefficient may be quadratic in the variable Z (which is needed in the rest of the book). Chapter 10 presents various versions of the stochastic maximum principle for the control of various types of BSDEs. Finally, Chapter 11 describes the theory of "coupled" FBSDEs.

The rest of the book studies the three classes of Principal–Agent contracts. After an introduction which presents the problem with its three variants, and their solutions in examples of single-period versions of the problem, the next three parts study the three versions of the problem in continuous time. The Risk Sharing problem is presented first in the case of a linear model, and then in full generality. The next section studies the Hidden Action case. The first chapter of this section presents the general problem under various conditions upon the coefficients, detailing both the Agent's and the Principal's problems. The second chapter presents solutions in special cases, both by a direct approach and by the application of the general tools presented in the last part of the book. The third chapter presents an application to the optimal financing of a company. Finally, one additional chapter presents the hidden-type model, also called the "adverse selection," since the Agent can pretend to be of a different type than he really is, which can adversely affect the Principal's utility. The most explicit results are obtained in the case where both the Agent and the Principal are risk-neutral.

The present referee is in no way an expert on contract theory. However, the presentation in this book is really clear. It could also be considered as an exposition of both the theory and application of stochastic control, including the theory of BSDEs. The probably better-known aspect of the theory of stochastic control is that of Hamilton–Jacobi–Bellman PDEs, which applies to the control of Markov processes; see, in particular, [1]. The stochastic maximum principle theory is more recent (at least in the modern formulation using BSDEs) and has not been much exposed in books so far, with the notable exception of [3]. The present book presents a nice exposition of the theory of the stochastic maximum principle, starting with BSDEs, and of its applications to contract theory.

The reading of this book clearly requires an understanding of stochastic calculus. The mathematical treatment is rigorous, but can also be followed by less technical readers, the most tricky computations being deferred to the final section of Chapter 10. This makes the book nice to read. I recommend it to anyone working on or teaching the mathematical aspects of contract theory and/or stochastic control.

REFERENCES

[1] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd ed., Springer, New York, 2006.
[2] E. Pardoux and S. G. Peng, *Adapted solution of a backward stochastic differential equation*, Systems Control Lett., 14 (1990), pp. 55–61.
[3] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer, New York, 1999.

Etienne Pardoux
Aix–Marseille University

Stochastic Processes. *By Richard F. Bass.* Cambridge University Press, Cambridge, UK, 2011. $75.00. xvi+390 pp., hardcover. ISBN 978-1-107-00800-7.

Richard Bass's book on stochastic processes gives a fresh account of a classical topic. What stands out is the balance between rigor and accessibility. The book fills a gap in the library of standard texts which for the most part either focus on less technical stochastic processes in discrete time or are technical tomes on continuous time stochastic processes that are harder for beginners to absorb.

The focus of the book is from the start on stochastic processes in continuous time, with Brownian motion at the forefront in the first several chapters. Bass keeps the presentation as accessible and nontechnical as possible, however, without leaving out measure theoretic details. From a firm foundation for Brownian motion, Bass goes on to give an introduction to the most relevant topics in the current theory of stochastic processes. He covers martingales, stochastic analysis with stochastic integration, and stochastic differential equations. He includes processes with jumps such as Poisson (point) processes and Lévy processes and also considers stochastic analysis for discontinuous processes. Weak convergence of probability measures on general spaces and thus convergence of stochastic processes are also treated, although not in so much detail. General Markov processes and their description via infinitesimal generators appear interspersed and toward the latter half of the book. Here, it should be mentioned that if you want to build basic intuition on Markov processes by first considering discrete time Markov chains, it would probably be wise to look elsewhere beforehand.

Advanced material is also covered, including general concepts of stochastic process theory as well as local times, Brownian excursions, and Ray–Knight theorems. Some chapters on applications are focused on financial mathematics (for example, the Black–Scholes formula and stochastic control) as well as filtering (in particular, the Kalman–Bucy filter). In order to keep the text readable, Bass does not aim to present the most general setup or results, and he keeps the chapters intentionally rather short with ample and quite doable exercises (without solutions).

Bass's book is well written with very few typos and a nice layout that is not too crammed and dense. Short introductions to the chapters provide background and general ideas of the material ahead as well as pointers about what is technical and may be skipped (on first reading) or what is really needed later on in the book. Notes at the end of chapters—in particular, the more advanced chapters—give hints for further reading as well as for alternative sources or proofs. There are also some interesting comments on the history of the material, pointing, in particular, to modern results, as well as modern improvements to classical results and proof ideas, a number of which are due to the author himself.

For graduate students in mathematics with a background in basic measure theoretic probability theory, this book serves as a good first resource for studying stochastic processes. A nice review of essential probability topics, including proofs, is provided in the appendix, but does not (and is not meant to) replace prior study of the subject. Readers are expected, for instance, to be already proficient in the basics of measure theory, which is not covered in the appendix. Given the appropriate background, and possibly some supplementary basic text on (discrete time) Markov chains (discrete time martingales can be found in the appendix), the book should also be quite suitable for self-study. The author himself acknowledges that, even for a year-long lecture course, a selection from the extensive material would have to be made. Suggestions for a manageable path through the text can be found in the preface to the book.

Overall, this new book on stochastic processes is an excellent read that provides a great basis for further study and research in this very active field.

ANJA STURM
University of Göttingen

SIAM Review **Vol. 57, Issue 3 (September 2015)**

Book Reviews

BOOK REVIEWS

The greatest challenge of this job is to find just the right reviewer for each book. Personal contacts get one only so far; I know lots of people in or near my own research area, but not so many in the larger applied mathematics community. So where do I look for reviewers? The Internet, of course! By searching the web I can find lots of names. But this approach has a drawback: the names I am most likely to find are those of very senior people, who are often too busy to write a review. Up-and-coming younger workers, who might have the time and might benefit from the exposure, are much less visible. I would like to find you.

Writing book reviews can be fun and rewarding. You spend some time with the book, get to know it well, then try to write something interesting and informative for the community. This is not a waste of time. You will surely learn something in the process. Moreover, if you write good reviews, they will be noticed.

If you (young *or* old) think that you would like to try your hand at writing reviews, please let me know (or contact one of the other members of the editorial board). Send me an email or introduce yourself at a meeting. Tell me (preferably in not too much detail) what interests you; perhaps I'll be able to find something for you.

In this issue we are pleased to offer you a timely review by Nick Trefethen of a large tome that is scheduled to appear this month, namely, *The Princeton Companion to Applied Mathematics*, edited by Nicholas J. Higham. This ambitious project is in the same spirit and style as *The Princeton Companion to Mathematics*, edited by Timothy Gowers, which appeared seven years ago. I hope you enjoy Nick's review.

In addition we have reviews of books on a wide variety of topics, including numerical linear algebra, risk and portfolio analysis, stochastic chemical kinetics, quantum mechanics, and computational complexity theory.

David S. Watkins
Section Editor
siam.book.review@gmail.com

SIAM Review
Vol. 57, No. 3, pp. 469–480

Book Reviews

Edited by David S. Watkins

Featured Review: The Princeton Companion to Applied Mathematics. *Edited by Nicholas J. Higham.* Princeton University Press, Princeton, NJ, 2015. xx+988 pp., hardcover. ISBN 978-0-6911-5039-0.

What is applied mathematics? How does it relate to pure mathematics, or should we simply say, to mathematics? With the appearance of the *Princeton Companion to Applied Mathematics*, we have two magnificent data points, 1000 pages each, to help us reflect on these questions.

The first thing one feels on looking at this volume is, quite simply, pleasure. Seven years ago *The Princeton Companion to Mathematics* was published to wide acclaim, and it was clear that a similar work on applied mathematics might be a good idea. Now it has appeared. The look and feel are the same, and this is highly satisfying. Figure 1 shows maps of the two volumes, which I will have more to say about in a moment. In each case the editors divided the collection into eight parts featuring pieces of differing lengths and flavors. The formatting and the typesetting are closely matched, and the new *Companion* is a perfect companion to the earlier one.

PCM was masterminded by Timothy Gowers of the University of Cambridge, and *PCAM* has been created by Nick Higham of the University of Manchester. These must be two of the most capable editors on earth. Anyone who knows Gowers and Higham will be aware of their combination of broad mathematical vision with phenomenal attention to detail. Higham, the Richardson Professor of Applied Mathematics at Manchester and a Fellow of the Royal Society, is celebrated not just for his research in numerical analysis but also for his outreach activities, including his *Handbook of Writing for the Mathematical Sciences*, the *MATLAB Guide* (coauthored with brother Des), and his blog. Like Gowers, he is a leader who cares deeply about his field. Princeton's appointment of him as editor was the perfect choice.

Of course, it takes a village. Like *PCM* before it, *PCAM* has a board of associate editors who helped shape the volume and contributed some of the articles: Mark Dennis, Paul Glendinning, Paul Martin, Fadil Santosa, and Jared Tanner. (I am pleased to note that five of the six editors are connected with England.) It has 165 authors, experts in their topics, many of them very eminent. A key person in the back office was Sam Clark of T&T Productions Ltd, who as project manager for both *PCM* and *PCAM* was involved in all the details and deserves much of the credit for making them such a comfortable pair.

So, what can you do with a book so big that its weight is measured in kilograms? With a million words of first-rate applied mathematics?

One possibility is to read it cover to cover. I more or less did that, but I doubt you will. (I did it as much to learn about myself as to learn about applied mathematics, which brings us to our first difference between mathematics and applied mathematics. I could not have read *PCM* cover to cover.)

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *Market St.,* 6^{th} *Floor, Philadelphia, PA* 19104-2688.

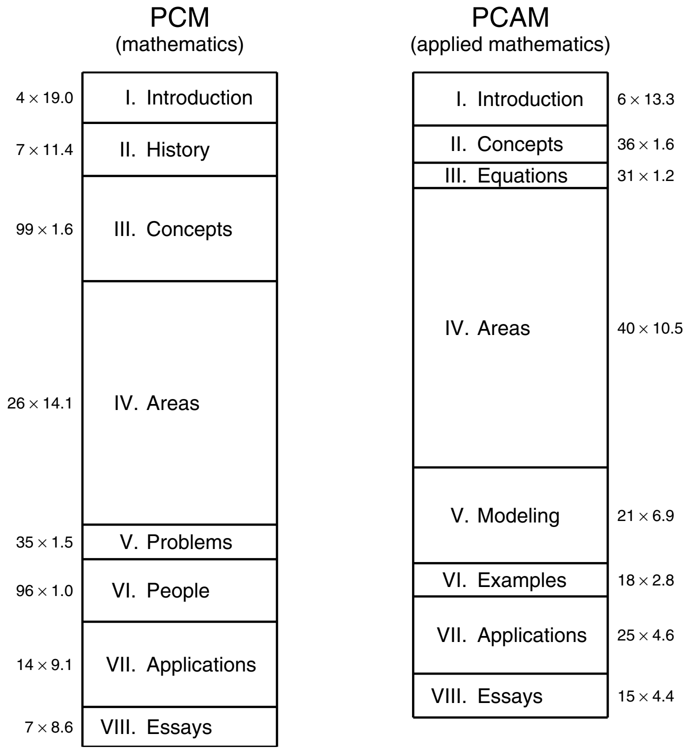

Fig. 1 *The eight parts of the two Princeton* Companions*, with shortened names. Annotations indicate numbers and average lengths of articles. For example,* PCAM *has* 18 *articles in Part VI, on Example Problems, and their average length is* 2.8 *pages.*

Another approach is to dip in and out at whim, as Higham et al. encourage.

> *Despite the careful organization, the editors expect that many readers will flick through the book to find something interesting, start reading, and by following cross-references navigate the book in an unpredictable fashion. This approach is perfectly reasonable.*

You bet it's reasonable! Your eye will be caught by Barbara Keyfitz on conservation laws, by Berry and Howls on divergent series, or by Jane Wang on insect flight. Maybe by Phil Holmes on dynamical systems, or Jack Dongarra on high-performance computing, or Andreas Griewank on automatic differentiation. The treasures go on and on.

I loved some of the accounts of things I hadn't known about. Ken Golden on the mathematics of sea ice, showing us how percolation theory applies to actual percolation—fascinating. Villani and Mouhot's article on kinetic theory—idiosyncratic and thoughtful, including a summary of 50 important papers in the field from 1912 to 2013. Donald Saari's beautifully simple explanation of why physicists believe the universe is full of dark matter. Doug Arnold's flight of a golf ball, a perfect example of how, by focusing on something small, we can see things that are big.

A third approach to this book is to use it for reference on smaller subjects or serious learning of bigger ones. I think the potential here is very great. All of us have areas we've touched upon but not immersed ourselves in properly, and some of these pieces offer outstanding opportunities for taking that next step. For example, I was grabbed by David Tong's article on classical mechanics, whose clarity is illustrated by its opening lines.

> *Classical mechanics is an ambitious subject. Its purpose is to predict the future and reconstruct the past, to determine the history of every particle in the universe.*

There are dozens of truly deep and expert survey articles in *PCAM,* such as Brian Davies on spectral theory, Stephen Wright on continuous optimization, Hairer and Lubich on the numerical solution of ODEs, David Griffiths on quantum mechanics, and Emily Shuckburgh on the dynamics of the Earth's ocean and atmosphere. Graduate students and established researchers will be profitably reading these articles for many years.

But *eight parts*?? What's going on here?

Those of us of a certain age remember when the *Encyclopedia Britannica* raised eyebrows with its 15th edition in the 1970s. Instead of the traditional flat collection of articles, they brought out 28 volumes divided into the *Propedia,* the *Macropedia*, and the *Micropedia*. Was this controversial organizational principle a success?

If you take a look at the eight parts of *PCAM,* charted in Figure 1, the presence of some of them seems self-explanatory. Of course there is going to be an Introduction to Applied Mathematics, which here consists of six articles on foundational material. (Five are by Higham, with a generally numerical viewpoint, but this lack of diversity in the opening 55 pages is unrepresentative. Overall the book is very balanced, not at all dominated by numerics.) And following *PCM's* successful model, it is satisfying to find a final Part VIII of Final Perspectives. This assortment begins with opinion pieces by Gowers and Higham themselves and moves on to Ian Stewart, David Donoho and Victoria Stodden, David Bailey and Jonathan Borwein, Heather Mendick, David Acheson, Peter Turner, Gil Strang, Rachel Levy, Ya-xiang Yuan, Maria Esteban, Jim Crowley, and Alistair Fitt. One of my favorites is Stewart's zestful essay on "How to Write a General Interest Mathematics Book."

> *I have called mathematics the Cinderella science. It does all the hard work but never gets to go to the ball.*

PCAM's organizational confusion lies in the middle 800 pages, Parts II to VII, which are devoted to Concepts, Equations, Areas, Modeling, Examples, and Applications. Though one can only admire the impulse to classify, I am afraid it is very difficult to keep these six headings straight. Why is Benford's Law an Equation and the traveling salesman problem an Example? Why does modern optics belong to Modeling and control theory to Areas of Applied Mathematics? It all feels rather arbitrary.

Naturally you get curious and wonder, were the middle six parts of the earlier *PCM* equally hard to keep straight? On inspection it turns out that no, they were not, because three of them had specially memorable themes. Part II was on history, a category that *PCAM* does not repeat. Part VI was on mathematicians, 96 one-page mini-biographies from Pythagoras to Bourbaki, which *PCAM* also does not repeat. (Too many of the best applied mathematicians had already been covered as mathematicians, I suspect, though Bourbaki, I hasten to add, was certainly not applied, quite apart from the question of existence.) And Part VII of the original *Companion* was also special, being devoted to applications ("The Influence of Mathematics"). Perplexingly, this is a category that *PCAM does* repeat, giving us effectively Applications of Applied Mathematics. (One muses about GNU's Not Unix and turtles all the way down.)

So the organization of *PCAM* is unconvincing, and Princeton could have maintained the *PCM* look and feel with five parts instead of eight, but to tell the truth, it doesn't matter. No *Britannica* reader worried much about the *Propedia* back in the 1970s, and nobody's going to lose sleep over *PCAM's* structure today. The gold is in the individual pieces, not their organization.

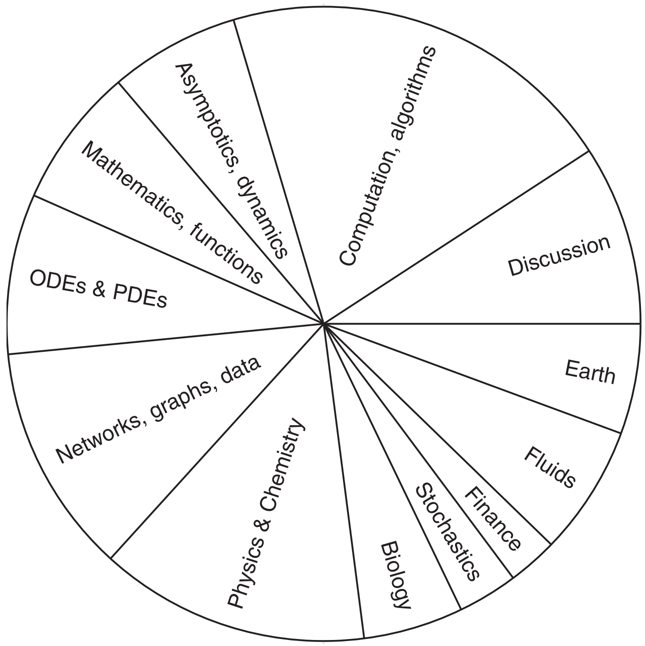

Fig. 2 *Rough breakdown of the pages of* PCAM.

Which brings us back to our opening question. What is applied mathematics? What view of the field does *PCAM* convey? To explore these matters it may be helpful to consider Figure 2, which outlines roughly where *PCAM* spends its pages.

One thing you notice is that there is a lot of physics and fluid mechanics here. These are the oldest, best established parts of applied mathematics, as important as ever, and their strength is powerfully displayed in articles on magnetohydrodynamics, quantum mechanics, optics, gravitation, and many other topics. Newer fields like finance, network theory, and biology are also well represented, although one can't help noting that whereas an article on a physics topic like kinetic theory or solid mechanics, say, will most likely be found in Areas of Applied Mathematics, a biological topic like physiology or biomechanics is more likely to appear in Modeling. It would seem that applied mathematics has terrain it has conquered and terrain it is still exploring. A century from now, will the boundaries have shifted?

It is striking that applied mathematics as displayed in *PCAM* does not define itself by its relationship to (pure?) mathematics, not at all. The book robustly stands on its own, and the proverbial Martian, if he stepped out of his flying saucer and read this volume of mathematics, would not suspect that earthlings knew any other kind.

PCAM shows us that applied mathematics is *vast* and it is *confident*. We see here a discipline engaged in every corner of the human enterprise, from cosmology to the spread of infectious diseases, from pattern formation to aircraft design, from financial portfolio optimization to the ranking of movie preferences. As Strang writes, "Our subject is extremely large!"

LLOYD N. TREFETHEN
University of Oxford

Numerical Linear Algebra with Applications: Using MATLAB. *By William Ford.* Academic Press, San Diego, CA, 2015. $120.00. xxvi+628 pp., hardcover. ISBN 978-0-12-394435-1.

Numerical linear algebra is a course often offered to upper-level undergraduates or early graduate students from a variety of fields, including mathematics and computer science as well as engineering and physical science disciplines. Compared to the other available texts on this subject, this book is meant to be "an entry point" to encyclopedic treatments like Golub and Van Loan's [2] or Higham's [3] as well as to more advanced texts like Demmel's [1] or Trefethen and Bau's [4]. Ford's approach is more closely aligned with Trefethen and Bau's focus on mathematical foundations over Demmel's consideration of efficiency of algorithms and implementations.

An important distinction of this book is the inclusion of the first six chapters on (nonnumerical) linear algebra. One intended audience comprises engineers and scientists who do not have the mathematical background typically supplied by an undergraduate course in linear algebra; the first section of the book is aimed at bringing those readers up to speed before delving into numerical computations. While I appreciate the convenience of having useful background material in the same book (as opposed to referencing a separate linear algebra text), I would like the author to have done more to integrate numerical ideas into the first six chapters. Forward pointers and other references to (the exciting!) numerical ideas to come later could better motivate the material, instead of presenting the ideas more as a stand-alone primer.

The material is fairly comprehensive, spanning nearly the same set of topics as the advanced textbooks [1, 4], but the presentation is certainly gentler. Aside from the initial linear algebra refresher chapters, the book is organized like most others on the topic. It is roughly divided into the following six sections (each given more or less equal weight): introduction to (nonnumerical) linear algebra, introduction to numerical algorithms, solving linear systems (LU factorization and its variants), solving least squares problems (with QR decompositions and the SVD), computing eigenvalue decompositions, and iterative methods. Much more time and space is spent on presenting concrete examples than on in-depth topics; for example, Cholesky decomposition is presented with multiple examples including MATLAB input and output, but the reader is referred elsewhere for the proof of the backwards stability.

Perhaps the biggest gap is a lack of information about available software. Just as the mathematical foundations of linear algebra are well established, algorithms and techniques for achieving high performance in matrix computations are also mature. Software libraries for linear algebra have been evolving with computer architectures, and it's important for engineers and scientists to be aware of the general-purpose, efficient, and up-to-date software already available so they don't waste time reinventing the wheel (or more likely, failing to do so).

I think the ideal reader is a scientist or engineer who takes a class based on this book and then uses it as a reference later in his or her career. It is an accessible introduction to the fundamental matrix computations, particularly to students lacking some of the mathematical background. While a single course is not sufficient to cover the breadth of material, the reader can later revisit sections in order to brush up on the standard techniques for solving the problems of interest and find pointers to more details and information about software in the literature.

REFERENCES

[1] J. Demmel, *Applied Numerical Linear Algebra*, SIAM, Philadelphia, 1997.

[2] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2012.

[3] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, 2nd ed., SIAM, Philadelphia, 2002.

[4] L. N. Trefethen and D. Bau, *Numerical Linear Algebra*, SIAM, Philadelphia, 1997.

Grey Ballard
Sandia National Laboratories

Risk and Portfolio Analysis: Principles and Methods. *By Henrik Hult, Filip Lindskog, Ola Hammarlid, and Carl Johan Rehn.* Springer, New York, 2012. $79.95. xiv+338 pp., hardcover. IBSN 978-1-4614-4102-1.

Investment and risk management problems are fundamental problems for financial institutions. A structured approach to these problems naturally leads one to the field of applied mathematics and statistics in order to translate subjective probability beliefs and attitudes toward risk and reward into actual decisions. These fields of applied mathematics and statistics form a natural basis for quantitatively analyzing the consequences of different investment and risk management decisions. Finance being largely a behavioral science, financial decisions strongly depend on subjective probabilities of the future values of financial instruments and investment choices. As such, financial decisions are often suboptimal, making it even difficult to specify a criterion for a desired trade-off between risk and potential reward in an investment situation. Applied mathematics and statistics can, however, assist in translating a probability distribution and an attitude toward risk and reward into a portfolio choice in a consistent way.

This book presents sound principles and useful methods for making investment and risk management decisions using standard principles, methods, and models. The authors combine useful practical insights with rigorous yet elementary mathematics. The material progresses systematically, and topics such as the pricing and hedging of derivative contracts, investment and hedging principles from portfolio theory, and risk measurement and multivariate models from risk management are covered appropriately.

The chapters have many real-world examples followed by several exercises to help reinforce the text and provide insight. The book is organized into two parts, as follows:

- Part I (principles) is composed of principles of portfolio analysis with a chapter on risk management.
- Part II (methods) covers risk measurement methods and multivariate models.

Chapter 1, on interest rates and financial derivatives, is very brief for a subject so broad. However, the principle of no-arbitrage for valuation of financial derivatives is well presented. Many of the investment and hedging problems that we encounter can be formulated as a minimization of a function over a set determined by the investor's risk and budget constraints and other restrictions on the type of positions that the investor can take. This and related issues are the focus of Chapters 2 to 5, which are on hedging and portfolio optimization.

Having taught risk management in a graduate program and with my experience as a practitioner, I found Chapter 6, on risk measurement principle, to be particularly well presented. The discussion on risk measurement properties is formal, yet accessible, and is followed by real-world examples. This chapter, like all of the chapters, ends with very practical examples that practitioners as well as academics will find very instructive. Chapters 7, 8, and 9 are on empirical models.

Considerations are also made on parametric family of distributions for a random variable and approaches to estimating the parameters. The book also discusses multivariate models for the joint distribution of several risk factors such as returns or log returns for different assets, zero rate changes for different maturity times, changes in implied volatility, and losses due to defaults on risky loans.

The material of this book is based on university lecture notes; as such the organization and structure of the material presented will well serve advanced undergraduate and graduate students. This book will also be beneficial to practitioners in insurance and finance, as well as to regulators. Prerequisites include undergraduate-level courses in linear algebra, analysis, statistics, and probability.

BLESSING MUDAVANHU
University of Witwatersrand &
African Banking Corporation

Stochastic Chemical Kinetics: Theory and (Mostly) Systems Biology Applications. *By Péter Érdi and Gábor Lente.* Springer, New York, 2014. $109.00. xvi+162 pp., hardcover. ISBN 978-1-4939-0386-3.

Compared to the relatively long history of calculus and differential equations going back to Newton and Leibniz, the modern mathematical theory of stochastic processes has a rather young age which can be traced back to A. N. Kolmogorov's foundational work on probability and stochastic processes in the 1930s [1, 2]. While stochastic-process models were widely used in physics following the works of Einstein, Smolochowski, and Langevin, they were not widely taught in chemistry, and even less so in biochemistry. Yet two fundamental papers already appeared in 1940: First, H. A. Kramers [3], the "K" in the WKB method, developed his theory of Brownian motion in a force field with double-well potential and obtained the celebrated "barrier crossing" rate formula that was named after him—using Laplace's asymptotic method for integrals. (This problem is treated briefly in section 3.2.2.2.) Kramers' theory showed how to understand, and compute, the rate constant for a discrete molecular reaction, as a rare event, based on a potential energy function of a collection of interacting atoms immersed in an aqueous "noisy" medium. This theory has become one of the most important items of condensed matter physics and chemistry [4]. In that same year, M. Delbrück, a quantum physicist turned Nobel Laureate in Physiology or Medicine (1969), published a paper [5] in which a continuous-time Markov jump process was used to represent the stochastic dynamics of a small chemical reaction system, assuming the rate constants for each and every reaction are known. This latter theory was later called the *chemical master equation*, and a review article was published in 1967 in the *Journal of Applied Probability* [6]. Together, these two papers paved the computational route from atomic physics to cellular biochemistry.

In 1970s, D. T. Gillespie published several papers, independent of the earlier work, on the stochastic Markov model of discrete chemical reactions, together with the algorithm that generates the exact sample trajectories [7]. This Delbrück–Gillespie approach to chemical reaction systems in a small volume became a major thrust of computational (sometimes called systems) cellular biochemistry in the late 1990s [8]. As a teacher in mathematical biology, I had been stitching together McQuarrie's review [6], a more recent review by Gillespie [9], and a brief Chapter 11 in my own book [10] into one part for my course readings on "Mathematical Theory of Cellular Dynamics." All this time, I have been wishing for a more coherent textbook.

The book under review, therefore, fills an urgent need and will receive warm welcome. I am glad to report that this little book is nicely done, and I will certainly use it next time when I again teach the subject—the only hesitation I have is its steep price of $109!

In many areas of science and mathematics, we in the U.S. have been guilty of ignoring concurrent or even preceding developments in other parts of the world. In this case, as told in the book, M. A. Leontovich had published a theory of chemical kinetics based on random processes as early as 1935 [11]. Furthermore, following an earlier work of A. Rényi in 1953, there is actually a "Hungarian school," centered around Eötvös Loránd University, which has been continuously researching the subject! The torch has been passed along to P. Arányi, J. Tóth, and the two authors of this book, Professors Érdi and Lente. They have brought a wealth of insight and a very unique perspective to their writing. The product is a pleasure to read. (There was also an earlier book by Érdi and Tóth published in 1989 [12].)

The book has three chapters plus a closing "Retrospect and Prospect." Chapter 1, "Stochastic Kinetics: Why and How," in a fairly accessible and relaxed fashion in 20 pages, breezes through the essential notions of deterministic chemical kinetics, Brownian motion, Markov jump processes in terms of master equations, and some very interesting historical remarks. I don't know how a biochemist will react, but anyone with a undergraduate education in physical sciences will get the big picture pretty well.

Chapter 2 is where one learns the mathematics. Some familiarity with probability and differential equations are certainly required to really understand the material. The main approach is what is known as the multi-dimensional birth-and-death process. One of the most important intellectual insights one gains in this approach to chemical kinetics is that deterministic, nonlinear dynamic behavior is an emergent property of an inherently stochastic process.

More specifically, the stochastic mathematical theory lays the foundation for the system of nonlinear differential equations (1b) describing chemical kinetics of m species involved in n reactions (1a) based on the law of mass action:

$$\begin{aligned}\text{(1a)}\qquad & \nu_1^j X_1 + \nu_2^j X_2 + \cdots + \nu_j^m X_m \\ & \underset{k^{j-}}{\overset{k^{j+}}{\rightleftharpoons}} \mu_1^j X_i + \mu_2^j X_2 + \cdots + \mu_m^j X_m,\end{aligned}$$

$$\text{(1b)}\qquad \begin{aligned}\frac{\mathrm{d}x_i}{\mathrm{d}t} &= \sum_{j=1}^{n}\left(\kappa_i^j - \nu_i^j\right) \\ &\cdot \left(k^{j+}\prod_{k=1}^{m} x_k^{\nu_k^j} - k^{j-}\prod_{k=1}^{m} x_k^{\mu_k^j}\right)\end{aligned}$$

$(j = 1, 2, \ldots, n)$, in which x_i is the concentration of chemical species X_i; ν_i^j and μ_i^j are called stoichiometric coefficients; and k^{j+} and k^{j-} are the rate constants for the jth reaction in the forward and backward directions. Corresponding to $\mathbf{x}(t) \in \mathbb{R}^m$, the stochastic dynamics is a Markov jump process $\mathbf{n}(t) \in \mathbb{Z}^m$, where $n_i = Vx_i$ represents the number of X_i in a volume V. A fundamental difference in studying deterministic dynamical systems like (1b) and such a stochastic process is the possibility of dual perspective: One can either study the stochastic sample trajectories $\mathbf{n}(t)$ or its probability distribution $p(\boldsymbol{\ell}, t) = \Pr\{\mathbf{n}(t) = \boldsymbol{\ell}\}$. The latter satisfies a linear differential equation.

Section 2.3.3 contains a nice discussion on the relationship between the deterministic system in (1b) and the stochastic process. Here T. G. Kurtz's "fundamental theorem of stochastic chemical kinetics" is mentioned. It would be nice to make it clear that the linear differential equation for the probability distribution is in a *function space*, which is usually infinite, while the nonlinear dynamics lives in a much lower dimensional space. In other words, as an analogy, $\dot{x} = f(x)$ in $\mathbb{R}^n$ always has a corresponding linear partial differential equation (Liouville) in phase space $\partial_t u = -\nabla_x\big(f(x)u\big)$.

Section 2.3.4 presents new results on stochastic mapping, developed very recently by one of the authors, which identifies in parameter space regions where a stochastic model is required.

Chapter 3 provides an assorted applications of the stochastic chemical kinetic approach. As clearly indicated in the subtitle of the book, they are mostly in systems cellular biochemistry, ranging from applications to membrane noise analysis and fluorescence fluctuation spectroscopy to olfactory neurobiology. The nonlinear bifurcation problem has added new features in stochastic dynamics; population extinction can occur as a rare event, while a differential equation shows $x = 0$ to be an unstable fixed point. Thinking dynamics with multiple time scales are essential. Enzyme kinetics, cellular signaling processes, stochastic gene expression, and chiral symmetry breaking are extensively covered. There is also a section on parameter estimation in stochastic kinetics models, as well as a very brief discussion of stochastic resonance (SR) in chemical systems. It is a pity the authors did not adopt the viewpoint that SR is a consequence of the breakdown of detailed balance—that would have provided a fresh and accessible introduction to the very popular but rather technical subject.

The last section of the chapter contains the novel idea of using stochastic kinetics in the theory of computation. One can sense the influence of J. J. Hopfield, who turned a "memory problem" into *designing* a differential equation with a large number of attractors with prescribed locations.

Not to diminish the virtue of the book, but there are several places where improvements or corrections can be made. For example, the exponential part of Kramers' formula given in equation (3.25) followed the statement "By evaluating the integral. . . ." This could be made more clear by actually showing the integration process. On p. 94, the mean time of the Michaelis–Menten single enzyme turnover could benefit from a parallel mean first-passage-time computation, which might be more illuminating. Sometime jargon slips in without a clear definition first: "infinitesimal generator" on p. 15; "CDS" and "CCS" on p. 18—though they are later defined on p. 27.

The book could benefit from an online erratum, which certainly will come handy in the second print: On p. 16, anomalous diffusion need not to be non-Gaussian, as the theory of fractional Gaussian processes testifies. Equation (1.30) should read $\mathrm{d}x = \sqrt{2D}\mathrm{d}W(t)$; and the time scale for extinction, on p. 74, must be proportional to e^{cV}, where $c > 0$. Extraneous symbols and characters appear in various places, which is more likely a reflection of the copyediting: An ${}^{n}p_{ij}^{m}$ in equation (2.8); $\mathrm{Var}[\xi(t)] = 2\lambda t$ on p. 73; reference [45] on p. 141 should not be "der Chen Y"; a "618" in reference [108] on p. 143 and again in reference [20] on p. 157.

In summary, this is the first book in a new, highly exciting, growing area of applied mathematics, with applications to modern cell biology. The level is very accessible to a wide range of readers from the physical sciences to biology and the life sciences. The subject area is very important. The book is lucidly written and has an extensive, scholarly researched bibliography which will certainly be useful to anyone who wants to go deeper into the subject.

REFERENCES

[1] A. N. Kolmogorov, *Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung*, Math. Ann., 104 (1931), pp. 415–458.

[2] A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer-Verlag, Berlin, 1933; English transl., *Foundations of the Theory of Probability*, Chelsea, New York, 1950.

[3] H. A. Kramers, *Brownian motion in a field of force and the diffusion model of chemical reactions*, Physica, 7 (1940), pp. 284–304.

[4] P. Hänggi, P. Talkner, and M. Borkovec, *Reaction-rate theory: Fifty years after Kramers*, Rev. Modern Phys., 62 (1990), pp. 251–341.

[5] M. Delbrück, *Statistical fluctuations in autocatalytic reactions*, J. Chem. Phys., 8 (1940), pp. 120–124

[6] D. A. McQuarrie, *Stochastic approach to chemical kinetics*, J. Appl. Probability, 4 (1967), pp. 413–478.

[7] D. T. Gillespie, *Exact stochastic simulation of coupled chemical reactions*, J. Phys. Chem., 81 (1977), pp. 2340–2361.

[8] Y. Cao, H. Li, and L. R. Petzold, *Efficient formulation of the stochastic simulation algorithm for chemically reacting systems*, J. Chem. Phys., 121 (2004), pp. 4059–4067.

[9] D. T. Gillespie, *Stochastic simulation of chemical kinetics*, Annu. Rev. Phys. Chem., 58 (2007), pp. 35–55.

[10] D. A. Beard and H. Qian, *Chemical Biophysics: Quantitative Analysis of Cellular Systems*, Cambridge University Press, Cambridge, UK, 2008.

[11] M. A. Leontovich, *Basic equations of kinetic gas theory from the viewpoint of the theory of random processes*, J. Exp. Theoret. Phys., 5 (1935), pp. 211–231.

[12] P. Érdi and J. Tóth, *Mathematical Models of Chemical Reactions. Theory and Applications of Deterministic and Stochastic Models*, Manchester University Press, Manchester, UK, 1989.

HONG QIAN
University of Washington

Quantum Theory for Mathematicians. *By Brian Hall.* Springer, New York, 2013. $89.95. xvi+554 pp., hardcover. ISBN 978-1-4614-7115-8.

This book is an introduction to quantum mechanics intended for mathematicians and mathematics students who do not have a particularly strong background in physics. I suspect there are many in this category. I myself was told as an undergraduate that I would not need any physics for what I was going to be doing. I've been playing catch up ever since. This book, which consists of 23 chapters and an appendix (in about 550 pages), will surely prove useful for this audience and others as well.

The book is, of course, written in mathematical language and aims for a level of precision that one would expect in a graduate textbook. In order to make the book accessible to as large an audience as possible, the author has striven to keep the prerequisites to a minimum. The reader is assumed only to have had at least a first course in real analysis, including some Hilbert space theory. The spectral theory of self-adjoint operators is not assumed; it is covered in the text.

But this is not just a mathematics book. The physical consequences of the mathematical results are discussed at every opportunity. The first chapter discusses briefly the physical experiments that forced scientists to look for a new theory. Chapter 2 is an introduction to classical mechanics in the simplest possible terms, starting with Newton's second law for a single particle in $\mathbb{R}^1$. It then moves on to multiple particles in $\mathbb{R}^n$, then to Hamiltonian mechanics, Poisson brackets, and conservation laws.

The study of quantum theory proper begins with Chapter 3. Wave functions and their probabilistic interpretation are introduced, followed by position and momentum operators. A list of axioms is drawn up, including the Schrödinger equation of motion, motivated by the famous $E = \hbar\omega$. The first example is a particle in a box. Chapter 4 discusses the free Schrödinger equation, solution by Fourier transform, and propagation of wave packets. Chapter 5 analyzes a particle in a square well and tunneling.

This is as far as the author cares to go without the spectral theorem. Chapters 6 through 10 constitute a 100-page chunk of the book in which the spectral theorem for unbounded self-adjoint operators on Hilbert space is formulated and proved. The presentation is quite detailed; the author advises the reader to read as much or as little as (s)he needs. The discussion begins with the finite-dimensional result: every Hermitian matrix is unitarily similar to a diagonal matrix. The difficulties associated with the infinite-dimensional case are discussed, as are the goals of spectral theory. A chapter is devoted to the formulation of the spectral theorem for bounded Hermitian operators on a Hilbert space, and the following chapter provides the proof. Then unbounded operators are discussed, and the spectral theorem for unbounded self-adjoining operators is formulated and proved.

With the spectral theorem in hand, the next chapters cover the quantum harmonic oscillator in one dimension, the Heisenberg uncertainty principle, quantization schemes, the Stone–von Neumann theorem, and the WKB approximation. The Stone–von Neumann theorem says, roughly speaking, that if you have two (or $2n$) operators that interact like position and momentum operators, then they *are* position and momentum operators.

Chapter 16 is a brief presentation of the representation theory of Lie groups and Lie algebras. This is preparation for the discussion of angular momentum and spin in Chapter 17. In fact the quantum mechanical theory of angular momentum is the same as the representation theory of the Lie algebra $su(2) \simeq so(3)$. Spherical harmonics are generated, and these are used in Chapter 18 to solve the Schrödinger equation for the hydrogen atom.

The last few chapters of the book are a bit sketchy, and the mathematical prerequisites are higher. The objective is to give the reader a glimpse of each subject. Chapter 19 discusses multiple particle systems and subsystems and introduces the density matrix. Chapter 20 presents the path integral formulation of quantum mechanics. Chapter 21 introduces (classical) Hamiltonian mechanics on manifolds in preparation for the final two chapters, which are on geometric quantization on Euclidean space and on manifolds, respectively.

A well-qualified graduate student can learn a lot from this book. I found it to be clear and well organized, and I personally enjoyed reading it very much. I already knew a fair bit of this stuff beforehand, but I was surprised at how many details were filled in for me. One warning: the book contains a fair number of typographical errors, so be on your guard.

DAVID S. WATKINS
Washington State University

Boolean Function Complexity: Advances and Frontiers. *By Stasys Jukna.* Springer, New York, 2012. $84.95. xvi+617 pp., hardcover. ISBN 978-3-642-24507-7.

The two main branches of theoretical computer science are algorithms and computational complexity theory. Both of these branches serve the same purpose of understanding the true limits of efficient computation. While algorithm researchers are devising faster ways to solve various computational tasks such as linear programming, matrix multiplication, and sorting, complexity theorists are working in the opposite direction and focusing on *lower bounds*, i.e., establishing that the same computational tasks inherently require a certain nontrivial amount of computational resources such as time, memory, and number of processing units. Apart from the noble challenge of uncovering important scientific truths, the work of complexity theorists has a strong motivation in modern cryptography that is largely based on computational assumptions like hardness of integer factorization, hardness of finding short vectors in lattices, etc. Perhaps surprisingly it turns out that, unlike progress in algorithm, design, progress in computational complexity has been extremely slow. Existing lower bounds are either very weak quantitatively or apply only to very restricted model of computation that do not come close to capturing the power of real computing devices.

Akin to combinatorics and number theory, most problems in computational complexity theory are easy to state and can often be explained even to an eager high school student. Most existing results require a clever insight rather than fancy mathematical tools. The tools used tend to come from algebra, combinatorics, probability theory, or analysis. Complexity theory originated in the 1950s and, despite its relatively young age, has already gained considerable prominence among other mathematical disciplines. One way to recognize its significance is to note that the central question of computational complexity theory, namely, the $\mathbb{P}$ vs. $\mathbb{NP}$ question, is now often considered one of the few most important open problems in mathematics together with the Riemann hypothesis.

Boolean Function Complexity reviews that state of the art in complexity theory focusing on concrete lower bounds. The publication is timely. The only recent book that has extensive coverage of lower bounds is [1, Part 2]. However, there the coverage is far more limited. By contrast, while the current book also does not cover some areas of lower bound work, e.g., models of quantum or algebraic computation, it does cover all major results and models dealing with classical circuit models and does so in a truly encyclopedic manner. To demonstrate the breadth of coverage of material let me mention that the book includes a result of Razborov from late 1980s on application of rigid matrices to separating classes in the communication complexity hierarchy (a result that was circulated as a manuscript and never published [3]). The book also covers some very recent developments such as the breakthrough by Williams [5] separating $\mathbb{NEXP}$ from $\mathbb{ACC}$.

The book is arranged into six parts. The first part is introductory. The second part deals with communication complexity, which is an information theoretic measure of complexity of functions that is useful to prove circuit lower bounds. The presentation here extends the treatment in [2] and covers all the main highlights, e.g., the log-rank conjecture, the number-in-hand and number-on-forehead models, and their applications. The third and fourth parts constitute the core of the book. Here the main lower bound gems such as lower bounds for the size of monotone circuits, constant depth circuits, and formulas are presented. The treatment is very careful and detailed. For instance, the whole of Chapter 10 is dedicated to a discussion of why proving lower bounds against monotone circuits is so much easier than against general circuits, and various quantitative aspects of this. These parts also cover some less-studied models of computations such as span programs.

Part five of the book covers the model of computation by branching programs. This model captures in a natural way the deterministic space (memory), whereas nondeterministic branching programs do the same for the nondeterministic model of computation. The chapter covers all the main classes of branching programs such as decision trees, read-once programs, and oblivious branching programs. Both lower bounds and upper bounds (Barrington's theorem) are presented. Part six deals with proof complexity. The main goal of this area is to show that some unsatisfiable formulae require long proofs in a formal proof system, as the opposite of this would imply that $\mathbb{NP} = \text{co}-\mathbb{NP}$. The chapter summarizes the state of the art in resolution and cutting plane proof systems including the size vs. width trade-offs for resolution proofs and applications of cutting plane proofs to integrality gaps of integer linear programs.

The last chapter of the book discusses the difficulties in proving lower bounds and focuses on the natural proofs barrier of Razborov and Rudich [4]. The chapter also contains an interesting discussion regarding the possibility that all of the class $\mathbb{P}$ might have linear size circuits. While currently this is considered quite unlikely, apparently in the early days of complexity theory an opposite viewpoint was expressed by the great mathematician Andrey Kolmogorov.

All material in the book is accessible to graduate or even undergraduate students of computer science, mathematics or electrical engineering. The book is self-contained. All necessary mathematical machinery is summarized in the appendix. I gladly recommend this book to beginning students, who will find this book a good starting point in exploring the field of complexity theory, as well as to mature researchers who would like to bring themselves up-to-date on some aspect of the theory. Finally, let me mention that the book contains a large number of open research problems, including some really enticing ones!

REFERENCES

[1] S. Arora and B. Barak, *Computational Complexity: A Modern Approach*, Cambridge University Press, Cambridge, UK, 2009.

[2] E. Kushilevitz and N. Nisan, *Communication Complexity*, Cambridge University Press, Cambridge, UK, 1997.

[3] A. Razborov, *On Rigid Matrices*, manuscript, 1989 (in Russian).

[4] A. Razborov and S. Rudich, *Natural proofs*, J. Comput. System Sci., 55 (1997), pp. 24–35.

[5] R. Williams, *Nonuniform ACC circuit lower bounds*, J. ACM, 61 (2014), article 2.

Sergey Yekhanin
Microsoft Research