Dynamics of metamaterials

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Problem Statement: Version-1

Consider a one-dimensional chain of N coupled oscillators with generalized displacements $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_N(t)]^T$. The i-th oscillator has a tunable lumped mass $m_i > 0$ and is connected to its nearest neighbors through identical linear springs of stiffness $k > 0$. In addition, the first (base) oscillator ($i = 1$) is attached to ground through a nonlinear hardening spring. The hardening spring force is modeled as

$$f_{\mathrm{nl}}(u_1) = k_0\, u_1 + \alpha\, u_1^3, \qquad k_0 \geq 0,\ \alpha > 0, \tag{1}$$

where k_0 is an optional linear grounding stiffness and α controls the hardening nonlinearity.

The equations of motion are written as

$$\mathbf{M}(\mathbf{m})\, \ddot{\mathbf{u}}(t) + \mathbf{K}\, \mathbf{u}(t) + \mathbf{f}_{\mathrm{nl}}(\mathbf{u}(t)) = \mathbf{0}, \tag{2}$$

where $\mathbf{M}(\mathbf{m}) = \mathrm{diag}(m_1, \ldots, m_N)$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix induced by the nearest-neighbor coupling springs, and

$$\mathbf{f}_{\mathrm{nl}}(\mathbf{u}) = \begin{bmatrix} f_{\mathrm{nl}}(u_1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3}$$

The objective is to tune the mass distribution $\mathbf{m}$ so that the resulting natural-frequency spectrum exhibits one or more band gaps, i.e., contiguous frequency intervals

$$\mathcal{B} = \bigcup_{r=1}^{R} [\omega_r^-, \omega_r^+] \tag{4}$$

over which the chain admits no admissible (linearized) normal modes (or, more generally, no sustained periodic responses in the presence of the hardening nonlinearity), subject to practical design constraints on the masses.

Formally, determine $\mathbf{m}$ such that a prescribed band gap is achieved or optimized, subject to

$$m_i^{\min} \le m_i \le m_i^{\max}, \quad i = 1, \ldots, N, \qquad \text{and optionally } \sum_{i=1}^{N} m_i = M_{\text{tot}} \text{ (or) } \sum_{i=1}^{N} m_i \le M_{\text{tot}}. \tag{5}$$

The problem setting includes the influence of the base hardening spring on the frequency content and seeks a mass-tuning strategy that yields a controllable band gap in the natural-frequency spectrum.