Scribe Notes: Optimization Class

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1 Convex Functions and Local Minima

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **convex** if for all $x, y \in \mathbb{R}^n$ and for any $\lambda \in [0, 1]$, we have:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1}$$

One fundamental property of convex functions is that they always achieve a single local minimum, and that local minimum is also the global minimum. This means that gradient-based optimization techniques work effectively for convex functions, as there are no local minima traps.

1.1 Strictly Convex Function

A function f is **strictly convex** if for all $x, y \in \mathbb{R}^n$, $x \neq y$, and for any $\lambda \in (0, 1)$, we have:

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y). \tag{2}$$

Strict convexity ensures that there is a unique global minimum.

Example: The function $f(x) = e^x$ is strictly convex because its second derivative is always positive.

1.2 Strict Convexity Proofs

1.2.1 Strict Convexity of $f(x) = x^2$

To prove that $f(x) = x^2$ is strictly convex, we check the strict convexity condition:

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &= (\lambda x + (1 - \lambda)y)^2 \\ &= \lambda^2 x^2 + 2\lambda(1 - \lambda)xy + (1 - \lambda)^2 y^2. \end{aligned}$$

Since $2\lambda(1 - \lambda)xy$ is strictly less than $\lambda x^2 + (1 - \lambda)y^2$ for $x \neq y$, we get:

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y). \tag{3}$$

Thus, $f(x) = x^2$ is strictly convex.

1.2.2 Strict Convexity of $f(x) = x^3$ for $x > 0$

We check the strict convexity condition:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= (\lambda x + (1-\lambda)y)^3 \\ &= \lambda^3 x^3 + 3\lambda^2(1-\lambda)x^2 y + 3\lambda(1-\lambda)^2 x y^2 + (1-\lambda)^3 y^3. \end{aligned}$$

For $x, y > 0$ and $x \neq y$, using the properties of the cubic function, we obtain:

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y). \tag{4}$$

Thus, $f(x) = x^3$ is strictly convex for $x > 0$.

1.2.3 Strict Convexity of $f(x) = |x|$ for all x

For $x, y \in \mathbb{R}$ and $x \neq y$, we check the strict convexity condition:

$$f(\lambda x + (1-\lambda)y) = |\lambda x + (1-\lambda)y|.$$

Since $|x|$ is piecewise linear, we analyze two cases:

- If x and y have the same sign, then $|x|$ is linear, meaning the equality holds.
- If x and y have opposite signs, then $|x|$ is strictly convex as:

$$|\lambda x + (1-\lambda)y| < \lambda |x| + (1-\lambda)|y|. \tag{5}$$

Thus, $f(x) = |x|$ is strictly convex for all x.

1.3 Theorem: Local Minima of a Convex Function is also a Global Minima

Theorem: Any local minimum x^* of a convex minimization problem is also a global minimum.

Proof:

Let $f : S \to \mathbb{R}$ be a convex function defined over a convex set $S \subset \mathbb{R}^n$. Suppose $x^* \in S$ is a local minimum of f. This means that there exists some $\epsilon > 0$ such that:

$$f(x^*) \leq f(x), \quad \forall x \in B[x^*, \epsilon] \cap S, \tag{6}$$

where $B[x^*, \epsilon]$ is an open ball centered at x^* with radius ϵ. That is, within a small neighborhood around x^*, the function does not attain a smaller value than at x^*.

Now, we need to show that x^* is also a global minimum, i.e.,

$$f(x^*) \leq f(x), \quad \forall x \in S. \tag{7}$$

Consider any arbitrary $x \in S \setminus B[x^*, \epsilon]$. Define the point:

$$y = \lambda x^* + (1-\lambda)x, \quad \text{for } \lambda \in (0, 1). \tag{8}$$

Since S is convex, $y \in S$. By the convexity of f, we have:

$$f(y) \leq \lambda f(x^*) + (1-\lambda) f(x). \tag{9}$$

From the assumption that x^* is a local minimum, we know that for sufficiently small λ, the point y remains within $B[x^*, \epsilon] \cap S$, meaning:

$$f(x^*) \leq f(y). \tag{10}$$

Combining the two inequalities, we obtain:

$$f(x^*) \leq \lambda f(x^*) + (1-\lambda) f(x). \tag{11}$$

Rearranging,

$$f(x^*) - \lambda f(x^*) \leq (1-\lambda) f(x). \tag{12}$$

Dividing by $(1-\lambda)$ (which is positive for $\lambda \in (0,1)$), we get:

$$f(x^*) \leq f(x). \tag{13}$$

Thus, for all $x \in S$, we conclude that $f(x^*) \leq f(x)$, proving that x^* is a global minimum. $\square$

1.4 Theorem: Set of Global Minima is Convex

Theorem: Let $S \subseteq \mathbb{R}^n$ be a convex set and let $f : S \to \mathbb{R}$ be a convex function. Consider the set of all global minima:

$$C = \{x^* \in S \mid f(x^*) \leq f(x) \text{ for all } x \in S\}. \tag{14}$$

Then, C is a convex set.

Proof: To prove that C is convex, we must show that for any two points $x_1, x_2 \in C$, and for any $\lambda \in [0,1]$, the point $x_\lambda = \lambda x_1 + (1-\lambda) x_2$ also belongs to C.

Since $x_1, x_2 \in C$, we have:

$$f(x_1) \leq f(x) \quad \text{and} \quad f(x_2) \leq f(x), \quad \forall x \in S. \tag{15}$$

By the convexity of f, we obtain:

$$f(x_\lambda) \leq \lambda f(x_1) + (1-\lambda) f(x_2). \tag{16}$$

Since both x_1 and x_2 are global minima, we know that $f(x_1) = f(x_2) = f^*$ for some constant f^*. Substituting this, we get:

$$f(x_\lambda) \leq \lambda f^* + (1-\lambda) f^* = f^*. \tag{17}$$

Thus, $f(x_\lambda) \leq f(x)$ for all $x \in S$, which implies that $x_\lambda \in C$. Therefore, C is a convex set. $\square$

2 Epigraph of a Function

Definition: The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as the set:

$$\mathrm{epi}(f) = \{(x, y) \in \mathbb{R}^{n+1} \mid y \geq f(x)\}. \tag{18}$$

It consists of all points lying on or above the graph of f.

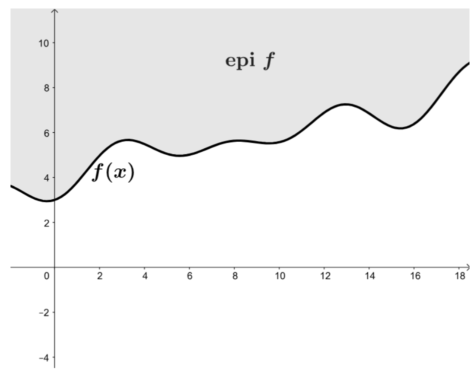

Figure 1: Epigraph

2.1 Observations on Convexity using Epigraph

A function f is convex if and only if its epigraph is a convex set. This characterization provides an alternative way to study convexity geometrically.

Example 1: Consider $f(x) = x^2$. The epigraph of f consists of all points (x, y) where $y \geq x^2$. The set of points above the parabola $y = x^2$ is convex, thus confirming that $f(x) = x^2$ is convex.

Example 2: Consider $f(x) = |x|$. The epigraph of f consists of points (x, y) where $y \geq |x|$. This region forms a convex set, reinforcing that $f(x) = |x|$ is convex.

2.2 Theorem: A Function is Convex if and only if its Epigraph is Convex

Theorem: A function $f : S \to \mathbb{R}$ is convex if and only if its epigraph is a convex set.

Proof: (Forward Direction: If f is convex, then epi(f) is convex.)

Consider two points $(x_1, y_1), (x_2, y_2) \in \mathrm{epi}(f)$. By definition of the epigraph, we have:

$$y_1 \geq f(x_1), \quad y_2 \geq f(x_2). \tag{19}$$

For any $\lambda \in [0, 1]$, define the convex combination:

$$x_\lambda = \lambda x_1 + (1 - \lambda)x_2, \quad y_\lambda = \lambda y_1 + (1 - \lambda)y_2. \tag{20}$$

Using the convexity of f, we get:

$$f(x_\lambda) \leq \lambda f(x_1) + (1-\lambda) f(x_2). \tag{21}$$

Since $y_1 \geq f(x_1)$ and $y_2 \geq f(x_2)$, we obtain:

$$f(x_\lambda) \leq \lambda y_1 + (1-\lambda) y_2 = y_\lambda. \tag{22}$$

Thus, $(x_\lambda, y_\lambda) \in \text{epi}(f)$, proving that $\text{epi}(f)$ is convex.

(Reverse Direction: If epi(f) is convex, then f is convex.)

Suppose that $\text{epi}(f)$ is a convex set. This means that for any two points $(x_1, y_1), (x_2, y_2) \in \text{epi}(f)$, the convex combination:

$$(x_\lambda, y_\lambda) = (\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \tag{23}$$

remains in $\text{epi}(f)$, implying:

$$y_\lambda \geq f(x_\lambda). \tag{24}$$

Choosing $y_1 = f(x_1)$ and $y_2 = f(x_2)$, we get:

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(x_\lambda). \tag{25}$$

This confirms the convexity of f. $\square$

3 Level Set

Definition: The level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ corresponding to a scalar $c \in \mathbb{R}$ is defined as:

$$L_c = \{x \in \mathbb{R}^n \mid f(x) \leq c\}. \tag{26}$$

The level set consists of all points where the function value does not exceed a given threshold.

3.1 Example of a Level Set

Consider the function $f(x, y) = x^2 + y^2$. The level set L_c for a given c is:

$$x^2 + y^2 \leq c. \tag{27}$$

This represents a closed disk of radius $\sqrt{c}$ centered at the origin, which is a convex set.

3.2 Theorem: Convexity of Level Sets of a Convex Function

Theorem: Let $S \subset \mathbb{R}^n$ be a convex set, and let $f : S \to \mathbb{R}$ be a convex function. Then, any level set $L_c = \{x \in S \mid f(x) \leq c\}$ is a convex set.

Proof: Consider any two points $x_1, x_2 \in L_c$, meaning that:

$$f(x_1) \leq c, \quad f(x_2) \leq c. \tag{28}$$

For any $\lambda \in [0, 1]$, consider the convex combination:

$$x_\lambda = \lambda x_1 + (1 - \lambda)x_2. \tag{29}$$

Using the convexity of f, we have:

$$f(x_\lambda) \leq \lambda f(x_1) + (1 - \lambda)f(x_2). \tag{30}$$

Since $f(x_1) \leq c$ and $f(x_2) \leq c$, it follows that:

$$\lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda c + (1 - \lambda)c = c. \tag{31}$$

Thus, $f(x_\lambda) \leq c$, which implies that $x_\lambda \in L_c$.

Since x_λ remains in L_c for any convex combination of x_1 and x_2, the level set L_c is convex. □