Now that we have calculated the cross-section and decay width we can move on to find the collision term.

Lets consider a reaction as follows

$$\psi + a + b + \cdots \to i + j + k + \cdots$$

From Fermi's Golden Rule, the transition rate for a process like the above is proportional to

$$|\mathcal{M}|^2 \times (\text{phase space}) \times \delta^4(\text{momentum conservation})$$

The lorentz phase space measure of i^{th} final state is

$$d\Pi_i = \frac{d^3 p_i}{(2\pi)^3 \underbrace{2E_i}_{\text{using relativistic normalization}}}$$

Now enforcing the conservation laws

$$(2\pi)^4 \delta^4(p_\psi + p_a + p_b + \cdots - p_i - p_j - p_k \cdots)$$

Statistical occupancy and quantum statistical effects

$$(1 \pm f_i)$$

where $f_i \to$ particle phase-space distribution

$\pm \to$ encodes quantum statistical effects

- $+$ for boson (Boson enhancement)
- $-$ for fermions (Pauli blocking)

Gain - loss structure of collision term

→ gain : particles scatter into the state with momentum $\vec{p}$

→ loss : particles scatter out of the state with momentum $\vec{p}$

$$C[f] = (\text{rate in}) - (\text{rate out})$$

$$\Rightarrow \frac{g}{(2\pi)^3}\int \frac{d^3p}{E} C[f] = -\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots \times (2\pi)^4 \delta^4(p_\psi + p_a + \cdots - (p_i + p_j + \cdots))$$

$$\times [|M|_1^2 f_\psi f_a f_b \cdots (1 \pm f_i)(1 \pm f_j) \cdots - |M|_2^2 f_i f_j \cdots (1 \pm f_\psi)(1 \pm f_a)(1 \pm f_b) \cdots]$$

where $|M|_1^2$ is for $\psi + a + b + \cdots \rightarrow i + j + k + \cdots$

$|M|_2^2$ is for $i + j + k + \cdots \rightarrow \psi + a + b + \cdots$

the matrix element M depends on the specific dark matter interaction

Assumpting CP(T) invariance

CP(T) invariance stands for the combined symmetry of Charge Conjugation (C), Parity (P), Time Reversal (T) applied to physical laws:

Charge Conjugation → changes particles into their anti-particles

Parity (P) → flips spatial coordinates

Time Reversal (T) → reverses the direction of time

And if a law is CP(T) invariant it should remain same under these changes

Now applying the assumption:

$|\mathcal{M}|_1^2 = |\mathcal{M}|_2^2 = |\mathcal{M}|^2$ (lets say)

The second critical assumption is to use Maxwell-Boltzmann distribution as the temperature is very high so the particles (bosons + fermions) are in classical limit.

$\therefore$ The quantum statistical effects

- Boson enhancement
- Fermion blocking

can be ignored, as in classical limit, the quantum processes cease to exist (take place)

$\Rightarrow$ $(1 \pm f_i)$ terms can be neglected

$$\Rightarrow \frac{g}{(2\pi)^3} \int \frac{d^3p}{E} C[f] = -\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots \times (2\pi)^4 \delta^4(p_\psi + p_a + p_b + \cdots - p_i - p_j - \cdots) \times |\mathcal{M}|^2 [f_a f_b \cdots f_\psi - f_i f_j \cdots]$$

now substituting in Boltzmann equation's simplified form

$$\underbrace{\dot{n}_\psi}_{\frac{dn_\psi}{dt}} + 3Hn_\psi = -\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots \times (2\pi)^4 \delta^4(p_\psi + p_a + p_b + \cdots - (p_i) - p_j - \cdots) \times |\mathcal{M}|^2 [f_a f_b \cdots f_\psi - f_i f_j \cdots]$$

$\because (1 \pm f_i) \to 1$ because of classical limit

and the distribution

$$f_i(E_i) = \exp[-(E_i - \mu_i)/T]$$

As it is assumed that dark matter is in equilibrium with standard model particles in the early universe via $2 \leftrightarrow 2$ interactions we can consider a reaction

$$\psi\bar{\psi} \rightarrow \chi\bar{\chi}$$

now for such a process the Boltzmann equation will become

$$\dot{n}_\psi + 3Hn_\psi = -\int d\Pi_\psi d\Pi_{\bar{\psi}} d\Pi_\chi d\Pi_{\bar{\chi}} (2\pi)^4 \delta^4(p_\psi + p_{\bar{\psi}} - p_\chi - p_{\bar{\chi}}) |M|^2_{\psi\bar{\psi}\rightarrow\chi\bar{\chi}} (f_\psi f_{\bar{\psi}} - f_\chi f_{\bar{\chi}})$$

as the particles are in thermal and chemical equilibrium

$$\mu_\psi + \mu_{\bar{\psi}} = \mu_\chi + \mu_{\bar{\chi}}$$

$\Rightarrow$ we can ignore the chemical potential in the distribution function

$$f_i(E_i) = \exp(-E_i/T)$$

$\therefore$ $f_\psi^{EQ} = \exp(-E_\psi/T)$ and $f_{\bar{\psi}}^{EQ} = \exp(-E_{\bar{\psi}}/T)$

and due to energy conservation

$$E_\psi + E_{\bar{\psi}} = E_\chi + E_{\bar{\chi}}$$

$$\Rightarrow f_\chi f_{\bar{\chi}} = f_\psi^{EQ} f_{\bar{\psi}}^{EQ}$$

put this in above equation

$$\dot{n}_\psi + 3Hn_\psi = -\int d\Pi_\psi d\Pi_{\bar{\psi}} d\Pi_\chi d\Pi_{\bar{\chi}} (2\pi)^4 \delta^4(p_\psi + p_{\bar{\psi}} - p_\chi - p_{\bar{\chi}}) |M|^2_{\psi\bar{\psi}\rightarrow\chi\bar{\chi}} [f_\psi f_{\bar{\psi}} - f_\psi^{EQ} f_{\bar{\psi}}^{EQ}]$$

We are now going to introduce thermal average cross-section as, in hot thermal bath, particles have momenta distribution near equilibrium, so averaging over these distributions captures the typical interaction rate helping us simplify the collision term while keeping the physics accurate.

Apply thermal average cross section.

we can write the difference in number densities as

$$f_\psi f_{\bar\psi} - f_\psi^{EQ} f_{\bar\psi}^{EQ} \approx \frac{n_\psi n_{\bar\psi}}{(n_\psi^{eq})^2}\left(f_\psi^{EQ} f_{\bar\psi}^{EQ}\right) - f_\psi^{EQ} f_{\bar\psi}^{EQ}$$

$$= f_\psi^{EQ} f_{\bar\psi}^{EQ}\left(\frac{n_\psi n_{\bar\psi}}{(n_\psi^{EQ})^2} - 1\right)$$

thermally averaged annihilation cross-section is defined as

$$\langle\sigma|v|\rangle = \frac{\int d\Pi_\psi d\Pi_{\bar\psi} d\Pi_X d\Pi_{\bar X} (2\pi)^4 \delta^4(p_\psi + p_{\bar\psi} - p_X - p_{\bar X})\, |M|^2 f_\psi^{EQ} f_{\bar\psi}^{EQ}}{\int d\Pi_\psi d\Pi_{\bar\psi} f_\psi^{EQ} f_{\bar\psi}^{EQ}}$$

$$= \frac{\int d\Pi_\psi d\Pi_{\bar\psi} d\Pi_X d\Pi_{\bar X} (2\pi)^4 \delta^4(p_\psi + p_{\bar\psi} - p_X - p_{\bar X})\, |M|^2 f_\psi^{EQ} f_{\bar\psi}^{EQ}}{(n_\psi^{EQ})^2}$$

$$\because \int d\Pi_\psi f_\psi^{EQ} = n_\psi^{EQ}$$

$\Rightarrow$ we can write the RHS part as

$$\langle\sigma|v|\rangle \left(n_\psi n_{\bar\psi} - n_\psi^{EQ} n_{\bar\psi}^{EQ}\right)$$

$\therefore$ Boltzmann equation can be written as

$$\boxed{\dot{n}_\psi + 3Hn_\psi = -\langle\sigma|v|\rangle\left(n_\psi n_{\bar\psi} - n_\psi^{EQ} n_{\bar\psi}^{EQ}\right)}$$

where

$$|v| = \frac{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}$$

i.e. Mollar velocity

where 1 and 2 repreasented as ψ and $\bar\psi$

as they are the interacting particles

Hence the Boltzmann equation is its simplified form is

$$\boxed{\dot{n}_\psi + 3Hn_\psi = -\langle\sigma|v|\rangle\left(n_\psi n_{\bar\psi} - n_\psi^{EQ} n_{\bar\psi}^{EQ}\right)} \;-\!\cdot$$

Now we should solve this equation to evaluate dark matter freeze out.

Now before we move further lets solve $\langle\sigma|v|\rangle$ for $m_1 = m_2 = m$

We know

$$\langle\sigma|v|\rangle = \frac{\int d^3p_\psi\, d^3p_{\bar\psi}\, \sigma\, v_{rel}\, e^{-(E_\psi/T)} e^{-(E_{\bar\psi}/T)}}{\int d^3p_\psi\, d^3p_{\bar\psi}\, e^{-(E_\psi/T)} e^{-(E_{\bar\psi}/T)}}$$

where $E_i = \sqrt{p_i^2 + m_i^2}$

$\because m_\psi = m_{\bar\psi} = m$

$$\therefore E_i = \sqrt{p_i^2 + m^2}$$

The Mandelstam variable s for two particles with 4-momenta $p_\psi, p_{\bar\psi}$ is

$$s = (p_\psi + p_{\bar\psi})^2$$

where s ranges from $4m^2$ to ∞

$\because p_i = (E_i, \vec{p}_i)$

if the particle is at rest

$\vec{p}_i = 0$

$E_i = m_i$

Now we try to express moler velocity (relative velocity) in terms of s

$$|v| = \frac{\sqrt{(p_\psi \cdot p_{\bar\psi})^2 - m_\psi^2 m_{\bar\psi}^2}}{E_\psi E_{\bar\psi}}$$

as $s = (p_\psi + p_{\bar\psi})^2 = p_\psi^2 + p_{\bar\psi}^2 + 2(p_\psi \cdot p_{\bar\psi})$

$\because$ particles are relativistic

$p_\psi^2 = m_\psi^2$

as $m_\psi = m_{\bar\psi} = m$

$s = 2m^2 + 2(p_\psi \cdot p_{\bar\psi})$

$\Rightarrow (p_\psi \cdot p_{\bar\psi}) = \frac{s - 2m^2}{2}$

$\therefore$ the numerator will becom

$$\sqrt{(p_\psi \cdot p_{\bar\psi})^2 - m^4} = \sqrt{\left(\frac{s^2 + 4m^4 - 4m^2 s}{4}\right) - m^4}$$

$$= \frac{\sqrt{s(s - 4m^2)}}{2}$$

$$\Rightarrow |v| = \frac{\sqrt{s(s-4m^2)}}{2 E_\psi E_{\bar\psi}}$$

now expressing the integration variables to center of mass quantities using Lorentz - invariant phase space measure

$$d^3 p_\psi \, d^3 p_{\bar\psi} = \frac{|\vec{p}_\psi|^2 d|\vec{p}_\psi| d\Omega_1}{(2\pi)^3} \quad \frac{|\vec{p}_{\bar\psi}|^2 d|\vec{p}_{\bar\psi}| d\Omega_2}{(2\pi)^3}$$

now change variables to

→ Total CM momentum $P = p_\psi + p_{\bar\psi}$

→ relative momentum or invariants s

→ scattering angles

$$\int \frac{ds}{\sqrt{s}} \int d\Omega \,(\text{kinematic factors})\, \sigma(s)\, v_{rel}\, e^{-\sqrt{s}/T}$$

using Maxwell - Boltzmann distribution and Bessel functions

→ we are integrating over momenta with the exponential distribution $e^{-E/T}$ this leads to modified Bessel functions K_n

over total energy $\equiv K_1(\sqrt{s}/T)$

normalizations $\equiv K_2(m/T)$

Now changing the integral measure and integration limits

→ we are integrating over s

$\therefore$ limits should be $4m^2$ to ∞

→ full measure includes flux and momentum factors proportional to $(s-4m^2)\sqrt{s}$

$\therefore$ integral becomes $\displaystyle\int_{4m^2}^{\infty} ds\, \sigma(s)\,(s-4m^2)\sqrt{s}\, K_1(\sqrt{s}/T)$

Now solving further we get

$$\langle\sigma|v|\rangle = \left(4\pi m^2 T K_2\left(\frac{m}{T}\right)\right)^{-2} (2\pi^2 T) \int_{4m^2}^{\infty} \sigma(s)(s-4m^2)\sqrt{s}\, K_1\left(\frac{\sqrt{s}}{T}\right) ds$$

Now coming back to Boltzmann equation

$$\dot{n}_\psi + 3Hn_\psi = -\langle\sigma|v|\rangle \left(n_\psi n_\phi - n_\psi^{EQ} n_\phi^{EQ}\right)$$

We now proceed and apply the concept of Thermal Freeze Out to solve it

To do so, from the concept of yield, we convert above equation in terms of yield.

Yield is defined as ratio of the number density of a particle species to the entropy density of the universe (s).

$$Y = \frac{n_\psi}{s}$$

where s the entropy density of the universe is

$$s = \frac{2\pi^2}{45} g_*^s (m/x)^3$$

g_*^s is degrees of freedom of matter in terms of entropy

s can be derived from cosmology and thermodynamics of early universe

$x = m/T$

where T is temperature of thermal bath

Now from cosmology of early universe we can say that

$$sR^3 = \text{constant}$$

differentiating with respect to 't' we get

$$\left(\frac{\partial s}{\partial t}\right)R^3 + s\left(\frac{\partial R^3}{\partial t}\right) = 0$$

$$\dot{s}R^3 + s(3\dot{R}R^2) = 0$$

$$\dot{s} = -3s\left(\frac{\dot{R}}{R}\right)$$

where $\dot{R}/R = H$

$$\boxed{\dot{s} = -3sH}$$

as we know

$$Y = \frac{n_\psi}{s}$$

$$n_\psi = Ys$$

$$\therefore \dot{n}_\psi = \dot{Y}s + Y\dot{s}$$

$$= \dot{Y}s - 3YsH$$

$$\boxed{\dot{n}_\psi + 3Hn_\psi = \dot{Y}s}$$

$\therefore$ The LHS of Boltzmann equation is rewritten in terms of yield.

Now we should focus on converting the RHS / collision term of Boltzmann equation into yield terms

During radiation dominated epoch of the universe all standard particle production and freeze-out processes occur.

We hence use the relation between time (t) and scale factor (x) that relates them during radiation dominated epoch.

We can derive it using Friedmann equation for radiation dominated epoch

Using first Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho$$

for radiation dominated epoch

$$\rho = \frac{\pi^2}{80} g_* T^4$$

$H = \dot{R}/R$

in natural units $\hbar = c = 1$

$\Rightarrow m_{Pl} = 1/\sqrt{G}$

where m_{Pl} is Planck mass

$$\Rightarrow (\dot{R}/R)^2 = \frac{8\pi G}{3}\left(\frac{\pi^2}{30} g_* T^4\right)$$ [Plugging them back into the equation]

$$\dot{R}/R = \sqrt{\frac{8\pi^3 g_*}{90}}\left(\frac{T^2}{1/\sqrt{G}}\right)$$

$$\boxed{\dot{R}/R = \sqrt{\frac{8\pi^3 g_*}{90}}\left(T^2/m_{Pl}\right)}$$

For radiation dominated era

$T \propto 1/R$

$R \propto t^{1/2}$

$\Rightarrow t \propto 1/T^2$

Quantitatively

$$t = 1/2H$$

$$= \frac{1}{2}\left(\sqrt{\frac{90}{8\pi^3 g_*}}\left(\frac{m_{Pl}}{T^2}\right)\right)$$

$$t = \sqrt{\frac{90}{32\pi^3}}\, g_*^{-1/2}\left(\frac{m_{Pl}}{T^2}\right)$$

$$\therefore\ t = 0.301\, g_*^{-1/2}\, \frac{m_{Pl}}{T^2}$$

we know $1/T = x/m$

$$t = 0.301\, g_*^{-1/2}\left(\frac{m_{Pl}}{m^2}\right)x^2$$

where g_* is effective massless degrees of freedom

$\frac{dY}{dt}$ shows how the number of particles n_ψ changes with respect to time due to collision, annihilation, production processes.

It basically gives change in abundance with time

$$\frac{dY}{dt} = \frac{dY}{dx}\left(\frac{dx}{dt}\right)$$

$$dt = 2\,(0.301)\, g_*^{-1/2}\left(\frac{m_{Pl}}{m^2}\right)(x)\, dx$$

$$\frac{dx}{dt} = \frac{g_*^{1/2}\, m^2}{0.602\, m_{Pl}\, x}$$

$$\Rightarrow \frac{dY}{dt} = \frac{dY}{dx}\left(\frac{g_*^{1/2}\, m^2}{0.602\, m_{Pl}\, x}\right)$$

We know that

$$Y = \frac{n_\psi}{s}$$

$$n_\psi = Ys$$

$$\Rightarrow n_{\bar\psi} = Ys$$

let Y at equilibrium be Y_{EQ}

so $n_\psi^{EQ} = n_{\bar\psi}^{EQ} = Y_{EQ}s$

and we already derived that

$$\dot n_\psi + 3Hn_\psi = \dot Y s$$

Now substitute all this into the simplified version of Boltzmann equation i.e.

$$\dot n_\psi + 3Hn_\psi = -\langle\sigma|v|\rangle\left(n_\psi n_{\bar\psi} - n_\psi^{EQ} n_{\bar\psi}^{EQ}\right)$$

$$\dot Y s = -\langle\sigma|v|\rangle s^2\left(Y^2 - Y_{EQ}^2\right)$$

$$\frac{dY}{dt} = -\langle\sigma|v|\rangle s\left(Y^2 - Y_{EQ}^2\right)$$

$$\Rightarrow \frac{dY}{dx}\left(\frac{g_*^{1/2}\, m^2}{0.602\, m_{Pl}\, x}\right) = -\langle\sigma|v|\rangle s\left(Y^2 - Y_{EQ}^2\right)$$

$$\frac{dY}{dx} = -\left(\frac{0.602\, m_{Pl}}{g_*^{1/2}\, m^2}\right) x\,\langle\sigma|v|\rangle s\left(Y^2 - Y_{EQ}^2\right)$$

let $H(m) = \dfrac{1.67\, g_*^{1/2}\, m^2}{m_{Pl}}$

$$\boxed{\frac{dY}{dx} = -\frac{x\,\langle\sigma|v|\rangle s}{H(m)}\left(Y^2 - Y_{EQ}^2\right)}$$

Hence we converted the Boltzmann equation in terms of Y and x

The equation is

$$\boxed{\frac{dY}{dx} = -\frac{x\langle\sigma|v|\rangle s}{H(m)}\left(Y^2 - Y_{EQ}^2\right)}$$

$\left(\frac{\partial}{\partial x}\right)$

where $H(m) = \dfrac{1.67\, g_*^{1/2}\, m^2}{m_{Pl}}$

converting the coefficients into Γ and H

we know $H = H(m)/x^2$

and $\Gamma = n^{EQ}\langle\sigma|v|\rangle$

$$\Rightarrow x\frac{dY}{dx} = \frac{-x^2\langle\sigma|v|\rangle\left(\frac{n^{EQ}}{Y_{EQ}}\right)}{H(m)} \cdot Y_{EQ}^2\left(\left(Y/Y_{EQ}\right)^2 - 1\right)$$

$$\boxed{\frac{x}{Y_{EQ}}\frac{dY}{dx} = -\frac{\Gamma}{H}\left[\left(Y/Y_{EQ}\right)^2 - 1\right]}$$

Boltzmann equation in terms of Y, x, Γ and H :

$\therefore$ The final equation to solve is

$$\boxed{\frac{x}{Y_{EQ}}\frac{dY}{dx} = \frac{-\Gamma}{H}\left[\left(Y/Y_{EQ}\right)^2 - 1\right]}$$

We know that for

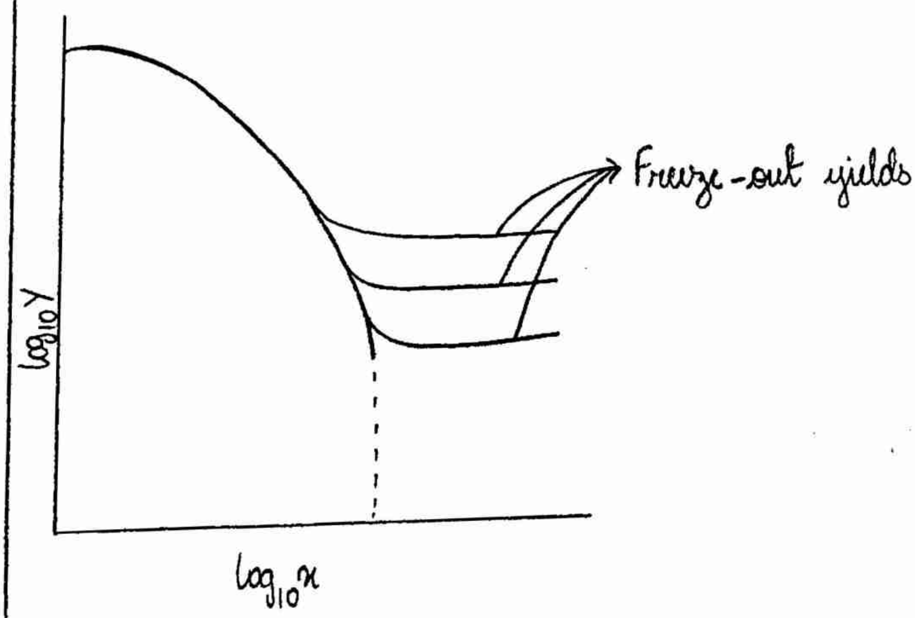
$\Gamma \gg H$: particles are in equilibrium with standard model particles

$$\Rightarrow Y(x < x_{FO}) = Y_{EQ}$$

$\Gamma \sim H$: marginal condition for freeze-out

$\Gamma \ll H$: particle freezes-out

$$\Rightarrow Y(x > x_{FO}) = Y(x = x_{FO})$$



Freeze-out yield is inversely proportional to annihilation cross-section. Larger the cross-section longer the DM remains in the thermal bath and hence smaller is the freezeout yield.

Numerically solving the Boltzmann equation

$$\frac{x}{Y_{EQ}} \frac{dY}{dx} = -\frac{\Gamma}{H} \left[\left(Y/Y_{EQ} \right)^2 - 1 \right]$$

$$\frac{dY}{dx} = \frac{-\Gamma Y_{EQ}}{H x} \left[\left(Y/Y_{EQ} \right)^2 - 1 \right]$$

$$\frac{dY}{dx} = \frac{s\langle\sigma|v|\rangle}{Hx}\left(Y^2 - Y_{EQ}^2\right)$$

we can get m and $\langle\sigma|v|\rangle$ from theoretical considerations that match the cosmological observations

$$m = 100\,\text{GeV}$$

$$\langle\sigma|v|\rangle = 3\times10^{-26}\,\text{cm}^3/\text{s}$$

the temperature at freeze out is

$$T \sim 4\,\text{GeV}$$

now $x = \frac{100}{4}$

$$\boxed{x \sim 25}$$

$$s = \frac{2\pi^2}{45} g_*^s T^3$$

$g_*^s = 80$ (at freeze-out temperature)

$$s = \frac{2\pi^2}{45}(80)[4]^3$$

$$= \frac{2\pi^2}{45}(80)(64)$$

$$\boxed{s \sim 2248\,\text{GeV}^3}$$

$$H = \sqrt{\frac{8\pi^3 G}{90} g_*^s T^4}$$

$$= \left[\sqrt{\frac{8\pi^3 G}{90}(80)}\right]16$$

$$H \sim 1.66\,(8.944)\,(1.31)\times10^{-18}\,\text{GeV}$$

$$H \sim 1.94\times10^{-17}\,(1.52\times10^{24})\,s^{-1}$$

$$\boxed{H \sim 3\times10^{7}\,1/s}$$

$Y_{EQ} = \frac{n_{EQ}}{s}$

$n_{EQ} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-100/4}$

$= 2\left(\frac{400}{2\pi}\right)^{3/2} e^{-25}$

$n_{EQ} \sim 1.41\times10^{-8}\ GeV^3$

$\Rightarrow Y_{EQ} \sim 6.4\times10^{-12}$

Now finding limits

$x \sim 1$

$Y(1) = Y_{EQ}(1)$

$= \frac{45}{2\pi^4}\left(\pi/8\right)^{1/2}\left(\frac{2}{100}\right)e^{-1}$

$Y(1) \sim 0.0011$

Now we cannot solve the Boltzmann equation analytically

$\therefore$ We use semianalytic method

In this method we first estimate the value of x_f iteratively

$$x_f = \ln\left(c(c+2)\sqrt{\frac{45}{8}}\ \frac{g\,m}{2\pi^3}\ \frac{m_{PL}\langle\sigma|v|\rangle}{\sqrt{g_*^{3}x_f}}\right)$$

generally c is taken 0.5-1

this gives $x_f \sim 20-30$

In semi-analytic method

$Y_{relic} = x_f/\lambda$

where $\lambda = 0.264 \dfrac{g_*^s}{\sqrt{g_*}} m_{Pl} m \langle\sigma|v|\rangle$

$$= 0.264 \frac{80}{\sqrt{80}} \times (1.2209\times10^{19}) \times (100) \times (2.5698\times10^{-9})$$

↳ in terms of GeV

$$\lambda \sim 7.41 \times 10^{12}$$

$$\Rightarrow Y_{relic} = x_f/\lambda$$

$$= 20/7.41\times10^{12}$$

$$Y_{relic} \sim 2.6 \times 10^{-12}$$

we know from other cosmological observations and models

$$\boxed{Y_{relic} \sim 10^{-12} - 10^{-10}}$$

This is the numerical solution of Boltzmann equation

Relic density : Ωh^2

$$\Omega = \frac{\rho}{\rho_c}$$

$$\rho = m n_0$$
$$= m Y_{relic} s_0$$

$\rho_c \equiv$ critical density of universe

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

$$\rho_c/h^2 = \frac{3H_0^2 m_{Pl}^3\, 0.26\, g_*^s}{\sqrt{g_*}} \qquad [h = H/100]$$

$\therefore$ Relic density is

$$\boxed{\Omega h^2 = \frac{m s_0 \sqrt{g_*}}{3H_0^2 m_{Pl}^3\, 0.26 g_*^s} Y_\infty} \ldots$$

where Y_∞ is Y_{relic}

now but putting all values we get

$$\Omega h^2 \simeq \frac{2.4 \times 10^{-10}\, GeV^{-2}}{\langle \sigma |v| \rangle}$$

Now from WMAP and PLANCK data at 67% CL we get

$$0.1133 \leq \Omega h^2 \leq 0.1189$$

Now from here we get

$$0.1189 \simeq \frac{2.4 \times 10^{-10}\, GeV^{-2}}{\langle \sigma |v| \rangle}$$

$$\langle \sigma |v| \rangle \simeq \frac{2.4 \times 10^{-10}}{0.1189}\, GeV^{-2}$$

$$\simeq 20.18 \times 10^{-10}$$

$$\langle \sigma |v| \rangle \simeq 2.018 \times 10^{-9}$$

we are getting $\langle \sigma |v| \rangle$ of the order 10^{-9}

Which is same for Weak Interaction cross-section (also according to the model from which we solved GEQ)

$\therefore$ Thermal freeze out of dark matter that provides correct relic density are termed as "Weakly Interacting Massive Particles" (WIMPS)

Model and Phenomology

DM - SM interaction

$$\mathcal{L}_{EFF} = \mathcal{L}_{SM} + \sum_n c_n \frac{O_{SM-DM}}{\Lambda^{(n-4)}}$$

where :

O_{SM-DM} is Dark Matter and Standard Model particles interaction which consists of DM and SM fields

$\Lambda^{(n-4)}$ is a new physics scale, which makes supression by appropriate dimension

$G_{SM} \times G_{DM}$ invariance

G_{SM} is standard model gauge group

$SU(3)_c \times SU(2)_L \times U(1)_Y$

This gives the fundamental gauge symmetries governing strong (colour), weak and hyper-charge interactions

G_{DM} is generalised dark matter gauge invariance

∴ The function or lagrangian which is invariant wrt both standard model gauge and dark matter gauge is said to have $G_{SM} \times G_{DM}$ invariance

G_{DM} has a discrete dark matter gauge symmetry Z_2

O_{SM-DM} is both Lorentz invariant and $G_{SM} \times G_{DM}$ invariant

$\mathcal{O}_{SM-DM} \sim \mathcal{O}_{DM}\mathcal{O}_{SM}$

where $\mathcal{O}_{SM}$ constitutes SM fields

$\mathcal{O}_{DM}$ constitutes DM fields

Lagrangian describing the interaction between SM and DM fields can be expressed as

$$\mathcal{L}_{DM-SM} \sim 1/\Lambda^{(n-4)}\, \mathcal{O}_{DM}\mathcal{O}_{SM}$$

For factorizing we assume that DM does not possess SM charge and vice - versa (only for simplification, not a necessity).

For renormalizable interactions $n=4$ (simplest DM-SM operator)

Now
SM gauge invariant $\mathcal{O}_{SM}$ can be written as

$$\mathcal{O}_{SM} = H^{\dagger}H$$

where H is Higgs doublet

$$H = \begin{pmatrix} H^{+} \\ H^{0} \end{pmatrix}$$

The mass dimension is $[M]:2$

ϕ is a scalar singlet of Dark Matter (possess no EM charge)

Now we apply the concept of Higgs Portal Interaction.

Higgs Portal Interaction is a theoretical mechanism by which dark matter particles are assumed to interact with standard model particles through coupling to the Higgs boson field

$\mathcal{L}_{DM-SM} \sim H^\dagger H \phi^2$

The stability of DM is ensured by a Z_2 symmetry

$\Rightarrow$ if ϕ is replaced by $-\phi$

$$\phi \rightarrow -\phi$$

$\Rightarrow$ $\lambda_1 \phi^2 H^\dagger H$ will be the Lagrangian

$$\mathcal{L} = \mathcal{L}_{SM} + 1/2\,(\partial^\mu \phi)(\partial_\mu \phi) - V(H,\phi)$$

$$V(H,\phi) = \mathcal{L}_{SM} + 1/2\,(\partial^\mu \phi)(\partial_\mu \phi) - \mathcal{L}$$

The requirements for $V(H,\phi)$ are

- renormalizability
- gauge invariance
- dark matter stability

Taking into account Higgs sector potential

$$V(H) = -\mu_H^2 H^\dagger H + \lambda_H (H^\dagger H)^2$$

where

- $-\mu_H^2 H^\dagger H$ term triggers electroweak symmetry breaking
- $\lambda_H (H^\dagger H)^2$ is the quartic () self-interaction term for the Higgs

Now for scalar singlet

As ϕ is a real scalar singlet under all SM gauge groups and odd under Z_2 symmetry

- the mass term should be quadratic : $1/2\, \mu_\phi^2 \phi^2$
- the quartic self-interaction term : $1/4!\, \lambda_\phi \phi^4$

The portal interaction term

$$V_{portal} = \frac{1}{2} \lambda_1 H^{\dagger} H \phi^2$$

It is

- gauge invarient since $H^{\dagger}H$ and ϕ^2 are both singlets
- Z_2 symmetric
- renormalizable (dimension 4)
- The Higgs portal coupling λ_1 controls the interaction strength

$\Rightarrow$ Full scalar potential is

$$\boxed{V(H,\phi) = -\mu_H^2 H^{\dagger}H + \lambda_H (H^{\dagger}H)^2 + \frac{1}{2}\mu_\phi^2 \phi^2 + \frac{1}{4!}\lambda_\phi \phi^4 + \frac{1}{2}\lambda_1 H^{\dagger} H \phi^2}$$

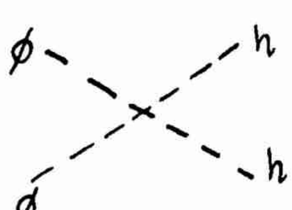
$$\therefore \mathcal{L} = \mathcal{L}_{SM} + \frac{1}{2}(\partial^{\mu}\phi)(\partial_{\mu}\phi) - V(H,\phi)$$

where $\mathcal{L}_{SM} \rightarrow$ contains all SM fields

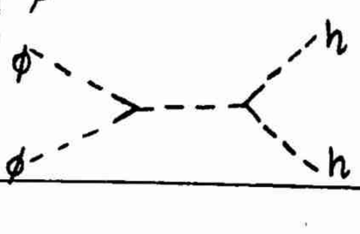
$\frac{1}{2}(\partial^{\mu}\phi)(\partial_{\mu}\phi) \rightarrow$ kinetic term for dark matter scalar singlet

$V(H,\phi) \rightarrow$ total scalar potential

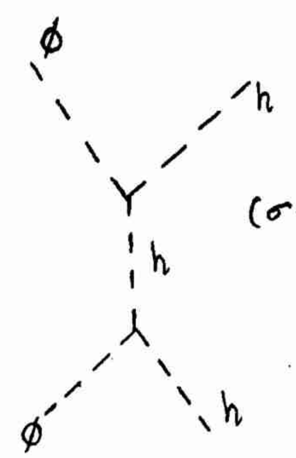
Annihilation cross - section & Relic density



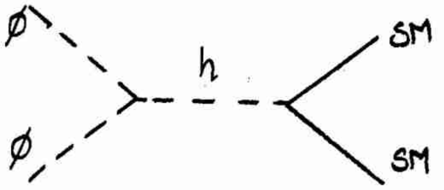
$$(\sigma v)_{\phi_1\phi_1 \rightarrow f\bar{f}} = \frac{1}{4\pi s\sqrt{s}} \frac{N_c \lambda_1^2 m_f^2}{(s-m_h^2)^2 + m_h^2 \Gamma_h^2} (s-4m_f^2)^{3/2}$$



$$(\sigma v)_{\phi_1\phi_1 \rightarrow W^+W^-} = \frac{\lambda_1^2}{8\pi} \frac{s}{(s-m_h^2)^2 + m_h^2 \Gamma_h^2} \left(1 + \frac{12 m_W^4}{s^2} - \frac{4m_W^2}{s}\right) \left(1 - \frac{4m_W^2}{s}\right)^{1/2}$$

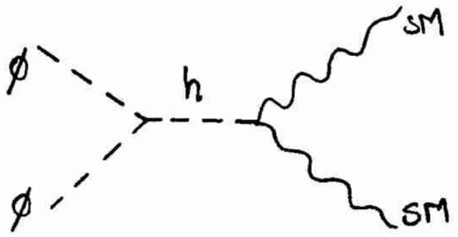


$$(\sigma v)_{\phi_1\phi_1\to ZZ} = \frac{\lambda_1^2}{16\pi}\,\frac{s}{(s-m_h^2)^2+m_h^2\Gamma_h^2}\left(1+\frac{12m_Z^4}{s^2}-\frac{4m_Z^2}{s}\right)\left(1-\frac{4m_Z^2}{s}\right)^{1/2}$$



$$(\sigma v)_{\phi_1\phi_1\to hh} = \frac{\lambda_1^2}{16\pi s}\left[1+\frac{3m_h^2}{(s-m_h^2)}-\frac{4\lambda_1 v^2}{(s-2m_h^2)}\right]^2\left(1-\frac{4m_h^2}{s}\right)^{1/2}$$

$\therefore$ the total area of annihilation cross-section

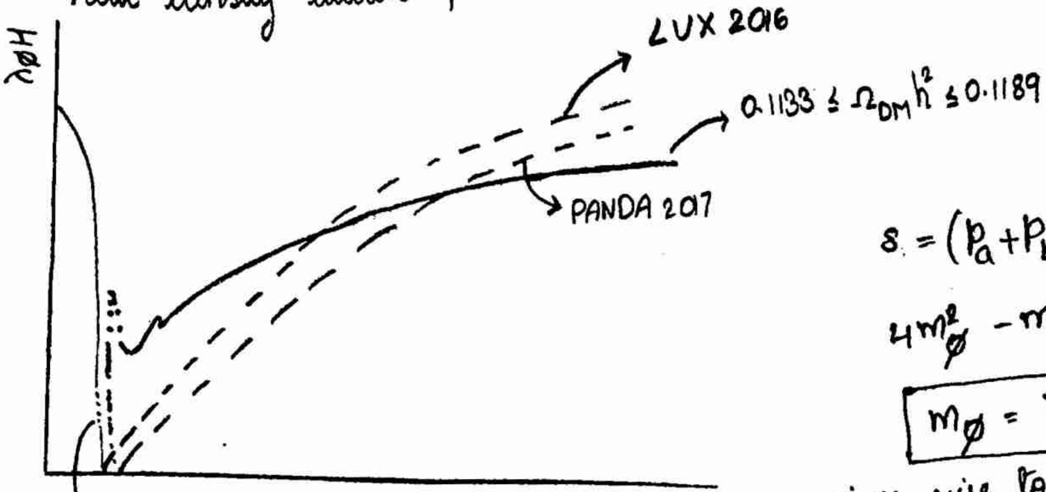


$$(\sigma v)_{\phi_1\phi_1\to SM} = (\sigma v)_{\phi_1\phi_1\to f\bar{f}} + (\sigma v)_{\phi_1\phi_1\to W^+W^-} + (\sigma v)_{\phi_1\phi_1\to ZZ} + (\sigma v)_{\phi_1\phi_1\to hh}$$

We know that Relic density is

$$\Omega h^2 \sim \frac{2.4\times 10^{-10}\ \text{GeV}^{-2}}{(\sigma v)_{\phi\phi\to SM}}$$

Relic density allowed parameter space



Resonance drop at 62.5 GeV = $m_h/2$

$$s = (p_a+p_b)^2 \simeq 4m_\phi^2$$

$$4m_\phi^2 - m_h^2 = 0$$

$$\boxed{m_\phi = m_h/2}$$

gives rise to to singularity or blows up the cross-section

This model we discussed above is called "Real scalar singlet Dark Matter Model".

It is the simplest real scalar dark matter model.

In this model we extend the standard Model by introducing a new field i.e. a real scalar singlet ϕ. which is neutral under SM gauge interactions

ϕ is also made stable via Z_2 symmetry, ensuring stability of dark matter particles

These two fields interact with one another using Higgs field as a medium (portal)

Thus the potential contains the terms of

- SM - SM interactions
- DM - DM interactions
- SM - DM interactions

Now let us move to a model that describes dark matter as a complex scalar field.

→ Complex scalar Singlet Dark Matter Model:

The key characteristics

- The dark matter candidate is a complex scalar field S, which can be decomposed into two real components.
- It can be charged either under a global or gauged U(1) symmetric conditions
- stabilization is often ensured by a reflection or discrete symmetry

acting on S.

- The potential contains the terms of

SM - SM interactions

SM - DM interactions

DM - DM interactions

The lagrangian of interaction can be written as

$$\mathcal{L}_{int} \sim |S|^2 H^\dagger H$$

where H is Higgs doublet

S is complex scalar singlet of dark matter

$S = 1/\sqrt{2}(S_1 + iS_2) \rightarrow$ can be written in this form, S_1 & S_2 are real

The interaction is invariant under both the $\underbrace{\text{dark sector U(1)}}_{\text{due to } |S|^2}$ symmetry and SM $\underbrace{\text{gauge symmetries}}_{\text{due to } H^\dagger H}$

it is renormalizable and is consistent with gauge invariance principles

$\therefore$ Lagrangian

$$\mathcal{L} = \mathcal{L}_{SM} + (\partial^\mu S)^\dagger(\partial_\mu S) - V(H,S)$$

where V(H,S) is scalar potential

Now building the potential

Due to Higgs doublets (SM - SM interaction)

$$V_H = -\mu_H^2 H^\dagger H + \lambda_H (H^\dagger H)^2$$

Due to complex scalar singlet : (DM-DM interaction)

$$V_S = \mu_S^2 |S|^2 + \lambda_S |S|^4$$

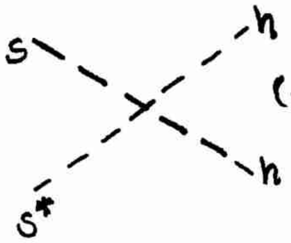
Due to interaction between SM and DM :

$$V_{int} = \lambda_{HS} |S|^2 H^\dagger H$$

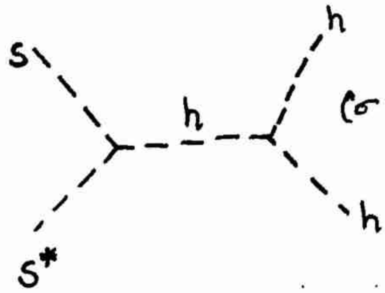
$$\therefore\ V(H,S) = V_H + V_S + V_{int}$$

$$\boxed{V(H,S) = -\mu^2 H^\dagger H + \lambda_H (H^\dagger H)^2 + \mu_S^2 |S|^2 + \lambda_S |S|^4 + \lambda_{HS} |S|^2 H^\dagger H}$$

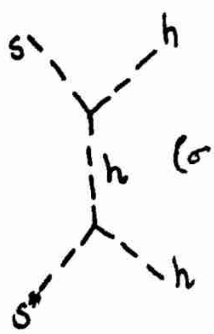
Annihilation cross section & Relic density



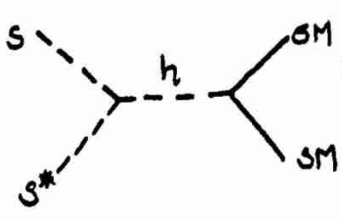
$$(\sigma v)_{SS^* \to f\bar{f}} = \frac{N_c \lambda_{HS}^2 m_f^2}{4\pi s} \frac{(s - 4m_f^2)^{3/2}}{[(s - m_h^2)^2 + m_h^2 \Gamma_h^2]}$$



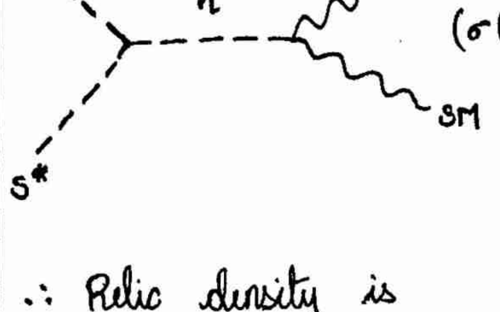
$$(\sigma v)_{SS^* \to W^+W^-} = \frac{\lambda_{HS}^2}{8\pi} \frac{s}{(s - m_h^2)^2 + m_h^2 \Gamma_h^2} \sqrt{\frac{s - 4m_W^2}{s}} \left(1 + \frac{12 m_W^4}{s^2} - \frac{4 m_W^2}{s}\right)$$



$$(\sigma v)_{SS^* \to ZZ} = \frac{\lambda_{HS}^2}{16\pi} \frac{s}{(s - m_h^2)^2 + m_h^2 \Gamma_h^2} \sqrt{1 - \frac{4m_Z^2}{s}} \left(1 + \frac{12 m_Z^4}{s^2} - \frac{4 m_Z^2}{s}\right)$$



$$(\sigma v)_{SS^* \to hh} = \frac{\lambda_{HS}^2}{16\pi s} \sqrt{1 - \frac{4m_h^2}{s}} \left[1 + \frac{3m_h^2}{s - m_h^2} - \frac{4\lambda_{HS} v^2}{s - 2m_h^2}\right]^2$$

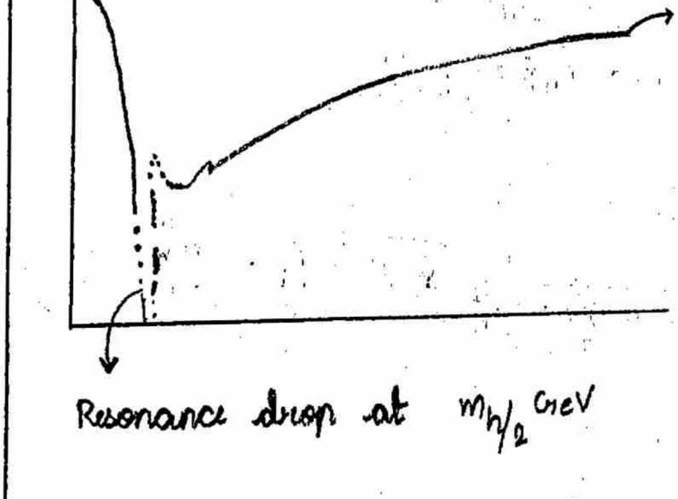


$$(\sigma(|v|))_{SS^* \to SM} = (\sigma v)_{SS^* \to f\bar{f}} + (\sigma v)_{SS^* \to W^+ W^-} + (\sigma v)_{SS^* \to ZZ} + (\sigma v)_{SS^* \to hh}$$

$\therefore$ Relic density is

$$\Omega h^2 \sim \frac{2.4 \times 10^{-10}\ \text{GeV}^{-2}}{(\sigma v)_{SS^* \to SM}}$$

Relic density allowed parameter space



$\to 0.1183 \leq \Omega_{DM} h^2 \leq 0.1189$

$$s = (p_a + p_b)^2 = 4 m_S^2$$

$$4 m_S^2 - m_h^2 = 0$$

$$\boxed{m_S = m_h/2}$$

Resonance drop at $m_h/2$ GeV

The graphs of real scalar singlet model and that of complex scalar singlet model are qualitatively similar they just vary quantitatively.

$\therefore$ The Relic density allowed parameter space of both models is same / similar.