⑥ $\mu=1, \nu=2$

$$\Gamma^3_{12} = \tfrac{1}{2} g^{33}\left(\partial_1 g_{32} + \partial_2 g_{13} - \partial_3 g_{12}\right)$$

$$= \tfrac{1}{2} g^{33} \left(\frac{\partial}{\partial r}(0) + \frac{\partial}{\partial \theta}(0) - \frac{\partial}{\partial \phi}(0)\right)$$

$$= 0$$

$$\boxed{\Gamma^3_{12} = 0}$$

$$\Rightarrow \boxed{\Gamma^3_{21} = 0}$$

⑦ $\mu=1, \nu=3$

$$\Gamma^3_{13} = \tfrac{1}{2} g^{33}\left(\partial_1 g_{33} + \cancel{\partial_3 g_{13}} - \cancel{\partial_3 g_{13}}\right)$$

$$= \tfrac{1}{2} g^{33}\left(\frac{\partial}{\partial r}\left(-R^2(t) r^2 \sin^2\theta\right)\right)$$

$$= \tfrac{1}{r}$$

$$\Rightarrow \boxed{\Gamma^3_{13} = \tfrac{1}{r}}$$

hence $\boxed{\Gamma^3_{31} = \tfrac{1}{r}}$

⑧ $\mu=2, \nu=2$

$$\Gamma^3_{22} = \tfrac{1}{2} g^{33}\left(\partial_2 g_{32} + \partial_2 g_{23} - \partial_3 g_{22}\right)$$

$$= \tfrac{1}{2} g^{33} \left(\frac{\partial}{\partial \theta}(0) + \frac{\partial}{\partial \theta}(0) - \underbrace{\frac{\partial}{\partial \phi}\left(-R^2(t) r^2\right)}_{0}\right)$$

$$= 0$$

$$\boxed{\Gamma^3_{22} = 0}$$

⑨ $\mu=2, \nu=3$

$$\Gamma^3_{23} = \tfrac{1}{2} g^{33}\left(\partial_2 g_{33} + \cancel{\partial_3 g_{23}} - \cancel{\partial_3 g_{23}}\right)$$

$$= \tfrac{1}{2}\left(-\frac{1}{R^2(t) r^2 \sin^2\theta}\right)\left(\frac{\partial}{\partial \theta}\left(-R^2(t) r^2 \sin^2\theta\right)\right)$$

$$= \frac{\sin\theta\cos\theta}{\sin^2\theta} = \cot\theta$$

$$\boxed{\Gamma^3_{23} = \cot\theta}$$

$$\hookrightarrow \boxed{\Gamma^3_{32} = \cot\theta}$$

⑩ $\mu = 3,\ \nu = 3$

$$\Gamma^3_{33} = \tfrac{1}{2} g^{33}(\partial_3 g_{33} + \cancel{\partial_3 g_{33}} - \cancel{\partial_3 g_{33}})$$

$$= \tfrac{1}{2} g^{33} \Big(\underbrace{\frac{\partial}{\partial\phi} (-R^2(t) r^2 \sin^2\theta)}_{0}\Big) = 0$$

$$\boxed{\Gamma^3_{33} = 0}$$

We can write a general form for $\lambda = 0$

$$\Gamma^0_{ij} = \frac{\dot{R}}{R} h_{ij}$$

where h_{ij} can be found from ds^2

$$ds^2 = h_{ij} dx^i dx^j$$

Now using Liouville Operator in FRW metric

→ for us to apply FRW metric to Liouville operator we should consider homogeneity and isotropic nature of universe, this makes phase-space independent of (homogeneous) spatial coordinates.

∴ The time evolution is the only factor we should focus on.

we know $p^\mu = (E, \vec{p})$

and $x^\mu = (t, \vec{x})$

∵ independent of spatial coordinates

$$f(p^\mu, x^\mu) \longrightarrow f(E, t)$$

Now applying the same logic to Liouville operator

$$\hat{L} = p^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} p^j p^k \frac{\partial}{\partial p^i}$$

will be similar to

$$\hat{L} = E \frac{\partial}{\partial t} - \Gamma^0_{jk} p^j p^k \frac{\partial}{\partial E}$$

due to spatial independence

substituting the general form of Γ^0_{jk}

$$\hat{L} = E \frac{\partial}{\partial t} - \frac{\dot{R}}{R} \underbrace{(h_{jk}) p^j p^k}_{|\vec{p}|^2} \frac{\partial}{\partial E}$$

$$\hat{L} = E \frac{\partial}{\partial t} - \frac{\dot{R}}{R} \frac{\cancel{(h_{jk}) p^j p^k}}{\cancel{(h_{jk}) p^j p^k}} (|\vec{p}|^2) \frac{\partial}{\partial E}$$

$$\hat{L} = E \frac{\partial}{\partial t} - \frac{\dot{R}}{R} |\vec{p}|^2 \frac{\partial}{\partial E}$$

We know that the number density of gas particles with internal degrees of freedom (g) is given by

$$n(t) = \frac{g}{(2\pi)^3} \int d^3p \, f(E,t)$$

where $f(E,t) \rightarrow$ phase-space density / distribution function

This allows us to link the solution of Boltzmann equation to the cosmological relic abundance.

Now calculating dn/dt, which describes how the number of particles per unit ~~(area)~~ volume changes over time in the expanding universe.

$$\frac{dn}{dt} = \frac{g}{(2\pi)^3} \int d^3p \, \frac{\partial f}{\partial t}$$

multiply and divide by E

$$\frac{dn}{dt} = \frac{g}{(2\pi)^3} \int \frac{d^3p}{E} \left(E \frac{\partial f}{\partial t} \right)$$

we know

$$E \frac{\partial f}{\partial t} - \frac{\dot{R}}{R} |(\vec{p})|^2 \frac{\partial f}{\partial E} = \hat{C}[f]$$

where $\hat{C}[f]$ is collision term

$$\Rightarrow E \frac{\partial f}{\partial t} = \hat{C}[f] + \frac{\dot{R}}{R} |\vec{p}|^2 \frac{\partial f}{\partial E}$$

substitute in dn/dt

$$\frac{dn}{dt} = \frac{g}{(2\pi)^3} \int \frac{d^3p}{E} C[f] + \frac{g}{(2\pi)^3} \int \frac{d^3p}{E} \left(\frac{\dot{R}}{R} |\vec{p}|^2 \frac{\partial f}{\partial E} \right)$$

solving $\int \frac{d^3p}{E} \left(\frac{\dot{R}}{R} |\vec{p}|^2 \frac{\partial f}{\partial E} \right)$

$$\int \frac{d^3p}{E} \left(\frac{\dot{R}}{R} |\vec{p}|^2 \frac{\partial f}{\partial E} \right) = \frac{\dot{R}}{R} \int d^3p \, \frac{(p_x^2 + p_y^2 + p_z^2)}{E} \left(\frac{\partial f}{\partial E} \right)$$

due to the symmetry, contributions from all the components will be equal

$$\Rightarrow \int \frac{d^3p}{E} \left(\frac{\dot{R}}{R} |\vec{p}|^2 \frac{\partial f}{\partial E} \right) = \frac{3\dot{R}}{R} \int d^3p \left(\frac{p_x^2}{E} \right) \frac{\partial f}{\partial E}$$

from $E = \sqrt{|\vec{p}|^2 + m^2}$

$$\frac{\partial E}{\partial p_x} = \frac{p_x}{E}$$

$$\Rightarrow \int \frac{d^3p}{E}\left(\frac{\dot{R}}{R}|\vec{p}|^2 \frac{\partial f}{\partial E}\right) = \frac{3\dot{R}}{R}\int d^3p\,(p_x)\frac{\partial E}{\partial p_x}\frac{\partial f}{\partial E}$$

$$= \frac{3\dot{R}}{R}\int d^3p \frac{\partial f}{\partial p_x}$$

$$= \frac{3\dot{R}}{R}\int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z\, p_x^2 \frac{\partial f}{\partial E}$$

$$= \frac{3\dot{R}}{R}\left[\cancelto{0}{\iint dp_y dp_z (p_x f)\Big|_{-\infty}^{\infty}} - \int dp_y dp_z dp_x f(E,t)\right]$$

$$\int \frac{d^3p}{E}\left(\frac{\dot{R}}{R}|\vec{p}|^2\frac{\partial f}{\partial E}\right) = -\frac{3\dot{R}}{R}\int (dp)^3 f(E,t)$$

$$\Rightarrow \frac{dn}{dt} = \frac{g}{(2\pi)^3}\int \frac{d^3p}{E} C[f] + \frac{g}{(2\pi)^3}\left[-\frac{3\dot{R}}{R}\int d^3p\,(f(E,t))\right]$$

$$\frac{\dot{R}}{R} = H$$

where H is Hubble parameter, i.e, expansion rate

$$\frac{g}{(2\pi)^3}\int d^3p\, f(E,t) = n(t)$$

$$\Rightarrow \frac{dn}{dt} = \frac{g}{(2\pi)^3}\int \frac{d^3p}{E} C[f] + (-3Hn(t))$$

$$\therefore \frac{dn}{dt} = \frac{g}{(2\pi)^3}\int\frac{d^3p}{E} C[f] - 3Hn(t)$$

$$\boxed{\frac{dn}{dt} + 3Hn(t) = \frac{g}{(2\pi)^3}\int \frac{d^3p}{E} C[f]} \rightarrow \text{Liouville operator}$$

We must use concepts of particle physics to solve collision term C

② Collision term :

It gives change in phase space due to particle creation / annihilation and scattering.

As collision term of Boltzmann equation encodes how particle interaction affect the distribution f and then the number density $n(t)$,

we calculate

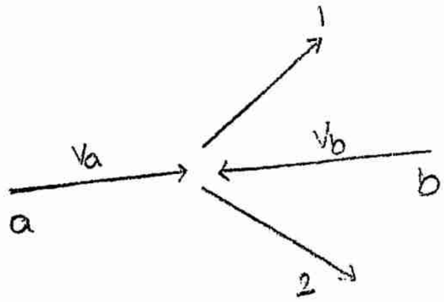
- cross-sections : to determine the rate at which particles scatter or annihilate
- decay widths : to get the rate at which unstable particles decay into stable ones, contributing to change in particle number

① Cross Section : cross section of a praticle collision determines how many desired final states can be produced given an intial set of particles.

Depends on three quantities

- initial flux
- matrix element of interaction
- phase-space

(all are Lorentz invariant)



$$\sigma = \frac{\text{no. of interactions per unit time per target particle}}{\text{incident flux (per unit area per unit time)}} \quad \{\text{general}\}$$

we know that differential cross-section in centre of mass frame is

$$d\sigma = \frac{1}{4|\vec{P}_i|\sqrt{s}} |M|^2 d\phi_f$$

where $d\phi_f$ is the final state Lorentz-invariant phase-space element

$M \rightarrow$ scattering amplitude/ matrix element of the process

$|M|^2$ gives the probability density for the transition

$\vec{P}_i \rightarrow$ initial moment vector (3D) of one of the incoming particle

$s \rightarrow$ Mandelstam variable s

it is square of the total energy in the center of mass frame

$$s = (P_1 + P_2)^2$$

where P_1 and P_2 are 4 momenta of two incoming particles

$\sqrt{s}$ is total available energy for the reaction in centre of mass frame

For a 2-body final state with particles of definite masses

$$d\phi_2 = \frac{|\vec{P}_f|}{16\pi^2\sqrt{s}} d\Omega$$

where $\vec{P}_f \rightarrow$ final 3 momentum vector of scattered/ outgoing particle

$d\Omega \rightarrow$ differential solid angle

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left(\frac{|\vec{P}_f|}{|\vec{P}_i|}\right)|M|^2$$

$$\therefore \boxed{\sigma = \int \frac{1}{64\pi^2 s}\left(\frac{|\vec{P}_f|}{|\vec{P}_i|}\right)|M|^2 \, d\Omega}$$

where the momenta are 3 momenta in centre of mass frame

$d\Omega$ is also wrt centre of mass

$\therefore$ I can write it as

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|^*}{|\vec{p}_i|^*} \int |\mathcal{M}|^2 \, d\Omega^*$$

where – * represents a quantity taken in center of mass reference

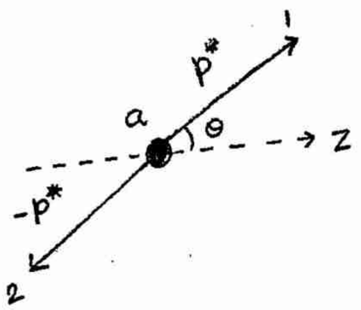
$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\left(\frac{|\vec{p}_f|^*}{|\vec{p}_i|^*}\right)|\mathcal{M}|^2$$

② Decay width of particles

→ It depends on

- matrix element of process
- phase space

(all are Lorentz Invariant)



general formula for decay width

$$\Gamma = \frac{1}{2m_a} \int d\phi_2 \, |\mathcal{M}_{fi}|^2$$

where m_a is mass of the decaying particle

$d\phi_2$ is Lorentz invariant two-body space phase element

$$d\phi_2 = (2\pi)^4 \delta^4(p_a - p_b - p_c) \frac{d^3\vec{p}_b}{(2\pi)^3 2E_b} \frac{d^3\vec{p}_c}{(2\pi)^3 2E_c}$$

where p_a is 4 momentum of initial particle

p_b, p_c 4 momenta of final particles

E_a is energy of initial particle

E_b, E_c are energies of final particles considering frame of refernce of a

$\Rightarrow \vec{p}_a = 0$

$E_a = m_a$

$\therefore p_a = (m_a, \vec{0})$

conservation of energy and momentum wrt a

$$\vec{p}_{ba} = -\vec{p}_{ca}$$

where $\vec{p}_{ba}$ and $\vec{p}_{ca}$ are momenta of particles b and c in rest frame of a

let $|\vec{p}_{ba}| = |\vec{p}_{ca}| = p^*$

$\therefore$ Energies will be

$$E_{ba} = \sqrt{|p^*|^2 + m_b^2} \quad , E_{ca} = \sqrt{|p^*|^2 + m_c^2} \quad \text{and} \quad m_a = E_{ba} + E_{ca}$$

now $\delta^4(p_a - p_b - p_c)$ will become

$$\delta^4(p_a - p_b - p_c) = \delta(E_a - E_b - E_c) \cdot \delta^3(\vec{p}_a - \vec{p}_c - \vec{p}_b)$$

which in rest frame of a will become

$$= \delta(m_a - E_{ba} - E_{ca}) \cdot \delta^3(0 - \vec{p}_{ca} - \vec{p}_{ba})$$

as we know

$$\int d^3\vec{p}_c \, \delta^3(-\vec{p}_{ca} - \vec{p}_{ba}) f(\vec{p}_{ca}) = f(-\vec{p}_{ba})$$

$$\Rightarrow \phi_2 = (2\pi)^4 \delta(\underbrace{E_{aa}}_{m_a} - E_{ba} - E_{ca}) \frac{d^3\vec{p}_{ba}}{(2\pi)^3 2E_{ba}} \cdot \frac{1}{(2\pi)^3 2E_{ca}}$$

we can write

$$d^3\vec{p}_{ba} = |\vec{p}_{ba}|^2 \, d|\vec{p}_{ba}| \, d\Omega$$
$$= p^{*2} \, dp^* d\Omega$$

now

$$\int dp^* \, \delta(m_a - E_{ba} - E_{ca}) = \frac{1}{\left(\frac{d}{dp^*}(E_{ba}+E_{ca})\right)}$$
$$= \frac{1}{p^*(1/E_{ba} + 1/E_{ca})} = \frac{E_{ba}E_{ca}}{p^*(E_{ba}+E_{ca})}$$

$$\Rightarrow d\phi_2 = \cancel{(2\pi)^4} \frac{(p^*)^{\cancel{2}} d\Omega}{(2\pi)^{\cancel{6}2} 4 \cancel{E_{ba}}\cancel{E_{ca}}} \left(\frac{\cancel{E_{ba}}\cancel{E_{ca}}}{\cancel{p^*} m_a} \right)$$

$$= \frac{p^* d\Omega}{16\pi^2 m_a}$$

$$\therefore \boxed{\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 d\Omega} \rightarrow$$ decay width of a single unstable particle

if there are different final states

$$\Gamma = \sum_j \Gamma_j$$

branching ratios

$BR(j) = \Gamma_j/\Gamma$ (probability that a will decay into final state j)

and decay life time (τ) $= 1/\Gamma$

Now that we have calculated the cross-section and decay width we can move on to find the collision term.

Lets consider a reaction as follows

$$\psi + a + b + \cdots \rightarrow i + j + k + \cdots$$

From Fermi's Golden Rule, the transition rate for a process like the above is proportional to

$$|\mathcal{M}|^2 \times (\text{phase space}) \times \delta^4(\text{momentum conservation})$$

The lorentz phase space measure of i^{th} final state is

$$d\Pi_i = \frac{d^3p_i}{(2\pi)^3 \underbrace{2E_i}_{\text{using relativistic normalization}}}$$

Now enforcing the conservation laws

$$(2\pi)^4 \delta^4(p_\psi + p_a + p_b + \cdots - p_i - p_j - p_k \cdots)$$

Statistical occupancy and quantum statistical effects

$$(1 \pm f_i)$$

where $f_i \rightarrow$ particle phase-space distribution

$\pm \rightarrow$ encodes quantum statistical effects

- $+$ for boson (Boson enhancement)
- $-$ for fermions (Pauli blocking)

Gain - loss structure of collision term

→ gain : particles scatter into the state with momentum $\vec{p}$

→ loss : particles scatter out of the state with momentum $\vec{p}$

$$C[f] = (\text{rate in}) - (\text{rate out})$$

$$\Rightarrow \frac{g}{(2\pi)^3}\int \frac{d^3p}{E} C[f] = -\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots \times (2\pi)^4 \delta^4(p_\psi + p_a + \cdots - (+p_i + p_j + \cdots))$$
$$\times [|M|_1^2 f_\psi f_a f_b \cdots (1 \pm f_i)(1 \pm f_j) \cdots - |M|_2^2 f_i f_j \cdots (1 \pm f_\psi)(1 \pm f_a)(1 \pm f_b) \cdots]$$

where $|M|_1^2$ is for $\psi + a + b + \cdots \to i + j + k + \cdots$

$|M|_2^2$ is for $i + j + k + \cdots \to \psi + a + b + \cdots$

The matrix element M depends on the specific dark matter interaction

Assumpting CP(T) invariance

CP(T) invariance stands for the combined symmetry of Charge Conjugation (C), Parity (P), Time Reversal (T) applied to physical laws:

Charge Conjugation → changes particles into their anti-particles

Parity (P) → flips spatial coordinates

Time Reversal (T) → reverses the direction of time

And if a law is CP(T) invariant it should remain same under these changes

Now applying the assumption:

$|\mathcal{M}|_1^2 = |\mathcal{M}|_2^2 = |\mathcal{M}|^2$ (lets say)

The second critical assumption is to use Maxwell-Boltzmann distribution as the temperature is very high so the particles (bosons + fermions) are in classical limit.

∴ The quantum statistical effects

- Boson enhancement
- Fermion blocking

can be ignored, as in classical limit, the quantum processes cease to exist (take place)

$\Rightarrow$ $(1 \pm f_i)$ terms can be neglected

$$\Rightarrow \frac{g}{(2\pi)^3}\int \frac{d^3p}{E} C[f] = -\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots \times (2\pi)^4 \delta^4(p_\psi + p_a + p_b + \cdots - p_i - p_j - \cdots) \times |\mathcal{M}|^2 [f_a f_b \cdots f_\psi - f_i f_j \cdots]$$

now substituting in Boltzmann equation's simplified form

$$\underbrace{\dot{n}_\psi}_{\frac{dn_\psi}{dt}} + 3Hn_\psi = -\int d\Pi_\psi d\Pi_a d\Pi_b \cdots d\Pi_i d\Pi_j \cdots \times (2\pi)^4 \delta^4(p_\psi + p_a + p_b + \cdots - (p_i) - p_j - \cdots) \times |\mathcal{M}|^2 [f_a f_b \cdots f_\psi - f_i f_j \cdots]$$

∵ $(1 \pm f_i) \to 1$ because of classical limit

and the distribution

$$f_i(E_i) = \exp[-(E_i - \mu_i)/T]$$

As it is assumed that dark matter is in equilibrium with standard model particles in the early universe via $2 \leftrightarrow 2$ interactions we can consider a reaction

$$\psi \bar{\psi} \rightarrow \chi \bar{\chi}$$

now for such a process the Boltzmann equation will become

$$\dot{n}_\psi + 3Hn_\psi = - \int d\Pi_\psi \, d\Pi_{\bar{\psi}} \, d\Pi_\chi \, d\Pi_{\bar{\chi}} \, (2\pi)^4 \delta^4 (p_\psi + p_{\bar{\psi}} - p_\chi - p_{\bar{\chi}}) \, |M|^2_{\psi\bar{\psi} \rightarrow \chi\bar{\chi}} \, (f_\psi f_{\bar{\psi}} - f_\chi f_{\bar{\chi}})$$

as the particles are in thermal and chemical equilibrium

$$\mu_\psi + \mu_{\bar{\psi}} = \mu_\chi + \mu_{\bar{\chi}}$$

$\Rightarrow$ we can ignore the chemical potential in the distribution function

$$f_i(E_i) = \exp(-E_i/T)$$

$\therefore$ $f_\psi^{EQ} = \exp(-E_\psi/T)$ and $f_{\bar{\psi}}^{EQ} = \exp(-E_{\bar{\psi}}/T)$

and due to energy conservation

$$E_\psi + E_{\bar{\psi}} = E_\chi + E_{\bar{\chi}}$$

$$\Rightarrow f_\chi f_{\bar{\chi}} = f_\psi^{EQ} f_{\bar{\psi}}^{EQ}$$

put this in above equation

$$\dot{n}_\psi + 3Hn_\psi = - \int d\Pi_\psi \, d\Pi_{\bar{\psi}} \, d\Pi_\chi \, d\Pi_{\bar{\chi}} \, (2\pi)^4 \delta^4 (p_\psi + p_{\bar{\psi}} - p_\chi - p_{\bar{\chi}}) \, |M|^2_{\psi\bar{\psi} \rightarrow \chi\bar{\chi}} \, [f_\psi f_{\bar{\psi}} - f_\psi^{EQ} f_{\bar{\psi}}^{EQ}]$$

We are now going to introduce thermal average cross-section as, in hot thermal bath, particles have momenta distribution near equilibrium, so averaging over these distributions captures the typical interaction rate helping us simplify the collision term while keeping the physics accurate.

Apply thermal average cross section.

we can write the difference in number densities as

$$f_\psi f_{\bar\psi} - f_\psi^{EQ} f_{\bar\psi}^{EQ} \approx \frac{n_\psi n_{\bar\psi}}{(n_\psi^{eq})^2}\left(f_\psi^{EQ} f_{\bar\psi}^{EQ}\right) - f_\psi f_{\bar\psi}^{EQ}$$

$$= f_\psi^{EQ} f_{\bar\psi}^{EQ}\left(\frac{n_\psi n_{\bar\psi}}{(n_\psi^{EQ})^2} - 1\right)$$

thermally averaged annihilation cross-section is defined as

$$\langle\sigma|v|\rangle = \frac{\int d\Pi_\psi d\Pi_{\bar\psi} d\Pi_X d\Pi_{\bar X} (2\pi)^4 \delta^4(p_\psi + p_{\bar\psi} - p_X - p_{\bar X})\, |M|^2 f_\psi^{EQ} f_{\bar\psi}^{EQ}}{\int d\Pi_\psi d\Pi_{\bar\psi} f_\psi^{EQ} f_{\bar\psi}^{EQ}}$$

$$= \frac{\int d\Pi_\psi d\Pi_{\bar\psi} d\Pi_X d\Pi_{\bar X} (2\pi)^4 \delta^4(p_\psi + p_{\bar\psi} - p_X - p_{\bar X})\, |M|^2 f_\psi^{EQ} f_{\bar\psi}^{EQ}}{(n_\psi^{EQ})^2}$$

$$\because \int d\Pi_\psi f_\psi^{EQ} = n_\psi^{EQ}$$

$\Rightarrow$ we can write the RHS part as

$$\langle\sigma|v|\rangle \left(n_\psi n_{\bar\psi} - n_\psi^{EQ} n_{\bar\psi}^{EQ}\right)$$

$\therefore$ Boltzmann equation can be written as

$$\boxed{\dot{n}_\psi + 3Hn_\psi = -\langle\sigma|v|\rangle\left(n_\psi n_{\bar\psi} - n_\psi^{EQ} n_{\bar\psi}^{EQ}\right)}$$

where

$$|v| = \frac{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}$$

i.e. Mollar velocity

where 1 and 2 repreasented as ψ and $\bar\psi$

as they are the interacting particles

Hence the Boltzmann equation is its simplified form is

$$\boxed{\dot{n}_\psi + 3Hn_\psi = -\langle\sigma|v|\rangle\left(n_\psi n_{\bar\psi} - n_\psi^{EQ} n_{\bar\psi}^{EQ}\right)} \ \text{—.}$$

Now we should solve this equation to evaluate dark matter freeze out.

Now before we move further lets solve $\langle\sigma|v|\rangle$ for $m_1 = m_2 = m$

We know

$$\langle\sigma|v|\rangle = \frac{\int d^3p_\psi\, d^3p_{\bar\psi}\, \sigma\, v_{rel}\, e^{-(E_\psi/T)} e^{-(E_{\bar\psi}/T)}}{\int d^3p_\psi\, d^3p_{\bar\psi}\, e^{-(E_\psi/T)} e^{-(E_{\bar\psi}/T)}}$$

where $E_i = \sqrt{p_i^2 + m_i^2}$

$\because m_\psi = m_{\bar\psi} = m$

$E_i = \sqrt{p_i^2 + m^2}$

The Mandelstam variable s for two particles with 4-momenta p_ψ, $p_{\bar\psi}$ is

$$s = (p_\psi + p_{\bar\psi})^2$$

where s ranges from $4m^2$ to ∞

$\because p_i = (E_i, \vec{p}_i)$

if the particle is at rest

$\vec{p}_i = 0$

$E_i = m_i$

Now we try to express molar velocity (relative velocity) in terms of s

$$|v| = \frac{\sqrt{(p_\psi \cdot p_{\bar\psi})^2 - m_\psi^2 m_{\bar\psi}^2}}{E_\psi E_{\bar\psi}}$$

as $s = (p_\psi + p_{\bar\psi})^2 = p_\psi^2 + p_{\bar\psi}^2 + 2(p_\psi \cdot p_{\bar\psi})$

$\because$ particles are relativistic

$p_\psi^2 = m_\psi^2$

as $m_\psi = m_{\bar\psi} = m$

$s = 2m^2 + 2(p_\psi \cdot p_{\bar\psi})$

$\Rightarrow (p_\psi \cdot p_{\bar\psi}) = \frac{s - 2m^2}{2}$

$\therefore$ the numerator will becom

$$\sqrt{(p_\psi \cdot p_{\bar\psi})^2 - m^4} = \sqrt{\left(\frac{s^2 + 4m^4 - 4m^2 s}{4}\right) - m^4}$$

$$= \frac{\sqrt{s(s-4m^2)}}{2}$$

$$\Rightarrow |v| = \frac{\sqrt{s(s-4m^2)}}{2E_\psi E_{\bar\psi}}$$

now expressing the integration variables to center of mass quantities using Lorentz - invariant phase space measure

$$d^3 p_\psi \, d^3 p_{\bar{\psi}} = \frac{|\vec{p}_\psi|^2 d|\vec{p}_\psi| d\Omega_1}{(2\pi)^3} \; \frac{|\vec{p}_{\bar{\psi}}|^2 d|\vec{p}_{\bar{\psi}}| d\Omega_2}{(2\pi)^3}$$

now change variables to

→ Total CM momentum $P = p_\psi + p_{\bar{\psi}}$

→ relative momentum or invariants s

→ scattering angles

$$\int \frac{ds}{\sqrt{s}} \int d\Omega \,(\text{kinematic factors})\, \sigma(s)\, v_{rel}\, e^{-\sqrt{s}/T}$$

using Maxwell - Boltzmann distribution and Bessel functions

→ we are integrating over momenta with the exponential distribution $e^{-E/T}$ this leads to modified Bessel functions K_n

over total energy $\equiv K_1(\sqrt{s}/T)$

normalizations $\equiv K_2(m/T)$

Now changing the integral measure and integration limits

→ we are integrating over s

∴ limits should be $4m^2$ to ∞

→ full measure includes flux and momentum factors proportional to $(s-4m^2)\sqrt{s}$

∴ integral becomes $\displaystyle\int_{4m^2}^{\infty} ds\, \sigma(s)\,(s-4m^2)\sqrt{s}\, K_1(\sqrt{s}/T)$