Lecture 6

Contents

- Sysrems of linear equations
- Matrix decomposition
- Matrix norms(optional)
- 数值计算
- 优化

Vector and Matrix Norms

Norms

Definition :

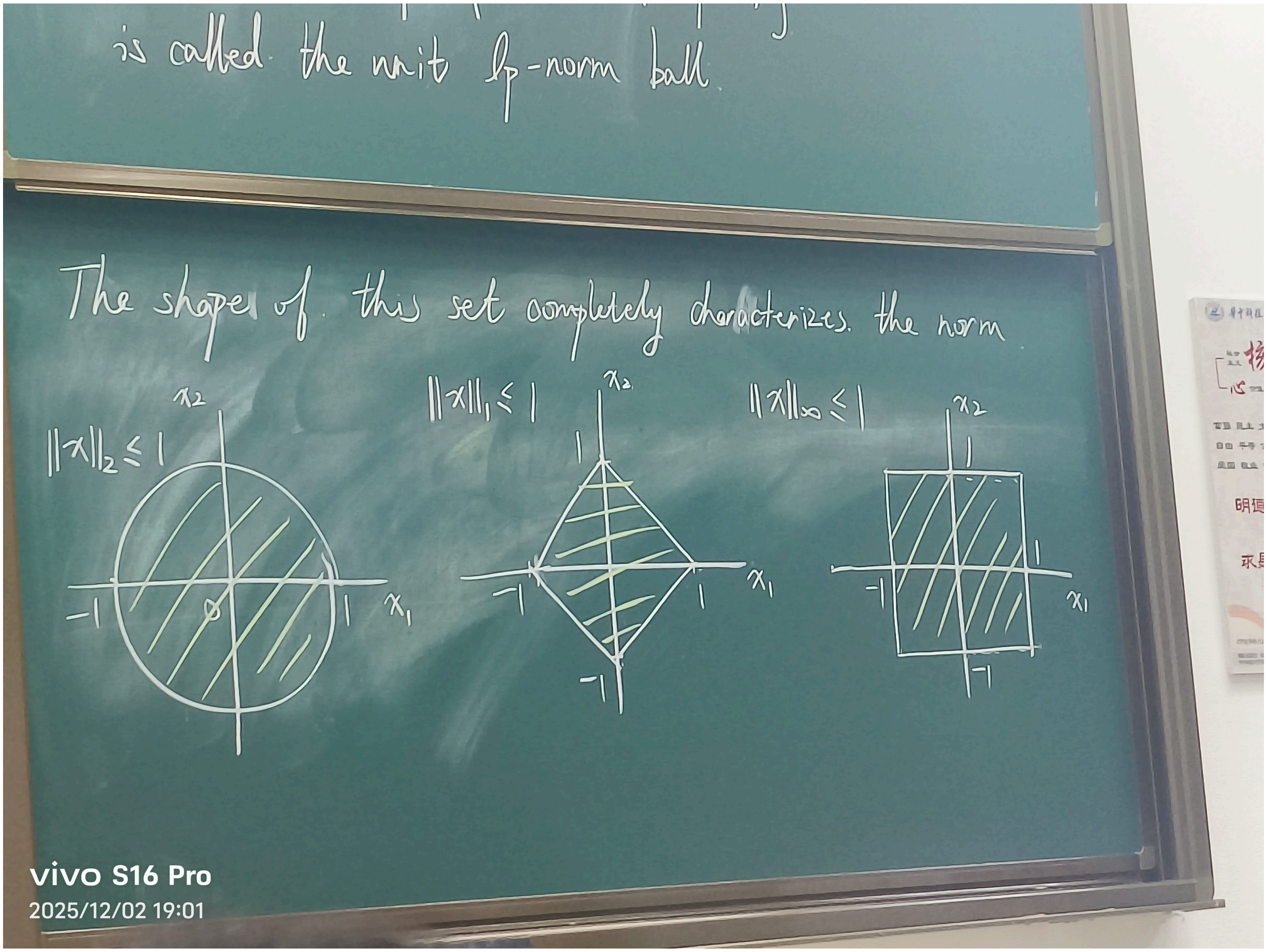
A function $||.||$

- $||x|| > 0 \, and \, ||x|| = 0 \, only \, x = 0$
- $||\alpha x|| = |\alpha| ||x|| \, for \, any$
- $||x + y|| \le ||x|| + ||y||$
  如果在n维空间中，则有
- **standard Enclidean length（欧几里得距离）**: $||x||_2 = (\sum_{i=1}^{n} {x_i}^2)^{\frac{1}{2}} \, (l_2 norm)$
- **sum-of absolute -values length（曼哈顿距离）**: $||x||_1 = \sum_{i=1}^{n} |x_i| (l_1 norm)$
- **the limit case $p = \infty$ define the l_∞ norm（求极限的结果）**:
  $||x||_\infty = \max_{i=1,\dots,n} |x_i|$
  It is **a.k.a max-absolute-value norm and Chebyshev norm（切比雪夫范数）**

> The $'l_0 norm'$,which is NOT formally a norm:
> $$||x||_0 = \sum_{i=1}^{n} H(x_i)(x_i \neq 0) \, where \, H(x_i)(x_i \neq 0) = \begin{cases} 1, & if \, x_i \neq 0 \\ 0, & otherwise \end{cases}$$

Norm balls

The set of all vectors with $l_p$norm less than or equal to $B_p = \{x \in R^n : \|x\|_p \leq 1\}$
The shape of this set completely characterizes the norm
二维的情况如下：



Matrices

- $Trace\,of\,A \in R^{n*n}$（迹） $: trace(A) = \sum_{k=1}^{n} a_{kk}$
- **Standard inner product（矩阵内积）** in R^{n*n}:$< A, B >= trace(A^T B)$
- **Frobenius norm**:$\|A\|_F = \sqrt{< A, A >} = \sqrt{trace(A^T A)} = \sqrt{\sum_{i,j} a_{ij}^2}$
- l_p**-induced matrix norm**($p = 1, 2, \infty$)
  $\|A\|_p = \max\limits_{\|u\|_p \neq 0} \frac{\|A_u\|_p}{\|u\|_p} = \max\limits_{\|u\|_p = 1} \|Au\|_p$
  - l_1**-induced norm**:the largest absolute column sum $\|A\|_1 = \max\limits_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}|$

- l_∞**-induced norm**:the largest absolute row sum$\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$
- l_2**-induced norm**:the squre-root of the largest eigenvalue $\lambda_{max}\, of\ A^T A$:$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)} = \sigma_{max}(A), a.k.a\ spectral\ norm, operator\ norm$

- **Nuclear norm（核范数）**:the sum of ites singular values of A
  $\|A\|_* = \sum_i \sigma_i(A)$
  a.k.a trace norm（迹范数）

Numerical Differentiation（数值微分）

给定一个函数$f(t)$，定义$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$,
泰勒展开Taylor series:$f(t+\Delta t) = f(t) + \Delta t \frac{df}{dt}(t) + \frac{(\Delta t)^2}{2!} \frac{d^2 f}{dt^2}(t) + o(\Delta t^2)$
$f(t-\Delta t) = f(t) - \Delta t \frac{df}{dt}(t) + \frac{(\Delta t)^2}{2!} \frac{d^2 f}{dt^2}(t) + o(\Delta t^2)$

- Forward Difference（前项差分）: $\frac{f(t+\Delta t) - f(t)}{\Delta t}$
- Backward Differece（后项差分）: $\frac{f(t) - f(t-\Delta t)}{\Delta t}$
  而前项差分、后项差分用泰勒估计为$\frac{df}{dt}(t) + O(\Delta t)$，所以误差是$O(\Delta t)$的
- Central Difference（中心差分）:两个泰勒展开式相减有$\frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t}$，这个误差的量级是$O(\Delta t^2)$
  - not possible when computing $f'(t)$ at boundaries of T data
  - real time