Lecture 9

- 算法的精度?
- 如何理解复杂度?

Error Analysis

> Forward Euler:$(y_{k+1} = y_k + \Delta t f(y_k)) \approx (y(t_{k+1}) = y(t_k + \Delta t))$

利用泰勒展开：
$y(t_k + \Delta t) = y(t_k) + \Delta t \frac{dy}{dt}(t_k) + \frac{\Delta t^2}{2!} \frac{d^2 y}{dt^2}(c), c \in [t_k, t_{k+1}]$
假设y_k完美估计了$y(t_k)$，也就是不考虑累积的影响，则有
$Error = y_k + \Delta t f(y_k)$，量级为$O(\Delta t^2)$
于是称$Local\, error: \epsilon_{k+1} = y(t_k + \Delta t) - y_{k+1} = \frac{\Delta t^2}{2} y''(c)$
$Global\, error: E_k = \sum_{j=0}^{k} \epsilon_k = (b-a) \frac{\Delta t}{2} y''(c)$，但是量级$O(\Delta t)$

Stability

Example 1

$y' = \lambda y, y(0) = y_0, \lambda \in R$
$solution: y(t) = e^{\lambda t} y_0$

> $F.E.: y_{k+1} = (1 + \lambda \Delta t) y_k$
> $B.E.: y_{k+1} = (1 - \lambda \Delta t)^{-1} y_k$

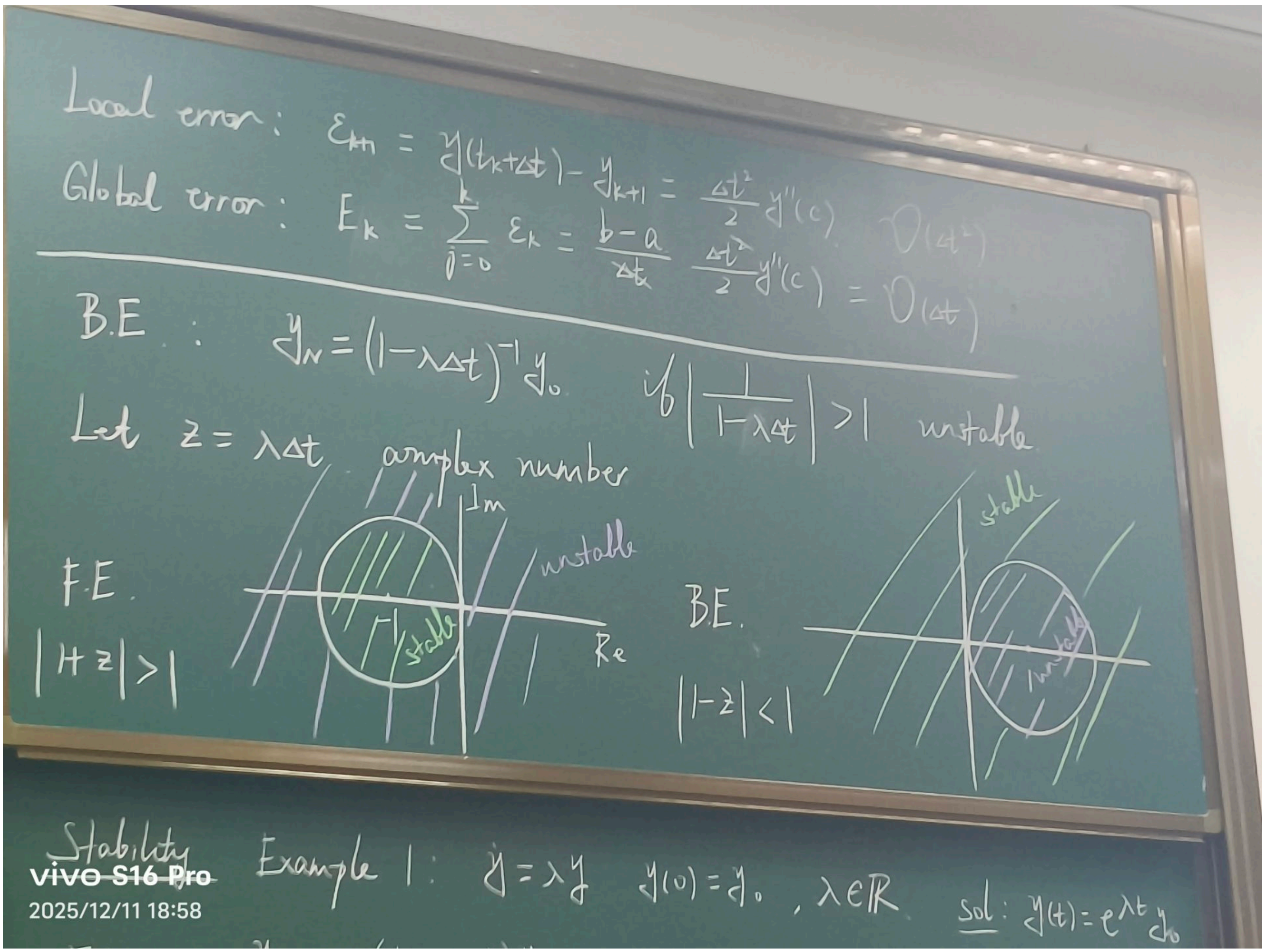
于是从$y_0 \to y_1 \to y_2 \to ... \to y_N$有$y_N = (1 + \lambda \Delta t)^N y_0$
假设初始出现了噪声，则有$round - off\, error: y_N = (1 + \lambda \Delta t)^N (y_0 + \delta)$
而$(E_N)^\delta = (1 + \lambda \Delta t)^N \delta$
$if |1 + \lambda \Delta t| > 1$，则误差发散（指数爆炸）
相应地，$B.E.$中$y_N = (1 + \lambda \Delta t)^{-N} y_0$
$if |\frac{1}{1 + \lambda \Delta t}| > 1$，则误差发散

Example 2

$y' = Ay, y(0) = y_0, Exact\,solution : y(t) = e^{At}y_0$

> $F.E. : y_{k+1} = (1 + \lambda\Delta t)y_k$
> $B.E. : y_{k+1} = (1 - \lambda\Delta t)^{-1}y_k$

根据$F.E.$分析的结果是$y_N = (1 + \Delta tA)^N y_0$
也就是$\forall |eig(I + \Delta tA)| < 1$
根据$B.E.$分析的结果是$y_N = (1 + \Delta tA)^{-N} y_0$
也就是$\forall |eig(I - \Delta tA)^{-1}| < 1$



更复杂的方程：$y' = f(t, y)$

> $F.E : x_{k+1} = x_k + \Delta t f(x_k, y_k)$
> $B.E. : x_{k+1} = x_k + \Delta t f(x_{k+1}, t_{k+1})$

- Generic integrator:$y_{k+1} = y_k + \Delta t\phi$
- We choose ϕ to reduce error
- **Idea**:Instead of phi being the slope at $(t_k, y_k)(F.E.)$,or $(t_{k+1}, y_{k+1})(B.E.)$,use the slope at (t_k, y_k) and at **another** point $(t_k + P\Delta t, y_k + Q\Delta t f(t_k, y_k))$
  $y_{k+1} = y_k + \Delta t[Af(t_k, y_k) + Bf(t_k + P\Delta t, y_k + Q\Delta t f(t_k, y_k))]$
  We get to choose A,B,P,Q to match Taylor series $y(t_k + \Delta t)\ if P = Q, then (t_k + P\Delta t, y_k + P\Delta t f(t_k, y_k))$is a small $P\Delta t\ F.E.$ step.

Calculation

$f(Af(t_k, y_k) + Bf(t_k + P\Delta t, y_k + P\Delta t f(t_k, y_k))) = f(y_k, t_k) + P\Delta t\frac{\partial f}{\partial t}(t_k, y_k) + P\Delta t\frac{\partial f}{\partial y}(t_k, y_k)f(t_k, y_k) + O(\Delta t^2)$
代入上述方程有$y_{k+1} = y_k + \Delta t(A + B)f(t_k, y_k) + PB\Delta t^2 f_t(t_k, y_k)f(t_k, y_k) + O(\Delta t^3)$
与原式的泰勒展开比较：
$y(t_k + \Delta t) = y(t_k) + \Delta t\frac{dy}{dt}(t_k) + \frac{\Delta t^2}{2!}\frac{d^2y}{dt^2}(t_k) + O(\Delta t^3) = y_k + \Delta t\frac{dy}{dt}(t_k, y_k) + \frac{\Delta t^2}{2!}\frac{d}{dt}f(t_k, y_k) + O(\Delta t^3)$
由链式法则有$y_k + \Delta t\frac{dy}{dt}(t_k, y_k) + \frac{\Delta t^2}{2!}[f_t(t_k, y_k) + f_y(t_k, y_k)f(y_k, y_k)] + O(\Delta t^3)$
运用待定系数法有

- $A + B = 1$
- $PB = \frac{1}{2}$
- $P = Q$
  这个方程组有无穷个解，最常用的是$A = 0, B = 1, P = \frac{1}{2}$
  故$y_{k+1} = y_k + \Delta t f(t_k + \frac{\Delta t}{2}, y_k + \frac{\Delta t}{2}f(t_k, y_k))$
  称为**Second-order Runge-Kutta**

> 在matlab中为'ode23'函数

> 'Forth-order Runge-Kutta'('ode45')
> $y_{k+1} = y_k + \frac{\Delta t}{6}[f_1 + 2f_2 + 2f_3 + f_4]$
> $where\ f_1 = f(t_k, y_k)$
> $f_2 = f(t_k + \frac{\Delta t}{2}, y_k + \frac{\Delta t}{2})f_1$
> $f_3 = f(t_k + \frac{\Delta t}{2}, y_k + \frac{\Delta t}{2}f_2)$
> $f_4 = f(t_k + \Delta t, y_k + \frac{\Delta t}{2}f_3)$