

1 Introduction

This document is a summary of work so far on defining security notion for searchable encryptions and finding good/bad constructions with respect to the notions. In particular, we have considered using idea of g-leakage to measure security of a given scheme with respect to a target (gain) function and shown it is equivalent to an experiment-based adversary. We have studied three schemes we proposed, and shown that even though all of the schemes are secure against keyword guessing attack, they are not equally secure against other types of attacks. This demonstrates that security against keyword guessing attack is not sufficient for an encryptions scheme to be secure. For instance, one of the scheme we proposed is secure against inference attack but it leaks all keywords in all documents information-theoretically.

2 Notation

The following notations are used for sets we used to describe a searchable database.

Notation	Explanation
DB	Set of searchable databases
EDB	Set of encrypted searchable databases
\mathcal{W}	Set of keywords in a database
\mathcal{EW}	Set of encrypted keywords in a encrypted searchable databases
\mathcal{EI}	Set of encrypted indices for keyword searching
F	Set of files
EF	Set of encrypted files
Q	Set of queries on the database
EQ	Set of encrypted queries on encrypted database
CST	Set of states of the client
R	Set of responses from a query
ER	Set of encrypted responses from an encrypted query

The following notations are used to describe realisations of a database.

Notation	Explanation
db	Database that has not been encrypted
edb	Encrypted database
W	Keyword set associated to some file
w	A keyword from keyword set \mathcal{W}
EW	Encrypted set associated to some encrypted file
ew	An encrypted keyword from encrypted keyword set \mathcal{EW}
EI	An encrypted index associated to an encrypted database
f	A file in the database
ef	An encrypted file
q	A query on the database
eq	An encrypted query on the encrypted database
cst	A state of the client
r	A response from a query
er	An encrypted response from an encrypted query

We use Π_i to mean projection of a product of sets to its i -th component. For example, $\Pi_1(A \times B) = A$. We may also use it to return the i -th element in a tuple. For example, $\Pi_1((A, B)) = A$. We abuse $W(f)$

and $F(W)$ to mean set of keywords associated to file f and the set of files associated to keyword set W . Similarly, we abuse $EW(ef)$ and $EF(W)$ to mean set of encrypted keywords associated to encrypted file ef and the set of encrypted files associated to keyword set W .

3 g-leakage

G-leakage is first studied by Alvim et al. in [1]. To understand how it works, we first define (classic) vulnerability, leakage and capacity.

Definition 1 (Information Channel). *A channel is a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{C})$, where \mathcal{X} and \mathcal{Y} are finite sets and \mathcal{C} is a channel matrix, an $|\mathcal{X}| \times |\mathcal{Y}|$ matrix whose entries are between 0 and 1 and whose rows each sum to 1.*

Definition 2 (Vulnerabilities). *Given prior distribution π on \mathcal{X} and channel C , the prior vulnerability is given by:*

$$V(\pi) = \max_{x \in \mathcal{X}} \pi[x], \quad (1)$$

and the posterior vulnerability is given by:

$$V(\pi, \mathcal{C}) = \sum_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} \pi[x] \mathcal{C}[x, y]. \quad (2)$$

Definition 3 (Leakage and Capacity). *We define min-entropy leakage $\mathcal{L}(\pi, \mathcal{C})$ and min-capacity $\mathcal{ML}(\mathcal{C})$ to be:*

$$\mathcal{L}(\pi, \mathcal{C}) = -\log V(\pi) + \log V(\pi, \mathcal{C}) = \log \frac{V(\pi, \mathcal{C})}{V(\pi)} \quad (3)$$

$$\mathcal{ML}(\mathcal{C}) = \sup_{\pi} \mathcal{L}(\pi, \mathcal{C}). \quad (4)$$

G-leakage is different from the classic definition by allowing the adversary to guess part of x from observation y , and his gain is measured by a gain function g . We define g-leakage with the following definitions.

Definition 4 (Gain function). *Given a set \mathcal{X} of possible secrets and a finite, non-empty set \mathcal{W} of allowable guesses, a gain function is a function $g : \mathcal{W} \times \mathcal{X} \rightarrow [0, 1]$.*

There is no restriction on how the gain function behaves from this definition. In particular, it is possible to set the gain to be 0 even if the adversary makes a perfect guess from what he has observed. Therefore, in our work, we need to define a class of well-behaved gain functions in order to talk about semantic meaning of the leakage. (**Note: not done yet.**)

Definition 5 (Prior g-vulnerability). *Given gain function g and prior π , the prior g-vulnerability is*

$$V_g(\pi) = \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \pi[x] g(w, x). \quad (5)$$

Definition 6 (Posterior g-vulnerability). *Given gain function g , prior π , and channel C , the posterior g-vulnerability is*

$$V_g(\pi, \mathcal{C}) = \sum_{y \in \mathcal{Y}} \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \pi[x] \mathcal{C}[x, y] g(w, x). \quad (6)$$

Definition 7 (g-leakage and g-capacity). *G-leakage $\mathcal{L}_g(\pi, \mathcal{C})$ and g-capacity $\mathcal{ML}_g(\mathcal{C})$ are defined as:*

$$\mathcal{L}_g(\pi, \mathcal{C}) = -\log V_g(\pi) + \log V_g(\pi, \mathcal{C}) = \log \frac{V_g(\pi, \mathcal{C})}{V_g(\pi)}$$

$$\mathcal{ML}_g(\mathcal{C}) = \sup_{\pi} \mathcal{L}_g(\pi, \mathcal{C}).$$

For application to searchable databases, \mathcal{X} in the definition is the database and other secrets the adversary tries to guess, and \mathcal{Y} is the encrypted database he observes. The adversary may not only be interested in recover some information about the unencrypted database but also some relation between the unencrypted database and its encryption. For instance, the adversary may want to know what is the encryption of each keyword, i.e. keyword guessing attack. Hence, we extend the gain function as:

Definition 8 (Extended gain function). *Given a set \mathcal{X} of possible secrets, a set \mathcal{Y} derived from \mathcal{X} and a finite, non-empty set \mathcal{W} of allowable guesses, a gain function is a function $g : \mathcal{W} \times \mathcal{X} \times \mathcal{Y} \rightarrow [0, 1]$.*

In our application, the prior vulnerability has no operational meaning so we decide to look at posterior vulnerability only. The posterior vulnerability with the new gain function is defined as follows.

Definition 9 (Extended posterior g-vulnerability). *Given gain function g , prior π , and channel C , the extended posterior g-vulnerability is*

$$EV_g(\pi, C) = \sum_{y \in \mathcal{Y}} \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \pi[x] C[x, y] g(w, x, y). \quad (7)$$

4 Syntax of Searchable Encrypted Database

In this section, we define syntax of searchable encrypted database and then refine it for keyword search.

4.1 Database and Encrypted Database

We define database db in the most general setting to be $db \in \{0, 1\}^*$. The database supports queries via function $\text{Query} : DB \times Q \rightarrow DB \times R$, where Q is a set of queries on the database and R is some response.

Similarly, the encrypted database edb is $edb \in \{0, 1\}^*$. The encrypted database supports encrypted queries via function $\text{EQuery} : EDB \times EQ \rightarrow EDB \times ER$.

4.2 Functions between Database and Encrypted Database

A mechanism for searchable encrypted database supports the following operations:

- **Key generation:** $KGen : 1^n \rightarrow \mathcal{K}$. The key generation algorithm takes in the security parameter and outputs keys used for the mechanism.
- **Initialisation:** $\text{Init} : 1^n \times \mathcal{K} \rightarrow CST$. The initialisation function takes in the security parameter and the key, and outputs the initial state of the client.
- **Encryption:** $\text{Enc} : 1^n \times \mathcal{K} \times DB \times CST \rightarrow EDB \times CST$. The encryption scheme takes in security parameter, a key, a database and the state of the client, and produce an encrypted database and a state.
- **Decryption:** $\text{Dec} : 1^n \times \mathcal{K} \times EDB \times CST \rightarrow DB$. The decryption scheme takes in security parameter, a key, an encrypted database and a state, and produce a database.
- **Query:** $\text{Query} : DB \times Q \rightarrow DB \times R$. A query function takes in a database and a query and outputs a response.
- **Query encryption:** $\text{EncQ} : \mathcal{K} \times Q \times CST \rightarrow EQ \times CST$. The query encryption function takes in a key and a query, and produces an encrypted query and a state.

- **Encrypted queries:** $E\text{Query} : EDB \times EQ \rightarrow EDB \times ER$. An encrypted query takes an encrypted database and an encrypted query, and returns an encrypted database and an encrypted response.
- **Response decryption:** $\text{DecR} : \mathcal{K} \times ER \times CST \rightarrow R \times CST$. The response decryption function takes a key, an encrypted response, and a state, and returns a response and a state.

When the security parameter and the key are obvious, we may omit them from the notation. We may abuse $\text{Enc}(\cdot)$ and $\text{Dec}(\cdot)$ as encryption and decryption functions on keywords, for example, $ew = \text{Enc}(w)$.

4.3 Correctness Requirements

Given key k , database db_0 , and queries q_0, \dots, q_l , and the following procedure:

1. $cst \leftarrow \text{Init}(1^n, k)$,
2. $(edb_0, cst_0) \leftarrow \text{Enc}(1^n, k, db_0, cst)$,
3. For $i = 0, \dots, l$,
 $(eq_i, cst'_i) \leftarrow \text{EncQ}(k, q_i, cst_i); (edb_{i+1}, er_i) \leftarrow E\text{Query}(edb_i, eq_i); (r_i, cst_{i+1}) \leftarrow \text{DecR}(k, er_i, cst'_i)$.

And a procedure:

1. For $i = 0, \dots, l$,
 $(db_{i+1}, r'_i) \leftarrow \text{Query}(db_i, q_i)$.

We require that for all security parameter 1^n , key k , database db_0 , and sequence of queries $q_0, \dots, q_l \in Q^*$ the following conditions hold:

1. **Correctness of encryption and execution of queries on databases:** $db_i = \text{Dec}(edb_i, cst_i)$.
2. **Correctness of query responses:** $r_i = r'_i$.

4.4 Refinement to Keyword Search

A database db for keyword search consists of a set of keywords \mathcal{W} and a set of files F , so $db = (\mathcal{W}, F)$. Each file $f \in F$ contains some keywords specified by the user. For simplicity of notation, we denote keywords associated to a file $W = \mathcal{W}(f)$. Since we are talking about encryptions for keyword search, searching of keywords must be one of the supported query types. In terms of notation, we may write $\mathcal{W}(db)$ to mean the set of keywords associated to the database db , i.e. $\mathcal{W}(db) = \Pi_1(db)$.

An encrypted database edb for keyword search consists of a set of encrypted keywords \mathcal{EW} , a set of encrypted files EF and an encrypted index \mathcal{EI} for encrypted queries, so $edb = (\mathcal{EW}, EF, \mathcal{EI})$. Each encrypted file $ef \in EF$ contains some encrypted keywords which are encryptions of corresponds keywords for the plain files. Similar to unencrypted databases, we denote encrypted keywords associated to an encrypted file by $EW = \mathcal{EW}(ef)$. Encrypted keyword search must be one of the supported encrypted query types. In terms of notation, we may write $\mathcal{EW}(edb)$ to mean the set of keywords associated to the encrypted database edb , i.e. $\mathcal{EW}(edb) = \Pi_1(edb)$.

5 Security Notions

In this section, we discuss why previous notions are not sufficient in defining security of searchable encryption schemes. We then motivates with a security notion for keyword guessing attack on searchable encrypted databases. We modify the experiment by considering a leakage function instead of the real encryption function. After that, we generalise the experiment to allow for arbitrary adversarial goals.

5.1 Previous Notions

Several security notions have been proposed in the literature. The first notion is based on an indistinguishability game which tries to capture the idea that the encrypted index does not leak any information about the underlying files. The notion is known as indistinguishability under chosen keyword attack (IND-CKA) is proposed by Goh in [2]. A slightly stronger notion known as IND2-CKA has been proposed later in the same paper. Chang and Mitzenmacher [3] proposed a simulation-based security definition which requires the trapdoors to be secure, in addition to the security requirement in IND2-CKA. However, both security notions are shown to be insufficient by Curtmola et al. in [4]. In the same paper, the authors improved on the notions and proposed non-adaptive and adaptive indistinguishability notions and semantic notions. The idea is that a leakage is associated to the scheme and no adversary should learn more than what the leakage leaks. They proved that in non-adaptive and adaptive settings respectively, their indistinguishability notions and semantic notions are equivalent. A slightly weaker notion, known as IND-CQA2 is proposed by Chase et. al.[5]. The difference between IND-CKA2 and IND-CQA2 is that the later adversary is only required to output a bit. The notion is further refined to allow for arbitrary leakages, and the leakages are used as random oracles [6, 7, 8]. Independent of this, Kurosawa et. al.[9] defined a notion that is stronger than IND-CQA2 by requiring a simulator to work for all adversaries.

However, security with leakage does not imply security in practice. One needs to understand what those leakages really reveals about the underlying database. In some cases, that is obvious. For example, if the leakage function is everything of the database then we know the encryption scheme is not secure, and if the leakage function reveals nothing then the scheme is secure. But with anything in between, we are not able to draw conclusion immediately. In fact, many types of leakages can be abused to recover information about the underlying database. If a scheme leaks repetition of keywords then it may be vulnerable to frequency-based statistical attack [10]. If a scheme leaks co-occurrence of keywords then it can be exploited by IKK attack [11] and count attack [12]. Notably, co-occurrence is part of search pattern and access pattern and those are leaked by most of the schemes. More complicated leakage pattern involving bloom filter can also be misused [13]. Therefore, in order to prove that a scheme is secure for practical use, we have to show that the leakage does not reveal information of the underlying database we try to hide.

The first step towards this goal is to consider a snapshot adversary who only has access to the encrypted database in the static manner and tries to recover some information. To make it more concrete, we consider an adversary who tries to recover some encrypted keywords. We call the security notion security against keyword guessing attack.

5.2 Security Notion for (Passive) Keyword Guessing Attack

There is a multitude of ways to evaluate how well the adversary has recovered information through keyword guessing attack. We will present with one of the simplest here. Namely, the adversary wins if he recovers a correct pair of keyword and its encryption (with one guess).

For simplicity, we consider snapshot attack here so the adversary is given the leakage of the scheme and there is no interaction between the encrypted database and the adversary. We may abuse $\text{Dec}(\cdot)$ to be the function that decrypts an encrypted document or a set of encrypted keywords associated to an encrypted

document, and outputs the set of real keywords.

Definition 10 (Security against Keyword Guessing Attack). *We define a keyword guessing attack adversary to be $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$, where \mathcal{A}_0 is a probabilistic polynomial time (PPT) adversary which takes in the security parameter and generates a database db and a state $st_{\mathcal{A}}$, and \mathcal{A}_1 is a PPT adversary which takes in the security parameter, encrypted database edb and state $st_{\mathcal{A}}$, and returns a pair of keyword and encrypted keyword. The experiment can be described as follows:*

$$\begin{aligned} & \text{Exp}_{\mathcal{A}}^{\text{Keyword-Guessing}}(n) \\ 1: & \quad k \leftarrow \text{KGen}(1^n) \\ 2: & \quad (db, st_{\mathcal{A}}) \leftarrow \mathcal{A}_0(1^n) \\ 3: & \quad edb \leftarrow \text{Enc}(1^n, k, db) \\ 4: & \quad (m, c) \leftarrow \mathcal{A}_1(1^n, edb, st_{\mathcal{A}}) \\ 5: & \quad \text{if } (m \in \mathcal{W}(db)) \wedge (c \in \mathcal{EW}(edb)) \wedge (\text{Dec}(c) = m) \\ 6: & \quad \text{return } 1 - f(m, db) \\ 7: & \quad \text{else return } -f(m, db) \end{aligned}$$

where $f(m, db)$ is a function that returns the frequency of keyword m in database db .

Advantage $Adv_{\mathcal{A}}^{\text{Keyword-Guessing}}(n)$ of an inference attack adversary \mathcal{A} is defined to be

$$Adv_{\mathcal{A}}^{\text{Keyword-Guessing}}(n) = \mathbb{E} \left[\text{Exp}_{\mathcal{A}}^{\text{Keyword-Guessing}}(n) \right]. \quad (8)$$

We say that a scheme is secure against keyword guessing attack if $Adv_{\mathcal{A}}^{\text{Keyword-Guessing}}(n)$ is less than some negligible function in n .

Intuitively, a scheme is secure against keyword guessing attack if there is no way he can guess any plaintext-ciphertext pair with probability greater than the frequency of the plaintexts. There are other possible definitions, for instance, one can look at the probability the adversary guessing all pairs of plaintexts and ciphertexts correctly.

This security notion is hard to work with because the database the adversary can generate is literally anything and the encryption scheme can generate any ciphertext. But in our game, we really do not care if keywords are ‘1’ and ‘2’ or ‘one’ and ‘two’, nor do we care how the ciphertexts look like. To simplify the problem, we want to work with idealised databases and encryption schemes which allows us to ignore information that are not relevant to our security game. Due to necessity of efficient searchability, encryption schemes for searchable encryption cannot achieve IND-CPA security, and the ‘insecure part’ of encryption function and query functions are understood in terms of leakage functions. Informally speaking, leakage function associated to a scheme captures what is leaked about the unencrypted database by looking at its encryption. For example, number of documents is part of leakage for any scheme that does not pad fake documents because after enough queries, all documents will be returned at least once, unless part of the database is never going to be returned by any query, but that defeats the purpose of putting them there in the first place.

In the next experiment, the adversary is only given the leakage associated to the mechanism applied to the generated database and queries challenged by the adversary. We argue that by choosing the leakage carefully, the advantage for the experiment will be the same as the previous one, up to an additive negligible term. We begin by defining leakage of searchable encryption.

Definition 11 (Leakage of Encryption Schemes for Searching). *Let Enc be the encryption algorithm for some searchable encryption scheme. We say $\mathcal{L}_{\mathcal{M}} : \{0, 1\}^* \times \mathcal{K} \times db \times Q^* \rightarrow \{0, 1\}^*$ is a sufficient leakage function for the mechanism if for all adversaries \mathcal{A} there exists a simulator \mathcal{S} such that for any PPT distinguishers \mathcal{D} , the following two games are indistinguishable.*

$Real_{\mathcal{A}}(n)$	$\text{Sim}_{\mathcal{A}, \mathcal{S}}(n)$
1 : $k \leftarrow_{\$} \text{KGen}(1^n)$	1 : $k \leftarrow_{\$} \text{KGen}(1^n)$
2 : $(db, \vec{q}, \mathbf{st}_{\mathcal{A}}) \leftarrow \mathcal{A}(1^n)$	2 : $(db, \vec{q}, \mathbf{st}_{\mathcal{A}}) \leftarrow \mathcal{A}(1^n)$
3 : $(edb, \vec{e}\vec{q}) \leftarrow \text{Enc}(1^n, k, db, \vec{q}, \mathbf{st}_{\mathcal{A}})$	3 : $l \leftarrow \mathcal{L}_{\mathcal{M}}(1^n, k, db, \vec{q})$
4 : return $(edb, \vec{e}\vec{q}, \mathbf{st}_{\mathcal{A}})$	4 : $(edb, \vec{e}\vec{q}) \leftarrow \mathcal{S}(1^n, l, \mathbf{st}_{\mathcal{A}})$
	5 : return $(edb, \vec{e}\vec{q}, \mathbf{st}_{\mathcal{A}})$

$\mathcal{L}_{\mathcal{M}}$ is said to be minimal if there exists a simulator \mathcal{S} such that for all adversaries \mathcal{A} , no PPT distinguishers \mathcal{D} can distinguish the two games above.

Now we are ready to define security against idealised keyword guessing attack.

Definition 12 (Security against Idealised (Passive) Keyword Guessing Attack). We define an idealised keyword guessing attack adversary to be $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$, where \mathcal{A}_0 is a PPT adversary which takes in the security parameter and generates a database db and a state $\mathbf{st}_{\mathcal{A}}$, and \mathcal{A}_1 is any adversary (not necessarily polynomial time bounded) which takes in the security parameter, leakage l generated by a sufficient leakage function $\mathcal{L}_{\mathcal{M}}$ and a state $\mathbf{st}_{\mathcal{A}}$, and returns a pair of keyword and encrypted keyword. The experiment can be described as follows:

$Exp_{\mathcal{A}}^{\text{Keyword-Guessing-Ideal}}(n)$
1 : $k \leftarrow_{\$} \text{KGen}(1^n)$
2 : $(db, \mathbf{st}_{\mathcal{A}}) \leftarrow \mathcal{A}_0(1^n)$
3 : $l \leftarrow \mathcal{L}_{\mathcal{M}}(1^n, k, db)$
4 : $(m, c) \leftarrow \mathcal{A}_1(1^n, l, \mathbf{st}_{\mathcal{A}})$
5 : if $(m \in \mathcal{W}(db)) \wedge (c \in \mathcal{E}\mathcal{W}(edb)) \wedge (\text{Dec}(c) = m)$
6 : return $1 - f(m, db)$
7 : else return $-f(m, db)$

where $f(m, db)$ is a function that returns the frequency of keyword m in database db .

Advantage $Adv_{\mathcal{A}}^{\text{Keyword-Guessing-Ideal}}(n)$ of an idealised inference attack adversary \mathcal{A} is defined to be

$$Adv_{\mathcal{A}}^{\text{Keyword-Guessing-Ideal}}(n) = \mathbb{E} \left[Exp_{\mathcal{A}}^{\text{Keyword-Guessing-Ideal}}(n) \right]. \quad (9)$$

We say that a scheme is secure against idealised keyword guessing attack if $Adv_{\mathcal{A}}^{\text{Keyword-Guessing-Ideal}}(n)$ is less than some negligible function in n .

Notice that in the definition, \mathcal{A}_1 does not need to be PT bounded. This is because we are in the idealised world where the cryptographic part has already been dealt with in the leakage. We will see later that with a well-defined leakage function $\mathcal{L}_{\mathcal{M}}$, advantage of the idealised experiment can be used to upper bound that of the real experiment.

5.3 Moving from Fixed Target Function to Generic Gain Function

In the two games for keyword guessing attack we defined above, the content returned by the games are fixed so proving security with respect to the games only says the scheme is secure against this specific adversary who tries to recover this specific information. This is very inflexible as a security notion for application in searchable encryption. In addition, wrong guesses in these games are completely punished in this setting in the sense that they will never show up in the advantage term, even though wrong guesses can still reveal

some information. Consider the case where the adversary always tries to guess decryption of some ciphertext c and he always fails because his guess is some m such that $\text{Dec}(c) \neq m$. But ultimately, the goal of the adversary can be to recover some keywords in some documents. If m always appears together with $\text{Dec}(c)$ in all documents, then in fact, the adversary recovers a significant amount of information about the underlying database, even though the guess itself is wrong. So we need to reward those guesses that are not completely correct but still reveals information about the database appropriately.

To do so, we use idea of gain function from g-leakage paper [1]. To put it in simple words, instead returning something from the experiment and analyse it in the advantage term, we compute the gain at the end of the experiment, and the advantage is defined to be the expectation of the result of the experiment for the best adversary. As a starting point of understanding leakages, we define the games with gain functions for non-adaptive adversaries.

Definition 13 (Non-adaptive Real Game with Gain Function). *We define a real adversary with gain function g to be $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$, where \mathcal{A}_0 is a PPT adversary which takes in the security parameter and generates a database db , a set of queries \vec{q} and a state $\mathbf{st}_{\mathcal{A}}$, and \mathcal{A}_1 is a PPT adversary which takes in the security parameter, encrypted database edb , encrypted queries $\vec{e}\vec{q}$ and state $\mathbf{st}_{\mathcal{A}}$, and returns some guess w in the set of allowed guesses \mathcal{W} . The experiment can be described as follows:*

Real _{\mathcal{A}} ^{g} (n)

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1:    $k \leftarrow_{\$} \mathsf{KGen}(1^n)$ 
2:    $(db, \vec{q}, \mathbf{st}_{\mathcal{A}}) \leftarrow \mathcal{A}_0(1^n)$ 
3:    $(edb, \vec{e}\vec{q}) \leftarrow \mathsf{Enc}(1^n, k, db, \vec{q})$ 
4:    $w \leftarrow \mathcal{A}_1(1^n, edb, \vec{q}, \mathbf{st}_{\mathcal{A}})$ 
5:   return  $g(w, (k, db, \vec{q}, \mathbf{st}_{\mathcal{A}}), (edb, \vec{e}\vec{q}, \mathbf{st}_{\mathcal{A}}))$ 

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Advantage $Adv_{\mathcal{A},g}^{Real}(n)$ is defined to be:

$$Adv_{\mathcal{A},g}^{Real}(n) = \mathbb{E}[Real_{\mathcal{A}}^g(n)]. \quad (10)$$

Definition 14 (Non-adaptive Ideal Game with Gain Function). *We define an ideal adversary with gain function g to be $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$, where \mathcal{A}_0 is a PPT adversary which takes in the security parameter and generates a database db , and a set of queries \vec{q} , and \mathcal{A}_1 is an adversary which takes in the security parameter, and a sufficient leakage function $\mathcal{L}_{\mathcal{M}}$, and returns some guess w in the set of allowed guesses \mathcal{W} . The experiment can be described as follows:*

Ideal _{\mathcal{A}} ^{$\mathcal{L}_{\mathcal{M}},g$} (n)

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1:    $k \leftarrow_{\$} \mathsf{KGen}(1^n)$ 
2:    $(db, \vec{q}, \mathbf{st}_{\mathcal{A}}) \leftarrow \mathcal{A}_0(1^n)$ 
3:    $l \leftarrow \mathcal{L}_{\mathcal{M}}(1^n, k, db, \vec{q})$ 
4:    $w \leftarrow \mathcal{A}_1(1^n, l, \mathbf{st}_{\mathcal{A}})$ 
5:   return  $g(w, (k, db, \vec{q}, \mathbf{st}_{\mathcal{A}}), (l, \mathbf{st}_{\mathcal{A}}))$ 

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Advantage $Adv_{\mathcal{A},\mathcal{L}_{\mathcal{M}},g}^{Ideal}(n)$ is defined to be:

$$Adv_{\mathcal{A},\mathcal{L}_{\mathcal{M}},g}^{Ideal}(n) = \mathbb{E}[Ideal_{\mathcal{A}}^{\mathcal{L}_{\mathcal{M}},g}(n)]. \quad (11)$$

Note that we are not putting any condition to quantify when a scheme is secure under these notions, because the advantage really depends on the choice of g . But with well-defined g , the advantage will have an operational meaning and we can understand the level of security offered by our scheme from there.

5.4 Games with Gain Function are equivalent to some Extended Posterior g-vulnerabilities

In this subsection, we prove that in fact, we can look at the advantage from the games or some corresponding extended posterior g-vulnerability exchangeably. To do so, we need a slight tweak to the games above.

Let π be the prior on the joint distribution of keys, databases, queries and states, with the restriction that probability of observing any key is independent from the rest. Then the games above is equivalent to the following.

$\text{Real}_{\mathcal{A}}^{\pi,g}(n)$	$\text{Ideal}_{\mathcal{A}}^{\pi,\mathcal{L}_{\mathcal{M}},g}(n)$
1 : $(k, db, \vec{q}, \mathbf{st}_{\mathcal{A}}) \leftarrow_{\$} \pi$	1 : $(k, db, \vec{q}, \mathbf{st}_{\mathcal{A}}) \leftarrow_{\$} \pi$
2 : $(edb, \vec{e}\vec{q}) \leftarrow \mathsf{Enc}(1^n, k, db, \vec{q})$	2 : $l \leftarrow \mathcal{L}_{\mathcal{M}}(1^n, k, db, \vec{q})$
3 : $w \leftarrow \mathcal{A}_1(1^n, edb, \vec{e}\vec{q}, \mathbf{st}_{\mathcal{A}})$	3 : $w \leftarrow \mathcal{A}_1(1^n, l, \mathbf{st}_{\mathcal{A}})$
4 : $\mathbf{return} g(w, (k, db, \vec{q}, \mathbf{st}_{\mathcal{A}}), (edb, \vec{e}\vec{q}, \mathbf{st}_{\mathcal{A}}))$	4 : $\mathbf{return} g(w, (k, db, \vec{q}, \mathbf{st}_{\mathcal{A}}), (l, \mathbf{st}_{\mathcal{A}}))$

Theorem 15 (Ideal game is equivalent to an extended posterior g-vulnerability). *Let π be the prior distribution of keys, databases, set of queries and state. Let $\mathcal{X} = \mathcal{K} \times DB \times Q^* \times \{0, 1\}^*$. Let $\mathcal{L}_{\mathcal{M}}$ be a sufficient leakage function. Let $\mathcal{Y} = \{\mathcal{L}_{\mathcal{M}}(1^n, k, db, \vec{q}) \mid k \in \mathcal{K}, db \in DB, \vec{q} \in Q^*\} \times \{0, 1\}^*$. Let $\mathcal{C}_{\mathcal{L}_{\mathcal{M}}}$ be the channel matrix from \mathcal{X} to \mathcal{Y} . Let $g : \mathcal{W} \times \mathcal{X} \times \mathcal{Y} \rightarrow [0, 1]$. Then*

$$\sup_{\pi \in \text{Dist}} \sup_{\mathcal{A}} \text{Adv}_{\mathcal{A}, \pi, \mathcal{L}_{\mathcal{M}}, g}^{\text{Ideal}}(n) = EV_g(\pi, \mathcal{C}_{\mathcal{L}_{\mathcal{M}}}). \quad (12)$$

where Dist is the set of distributions on the database.

Proof. We fist assume Dist is a fixed distribution, so $|\text{Dist}| = 1$. That corresponds to some distribution on $(k, db, \vec{q}, \mathbf{st}_{\mathcal{A}})$. It suffices to look at the expression $\sup_{\mathcal{A}} \text{Adv}_{\mathcal{A}, \pi, \mathcal{L}_{\mathcal{M}}, g}^{\text{Real}}(n)$ with $\mathcal{L}_{\mathcal{M}}$ being a sufficient leakage. Suppose $\mathcal{L}_{\mathcal{M}}$ is indeed a sufficient leakage, then

$$\sup_{\mathcal{A}} \text{Adv}_{\mathcal{A}, \pi, \mathcal{L}_{\mathcal{M}}, g}^{\text{Ideal}}(n) \quad (13)$$

$$= \sup_{\mathcal{A}} \mathbb{E}_{\mathcal{A}} \left[\text{Ideal}_{\mathcal{A}}^{\mathcal{L}_{\mathcal{M}}, g}(n) \right] \quad (14)$$

$$= \mathbb{E}_{\mathcal{L}_{\mathcal{M}}} \left[\max_{w \in \mathcal{W}} \mathbb{E}_{\pi} [g(w, (db, k), l \mid l)] \right] \quad (15)$$

$$= \mathbb{E}_{\mathcal{Y}} \left[\max_{w \in \mathcal{W}} \mathbb{E}_{\pi} [g(w, \mathcal{X}, y) \mid y] \right] \quad (16)$$

$$= \sum_{y \in \mathcal{Y}} \Pr[\mathcal{Y} = y] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[\mathcal{X} = x \mid \mathcal{Y} = y] g(w, x, y) \quad (17)$$

$$= \sum_{y \in \mathcal{Y}} \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[\mathcal{X} = x, \mathcal{Y} = y] g(w, x, y) \quad (18)$$

$$= \sum_{y \in \mathcal{Y}} \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \pi[x] \mathcal{C}_{\mathcal{L}_{\mathcal{M}}}[x, y] g(w, x, y) \quad (19)$$

$$= EV_g(\pi, \mathcal{C}_{\mathcal{L}_{\mathcal{M}}}). \quad (20)$$

In the proof, equation 14 is just writing out the adversary explicitly. Equation 15 uses the fact that for any given leakage, there is a deterministic guess that maximises the conditional expectation and the best adversary overall does this for any leakage l he sees. The next few lines of the proof are just re-writing the expectation into algebraic form and re-arranging the terms.

To extend the proof to arbitrary set of distributions Dist , \square

We can use the theorem above to draw some immediate conclusion for the real game.

Proposition 16 (Real game is equivalent to an extended posterior g-vulnerability). *Let π be the prior distribution of keys, databases, set of queries and state. Let $\mathcal{X} = \mathcal{K} \times DB \times Q^* \times \{0,1\}^*$. Let $\mathcal{Y} = EDB \times EQ \times \{0,1\}^*$. Let \mathcal{C}_M be the channel matrix from \mathcal{X} to \mathcal{Y} . Let $g : \mathcal{W} \times \mathcal{X} \times \mathcal{Y} \rightarrow [0,1]$. Then*

$$\sup_{\pi \in Dist} \sup_{\mathcal{A}} Adv_{\mathcal{A}, \pi, g}^{Real}(n) = EV_g(\pi, \mathcal{C}_M). \quad (21)$$

where $Dist$ is the set of valid prior for the database.

Proof. Let \mathcal{L}_M in the ideal experiment be the encryption function $\text{Enc}(\cdot)$ with a simulator that generates $\text{st}_{\mathcal{A}}$, then the ideal game is exactly the real game. The result follows immediately. \square

5.5 Bound the Advantage in the Real World using Advantage in the Idealised World

It would not make sense to look at the advantage in the idealised world if it has no connection to the advantage in the real world. In this subsection, we prove that by picking appropriate gain function in the idealised world, we can use the advantage in the idealised world to bound that in the real world.

Theorem 17. *Let π be the prior distribution of keys, databases, set of queries and state. Let $\mathcal{X} = \mathcal{K} \times DB \times Q^* \times \{0,1\}^*$. Let $\mathcal{Y}_{Real} = EDB \times EQ \times \{0,1\}^*$. Let \mathcal{C}_M be the channel matrix from \mathcal{X} to \mathcal{Y}_{Real} . Let $g_{Real} : \mathcal{W} \times \mathcal{X} \times \mathcal{Y}_{Real} \rightarrow [0,1]$.*

Let \mathcal{L}_M be a valid leakage function and Sim an simulator associated to \mathcal{L}_M . Let $\mathcal{Y}_{Ideal} = \{\mathcal{L}_M(1^n, k, db, \vec{q}) \mid k \in \mathcal{K}, db \in DB, \vec{q} \in Q^\}$. Let $\mathcal{C}_{\mathcal{L}_M}$ be the channel matrix from \mathcal{X} to \mathcal{Y}_{Ideal} . Let $g_{Ideal} : \mathcal{W} \times \mathcal{X} \times \mathcal{Y}_{Ideal} \rightarrow [0,1]$ such that $g_{Ideal}(w, x, y) = g_{Real}(w, x, \text{Sim}(y))$. Then*

$$EV_{g_{Real}}(\pi, \mathcal{C}_M) \leq EV_{g_{Ideal}}(\pi, \mathcal{C}_{\mathcal{L}_M}) \quad (22)$$

with overwhelming probability.

Proof. The proof takes two steps. We will first relate the extended g-vulnerability in the idealised world to some other extended g-vulnerability. We then argue that the game related to the extended g-vulnerability we found is indistinguishable from that of the real world. Therefore, concluding the desired inequality.

We begin by re-writing extended g-vulnerability in the idealised world in the following way:

$$EV_{g_{Ideal}}(\pi, \mathcal{C}_M) \quad (23)$$

$$= \sum_{y \in \mathcal{Y}_{Ideal}} \Pr[y] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x \mid y] g_{Ideal}(w, x, y) \quad (24)$$

$$= \sum_{y \in \mathcal{Y}_{Ideal}} \Pr[y] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x \mid y] g_{Real}(w, x, \text{Sim}(y)) \quad (25)$$

$$= \sum_{\text{Sim}(y) \in \text{Sim}(\mathcal{Y}_{Ideal})} \sum_{y \in \mathcal{Y}_{Ideal}} \Pr[\text{Sim}(y)] \Pr[y \mid \text{Sim}(y)] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x \mid y] g_{Real}(w, x, \text{Sim}(y)) \quad (26)$$

$$\geq \sum_{\text{Sim}(y) \in \text{Sim}(\mathcal{Y}_{Ideal})} \Pr[\text{Sim}(y)] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}_{Ideal}} \Pr[x \mid y] \Pr[y \mid \text{Sim}(y)] g_{Real}(w, x, \text{Sim}(y)) \quad (27)$$

$$= \sum_{\text{Sim}(y) \in \text{Sim}(\mathcal{Y}_{Ideal})} \Pr[\text{Sim}(y)] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}_{Ideal}} \Pr[x, y \mid \text{Sim}(y)] g_{Real}(w, x, \text{Sim}(y)) \quad (28)$$

$$= \sum_{\text{Sim}(y) \in \text{Sim}(\mathcal{Y}_{Ideal})} \Pr[\text{Sim}(y)] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x \mid \text{Sim}(y)] g_{Real}(w, x, \text{Sim}(y)). \quad (29)$$

To go from equation 26 to equation 27, we used Jensen's inequality. To arrive at equation 28, we use the fact that $\mathcal{X} \rightarrow \mathcal{Y}_{\text{Ideal}} \rightarrow \text{Sim}(\mathcal{Y}_{\text{Ideal}})$ is a Markov chain. Finally, we get equation 29 due to law of total probability.

We observe that equation 29 is equivalent to the following game:

$\text{Simulated}_{\mathcal{A}}^{\text{Sim} \circ \mathcal{L}_{\mathcal{M}}, g_{\text{Real}}}(n)$
1 : $k \leftarrow \text{KGen}(1^n)$
2 : $(db, \vec{q}) \leftarrow \mathcal{A}_0(1^n)$
3 : $(edb, \vec{e}\vec{q}, \mathbf{st}_{\mathcal{A}}) \leftarrow \text{Sim}(\mathcal{L}_{\mathcal{M}}(1^n, k, db))$
4 : $w \leftarrow \mathcal{A}_1(1^n, edb, \vec{e}\vec{q}, \mathbf{st}_{\mathcal{A}})$
5 : $\text{return } g_{\text{Real}}(w, (k, db, \vec{q}), (edb, \vec{e}\vec{q}))$

But this game is indistinguishable from the real game as Sim is a simulator that makes $\text{Sim} \circ \mathcal{L}_{\mathcal{M}}(\cdot)$ indistinguishable from $\text{Enc}(\cdot)$ by definition. Therefore, we conclude that

$$EV_{g_{\text{Real}}}(\pi, \mathcal{C}_{\mathcal{L}_{\mathcal{M}}}) = \sum_{\text{Sim}(y) \in \text{Sim}(\mathcal{Y}_{\text{Ideal}})} \Pr[\text{Sim}(y)] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x \mid \text{Sim}(y)] g_{\text{Real}}(w, x, \text{Sim}(y)) \quad (30)$$

with overwhelming probability. Therefore, we conclude

$$EV_{g_{\text{Real}}}(\pi, \mathcal{C}_{\mathcal{L}_{\mathcal{M}}}) \leq EV_{g_{\text{Ideal}}}(\pi, \mathcal{C}_{\mathcal{M}}) \quad (31)$$

as desired. \square

5.6 Bounding Extended g-vulnerability using Extended g-vulnerability on Sub-channels

In this subsection, we highlight how to use extended g-vulnerability on sub-channels to bound extended g-vulnerability of some channel. It is well-known from information theory that if $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$ is a Markov chain, then $I(\mathcal{X}; \mathcal{Y}) \geq I(\mathcal{X}; \mathcal{Z})$, where $I(\cdot)$ is the mutual information. Similar result has been shown for g-leakage in [1, 14, 15] for gain functions of the form $g : \mathcal{W} \times \mathcal{X} \rightarrow [0, 1]$. However, it also holds for mutual information that $I(\mathcal{Y}; \mathcal{Z}) \geq I(\mathcal{X}; \mathcal{Z})$ but there is no corresponding result for g-vulnerability in the literature. We show that it is possible to obtain a similar result with our definition of extended g-vulnerability. We then extend our result to a channel consisting of more than two sub-channels.

Theorem 18 (Bounding extended g-vulnerability with first sub-channel). *Let $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$ be a Markov chain. Let π be the prior distribution on \mathcal{X} . Let $\mathcal{C}_{\mathcal{X}, \mathcal{Y}}$ be the channel matrix between \mathcal{X} and \mathcal{Y} . Let $\mathcal{C}_{\mathcal{X}, \mathcal{Z}}$ be the channel matrix between \mathcal{X} and \mathcal{Z} . Let \mathcal{W} be the set of allowed guesses. Let $g_{\mathcal{X}, \mathcal{Y}} : \mathcal{W} \times \mathcal{X} \times \mathcal{Y} \rightarrow [0, 1]$ be a gain function on sub-channel $\mathcal{X} \rightarrow \mathcal{Y}$, and $g^* : \mathcal{W} \times \mathcal{X} \times \mathcal{Z} \rightarrow [0, 1]$ be a gain function on the full channel. If*

$$g^*(w, x, z) \leq \sum_{y \in \mathcal{Y}} \Pr[y \mid x, z] g_{\mathcal{X}, \mathcal{Y}}(w, x, y) \quad (32)$$

for all $w \in \mathcal{W}, x \in \mathcal{X}$, and $z \in \mathcal{Z}$, then

$$EV_{g^*}(\pi, \mathcal{C}_{\mathcal{X}, \mathcal{Z}}) \leq EV_{g_{\mathcal{X}, \mathcal{Y}}}(\pi, \mathcal{C}_{\mathcal{X}, \mathcal{Y}}).$$

Proof. The proof is no more than just manipulating the probabilities.

$$\begin{aligned}
& EV_{g_{\mathcal{X},\mathcal{Y}}}(\pi, \mathcal{C}_{\mathcal{X},\mathcal{Y}}) \\
&= \sum_{y \in \mathcal{Y}} \Pr[y] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x | y] g_{\mathcal{X},\mathcal{Y}}(w, x, y) \\
&= \sum_{z \in \mathcal{Z}} \sum_{y \in \mathcal{Y}} \Pr[z] \Pr[y | z] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x | y] g_{\mathcal{X},\mathcal{Y}}(w, x, y) \\
&\geq \sum_{z \in \mathcal{Z}} \Pr[z] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \Pr[x | y] \Pr[y | z] g_{\mathcal{X},\mathcal{Y}}(w, x, y) \\
&= \sum_{z \in \mathcal{Z}} \Pr[z] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \Pr[x, y | z] g_{\mathcal{X},\mathcal{Y}}(w, x, y) \\
&= \sum_{z \in \mathcal{Z}} \Pr[z] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x | z] \sum_{y \in \mathcal{Y}} \Pr[y | x, z] g_{\mathcal{X},\mathcal{Y}}(w, x, y) \\
&\geq \sum_{z \in \mathcal{Z}} \Pr[z] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x | z] g^*(w, x, z) \\
&= EV_{g^*}(\pi, \mathcal{C}_{\mathcal{X},\mathcal{Z}}).
\end{aligned}$$

□

The result above can be seen as generalization of results in [1, 14, 15]. In fact, if we consider gain functions in the form of the aforementioned papers, we obtain the same data-processing inequality as they do.

Corollary 19 (Bounding extended g-vulnerability with simple gain functions). *Let $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$ be a Markov chain. Let π be the prior distribution on \mathcal{X} . Let $\mathcal{C}_{\mathcal{X},\mathcal{Y}}$ be the channel matrix between \mathcal{X} and \mathcal{Y} . Let $\mathcal{C}_{\mathcal{X},\mathcal{Z}}$ be the channel matrix between \mathcal{X} and \mathcal{Z} . Let \mathcal{W} be the set of allowed guesses. Let $g_{\mathcal{X},\mathcal{Y}} : \mathcal{W} \times \mathcal{X} \times \mathcal{Y} \rightarrow [0, 1]$ be a gain function on sub-channel $\mathcal{X} \rightarrow \mathcal{Y}$, and $g^* : \mathcal{W} \times \mathcal{X} \times \mathcal{Z} \rightarrow [0, 1]$ be a gain function on the full channel. If there exists some function $G : \mathcal{W} \times \mathcal{X}$ such that $g^*(w, x, z) = g_{\mathcal{X},\mathcal{Y}}(w, x, y) = G(w, x)$ for any $w \in \mathcal{W}, x \in \mathcal{X}, y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, then*

$$EV_{g^*}(\pi, \mathcal{C}_{\mathcal{X},\mathcal{Z}}) \leq EV_{g_{\mathcal{X},\mathcal{Y}}}(\pi, \mathcal{C}_{\mathcal{X},\mathcal{Y}}).$$

Proof. It suffices to verify equation 32 in theorem 18. We have

$$\begin{aligned}
& \sum_{y \in \mathcal{Y}} \Pr[y | x, z] g_{\mathcal{X},\mathcal{Y}}(w, x, y) \\
&= \sum_{y \in \mathcal{Y}} \Pr[y | x, z] G(w, x) \\
&= G(w, x) \sum_{y \in \mathcal{Y}} \Pr[y | x, z] \\
&= G(w, x) \\
&= g^*(w, x, z).
\end{aligned}$$

So the desired result follows. □

A limitation of previous results is that one can only bound extended g-vulnerability of a channel by the first sub-channel. We show that with our definition, we can use extended g-vulnerability of the last sub-channel to bound that of the channel.

Theorem 20 (Bounding extended g-vulnerability with last sub-channel). *Let $\mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z}$ be a Markov chain. Let π be the prior distribution on \mathcal{X} and $\pi_{\mathcal{Y}}$ be the posterior distribution on \mathcal{Y} . Let $\mathcal{C}_{\mathcal{Y},\mathcal{Z}}$ be the channel*

matrix between \mathcal{Y} and \mathcal{Z} . Let $\mathcal{C}_{\mathcal{X}, \mathcal{Z}}$ be the channel matrix between \mathcal{X} and \mathcal{Z} . Let \mathcal{W} be the set of allowed guesses. Let $g_{\mathcal{Y}, \mathcal{Z}} : \mathcal{W} \times \mathcal{Y} \times \mathcal{Z} \rightarrow [0, 1]$ be a gain function on sub-channel $\mathcal{Y} \rightarrow \mathcal{Z}$, and $g^* : \mathcal{W} \times \mathcal{X} \times \mathcal{Z} \rightarrow [0, 1]$ be a gain function on the full channel. If

$$g^*(w, x, z) \leq \sum_{y \in \mathcal{Y}} \Pr[y | x, z] g_{\mathcal{Y}, \mathcal{Z}}(w, y, z) \quad (33)$$

for all $w \in \mathcal{W}$, $x \in \mathcal{X}$, and $z \in \mathcal{Z}$, then

$$EV_{g^*}(\pi, \mathcal{C}_{\mathcal{X}, \mathcal{Z}}) \leq EV_{g_{\mathcal{Y}, \mathcal{Z}}}(\pi_{\mathcal{Y}}, \mathcal{C}_{\mathcal{Y}, \mathcal{Z}}).$$

Proof.

$$\begin{aligned} & EV_{g_{\mathcal{Y}, \mathcal{Z}}}(\pi_{\mathcal{Y}}, \mathcal{C}_{\mathcal{Y}, \mathcal{Z}}) \\ &= \sum_{z \in \mathcal{Z}} \Pr[z] \max_{w \in \mathcal{W}} \sum_{y \in \mathcal{Y}} \Pr[y | z] g_{\mathcal{Y}, \mathcal{Z}}(w, x, y) \\ &= \sum_{z \in \mathcal{Z}} \Pr[z] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \Pr[x, y | z] g_{\mathcal{Y}, \mathcal{Z}}(w, x, y) \\ &= \sum_{z \in \mathcal{Z}} \Pr[z] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x | z] \sum_{y \in \mathcal{Y}} \Pr[y | x, z] g_{\mathcal{Y}, \mathcal{Z}}(w, x, y) \\ &\geq \sum_{z \in \mathcal{Z}} \Pr[z] \max_{w \in \mathcal{W}} \sum_{x \in \mathcal{X}} \Pr[x | z] g^*(w, x, z) \\ &= EV_{g^*}(\pi, \mathcal{C}_{\mathcal{X}, \mathcal{Z}}) \end{aligned}$$

□

So far we have only considered a channel consists of two sub-channels but in reality we can have more. Correspondingly, we want to use extended g-vulnerability of some intermediate sub-channel to bound that of the channel. The next theorem does exactly that.

Theorem 21 (Bounding extended g-vulnerability with intermediate sub-channel). *Let $\mathcal{X}_0 \rightarrow \dots \rightarrow \mathcal{X}_n$ be a Markov chain with $n \geq 3$. Let $\pi_{\mathcal{X}_i}$ be the posterior distribution on \mathcal{X}_i , for $i = 0, \dots, n$. Let \mathcal{C} be the channel matrix between \mathcal{X}_0 and \mathcal{X}_n , and $\mathcal{C}_{i,j}$ be the channel matrix between \mathcal{X}_i and \mathcal{X}_j , for $i, j \geq 0, j > i$. Let winW be the set of allowed guesses. Let $g_{i,j} : \mathcal{W} \times \mathcal{X}_i \times \mathcal{X}_j \rightarrow [0, 1]$ be the set of gain functions. If*

$$g_{0,n}(w, x_0, x_n) \leq \sum_{x_i \in \mathcal{X}_i} \sum_{x_j \in \mathcal{X}_j} \Pr[x_i, x_j | x_0, x_n] g_{i,j}(w, x_i, x_j) \quad (34)$$

for all $w \in \mathcal{W}$, $x_0 \in \mathcal{X}_0$ and $x_n \in \mathcal{X}_n$, then

$$V_{g_{0,n}}(\pi, \mathcal{C}_{0,n}) \leq EV_{g_{i,j}}(\pi_i, \mathcal{C}_{i,j}). \quad (35)$$

Proof.

$$\begin{aligned}
& EV_{g_{i,j}}(\pi_i, \mathcal{C}_{i,j}) \\
&= \sum_{x_j \in \mathcal{X}_j} \Pr[x_j] \max_{w \in \mathcal{W}} \sum_{x_i \in \mathcal{X}_i} \Pr[x_i | x_j] g_{i,j}(w, x_i, x_j) \\
&= \sum_{x_n \in \mathcal{X}_n} \sum_{x_j \in \mathcal{X}_j} \Pr[x_n] \Pr[x_j | x_n] \max_{w \in \mathcal{W}} \sum_{x_0 \in \mathcal{X}_0} \sum_{x_i \in \mathcal{X}_i} \Pr[x_0, x_i | x_j] g_{i,j}(w, x_i, x_j) \\
&\geq \sum_{x_n \in \mathcal{X}_n} \Pr[x_n] \max_{w \in \mathcal{W}} \sum_{x_0 \in \mathcal{X}_0} \sum_{x_i \in \mathcal{X}_i} \sum_{x_j \in \mathcal{X}_j} \Pr[x_0, x_i | x_j] \Pr[x_j | x_n] g_{i,j}(w, x_i, x_j) \\
&= \sum_{x_n \in \mathcal{X}_n} \sum_{x_j \in \mathcal{X}_j} \Pr[x_n] \Pr[x_j | x_n] \max_{w \in \mathcal{W}} \sum_{x_0 \in \mathcal{X}_0} \sum_{x_i \in \mathcal{X}_i} \Pr[x_0, x_i | x_j] g_{i,j}(w, x_i, x_j) \\
&\geq \sum_{x_n \in \mathcal{X}_n} \Pr[x_n] \max_{w \in \mathcal{W}} \sum_{x_0 \in \mathcal{X}_0} \sum_{x_i \in \mathcal{X}_i} \sum_{x_j \in \mathcal{X}_j} \Pr[x_0, x_i, x_j | x_n] g_{i,j}(w, x_i, x_j) \\
&= \sum_{x_n \in \mathcal{X}_n} \sum_{x_j \in \mathcal{X}_j} \Pr[x_n] \Pr[x_j | x_n] \max_{w \in \mathcal{W}} \sum_{x_0 \in \mathcal{X}_0} \sum_{x_i \in \mathcal{X}_i} \Pr[x_0, x_i | x_j] g_{i,j}(w, x_i, x_j) \\
&\geq \sum_{x_n \in \mathcal{X}_n} \Pr[x_n] \max_{w \in \mathcal{W}} \sum_{x_0 \in \mathcal{X}_0} \Pr[x_0 | x_n] \sum_{x_i \in \mathcal{X}_i} \sum_{x_j \in \mathcal{X}_j} \Pr[x_i, x_j | x_0, x_n] g_{i,j}(w, x_i, x_j) \\
&\geq \sum_{x_n \in \mathcal{X}_n} \Pr[x_n] \max_{w \in \mathcal{W}} \sum_{x_0 \in \mathcal{X}_0} \Pr[x_0 | x_n] g_{0,n}(w, x_0, x_n) \\
&= V_{g_{0,n}}(\pi, \mathcal{C}_{0,n})
\end{aligned}$$

□

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