Updated Assignment # 2 (of 2) in EM4

Updates

This is the revised Assignment #2 for ENGE800 for 2020. The assignment is similar to the original assignment given, but some questions have been deleted, and is now worth 50% of your final grade.

Key points

This assignment is part of the overall course assessment. It has 22 points and is worth 50% of your final grade and is due on 9 June 2020.

The following general points apply to all work handed in for assessment:

- As per AUT rules, your assignment *must* include a signed cover sheet.
- In addition, please remember to include your name, student ID, date, course code, and assignment # on your assignment solutions, and if you use more than one sheet, number the pages and please staple them together.
- Note: Submit your the assignment electronically on AUTonline by 24:00 (midnight) on the 9revised) due date, 9 June 2020.
- You are allowed to discuss the problems with your colleagues, but you must hand in your own work.

Useful hints for all assignments are:

- Read the question carefully, and make sure you answer any specific questions. You may need to write a paragraph or two explaining your results or giving a context. Often this is more important than your numerical answer.
- You solutions must include all workings to show how you derived the answer. Simply giving the numerical solutions (as is given in the textbook in the drill exercises) is near-worthless.
- When including computer code, use an appropriate font such as `courier` that clearly distinguishes between the letter `l` and number 1 (one), the upper case letter O and the number zero, `0`, etc. Do not forget *meaningful* comments in your code.
- Don't ever print out long lists of numbers. Instead, give me a plot, it saves paper. Be concise.
- When including plots and figures, incorporate them in your submission with appropriate axis labels. *Always* label your plot axes with titles and units. Title your plot.
- Even though it is not always specifically asked for, general comments and a problem critique are always a good thing, in fact they are expected in a professional report. Some of these questions are deliberately vague where you are required to make some intelligent and professional decisions regarding issues like colour maps, axis limits and so forth. Make those decisions, and justify them.

A CO_2 model for Climate Simulation

What effect does the burning of fossil fuels have on the carbon dioxide in the earth's atmosphere? Even though today carbon dioxide accounts for only about 350 parts per million of the atmosphere, any increase has profound implications for our climate. An informative background article is available at a Web site maintained by the Lighthouse Foundation, `www.lighthouse-foundation.org`.

A dynamic CO_2 model

A model developed by J. C. G. Walker, (*Numerical Adventures with Geochemical Cycles*, [1]), simulates the interaction of the various forms of carbon that are stored in three regimes: the atmosphere, the shallow ocean, and the deep ocean. The derivation of the dynamics, and the units of the various constants are not important for this assignment, but further details are given in the original book, [1, Chapter 5].

The five principal variables in the model are all functions of time:

p, partial pressure of carbon dioxide in the atmosphere;
σ_s, total dissolved carbon concentration in the shallow ocean;
σ_d, total dissolved carbon concentration in the deep ocean;
α_s, alkalinity in the shallow ocean;
α_d, alkalinity in the deep ocean.

Three additional quantities are involved in equilibrium equations in the shallow ocean:
h_s, hydrogen carbonate (HCO_3^-, or bicarbonate) in the shallow ocean;
c_s, carbonate (CO_3^{2-}) in the shallow ocean;
p_s, partial pressure of gaseous carbon dioxide in the shallow ocean.

A diagram of the carbon cycle taken from [1, Fig 5-1, p50] is given in Fig. 1.

The rate of change of the five principal variables is given by five ordinary differential equations. The exchange between the atmosphere and the shallow ocean involves a constant characteristic transfer time d, and a source term $f(t)$,

$$\frac{dp}{dt} = \frac{p_s - p}{d} + \frac{f(t)}{\mu_1}$$

We will explore later using different scenarios for this source term.

The equations describing the exchange between the shallow and deep oceans involve v_s and

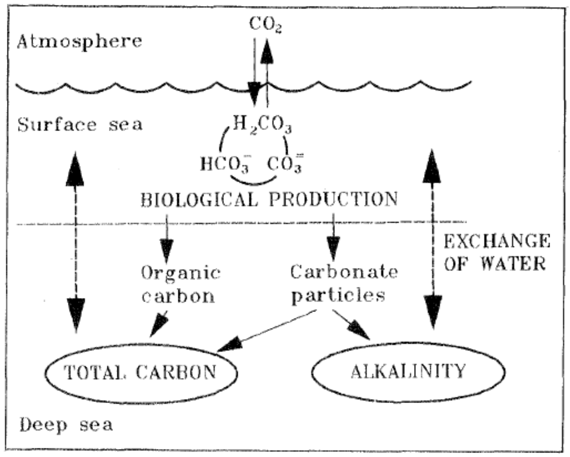

Figure 1: Settling particles of biological origin carry carbon and alkalinity into the deep sea. Carbonate equilibrium reactions in the surface sea affect the atmospheric pressure of carbon dioxide. Figure from [1, Fig 5-1].

v_d, the volumes of the two regimes:

$$\begin{aligned}
\frac{d\sigma_s}{dt} &= \frac{1}{v_s}\left((\sigma_d - \sigma_s)w - k_1 - \frac{p_s - p}{d}\mu_2\right) \\
\frac{d\sigma_d}{dt} &= \frac{1}{v_d}\left(k_1 - (\sigma_d - \sigma_s)w\right) \\
\frac{d\alpha_s}{dt} &= \frac{1}{v_s}\left((\alpha_d - \alpha_s)w - k_2\right) \\
\frac{d\alpha_d}{dt} &= \frac{1}{v_d}\left(k_2 - (\alpha_d - \alpha_s)w\right)
\end{aligned}$$

The equilibrium between carbon dioxide and the carbonates dissolved in the shallow ocean is described by three nonlinear algebraic equations:

$$\begin{aligned}
h_s &= \frac{\sigma_s - \left(\sigma_s^2 - k_3\alpha_s(2\sigma_s - \alpha_s)\right)^{1/2}}{k_3}, \\
c_s &= \frac{\alpha_s - h_s}{2}, \\
p_s &= k_4\frac{h_s^2}{c_s}.
\end{aligned}$$

The numerical values of the constants involved in the model are

$$
\begin{aligned}
d &= 8.64, \\
\mu_1 &= 4.95 \cdot 10^2, \\
\mu_2 &= 4.95 \cdot 10^{-2}, \\
v_s &= 0.12, \\
v_d &= 1.23, \\
w &= 10^{-3}, \\
k_1 &= 2.19 \cdot 10^{-4}, \\
k_2 &= 6.12 \cdot 10^{-5}, \\
k_3 &= 0.997148, \\
k_4 &= 6.79 \cdot 10^{-2}
\end{aligned}
$$

We will use a time interval that starts about a thousand years ago and extends a few thousand years into the future:

$$1000 \le t \le 5000.$$

The initial values of the 5 states at $t = 1000$ are:

$$
\begin{aligned}
p &= 1.00, \\
\sigma_s &= 2.01, \\
\sigma_d &= 2.23, \\
\alpha_s &= 2.20, \\
\alpha_d &= 2.26,
\end{aligned}
$$

and these represent preindustrial equilibrium and remain nearly constant as long as the source term $f(t)$ is zero.

The source term, $f(t)$, due to the burning of fossil fuels

The source term $f(t)$ describes the burning of fossil fuels in the modern industrial era. This is something that we as humanity have some control over. For example we could encourage the take-up of electric cars, or ban the use of coal-fired power stations. This project is going to investigate the effect of the CO_2 in the atmosphere due to various alternatives.

The following table, Table 1, describes one scenario for a source term $f(t)$ that models the release of carbon dioxide from burning fossil fuels, especially gasoline. The amounts begin to be significant after 1850, peak near the end of this century, and then decrease until the supply is exhausted.

Figure 2 shows this source term using the data from Table 1 and its effect on the atmosphere and the ocean. The three graphs in the lower half of the figure show the atmospheric, shallow ocean, and deep ocean carbon. (The two alkalinity values are not plotted at all because they

Table 1: The source term, $f(t)$, which models our burning of fossil fuels as a function of time for scenario #1.

year	rate, $f(t)$
1000	0.0
1850	0.0
1950	1.0
1980	4.0
2050	8.0
2100	10.0
2120	10.5
2150	10.0
2225	3.5
2300	2.0
2500	0.0
5000	0.0

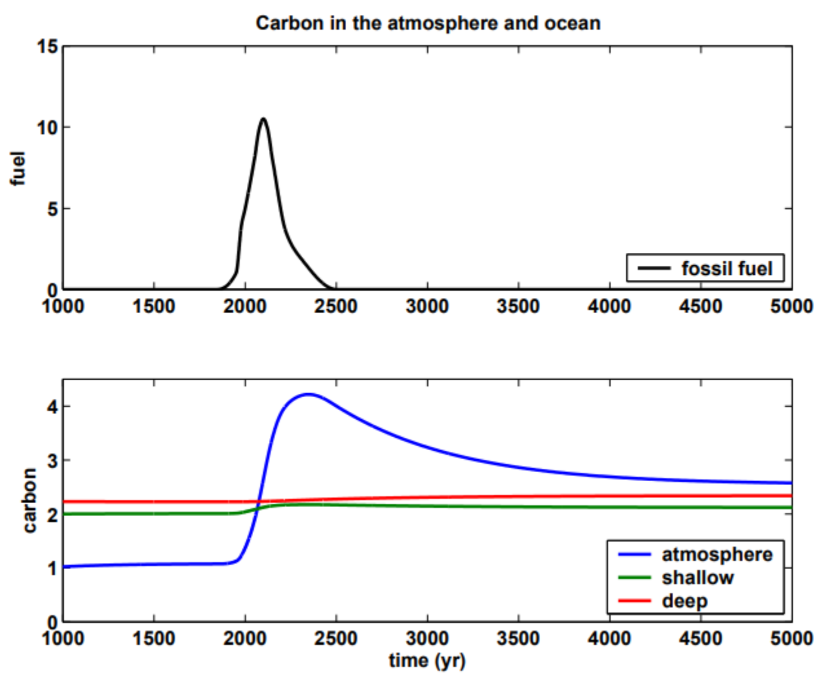

Figure 2: Carbon in the atmosphere and ocean

are almost constant throughout this entire simulation.) Initially, the carbon in the three regimes is nearly at equilibrium and so the amounts hardly change before 1850.

Over the period $1850 \le t \le 2500$, the upper half of Fig. 2 shows the additional carbon produced by burning fossil fuels entering the system, and the lower half shows the system response. The atmosphere is the first to be affected, showing more than a fourfold increase in 500 years. Almost half of the carbon is then slowly transferred to the shallow ocean and eventually to the deep ocean.

3 pts 1. In order to solve the ODEs later in question 2, we need to be able to fit a smooth function, (or interpolate somehow), to the discrete data given for the source term in table 1.

For example, we need some way to interpolate the data to give us the source term at intermediate years, such as say year 2020. In this case, from my plot given below in Fig. 3, you can see that $f(2020) \approx 6.52$. Note that my interpolating curve is 'smooth'.

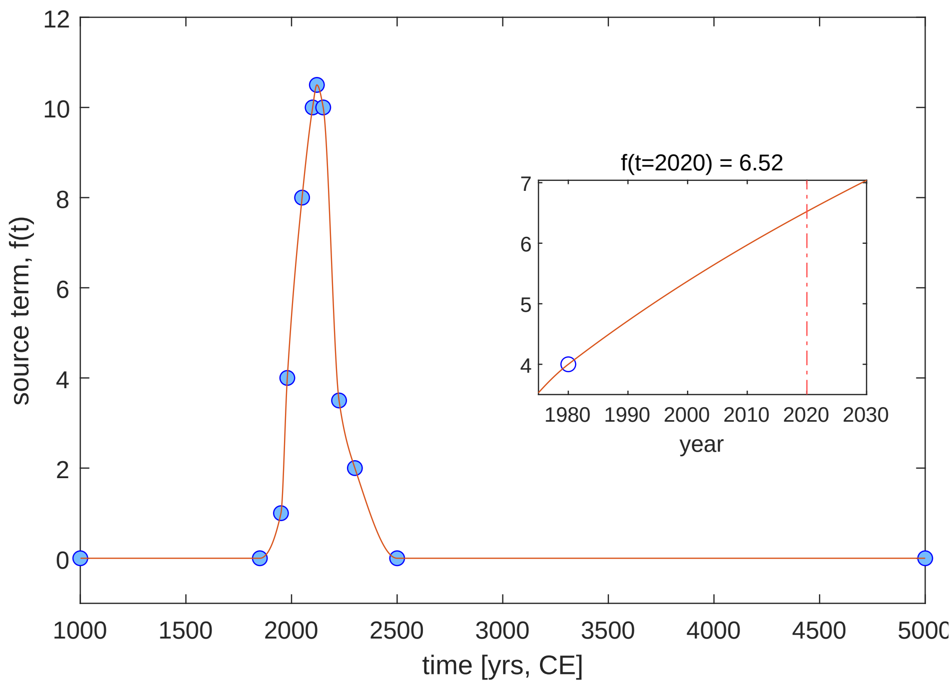

Figure 3: Source term, $f(t)$, interpolated from data given Table 1.

Write a function that takes a single argument time, t, and delivers the source term $f(t)$. Use something like the following:

```
>> t = 2020; % current year
>> f = sourceFossilFuels(t)
```

Ensure that your interpolating curve is reasonably smooth, and that it never goes negative. Also it is prudent to ensure that the curve extrapolates correctly out to the distant future.

Hint One option is to use `interp1` to interpolate the fuel table. of the various options, `pchip` is recommended. Why do you think this is so? Another option is to curve fit a smooth curve to the data.

See [2, §4.5.2] for some info on spline smoothing.

10 pts 2. (a) Reproduce the trends shown in Fig. 2 above by solving the ODEs using your routine to generate the source term $f(t)$ from question 1. Use `ode23` with the default tolerances to solve the differential equations.

Hint: Make sure you plot an attractive informative plot with axis labels, and sensible ranges — something like Fig. 4.

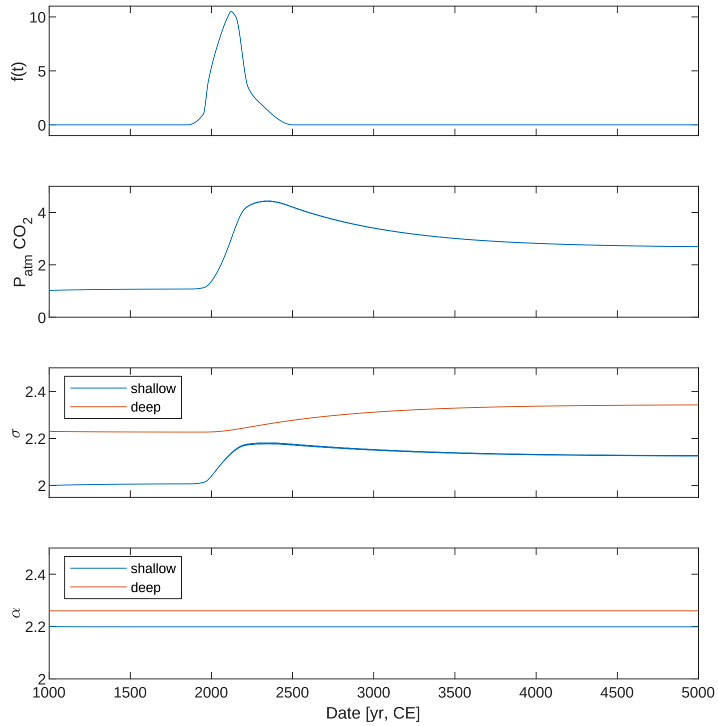

Figure 4: The state trends due to scenario #1.

(b) When does the atmospheric carbon dioxide reach its maximum? (Hint: You can see from the plot that it is somewhere around year 23000). Design an algorithm to find the maximum *automatically*, without too much manual intervention.

6 pts

3. (a) These equations are mildly stiff, because the various chemical reactions take place on very different time scales. If you zoom in on some portions of the graphs, you should see a characteristic sawtooth behaviour caused by the small time steps required by `ode23`. Find such a region and plot it at a suitable scale. What is the oscillation of the wobbles?

   Hint: See [2, §7.2.10] for some info stiff systems.

   (b) Experiment with other Matlab ordinary differential equation solvers, including `ode23`, `ode45`, `ode113`, `ode23s`, and `ode15s`. Try various tolerances and report computational costs by using an option something like:

```
odeset('RelTol',1.e-6,'AbsTol',1.e-6,'stats','on');
```

   Which method is preferable for this problem?

   Hint: See [2, §7.2.7] for some info on the various integrators available in MATLAB and their characteristics.

3 pts

4. Now we are ready to try various different scenarios to see what effect they have on the CO_2 level in the atmosphere over the next few centuries. (I am personally particularly interested in year 2075.) In Fig. 5 I have sketched 3 scenarios for the future burning of fossil fuels:

1. Nominal trajectory (as given in Table 1.)
2. Where we continue to burn fossil fuels at a rate greater than expected, (i.e. Trump wins in 2020!)
3. Where we invent in strategies over the next few years ways to capture and remove CO_2 from the atmosphere, so in effect we burn *negative* fuel.

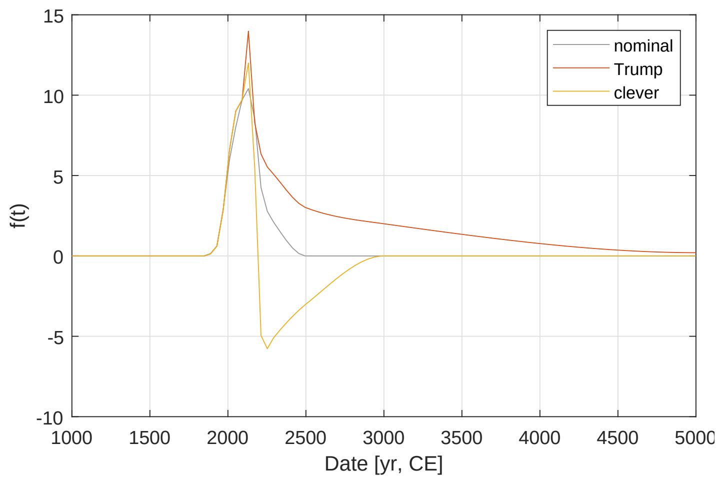

Figure 5: Various future scenarios for the source term $f(t)$.

Simulate the responses to these 3 scenarios and discuss the results and discuss your findings.

References

[1] James C. G. Walker. *Numerical Adventures with Geochemical Cycles.* Oxford University Press, 1991.

[2] David I. Wilson. *Numerical Analysis with Matlab for Engineers.* Auckland University of Technology, Auckland, New Zealand, 2020.