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MA2C01

1 Question 1

Prove that$(3n)! \leq 27^n(n!)^3$ for all $n \in N$

The formula holds for n=1. Suppose the formula holds for $m = n$ where m is some natural number.

$$(3m)! \leq 27^m(m!)^3$$

Then

$$(3(m+1))! \leq 27^{m+1}((m+1)!)^3$$
$$(3m)!(3m+3)(3m+2)(3m+1) \leq (27)(27)^m((m!)(m+1))^3$$
$$(3m)!(3m+3)(3m+2)(3m+1) \leq (27)(27)^m((m!)^3(m+1)^3)$$

But

$$(3m)! \leq 27^m(m!)^3$$

Which leaves

$$(3m+3)(3m+2)(3m+1) \leq 27(m+1)^3$$

$$27m^3 + 54m^2 + 33m + 6 \leq 27m^3 + 81m^2 + 81m + 27$$
$$0 \leq 27m^2 + 48m + 21$$

Which holds for all $m \in N$,$m > 0$. It now follows from the Principle of Mathematical Induction that this identity holds for all natural numbers m.

2 Question 2

Prove that

$$(A \cup B)\backslash(A \cap B) = (A\backslash B) \cup (B\backslash A)$$

Let $x \in (A \cup B)\backslash(A \cap B)$. Then $x \in A \cup B, x \notin A \cap B$ Then $x \in A$ or $x \in B$ but $x \notin A \cap B$.
If $x \in A$, then $x \notin B$, $x \in A\backslash B$. If $x \in B$, then $x \notin A$, $x \in B\backslash A$. Then if $x \in A$, then $x \notin B$ and $x \notin B\backslash A$. Similarly for $x \in B$. Thus $x \in (A\backslash B) \cap (B\backslash A)$
Now let $x \in (A\backslash B) \cup (B\backslash A)$. Then either $x \in A\backslash B$ or $x \in B\backslash A$. In both cases $x \in A \cup B$, $x \notin A \cap B$. Thus $x \in (A\backslash B) \cup (B\backslash A)$.

We have shown that the sets $(A \cup B)\backslash(A \cap B)$ and $(A\backslash B) \cup (B\backslash A)$ have the same elements, and thus that $(A \cup B)\backslash(A \cap B) = (A\backslash B) \cup (B\backslash A)$.

3 Question 3

3.1 Relation Q on N

1. The relation Q on N is not reflexive as

$$(1)^2 - (1)^2 = 2^k$$
$$1 - 1 = 2^k$$
$$0 = 2^k$$

   There is no $k \in N$ such that $2^k = 0$.

2. The relation Q on N is not symmetric.

$$(3)^2 - (1)^2 = 2^k$$
$$9 - 1 = 2^k$$
$$8 = 2^k$$
$$8 = 2^3$$

   But

$$(1)^2 - (3)^2 = 2^k$$
$$1 - 9 = 2^k$$
$$-8 = 2^k$$

   There exists no $k \in N$ such that $2^k = -8$.

3. xQy holds if $x > y$. But yQx is false if $x > y$. There is no element k if $x \leq y$.As xQy and yQx can never be true when $x, y \in N$ where $x, y > 0$. Therefore the function is anti-symmetric.

4. The relation Q is not transitive. $10Q6$ and $6Q2$ but $10 \neq 2$.

5. As the relation Q is not reflexive, symmetric or transitive it is not an equivalence relation or a partial order.

3.2 Relation R on N

1. The relation R on N is reflexive as

$$n^2/n^2 = 2^k$$

Where $n \in N$,$n > 0$

$$1 = 2^k$$

$$1 = 2^0$$

2. The relation R is not symmetric as

$$2^2/1^2 = 2^k$$

$$4/1 = 2^k$$

$$4 = 2^2$$

But

$$1^2/2^2 = 2^k$$

$$1/4 = 2^k$$

There exists no element $k \in N$ such that $2^k = 1/4$.

3. Let $x, y \in N$, $x, y > 0$ satisfy xRy. Then $x^2/y^2 = 2^k$ for some $k \in N$. Now let the same x, y satisfy yRx. Then $y^2/x^2 = 2^{-k}$, but $-k \not> 0$. Therefore for x, y to satisfy xRy and yRx, k must equal 0. $k = 0$ if and only if $x = y$. R is anti-symmetric.

4. Let $x, y, z \in N$ where $x, y, z > 0$ satisfy xRy and yRz. The there exists $k, l \in N$ such that $\frac{x^2}{y^2} = 2^k$ and $\frac{y^2}{z^2} = 2^l$. Assume $\frac{x^2}{z^2} = 2^m$.

$$x^2 = 2^k.y^2, z^2 = \frac{y^2}{2^l}$$

$$\frac{(2^k.y^2)}{\frac{y^2}{2^l}} = 2^m$$

$$\frac{2^k.y^2.2^k}{y^2}$$

$$2^k.2^l = 2^m$$

$$2^{k+l} = 2^m$$

Therefore the relation is transitive as all three terms are a factor of xRz.

5. The relation R is not symmetric, therefore it is not an equivalence relation. As it is not anti-symmetric, the relation R is not a partial order.

4 Question 4

Let $x, y \in [0, +\infty)$. If $x < y$ then $\frac{1}{1+x^2} > \frac{1}{1+y^2}$. If $x > y$ then $\frac{1}{1+x^2} < \frac{1}{1+y^2}$. If $x \neq y$ then either $\frac{1}{1+x^2} > \frac{1}{1+y^2}$ or $\frac{1}{1+x^2} < \frac{1}{1+y^2}$. It follows that $f(x) \neq f(y)$. Therefore the function is injective.

Given any $y \in (0, 1]$ there exists an $x \in [0, +\infty]$ such that $x = \sqrt{\frac{1}{y} - 1}$. As f has an inverse, it follows that f is also surjective.

As shown above the function has an inverse, it follows that the function is surjective and therefore bijective, as per theorem 2.4. It has an inverse, such that $f^{-1} = \sqrt{\frac{1}{y} - 1}$.