Chapter 2.3
Análisis y Diseño de Algoritmos (Algorítmica III)

-Heaps-
-Heap Sort-
-Priority Queues-

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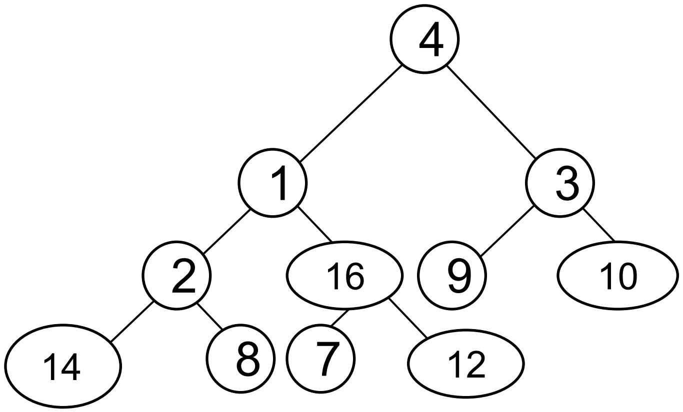
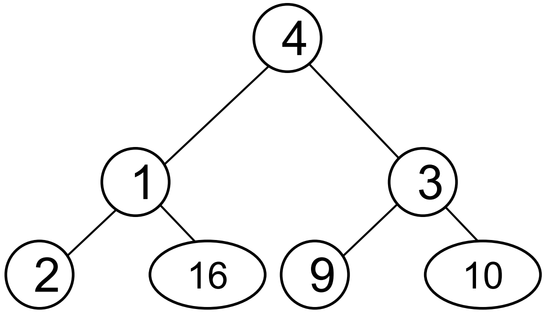
Special Types of Trees

- Def: **Full binary tree** = a binary tree in which each node is either a leaf or has degree exactly 2.
- Def: **Complete binary tree** = a binary tree in which all leaves are on the same level and all internal nodes have degree 2.



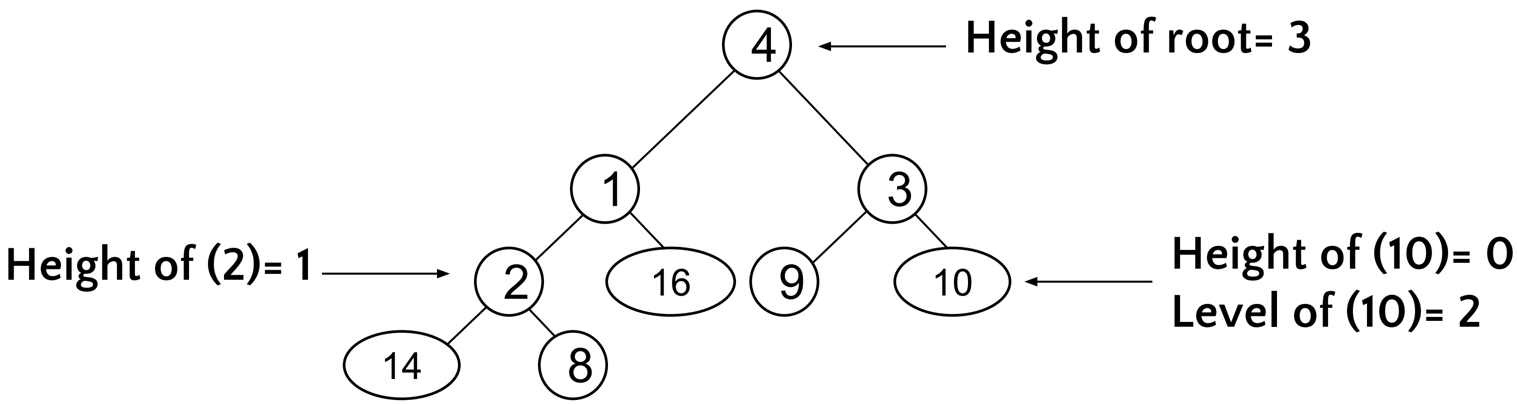
Full binary tree



Complete binary tree

Definitions

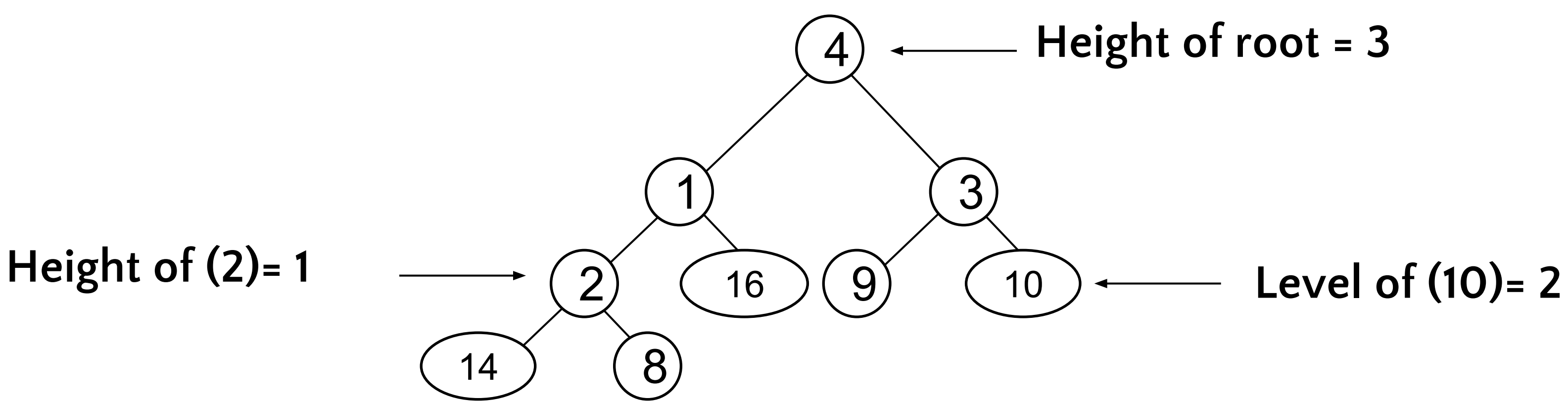
- **Height** of a node = the number of edges on the longest simple path from the node down to a leaf
- **Level** of a node = the length of a path from the root to the node
- **Height of tree** = height of root node



Useful Properties

- There are **at most 2^l** nodes at level (or depth) $\boldsymbol{l}$ of a binary tree
- A binary tree with height d has **at most 2^{d+1} -1** nodes
- A binary tree with $\boldsymbol{n}$ nodes has height ***at least*** $\lfloor \lg n \rfloor$

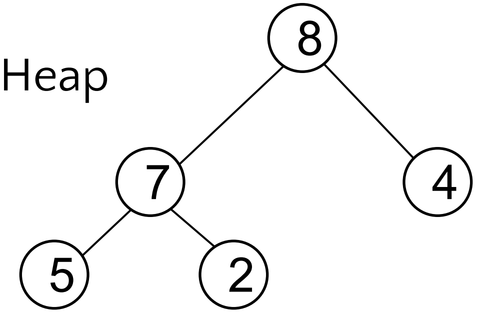
$$n \le \sum_{l=0}^{d} 2^l = \frac{2^{d+1}-1}{2-1} = 2^{d+1}-1$$



The Heap Data Structure

- Def: A **heap** is a nearly complete binary tree with the following two properties:
  - **Structural property:** all levels are full, except possibly the last one, which is filled from left to right
  - **Order (heap) property:** for any node x: **Parent(x) ≥ x**

Heap

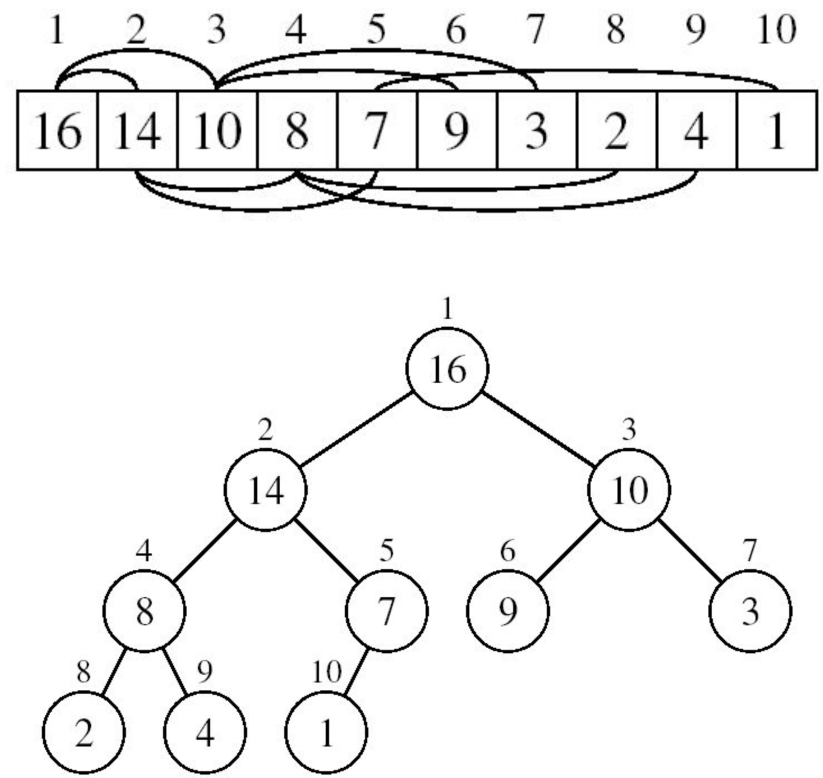


From the heap property, it follows that:
"The root is the maximum element of the heap!"

A heap is a binary tree that is filled in order

Array Representation of Heaps

- A heap can be stored as an array *A*.
  - Root of tree is A[1]
  - Left child of A[i] = A[2i]
  - Right child of A[i] = A[2i + 1]
  - Parent of A[i] = A[$\lfloor i/2 \rfloor$]
  - Heapsize[A] ≤ length[A]
- The elements <u>in the subarray A[($\lfloor n/2 \rfloor$+1) .. n] are leaves</u>

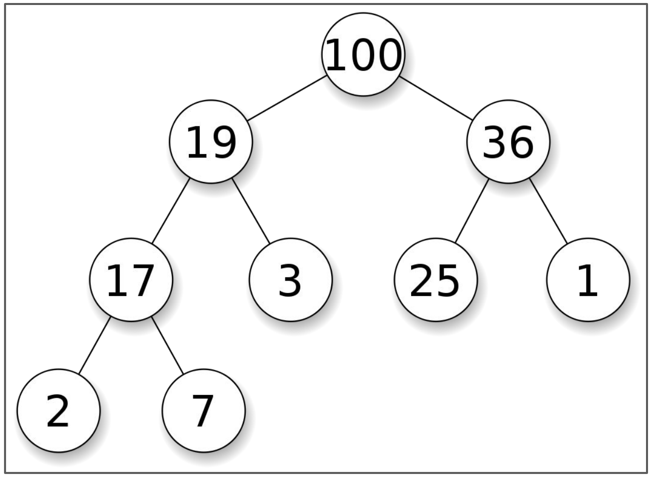


Heap Types

- **Max-heaps** (largest element at root), have the *max-heap property:*

  for all nodes i, excluding the root:

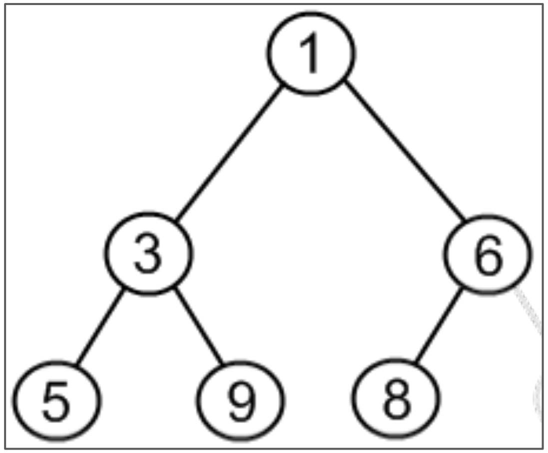
  $$A[\text{PARENT}(i)] \geq A[i]$$



- **Min-heaps** (smallest element at root), have the *min-heap property:*

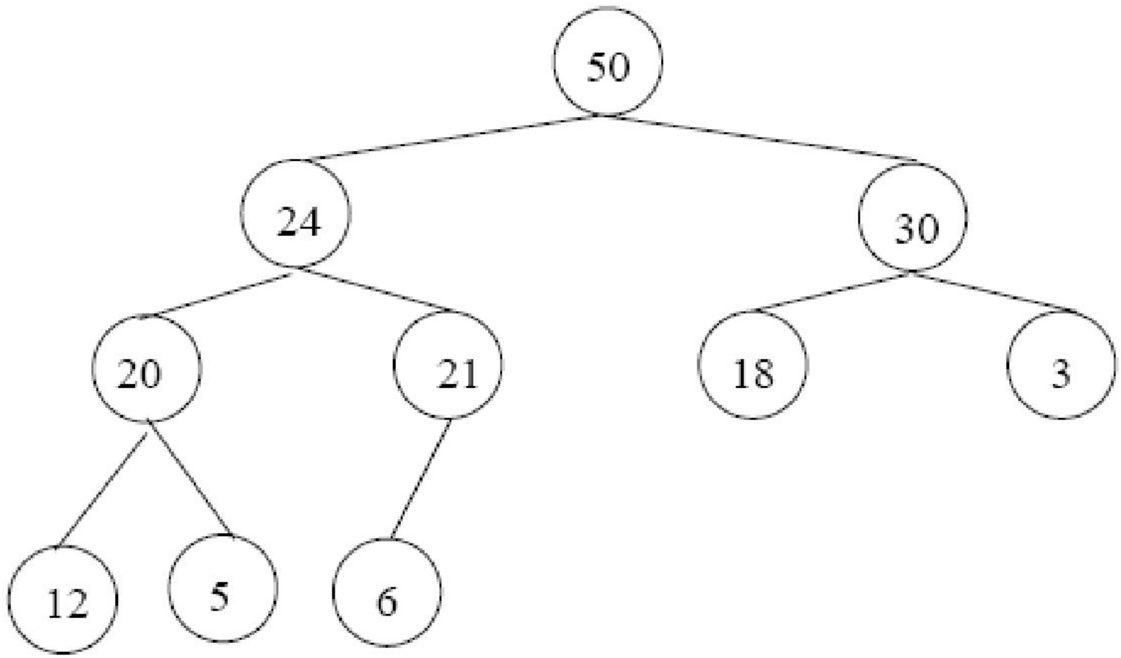
  for all nodes i, excluding the root:

  $$A[\text{PARENT}(i)] \leq A[i]$$



Adding/Deleting Nodes

- New nodes are always inserted at the bottom level (left to right)
- Nodes are removed from the bottom level (right to left)

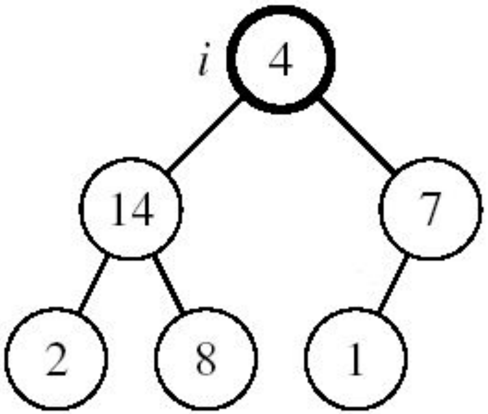


Operations on Heaps

- Maintain/Restore the max-heap property
  - MAX-HEAPIFY
- Create a max-heap from an unordered array
  - BUILD-MAX-HEAP
- Sort an array in place
  - HEAPSORT
- Priority queues

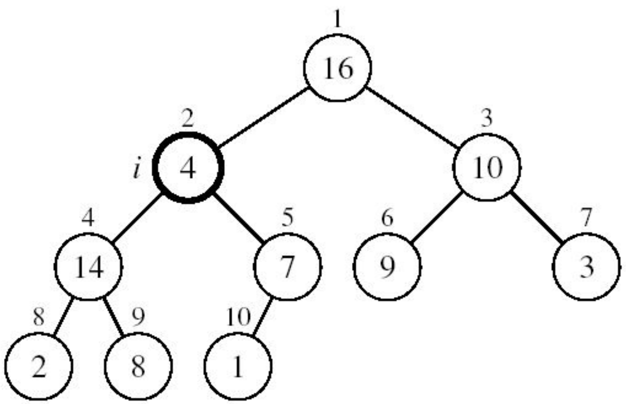
Maintaining the Heap Property

- Suppose a node is smaller than a child
  - Left and Right subtrees of *i* are max-heaps
- To eliminate the violation:
  1) Exchange with larger child
  2) Move down the tree
  3) Continue until node is not smaller than children



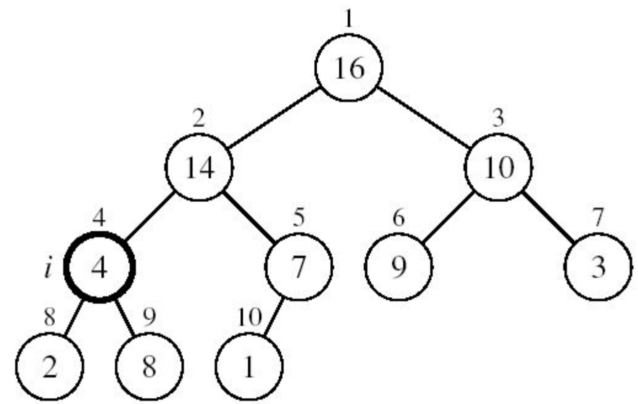
Example

MAX-HEAPIFY(A, 2, 10)



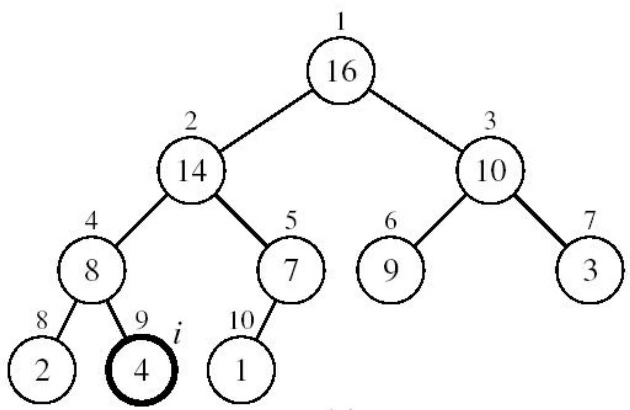
A[2] violates the heap property

A[2] ↔ A[4]



A[4] violates the heap property

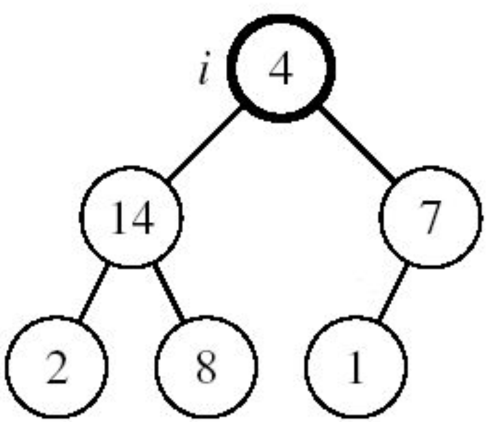
A[4] ↔ A[9]



Heap property restored

Maintaining the Heap Property

- Assumptions:
  - Left and Right subtrees of i are max-heaps
  - A[i] may be smaller than its children



Alg: **MAX-HEAPIFY(A, i, n)**

l ← LEFT(i)
r ← RIGHT(i)
if l≤n and A[l]>A[i] **then**
    largest← l
else
    largest← i
if r≤n and A[r]>A[largest] **then**
    largest← r
if largest ≠ i **then**
    exchange(A[i] ↔ A[largest])
    MAX-HEAPIFY(A, largest, n)

MAX-HEAPIFY Running Time

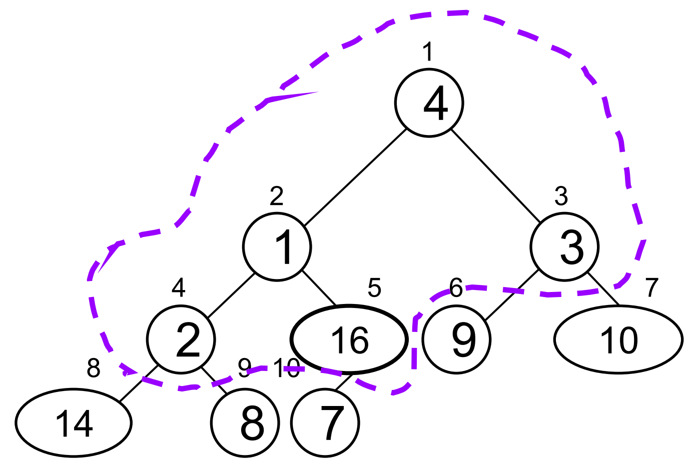
- **Intuitively:**
  - It traces a path from the root to a leaf (longest path *h*)
  - At each level, it makes exactly *2* comparisons
  - Total number of comparisons is *2h*
  - Running time is *O(h)* or *O(lg n)*
- **Running time of MAX-HEAPIFY is O(lgn)**
- **Can be written in terms of the height of the heap, as being O(h)**
  - Since the height of the heap is $\lfloor \lg n \rfloor$

Building a Heap

- Convert an array A[1 … n] into a max–heap (n = length[A])
- The elements in the subarray $A[(\lfloor n/2 \rfloor+1) .. n]$ are leaves
- Apply MAX–HEAPIFY on elements between 1 and $\lfloor n/2 \rfloor$

Alg: BUILD-MAX-HEAP(A)

```
n = length[A]
 for i ← ⌊n/2⌋ to 1 do
     MAX-HEAPIFY(A, i, n)
```

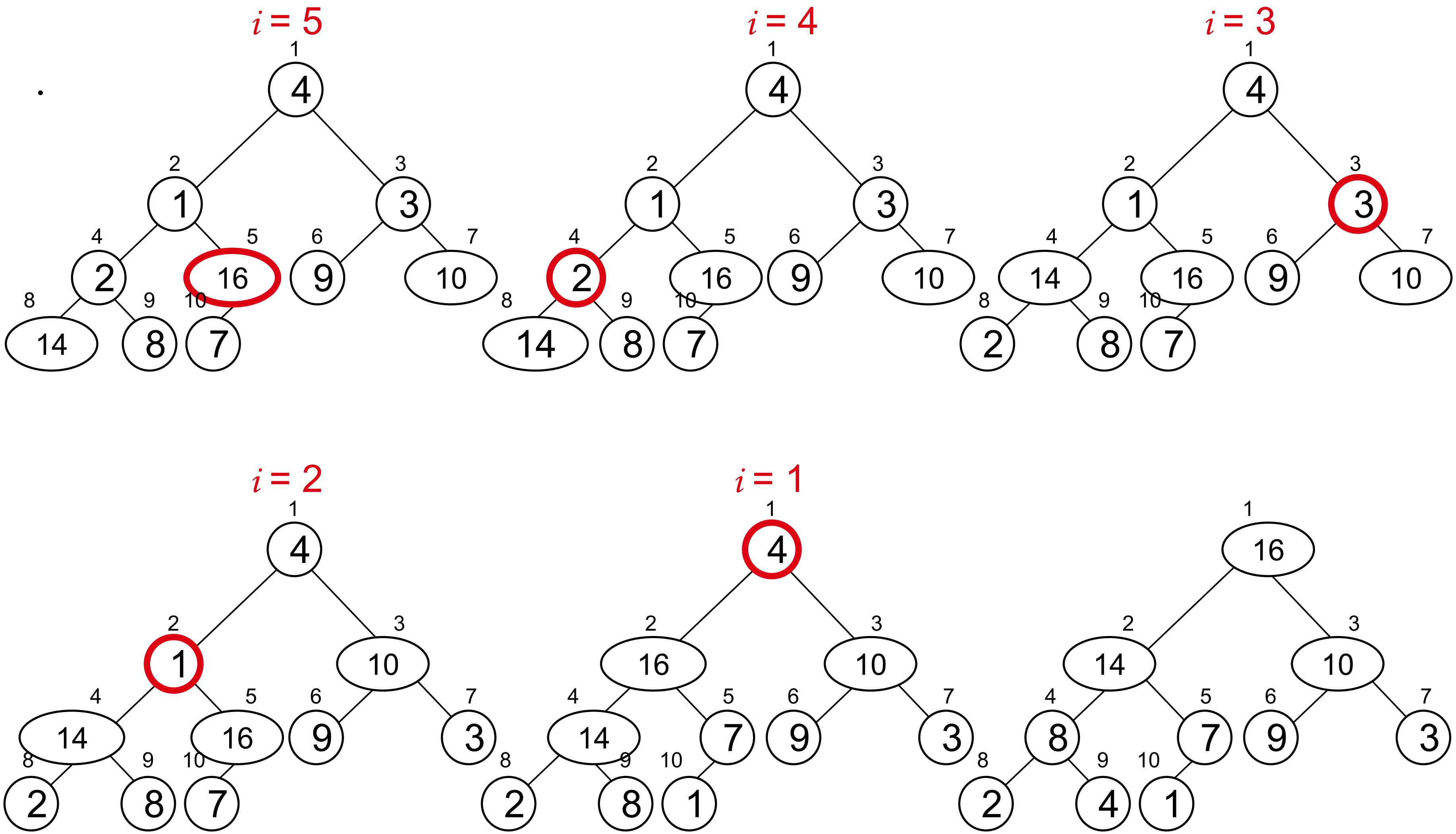


A:

4	1	3	2	16	9	10	14	8	7

Example: A =

4	1	3	2	16	9	10	14	8	7



Running Time of BUILD MAX HEAP

Alg: <u>**BUILD-MAX-HEAP(A)**</u>

n = length[A]

for i ← $\lfloor n/2 \rfloor$ **to** 1 **do**

MAX-HEAPIFY(A, i, n) O(lgn)

O(n)

⇒Running time: O($n$lg$n$)

- This is not an asymptotically tight upper bound

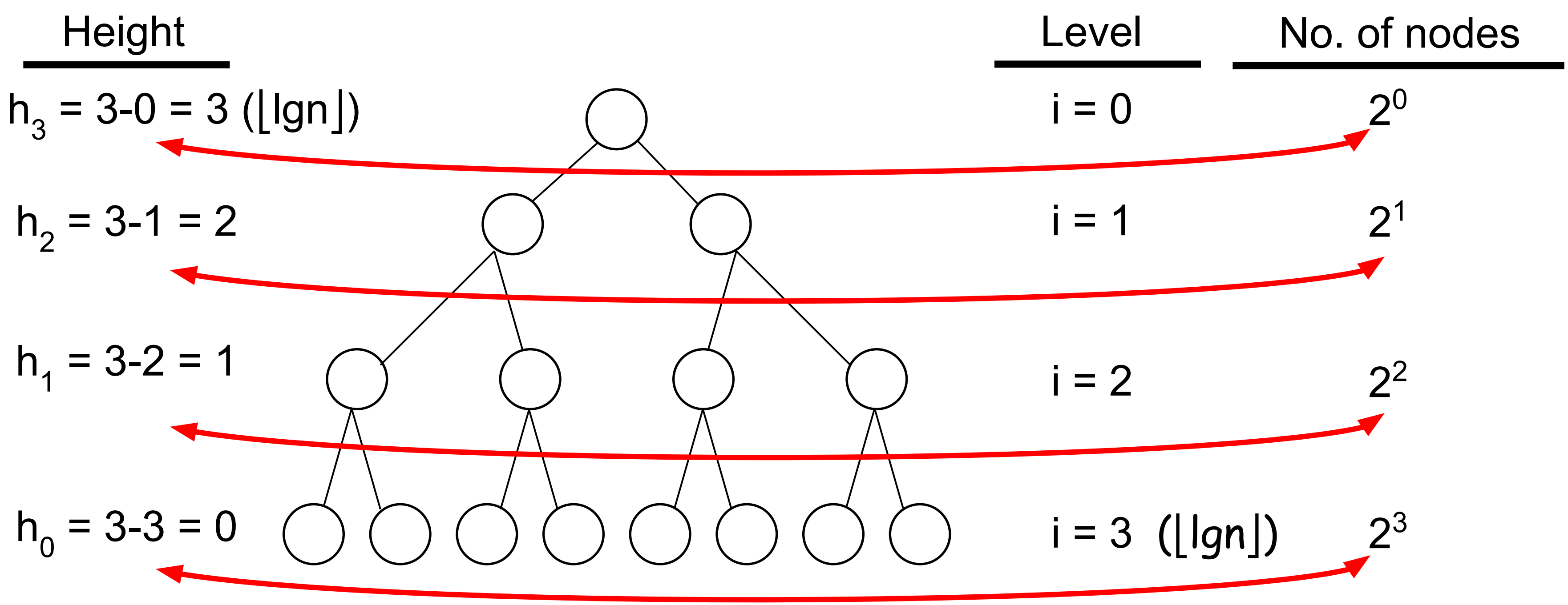
Running Time of BUILD MAX HEAP

- HEAPIFY takes $O(h)$ ⇒ the cost of HEAPIFY on a node *i* is proportional to the height of the node *i* in the tree

$$\Rightarrow T(n) = \sum_{i=0}^{h} n_i h_i = \sum_{i=0}^{h} 2^i (h-i) = O(n)$$

$h_i = h - i$ height of the heap rooted at level i

$n_i = 2^i$ number of nodes at level i



Running Time of BUILD MAX HEAP

$$T(n) = \sum_{i=0}^{h} n_i h_i$$

Cost of HEAPIFY at level i ∗ number of nodes at that level

$$= \sum_{i=0}^{h} 2^i (h-i)$$

Replace the values of n_i and h_i computed before

$$= \sum_{i=0}^{h} \frac{h-i}{2^{h-i}} 2^h$$

Multiply by 2^h both at the numerator and denominator and write 2^i as $\frac{1}{2^{-i}}$

$$= 2^h \sum_{k=0}^{h} \frac{k}{2^k}$$

Change variables: k = h – i

$$\leq n \sum_{k=0}^{\infty} \frac{k}{2^k}$$

The sum above is smaller than the sum of all elements to ∞ and h = lgn

$$= O(n)$$

The sum above is smaller than 2

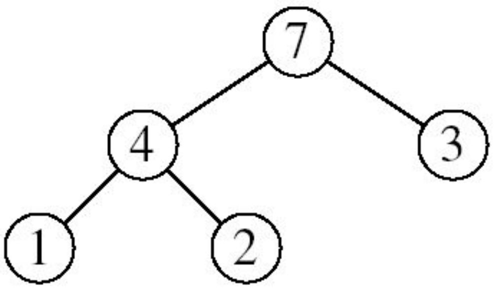
Running time of BUILD-MAX-HEAP: T(n) = O(n)

Heapsort

Heapsort

- **Goal:**

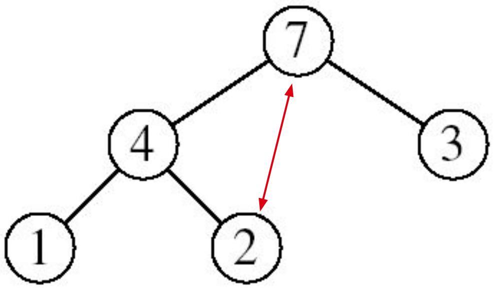
  Sort an array using heap representations



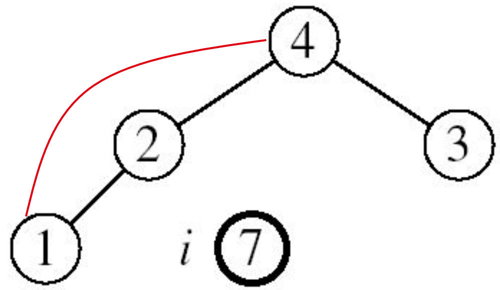
- **Idea:**

  1. Build a **max-heap** from the array
  2. Swap the root (the maximum element) with the last element in the array
  3. “Discard” this last node by decreasing the heap size
  4. Call MAX-HEAPIFY on the new root
  5. Repeat this process until only one node remains

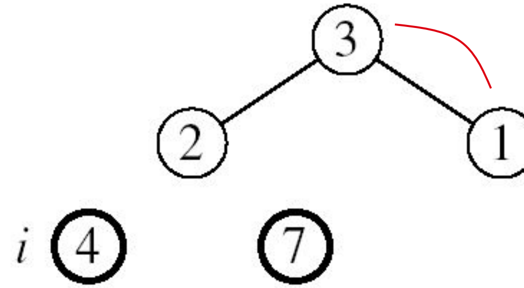
Example: A=[7, 4, 3, 1, 2]



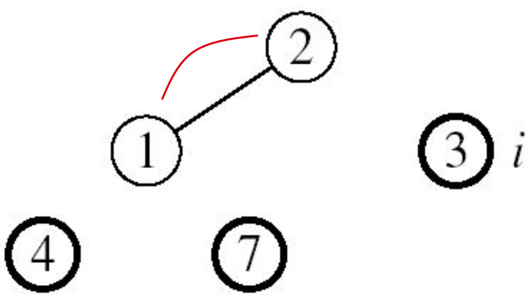
MAX-HEAPIFY(A, 1, 4)



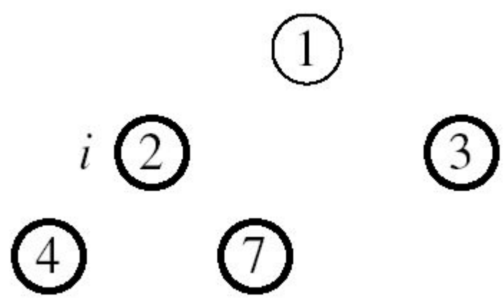
MAX-HEAPIFY(A, 1, 3)



MAX-HEAPIFY(A, 1, 2)



MAX-HEAPIFY(A, 1, 1)



A	1	2	3	4	7

Algoritmo Heapsort (A)

Alg: HEAPSORT(A)

```
BUILD-MAX-HEAP(A)                        O(n)
for i ← length[A] to 2 do
    exchange (A[1] ↔ A[i])                    n-1 times
    MAX-HEAPIFY(A, 1, i - 1)             O(lgn)
```

- **Running time:** $O(n\lg n)$ Can be shown to be $\Theta(n\lg n)$

Priority Queues

Priority Queues

Problem.

Let $S=\{(s_1,p_1), (s_2,p_2),\ldots,(s_n,p_n)\}$ where s(i) is a key and p(i) is the priority of s(i).

How to design a data structure/algorithm to support the following operations over S?

ExtractMin: Returns the element of S with minimum priority

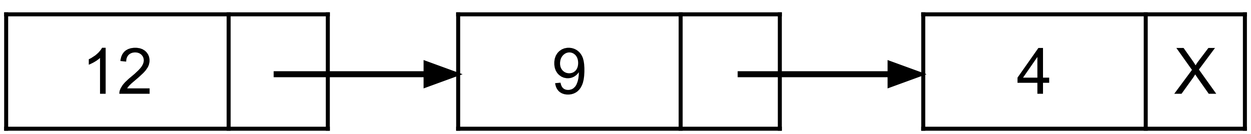
Insert(s,p): Insert a new element (s,p) in S

RemoveMin: Remove the element in S with minimum p

Priority Queues

Properties

- Each element is associated with a value (priority)
- The key with the highest (or lowest) priority is extracted first
- Major operations
  - ***Remove*** an element from the queue
  - ***Insert*** an element in the queue



Priority Queues

Solution 1. Used a sorted list

- **ExtractMin:** O(1) time
- **Insert:** O(n) time
- **DeleteMin:** O(1) time

Solution 2. Use a list with the pair with minimum p at the first position

- **ExtractMin:** O(1) time
- **Insert:** O(1) time
- **DeleteMin:** O(n) time

Can we do better? How?

Operations on Priority Queues

- **Max-priority queues support the following operations:**
  - **INSERT(S, x):** inserts element x into set S
  - **EXTRACT-MAX(S):** removes and returns element of S with largest key
  - **MAXIMUM(S):** returns element of S with largest key
  - **INCREASE-KEY(S, x, k):** increases value of element x's key to k (Assume k ≥ x's current key value)

HEAP-MAXIMUM

Goal:

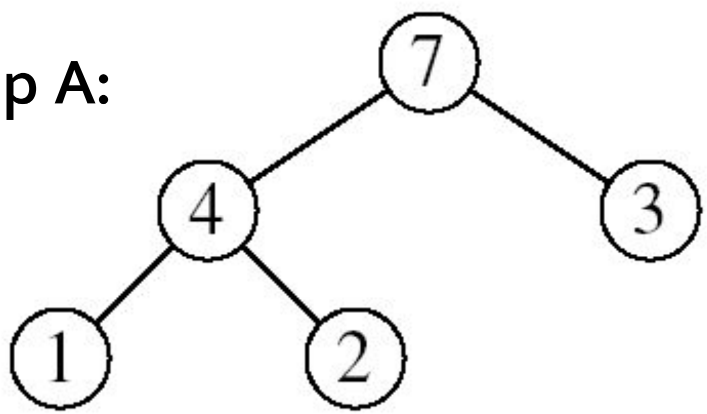
- Return the largest element of the heap

Alg: HEAP-MAXIMUM(A)

return $A[1]$

Running time: $O(1)$

Heap A:



Heap-Maximum(A) returns 7

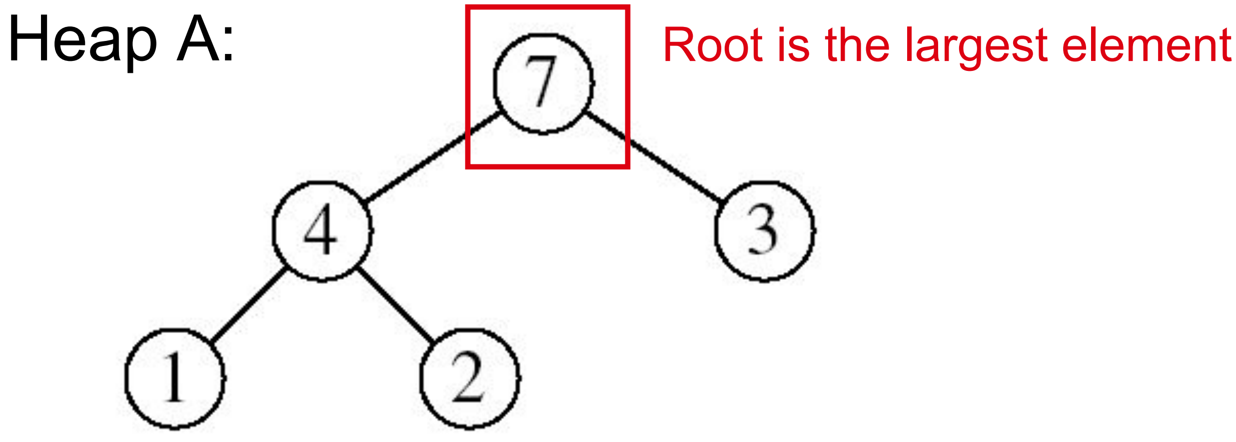
HEAP-EXTRACT-MAX

Goal:

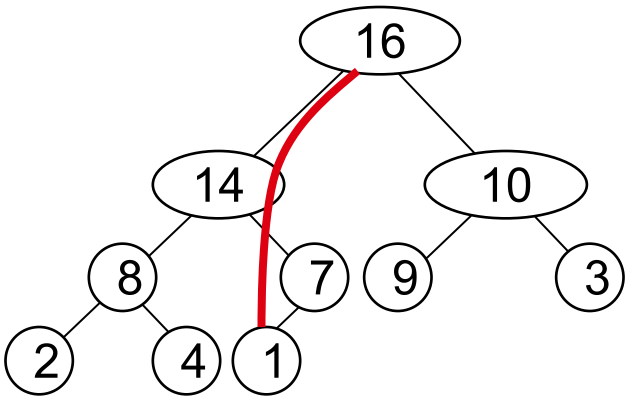
- Extract the largest element of the heap (i.e., return the max value and also remove that element from the heap

Idea:

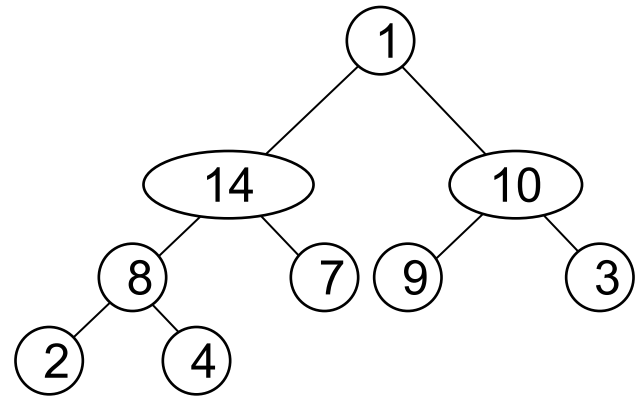
- Exchange the root element with the last
- Decrease the size of the heap by 1 element
- Call MAX-HEAPIFY on the new root, on a heap of size n-1



Example: HEAP-EXTRACT-MAX

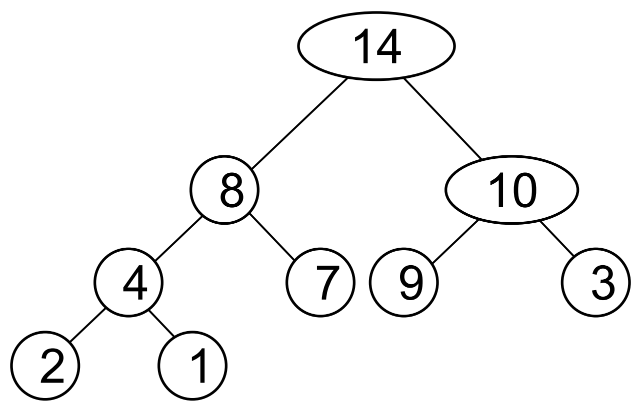


max = 16



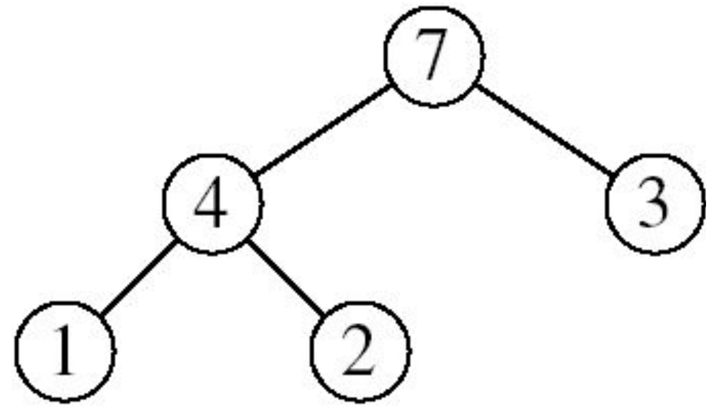
Heap size decreased with 1

Call MAX-HEAPIFY(A, 1, n-1)



HEAP-EXTRACT-MAX

Alg: HEAP-EXTRACT-MAX(A, n)




```
if n < 1
   then error "heap underflow"
max ← A[1]
A[1] ← A[n]
MAX-HEAPIFY(A, 1, n-1)
return max
```

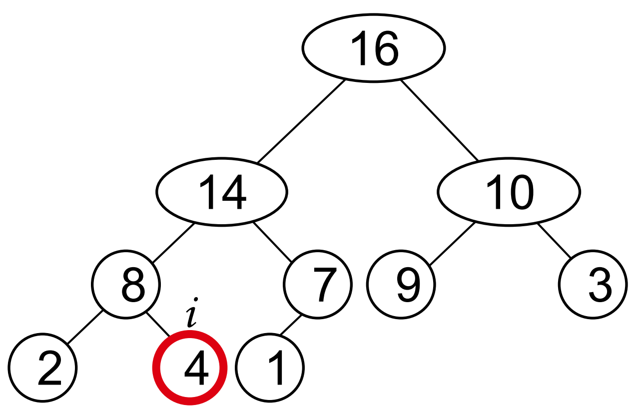
▷ remakes heap

Running time: **O(lgn)**

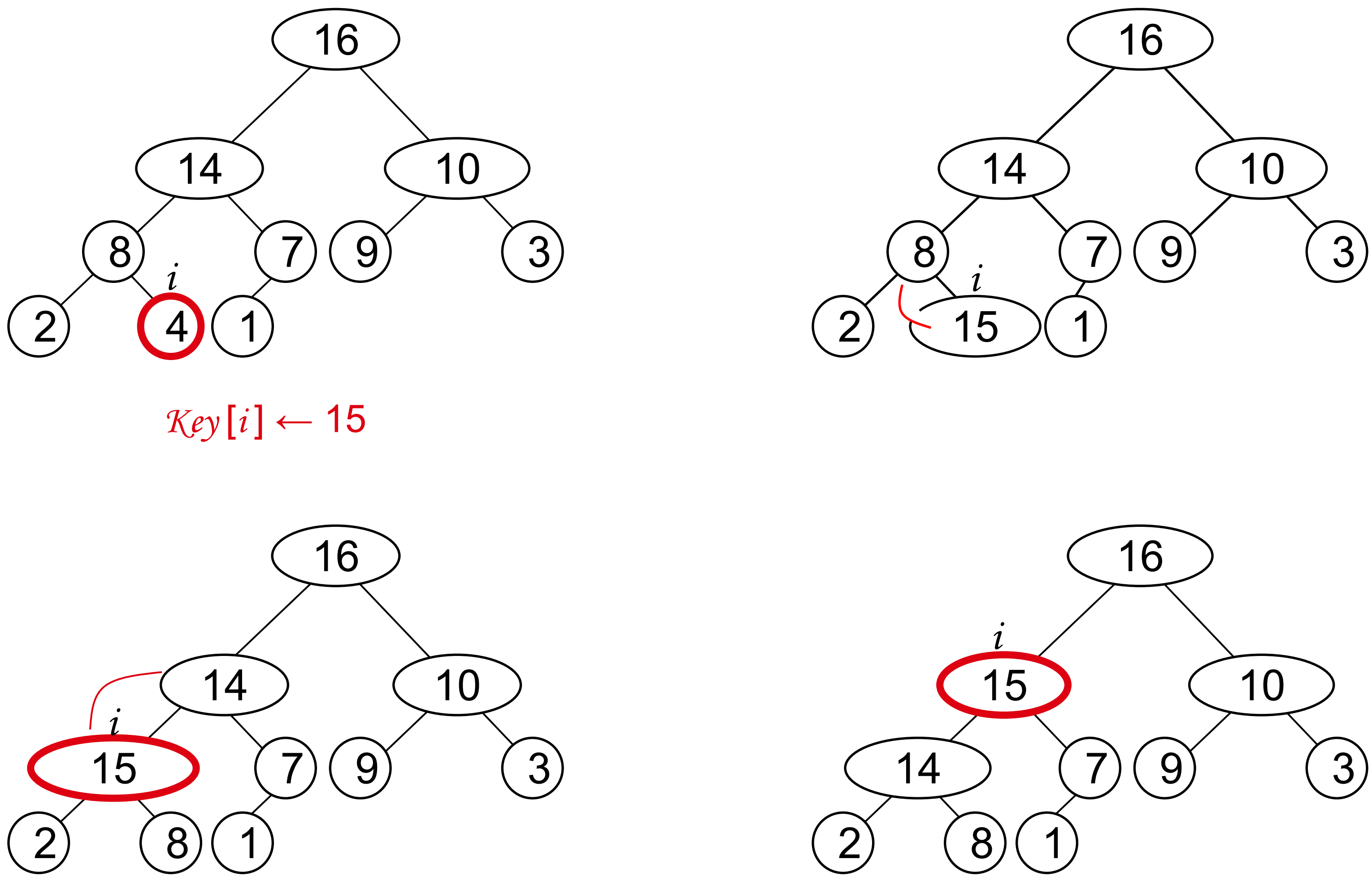
HEAP-INCREASE-KEY

- Goal:
  - Increases the key of an element i in the heap
- Idea:
  - Increment the key of A[i] to its new value
  - If the max-heap property does not hold anymore: traverse a path toward the root to find the proper place for the newly increased key

Key [i] ← 15



Example: HEAP-INCREASE-KEY

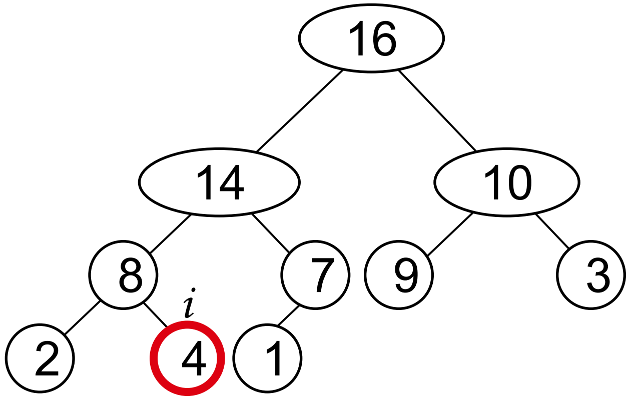


HEAP-INCREASE-KEY

Alg: HEAP-INCREASE-KEY(A, i, key)

```
if key < A[i]
   then error "new key is smaller than current key"
A[i] ← key
while i > 1 and A[PARENT(i)] < A[i]
  do exchange A[i] ↔ A[PARENT(i)]
     i ← PARENT(i)
```

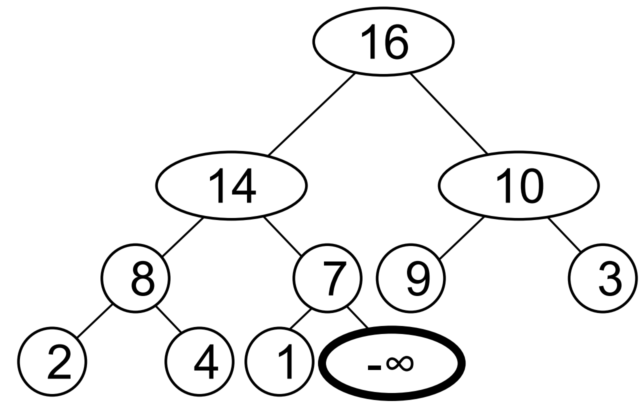
- Running time: $O(\lg n)$

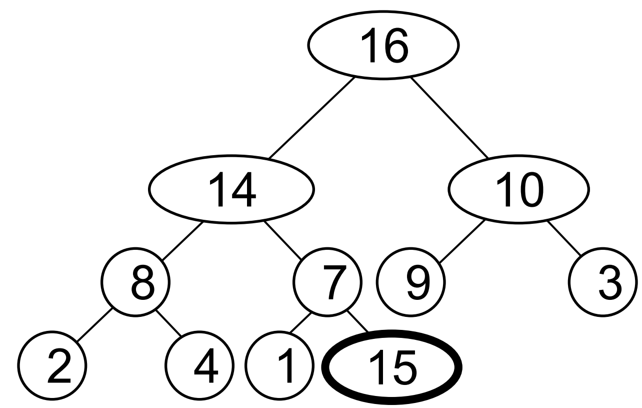


Key [i] ← 15

MAX-HEAP-INSERT

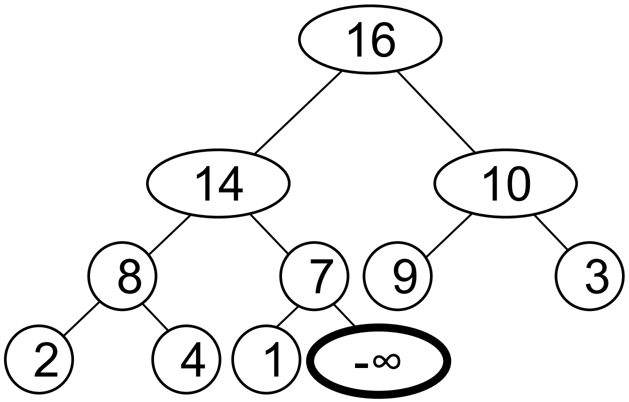
- **Goal:**
  - Inserts a new element into a max-heap
- **Idea:**
  - Expand the max-heap with a new element whose key is -∞
  - Calls HEAP-INCREASE-KEY to set the key of the new node to its correct value and maintain the max-heap property



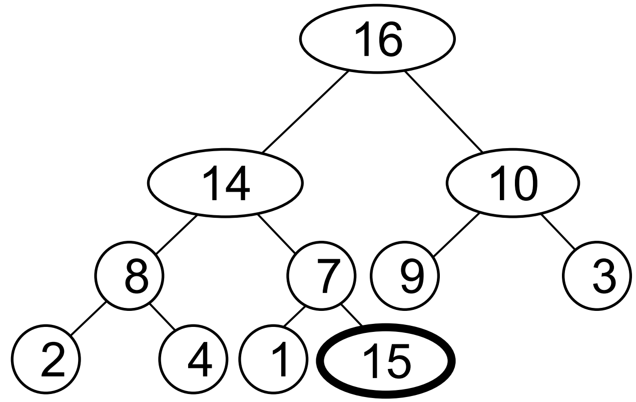


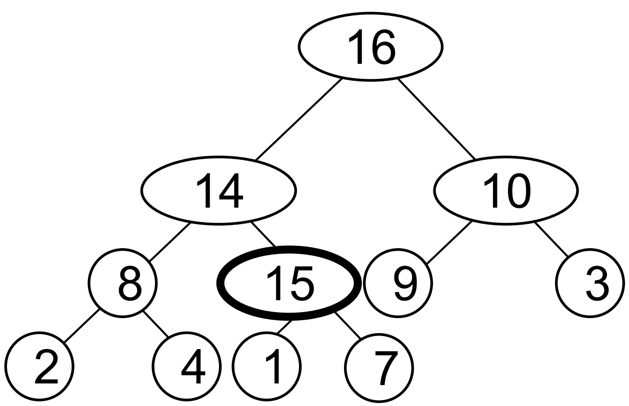
Example: MAX-HEAP-INSERT

Insert value 15:
- Start by inserting -∞

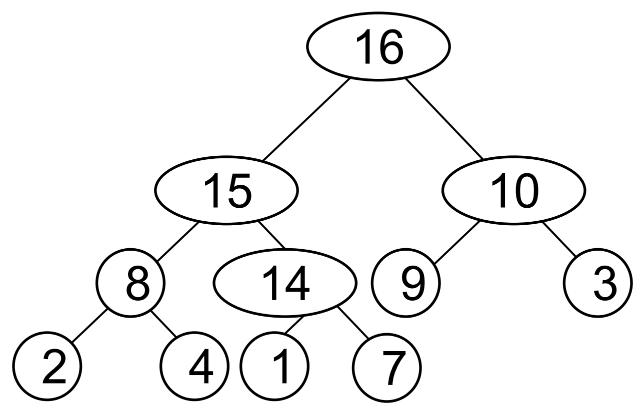


Increase the key to 15
Call HEAP-INCREASE-KEY on A[11] = 15





The restored heap containing
the newly added element



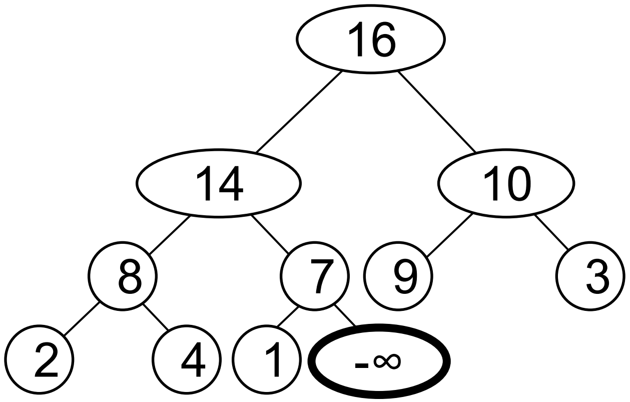
MAX-HEAP-INSERT

Alg: MAX-HEAP-INSERT(A, key, n)

heap-size[A] $\leftarrow$ n + 1

A[n + 1] $\leftarrow$ -∞

HEAP-INCREASE-KEY(A, n + 1, key)



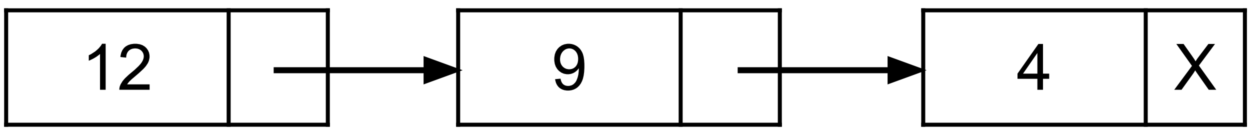
Running time: $O(\lg n)$

Summary

- We can perform the following operations on heaps:
  - MAX-HEAPIFY O(lgn)
  - BUILD-MAX-HEAP O(n)
  - HEAP-SORT O(nlgn)
  - MAX-HEAP-INSERT O(lgn)
  - HEAP-EXTRACT-MAX O(lgn)
  - HEAP-INCREASE-KEY O(lgn)
  - HEAP-MAXIMUM O(1)

Average O(lgn)

Priority Queue Using Linked List



Remove a key: O(1)

Insert a key: O(n)

Increase key: O(n)

Extract max key: O(1)

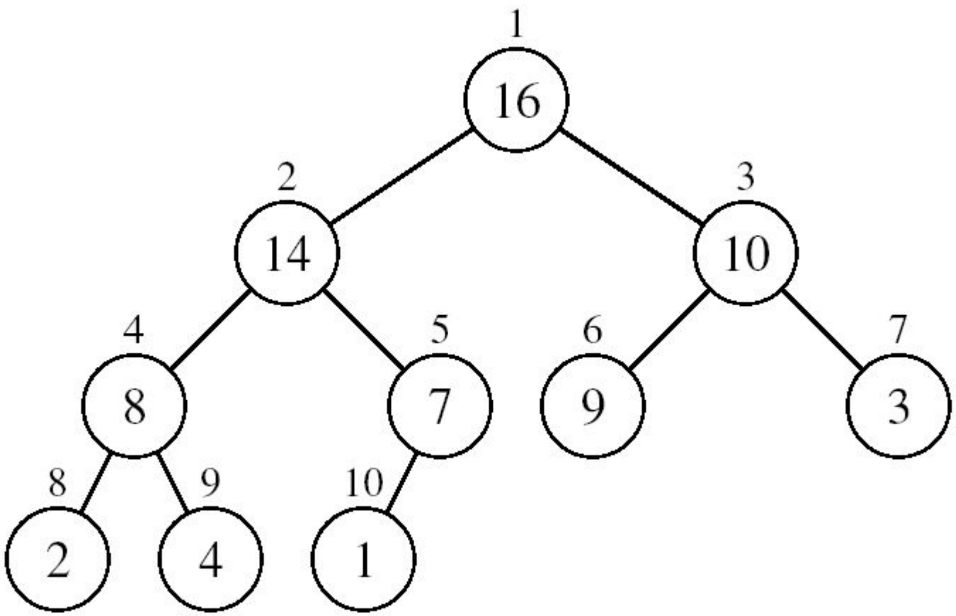
Average: O(n)

Applications of Priority Queue:

1) CPU Scheduling
2) Graph algorithms like Dijkstra's shortest path algorithm, Prim's Minimum Spanning Tree, etc
3) All queue applications where priority is involved.

Problems

Assuming the data in a max-heap are distinct, what are the possible locations of the second-largest element?



Problems

(a) What is the maximum number of nodes in a max heap of height h?

(b) What is the maximum number of leaves?

(c) What is the maximum number of internal nodes?

Problems

Demonstrate, step by step, the operation of Build-Heap on the array

$$A=[5, 3, 17, 10, 84, 19, 6, 22, 9]$$

Problems

Let A be a heap of size n. Give the most efficient algorithm for the following tasks:

A. Find the sum of all elements
B. Find the sum of the largest $\lg n$ elements