Chapter 2.1
Análisis y Diseño de Algoritmos (Algorítmica III)
– Ordenation Algorithms –

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Sorting

Sorting. Given n elements, rearrange in ascending order.

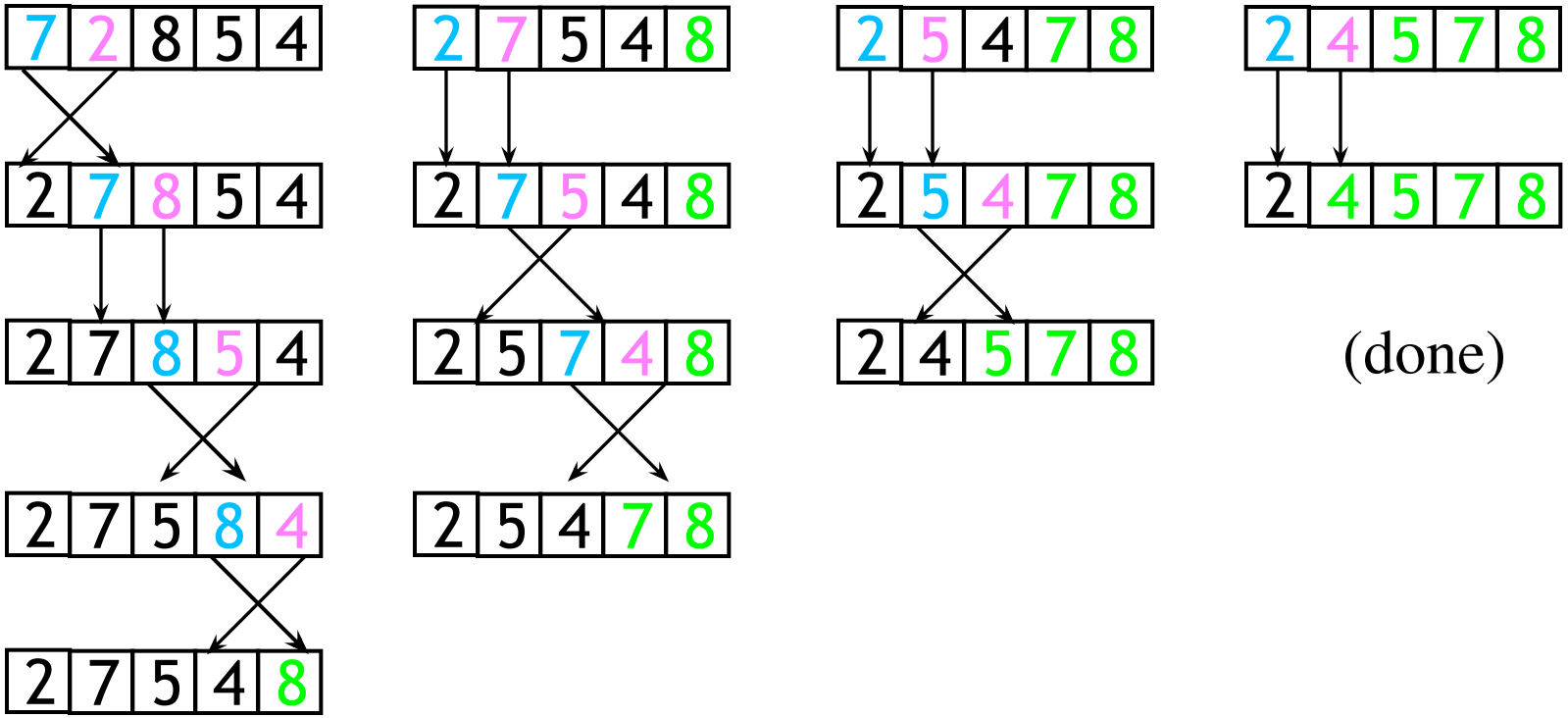
- Obvious sorting applications.
  - List files in a directory. Organize an MP3 library. List names in a phone book. Display Google PageRank results.
- Problems become easier once sorted.
  - Find the median. Find the closest pair.
  - Binary search in a database.
  - Identify statistical outliers. Find duplicates in a mailing list.
- Non-obvious sorting applications.
  - Data compression. Computer graphics. Interval scheduling. Computational biology. Minimum spanning tree. Supply chain management.
  - Simulate a system of particles. Book recommendations on Amazon.
  - Load balancing on a parallel computer.
  - . . .

1. Basic Algorithms: Bubble Sort and Selection Sort

Bubble sort

- **Compare each element (except the last one) with its neighbor to the right**
  - If they are out of order, swap them
  - This puts the largest element at the very end
  - The last element is now in the correct and final place
- **Compare each element (except the last two) with its neighbor to the right**
  - If they are out of order, swap them
  - This puts the second largest element next to last
  - The last two elements are now in their correct and final places
- **Compare each element (except the last three) with its neighbor to the right**
  - Continue as above until you have no unsorted elements on the left

Example of bubble sort



Analysis of bubble sort

Claim: The time complexity T(n) of Bubble Sort is $O(n^2)$

- n operations in the first scan
- (n–1) operations in the second scan
- .
- .
- .
- (n–(i–1)) operations in i–th scan

- Adding the number of operations over all scans we have

$$n+(n-1)+(n-2) + \ldots + 1 = O(n^2)$$

Analysis of bubble sort

Lower bound:
In a list sorted from the largest to the smallest element we spend $n+(n-1)+(n-2) + \ldots + 1 = \Omega(n^2)$

So Bubble Sort has time complexity $\theta(n^2)$

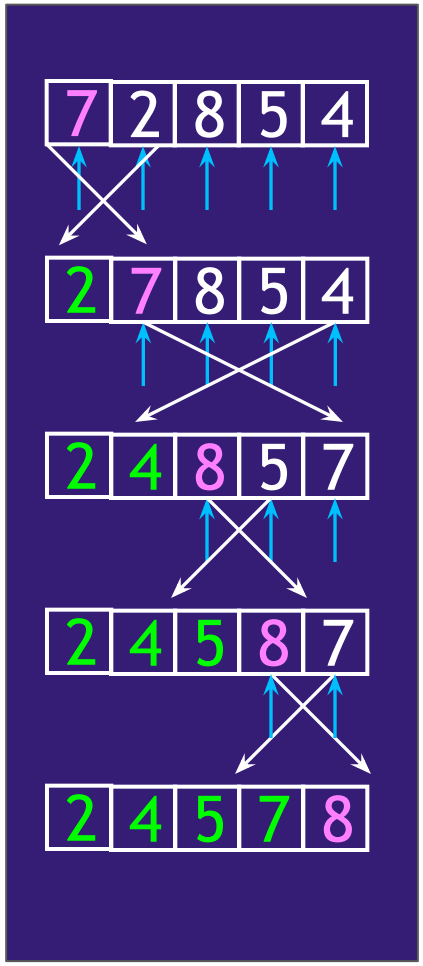
Obs: In the best case (sorted list) we just scan the list once: O(n) time

Selection sort

Given an array of length n,

- Search elements 1 through n and select the smallest
  - Swap it with the element in location 1
- Search elements 2 through n and select the smallest
  - Swap it with the element in location 2
- Search elements 3 through n and select the smallest
  - Swap it with the element in location 3
- Continue in this fashion until there's nothing left to search

Example and analysis of selection sort



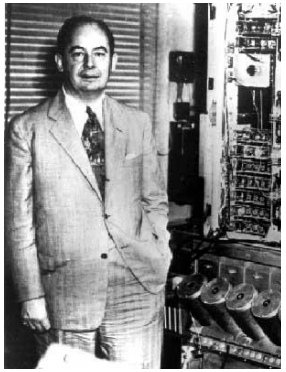
- The selection sort might swap an array element with itself--this is harmless, and not worth checking for
- Analysis:
  - The outer loop executes n-1 times
  - The inner loop executes about n/2 times on average (from n to 2 times)
  - Work done in the inner loop is constant (swap two array elements)
  - Time required is roughly (n-1)*(n/2)
  - You should recognize this as $O(n^2)$

2. Mergesort

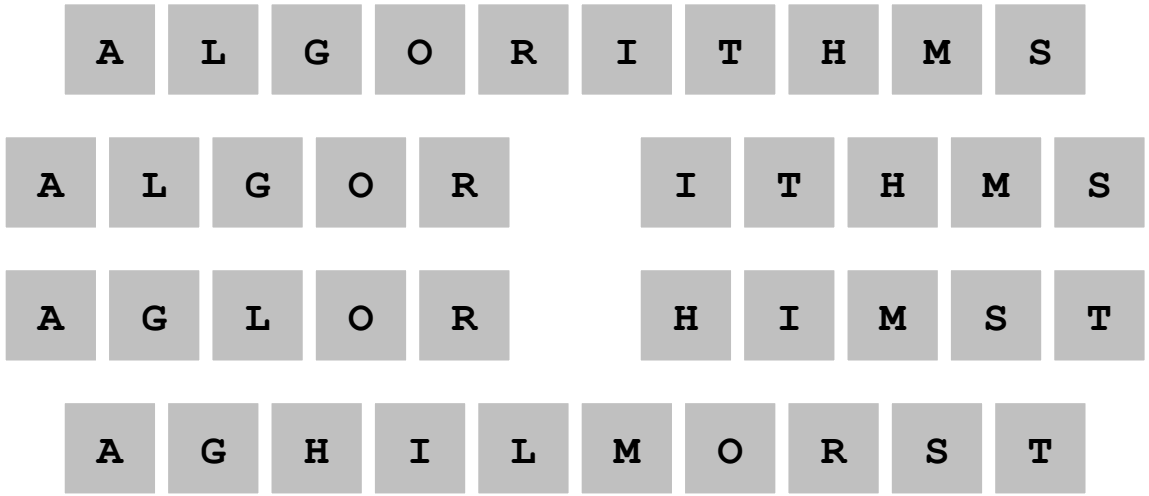
Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.



Jon von Neumann (1945)

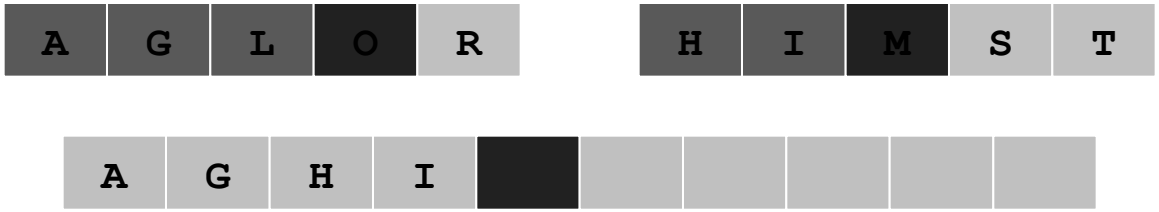


Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently?

- Linear number of comparisons.
- Use temporary array.



using only a constant amount of extra storage

Challenge for the bored. In-place merge. [Kronrud, 1969]

A Useful Recurrence Relation

Def. T(n) = number of comparisons to mergesort an input of size n.

Mergesort recurrence.

$$T(n) \le \begin{cases} 1 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

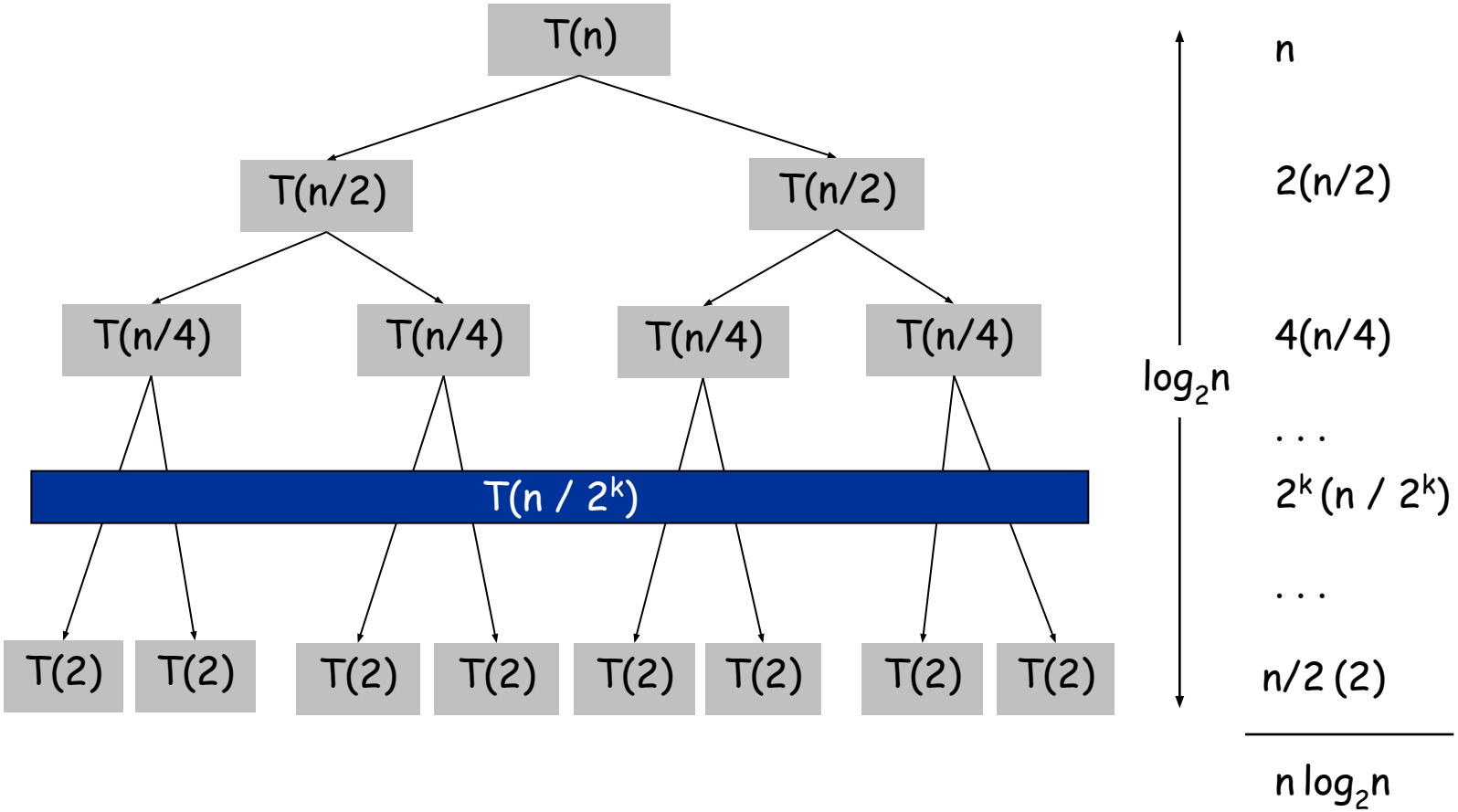
Solution. T(n) = $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence.

Initially we assume n is a power of 2 and replace ≤ with =.

Proof by Recursion Tree

$$T(n)=\begin{cases} 0 & \text{if } n=1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



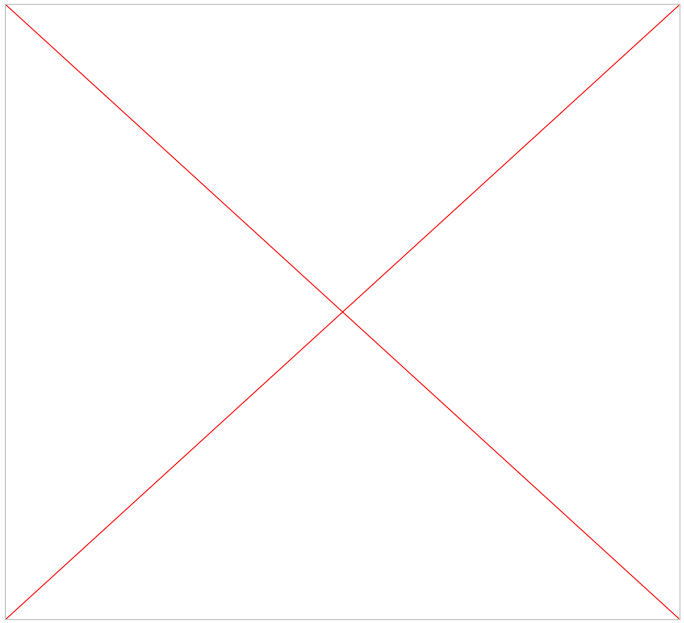
Proof by Telescoping

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

↑ assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. For n > 1:



Proof by Induction

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Inductive hypothesis: T(n) = n $\log_2$ n.
- Goal: show that T(2n) = 2n $\log_2$ (2n).

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= (2n/2)(\log_2(n/2)) + n \\ &= (2n/2)(\log_2(n) - 1) + n \\ &= n\log_2(n) \end{aligned}$$

Analysis of Mergesort Recurrence

Claim. If T(n) satisfies the following recurrence, then $T(n) \le n \lceil \lg n \rceil$.

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

$\log_2 n$

Pf. (by induction on n)

- Base case: n = 1.
- Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$.
- Induction step: assume true for 1, 2, ... , n–1.

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ &\le n_1 \lceil \lg n_1 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &\le n_1 \lceil \lg n_2 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &= n \lceil \lg n_2 \rceil + n \\ &\le n(\lceil \lg n \rceil - 1) + n \\ &= n \lceil \lg n \rceil \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \lceil 2^{\lceil \lg n \rceil} / 2 \rceil \\ &= 2^{\lceil \lg n \rceil} / 2 \\ \Rightarrow \lg n_2 &\le \lceil \lg n \rceil - 1 \end{aligned}$$

Mergesort

In conclusion, Mergesort take time O(n log n)

Interesting: Does not need to compare all pairs of items (that would be $\Omega(n^2)$)

2. Quicksort

Quicksort

- Sorts $\Theta(n \lg n)$ in the average case (we will not prove)
- Sorts $O(n^2)$ in the worst case

So why would people use it instead of Mergesort?

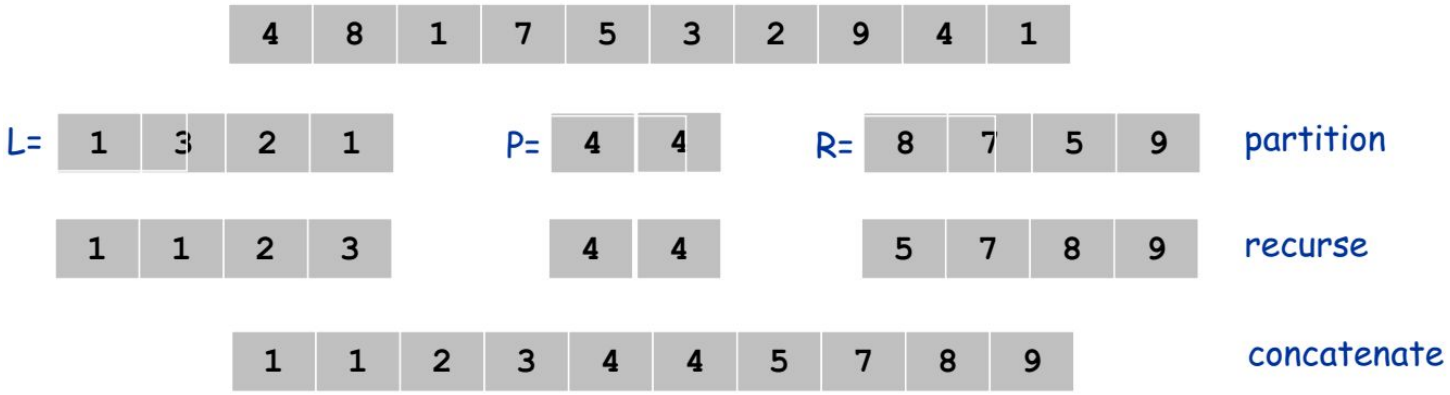
- Very simple to implement
- In practice is very fast (worst case is quite rare and constants are low)

Quicksort

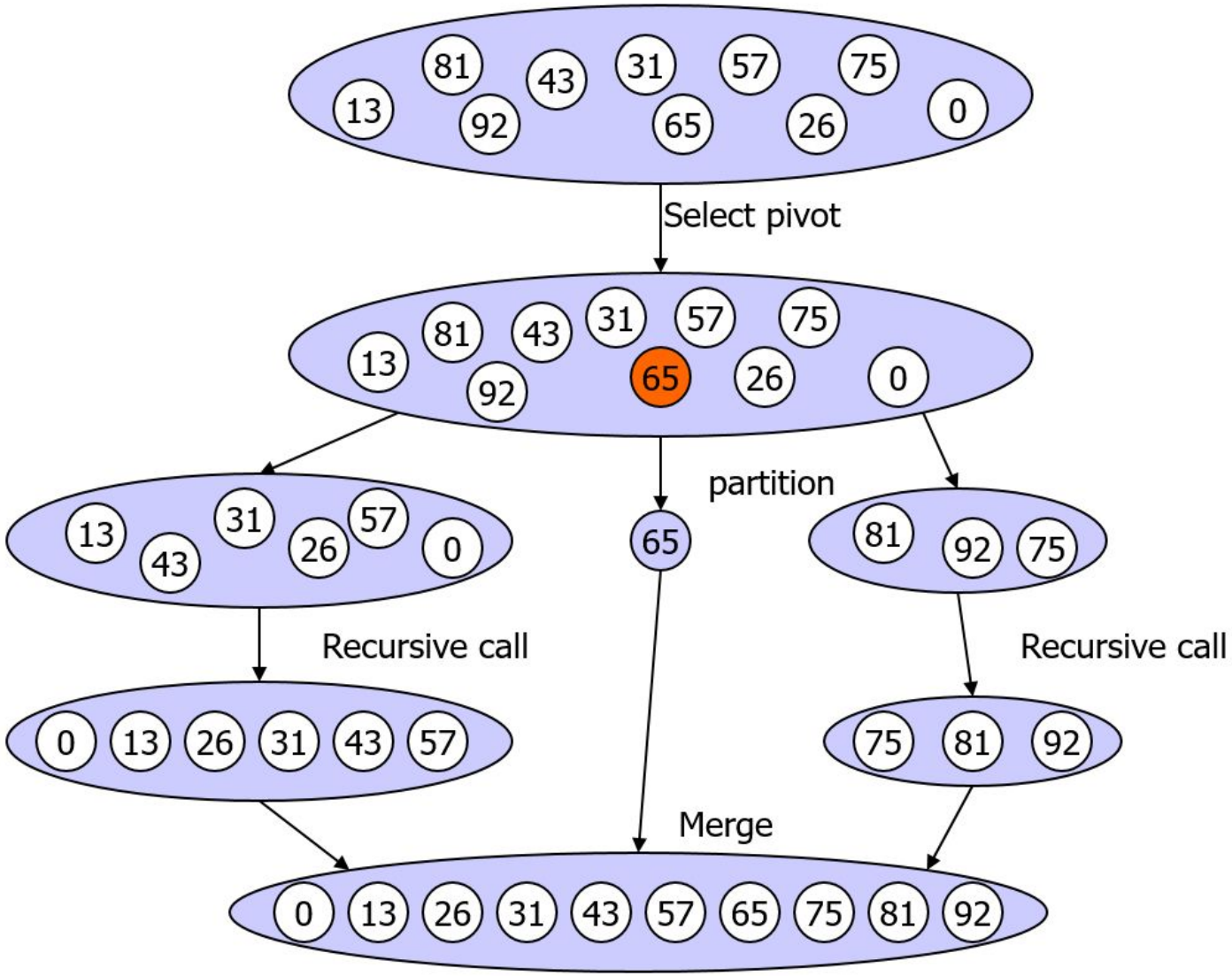
Input: list of numbers A[1], A[2], …, A[n]

Quicksort algorithm:

- Choose a number *pivot* in the list; lets say we always choose *pivot* = A[1]
- Partition the list A into the list L containing all numbers < *pivot* , the list R containing all numbers > *pivot*, and list P containing all numbers = *pivot*
- Sort the list L and R recursively using Quicksort
- Concatenate L, P, R (in this order)



Quicksort example



Quicksort

Input: list of numbers A[1], A[2], …, A[n]

Quicksort algorithm:

- Choose a number *pivot* in the list; lets say we always choose *pivot* = A[1]
- Partition the list A into the list L containing all numbers < *pivot* , the list R containing all numbers > *pivot* , and list P containing all numbers = *pivot*
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- Concatenate L, P, R (in this order)

```
Quicksort(A)
     if (A.len > 1)
          L, R, P = Partition(A);
          Quicksort(L); Quicksort(R);
      end if
     Return the concatenation of L,P,R
```

Partition

We can implement partition step in O(n) using a auxiliary vectors

Can even do *in place* without any auxiliary memory

Analyzing Quicksort

Claim: The time complexity of Quicksort is $O(n^2)$

Proof:

- The height of execution tree of QuickSort is at most n (lists L and R are strictly smaller than A; that's why we put pivot in P).

- At each level of the tree the algorithm spends O(n) due to the partition procedure

- So the algorithm spends at most cn^2 operations

Analyzing Quicksort

Second proof (by induction): We have the recurrence

$$T(0) = c$$
$$T(1) = c$$
$$T(n) <= \max_{i=0\ldots n-1} \{T(i) + T(n - i - 1) + cn\}$$

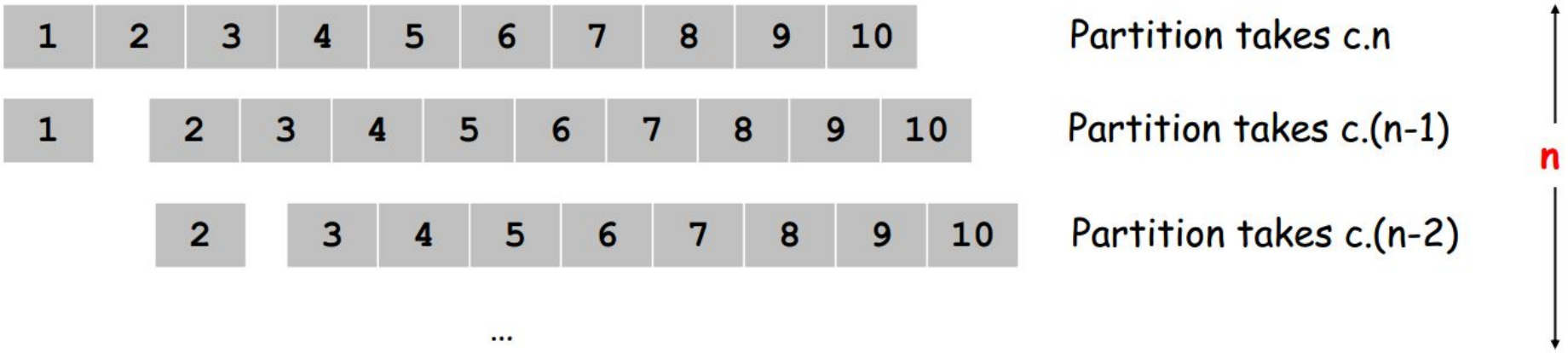
We do not know the size of the subarrays

Proof by induction that $T(n) <= cn^2$

- Base case n=0,1 is ok
- Let i* be the value of i that maximizes the expression. By induction $T(n) <= c(i^*)^2 + c(n-i^*-1)^2 + cn$
- The right-hand side is a parabola in i*. So it takes maximum value with i*=0 or i*=n-1. Plugging in these values we get the induction hypothesis

Analyzing Quicksort

Is this the best analysis analysis we can do?
Yes: If the input is sorted, the partition is always unbalanced: height n



Algorithm takes c.n + c.(n−1) + c.(n−2) + ... + c = c.n.(n+1)/2 = $\Omega(n^2)$
So Quicksort is $\Omega(n^2)$
So Quicksort is $\Theta(n^2)$

Analyzing Quicksort: Average Case

A(n): time complexity of Quicksort assuming instance is a random permutation

So partition generates splits (1:n-1, 2:n-2, 3:n-3, … , n-2:2, n-1:1) each with probability 1/n

So it is often balanced

$$A(n) = \frac{1}{n-1}\sum_{k=1}^{n-1}\left[A(k) + A(n-k)\right] + \Theta(n)$$

We can solve the recurrence to get that A(n) is O(n log n)

Improving Quicksort

The real liability of quicksort is that it runs in $O(n^2)$ on already– sorted input

Book discusses two solutions:

- Randomize the input array, OR
- Pick a random pivot element

How will these solve the problem?

- By ensuring that no particular input can be chosen to make quicksort run in $O(n^2)$ expected time

Exercício

Descreva um algoritmo com complexidade O(n log n) com a seguinte especificação

Entrada: Uma lista de *n* números reais

Saida: SIM se existem números repetidos na lista e NÃO caso contrário

Exercício – Solução

```
Ordene a lista lista L
Para i=1 até |L|–1
        Se L[i]=L[i+1] Devolva SIM
Fim Para
Devolva NÃO
```

- ANÁLISE
  - A ordenação da lista L requer O (n log n) utilizando o MergeSort
  - O loop **Para** requer tempo linear

Exercício

Descreva um algoritmo com complexidade O(n log n) com a seguinte especificação

Entrada: conjunto *S* de *n* números reais e um número real *x*

Saida: SIM se existem dois elementos distintos em *S* com soma x e NÃO caso contrário

Exercício – Solução

L <- conjunto S em ordem crescente
Enquanto a lista L tiver mais que um elemento **faça**
    Some o menor e o maior elemento de L
    Se a soma é x
        Devolva SIM
    Se a soma é maior que x
        Retire o maior elemento de L
    Se a soma é menor que x
        Retire o menor elemento de L
    Fim Enquanto
    Devolva NÃO

Exercício – Solução

L <- conjunto S em ordem crescente
Enquanto a lista L tiver mais que um elemento **faça**
    Some o menor e o maior elemento de L
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        Retire o maior elemento de L
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    Devolva NÃO

ANÁLISE

- A ordenação do conjunto S requer O (n log n) utilizando o MergeSort
- O loop enquanto requer tempo linear

3 How fast can we sort?

How Fast Can We Sort?

Can we sort a list of n numbers in constant time (independent of n)?
How can we prove that no algorithm can sort in constant time?

- This is very hard, research question. We don't know how to do it....

All of the sorting algorithms so far are *comparison sorts*

- The only operation used to gain ordering information about a sequence is the pairwise comparison of two elements

Easier question: How many comparisons are necessary for any comparison sort algorithm?

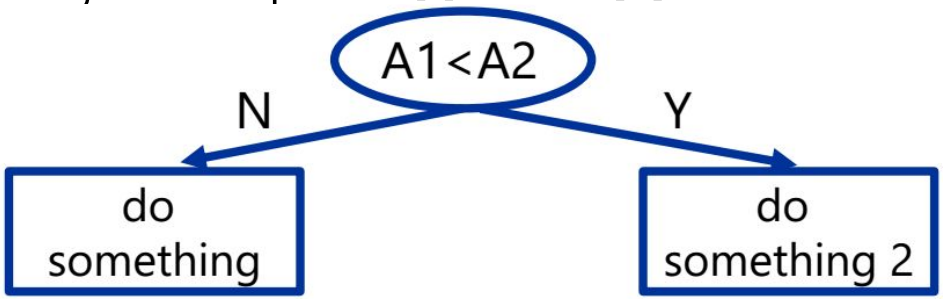
- Just counting comparisons, reordering of items is free

How Fast Can We Sort?

Ex: Can we sort 3 numbers with just 1 comparison?

A = | ? | ? | ?

Answer: No: Suppose just compare A[1] with A[2]



But consider A= | 2 | 3 | 4 | and A= | 2 | 3 | 1 |: both execute "do something 2"

- If "do something 2" moves A[3] then it makes a mistake in instance 2,3,4
- If "do something 2" does not move A[3] then it makes a mistake in instance 2,3,1

Need to make enough comparisons to find out order of the instance

How Fast Can We Sort?

We will show that no comparison algorithm can beat MergeSort

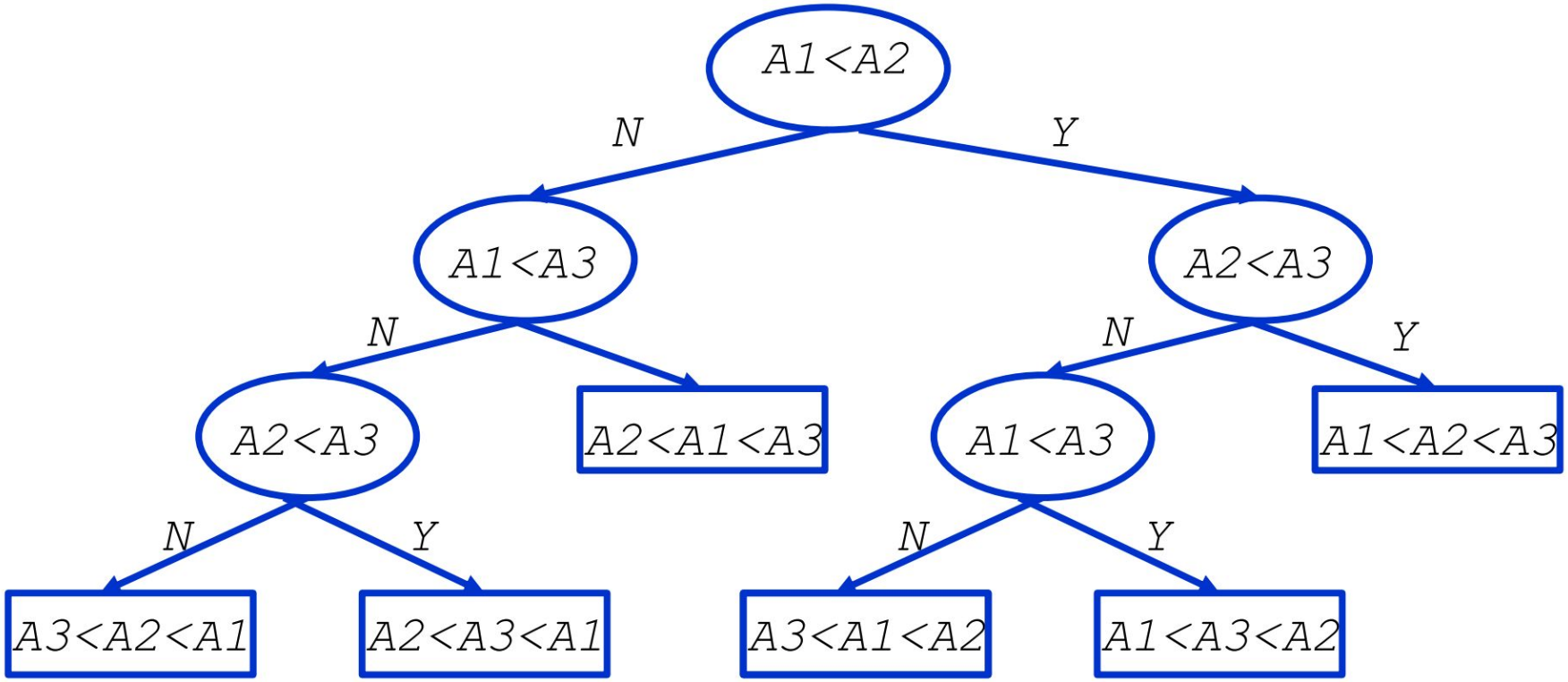
Theorem: Any comparison based sorting algorithm needs at least $\Omega(n \log n)$ comparisons

Decision Trees

- Decision trees provide an abstraction of comparison sorts
  - A decision tree represents the comparisons made by a comparison sort. Everything else is ignored
  - Each node corresponds to a comparison
  - Each branch corresponds to an outcome of the comparison
  - There is one leaf for each possible ordering of the numbers

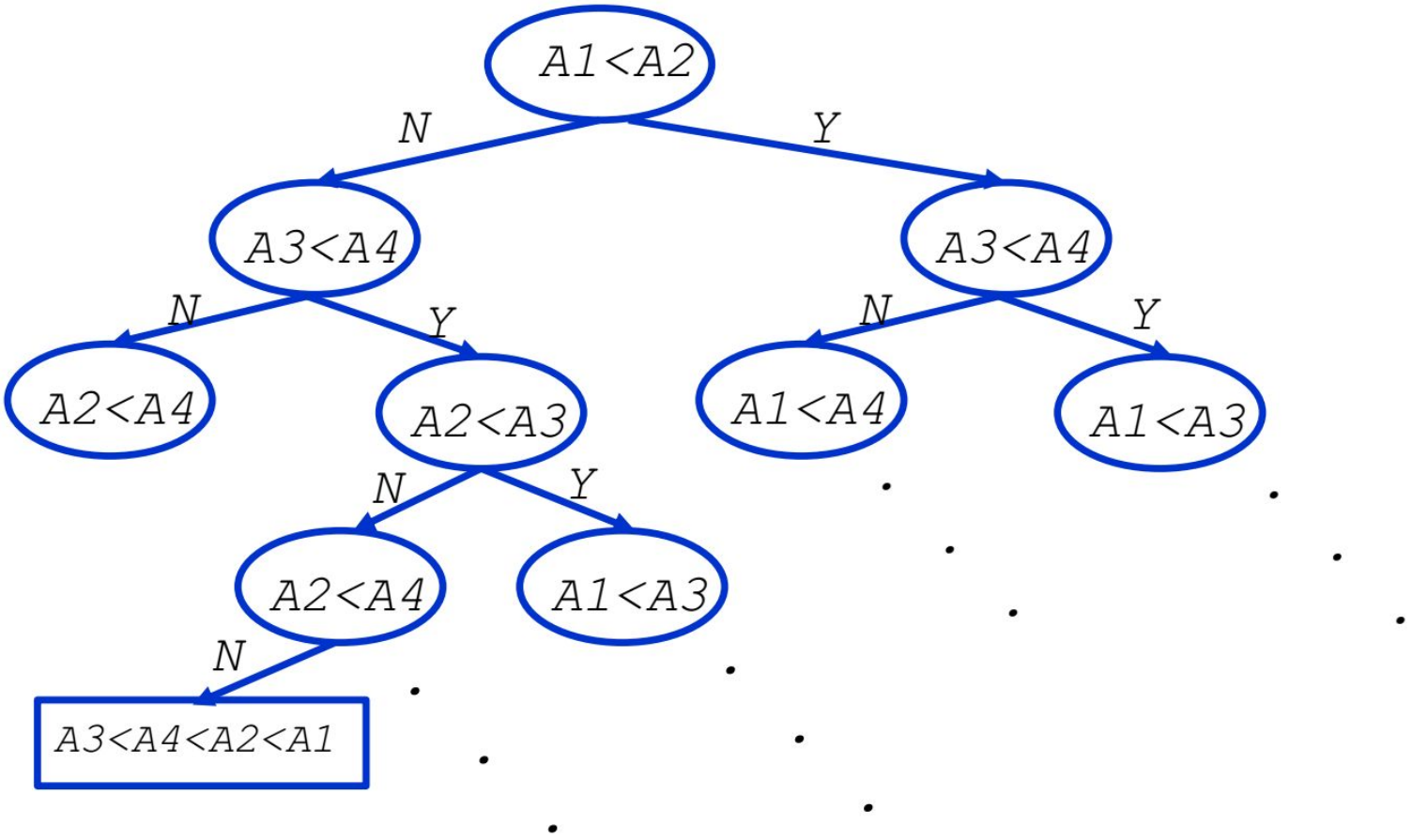
Decision tree for Bubble Sort

Bubble sort for 3 numbers A1, A2, A3



Decision tree for MergeSort

n=4



Decision Trees

- Decision trees provide an abstraction of comparison sorts

  Q: How many leaves does a decision tree have?

  Q: What does the height correspond to?

Decision Trees

- Decision trees provide an abstraction of comparison sorts

  Q: How many leaves does a decision tree have?

  A: n!

  Q: What does the height correspond to?

  A: Number of comparisons for worst case instance

What can we say about the height of any decision tree?

Lower bound for comparison sorting

Theorem: The height of any decision tree that sorts n numbers is at least $\Omega(n \log n)$

Proof:

- Any decision tree has n! leaves
- The height of a decision tree with n! leaves is at least log(n!) because the decision tree is a binary tree
- log(n!) ≥ (n/2) log n – (n/2):

log(n!) = log(n*(n–1)*...*2*1) = log n + log (n–1) + ... log 2 + log 1

≥ log n + log (n–1) + ... + log (n/2) >= (n/2) log (n/2)

= (n/2) log n – (n/2)

Lower bound for comparison sorting

Theorem: The height of any decision tree that sorts n numbers is at least $\Omega(n \log n)$

Corollary: No comparison sort can beat MergeSort asymptotically

5.4 Sorting in linear time

Sorting In linear time

- As notas do vestibular são números entre 1 e 100
- 10.000 alunos prestaram vestibular. Como ordenar os alunos conforme sua classificação?
- Versão reduzida para exemplo
  - 10 alunos com notas {1,2,2,3,4,1,2,2,4,3}

Sorting In linear time

We will see Counting sort

No comparisons between elements!

But...depends on assumption about the numbers being sorted

- We assume numbers are in the range $1..k$

Counting Sort

Input: List of integers A[1], A[2], ..., A[n] with values between 1 and k

Main idea: Compute vector C where C[i] is number of elements in the list at most i

How can we use this to sort?

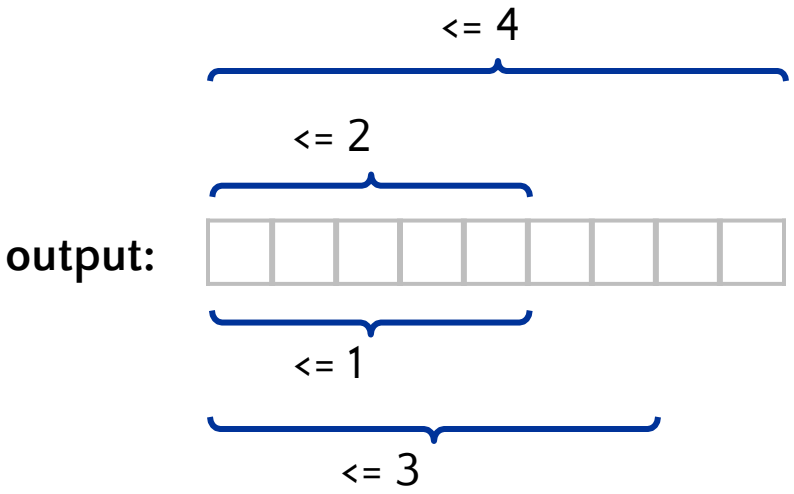
Ex 1: Consider distinct numbers A = [3, 1, 5, 10, 7]

- Counting vector is C = [1, 1, 2, 2, 3, 3, 4, 4, 4, 5]
- Put number “3” in position C[3], number “1” in position C[1], ...

Counting Sort

Ex 2: Consider non-distinct numbers A = [1,3,3,1,1,4,1,1,4]

- Counting vector is C = [5, 5, 7, 9]
- Put (say) second "3" in position C[3], put first "3" in position C[3] – 1,...



Counting Sort

```
1  CountingSort(A, B, k)
2       for i=1 to k
3             C[i]= 0;
4       for j=1 to n
5             C[A[j]] += 1;
6       for i=2 to k
7             C[i] = C[i]+C[i-1];
8       for j=n downto 1
9             B[C[A[j]]] = A[j];
10            C[A[j]] = C[A[j]]- 1;
```

Initialize

C[i] stores the number of elements equal

to i,i =1,…,k

C[i] stores the number of elements smaller or equal to i,i =1,…,k

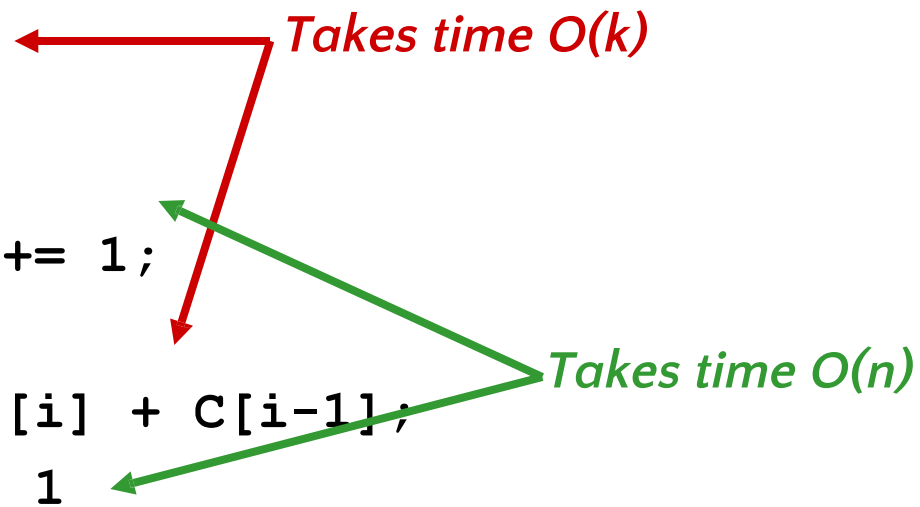
The position of element A[j] in B is equal to the number of integers that are at most A[j], which is C[A[j]]

Ex: A = {1,2,2,3,4,1,2,2,4,3}

Counting Sort

Q: What is the running time of Counting Sort?

```
1       CountingSort(A, B, k) for
2             i=1 to k
3                   C[i]= 0;
4             for j=1 to n
5                   C[A[j]] += 1;
6             for i=2 to k
7                   C[i] = C[i] + C[i-1];
8             for j=n downto 1
9                   B[C[A[j]]] = A[j];
10                  C[A[j]] -= 1;
```



Counting Sort

- Total time: $O(n + k)$
  - In many cases, $k = O(n)$; when this happens, Counting sort is $O(n)$
- But sorting is $\Omega(n \lg n)$!
  - No contradiction--this is not a comparison sort (in fact, there are *no* comparisons at all!)
- Notice that this algorithm is *stable*
  - If x and y are two identical numbers and x is before y in the input vector then x is also before y in the output vector (Important for Radix sort)
  - That is the only reason why the last "For" is in reverse order

Counting Sort

Cool! *Why don't we always use counting sort?*

Because it depends on range k of elements

Could we use counting sort to sort 32 bit integers? Why or why not?

Answer: no, k too large (2^{32} = 4,294,967,296)

Radix Sort

You have a list of numbers to sort

One idea is to sort them based on most significant digit, then the second m.s.d., etc.

Q: Does it work?

A: Not directly: consider input 91,19, 55, 54

- Sorting by most significant digit: 19, 55, 54, 91
- Then sorting by second most significant: 91, 54, 44, 91 **wrong order!**

Key idea: sort the least significant digit first

```
RadixSort(A, d) for i=1 to d
        Use a stable sorting algorithm to
        sort based on digit i
```

Radix Sort

Ex: A =

329	457	657	839	436	720

Step 1: third digit:

720	436	457	657	329	839

Step 2: second digit:

720	329	436	839	457	657

Step 3: first digit:

329	436	457	657	720	839

That is why we need a stable sorting algorithm

Radix Sort

- *Can we prove it will work?*

- Sketch of an inductive argument (induction on the number of passes):
  - Assume lower-order digits {j: j<i}are sorted
  - Show that sorting next digit i leaves array correctly sorted
    - If two digits at position i are different, ordering numbers by that digit is correct (lower-order digits irrelevant)
    - If they are the same, numbers are already sorted on the lower-order digits. Since we use a stable sort, the numbers stay in the right order

Radix Sort

What sort will we use to sort on digits?

Counting sort is obvious choice:

- Sort n numbers on digits that range from 0..9
- Time: O(n)

Each pass over n numbers with d digits takes time O(n), so total time O(dn)

Obs: Does not need to use digits (writing numbers in base 10), can use any other base

Radix Sort

In general, radix sort based on counting sort is:

- Fast
- Good worst-case guarantee O(nd)
- Simple to code
- A good choice

Radix Sort

Exercise: Suppose we want to sort a list of n numbers with values in {1,2, . . , n^2}. Show we can do this in time O(n).

Hint: Consider Radix Sort (with Counting Sort) but don't use digits to represent the numbers, use a different base (with more symbols).

A: Suppose numbers are represented in base k. Then Radix Sort with Counting Sort takes time

$$O((\log_k n^2)*(n + k)) = O((\log_k n) * (n + k))$$

Setting the base $k = n$, this becomes O(n).