Reduction of Collatz Conjecture through Modular Arithmetic

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1 The Conjecture

First proposed by Lothar Collatz in 1937, the Collatz conjecture presents a deceptively simple iterative process involving positive integers, yet its mathematical behavior remains enigmatic. The problem is defined as follows: for any positive integer n,

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

The conjecture states that iteratively applying this rule, i.e.,

$$C^{\circ n}(x),$$

where n represents the number of iterations, will eventually lead to a sequence that becomes periodic, specifically entering the loop

$$1 \to 4 \to 2 \to 1,$$

for all positive integers x. To clarify, $C^{\circ n}(x)$ denotes the composition of the function $C(x)$ applied n times to the initial value x. For instance,

$$C^{\circ 1}(x) = C(x), \quad C^{\circ 2}(x) = C(C(x)),$$

The significance of reaching the value 1 lies in its implications for the periodicity of the sequence. Once the sequence enters the value 1, it inevitably transitions through the cycle

$$1 \to 4 \to 2 \to 1,$$

confirming that the sequence does not diverge but instead stabilizes into a periodic loop. Establishing this periodicity for all positive integers is a key step in proving the conjecture. Due to the algorithmic nature of the conjecture, computational approaches are quite common. Computationally, the conjecture is true up to 2^{68}. However, due to the nature of the problem's domain, the natural numbers, rigorous mathematical proofs are required to ascertain if the conjecture holds. In 1970, mathematicians showed that almost all Collatz sequences lead to a number smaller than the original—a weak proof that the sequences converge. In 2023, however, Terence Tao used PDEs to make advances on the proof. Inspired by an idea on his blog to prove it for "almost all" numbers, he constructed a PDE system. This allowed him to draw conclusions along the lines of 99greater than 1 quadrillion eventually reaching a value below 200.

2 Methodology

My inspiration to solve this problem stems from the desire to reduce the sheer number required to prove the conjecture. What happens when you observe the way the unit digits change when the conjecture is applied to a number? Interestingly, irrespective of the unit digit with which you begin, you end up with a digit ending in 1. We demonstrate this below.

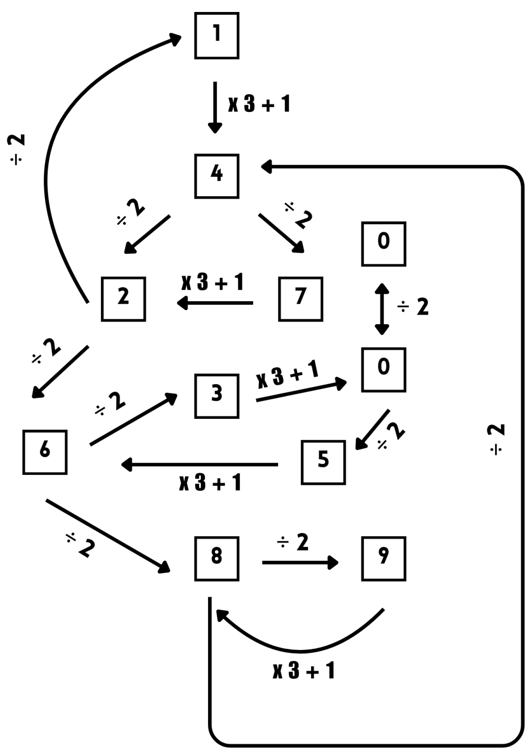

Figure 1: Flowchart demonstrating the behavior of unit digits in the Collatz conjecture.

3 Generalization

$$\forall n \in \mathbb{N}, \exists \omega \in \mathbb{N} \text{ such that } \omega \in [0, 9] \text{ and } \omega \equiv n \pmod{10}.$$

We define a self-sufficient cycle as follows: If $C^{\beta}(n)$ for some $\beta \in \mathbb{N}$ leads back to a number ending in ω before it leads back to 1, ω has a self-sufficient cycle. Note that we exclude powers of 2 (like 16, 8) from counting self-sufficient cycles, since Collatz is trivially proven for them.

Through the diagram, it is evident that three numbers have self-sufficient cycles: 1, 6, and 8. Therefore, if there has to be a number that does not fulfill the Collatz conjecture, it must get stuck in a self-sufficient cycle and loop around, not converging to 1. Thus, I claim if there exists a set of numbers that does not satisfy the conjecture, it contains a number ending in either 1, 6, or 8.

4 Computation

To ensure brevity, we have not included our code. However, this code and further computations can be found here.

We now try and map the average number of iterations it takes for each ω to reach a number ending in 1. Our findings are presented as follows:

Upper Bound of Range	Max Iteration Digit	Min Iteration Digit
10	6	2
100	3	4
1000 and above	3	2

Table 1: Iteration digit statistics for different ranges

This behavior is expected, as evident in the diagram:

3 lies in the middle of a self-sufficient cycle, while 4 has a direct path to 1 of the form:

$$4 \xrightarrow{\text{divided by 2}} 2 \xrightarrow{\text{divided by 2}} 1$$

We also try and map the number of iterations it takes for a given range of $\mathbb{N}$ to reach a 1.

Unlike the previous case, our findings are inconsistent here. They are presented as follows:

The graphs we obtain for a range up to 1 million are as follows:

Upper Bound of Range	**Max Iteration Digit**	**Min Iteration Digit**
10	9	1
100	1	0
1000	1	8
10000	4	9
100000	3	8
1000000	5	8

Table 2: Iteration digit statistics for different ranges

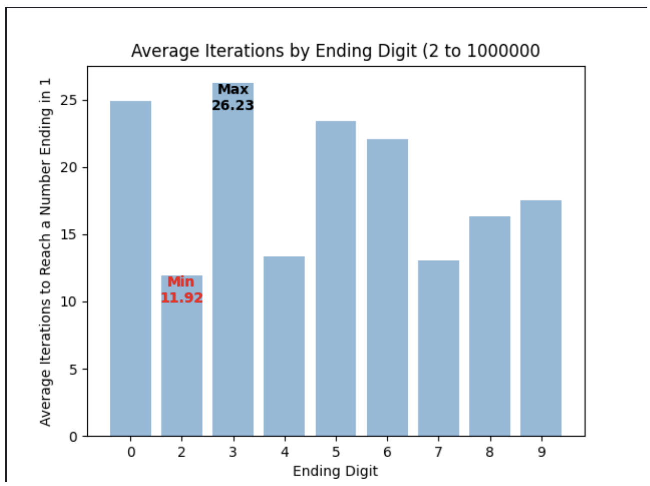

Figure 2: Average iterations for each last digit in the range 1–1 million to reach a number ending in 1 using the Collatz sequence.

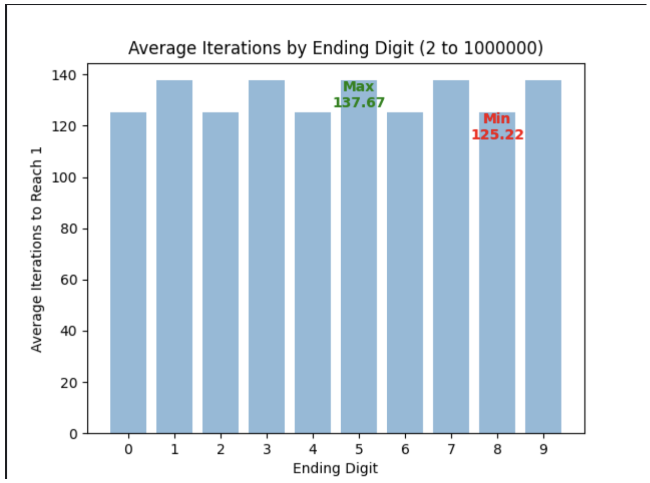

Figure 3: Average iterations for each last digit in the range 1–1 million to reach 1 using the Collatz sequence.

5 Conclusion

This paper is summarized by the following findings:

- Proving Collatz for all numbers ending in 1 is sufficient to prove Collatz for all natural numbers.
- If Collatz is untrue for a non-empty set of numbers, the set must contain a number ending in either 1, 6, or 8.
- Graphical exploration of unit digit variation further highlights the patterns within the Collatz conjecture.

6 References

1. Wikipedia contributors. "Collatz conjecture." *Wikipedia*, `https://en.wikipedia.org/wiki/Collatz_conjecture`, accessed December 10, 2024.

2. Klarreich, Erica. "Mathematician Proves Huge Result on Dangerous Problem." *Quanta Magazine*, `https://www.quantamagazine.org/mathematician-proves-huge-result-on` published December 11, 2019, accessed December 10, 2024.