Mathematics for Computers

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Unit 1 Relevance of Mathematics

1.0 Objectives

By the end of this Unit, learners should be able to understand:

- Importance of mathematics
- Need of studying different topics from mathematics
- Role of mathematics in computer science

1.1Introduction

Mathematics is often defined as the study of quantity, pattern, change and space. Some people call mathematics the study of "figures and numbers", but this is an oversimplification. It is the investigation of axiomatically defined abstract structures using logic and mathematical notation. In fact, it is the investigation of objects or concepts that exist independently of our reasoning about them. Due to its applicability in practically every scientific discipline, mathematics has been called "the language of science",

1.2 Purpose of the course

A mathematics course has more than one purpose. The main purpose of this course is to learn basic concepts from mathematics that are needed in the study of computer science. This course is a prerequisite for many of the computer science courses. The course is to be studied from a computer science viewpoint.

There is no specific knowledge required other than some elementary topics from school level mathematics. Almost all topics and techniques introduced here will be needed while studying computer science. This course will also help the students to develop their general ability to think abstractly and their problem solving skills also.

1.3 Some facts about Computers

A computer is a machine for manipulating data according to a list of instructions known as a program. Originally, the term "computer" referred to a person who performed numerical calculations, often with the aid of a calculating device.

Although the electronic computer is of recent origin (some 50 years), the idea of automating the process of computation was born long back, probably, when bookkeeping, accounting and astronomy became tedious.

The first actual calculating mechanism known to us is the abacus. The origins of the abacus are disputed, as many different cultures have been known to have similar tools. But these early computing machines were definitely used for numerical calculations.

Computer science has a much closer relationship with mathematics than many scientific disciplines. Early computer science was strongly influenced by the work of mathematicians such as Kurt Godel and Alan Turing.

Charles Babbage (1791 AD – 1871 AD) is many times referred as the "Father of computer ". He was an English mathematician, analytical philosopher, mechanical engineer who put forth the idea of a programmable computer.

Babbage's machine could be programmed to follow a series of steps, where each step could be a combination of four basic operations addition, subtraction, multiplication and division. But more important was the fact that the machine had decision-making capability. It could change the order of calculation depending on the value of a certain quantity, which it had computed. This first computer was also mainly useful for mathematical calculations.

1.4 Role of mathematics in computer science

Study of different branches of mathematics helps in studying computer science. In analysis and design of algorithms, Graph Theory is used more often than any other branch of mathematics. A weighted directed graph, in which vertex represents program block and each edge represents possible transfer of control from one block to another, is useful representation of a computer program. It is also used for Time analysis, segmentation of program and in detecting common type of errors

Cryptography is considered as a discipline of algebra concerned with information security, and related issues, particularly encryption, authentication and access control. Its purpose is to hide the meaning of a message rather than its existence. In modern times, it has also branched out into computer science.

1.5 The course outline

The mathematics course starts with fundamental concepts like Set Theory, Mathematical Induction, Number systems, Exponents, Logarithms, Permutations and combinations.
Then it covers the topics that are useful in computer science such as Logic, Relations, Functions and Graph Theory.
It also covers the topics like Vectors, Matrices and Determinants, Mensuration,

Linear Equations and Polynomials that will be helpful to the students.

Unit 2 Set Theory

2.0 Objectives

By the end of this Unit, learners should be able to:

- Understand concept of a set.
- Describe different types of sets.
- Perform various operations on sets.
- Explain different properties of sets.
- Represent sets using Venn diagrams.

2.1Introduction

Set Theory is the mathematical science in which properties of sets are studied. Sets are basic abstract objects that are used in the study of logic, discrete mathematics and computer science etc. Here we will discuss the basics of set, types of sets and different operations on sets.

2.2 Set notations

2.2.1 Set

A set is a well-defined collection of objects. The objects of a set are called elements or members.

The elements of a set can be anything. A set can be a collection of numbers, collection of names of persons, collection of letters of the alphabet or even collection of other sets. Sets are conventionally denoted with capital letters A, B, C, etc. The elements are generally denoted with lower case letters x, y, z etc. If x is a member of a set A, then symbolically it is denoted as x©A. If x is not a member of a set A then symbolically it is denoted as x ª A.

Various methods are used to indicate members of a set. The two methods that are more commonly used to represent sets are the Set listing method and the Set builder method.

1. <u>Listing Method / Roaster Method:</u>

In set listing method the elements of a set are explicitly listed within braces or curly brackets. If set contains only a small number of elements or if there is some particular pattern in the list of elements then this method is useful.

The order in which the elements of a set are listed is not important. And also the repetitions in the list do not change the contents of a set.

For example, {2,4,6, 8} = {6,2,8, 4} = {8,6, 4, 2} = {2,2,4,6,6,8}. For the sets with too many elements, more often an abbreviated list is used. This list is ended with three dots "…", which indicates that the list continues in the obvious way.

Examples:

‹ A = {1, 3, 5,7, 9}. Here 1, 3, 5,7 and 9 are the members of the set A. So we can write 1©A, 3©A, 5©A, 7©A and 9©A; but 2 ª A, 4ª A or 8ª A etc.

‹ B = {BASIC, C, C++, PASCAL}
In this case BASIC, C, C++ and PASCAL are the members of set B. So we can write BASIC © A, C © A, C++ © A, and PASCAL © A whereas COBOL ª A or FORTRAN ª A etc.

‹ The set of the first one hundred natural numbers can be described using listing method as A = {1, 2, 3, ..., 100}.

‹ Similarly the set of all odd natural numbers can be written using listing method as {1, 3, 5, 7, ...}.

Note that if we cannot observe any particular pattern in the list of members and we cannot guess at any stage what the next number in the list might be, then for such sets listing method is not useful.

2. <u>Set builder Method:</u>

More complicated sets are described by a different way. If a set contains a large number of elements and the members of the set follow some obvious pattern, then the set builder method is used to represent such set. In this method a set is described by indicating the properties that its members must satisfy.

The generalized form set-builder notation is {*x* : *p*(*x*)}. It denotes the set of every object *x* which satisfy the property p(x). In this description, the colon "**:**" means "such that" (Sometimes the notation "|" is used instead of the colon.).

Examples:

‹ A = {x : x is a natural number less than 101} . We read this as the set A is a set of all x, such that x is a natural number less than 101. This set A contains hundred numbers, which are all natural numbers from 1 to 100. This set can also be represented using listing method as A = {1, 2, 3, ..., 100}.

‹ B = {*x*: *x* is a natural number and 2 ®˙x^2 ®˙ $d'˙˙K Y˙fYUX˙h\]g˙Ug˙hY˙gYh6˙]g˙U˙gYhcZ all x, such that x is a natural number whose square is between 2 to 30. This set A can be written using set listing method as B = {2, 3, 4, 5 }.

2.3 Self-Test 1

Select the correct alternative from the given alternatives.

1. Which of the following represents the set A = {11, 13, 15, 17, 19}?
(a) A = { x : x is a natural number greater than 11}
(b) A = { x : x is an odd natural number greater than 11}
(c) A = { x : x is an odd natural number between 10 to 20 }
(d) A = { x : x is a natural number less than 20}

2. Which of the following represents the statement "The number 5 is not a member of the set A"?
(a) 5©A **(b)** 5ª A **(c)** A ª 5 **(d)** A © 5

3. Which of the following represents the statement "The number 10 is a member of the set B"?
(a) 10©B **(b)** 10ª B **(c)** B ª 10 **(d)** B © 10

4. Which of the following represents the set A = {1, 4, 9, 16, 25}?
(a) A = {x : x is a square of natural number and less than 30}
(b) A = { x : x is an odd natural number less than 30 }
(c) A = {x : x is an odd natural number between 1 to 30 }
(d) A = {x : x is a natural number less than 30}

5. Which of the following represents the set A = {a, e, i, o, u}?
(a) A = { x : x is alphabet }
(b) A = { x : x is English alphabet }
(c) A = { x : x is an English alphabet and a vowel}
(d) A = { x : x is an English alphabet and a consonant } .

6. Which of the following represents the set A = {4, 5, 6, 7, 8, 9 }?
(a) A = { x : x is a number less than 10}
(b) A = { x : x is an integer greater than 3}
(c) A = { x : x is an odd integer between 3 to 10 }
(d) A = { x : x is an integer and 3 O x O 10} .

7. Which of the following represents the set ,B = { x : x is an integer and 3x = 6 } ?
(a) B = { } **(b)** B = { 2 } **(c)** B = { 3 } **(d)** B = { 6 }

8. Which of the following represents the set, B = { x : x is an integer, $x^2 + 1 = 10$} ?
(a) B = { 3} **(b)** B = { – 3 , 3} **(c)** B = { } **(d)** B = {– 3 ,..., 3}

9. Which option represents the set B = { x : x is a vowel and x is not a or i } ?
(a) B = { a, e, i, o, u} **(b)** B = { a, i }
(c) B = { e, i, o, u} **(d)** B = { e, o, u} .

2.4 Types of sets

2.4.1 Empty Set or Null set

The empty set is a set that contains no elements. Usually empty set is represented using the symbol Z. Some times it is also represented as { }. But note that the set {Z} is not an empty set.

Examples:

‹ A is the set of natural numbers whose squares are negative. This set A is an empty set because there is no natural number whose square is negative.

‹ B = { x : x is a prime integer whose square is one}. This set B = Z, because there is no prime integer whose square is one.

2.4.2 Singleton Set

A singleton set is a set that contains only one element.

Examples:

‹ If A is the set of all natural numbers whose square is 100, then set A = { 10}. A is a singleton set because there is only one natural number whose square is 100.

‹ B = {x : x is an even prime integer} . This set in listing form can be written as B = {2}. B is a singleton set because there is only one prime number which is even.

2.4.3 Subset and Superset

If every member of the set A is also a member of the set B, then set A is said to be a subset of set B. It is written as A § B. It is also pronounced as "*A is contained in B*".

Equivalently, in this case we can write B ¤ A, and read it as "*B is a superset of A*", or "*B includes A*", or "*B contains A*".

Empty set is a subset of every set.

If *A* is a subset of B but not equal to *B*, then *A* is called a proper subset of *B*, it is written as A § B. In this case *B is a proper superset of A and* it can also be written as B ¤ A.

Examples:

‹ The set of all women is a proper subset of the set of all people.

‹ {2, 5 ,7} § {1,2, 5, 6, 7}

‹ {a, b, c, d} ¨ {a, b, c, d}. In fact every set is a subset as well as superset of itself.

‹ {black, yellow, blue, red} ¤ {red, yellow}

For any set A empty set is a subset of A and the set A itself is also a subset of A. These two subsets are called improper subsets of set A and all remaining subsets are called proper subsets of set A.

Example: *If A* = {a, b, c} then, the improper subsets of *A* are Z (the empty set) and A= {a, b, c} whereas the proper subsets of *A* are {a}, { b}, {c}, {a, b}, {a, c} and {b, c}.

2.4.4 Finite set and infinite set

A set is said to be a finite set if it contains a finite number of elements, all other sets are called infinite sets.

Examples:

‹ *A* = {*x*: *x* is a natural number and 2 ® x^2 ®' $q"+h]g U Z]b]hY gYh VYWUi gY]h Wcb hU]bg finite number of members which are 2 ,3,4 and 5.

‹ B = {a, b, c, d, e, f}. It is also a finite set because it contains 6 members only.

‹ C = {x: x is a prime integer}. This is an infinite set because the number of prime integers is not finite. Using set listing form this set can be written as C = {2, 3, 5, 7, 11, ...}

Some standard infinite sets of numbers, which are commonly used, are listed below:

‹ Set of all natural numbers, N = {1, 2, 3, ...}

‹ Set of all integers, Z = {... –3, –2, – 1, 0, 1, 2, 3, ...}

‹ Set of all rational numbers, Q = {p / q : p , q © Z, q ' 0 }

‹ Set of all real numbers, R = {x : – wu x u w}

2.4.5 Universal Set

A set that contains all the elements in the universe of discourse is called a universal set. It is generally denoted by *U*. Normally anything under consideration is part of the Universal Set. So any thing other than the universal set is an empty set.

Examples:

- If the sets involved in discussion are of numbers then the universal set can be R, the set of all real numbers.
- If we are discussing about people in different states of India, then the universal set is the set of all people in India.

2.4.6 Power set

*The set of all subsets of a set **A** is called the power set of **A** and is denoted by P(A) .*

If a set A contains n number of elements then its power set P(A) contains 2^n number of elements.

Examples:

- If A = {2,3} then all possible subsets of A are Z (the empty set), {2}, {3} and {2,3}. Therefore the power set of *A* is P(A)= {Z, {2}, {3}, {2, 3} }. We observe that set A contains 2 elements and P(A) contains 4 = 2^2 elements.
- If *B* = {a, b, c} then all subsets of *B* are as follows: Z, {a}, {b}, {c}, {a, b}, {a, c}, {b, c} and {a, b, c} . Therefore the power set of *B* is P(B)= {Z, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }. As the set B contains 3 elements, its power set contains 8 = 2^3 elements.

2.5 Self-Test 2

Select the correct alternative from the given alternatives.

1. Which of the following is a null (or empty) set?
(a) { x : x is a natural number and $x^2 + 1 = 10$}
(b) { x : x is a natural number and $x^2 = 121$}
(c) { x : x is a natural number and $x^2 = -10$ }
(d) { x : x is a natural number and x^2 u 100 }

2. Which of the following is a singleton set?
(a) { x : x is an integer and $x^2 = 16$} **(b)** { x : x is an integer and $x^2 - 1 = 120$}
(c) { x : x is a an integer and $x^2 = x$ } **(d)** { x : x is a an integer and 4x = 8}

3. If A = {1, 2, 3, 4, 5} and B = {2, 4, 5 }, then which of the following holds?
(a) A § B **(b)** B § A **(c)** A = B **(d)** A = B^c

4. Which of the following sets is a finite set?
(a) { x : x is an integer and x^2 2 0}
(b) A = {x :x is a prime number greater than 10}
(c) { x : x is a an integer and $x^2 = x$ }
(d) A = { x : x is an integer less than 20}

5. Which of the following can be an Universal set, for the sets
A = {1, 4, 9, 16,25}, B = {3, 7, 20} and C= {15, 20, 25} ?
(a) U = { x : x is a square of natural number less than 30}
(b) U = { x : x is an odd natural number less than 30 }
(c) U = { x : x is an odd natural number between 1 to 30 }

(d) U = { x : x is a natural number less than 30} .

6. Which of the following is true for the standard sets?
(a) R § Z **(b)** Z § Q **(c)** Z § N **(d)** Q § N

7. If we are dealing with the set of all computer programmers in the world, then which of the following can be an Universal set ?
(a) set of all men in the world **(b)** set of all women in the world
(c) set of all people in the world **(d)** set of all Indians in the world.

8. If A = {4, 5, 6, 7,9 }, then power set of A contains how many elements?
(a) 4 **(b)** 25 **(c)** 5 **(d)** 32

9. If A = {1, 3, 9 }, then which of the following is power set of A ?
(a) {1, 3}, {3, 9}, , {3, 1}, {9, 3}, {1, 9} }
(b) { { } , { 1}, { 3 }, { 9 }, {1, 3}, {1, 9}, {3, 9}, {1, 3, 9} }
(c) { Z, {1, 4}, {1, 9},{3, 7} , A }
(d) B = { {1, 3}, {3, 9} }.

10. Which of the following represents the power set of N?
(a) { {1, 1}, {1, 2}, {1, 3} ,...} **(b)** { A : A is a subset of N }
(c) { x : – wu x u w} **(d)** { {x } : x is a member of N }.

2.6 Set Operations

In arithmetic we study different operations of two numbers such as addition, multiplication, division etc. Similarly we define different operations on sets, which are union, intersection, subtraction Cartesian products etc.

2.6.1 Equality of sets: *Two sets A and B are said to be equal, if they contain the same elements*. It is written as A = B. Any two sets ***A*** and ***B***, are equal if and only if all members **x** of set A are such that **x** is also a member of set B. So if A ¨ B and B ¨ A, then A= B.

Examples:

‹ {1, 2, 3} = {3, 2, 1},

‹ If *A* = {*x***:** *x* is a natural number and 2 ®x^2 ®' $qUbX 6 1 o2,3, 4, 5 } then we observe that every member of set A is also a member of set B and every member of set B is also a member of set A. Therefore A = B.

2.6.2 Union of sets: *If A and B are any two sets, then the union of sets A and B, is the set of all the elements which are either from set A or from set B or from both sets.* It is denoted by *A* £ *B*.
Symbolically this set is defined as A £ *B* = { *x* | x©A or x©B }

Examples:

‹ If *A* = {1, 2, 3} and *B* = {1, 2, 4, 5}, then *A* £ *B* = {1, 2, 3, 4,5}.

‹ If *A* = {1, 2, 3} and *B* = {4, 5}, then A £ *B* = {1, 2, 3, 4,5}.

2.6.3 Intersection of sets: *If A and B are any two sets, then the intersection of sets A and B, is the set of all elements which are in both A and B (i.e. those elements which are common to both sets).* It is denoted by *A ¢ B.*
Symbolically this set is defined as *A ¢ B* = { *x* | x © A and x © B } .

Examples:

- If *A* = {1, 2, 3, 7 } and *B* = {1, 2, 4, 5 } ,then *A ¢ B* = {1, 2 } .
- If *A* = {1, 2, 3} and *B* = {4, 5 } , then *A ¢ B* = Z .
- If *A* = {1, 2, 3 } and *B* = {1, 2, 3 } ,then *A ¢ B* = {1, 2, 3 } .

If two sets A and B have no common elements then *A ¢ B* = Z, such sets are called mutually disjoint sets.

2.6.4 Difference of sets: *If A and B are any two sets, then the difference of set A from set B, is a set of all those elements of A which are not elements of B.* It is denoted by *A – B* (or *A* / *B*). Symbolically this set is defined as *A – B* = { *x* | x © A and x ª B} .

And similarly *B – A* = { *x* | x © B and x ª A} .Obviously *A – B* ' *B – A*.

Examples:

- If *A* = {1, 2, 3,7,9 } and *B* = {1, 2, 3,4, 5} ,then A – *B* = {7,9 } and *B – A* = {4, 5 } .
- If *A* = {1, 2, 3,…9 } and *B* = {1, 3, 5, 7, 9} ,then A – *B* = {2, 4, 6,8 } and *B – A* = Z .

2.6.5 Complement of a set: *If A is any set for which U is the universal set, then the Complement of the set A is the set, which contains those elements of U, which are not elements of A.* It is denoted by A^C *or A'*.

Symbolically this set is defined as A^C = *A'* ={ *x* | x © U and x ª A}. Obviously A^C = *U – A*.

Examples:

- If *U* = {1, 2, 3, …, 10 } and *A* = {1, 2, 3,4, 5 } ,then A^C = {6, 7, 8, 9, 10 } .
- If U: set of all integers and B: set of all even integers then B^C is the set of all odd integers.

For every set A, there are no common elements in sets A and A^C, i.e. *A ¢* A^C = Z. So every set and its compliment are disjoint sets.

2.6.6 Cartesian product: *If A and B are any two sets, then the set of all ordered pairs (a, b), where a is an element of A and b is an element of B, is called the Cartesian product of A and B. It is denoted by A X B.*

Cartesian product of two set A X B is formally defined as A X B= {(a, b)| (a ©A and b©B }
Similarly the Cartesian product of set B with set A is *B* X *A* = { (b, a) | b © B and a © A}.
Obviously *A* X *B* ' *B* X *A*

Examples:

- If A = {1, 2} and B = {a, b, c }, then
  A X B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c) } and
  B X A = { (a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2) }

‹ If A ={x, y, z} , then

A X A = { (x, x), (x, y), (x, z), (y, x), (y, y), (y, z), (z, x), (z, y), (z, z)}

2.7 Self-Test 3

Select the correct alternative from the given alternatives.

1. If A = {2, 3, 5,7,11, 13 }, then which of the following B is such that, A = B ?
(a) B = { x : x is a natural number between 1 and 13}
(b) B = { x : x is a natural number less than 13}
(c) B = { x : x is a prime number less than 15 }
(d) B = { x : x is a natural number greater than 1 }

2. If A = {1, 2, 3, 4, 5 }, B = { 1,3,5,7,9}, then which of the following is AUB ?
(a) {1, 2, 3, 4, 5, 7, 9} **(b)** {1, 2, 3, 4, 5} **(c)** {1, 3, 5, 7, 9} **(d)** {1, 3, 5}

3. If A = { x : x is an even natural number between 1 to 11 } and
B = { x : x is a prime number less than 15 }, then which of the following is AUB ?
(a) {1, 2, 3, 4, 5,7, 9} **(b)** {2, 4, 6,8, 10}
(c) {2, 3, 4, 5, 6, 7,8, 10, 11, 13} **(d)** {2, 3, 5,7,11, 13}.

4. If A = {1, 2, 3, 4, 5}, B = { 1,3,5,7,9}, then which of the following is A¢B ?
(a) {1, 2, 3, 4, 5,7, 9} **(b)** {1, 2, 3, 4, 5} **(c)** {1,3,5,7,9} **(d)** {1, 3, 5}

5. If A = { x : x is an odd natural number between 1 to 11 both inclusive} and
B = { x : x is a prime number less than 15 }, then which of the following is A¢B ?
(a) {2, 3, 5,7,11, 13} **(b)** {1, 2, 3, ..., 11} **(c)** {1, 3, 5,7,9,11} **(d)** {3, 5,7, 11}

6. If A = {1, 2, 3, ..., 10}, B = { 1,4,9, 16, 25}, then which of the following is A – B ?
(a) {1, 2, 3, 5,6,7,8,10} **(b)** {1, 2, 3, 5, 7, 16} **(c)** {1, 3,5,7,9} **(d)** {16, 25}

7. If A = { x : x is a natural number between 5 to 15 both inclusive} and
B ={x : x is a prime number less than 18}, then which of the following is B – A?
(a) {6, 8, 9, 10, 12, 14, 15, 16} **(b)** {5, 6, 7, ...,15}
(c) {1, 3, 5,7,9, 11} **(d)** {2, 3, 17}

8. If universal set is the set of all people in the world and A is the set of all Indians in the world, then which of the following is A^c?
(a) Set of all Americans in the world
(b) Set of all persons in the world who are not Indians
(c) Set of all people in the world
(d) Set of all Non Resident Indians in the world.

9. If A = {1, 4, 9 }, B = { 3, 7}, then which of the following is AXB ?
(a) {(3, 1), (3, 4), (3, 9), (7, 1), (7, 4), (7, 9)}
(b) {(1, 3), (1, 7), (4, 3), (4, 7) (9, 3), (9, 7)}
(c) {(1, 4), (1, 9),(3, 7) }
(d) B = { (1, 3), (4, 7) }.

10. Which of the following represents the Cartesian product R x R?

(a) {(1, 1), (1, 2), (1, 3), ...} **(b)** { (x, x) : x © R }

(c) {x : – w u x u w} **(d)** { (x, y) : x © R and y © R }.

2.8 Properties of set operations:

The operations on sets satisfy many algebraic properties. All of these properties can be proved using the definitions of the operations. These can also be proved using the diagrammatic representations of sets, which are Venn diagrams. We will study these Venn diagrams later.

All above defined operations on sets satisfy the following properties:

1. <u>Commutative properties:</u>

   For all sets A and B, we have

   (i) Union of sets is a commutative operation, i.e. A £ *B* = B £ *A*.

   (ii) Intersection of sets is a commutative operation, i.e. *A* ¢ *B* = *B* ¢ *A*.

2. <u>Associative properties:</u>

   For all sets A, B and C, we have

   (i) Union of sets is an associative operation, i.e. (A £ *B)* U C= A £ (*B* U C).

   (ii) Intersection of sets is an associative operation, i.e. *(A ¢ B) ¢ C = A ¢ (B ¢ C).*

3. <u>Distributive properties:</u>

   For all sets A, B and C, we have

   (i) Union of sets is a distributive operation over intersection of sets

   i.e. A £ (*B* ¢ C) = (A £ *B)* ¢ (A U C).

   (ii) Intersection of sets is a distributive operation over union of sets

   i.e. A ¢ (*B* £ C) = (A ¢ *B)* £ (A ¢ C).

4. <u>Idempotent laws:</u>

   For every set A, we have

   (i) A £ A = A. (ii) A ¢ A = A.

5. <u>DeMorgan's laws:</u>

   For all sets A and B, we have

   (i) (A £ B)^{c} = A^{c} ¢ B^{c}. (ii) (A ¢ B)^{c} = A^{c} £ B^{c}.

6. <u>Properties of complements:</u>

   For every set A , universal set U and null set Z , we have

   (i) ((A)^{c})^{c} = A. (ii) A £ A^{c} = U. (iii) A ¢ A^{c} = Z.

   (iv) Z^{c} = U. (v) U^{c} = Z.

7. <u>Properties of the universal set:</u>

   For every set A and universal set U , we have

   (i) A £ U = U. (ii) A ¢ U = A.

8. <u>Properties of the null set:</u>

   For every set A and null set Z, we have

   (i) A £ Z = A. (ii) A ¢ Z = Z.

2.9 Self-Test 4

For solving the exercises 1 to 5 below, consider the following sets:

U = {1,2,3,...10}, A= { x : x is a prime number less than 10 } ,

B ={2,4,6,8,10} C = { 1,4,9, 16, 25},

Select the correct alternative from the given alternatives.

1. Which of the following is the set, (A £ B)^c?
(a) {2, 3, 4, 5,6, 7,8,10 } **(b)** {1, 9 } **(c)** { 1,3,5,7,9} **(d)** {1, 3, 5, 9}

2. Which of the following is the set, A^c £ B^c?
(a) {1, 3, 4, 5, 6, 7, 8, 9, 10} **(b)** {1, 9} **(c)** {1,3,5,7,9} **(d)** {1, 2, 3, 4, 5}

3. Which of the following is the set, ((A)^c)^c?
(a) {1, 4,6, 8,9,10} **(b)** {1,2,3,...10} **(c)** {1, 2, 3, 5,7 } **(d)** {2, 3, 5,7 }

4. Which of the following is the set C £ A^c?
(a) {2, 3, 4, 5, 6, 7,8, 10, 16, 25} **(b)** {1, 2, 3, 4, 5}
(c) {1,3,5,7,9} **(d)** {1, 4,6, 8,9,10, 16, 25}

5. Which of the following is the set B^c?
(a) {2, 3, 5,7,9 } **(b)** {1, 2, 3, ..., 10 } **(c)** {1, 3, 5,7,9} **(d)** {3, 5, 7, 9}

6. Which of the following is an associative property?
(a) A ¢ (*B* £ C) = (A ¢ *B)* £ (A ¢ C) **(b)** A £ A = A.
(c) (A £ *B)* U C= A £ (*B* U C) **(d)** (A ¢ B)^c = A^c £ B^c.

7. Which of the following is a distributive property ?
(a) (A ¢ B)^c = A^c £ B^c **(b)** A ¢ (*B* £ C) = (A ¢ *B)* £ (A ¢ C)
(c) A £ A = A. **(d)** (A £ *B)* U C= A £ (*B* U C).

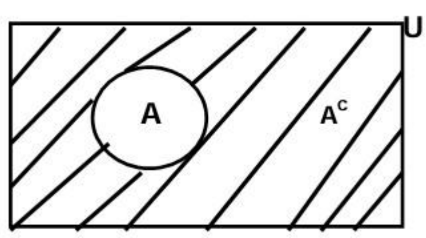
8. Which of the following is an idempotent law?
(a) A £ A = A. **(b)** (A ¢ B)^c = A^c £ B^c
(c) A ¢ (*B* £ C) = (A ¢ *B)* £ (A ¢ C) **(d)** (A £ *B)* U C= A £ (*B* U C)

9. Which of the following is a De Morgan's law?
(a) (A £ *B)* U C= A £ (*B* U C)

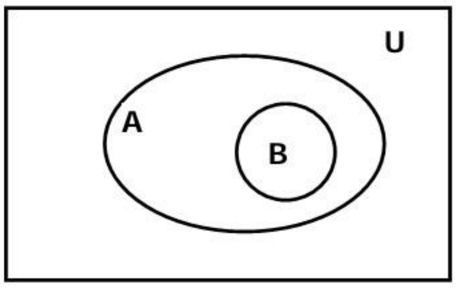
10. Which of the following is not true?
(a) A £ U = A **(b)** Z^c = U **(c)** A ¢ Z = Z **(d)** A £ A^c=U

2.10 Venn diagrams:

Sets can be represented, using Venn diagrams also. A Venn diagram is a pictorial representation of sets in a plane. In Venn diagrams circles, ellipses or closed curves and a rectangle represent sets. The Universal set is represented by the interior of a rectangle and the other sets are represented by circles or closed curves lying within the rectangle.



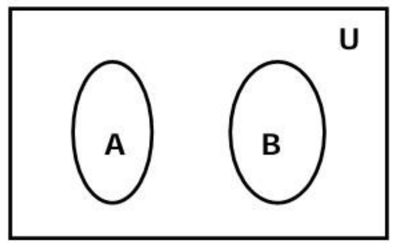
1. If U is universal set and U ¤ A, then U is divided into two subsets A and A^c, Because U = A U A^c. A Venn diagram of this situation can be as shown below:



Shaded portion in this diagram shows A^c.

If B § A, then inside a rectangle representing universal set a circle or ellipse representing set A is drawn and the circle or ellipse representing the set B is drawn entirely within set A. It is because of the fact, that in this case every element of set B is also an element of set A. A Venn diagram of B § A can be as shown in figure.

2. If A and B are disjoint sets i.e. A and B have no common elements then the circles or ellipses representing set A and set B are drawn separated within the rectangle representing universal set. A Venn diagram of this case can be as shown here.



3. If A and B are any two sets then it is possible that some elements of set U are in both sets A and B, some are only in set A but not in set B, some are only in set B but not in set A and some are neither in set A nor in set B. A Venn diagram of this case can be as shown here.

Shaded portion in the diagram1 shows the set AUB and the shaded portion in the diagram2 shows the set A ¢ B. In these diagrams the two ovals represent sets A and B respectively and the rectangles represents the universal set.

Diagram 1

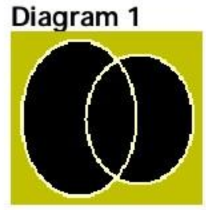
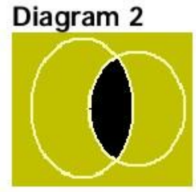


Diagram 2



2.11 Summary

In this unit learners studied the following topics in details:

1. Concept of set and how to write sets using set listing form and set builder form.
2. Different types of sets which are null or empty set, singleton set, subset, superset, universal set , finite/ infinite set and power set etc.
3. How to perform various operations on sets such as union, intersection, difference and Cartesian product.
4. What are different properties of sets such as Commutative properties, Associative properties, Distributive properties, Idempotent laws, DeMorgan's laws, Properties of complements, Properties of the universal set and Properties of the null set.
5. Representation of sets using Venn diagrams.

Unit 3: Mathematical Induction:

3.0 Objectives

By the end of this Unit, learners will be able to:

- Understand the first principle of mathematical induction.
- Write proofs of statements using the method of mathematical induction.

3.1 Introduction

Mathematical induction is a method of proving a particular class of conjectures. A conjecture in mathematics is a statement, which is not proved. Induction is generally used to establish that, a given statement is true for all natural numbers. The method can be extended to prove statements about more general structures; this generalization, known as structural induction or strong induction is used in Mathematical logic and computer science. In fact, mathematical induction is a form of deductive reasoning. The earliest traces of mathematical induction can be found in the Euclid's proof that the set of all primes is an infinite set.

A method of proving conjectures is needed because, in mathematics "disproof by counterexample" always works but "proof by example" seldom works. But if the conjecture is a statement about a small finite set then we can prove the conjecture by showing that for each member of the set it is true.

Example: Suppose we want to prove the following statement

"If a natural number from 1 to 13 is divisible by 6, then it is also divisible by 3."

One proof can be written as:

From 1 to 13, 6 is divisible by 6, it is also divisible by 3.

12 is divisible by 6 and it is also divisible by 3.

1,2,3,4,5,7,8,9,10,11,and 13 are not divisible by 6.

So we can say that above statement is proved, as it is checked for the two numbers in question.

But if we want to prove any statement for all natural numbers then this method is not useful as it is impossible to verify the statement for all natural numbers. So to prove a conjecture is true, we need some more formal methods of proof. One of these methods is of mathematical induction.

Principle of Mathematical Induction can be stated as:

"Show that something works for the first time.
Assume that it works for this time, and show that it will work for the next time.
Conclusion, it works all the time."

3.2 The first principle of mathematical induction

The simplest and most common form of mathematical induction is the first principle of mathematical induction.

Mathematically it is stated as:

A statement p(n), involving natural numbers n is true for all natural numbers n if we can show that ,

1. The statement p(n) is true when n = 1.
2. The statement p(n) is true when n = k+1,assuming that the statement is true for some natural number k .

So the proof of statement p(n) using the first principle of mathematical induction consists of two steps:

1.The basis step: In which we show that the statement holds when n = 1.

2.The inductive step: In which we show that ***if*** the statement holds for n = k, ***then*** it also holds for n = k + 1.

The proposition following the word "if" in the inductive step is called the induction hypothesis. To perform the inductive step, we assume the induction hypothesis and then use this assumption to prove the statement for n = k + 1.

Examples:

1. Suppose we wish to prove the statement that: "**The addition of the first n natural numbers is equal to (n (n+1))/2 ,for all natural numbers n.**"

Proof : The proof that this statement is true for all natural numbers *n* proceeds as follows:

We denote the statement to be proved by p(n).

So p(n) : $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Basis of Induction: Determine whether p(n) is a true statement for *n* = 1.

The sum of 1 and no other number is simply 1. And also $1(1 + 1) / 2 = 1$.

So the statement is true for *n* = 1.

Inductive step: Now we have to show that if the statement holds when *n* = k, then it also holds when *n* = k + 1.

Assume the statement is true for *n* = k, i.e. assume that, p(k) is true.

i.e. $1 + 2 + 3 + \ldots + k = k(k+1)/2$

Add (k + 1) which is the left-hand side's next term, to both sides. This does not change the equality:

P $1 + 2 + 3 + \ldots + k + (k + 1) = [k(k+1)/2] + (k + 1)$
$= [k(k+1) + 2(k + 1)]/2$
P $1 + 2 + 3 + \ldots + k + (k + 1) = [(k+1)(k + 2)]/2$ **P** p(k +1) holds.
So, p(k +1) is proved assuming p(k) to be true.
Hence this statement p(n) is true for all natural numbers n because of the first principle of mathematical induction.

2. Prove the following formula for the sum of consecutive cubes:

$\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3 = [n^2 (n+1)^2] / 4}$, for all natural numbers n.

Proof : We denote the statement to be proved by p(n).

P p(n) : $1^3 + 2^3 + 3^3 + \ldots + n^3 = [n^2 (n+1)^2]/4$

Basis of Induction: Determine whether p(n) is a true statement for *n* = 1.

The sum of 1^3 and no other number is simply 1. P L.H.S. of p(1) = 1

And also $1^2 (1 + 1)^2 / 2 = 1(2)^2/4.$

$= 4 / 4 = 1$ P R.H.S. of p(1) = 1

As L.H.S. = R.H.S., the statement is true for n = 1.

<u>Inductive step</u> : Assume the statement is true for n = k, i.e. assume that, p(k) is true.

p(k) : $1^3 + 2^3 + 3^3 +\ldots +k^3 = [k^2 (k+1)^2] /4$

Then we have to show that the statement is true for its successor, k + 1.

P Add $(k + 1)^3$ which is the left-hand side's next term, to both sides of p(k). This addition does not change the equality.

$$1^3 + 2^3 + 3^3 +\ldots +k^3 +(k + 1)^3 = [k^2 (k+1)^2] /4 +(k + 1)^3$$
$$= [(k+1)^2 (k^2 + 4 (k + 1))] / 4$$
$$= [(k+1)^2 (k^2 + 4k + 4)] / 4$$
$$= (k+1)^2 (k+ 2)^2) / 4$$

P $1^3 + 2^3 + 3^3 +\ldots +k^3 +(k + 1)^3 = (k+1)^2 (k+ 2)) / 4$ **P** p(k +1) holds.

So, p(k +1) is proved by assuming that p(k) to be true.

Hence this statement p(n) is true for all natural numbers n, by the first principle of mathematical induction.

<u>**3.**</u> Prove that: The sum of the first n odd numbers is equal to the nth square:

i.e. **1 + 3 + 5 + 7 + . . . + (2n %Ł1 b²**, for all natural numbers n.

<u>Basis of Induction:</u> Determine whether p(n) is a true statement for n = 1.

On the left hand side , the sum of 1and no other number is simply 1.

And on the right hand side, $1^2 = 1$. P L.H.S. of p(1) = R.H.S. of p(1) = 1

So the statement is true for n = 1.

<u>Inductive step</u> : Assume the statement is true for n = k, i.e. assume that, p(k) is true.

p(k) : 1 + 3 + 5 + 7 + . . . + (2k %Ł1 _²

Then we have to show that the statement is true for k + 1.

P Add [2(k+1) – 1] which is the left-hand side's next term, to both sides of p(k). This addition does not change the equality.

P 1 + 3 + 5 + 7 + . . . + (2k %ŁŽ Œ&fl_Ž%Ł– 1] = k^2 + [2(k+1) – 1]

P 1 + 3 + 5 + 7 + . . . + (2k %ŁŽ &_Ž % .... = k^2 + [2k+1] = $(k + 1)^2$

This is the required statement p(k+1).

So, p(k +1) is proved by assuming that p(k) is true.

P 1 + 3 + 5 + 7 + . . . + (2n %Ł1 b² is true for all natural numbers n, by the principle of mathematical induction.

<u>**4.**</u> Prove that: ***$1 + 4 + 9 + \ldots + n^2 = n (n + 1) (2n + 1) / 6$*** for all positive integers n.

Proof: Here the statement to be proved is ,

p(n) : $1^2 + 2^2 + 3^2 +\ldots +n^2 = n (n + 1) (2n + 1) / 6.$

<u>Basis of Induction:</u> Determine whether p(n) is a true statement for n = 1.

On the left hand side, the sum of 1^2 and no other number is simply 1.

And on the right hand side, value is,

$1\ (1 + 1)\ (2ˆ 1 + 1) / 6 \ = \ (1 ˆ 2 ˆ 3) /6 \ = \ 6 / 6 \ = 1.$

P L.H.S. of p(1) = R.H.S. of p(1)

So the statement is true for n = 1.

Inductive step: Assume the statement is true for n = k, i.e. assume that, p(k) is true.

p(k) : $\quad 1^2 + 2^2 + 3^2 + \ldots + k^2 = k\ (k + 1)\ (2k + 1) / 6.$

Then we have to show that the statement is true for k + 1.

P Add $[(k+1)^2]$ which is the left-hand side's next term, to both sides of p(k). This addition does not change the equality.

$$\text{P } 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = [(k)(k+1)(2k+1)/6] + (k+1)^2$$
$$= [(k+1)(k+2)(2k+3)] / 6.$$

This is the required statement p(k+1).

So, p(k +1) is proved by assuming that p(k) is true.

P $1^2 + 2^2 + 3^2 + \ldots + n^2 = n\ (n + 1)\ (2n + 1) / 6$, is true for all positive integers n, by the principle of mathematical induction.

5. Prove that: ***n! ‡*** 2^{n-1} for all positive integers n ‡ 1.

$n! = nˆ\ (n - 1)\ ˆ\ (n - 2)\ ˆ\ (n - 3)\ ˆ \ldots ˆ\ 3\ ˆ\ 2\ ˆ\ 1.$

Proof: Here the statement to be proved is, $\quad$ p(n) : $n!$ ‡ 2^{n-1}.

Basis of Induction: Determine whether p(n) is a true statement for n = 1.

On the left hand side, n! = 1! = 1.

And on the right hand side, $2^{n-1} = 2^{1-1} = 2^0 = 1$.

P L.H.S. ‡ R.H.S. of p(1) = 1. In fact it is equal. So the statement is true for n = 1.

Inductive step: Assume the statement is true for n = k, i.e. assume that, p(k) is true, where $\quad$ p(k) : $\quad k!$ ‡ 2^{k-1}.

Then we have to show that the statement is true for k + 1.

P Multiply both sides of the inequality by (k+1) which is the left-hand side's next term. This multiplication does not change the inequality.

P (k+1) ˆ k! ‡ (k+1) ˆ 2^{k-1} .

P (k+1)! ‡ (k+1) ˆ 2^{k-1} $\qquad$ n! = n(n-1)!

‡ 2 ˆ 2^{k-1} .... because k + 1 ‡ 2 as k ‡ 1.

$= 2^{(k+1)-1} = 2^k$

P (k+1)! ‡ $2^{(k+1)-1}$. $\quad$ This is the required statement p(k+1).

So, p(k +1) is proved by assuming that p(k) is true.

P n! ‡ 2^{n-1}, is true for all all positive integers n ‡ 1, by the principle of mathematical induction.

6. Prove that: **7^n – 1 is divisible by 6** for all natural numbers n ‡ 1.

Proof: Here the statement to be proved is ,

p(n) : $\quad 7^n - 1$ is divisible by 6.

Basis of Induction: Determine whether p(n) is a true statement for n = 1.

For n = 1,

$7^n - 1 = 7^1 - 1 = 6$, it is divisible by 6.

So the statement is true for n = 1.

Inductive step : Assume the statement is true for $n = k$,

i.e. assume that, $7^k - 1$ is divisible by 6.

P $7^k - 1 = 6m$, for some integer m.

P $7^k = 6m + 1$ --------------------------------(I)

Then we have to show that the statement is true for k + 1. i.e. we have to show that $7^{(k+1)} - 1$ is divisible by 6.

Now, $7^{(k+1)} - 1 = (7$ ˆ $7^k) - 1$

$= (7$ ˆ $(6m + 1)) - 1$, using (I) above.

$= 42m + 7 - 1$

$= 42m + 6$, it is divisible by 6.

P $7^{(k+1)} - 1$ is divisible by 6 , if $7^k - 1$ is divisible by 6.

So, p(k +1) is proved by assuming that p(k) is true.

P $7^n - 1$ is divisible by 6, for all positive integers n ‡ 1, by the principle of mathematical induction.

3.3 Self-Test 1

Using principle of mathematical induction prove that the given statements are true for all natural numbers n.

1. $2 + 7 + 12 + \ldots + (5n - 3) = [n(5n - 1)] /2$.
2. Sum of the first n even natural numbers is n(n+ 1).
3. $1(1!) + 2(2!) + 3(3!) + \ldots + n(n!) = (n+1!) - 1$.
4. $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$.
5. $1 + 7 + 7^2 + 7^3 + \ldots + 7^{n-1} = (7^n - 1) / 6$.
6. $2n^3 + 9n^2 + 13n + 7$ 2 0.
7. $n^2 + n$ is an even number.
8. $5^n - 1$ is divisible by 4.
9. $11^n - 6$ is divisible by 5.
10. $30 - 6 (5^n)$ is divisible by 24.

3.4 Summary

In this unit learners studied the following topics in details:

1. The first principle of Mathematical Induction.
2. Applications of the first principle of Mathematical Induction to prove different mathematical statements about natural numbers.

Unit 4 Exponents and Logarithms, Surds

4.0 Objectives

By the end of this Unit, learners should be able to:

- Understand Exponential form, integral exponents.
- State Laws of Exponents and solve problems using Laws of Exponents.
- Simplify fractional exponents and surds.
- Explain Logarithms and Laws of Logarithms.
- Convert Logarithm to a different base.
- Perform complex calculations by applying Logarithms.

4.1 Introduction

Exponentiation is a mathematical operation used in many fields other than mathematics such as economics, biology, chemistry, physics and computer science. It is used in the calculations involving compound interest, population growth, chemical reaction kinetics, wave theory and public key cryptosystems.

Exponentiation where the exponent is a matrix is used for solving system of linear differential equation. Exponentiation is defined for even complex exponents. The special exponentiation function e^x is example of it. It is used to express the trigonometric functions as exponentiations.

Exponentiation is nothing but the multiplication by the same factor. The inverse of this operation gives us two operations, which are extracting roots and taking logarithms.

The logarithm is perhaps the single, most useful arithmetic concept used in all sciences. Understanding logarithm is essential to understand many scientific ideas. Logarithms may be defined and introduced in several different ways. Previously when there were no calculators, arithmetic calculations involving large multiplications, divisions and powers were time consuming. Logarithms were invented to reduce the amount of work involved in such complex and difficult calculations. Even in these days of calculators and computers logarithm is still an important working tool in mathematics.

Another inverse operation of exponentiation is nothing but obtaining roots. In this unit we will study about special types of n-th roots of positive integers, which are surds.

4.2 Exponential form and Laws of Exponents

Exponentiation is a mathematical operation written as a^n, which corresponds to the repeated multiplication if n is a natural number. This operation is defined as, $a^n = a \times a \times a \times a \times \ldots \times a$ (n times). In this case 'a' is called the base and 'n' is called the exponent.

Also, it is defined that $a^0 = 1$ and $a^1 = a$.

The exponent is generally shown as a superscript to the right of the base. The exponentiation a^n can be read as "a raised to the n^{th} power" or "a to the n^{th} power" or in short "a to the n". But a^2 is read as "a squared" and a^3 is read as "a cubed". As mentioned earlier any non zero number raised to the power 0 is 1 and any number raised to the power 1 is itself. The exponentiation, a^n is also defined when the exponent is a negative integer.

4.2.1 Positive integer exponents: The exponent is the number, that many times the base is multiplied together. Hence in general a^n = aˆ a ˆ a ˆ a ˆ … ˆ a, where n copies of 'a' are multiplied together.

If a is any real number then,

a^0= 1, if a is not equal to 0.

a^1= a.

a^2= aˆ a. It is also read as "a squared" because one can relate it with the area of a square whose sides are of length a.

a^3 = aˆ a ˆ a. It is also read as "a cubed" because one can relate it with the volume of a cube whose sides are of length a.

Examples:

‹ 2^0= 1. **P** 2 to the power 0 is 1.

‹ 6^1= 6.**P** 6 to the power 1 is 6.

‹ 3^2 = 3 ˆ 3 = 9. **P** 9 is 3 squared.

‹ 5^3 = 5 ˆ 5 ˆ 5 = 125. **P** 125 is 5 cubed.

‹ 2^5 = 2ˆ 2 ˆ 2 ˆ 2 ˆ 2 = 32. Here the base 2 is multiplied with itself 5 times as 5 is the exponent. **P** 32 is the 5^{th} power of 2 or 2 raised to the 5^{th} power.

4.2.2 Negative integer exponents: Any nonzero number raised to the –1 power is defined to be equal to the reciprocal.

P a^{-1}= 1 / a and hence

a^{-n} = 1 / (a^n) = 1 / [aˆ a ˆ a ˆ a ˆ … ˆ a (n times multiplication)].

Examples:

‹ 2^1= 2. **P** 2^{-1}= 1 / 2.

‹ 3^2 = 3 ˆ 3 = 9. **P** 3^{-2} = 1 / 3^2 =1 / (3 ˆ 3) = 1/ 9.

‹ 5^3 = 5 ˆ 5 ˆ 5 = 125. **P** 5^{-3} = 1/ 5^3 = 1 / (5 ˆ 5 ˆ 5) = 1/ 125.

‹ 2^6 = 2ˆ 2 ˆ 2 ˆ 2 ˆ 2ˆ 2 = 64.

P 2^{-6} = 1 /2^6 = 1 / (2ˆ 2 ˆ 2 ˆ 2 ˆ 2ˆ 2) = 1/64.

If the exponent n is positive then, then the n^{th} power of zero is zero i.e. 0^n= 0. If the exponent is zero then 0^0 is undefined. Zero raised to the –1 power is also undefined because it is division by zero. So 0 raised to any negative power is undefined.

All the integer powers of 1 are 1 i.e. 1^n = 1, for positive as well as negative n.

If the exponent is even number then, the power of –1 is 1 i.e. $(-1)^{2n}$ = 1 and if the exponent is odd number then, the power of –1 is –1 i.e. $(-1)^{2n+1}$ = –1.

In computer science, binary number system is very useful. In this number system positive powers of 2 are of much importance.

Exponentiation with base 10 is used to write very large or very small numbers. This method of writing numbers is called scientific notation method. To write speed of light, distances between stars, dimensions of bacteria or viruses numbers are written using scientific notations. E.g. the speed of light in vacuum is

299792458meters per second. Using scientific notations this number is written as 2.99792458×10^8 which is approximately equals to 2.998×10^8 or 3×10^8. Integer powers of 10 are also used to represent the numbers as $100000 = 10^5$, $100,000,000 = 10^8$ or $0.00001 = 10^{-5}$ etc.

4.2.3 Laws of Exponents: If a, b are real numbers and m and n are integers, then the following properties of exponents hold.

1. $\mathbf{a^0 = 1, a \neq 0}$

   Examples:

   - $3^0 = 1$
   - $(1/2)^0 = 1$

2. $\mathbf{a^m \times a^n = a^{m+n}}$

   Examples:

   - $3^2 \times 3^4 = (3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^6 = 3^{2+4}$.
   - $(1/2)^4 \text{ x } (1/2)^3 = (1/2) \text{ x } (1/2) \text{ x}(1/2) \text{ x } (1/2) \text{ x } (1/2) \text{ x}(1/2) \text{ x } (1/2) = (1/2)^7$

3. $\mathbf{(a^m)^n = a^{m \times n}}$

   Examples:

   - $(3^2)^4 = 3^2 \times 3^2 \times 3^2 \times 3^2 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^8 = 3^{2 \times 4}$.
   - $[(1/2)^4]^3 = (1/2)^4 \text{ x } (1/2)^4 \text{ x } (1/2)^4 = (1/2)^{12} = (1/2)^{4\text{x}3}$

4. $\mathbf{(a \times b)^m = a^m \times b^m}$

   Examples:

   - $(2 \times 3)^4 = (2 \times 3) \times (2 \times 3) \times (2 \times 3) \times (2 \times 3) = (2 \times 2 \times 2 \times 2) \times (3 \times 3 \times 3 \times 3) = 2^4 \times 3^4$
   - $(7 \times 4)^3 = (7 \times 4) \times (7 \times 4) \times (7 \times 4) = (7 \times 7 \times 7) \times (4 \times 4 \times 4) = 7^3 \times 4^3$

5. $\mathbf{a^m / a^n = a^{m-n}}$

   Examples:

- $2^5 / 2^2 = (2 \times 2 \times 2 \times 2 \times 2) / (2 \times 2) = 2^3 = 2^{5-2}$
- $4^9 / 4^5 = (4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4) / (4 \times 4 \times 4 \times 4 \times 4) = 4^4 = 4^{9-5}$

6. $\mathbf{(a / b)^m = a^m / b^m}$

   Examples:

   - $(2 / 3)^4 = (2/3) \times (2/3) \times (2/3) \times (2/3) = (2 \times 2 \times 2 \times 2) / (3 \times 3 \times 3 \times 3) = 2^4 / 3^4$
   - $(7 \times 5)^3 = (7 \times 5) \times (7 \times 5) \times (7 \times 5) = (7 \times 7 \times 7) \times (5 \times 5 \times 5) = 7^3 \times 5^3$

7. $\mathbf{a^{-m} = 1 / a^m, a \neq 0}$

   Examples:

   - $2^{-3} = 1 / 2^3 = (1/2) \text{ x } (1/2) \text{ x}(1/2) = (1/2)^3$
   - $4^{-5} = 1 / 4^5 = (1/4) \text{ x } (1/4) \text{ x}(1/4) \text{ x } (1/4) \text{ x}(1/4) = (1/4)^5$

Here we must note that the multiplication of real numbers is a commutative and associative operation, but exponentiation is not both commutative as well as associative operation.

Multiplication of real numbers is a commutative operation i.e. $a \times b = b \times a$.

But $a^b \neq b^a$. e.g. $3^4 = 3 \times 3 \times 3 \times 3 = 81$ and $4^3 = 4 \times 4 \times 4 = 64$.

$\therefore$ Exponentiation is not commutative operation.

Multiplication of real numbers is an associative commutative operation

i.e. $a \times (b \times c) = (a \times b) \times c$.

But $a^{(bc)} \neq (a^b)^c$, $\therefore$ exponentiation is not associative operation.

e.g. 3 raised to $(2^4)^{th}$ power is 3 raised to 16^{th} power $= 3^{16} = 43046721$

and 3^2 raised to 4^{th} power is 9 raised to 4^{th} power $= 9^4 = 6561$.

$\therefore 3^{(24)} \neq (3^2)^4$.

4.3 Self-Test 1

Select the correct alternative from the given alternatives.

1. What is the exponential form of $4 \times 4 \times 4 \times 4 \times 4 \times 4$?

(a) 4^3 **(b)** 4^4 **(c)** 4^6 **(d)** 4^8

2. What is the exponential form of $1 / (5 \times 5 \times 5 \times 5)$?

(a) 5^4 **(b)** 5^{-4} **(c)** 4^5 **(d)** 4^{-5}

3. What is the exponential form of $-5 \times -5 \times -5 \times -5$?

(a) $-(5^4)$ **(b)** 5^{-4} **(c)** $(-5)^4$ **(d)** 4^{-5}

4. What is the exponential form of $(1/3) \times (1/3) \times (1/3) \times (1/3) \times (1/3)$?

(a) $-(3)^5$ **(b)** 3^{-4} **(c)** $(1/3)^5$ **(d)** 5^{-3}

5. What is the value of a^0 if $a \neq 0$?

(a) 0 **(b)** 1 **(c)** a **(d)** undefined quantity

6. What is the value of 0^0 ?

(a) 0 **(b)** 1 **(c)** –1 **(d)** undefined quantity

7. What is the value of $(-1)^5$?

(a) 0 **(b)** 1 **(c)** –1 **(d)** undefined quantity

8. What is the value of 10^{-4}?

(a) 10000 **(b)** 0.0001 **(c)** 0.00001 **(d)** 10

9. What is the simplification of $(3^7 \times 3^{-2} \times 3^0) / 3^4$?

(a) 1 **(b)** 3 **(c)** 3^2 **(d)** 0

10. What is the simplification of $(2^3 \times 2^{-6} \times 2^6 \times 2^{-7}) / (2^4 \times 2^{-5})$?

(a) 2^0 **(b)** 2^4 **(c)** 2^3 **(d)** 2^{-3}

4.4 Fractional exponents and surds

One inverse operation of exponentiation is nothing but obtaining roots of a number. If a and b are two numbers such that, b is exponential form a^n, i. e. $b = a^n$, then a is defined to be an n-th root of b. Equivalently it is also written as $a = b^{1/n}$. So the nth roots of a number involve fractional exponents. These are also denoted using radical symbol or the root symbol i.e. if $a = b^{1/n}$ then we write $a = \sqrt[n]{b}$. We will study the n-th roots of positive real numbers only.

Generally a second root of a number is called as its square root and a third root is called as a cube root of a number.

Examples:

- $2^5 = 32 \Rightarrow 2$ is a 5^{th} root of 32. It is written as $2 = (32)^{1/5} = \sqrt[5]{32}$.
- $3^2 = 3 \times 3 = 9$. $\Rightarrow$ 3 is a square root of 9, hence $3 = 9^{1/2}$.

  Also $(-3)^2 = -3 \times -3 = 9$. $\Rightarrow$ -3 is another square root of 9.

  Symbolically is written as $\sqrt{9} = \pm 3$.
- $5^3 = 5 \times 5 \times 5 = 125$. $\Rightarrow$ 5 is a cube root of 125 or $5 = (125)^{1/3}$.

  Symbolically is written as $5 = \sqrt[3]{125}$.

The earlier studied laws of exponents are applicable even to fractional exponents.

A special class numbers involving fractional exponents is of surds. Surds are numbers written in radical form, which are irrational numbers of a particular type.

4.4.1 Laws of fractional exponents: If a, b are positive real numbers and m and n rational numbers, then we have,

1. $a^m \times a^n = a^{m+n}$
2. $(a / b)^m = a^m / b^m$
3. $(a^m)^n = a^{m \times n}$

Note:

1. A number which can be written in the form **p/q** where **p** and **q** are integers and **q** $\neq$ 0, is called a rational number.
2. The numbers, which are not rational numbers, are called as irrational number. For irrational numbers the decimal representation is non-terminating and non recurring.

   Examples: All integers are rational numbers. Also the numbers ½, 4/7, 25/125 and $\sqrt{4}$ etc. are rational numbers. But $\sqrt{2}, \sqrt{3}, \sqrt[4]{5}$ etc. are not rational numbers. Also the constant π is an irrational number, where π is the ratio of circumference of any circle to its diameter.

4.4.2 Surd: A surd is a number $\sqrt[n]{a} = a^{1/n}$ if and only if

1. it is an irrational number ,
2. $n \neq 1$ is a natural number and a is positive rational number .

Examples:

- $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$ are all surds but $\sqrt{4}$ is not a surd as it equals to 2, which is a rational number.
- $\sqrt[3]{2}$, $\sqrt[5]{8}$, $\sqrt[4]{20}$ are all surds but $\sqrt[3]{125}$ is not a surd as it equals to 5 which is a rational number.

- $\sqrt{5+\sqrt{3}}$ is not a surd as $5+\sqrt{3}$ is not a rational number, also $\sqrt{d}$ is not a surd because d is an irrational number.

Using rules of operations of radicals, we can determine that whether a given number is a surd or not.

Examples:

- $\sqrt{10} \times \sqrt{90} = \sqrt{900} = \sqrt{30 \times 30} = \left(\sqrt{30}\right)^2 = 30$ it is a rational number. Rational numbers are not suds. **∴** $\sqrt{10} \times \sqrt{90}$ is not a surd . But $\sqrt{10}$ and $\sqrt{90}$ are surds.
- $\sqrt[3]{128} = \sqrt[3]{4 \times 4 \times 4 \times 2} = \sqrt[3]{64} \times \sqrt[3]{2} = 4\sqrt[3]{2}$ is a surd as it is an irrational number which satisfy the definition of a surd .
- $\sqrt{\sqrt{7}} = (7^{½})^{½} = 7^{¼} = \sqrt[4]{7}$ it is a surd, by the definition of a surd.

4.4.3 Rules of operations with surds

1. $(\sqrt[n]{a})^n = a.$

Examples: $(\sqrt[3]{2})^3 = 2.$ $\qquad$ $(\sqrt[5]{4})^5 = 4.$

2. $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$, $a \neq 0$ and $b \neq 0$.

Example: $\sqrt[3]{20} = \sqrt[3]{4} \times \sqrt[3]{5}$

3. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, $a \neq 0$ and $b > 0$.

Example: $\sqrt[3]{\frac{4}{5}} = \frac{\sqrt[3]{4}}{\sqrt[3]{5}}.$

4. $\sqrt[n]{a^m} = (\sqrt[n]{a})^m = (a^{1/n})^m = a^{m/n}$

Examples:

- $\sqrt[3]{2^6} = (\sqrt[3]{2})^6 = (2^{1/3})^6 = 2^{6/3} = 2^2 = 4.$
- $\sqrt[3]{5^4} = (\sqrt[3]{5})^4 = (5^{1/3})^4 = 5^{4/3}.$

5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$

Examples:

- $\sqrt[2]{\sqrt[3]{7}} = \sqrt[6]{7} = \sqrt[3]{\sqrt[2]{7}}$
- $\sqrt[3]{\sqrt[5]{21}} = \sqrt[15]{21} = \sqrt[5]{\sqrt[3]{21}}$

4.4.4 Order of a surd: In the surd $b\sqrt[n]{a}$, b is called coefficient of the surd, the index n is called the order of the surd and a is called radicand. When the coefficient of a surd is not written, it is assumed to be 1. When the order n of a surd is not mentioned it is taken as 2.

Examples:

- In the surd $5\sqrt[3]{2}$, the coefficient of the surd is 5, the order of the surd is 3 and the radicand is 2.
- In the surd $\sqrt[4]{5}$ the order of the surd is 4 and the radicand is 5 and coefficient is 1.

4.4.5 Forms of surds

1. **Pure Surd:** A surd of the form $\sqrt[n]{a}$ is called a pure surd.

   Examples: $\sqrt{10}$, $\sqrt[3]{12}$, $\sqrt[4]{7}$

2. **Mixed Surd:** A surd of the form $b\sqrt[n]{a}$ is called a mixed surd where b is a rational number and $b \neq 1$.

   Examples: $5\sqrt{10}$, $2\sqrt[3]{12}$, $8\sqrt[4]{7}$

   <u>Note:</u> A mixed surd may be expressed as a pure surd as well as a pure surd may be expressed as a mixed surd.

   Examples:

   - $5\sqrt[3]{10} = \sqrt[3]{5 \times 5 \times 5 \times 10} = \sqrt[3]{1250}$
   - $\sqrt{112} = \sqrt{16 \times 7} = \sqrt{4 \times 4} \times \sqrt{7} = 4\sqrt{7}$

3. **Similar or like surds:** The surds of the form $p\sqrt[n]{a}$ and $q\sqrt[n]{a}$ are called similar surds or like surds, where p and q are rational numbers.

Examples: Following are similar surds

- $5\sqrt[3]{10}$, $\sqrt[3]{10}, 6\sqrt[3]{10}$
- $\sqrt{7}$, $100\sqrt{7}, 5\sqrt{7}, 4\sqrt{7}$

4. **Simplest form of a Surd:** A surd $\sqrt[n]{a}$, is said to be in its simplest form if
   - the radicand "a" has no divisor which is n^{th} power of a rational number.
   - the radicand "a" is not a fraction and
   - "n" is the least such power.

   Examples:

   - $\sqrt[3]{1250} = \sqrt[3]{125 \times 10} = \sqrt[3]{125} \times \sqrt[3]{10} = 5\sqrt[3]{10}$,

     $\therefore$ The surd $\sqrt[3]{1250}$ has the simplest form $5\sqrt[3]{10}$

   - $\sqrt[4]{1875} = \sqrt[4]{625 \times 3} = \sqrt[4]{25 \times 25 \times 3} = \sqrt[4]{5 \times 5 \times 5 \times 5} \times \sqrt[4]{3} = 5\sqrt[4]{3}$

     $\therefore$ The surd $\sqrt[4]{1875}$ has the simplest form $5\sqrt[4]{3}$.

4.5 Self-Test 2

Select the correct alternative from the given alternatives.

1. What is the order of the surd $\sqrt[3]{15}$?

   (a) 15 **(b)** 2 **(c)** 3 **(d)** 5

2. What is the radicand of the surd $\sqrt[3]{15}$?

 (a) 15 (b) 2 (c) 3 (d) 5

3. Which of the following radicals is a surd?

 (a) $\sqrt[3]{125}$ (b) $\sqrt[4]{15}$ (c) $\sqrt{25}$ (d) $\sqrt[4]{81}$

4. Which of the following is a surd?

 (a) $\sqrt[3]{1+\sqrt{8}}$ (b) $\sqrt[3]{8}$ (c) $\sqrt{\sqrt{5}}$ (d) $\sqrt[4]{16}$

5. Which of the following is a pure surd?

 (a) $\sqrt[3]{125}$ (b) $\sqrt[4]{15}$ (c) $2\sqrt{5}$ (d) $5\sqrt[4]{81}$

6. Which of the following is a mixed surd?

 (a) $\sqrt[3]{125}$ (b) $\sqrt[4]{15}$ (c) $2\sqrt{5}$ (d) $5\sqrt[4]{81}$

7. Which of the following is a pair of similar surds?

 (a) $\sqrt[3]{12}, \sqrt{12}$ (b) $3\sqrt[5]{7}, 9\sqrt[5]{7}$

 (c) $2\sqrt{5}, \sqrt[4]{5}$ (d) $5\sqrt[4]{8}, 4\sqrt{8}$.

8. What is the simplest form of the surd $\sqrt[3]{135}$?

 (a) $\sqrt[3]{5\times 27}$ (b) $3\sqrt[3]{27}$ (c) $\sqrt[3]{135}$ (d) $\sqrt[3]{5\times 3\times 3\times 3}$

9. What is the simplest form of the surd $\sqrt{343/45}$?

 (a) $(7/15)\sqrt{35}$ (b) $(7/5)\sqrt{49/9}$

 (c) $(7/3)\sqrt{7/5}$ (d) $\sqrt{(49\text{x}7)/(9\text{x}5)}$

10. What is the simplification of $3\sqrt{8}+\sqrt{32}-5\sqrt{2}$?

 (a) $-5\sqrt{2}$ (b) $-2\sqrt{20\times 2}$

 (c) $5\sqrt{2}$ (d) $3\sqrt{4\times 2}+\sqrt{16\times 2}-5\sqrt{2}$

4.6 Logarithms and Laws of Logarithms

4.6.1:Logarithm: If a and b are two positive real numbers such that, a is exponential form b^n, i.e. $a = b^n$ and $b \neq 1$, then $n = \log_b a$. i.e. the logarithm of b to the base a is n.

So the logarithmic equivalent form of the exponential equation $a = b^n$ is $n = \log_b a$.

Examples:

- $7^0 = 1 \Rightarrow 0 = \log_7 1$.
- $4^1 = 4 \Rightarrow 1 = \log_4 4$.
- $2^5 = 32. \Rightarrow 5 = \log_2 32$ i.e. the logarithm of 32 to the base 2 is 5.
- $5^3 = 5\times 5\times 5 = 125. \Rightarrow 3 = \log_5 125$, i.e. the logarithm of 125 to the base 5 is 3.
- $10^4 = 10000. \Rightarrow 4 = \log_{10} 10000$, i.e. the logarithm of 10000 to the base 10 is 4.

<u>Note that:</u>

1. $\log_a 1 = 0$ for all $a > 0$, because $a^0 = 1$.
2. $\log_a a = 1$ for all $a > 0$, because $a^1 = a$.

3. Logarithms $\log_b a$ are defined only for positive values of a and b, when $b \neq 1$.
4. The logarithm of a to the base 10 i.e. $\mathbf{\log_{10} a}$ is also called as common logarithm and it is written as log a.

   Examples:

   - Since $10 = 10^1$. $\therefore$ $\log 10 = 1$.
   - Since $1000 = 10^3$. $\therefore$ $\log 1000 = 3$.
   - Since $0.1 = 10^{-1}$. $\therefore$ $\log 0.1 = -1$.
   - Since $0.0001 = 10^{-4}$. $\therefore$ $\log 0.0001 = -4$.

5. The logarithm of a to the base e is also called as natural log of a, where e has approximate value 2.7182 and it is an irrational number.

   Examples:

   - Since $e = e^1$. $\therefore$ $\log_e e = 1$, similarly $\log_e e^2 = 2$ etc.
   - Since $1 / (e^2) = e^{-2}$. $\therefore$ $\log_e (1 / (e^2)) = -2$.

4.6.2 Laws of Logarithms: If a, b are positive real numbers, x is a positive real number such that $x \neq 1$,and n is a real number, then the following properties of Logarithms hold.

1. $\log_x (a \times b) = \log_x a + \log_x b$.

   Example: $\log_5 (25 \times 125) = \log_5 25 + \log_5 125 = \log_5 5^2 + \log_5 5^3 = 2 + 3 = 5$.
2. $\log_x (a/b) = \log_x a - \log_x b$.

   Example: $\log_5 (125/25) = \log_5 125 - \log_5 25 = \log_5 5^3 - \log_5 5^2 = 3 - 2 = 1$.
3. $\log_x (a^n) = n \times \log_x a$.

   Example: $\log_2 64 = \log_2 (4^3) = 3 \times \log_2 4 = 3 \times 2 = 6$.
4. $\log_x x^a = a$.

   Example: $\log_5 5^2 = 2$ and $\log_5 5^3 = 3$.
5. $x^{(\log_x a)} = a$.

   Example: $10^{\log_{10} 5} = 10^{0.69897} = 5$.

4.6.3 Antilogarithms: The process of finding antilogarithm is just the reverse of finding a logarithm. We know that $4 = \log_{10} 10000$, $\therefore$ $\text{antilog}_{10}\ 4 = 10000$. Antilogarithm is found using the same base as that of the corresponding logarithm. While doing complicated calculations we need antilogarithms to find final answer.

4.7 Conversion to a different Base

While doing calculations it is frequently needed to change the logarithm of a number to a different base. The following rule is used for conversion of the logarithm of a number to a different Base:

$$\log_b a = \log_x a / \log_x b$$

It is more common to use common logarithm (i.e. to the base 10) in place of natural logarithm (i.e. to the base e). In changing the base from e to 10 and vice versa we need the values of $\log_{10} e$ and $\log_e 10$, which are

$\log_{10} e \approx 0.4343$ and $\log_e 10 \approx 2.3026$.

Examples:

‹ $\log_e 100 = \log_{10} 100 / \log_{10} e = 2 / 0.4343 = 4.6051$

‹ We have seen above that $\log_2 64 = 6$. Also

$\log_2 10 = \log_e 10 / \log_e 2 = 2.3026 / 0.6931$ “ 3.3222

P $\log_{10} 64 = \log_2 64 / \log_2 10 = 6 / 3.3222$ “ 1.8060

4.8 Application of Logarithms in complex calculations

Logarithms and antilogarithms can be used to do laborious and tedious calculations involving very large or very small numbers. For such calculations laws of logarithms, logarithm table and antilogarithm table is used. Generally common logarithm or logarithm to the base 10 is used in calculations. It is common practice to write log x in place of $\log_{10} x$.

Examples:

1. Find $(1.57)^5$

Solution: let $x = (1.57)^5$.

P $\log(x) = \log((1.57)^5) = 5$ ˆ $\log(1.57) = 5$ ˆ $0.1959 = 0.9795$

P ˆ $=$ antilog $(0.9795) = 9.539$.

2. Find $\sqrt[3]{32}$

Solution: let $x = \sqrt[3]{32} = (32)^{1/3}$.

P $\log(x) = \log((32)^{1/3}) = (1/3)$ ˆ $\log(32) = (1/3)$ ˆ $1.5051 = 0.5017$

P $x =$ antilog $(0.5017) = 3.1746$.

3. Calculate $(45.4)^2 / ((3.2)^2$ ˆ $(5.6)^3)$

Solution: let $x = (45.4)^2 / ((3.2)^2$ ˆ $(5.6)^3)$

P $\log(x) = \log(45.4)^2 / ((3.2)^2$ ˆ $(5.6)^3)$

$= \log(45.4)^2 - \log((3.2)^2$ ˆ $(5.6)^3)$

$= \log(45.4)^2 - \{\log(3.2)^2 +$ ˆ $\log(5.6)^3\}$

$= 2$ ˆ $\log(45.4) - \{2$ ˆ $\log(3.2) + 3$ ˆ $\log(5.6)\}$

$= 2$ ˆ $1.6571 - \{2$ ˆ $0.5051 + 3$ ˆ $0.7482\}$

$= 0.0594$

P $x =$ antilog $(0.0594) = 1.1465$.

4.9 Self-Test 3

Select the correct alternative from the given alternatives.

1. What is the logarithmic form of the exponential equation $7^3 = 343$?

(a) $3 = \log_7 343$ **(b)** $7 = \log_3 343$ **(c)** $3 = \log_{10} 343$ **(d)** $3 = \log_e 343$

2. What is the logarithmic form of the exponential equation $\sqrt{16} = 4$?

(a) $½ = \log 16$ **(b)** $16 = \log_{1/2} 4$ **(c)** $4 = \log_2 16$ **d)** $½ = \log_{16} 4$

3. What is the exponential form of the logarithmic equation $\log_{10} 1000 = 3$?

(a) $3^{10} = 1000$ (b) $\sqrt[3]{1000} = 3$ (c) $10^3 = 1000$ (d) $10^{-3} = 1000$

4. What is the exponential form of the logarithmic equation $-2 = \log_3 (1/9)$?

(a) $3^9 = -2$ (b) $\sqrt{1/9} = 3$ (c) $1/9 = 2^{-3}$ (d) $1/9 = 3^{-2}$

5. Which of the following is the common logarithm of 1000000?

(a) 2 (b) 4 (c) 6 (d) 8

6. Which of the following is the common logarithm of 0.000001?

(a) 6 (b) – 5 (c) – 6 (d) – 4

7. What is the value of x if $\log_e x = 3$?

(a) $e = 2.7182$ (b) $e^2 = 7.3886$ (c) $e^3 = 20.0837$ (d) $e^{-2} = 0.1353$

8. What is the value of $\log_e 12$, if $\log_{10} 12 = 1.0792$?

(a) 0.4686 (b) 2.4849 (c) 0.4343 (d) 2.3026

9. What is the value of $\log_2 6$, if $\log_{10} 6 = 0.7782$ and $\log_{10} 2 = 0.3010$?

(a) 2.5853 (b) 2.4849 (c) 0.4343 (d) 2.3026

10. What is the value of $\sqrt[3]{\frac{3 \hat{} 71.34}{7.284}}$?

(a) 29.3822 (b) – 3.0857 (c) 3.0857 (d) 0.7071

4.10 Summary

In this unit learners studied the following topics in details:

1. What are Exponential forms, and Laws of Exponents for real numbers.
2. Positive integer exponents and Negative integer exponents and Laws of Exponents.
3. Fractional exponents, surds and Rules of operations with surds
4. Order of a surd, Pure surd, Mixed surd, and Similar or like surds and simplest form of a Surd.
5. Logarithms and Laws of Logarithms, Antilogarithms.
6. How to convert logarithm of a number to a different Base.
7. How to use Logarithms in doing complex calculations.

Unit 5 Number systems

5.0 Objectives

By the end of this Unit, learners should be able to:

- Understand need of different number systems.
- Convert a decimal number to binary number.
- Convert a binary number to decimal number.
- Add and subtract binary numbers.
- Convert a decimal number to octal number.
- Convert an octal number to decimal number.
- Convert a decimal number to hexadecimal number.
- Convert a hexadecimal number to decimal number

5.1Introduction

While doing any calculations in our day to day life, generally we use the decimal number system. The number 10 is called the basis or radix of this number system. The different individual symbols used to write numbers in the decimal system are called digits. The numbers 0 to 9 are the digits used for decimal system.

There are other number systems which are mostly used in computer and telecommunication fields. In computer science data is converted into information and information is expressed in words. A computer word is a sequence of combinations of only two digits which are 0 and 1. These two digits are called binary digits or in short bits. The fundamental grouping of bits is called a byte rather than a word. This 2- digit number system is ideal for coding purpose for the computers because of the two state natures of the electronic components, which are used in computers. This number system is called as binary number system.

In this unit we are going to study how to convert numbers from any system to another system. While studying different number systems, the basis using which the number is written, is indicated as a subscript to the lower right side e.g. $(125)_{10}$ denotes the decimal number 125, while $(125)_8$ denotes the octal number 125 and $(1100101)_2$ denotes a binary number.

5.2 The Binary number system

In decimal number system the base is equal to 10 because any position in a number can be occupied by one of the 10 digits which are 0, 1, 2, 3 …,9, while in binary number system the base is equal to 2 and the numbers are written using only 2 digits which are 0 and 1. This system has a carrying factor of 10 and each digit indicates a value which depends on the position it occupies.

For example, in the decimal number 23456 the digit 2 signifies 2 ˆ 10000, the digit 3 signifies 3 ˆ 1000, the digit 4 signifies 4 ˆ 100, the digit 5 signifies 5 ˆ 10 and the digit 6 signifies 6 ˆ 1 or 6 .

5.2.1 Conversion of a decimal number to a binary number

The procedure of converting a decimal number to its binary equivalent consists of dividing the decimal number by 2, until we get a quotient of zero. The remainders of these different divisions are written in the opposite order to that in which they are obtained. The number written in this way is the binary representation of the decimal number.

Examples:

1. Conversion of $(25)_{10}$ into its binary equivalent number is performed as follows:
We write the string of remainders from bottom to the top, it is the required binary equivalent.

Number	Quotient when divided by 2	Remainder
25	12	1
12	6	0
6	3	0
3	1	1
1	0	1

P $(25)_{10} = (11001)_2$.

2. Conversion of $(142)_{10}$ into its binary equivalent number:

We write the string of remainders from bottom to the top, it is the required binary equivalent.

Number	Quotient when divided by 2	Remainder
142	71	0
71	35	1
35	17	1
17	8	1
8	4	0
4	2	0
2	1	0
1	0	1

P $(142)_{10} = (10001110)_2$.

3. Conversion of $(30)_{10}$ into its binary equivalent number is performed as follows:

Write the string of remainders from bottom to the top, it is the required binary equivalent.

Number	Quotient when divided by 2	Remainder
30	15	0
15	7	1
7	3	1
3	1	1
1	0	1

P $(30)_{10} = (1111\ 0)_2$.

5.2.2 Conversion of a binary number to a decimal number

For binary number system the basis is 2 and every number is written using the bits i.e. 0 and 1. To convert a number from binary number system to decimal number system following procedure is used. We multiply the rightmost bit by 2^0, then second right bit by 2^1, the third right bit by 2^2 and continue in this manner till we reach the first bit. These products are then added to find the required decimal equivalent.

Examples:

1. Conversion of $(11001)_2$ into decimal equivalent number is performed as follows:

$$
\begin{aligned}
(11001)_2 &= (1 \,\hat{}\, 2^4) + (1 \,\hat{}\, 2^3) + (0 \,\hat{}\, 2^2) + (0 \,\hat{}\, 2^1) + (1 \,\hat{}\, 2^0) \\
&= (1 \,\hat{}\, 16) + (1 \,\hat{}\, 8) + 0 + 0 + (1 \,\hat{}\, 1) \\
&= 16 + 8 + 1 \\
&= (25)_{10}.
\end{aligned}
$$

2. Conversion of $(10001110)_2$ into decimal equivalent number:

$$(10001110)_2 = (1 \times 2^7)+(0\times 2^6)+(0\times 2^5)+(0\times 2^4)+(1\times 2^3)+(1\times 2^2)+(1 \times 2^1)+(0 \times 2^0)$$
$$= (1 \times 128) + 0 + 0 + 0\ (1 \times 8) + (1 \times 4) + (1 \times 2) + (0 \times 1)$$
$$= 128 + 8 + 4 + 2$$
$$= (142)_{10}\ .$$

3. Conversion of $(11110)_2$ into decimal equivalent number:

$$(11110)_2 = (1 \times 2^4) + (1 \times 2^3) + (1\times 2^2) + (1 \times 2^1) + (0 \times 2^0)$$
$$= (1 \times 16) + (1 \times 8) + (1 \times 4) + (1 \times 2) + 0$$
$$= 16 + 8 + 4 + 2$$
$$= (30)_{10}\ .$$

5.3 Self-Test 1

Select the correct alternative from the given alternatives.

1. What is the basis for decimal number system?
(a) 2 **(b)** 8 **(c)** 10 **(d)** 16

2. What is the basis for binary number system?
(a) 2 **(b)** 8 **(c)** 10 **(d)** 16

3. What is the significant value of the digit 4 in the decimal number 5436?
(a) 4×10 **(b)** 4×1 **(c)** 4×100 **(d)** 4×1000

4. What is the binary equivalent of the decimal number 12?
(a) 1010 **(b)** 1100 **(c)** 1101 **(d)** 1110

5. What is the binary equivalent of the decimal number 59?
(a) 111011 **(b)** 11100 **(c)** 11101 **(d)** 11110

6. What is the binary equivalent of the decimal number 184?
(a) 1010111 **(b)** 1111100 **(c)** 11101 **(d)** 10111000

7. What is the decimal equivalent of the binary number 1110?
(a) 12 **(b)** 13 **(c)** 14 **(d)** 15

8. What is the decimal equivalent of the binary number 11111?
(a) 21 **(b)** 31 **(c)** 41 **(d)** 55

9. What is the decimal equivalent of the binary number 101010?
(a) 42 **(b)** 43 **(c)** 44 **(d)** 45

10. What is the decimal equivalent of the binary number 1001000?
(a) 71 **(b)** 72 **(c)** 73 **(d)** 75

5.4 Addition and subtraction of binary numbers

In machine language number are represented using binary number system. So when arithmetic operations such as addition and subtraction are performed, the result is also binary numbers. We can perform addition, subtraction, multiplication and division of binary numbers. Here we will study only the addition and subtraction of binary numbers.

5.4.1 Addition of binary numbers

The addition of binary numbers is done in a similar way, to that for decimal numbers, except that a 1 is carried to the next left column when two 1s are added.

Following are the basic rules for binary addition:

0 + 0 = 0 i.e. when 0 is added to 0 the addition is 0.

1 + 0 = 1 i.e. when 1 is added to 0 the addition is 1.

0 + 1 = 1 i.e. when 0 is added to 1 the addition is 1.

1+1=10 i.e. when 1 is added to 1 the addition is 10, but it is written as 0 with a carry of 1.

Examples:

1. Let us find $(11)_2 + (01)_2$.

$$\begin{array}{rrrl} 1 & 1 & & \text{carry} \\ & 1 & 1 & \\ + & 0 & 1 & \\ \hline 1 & 0 & 0 & \end{array}$$

While doing this addition, in the rightmost column we get 1 +1 = 1 0 . So 0 is written with a carry 1 to the 2nd column from right side.

In the 2nd column from the right side the carry 1 is added to 1 + 0 i.e. value for this column is 1+(1 + 0) = 1+ 1 = 1 0.

P In the 2nd column from the right side addition is 0 with carry 1 to the next column.

P In the addition in left most column the carry 1 is written.

So $(11)_2 + (01)_2 = (100)_2$.

2. We will find $(11)_2 + (111)_2$.

$$\begin{array}{rrrrl} & 1 & 1 & & \text{carry} \\ & & 1 & 1 & \\ + & 1 & 1 & 1 & \\ \hline 1 & 0 & 1 & 0 & \end{array}$$

While doing this addition, in the rightmost column we get 1 +1 = 10.

So 0 is written in the rightmost column, with a carry 1 to the 2nd column from right side.

In the 2nd column from the right side the carry 1 is added to 1 + 1 i.e. value for this column is 1+(1 + 1) = 1+ 10 = 11.

- **P** In the 2nd column from the right side addition is 1 with carry 1 to the next column.

In the 3rd column from the right side the carry 1 is added to 1 + 1 i.e. value for this column is 1+(0 + 1) = 1+ 1 = 10.

- **P** In the 3rd column from the right side addition is 0, with carry 1 to the next column.

- **P** In the addition, left most column entry is the carry 1.

So $(11)_2 + (111)_2 = (1010)_2$.

3. Consider the addition $(11100)_2 + (10011)_2$.

```
 1                 carry
   1  1  1  0  0
+  1  0  0  1  1
------------------
 1  0  1  1  1  1
```

While doing this addition, in the rightmost column we get 0 +1 = 1.

In the 2nd column from the right side the addition is 0 +1 = 1.

In the 3rd column from the right side the addition is 1 + 0 = 1.

In the 4th column from the right side the addition is 1 + 0 = 1.

In the 5th column from the right side the addition is 1 + 1 = 1 0. So 0 is written in the In the 5th column from the right side, with carry 1 to the next column.

- **P** In the addition, left most column entry is the carry 1.

So $(11100)_2 + (10011)_2 = (101111)_2$.

5.4.2 Subtraction of binary numbers

The subtraction of binary numbers is also done in a similar way , to that for decimal numbers. Following are the basic rules for binary addition:

0 – 0 = 0 i.e. when 0 is subtracted from 0 the subtraction is 0.

1 – 0 = 1 i.e. when 0 is subtracted from 1 the subtraction is 1.

1 – 1 = 0 i.e. when 1 is subtracted from 1 the subtraction is 0.

10 – 1 = 1 i.e. when 1 is subtracted from 10 the subtraction is 1.

Examples:

1. Subtract $(01)_2$ from $(11)_2$.

```
   1  1
–  0  1
--------
   1  0
```

While doing this subtraction, in the rightmost column we get 1 –1 = 0. In the 2nd column from the right side we get 1 – 0 = 1.

So $(1\ 1)_2 - (1\ 0)_2 = (1\ 0)_2$.

2. Find $(1001)_2 - (110)_2$.

$$\begin{array}{r} {}^{1} \quad\quad \text{carry} \\ 1\ 0\ 0\ 1 \\ -\quad 1\ 1\ 0 \\ \hline 0\ 0\ 1\ 1 \end{array}$$

While doing this subtraction, in the rightmost column we get 1 – 0 = 1.

In the 2nd column from the right side 1 cannot be subtracted from 0, so we borrow 1 from the 3rd column from the right side. Then in the 2nd column we obtain the value 10 – 1 = 1.

In the 3rd column from the right side we want 0 – (1+1), because the borrowed 1 is to be subtracted. So we borrow 1 from the 4th column from the right side. Then in these two columns we obtain the value

10 – (1 +1) = 10 – 10 = 00

P $(1001)_2 - (110)_2 = (011)_2$

3. Consider the subtraction $(11100)_2 - (10\,011)_2$.

$$\begin{array}{r} 1\,1\,1\,0\,0 \\ -\ 1\,0\,0\,1\,1 \\ \hline 0\,1\,0\ 0\,1 \end{array}$$

While doing this subtraction, in the rightmost column 1 can't be subtracted from 0, so we borrow 1 from the 2nd column from the right side and obtain the value 10 – 1 = 1.

In the 2nd column from the right side 1 + borrowed 1 = 1+1 = 10 cannot be subtracted from 0, so we borrow 1 from the 3rd column from the right side and obtain the value 10 – 10 = 0.

In the 3rd column from the right side the subtraction is 1 – 0 = 1.

In the 4th column from the right side the subtraction is 1 – 0 = 1.

In the 5th column from the right side the subtraction is 1 – 1 = 0.

P $(11100)_2 - (10011)_2 = (1001)_2$.

5.5 Self-Test 2

Select the correct alternative from the given alternatives.

1. What is the value of 1 + 1, in binary number system?

   (a) 0 **(b)** 1 **(c)** 10 **(d)** 11

2. What is the value of 11 + 11, in binary number system?

   (a) 110 **(b)** 111 **(c)** 101 **(d)** 011

3. What is the value of 1010 +1111, in binary number system?

   (a) 10001 **(b)** 11001 **(c)** 1001 **(d)** 1101

4. What is the value of 11001 + 10101, in binary number system?

(a) 1011110 **(b)** 11111 **(c)** 101110 **(d)** 101011

5. What is the value of 1110 + 110111, in binary number system?

(a) 111110 **(b)** 111000 **(c)** 101010 **(d)** 1000101

6. What is the value of 1 0 – 1, in binary number system?

(a) 0 **(b)** 1 **(c)** 10 **(d)** 11

7. What is the value of 111 – 10, in binary number system?

(a) 110 **(b)** 111 **(c)** 101 **(d)** 011

8. What is the value of 10100 – 11 11, in binary number system?

(a) 101 **(b)** 1001 **(c)** 1010 **(d)** 110

9. What is the value of 11001 – 10101, in binary number system?

(a) 101 **(b)** 111 **(c)** 100 **(d)** 1010

10. What is the value of 111001 – 110111, in binary number system?

(a) 11110 **(b)** 1110 **(c)** 110 **(d)** 10

5.6 The Octal number system

In octal number system the base is equal to 8. The numbers are written using 8 different symbols, which are the digits from 0 to 7. We call these digits by their usual decimal names i.e. zero, one, two etc. Each digit indicates a value, which depends on the position it occupies.

5.6.1 Conversion of a decimal number to a octal number

The procedure of converting a decimal number to its octal equivalent is similar to the procedure of converting a decimal number to its binary equivalent. This procedure consists of dividing the decimal number by 8, until we get a quotient of zero. The remainders of these different divisions are written in the opposite order to that, in which they are obtained. The number written in this way is the octal representation of the decimal number.

Examples:

1. Conversion of $(25)_{10}$ into its octal equivalent number is performed as follows:

Number	Quotient when divided by 8	Remainder
25	3	1
3	0	3

We write the remainders from bottom to the top, it is the required octal equivalent.
P $(25)_{10} = (31)_8$.

2. Conversion of $(111)_{10}$ into its octal equivalent number:

Number	Quotient when divided by 8	Remainder
111	13	7
13	1	5
1	0	1

We write the remainders from bottom to the top, it is the required octal equivalent. **P** $(111)_{10} = (157)_8$.

3. Conversion of $(845)_{10}$ into its octal equivalent number:

Number	Quotient when divided by 8	Remainder
845	105	5
105	13	1
13	1	5
1	0	1

Write the f remainders from bottom to the top, it is the required octal equivalent.
P $(845)_{10} = (1515)_8$.

4. Conversion of $(2088)_{10}$ into its octal equivalent number:

Number	Quotient when divided by 8	Remainder
2088	261	0
261	32	5
32	4	0
4	0	4

Write the f remainders from bottom to the top, it is the required octal equivalent.
P $(2088)_{10} = (4050)_8$.

5.6.2 Conversion of an octal number to a decimal number

To convert a number from octal number system to decimal number system following procedure is used. We multiply the rightmost bit by 8^0, then second right bit by 8^1, the third right bit by 8^2 and continue in this manner till we reach the leftmost bit. These multiplications are then added to find the required decimal equivalent.

Examples:

1. Conversion of $(31)_8$ into decimal equivalent number is performed in following steps:

$$\begin{aligned}(31)_8 &= (3\,\hat{}\,8^1) + (1\,\hat{}\,8^0)\\ &= (3\,\hat{}\,8) + (1\,\hat{}\,1)\\ &= 24 + 1 \quad = (25)_{10}.\end{aligned}$$

2. Conversion of $(157)_8$ into decimal equivalent number:

$$\begin{aligned}(157)_8 &= (1\,\hat{}\,8^2) + (5\,\hat{}\,8^1) + (7\,\hat{}\,8^0)\\ &= (1\,\hat{}\,64) + (5\,\hat{}\,8) + (7\,\hat{}\,1)\\ &= 64 + 40 + 7\\ &= (111)_{10}.\end{aligned}$$

3. Conversion of $(1515)_8$ into decimal equivalent number:

$$\begin{aligned}(1515)_8 &= (1\,\hat{}\,8^3) + (5\,\hat{}\,8^2) + (1\,\hat{}\,8^1) + (5\,\hat{}\,8^0)\\ &= (1\,\hat{}\,512) + (5\,\hat{}\,64) + (1\,\hat{}\,8) + (5\,\hat{}\,1)\\ &= 512 + 320 + 8 + 5\\ &= (845)_{10}.\end{aligned}$$

4. Conversion of $(4050)_8$ into decimal equivalent number:

$$\begin{aligned}(4050)_8 &= (4\,\hat{}\,8^3) + (0\,\hat{}\,8^2) + (5\,\hat{}\,8^1) + (0\,\hat{}\,8^0)\\ &= (4\,\hat{}\,512) + 0 + (5\,\hat{}\,8) + 0\\ &= 2048 + 40\\ &= (2088)_{10}.\end{aligned}$$

5.7 Self-Test 3

Select the correct alternative from the given alternatives.

1. What is the basis for octal number system?

(a) 2 **(b)** 8 **(c)** 10 **(d)** 16

2. Which of the following is not a valid octal number ?

(a) 2467 **(b)** 3489 **(c)** 1000 **(d)** 756

3. What is the octal equivalent of the decimal number 12?

(a) 14 **(b)** 11 **(c)** 101 **(d)** 10

4. What is the octal equivalent of the decimal number 59?

(a) 71 **(b)** 72 **(c)** 73 **(d)** 74

5. What is the octal equivalent of the decimal number 184?

(a) 275 **(b)** 267 **(c)** 270 **(d)** 184

6. What is the octal equivalent of the decimal number 1882?

(a) 3532 **(b)** 3235 **(c)** 3625 **(d)** 1000

7. What is the decimal equivalent of the octal number 110?

(a) 72 **(b)** 73 **(c)** 74 **(d)** 75

8. What is the decimal equivalent of the octal number 1111?

(a) 558 **(b)** 855 **(c)** 585 **(d)** 888

9. What is the decimal equivalent of the octal number 1631?

(a) 129 **(b)** 913 **(c)** 921 **(d)** 219

10. What is the decimal equivalent of the octal number 7714?

(a) 4404 **(b)** 4440 **(c)** 4440 **(d)** 4044

5.8 The Hexadecimal number system

For the hexadecimal number system the base is equal to 16. In this system the numbers are written using 16 different symbols. The symbols used are the digits from 0 to 9 and the letters A, B, C, D, E, and F. Therefore in the hexadecimal number system the letter "A" has the value "10", "B" has the value "11", "C" has the value "12", "D" has the value "13", "E" has the value "14", and "F" has the value "15". We call these symbols by their common names.

5.8.1 Conversion of a decimal number to a hexadecimal number

The procedure of converting a decimal number to its hexadecimal equivalent is similar to the procedure of converting a decimal number to its octal equivalent. This procedure consists of dividing the decimal number by 16, until we get a quotient of zero. The remainders of these different divisions are written in the opposite order to that, in which they are obtained. The number written in this way is the hexadecimal representation of the decimal number.

Examples:

1. Conversion of $(25)_{10}$ into its hexadecimal equivalent number is performed as follows:

Number	Quotient when divided by 16	Remainder in decimal digits	Symbol for Remainder
25	1	9	9
1	0	1	1

We write the remainders from bottom to the top, it is the required hexadecimal equivalent of 25.

P $(25)_{10} = (19)_{16}$.

2. Conversion of $(22850)_{10}$ into its hexadecimal equivalent number:

Number	Quotient when divided by 16	Remainder in decimal digits	Symbol for Remainder
22850	1428	2	2
1428	89	4	4
89	5	9	9
5	0	5	5

We write the remainders from bottom to the top, it is the required hexadecimal equivalent.

P $(22850)_{10} = (5942)_{16}$.

3. Conversion of $(43981)_{10}$ into its hexadecimal equivalent number:

Number	Quotient when divided by 16	Remainder in decimal digits	Symbol for Remainder
43981	2784	13	D
2784	174	12	C
174	10	11	B
10	0	10	A

Write the remainders from bottom to the top, it is the required hexadecimal equivalent.

P $(43981)_{10} = (ABCD)_{16}$.

4. Conversion of $(10895)_{10}$ into its hexadecimal equivalent number:

Number	Quotient when divided by 16	Remainder in decimal digits	Symbol for Remainder
10895	680	15	F
680	42	8	8
42	2	10	A
2	0	2	2

Write the f remainders from bottom to the top, it is the required hexadecimal equivalent.

P $(10895)_{10} = (2A8F)_{16}$.

5.8.2 Conversion of an hexadecimal number to a decimal number

To convert a number from hexadecimal number system to decimal number system following procedure is used. We multiply the rightmost bit by 16^0, then second right bit by 16^1, the third right bit by 16^2 and continue in this manner till we reach the leftmost bit. These multiplications are then added to find the required decimal equivalent.

Examples:

1. Conversion of $(19)_{16}$ into decimal equivalent number is performed in following steps:

   $(19)_{16}$ = (1 ˆ 16^1) +(9 ˆ 16^0)

   = 16 + 9 = $(25)_{10}$

2. Conversion of $(5942)_{16}$ into decimal equivalent number:

   $(5942)_{16}$ =(5 ˆ 16^3) + (9 ˆ 16^2) + (4 ˆ 16^1) + (2 ˆ 16^0)

   = (5 ˆ 4096) + (9 ˆ 256) + (4 ˆ 16) +(2 ˆ 1)

   = 20480 + 2304 + 64 + 2

   = $(22850)_{10}$

3. Conversion of $(A\ B\ C\ D)_{16}$ into decimal equivalent number:

   $(ABCD)_{16}$ = (A ˆ 16^3) + (B ˆ 16^2) + (C ˆ 16^1) + (D ˆ 16^0)

   Now, we know that in decimal system "A" has the value "10", "B" has the value "11", "C" has the value "12", "D" has the value "13".

   P $(ABCD)_{16}$ = (10 ˆ 16^3) + (11 ˆ 16^2) + (12 ˆ 16^1) + (13 ˆ 16^0)

   = (10 ˆ 4096) + (11 ˆ 256) + (12 ˆ 16) + (13 ˆ 1)

   = 40960 + 2816 + 192 + 13

   = $(43981)_{10}$

4. Conversion of $(2\ A\ 8\ F)_{16}$ into decimal equivalent number:

   $(2A8F)_{16}$ = (2 ˆ 16^3) + (A ˆ 16^2) + (8 ˆ 16^1) + (F ˆ 16^0)

   Now, we know that in decimal system "A" has the value "10", "F" has the value "15".

   P $(2A8F)_{16}$ = (2 ˆ 16^3) + (A ˆ 16^2) + (8 ˆ 16^1) + (F ˆ 16^0)

   = (2 ˆ 4096) + (10 ˆ 256) + (8 ˆ 16) + (15 ˆ 1)

   = 8192 + 2560 + 128 + 15

   = $(0895)_{10}$

5.9 Self-Test 4

Select the correct alternative from the given alternatives.

1. What is the basis for hexadecimal number system?

   (a) 2 **(b)** 8 **(c)** 10 **(d)** 16

2. Which of the following is not a valid hexadecimal number ?

(a) 2D67 (b) H3A8C (c) 1CEFD (d) 1756

3. What is the hexadecimal equivalent of the decimal number 12?

   (a) 14 (b) B (c) 12 (d) C

4. What is the hexadecimal equivalent of the decimal number 59?

   (a) 3B (b) 4B (c) 3A (d) 3C

5. What is the hexadecimal equivalent of the decimal number 184?

   (a) A8 (b) B8 (c) C8 (d) B7

6. What is the hexadecimal equivalent of the decimal number 6699?

   (a) 1A2B (b) A1B2 (c) AB12 (d) 12BA

7. What is the decimal equivalent of the hexadecimal number 8E?

   (a) 144 (b) 140 (c) 142 (d) 442

8. What is the decimal equivalent of the hexadecimal number 6DCE?

   (a) 28172 (b) 28731 (c) 28110 (d) 28175

9. What is the decimal equivalent of the hexadecimal number F1AB?

   (a) 55867 (b) 61867 (c) 61585 (d) 67861

10. What is the decimal equivalent of the hexadecimal number BCA?

    (a) 3029 (b) 3913 (c) 3018 (d) 3219

5.6 Summary

In this unit learners studied the following topics in details:

1. What is the Binary number system?
2. How to convert a decimal number to its binary equivalent.
3. How to convert a binary number to its decimal equivalent.
4. Addition and subtraction of binary numbers.
5. The Octal number system, Conversion of a decimal number to a octal number and vice versa.
6. The Hexadecimal number system, Conversion of a decimal number to a hexadecimal number and vice versa.

Unit 6: Permutations and Combinations

6.0 Objectives

By the end of this Unit, learners should be able to:

- Understand Addition principle of counting.
- State and apply Multiplication principle of counting.
- Explain factorial of a number.
- Understand the concept of permutations and their applications.
- Understand the concept of combinations and their applications.

6.1Introduction

In day-to-day life we come across problems of counting the number of ways of doing certain things. Such counting problems arise throughout mathematics and Computer Science also. In this unit we will study some techniques of counting without actually listing all possible answers. One such technique is to use principles of counting such as addition principle, multiplication principle, principle of Inclusion and Exclusion and Pigeonhole principle etc. In this unit we will study only two of these principles. We will also study that how to use concept of permutations and combinations in computing problems. We will also study the concept of factorial of a number, as it is used in solving problems related with permutations and combinations.

6.2 Addition and Multiplication principles of counting:

As stated above there are different principles used to solve problems related with counting. Here we will study two simplest principles of counting which are addition principle and multiplication principle.

6.2.1 Addition principle

In mathematics, addition principle or the rule of sum is a basic counting principle. Stated simply, it is the idea that, if we have 'a' number of ways of doing something and 'b' number of ways of doing another thing and we cannot do both things at the same time, then there are 'a + b' ways to do one of the actions.

More formally, the addition principle can be stated as a fact about set theory as follows.

Addition principle: If A and B are two finite disjoint sets then the number of elements in their union, A £ B is the addition of the number of elements in A and the number of elements in B. Symbolically we can write

|A £ B| = |A| +|B|, where |A | denotes the number of elements in the set A.

Examples:

- If there are 5 varieties of fruits and 3 types of sweets and 8 types of ice creams for a Dinner party. What is the number of different ways a person can select any one of these food items?

Solution: Here at a time one can select any one of the items not all or more than one, so here selection process is based on addition principle. Total number of ways of selection is 5+3+8 = 16.

- If there are 10 different mathematics books, 15 different computer science books and 8 different management books on the shelf. In how many ways one can select one of these books from the shelf?

  Solution: Here at a time one can select any one of the books and not more than one, so here also selection process is based on addition principle. Total number of books is 10 + 15 + 8 = 33. So one can select any one of these books from the shelf, in 33 possible ways.

- If the available computer software programming courses are C, C++, Java and Visual BASIC and Mathematics courses are Algebra, Discrete mathematics and calculus. Each student is allowed to offer one software-programming course or one Mathematics course. In how many ways one student can offer one of these courses?

  Solution: Here at a time one student can offer any one course, out of 4 different software-programming courses and 3 different Mathematics courses. So here answer is based on addition principle. Total number of ways of offering a course is 4 +3 = 7.

6.2.2 Multiplication principle

In mathematics the multiplication principle or the rule of product is another basic counting principle. Stated simply, it is the idea that if we have 'a' ways of doing something and 'b' ways of doing another thing, then there are '$a \times b$' ways of performing both actions simultaneously. It is also mentioned as the fundamental principle of counting.

More formally, the multiplication principle is about the number of elements in the Cartesian product of 2 sets.

Multiplication principle: If A and B are any two finite sets then the number of elements in the Cartesian product $A \times B$, of sets A and B is the multiplication of the number of elements in set A and the number of elements in set B . Symbolically we can write

$|A \times B| = |A| \times |B|$, where $|A|$ denotes the number of elements in the set A.

Examples:

- How many two symbol labels can be formed if the first symbol can be any capital letter from English alphabet and the second can be any digit?

  Solution: In this case to form these labels one can select any capital letter in 26 ways, and for each of these 26 letters the 2nd symbol can be selected from the 10 digits, which are from 0 to 9. Both selections are simultaneously done. Hence here multiplication principle is to be used. So the total number of forming the labels is $26 \times 10 = 260$.

  In terms of set theory we can write it as,

  E = set of English letters = {A,-- - - Z } and

  D = set of digits = {0 to 9}.

  $|E| = 26$ and $|D| = 10$ and the required no of labels= $|E| \times |D| = 26 \times 10 = 260$

- A man has 3 hats, 6 shirts and 2 pants with him. How many different outfits he can assemble?

  Solution: In this case the man can select any 1 hat from the 3 in the wardrobe, for each selection of hat; he can select 1 shirt from the 6 in the wardrobe. So there are total $3 \times 6 = 18$ ways of selection of a hat and a shirt. For each of these 18 selections he can select from the 2 pants. Hence by multiplication principle the total number of outfits he can assemble = $3 \times 6 \times 2 = 36$.

- If a password for a computer system must consists of a sequence of four different letters from English alphabet followed by any two digits from 0 through 9, then how many different passwords can be formed?

  Solution: Since there are 26 letters in English alphabet, the first letter in the password can be any of these 26 letters.

  After the selection of the first letter, the 2^{nd} letter in the password can be any of the remaining 25 letters as all letters are different.

  After the selection of the first two letters, the 3^{rd} letter in the password can be any of the remaining 24 letters as all letters are different.

  After selecting the first three letters, the 4^{th} letter in the password can be any of the remaining 23 letters as all letters are different.

  After selecting the 4 letters, then for the next place any digit from 0 through 9 can be selected in 10 ways.

  Now for the last place in the password again any of the 10 digits can be used as the two digits need not be different.

  Hence by the multiplication principle,

  The number of passwords formed = $26 \times 25 \times 24 \times 23 \times 10 \times 10 = 3, 58, 80,000$.

6.3 Factorial of a number

In problems of counting it is often needed to find the multiplication of consecutive nonnegative integers. Such multiplications are expressed using factorial notations. Factorials are used in problems of counting which involves permutations and combinations. These are also used in probability theory and number theory.

6.3.1 Factorial: If n is a nonnegative integer, then factorial of n is defined as the product of all natural numbers from 1 to n. For number 0, its factorial is defined to be 1. Factorial of n is denoted by the notation n!. It is read as "n factorial".

So we have, $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ and $0! = 1$.

Examples:

- $1! = 1$
- $3! = 3 \times 2 \times 1 = 6$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$
- $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40, 320$

Note that:

1. From the above calculations we observe that $8! = 8 \times 7!$.

   This result is in general true i.e. $n! = n \times (n-1)!$.

2. We also observe that,

$$8 \times 7 \times 6 \times 5 \times 4 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{8!}{3!}$$

In general this result is written as,

$$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$$

$$= \frac{n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times (n-r+1)}{1} \times \frac{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-r)!}.$$

6.4 Permutations

Permutation of a set of distinct objects is an ordered arrangement of these objects. Suppose that we have n distinct objects, in how many distinct ways can we rearrange these n objects, in a row?

If we have n distinct objects, and we want to arrange all of them in a row, then we have n ways of placing the first object in the row. We have $(n-1)$ ways of placing an object for the second place in the row as one object is already placed. We then have $(n-2)$ ways of placing the third object in row because two objects are already placed. Arranging in the similar manner we have 1 last object for the n^{th} place. Each possible choice for arranging the n distinct elements is called a permutation.

Now, out of these n distinct objects if we want to arrange r objects only, then we have n choices for the first position, $(n-1)$ remaining choices for the second position, $(n-2)$ choices for the third position, and so on for r^{th} position the possible choices of objects are $(n-(r-1)) = (n-r+1)$ in number.

So the number of arrangements or permutations of n objects taken r at a time is,

$$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1).$$

The two key things to notice about permutations are that there is no repetition of objects allowed and that the order is important.

6.4.1 Permutations: Given n different objects, the number of permutations that can be formed by taking r ($r \le n$) objects at a time, denoted by $P(n, r)$ or $^{n}P_{r}$ is defined as

$$P(n, r) = {}^{n}P_{r} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) =$$

$$= \frac{n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times (n-r+1)}{1} \times \frac{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-r)!}$$

We conclude that the total number of permutations of n distinct objects taken all at a time is,

$$P(n, n) = {}^{n}P_{n} = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1.$$

And using the formula explained above we can write it as, ${}^{n}P_{r} = \dfrac{n!}{(n-r)!}$.

Note that:

1. $${}^{n}P_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1.$$

2. Since a permutation is the number of ways in which you can arrange objects, it will always be a natural number. The denominator in the formula will always divide the numerator.

Examples:

- List all possible arrangements of the letters in the word "ONE". How many arrangements are possible?

  Solution: The list of all such arrangements is, ONE, OEN, NEO, NOE, EON, and ENO. There are 6 arrangements possible as listed, and by the above formula also we can find this number.

  It is the number of arrangements of 3 different objects taken all at a time

  $$= {}^{3}P_{3} = 3! = 3 \times 2 \times 1 = 6.$$

- What is the number of all possible passwords for a computer system, if a password must consists of a sequence of five different letters from English alphabet?

  Solution: In this case to write the list of all such passwords is obviously not possible. What we want to find is, the number of possible arrangements of 5 letters out of the 26 letters in English alphabet, or it is the number of permutations of 26 objects taken 5 at a time. And by the above formula this number $= {}^{26}P_{5}$

  $$= \frac{26!}{(26-5)!} = \frac{26!}{21!} = 26 \times 25 \times 24 \times 23 \times 22 = 78{,}93{,}600.$$

- A debating team consists of 4 boys and 3 girls. Find the number of ways they can sit in a row.

  Solution: There are total 7 persons and the required number is the number of permutations of 7 objects taken all at a time.

  By the above formula this number $= {}^{7}P_{7} = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

6.5 Combinations

How many ways are there to select 3 different letters from the set of letters {a, b, c, d}?

This problem is different than the arrangement problem or permutations. In permutations, the order is all-important. While counting arrangements we count the arrangement "a b c" as different from the arrangement "b c a" and these two arrangements are different from the arrangement "a c b". But when we count selections, we are concerned about the fact that, whether a, b and c have been selected or not. So the above 3 arrangements are the same selection.

Hence all the selections of the letters {a, b, c, d} taken three at a time are {a, b, c}, {a, b, d}, {a, c, d} and {b, c, d} .These are four combinations. We call this the number of combinations of 4 things taken 3 at a time.

6.5.1 <u>Combinations</u>: Given n different objects, the number of combinations (subsets) that can be formed by taking r ($r \le n$) objects at a time, denoted by C(n , r) or ^nC_r is defined as

$$C(n, r) = {}^nC_r = \frac{n!}{r!(n-r)!}.$$

Hence using this definition the above number of ways to select 3 different letters from the set of letters { a, b, c, d, } is denoted as $^4C_3$ or C(4,3).

and $^4C_3 = \dfrac{4!}{3!(4-3)!} = \dfrac{4}{1!} = 4$

Properties of Combinations: Some basic properties of combinations are listed below. These properties can be proved using above formula of combination or by other techniques also.

1. $^nC_0 = 1 = {}^nC_n$
2. $^nC_1 = n = {}^nC_{n-1}$
3. $^nC_r = {}^nC_{n-r}$
4. $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Examples:

- If a group of 30 people have been trained for a particular computer software project. How many ways are there to select 6 people from this group for advance training?

  <u>Solution:</u> This problem is different than the problems related with permutations. In permutations, the order is the only important aspect. While considering the permutations we count the permutations of persons sitting in row, in the order a, b, c, d, e, f as different from the permutation a, b, d, c, e, f. But the above 2 permutations are the same combinations.

  The required number in this case is, the number of combinations of 20 people taken 6 at a time

  $$= {}^{20}C_6 = \frac{20!}{6!(20-6)!} = \frac{20!}{6! \times 14!} = \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 38,760$$

- Some computer monitors can display any of 4096 different shades of colours. If only 12 shades of colours can be displayed at a time, how many groups of 12 shades can be displayed.

  <u>Solution:</u> This problem is of calculating the number of combinations of 4096 shades of colours taken 12 at a time,

$$= {}^{4096}C_{12} = \frac{4096!}{12!\,(4096 - 12)!} = \frac{4096!}{12! \times 4084!}.$$

It is too big value to calculate even if we use calculators. In such cases it is common practice to write answers in terms of factorial notations.

- How many ways are there to form a committee of 5 from four men and six women ?

  Solution: To form a committee of 5 from four men and six women is same as to select 5 persons from four men and six women i.e. from total 10 persons. So, the required number in this case is, the number of combinations of 10 people taken 5 at a time

  $$= {}^{10}C_5 = \frac{10!}{5!\,(10 - 5)!} = \frac{10!}{5! \times 5!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} = 252 .$$

- How many ways are there to form a committee from four men and six women with at least 2 men and at least twice as many women as men?

  Solution: In each committee to be formed there must be minimum 2 men and at least twice as many women as men. So, there are different possible cases as listed below:

  Case 1: A committee can be formed of 2 men and 4 women.

  In this case the 2 men can be selected from the 4 in ${}^4C_2$ number of ways, and the 4 women can be selected from the 6 in ${}^6C_4$ number of ways. And using multiplication principle the number of such committees = ${}^4C_2 \times {}^6C_4$ =

  $$= \frac{4!}{2!\,(4-2)!} \times \frac{6!}{4!\,(6-4)!} = \frac{4!}{2! \times 2!} \times \frac{6!}{4! \times 2!} = \frac{4 \times 3}{2 \times 1} \times \frac{6 \times 5}{2 \times 1} = 6 \times 15 = 90.$$

  Case 2: A committee can be formed of 2 men and 5 women.

  In this case the 2 men can be selected from the 4 in ${}^4C_2$ number of ways, and the 5 women can be selected from the 6 in ${}^6C_5$ number of ways. And by multiplication principle the number of such committees = ${}^4C_2 \times {}^6C_5$

  $$= \frac{4!}{2!\,(4-2)!} \times \frac{6!}{5!\,(6-5)!} = \frac{4!}{2! \times 2!} \times \frac{6!}{5! \times 1!} = \frac{4 \times 3}{2 \times 1} \times \frac{6}{1} = 6 \times 6 = 36.$$

  Case 3: A committee can be formed of 2 men and 6 women.

  In this case the 2 men can be selected from the 4 in ${}^4C_2$ number of ways, and the 6 women can be selected from the 6 in ${}^6C_6$ number of ways. And by multiplication principle the number of such committees = ${}^4C_2 \times {}^6C_6$

  $$= \frac{4!}{2!\,(4-2)!} \times \frac{6!}{6!\,(6-6)!} = \frac{4!}{2! \times 2!} \times \frac{6!}{6! \times 0!} = \frac{4 \times 3}{2 \times 1} \times \frac{1}{1} = 6 \times 1 = 6.$$

  Case 4: A committee can be formed of 3 men and 6 women.

  In this case the 3 men can be selected from the 4 in ${}^4C_3$ number of ways, and the 6 women can be selected from the 6 in ${}^6C_6$ number of ways. And by multiplication principle the number of such committees = ${}^4C_3 \times {}^6C_6$

$$= \frac{4!}{3!\,(4-3)!} \times \frac{6!}{6!\,(6-6)!} = \frac{4!}{3! \times 1!} \times \frac{6!}{6! \times 0!} = \frac{4}{1} \times \frac{1}{1} = 4 \times 1 = 4.$$

Now the committee may be consisting of 2 men and 4 women or of 2 men and 5 women or of 2 men and 6 women or of 3 men and 6 women. Hence by addition principle the total number of such committees formed = 90 + 36 + 6 + 4 = 136.

‹ Consider 4 vowels a, e, o, u and eight consonants b, c, d, p, q, r, s , t from English alphabet . Find the number of five lettered words (meaningful or meaningless), containing 2 different vowels and 3 different consonants, from above 12 letters.

Solution: The 2 different vowels can be selected from the given four vowels in ${}^{4}C_{2}$ ways and the 3 different consonants can be selected from the given eight consonants in ${}^{8}C_{3}$ ways. Further more each 5 lettered word is an arrangement of the letters i.e. a permutation, number of these permutations is ${}^{5}P_{5} = 5!$ Thus by multiplication principle, the number of five lettered words, containing 2 different vowels and 3 different consonants

$$= {}^{4}C_{2} \times {}^{8}C_{3} \times 5! = \frac{4!}{2!\,(4-2)!} \times \frac{8!}{3!\,(8-3)!} \times 5! = \frac{4!}{2! \times 2!} \times \frac{8!}{3! \times 5!} \times 5!$$

$$= \frac{4 \times 3}{2 \times 1} \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times 5! = 6 \times 56 \times 120 = 40,320$$

6.6 Summary

In this unit learners studied the following topics in details:

1. Statement of the addition principle and its applications
2. Statement of the multiplication principle and how to use it in problems.
3. Concept of a factorial of a natural number n.
4. Concept of permutations of n objects taken r at a time.
5. Concept of combinations of n objects taken r at a time.

Unit 7: Mathematical Logic

7.0 Objectives

By the end of this Unit, learners should be able to:

- Define statements in accordance with mathematical logic.
- Find the truth-value of a statement.
- Construct the truth tables for different propositional forms.
- Explain various logical connectives and compound statement.
- Define logical equivalence.
- Prove the logical equivalence of statement patterns.
- Understand tautology and contradiction.
- Find the converse, inverse and contrapositive of a conditional statement.

7.1Introduction

In this unit we will discuss about one of the oldest branches of mathematics, which is known as Logic. Logic is considered as the science of reasoning. The earliest use of mathematics and geometry in relation to logic and philosophy goes back to the ancient Greeks such as Euclid, Plato, and Aristotle.

The field of logic ranges from core topics such as the study of fallacies and paradoxes to the specialized analysis of reasoning using probability. Traditionally, logic is studied as a branch of philosophy. And in modern era we know that the development of formal logic and its implementation in computing machinery is the foundation of computer science.

7.2 Statements in Mathematical Logic

While studying Mathematical logic we deal with a discrete mathematical object, which is a statement or proposition.

7.2.1 Statement: A statement is defined as a declarative sentence, which is either true or false, but not both at a time.

It is also known as a proposition or an atomic statement.

Examples:

Consider the following sentences,

- "The earth rotates around the sun."

  This sentence is declarative as it is giving us some information. It is a true statement. Hence it is a logical statement or a proposition.

- "It is not raining today. "

  This sentence is also giving some information. It is either true or false, but of course, not both at any particular time. Hence it is a Statement.

- "Give me that book."

This sentence is not giving any information. Also we cannot decide whether it is true or false. Hence it is not a Statement.

- "Are you interested in Cricket?"

  This question is not giving any information. Also we cannot decide whether it is true or false. Hence it is not a statement.

- "The positive divisors of 10 are 1, 2, 5 and 10 only."

  This sentence is declarative as it is giving some information. It is a true statement. Hence it is a logical Statement or a proposition.

- " 4 + 15 > 20."

  This sentence is giving information about the addition of the numbers 4 and 15. It is false as 4 +15 = 19 O 20. Hence it is a Statement.

- "19 is a prime number."

  This sentence is giving information about the number 19. It is a true sentence, as 19 is in fact a prime number. Hence it is a Statement.

- "How nice!"

  This sentence is not giving information about anything. Hence it is not a statement.

- "She is tall."

  This is not a statement because, it is not giving any particular information as we do not know that who 'she' is.

  So we know that Imperative sentences, Exclamatory sentences and Interrogative sentences are not statements in Logic. The fundamental identities are statements in Logic.

7.2.2 Truth-value of a statement: The truth-ness or falsity of the statement is known as the truth-value of the statement.

If the statement is true, we say that its truth-value is true and denote it by the letter 'T' or by the number "1". If the statement is false, we say that its truth-value is False and it is denoted by the letter 'F' or by the number '0'.

Examples:

Consider the following sentences,

- "The earth rotates around the sun".

  This statement is true. Hence it is a logical Statement, which has truth-value "T".

- "The positive divisors of 10 are 1, 2, 5 and 10 only. "

  This statement is true. So it has truth-value "T".

- "4 + 15 > 20."

  This statement is false. So it has truth-value "F".

- "18 is a prime number."

  This statement is false. So it has truth-value "F".

7.3 Self-Test 1

Select the correct alternative from the given alternatives.

1. Which of the following is a logical statement?

(a) Mumbai is in India. **(b)** What are you doing?

(c) He is a kind person. **(d)** $x + 9 = 12$.

2. Which of the following is not a logical statement?

(a) Mumbai is in India. **(b)** Put that pen on the table.

(c) It is raining. **(d)** $8 + 9 = 12$.

3. Which of the following is a logical statement?

(a) If x is tall then y is handsome **(b)** Please, be seated.

(c) Kapil Deo is a great cricketer. **(d)** $x + 9y = 12$.

4. Which of the following is not a logical statement?

(a) Mumbai is in India and 19 ‡ 12. **(b)** How are you?

(c) It is raining or it is not raining. **(d)** 8 is a square of 4.

5. Which of the following is a true logical statement?

(a) Mumbai is in England. **(b)** Are you going home?

(c) The moon rotates around the earth. **(d)** 8 is a square of 4.

6. Which of the following is a false logical statement?

(a) London is in England. **(b)** Are you going home?

(c) The earth rotates around the sun. **(d)** 8 is the cube of 4.

7. What is the truth value of the statement,"1 is a prime integer." ?

(a) T **(b)** F **(c)** Both T and F **(d)** Neither T nor F

8. What is the truth value of the statement, "1 is an odd integer." ?

(a) T **(b)** F **(c)** Both T and F **(d)** Neither T nor F

9. What is the truth value of the statement, "x is a prime integer." ?

(a) T **(b)** F **(c)** Both T and F **(d)** it is not a statement.

10. What is the truth value of the statement, " Tiger is a wild animal." ?

(a) T **(b)** F **(c)** Both T and F **(d)** Neither T nor F

7.4Logical connectives

In our day-to-day language, we form new sentences with two or more sentences. Similarly in mathematical logic we use some words to form new statements. Such words are called logical connectives. Examples of logical connectives are words such as 'and', 'or' and 'not'. Also the phrases such as if - then and if and only if are examples of logical connectives.

7.4.1 Types of logical statements

Simple statement: A logical statement in which no connective is used is called a "simple statement "or a proposition or an atomic statement . Generally small case letters are use to denote simple statement.

Examples:

p: London is in England.

q: 2 + 7 = 10.

r: 21 is a multiple of 9.

s: 11 2 121

Compound statement: If a statement is formed by joining 2 or more simple statements using logical connectives then it is called a "compound statement" or "a propositional form.

Examples:

p: Mumbai is in England and it is raining.

q: 2 + 7 = 10 or 1 is a prime number.

r: If 21 is a multiple of 9 then 2+ 2= 4.

s: It is not true that , 11 2 121.

7.4.2 Types of Compound statements

The important compound statements are negation, conjunction, disjunction, conditional and biconditional statements. The truth-values of these connectives are defined by means of the truth tables.

A truth table is a computational device by which we can determine the truth-values of compound statements. We tabulate all possible values of the simple statements in it. If the compound statement contains n variables i.e. n simple statements then the truth table contains 2^n rows.

1. Negation: Negation is the compound statement in which the word "not" is the connective used. We always use negations of the sentences to indicate exactly opposite meaning.

If 'p' is a simple statement, then the statement 'not p' is known as the **negation of p**. It is denoted symbolically as 'r p'.

Clearly, if p is true statement then its negation is false and if p is false then ~p is true statement. So the truth-values of the negation are as per the truth table.

p	r p
T	F
F	T

Examples:

p	**r p**
Logic is easy.	Logic is not easy
2 is not a rational number.	2 is a rational number.
The earth rotates around the moon.	The earth does not rotate around the moon.

Note that, while writing the negations we use the logical connective "not" and do not write the words of opposite meaning.

2. **Conjunction** is the compound statement in which connective used is the word "and". Suppose 'p' and 'q' are any two statements. The conjunction of 'p' and 'q' is defined as the statement 'p and q'. It is denoted symbolically as p ´ q.

Example: If **p :** Sunday is a holiday,

q : Every day I study for at least 4 hours,

then the conjunction of p and q is

p ´ q : Sunday is a holiday and Every day I study for at least 4 hours.

In case of Conjunction, the statement p ´ q is true if both p as well as q are true statements else p ´ q is a false statement. Therefore the truth table for conjunction is:

p	**q**	**p ´ q**
T	T	T
T	F	F
F	T	F
F	F	F

3. **Disjunction** is the compound statement in which the connective "or" is used. Suppose p and q are two statements then the disjunction of 'p' and 'q' is the statement 'p or q'. It is denoted symbolically as pµq.

Example: If **p :** It is raining now. **q :** The air is fresh now.

then the disjunction of p and q is

pµq : It is raining now or the air is fresh.

In case of Disjunction, the statement p µ q is true if either p or q is true and if both p and q are false the Disjunction is false. Therefore the truth table of disjunction is:

p	**q**	**p µ q**
T	T	T
T	F	T
F	T	T
F	F	F

4. **Conditional Statement or Implication:** It is a compound statement in which, we combine two statements by the phrase "If – then" or "implies that". This gives us a new statement. Suppose p and q are two statements then the compound statement "If p, then q" is called a conditional statement. We denote this conditional statement symbolically as p , q.

The statement 'p' is known as the hypothesis or antecedent. The statement 'q' is known as the conclusion or consequent.

The conditional statement "if p then q" can be phrased in different ways as:

"p only if q " or "q, provided that p" or

"q if p" or "p implies q" or

" p is a sufficient condition for q" or "q is a necessary condition for p".

Examples:

‹ If two simple statements are,

p: The student has a quest for knowledge. q: The student will take up new courses.

then the conditional statement is :"If the student has a quest for knowledge, then he will take up new courses"

‹ Suppose two simple statements are, p: 3 +5 = 9 and q: 5 is even number. Then the conditional statement is: "5 is even number if, 3 +5 = 9".

A conditional statement is False only when hypothesis 'p' is true, but the conclusion 'q' is False. It is true for all remaining combinations of the truth-values of p and q. Therefore the truth table of a conditional statement is:

p	**q**	**p , q**
T	T	T
T	F	F
F	T	T
F	F	T

5. Biconditional Statement or Double Implication: It is a compound statement in which, we combine two statements by the phrase "If and only if". Suppose p and q are any two statements, then the compound statement "p if and only if q" is called a biconditional statement. It is denoted by p } q

Examples:

‹ If two simple statements are,

p: The student has a quest for knowledge. q: The student will take up new courses.

Then the biconditional statement is: "The student has a quest for knowledge if and only if he will take up new courses"

‹ If two simple statements are, p: 3 +5 = 9 and q: 5 is even number.

Then the conditional statement is: " 3 +5 = 9 if and only if 5 is even number".

The biconditional statement "p if and only if q" can be phrased in different words as "p is a necessary and sufficient condition for q". The bi-conditional statement p } q, is true if both p and q are having the same truth values.

The truth table of the biconditional statement p } q is as follows:

p	**q**	**p } q**
T	T	T
T	F	F
F	T	F
F	F	T

Using these 5 connective different compound statement patterns can be formed. We can decide the truth-values of such statement patterns by preparing their truth tables.

Examples:

‹ The truth table for the compound statement p´ r q contains 4 columns which represent the statements p, q , r q and p ´ r q respectively. The table is as follows:

p	q	r q	p ´ r q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Similarly the truth table for the compound statement (pµ q)´ r p is as follows:

p	q	pµq	r p	(p µ q) ´ rp
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

‹ The truth table for the statement pattern (r p µ r q) µ p is as follows:

p	q	r p	r q	rpµrq	(rp µ rq) µ p
T	T	F	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

‹ The truth table for the compound statement r (p ´ q) µ r (q µ r) is as follows:

As this statement pattern contains 3 simple variables, there are 2^3 combinations of truth-values. Hence there are 2^3 = 8 rows in this truth table.

p	q	r	p´q	q µ r	r (p ´q)	r (q µr)	r (p ´q) µ r (q µr)
T	T	T	T	T	F	F	F
T	T	F	T	T	F	F	F
T	F	T	F	T	T	F	T
T	F	F	F	F	T	T	T
F	T	T	F	T	T	F	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	F	T
F	F	F	F	F	T	T	T

7.5 Self-Test 2

Exercise 1: Select the correct alternative from the given alternatives.

1. Which of the following is a simple statement?

(a) Mumbai is in India and Paris is in England. **(b)** It is not raining.
(c) If roses are red then violets are blue. **(d)** 9 ‡ 12.

2. Which of the following is a compound statement?

(a) 9 ‡ 12. **(b)** Paris is in England.
(c) It is raining or it is cold. **(d)** Roses are red.

3. Which of the following words is a connective in logic?

(a) yes **(b)** in fact **(c)** then **(d)** not

4. Which of the following words is not a connective in logic?

   (a) and **(b)** or **(c)** then **(d)** not

5. Which of the following is an example of a negation?

   (a) Mumbai is in India and Paris is in England. **(b)** It is not raining.
   (c) If roses are red then violets are blue. **(d)** 9 ‡ 12.

6. Which of the following is an example of a conjunction?

   (a) Paris is in England and London is in India. **(b)** It is raining or it is cold.
   (c) If roses are red then violets are blue. **(d)** 9 ‡ 12.

7. Which of the following is an example of a implication?

   (a) Mumbai is in India and Paris is in England. **(b)** It is not raining.
   (c) If roses are red then violets are blue. **(d)** 9 ‡ 12.

8. Which of the following is an example of a disjunction?

   (a) If roses are red then violets are blue. **(b)** It is raining or it is cold.
   (c) Paris is in England and London is in India. **(d)** 9 ‡ 12 if and only if 7 O16.

9. Which of the following is an example of a biconditional statement?

   (a) If roses are red then violets are blue. **(b)** It is raining or it is cold.
   (c) Paris is in England and London is in India. **(d)** 9 ‡ 12 if and only if 7 O16.

10. How many rows are there in the truth table, if the compound statement pattern contains 5 simple statements?

   (a) 2^4 **(b)** 4 **(c)** 8 **(d)** 2^5

Exercise 2: Write the truth table of each statement pattern given below.

1. r (p } q) **2.** r p µ r q **3.** (p ´ q) , (p µ r q)
4. (p , q) , r **5.** [r (p ´ q)] } (q µ r)

7.6 Logical Equivalence

In many cases two different compound statements, which are formed by the same simple statements (but in which different connectives appear), have the same truth-values. Such compound statements are logically equivalent statements. In such cases, we can replace one compound statement by another. For example, you may come across a complex logical condition in a computer program and you can replace it by another logically equivalent, but simpler condition. We define this concept formally as follows.

7.6.1Logically Equivalent statements: Suppose that P and Q are two compound statements made up of the same atomic statements $p_1, p_2, \ldots, p_n$. We say that P and Q are logically equivalent if for every combination of the truth values of the atomic statements $p_1, p_2, \ldots, p_n$; the truth values of P and Q are identical.

If P and Q are logically equivalent, we denote this by writing, P ′ Q.

To verify whether P and Q are logically equivalent, we construct the truth table for P and Q. If the columns which give truth-values of P and Q are identical, then P and Q are logically equivalent. Otherwise they are not logically equivalent.

Examples:

- Verify whether $p \rightarrow q \equiv \sim p \vee q$

  Proof: We construct the truth table as below

1	2	3	4	5
p	**q**	$p \rightarrow q$	$\sim p$	$(\sim p) \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

  From this table we observe that the column corresponding to $p \rightarrow q$ i.e. 3rd column in the table has same truth-values as that of the column corresponding to $\sim p \vee q$ i.e. 5th column in the table. Hence we conclude that $p \rightarrow q \equiv \sim p \vee q$.

- Verify whether $\sim(p \rightarrow q) \equiv p \wedge (\sim q)$

  Proof: We construct the truth table as below

1	2	3	4	5	6
p	**q**	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \wedge (\sim q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

  From this table we observe that the column corresponding to $\sim(p \rightarrow q)$ i.e. 3rd column in the table has same truth values as that of the column corresponding to $p \wedge (\sim q)$ i.e. 6th column in the table. Hence we conclude that $\sim(p \rightarrow q) \equiv p \wedge (\sim q)$.

7.6.2 Logical identities: There are many important logical equivalences called logical identities, few of them are as follows, where p, q and r denote any statements:

1. Double negation law: $\sim(\sim p) \equiv p$
2. Demorgan laws: $\sim(p \wedge q) \equiv \sim p \vee \sim q$

   $\sim(p \vee q) \equiv \sim p \wedge \sim q$
3. Commutative laws: $p \wedge q \equiv q \wedge p$

   $p \vee q \equiv q \vee p$
4. Associative laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

   $(p \vee q) \vee r \equiv p \vee (q \vee r)$
5. Distributive laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
6. $p \rightarrow q \equiv \sim q \rightarrow \sim p$
7. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

We can prove these equivalences by constructing their truth tables.

7.7 Tautology and contradiction

A compound statement, which is always true, is called a tautology and a compound statement, which is always false, is called a contradiction. A statement, which is neither a tautology nor a contradiction, is called a contingent statement.

Note that If p is a tautology, then ~p is a contradiction and if p is a contradiction then ~p is a tautology.

7.7.1 Tautology: If P is a compound statement made up of the atomic statements p_1, p_2, …., p_n and connectives; and if for every combination of the truth values of the atomic statements p_1, p_2, …., p_n; the truth value of P is True i.e T, then P is a tautology.

Example:

- Verify whether $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology or not.

p	**q**	**$p \rightarrow q$**	**$p \wedge (p \rightarrow q)$**	**$(p \wedge (p \rightarrow q)) \rightarrow q$**
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

As all values in the last column are T, the compound statement

$(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology.

7.7.2 Contradiction: If P is a compound statement made up of the atomic statements p_1, p_2, …., p_n and connectives; and If for every combination of the truth values of the atomic statements p_1, p_2, …., p_n; the truth value of P is False i.e., F then the compound statement P is a contradiction .

Example:

- Verify whether $(p \wedge q) \wedge \sim q$ is a contradiction or not.

p	**q**	**$p \wedge q$**	**$\sim q$**	**$(p \wedge q) \wedge \sim q$**
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	F	T	F

As all values in the last column are F, $(p \wedge q) \wedge \sim q$ is a Contradiction.

7.8 Converse, inverse and contrapositive

One important concept in mathematical logic is of converse, inverse and contrapositive of a conditional statement. Using these concepts many results in mathematics are proved easily.

7.8.1 Converse of a conditional statement: Converse of a conditional statement $p \rightarrow q$, is the conditional statement $q \rightarrow p$.

7.8.2 Inverse of a conditional statement: Inverse of a conditional statement $p \rightarrow q$, is the conditional statement $\sim p \rightarrow \sim q$.

7.8.3 Contrapositive of a conditional statement: Contrapositive of a conditional statement p , q, is the conditional statement r q, r p.

Examples:

‹ Consider a conditional statement, "If two triangles are congruent then their sides are equal." This is in p , q form where p: two triangles are congruent, and q: their sides are equal.

Converse: q , p: If their sides are equal then two triangles are congruent.

Inverse : r p, r q: If two triangles are not congruent then their sides are not equal.

Contrapositive: r q, r p : If their sides are not equal then two triangles are not congruent.

‹ If a conditional statement in p , q form is: "If a function is bijection then the inverse function exists. " For this statement we have,

Converse: q , p: If the inverse function exists then a function is bijection.

Inverse: r p, r q: If a function is not a bijection then the inverse function does not exist.

Contrapositive : r q, r p : If the inverse function does not exist then a function is not bijection.

‹ p , q: " If Sachin receives a scholarship then he will study further."

Converse: q , p: If Sachin will study further then he receives a scholarship.

Inverse: r p, r q: " If Sachin does not receive a scholarship then he will not study further.

Contrapositive : r q, r p : If Sachin will not study further then he does not receive a scholarship.

7.9 Self-Test 3

Exercise 1: Write the truth table of each of the following and determine whether it is a tautology or contradiction or a contingent statement.

i) p ´ r q

ii) (p ´ q) , q

iii) (p, q) ´ (p ´ r q)

iv) p ´ r p

v) p µ r p

vi) [(p } q) ´ q] , p

vii) r p ´ (p ´ q)

viii) (p ´ q) ´ r (p µ q)

ix) (p ´ q) µ(p ´ r)

x) [(p, q) ´ (q , r)] , (p, r)

Exercise 2: Using truth table determine whether following statement patterns are logically equivalent.

i) p ´ q and p µ q

ii) ~(p µ q) and ~p ´ ~q

iii) ~ (p ´ q) and ~p µ ~q

iv) ~(p ´ q) and ~p ´ ~q

v) p ´ (q µ r) and (p ´ q) µ (p ´ r)

vi) p , q and ~q , ~p

vii) p µ q and ~p µ ~q

viii) ~(p ´ q) and ~p ´ ~q

ix) p and p ´ (p µ q)

x) p µ q and q µ p

Exercise 3: Write the converse, inverse and contrapositive of each of the following conditional statements.

- **(i)** If interest rates are low then the economy is good.
- **(ii)** If you are good in logic then you are good in mathematics.
- **(iii)** A number is divisible by 2 implies that it is an even number.
- **(iv)** A sufficient condition for Meena to visit Agra is that she goes to Tajmahal.
- **(v)** Violets are blue if roses are red.

7.10 Summary

In this unit learners studied the following topics in details:

1. A logical statement, and the definition of its truth-value.
2. Simple and compound statements.
3. Important logical connectives, which are negation, conjunction, disjunction, conditional and biconditional, with their truth tables.
4. Logical equivalence and how to prove it.
5. A tautology, a contradiction and a contingent statement.
6. Converse, inverse and contrapositive of a conditional statement.

UNIT 8 RELATIONS

8.0 Objectives

By the end of this Unit, learners should be able to:

- Explain Cartesian product of two sets.
- Characterised relations in terms of set theory.
- Understand matrix representation of relations.
- Describe and discuss different types of relations
- Describe reflexive, symmetric and transitive relations
- Describe equivalence relations
- Understand reflexive, symmetric and transitive closures

8.1 Introduction

Once the set of objects is introduced it is required to understand interrelationship among its elements. Associated with the relation is the act of comparing elements, which are related to each other. In this unit we will first formalize the concept of a relation and then discuss types of relations. The fundamental requisite for the concept of relation is 'the Cartesian product of two sets'. Hence in this lesson we start with the revision of the concept of product of sets. We will also characterise relation in terms of set theory and represent a relation-using matrix. Then we will discuss important class of relations, which is equivalence relation. All these have useful applications in the design of digital computers and other sequential machines.

The notion of a relation between two sets of objects is quite common and basic concept. We use various relations in every day life as well as in mathematics. In every day life we always deal with human relationships. The word "relation" suggests some familiar examples of human relations such as "relation between a father and a daughter", "relation between brother and sister", "etc.

When we study relations, generally two sets are used and relation between their elements is considered.

Let M be the set of all men and W be the set of all women on the earth. Let "Ram" and "Rani" be the members of these two sets respectively. One can assume different relations between these two members as:

"Ram" *is father of* "Rani"

"Ram" *is a brother of* "Rani"

"Ram" *is husband of* "Rani"

"Ram" is *a son of* "Rani" etc.

In fact we can define relation between any two sets, which are not necessarily of persons. Let A = {eggs, milk, rice, bread} and B = {cows, goats, hens} be two sets. Assume that if an element x of set A is produced by an element y of set B, then "x is related to y"; otherwise "x is not related to y". Then we observe that eggs and hens are related and milk and cows are related. But rice and hens are not related also bread and goats are not related.

We can define relation between elements of any one set also. In arithmetic there are many commonly used relations between any two numbers, few of such are "greater than", "less than" or "equality" etc. In geometry we study the relation between the area of a circle and its radius, the relation between the volume of a cube and the length of its sides etc. We consider appropriate sets of objects on which these relations can be defined.

8.2 Cartesian product of sets

As mentioned above, every relation is essentially equal to a list of ordered pairs of some elements, and each ordered pair specifies that its first element is related to its second element. So while listing elements of two sets, which are related, we use pairs containing the first element of one set and the second element of the other set.

If three students Rani, Ram and Ganesh can offer either Mathematics or Biology for the examination, then we can list all possible choices of courses offered, by listing all possible pairs of the form (student, course). Obviously these pairs are 'ordered pairs' that means we consider the fixed sequence. We first consider the name of student and then a name of course he or she is offering. Then the following set of ordered pairs shows all possibilities of these two courses taken by these three students.

Choices = {(Rani, Mathematics), (Ram, Mathematics), (Ganesh, Mathematics),
(Rani, Biology), (Ram, Biology), (Ganesh, Biology)}.

This set named as "Choices" is a **product set** (or Cartesian product) of two sets, which are

1. the set of students = {Rani, Ram, Ganesh } and
2. the set of courses = { Mathematics, Biology}

8.2.1 Product set (or the Cartesian product): For any two nonempty sets A and B, the set of all ordered pairs *(a, b)*, where *a* is an element of *A* and *b* is an element of *B*, is called the Product set (or Cartesian product) of A and B and is denoted by $A \times B$. Thus, using the set builder form we can write, $A \times B = \{(a, b) / a \in A \text{ and } b \in B\}$.

The number of elements in $A \times B$ = the number of elements in set A $\times$ the number of elements in set B.

Examples:

- If A denotes the set of programmers and B denotes the set of computer languages, then $A \times B$ is a set of all possible ordered pairs (a, b) where $a \in A$ and $b \in B$. We can interpret this product set $A \times B$ as a set of all possible pairs of programmers and computer languages.
- Let L = {C, Pascal, COBOL} is a set of computer languages;

  and S = {Windows, UNIX, dos} is a set of operating systems.

  Then their product is

$L \times S$ = {(C, Windows), (Pascal, Windows), (COBOL, Windows), (C, UNIX),
(Pascal, UNIX), (COBOL, UNIX), (C, dos), (Pascal, dos), (COBOL dos)}

- If A = {a, b} and B = {a, c, d} are two sets then the product sets are

  $A \times B$ = {(a, a), (a, c), (a, d), (b, a), (b, c), (b, d)}
  $B \times A$ = {(a, a), (a, b), (c, a), (c, b), (d, a), (d, b)}

Note that, product of sets is not a commutative operation, hence $A \times B \neq B \times A$ always.

8.3 Relations

A relation is a rule, which assigns elements of one set to the elements of another set. Hence any set of ordered pairs defines a relation or a binary relation from one set to another set.

8.3.1 Relation: A relation R (or also called a binary relation R) from set A to set B is a subset R of the product set A ˆ B.

Examples:

- Consider the set, namely P = {(milk, cows), (eggs, hens)}, this set P is a relation as it is a set of ordered pairs. We can say that "a is related to b" if and only if "a is produced by b". In this case "a is produced by b" is a rule which assigns elements of set A = {eggs, milk, rice} to the elements of set B = {cows, hens}.
- Consider a set R = {(C, UNIX), (C, dos), (Pascal, dos), (COBOL, dos)}. This set R is of ordered pairs and it is a relation from a set L = {C, Pascal, COBOL} to a set S = {Windows, UNIX, dos}. Where L is a set of computer languages and S is a set of operating systems. In this case an element 'x' from set L is related to an element 'y' from set S, if the computer language 'x' is used with the help of the operating system 'y'.

If (a, b) © R then it means that an element 'a' from set A is related to an element 'b' from set B by the relation 'R' and it is also denoted by 'aRb'. Also if (x, y) ª R then it means that an element 'x' from set A is not related to an element 'y' from set B by the relation 'R' and it is denoted by 'a$\not R$b'.

The relation R can be described by any or all of the following three ways:

(i) listing all pairs belonging to set R or
(ii) by describing a membership rule for R or
(iii) by writing matrix of relation R

If R is any relation from set A to set B, then the set

{a © A / (a, b) © R for some b © B } is called as the **domain** of relation R.

And the set {b © B / (a, b) © R for some a © A} is called as the **range** of relation R. If A = B then R is called a relation (or a binary relation) **on** A, and in this case R is a relation such that R § A ˆ A. For a relation defined on set A the domain and range both are subsets of set A.

Examples:

- Let S be a set of students studying Computer science and N is a set of natural numbers between 101 to 120 representing Computers available for use.

Let R = {(Rani, 101), (Ram, 105), (Ganesh, 103), (Geeta, 101)}. R is a set of ordered pairs in which the first elements are the names of students and the second elements are the numbers showing which computer is used by the student. Hence R is a relation from set S to the set N.

This relation R can be described in words by the rule:

'For x © S and y © N, (x, y) © R if the student x is using the computer numbered y'.

For this relation S, the domain = {Rani, Ram, Ganesh, Geeta} and the range = {101, 105, 103}.

- If A = {3, 5, 8, 15} and B= {5, 10, 15}, then

A ˆ B = {(3 ,5), (3,10), (3,15), (5,5), (5,10), (5,15), (8 ,5), (8,10), (8,15), (15,5), (15,10), (15 ,15) }.

(i) If S = {(5,5), (15 ,15)}, then S is a relation from set A to set B as S § A ˆ B.

This relation S can be described in words by the rule:

For a © A and b © B, (a, b) © S if a 1 b.

For this relation S, the domain = {5, 15} and the range = {5, 15}.

(ii) If R = {(3, 5), (3,10), (3,15), (5,10), (5,15), (8,15) }, then R is a relation from set A to set B as R § A ˆ B .

This relation R can be described in words by the rule:

For a© A and b© B, (a, b) © R if a O b.

We observe that 3 is related to 5 by this relation but 8 is not related to 5 by this relation. For this relation R , the domain = {3, 5, 8} and the range = {5, 10, 15}

‹ If A = {a, b, c, d} and R is a set. R = {(a, a), (b, b), (b, c), (c, c), (c, b), (d, d)}. Then R is a relation on set A as R § A ˆ A.

This R is a relation, which cannot be described by some obvious membership rule as in previous example.

For this relation R, the domain = {a, b, c, d} and the range = {a, b, c, d}.

8.4 Self-Test 1

In each case, find the domain and range of the relation R. Also list all ordered pairs specifying the relation R, if it is not given as a set of ordered pairs.

1. A = {1, 2, 5 } and B = {1, 5, 9, 16} and relation R from A to B is,
R = {(1,5), (1,9), (1,16), (2,5), (2,9), (2,16), (5, 9), (5,6)}.

2. A = B = {1, 2, 3, 4} and relation R from A to B is, R = {(1, 1), (2, 2), (3, 3),(4, 4)}.

3. A = {1, 2, 3, 4, 5} and B = {1, 4, 9, 10, 16}. R is a relation from set A to set B defined as (a, b) © R if and only if b = a^2.

4. A = {1, 2, 3, 4} and B = {1, 4, 9, 16}. R is a relation from set A to set B defined as (a, b) © R if and only if b is a multiple of a.

5. A = {2, 4, 6, 8, 10} and B = {2, 3, 4, 5}. R is a relation from set A to set B defined as (a, b) © R if and only if a 2 b.

6. A = {1, 2, 3, 4, 5} and R is a relation defined on set A as (a, b) © R if and only if b – a = 1.

7. A = {1, 2, 3, 4, 5, 6} and R is a relation defined on set A as (a, b) © R if and only if a + b = 9.

8. A = {3, 6, 9, 12, 15} and R is a relation defined on set A as (a, b) © R if and only if a + 3 = b.

9. A = { –2, –1, 0, 1, 3} and R is a relation defined on set A as (a, b) © R if and only if | a | O | b |.

10. A = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (3, 1), (2, 2), (3, 3), (4, 1), (4,4)}.

8. 5 Types of Relations

In general when we define a relation R from a set A to a set B then every element of A need not be related to an element of set B. But if every element of A is related to some element of set B then there are different types of such relations. Sometimes such relations are also referred to as “correspondence”.

8.5.1 One to one relation: A relation R from set A to set B is a one to one relation if every element of A is related to a unique element of set B.

Examples:

- Let A = {1, 3, 7, 9} and B = {1, 9, 25, 49, 81, 100}. Define a relation R from set A to set B as (a, b) © R if and only if $b = a^2$.

  Then R as a set of ordered pairs can be written as

  R = { (1, 1), (3, 9), (7, 49), (9, 81)}

  We observe that every element of A is related to a unique element of set B, so this relation is one to one relation from set A to set B.

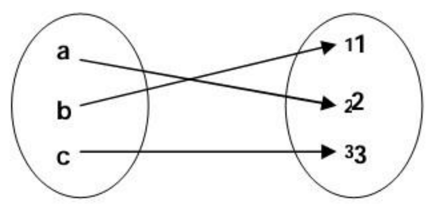
- Let A = {1, 3, 5, 7} and B = {1, 2, 3, …, 10}. Define a relation R from set A to set B as (a, b) © R if and only if a = b.

  Then R as a set of ordered pairs can be written as

  R = {(1, 1), (3, 3), (5, 5), (7, 7)}

  We observe that every element of A is related to a unique element of set B, so this relation is also one to one relation from set A to set B.

- Let A = {a, b, c} and B = {1, 2, 3 }. Define a relation R from set A to set B as R = {(a, 2), (b, 1), (c,3)} this relation is by definition one to one relation. This can also be understood by the Venn diagram representation as shown here.



One to one relation

8.5.2 One to many relation: A relation R from set A to set B is a one to many relation if every element of A is related to some element of set B and at least one element of A is related to two or more elements of set B.

Examples:

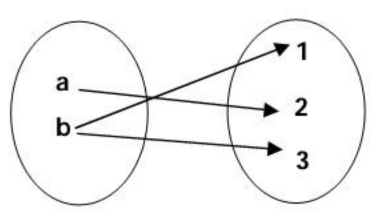
- Let A = {1, 3, 7, 9} and B = {1, 5, 15, 20}. Define a relation R from set A to set B as (a, b) © R if and only if a < b.

  Then R as a set of ordered pairs can be written as

  R = {(1, 5), (1,15), (1,20), (3, 5), (3, 15), (3, 20), (7, 15), (7, 20), (9, 15), (9, 20)}
  We observe that every element of A is related to two or more elements of set B, so this relation is one to many relation from set A to set B.

- Let A = {2, 4, 6, 8, 10} and B = {1, 2, 3, …, 9 }. Define a relation R from set A to set B as (a, b) © R if and only if a + b > 10.
  By this relation every element of A is related to two or more elements of set B, so this relation is one to many relation from set A to set B.

- Let A = {a, b} and B = {1, 2, 3}. Define a relation R from set A to set B as R = { (a, 2), (b,1), (b,3) } this relation is by definition one to many relation. This can be understood by the Venn diagram representation as shown here.



One to many relation

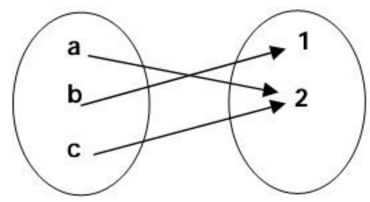
8.5.3 Many to one relation: A relation R from set A to set B is a many to one relation if every element of A is related to some element of set B and two or more elements of A are related to unique element of set B.

Example:

- Let A = {–2, –1, 1, 3} and B = {1, 4, 9, 25}. Define a relation R from set A to set B as (a, b) © R if and only if $a^2 = b$. Then R as a set of ordered pairs can be written as R = {(–2, 4), (–1,1), (1,1), (3, 9)}

  We observe that two elements –1 and 1 of A are related to unique element 1 of set B, so this relation is many to one relation from set A to set B.

- Let A = {a, b, c} and B = {1, 2}. Define a relation R from set A to set B as R = {(a, 2), (b, 1), (c, 2)} this relation is by definition many to one relation. This can be understood by the Venn diagram representation as shown here.



Many to one relation

8.5.4 Many to many relation: A relation R from set A to set B is a many to many relation if every element of A is related to some element of set B and two or more elements of A are related to two or more elements of set B.

Examples:

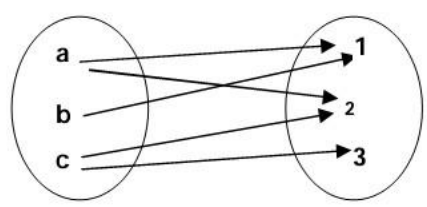
- Let A = {1, 3, 5, 7} and B = {1, 2, 3, ..., 10}. Define a relation R from set A to set B as (a, b) ©R if and only if a – b is even.

  By this relation more than one elements of A are related to two or more elements of set B, so this relation is a many to many relation from set A to set B.

- Let A = {2, 4, 6, 8, 10} and B = {1, 2, 3, ..., 9 }. Define a relation R from set A to set B as (a, b) © R if and only if a + b is a multiple of 5.

  By this relation more than one elements of A are related to two or more elements of set B, so this relation is also a many to many relation from set A to set B.

- Let A = { a, b, c } and B = {1, 2, 3 }. Define a relation R from set A to set B as R = {(a, 1), (a, 2), (b,1), (c,2), (c,3) } this relation is by definition many to many relation. This can be understood by the Venn diagram representation as shown here.



Many to many relation

8.6 Equivalence Relations and Equivalence Classes

When we define a relation on any set, then it satisfies different properties. According to some properties satisfied by relations we define different types of relations.

8.6.1 Reflexive relation: A relation R on a set A is a reflexive relation if every element in A is related to itself. R is a **reflexive** relation on A, if (a, a) © R for every a©A.

Examples:

- Let A be the set of all male Americans and let R be a relation defined on A as (a, b) ©R if 'a is a brother of b'. Then R is not a reflexive relation on A because any man cannot be a brother of himself.

- If $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (2,1), (3,1), (4,1), (4, 2), (3,3), (4,4)\}$ then R is not a **reflexive** relation on set A because every element of A is not related to itself. Here we observe that, $(1, 1) \in R$, $(3, 3) \in R$, $(4, 4) \in R$ but $(2, 2) \notin R$.
- If $A = \{a, b, c\}$ and R is a relation on set A, where $R = \{(a, a), (b , b), (b, c), (c, c), (c, b)\}$, then R is a **reflexive** relation on A because $(a, a) \in R$, $(b, b) \in R$ and $(c, c) \in R$.
- Let L be the set of straight lines in a plane. The relation R is defined as line l_1 is related to line l_2 if line l_1 is parallel to line l_2. Then 'to be parallel' is a reflexive relation on L because every line in plane is always parallel to itself. So every element in set L is related to itself.
- The relation R on $A = \{1, 2, 3, 4\}$ is defined by $(x, y) \in R$ if $x^2 \geq y$, then R is a reflexive relation as $x^2 \geq x$ for all $x \in A$, so $(x, x) \in R \ \forall x \in A$.

  If we write the relation R as a set of ordered pairs, then we observe that, $(1, 1) \in R$, $(2, 2) \in R$, $(3, 3) \in R$, $(4, 4) \in R$.

8.6.2 Symmetric relation: A relation R on a set A is symmetric relation if for all $a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$. In other words, R is a symmetric relation on A if we have (a, b) R as well as $(b, a) \in R$.

Examples:

- Let A be the set of all male Americans and let R be a relation defined on A as $(a, b) \in R$ if 'a is a brother of b'. Then R is a symmetric relation on A because if x is a brother of y then y is a brother of x.
- If $A = \{a, b, c\}$ and R is a relation on set A, where $R = \{(a, a), (b , b), (b, c), (c, c), (c, b)\}$, then R is a **symmetric** relation on A because $(a, a) \in R$, $(b, b) \in R$, $(c, c) \in R$, $(b, c) \in R$ and $(c, b) \in R$.. So, whenever $(x, y) \in R$ we observe that $(y, x) \in R$, for $x, y \in A$.
- Let L be the set of straight lines in a plane. The relation R is defined as line l_1 is related to line l_2 if line l_1 is parallel to line l_2 If line l_1 is parallel to line l_2 then line l_2 is parallel to line l_1. Therefore if $l_1 \, R \, l_2$ then $l_2 \, R \, l_1$ for all such lines l_1 and l_2 in L. Hence R i.e. to be parallel is a symmetric relation on set L.
- If $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (3,3),(4,4)\}$ then R is not a symmetric relation on A because here $(2,1) \in R$ but $(1, 2) \notin R$.
- If $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (1, 2), (1, 3), (1, 4), (2, 4)\}$. This relation R on A is a **symmetric** relation because we observe that $(y, x) \in R$, for all $x, y \in A$ such that $(x, y) \in R$.

8.6.3 Transitive relation: A relation R on a set A is a transitive relation if for all $a, b, c \in A$, whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. **or**

R is a transitive relation on A if we have $(a, b) \in R$, $(b, c) \in R$ as well as $(a, c) \in R$.

Examples:

- Let A be the set of all male Americans and let R be a relation defined on A as $(a, b) \in R$ if 'a is a brother of b'. Then R is not a transitive relation on A because, x is a brother of y and y is a brother of x but x is not a brother of x.

  i.e. If $(x, y) \in R$ and $(y, x) \in R$ then it is not true that $(x, x) \in R$.
- let $A = \{a, b, c\}$ and R is a relation on set A, where $R = \{(a, a), (b, b), (b, c), (c, c), (c, b)\}$. Here we observe that R satisfies the transitivity condition i.e. $(x, z) \in R$, for all $x, y, z \in A$ such that $(x, y) \in R$ and $(y, z) \in R$

For (a, a)© R, consider x = a, y = a and z = a, these x, y, z satisfy the transitivity condition.

For (b, b) © R, consider x = b, y = b and z = b, these x, y, z satisfy the transitivity condition.

For (c, c) © R, consider x = c, y = c and z = c, these x, y, z satisfy the transitivity condition.

Also

(b, c) © R, (c, b) © R and (b, b) © R.

(c, b) © R, (b, c) © R and (c, c) © R.

Thus relation R on a set A is a **transitive relation.**

- Let L be the set of straight lines in a plane. The relation R is defined as line l_1 is related to line l_2 if line l_1 is parallel to line l_2 . If line l_1 is parallel to line l_2 and line l_2 is parallel to line l_3 then line l_1 is parallel to the line l_3. Therefore if l_1 R l_2 and l_2 R l_3 then l_1 R l_3 for the lines l_1, l_2 and l_3 in L. Therefore R i.e. 'to be parallel' is a transitive relation on set L.
- If A = {1, 2, 3, 4} and R = {(1,1), (2,1), (3,1), (4,1), (2, 2), (4, 2), (3,3), (4,4), (1, 3),(4, 3), (2,3)}, then R is a transitive relation on A, because here (x, z) © R, for all x, y, z © A such that (x, y) ©R and (y, z) ©R .
- If A = {1, 2, 3, 4} and R = {(1,1), (2,1), (3,1), (4,1),(2, 2),(4, 2), (3,3),(4,4)} then R is a transitive relation on A because here (x, z) © R, for all x, y, z © A such that (x, y) ©R and (y, z) ©R.

8.6.4 Equivalence relation: A relation R on a set A is an equivalence relation if it is reflexive, symmetric and transitive relation. **or**

A relation R on a set A is an **equivalence relation** if it satisfies the following three properties:

(i) (a, a) © R for all a © A.
(ii) If (a, b) © R then (b, a) © R, for a, b © A.
(iii) If (a, b) © R and (b, c) © R then (a, c) © R, for all such a, b, c © A.

Examples:

- If A = {a, b, c} and R is a relation on set A, defined as a set of ordered pairs as R = {(a, a), (b , b), (b, c), (c, c), (c, b)}, then R is an **equivalence relation** on A because it is reflexive, symmetric and transitive relation on A.
- If L is the set of straight lines in a plane and the relation R is defined as o l_1 R l_2, if line l_1 is parallel to line l_2 , then R is an **equivalence relation** on L because it is reflexive, symmetric and transitive relation on L.
- Let A = {1, 2, 3, 4} and R = {(1,1), (2,1), (3,1), (4,1), (2, 2), (4, 2), (3,3), (4,4), (1, 3), (4, 3), (2,3)},

  R is a reflexive relation as, (1, 1) © R, (2, 2) © R, (3, 3) © R, (4, 4) © R.

  Also R is a transitive relation on A, because here (x, z) © R, for all x, y, z © A such that (x, y) © R and (y, z) © R.

  But it is not an equivalence **relation** on A because it is not a symmetric relation. R is not a symmetric relation as (4, 1) © R but (1, 4)ª R.

8.6.5 Equivalence class: If R is an equivalence relation defined on set A, and if a©A is any element of A, then equivalence class of a is defined as the set of all elements of A which are related to a. It is denoted as Ta or [a].

Using Set theory we can write, Ta = [a] ={x ©A / (a, x) ©R }

Example:

‹ If A = {1, 2, 3, 4} and R = {(1,1), (1, 3), (3,1), (3,3), (2, 2), (4, 4)} then R is an equivalence relation on A.

Then by definition, the equivalence class of 1 is defined as the set of all elements of A which are related to 1. Here 1 is related to 1 and 3.

P T1 = [1] ={1, 3}.

Similarly T2 = {2}, T3 = {1, 3} and T4 = { 4 }.

‹ If A = {a, b, c, d} and R is a relation on set A, defined as a set of ordered pairs as R = {(a, a), (b, b), (b, c), (c, c), (c, b), (a, d), (d, a), (d, d)}, then R is an equivalence relation on A. The equivalence class of a is defined as the set of all elements of A which are related to a. Here a is related to a and d.

P Ta = {a, d }. Similarly Tb = {b, c }, Tc = { b, c }, and Td = { a, d }.

8.7 Self-Test 2

Select the correct alternative from the given alternatives.

1. On set A = {1, 2, 3, 4} define relation R = {(1, 1), (1, 2), (3,1), (3, 2), (3, 3), (3, 4), (4,3), (4,4)}. What is the domain of relation R?

(a) {1, 4} **(b)** {1, 2, 3} **(c)** {1, 3, 4} **(d)** {1, 2, 3, 4}

2. On set A = {1, 2, 3, 4} define relation R = {(1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (3, 4), (4,4)}. What is the range of relation R?

(a) {1, 2, 4} **(b)** {1, 2, 3} **(c)** {1, 3, 4} **(d)** {1, 2, 3, 4}

3. Let A = {1, 2, 3, 4, 5} and R is a relation defined on set A as (a, b) © R if and only if a + b = 10. Then relation R is of what type?

(a) a reflexive relation **(b)** not a reflexive relation
(c) not a symmetric relation **(d)** an equivalence relation

4. Let A = {1, 2, 3, 4, 5} and R is a relation defined on set A as (a, b) © R if and only if a ‡ b. Then relation R is of what type?

(a) a symmetric relation **(b)** not a reflexive relation
(c) a reflexive and transitive relation **(d)** an equivalence relation

5. Let A = {1, 2, 3, 4, 5} and R is a relation defined on set A as aRb if and only if a – b is even. Then relation R is of what type?

(a) a reflexive relation **(b)** not a reflexive relation
(c) not a symmetric relation **(d)** an equivalence relation

6. What is an equivalence relation?

(a) A relation, which is reflexive and symmetric.
(b) A relation, which is symmetric and transitive.
(c) A relation, which is reflexive, symmetric and transitive.
(d) A relation, which is reflexive and transitive but not symmetric.

7. On set A = {a, b, c} define relation R = {(a, a), (b, b), (c, c)}. Then relation R is of what type?

(a) only a reflexive relation (b) not a reflexive relation
(c) not a symmetric relation (d) an equivalence relation

8. On set A = {a, b, c} define relation R = {(a, b), (b, a), (c, c)}. Then relation R is of what type?

(a) only a reflexive relation (b) not a reflexive relation
(c) not a symmetric relation (d) an equivalence relation

9. On set of all triangles in a plane, define a relation S as two triangles ÅABC and ÅPQR are related by S if and only if they are congruent triangles. Then relation S is of what type?

(a) only a reflexive relation (b) not a reflexive relation
(c) not a symmetric relation (d) an equivalence relation.

10. If A = {1, 10, 100, 1000} and B = {1, 4, 9, 16}, then How many elements are there in the product set A ˆ B?

(a) 11 (b) 12 (c) 9 (d) 16

8.8 Matrix of a Relation

When the number of elements in a set is more, the listing of relation on it in the form of ordered pairs is not easy. In such case it is more easy to use matrix representation of relation. It is useful for analysis of a relation by a computer.

If A and B are two finite sets containing m and n elements respectively and R is a relation from set A to set B, then the matrix of relation R is a matrix of order $m \times n$ containing entries as 0s and 1s only. It is the matrix in which rows correspond to the elements of set A and the columns correspond to the elements of set B. If a© A and b© B are two arbitrary elements and if (a, b) © R, then the entry in the row corresponding to the element 'a' and column corresponding to the element 'b' is equal to 1 and it is equal to 0 otherwise.

Note that:

1. The matrix on relation on set A is a square matrix. If orders of elements in the set A and /or B are changed then the Matrix of relation R is different.
2. The relation R is a reflexive relation if and only if the matrix of R has 1's on the main diagonal.
3. The relation R is a symmetric relation if and only if the matrix of R is a symmetric matrix.
4. It is not easy to decide whether R is a transitive relation by observing the matrix.

Examples:

‹ If A = { 3, 5, 8, 15 } and B= {5, 10, 15 }, and S is a relation from set A to set B defined as S = { (5,5), (15 ,15)} .

The matrix of relation S is a matrix of order 4 × 3. It is the matrix in which rows correspond to the elements 3, 5, 8 and 15 of set A (in this ordering); and the columns correspond to the elements 5, 10 and 15 of set B (in this ordering).

In the matrix of relation S the entry in the row corresponding to the element '5' and the column corresponding to the element '5' is equal to 1 because (5, 5) © S. Also (15, 15) © S, therefore the entry in the row corresponding to the element

'15' and the column corresponding to the element '15' is equal to 1. All other entries in the matrix are equal to 0.

Then the matrix of relation S for the given ordering of A and B is:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Note that</u>: If the ordering of elements in A and B is changed then the places of 0s and 1s in the corresponding matrix of relation are changed.

‹ If A = { 3, 5, 8, 15 } and B= {5, 10, 15 }, and R is a relation from set A to set B defined as R = {(3 ,5), (3,10), (3,15), (5,10), (5,15), (8,15) }

Then the matrix of relation R = $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

‹ If A = {1, 2, 3, 4} and R is a relation on set A listed as a set R = {(1,1), (2,1), (3,1), (4,1), (2, 2), (4, 2), (3,3), (4,4)}. Then the matrix of relation R is :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Here we observe that R is a reflexive relation and the matrix of R has 1's on the main diagonal.

‹ The relation R on A = {1, 2, 3, 4} is defined by: (x, y) © R if x^2 ‡ y.

This relation R can be written as a set of ordered pairs as:

R = {(1, 1), (2, 1), (3, 1), (4, 1), (2, 2), (3, 2), (4, 2), (3, 3), (3, 4), (4, 4)}

Then the matrix of relation R is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Here also R is a reflexive relation and the matrix of R has 1's on the main diagonal.

‹ If A = {a, b, c} and R is a relation on set A, where R = {(a, a), (b, b), (b, c), (c, c), (c, b)}. Then the matrix of relation R is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

<u>Note that:</u>

(i) R is a reflexive relation and the matrix of R has 1's on the main diagonal

(ii) R is symmetric and the matrix of R is also symmetric about its main diagonal.

8.9 Self-Test 3

Find the matrix of the given relation R.

1. Set A = {1, 2, 5} and B = {1, 5, 9, 16} and relation R from A to B is,

   R = {(1,5), (1,9), (1,16), (2,5), (2,9), (2,16), (5, 9), (5,6)}.

2. Set A =B = {1, 2, 3, 4} and relation R from A to B is,

   R ={(1, 1), (2, 2), (3, 3),(4, 4)}.

3. Set A = {1, 2, 3, 4, 5} and B = {1, 4, 9, 10, 16}. R is a relation from set A to set B defined as (a, b) © R if and only if b = a^2.

4. Set A = {1, 2, 3, 4} and B = {1, 4, 9, 16}. R is a relation from set A to set B defined as (a, b) © R if and only if b is a multiple of a.

5. Set A = {2, 4, 6, 8, 10} and B = {2, 3, 4, 5}. R is a relation from set A to set B defined as (a, b) © R if and only if a 2 b.

6. Set A = {1, 2, 3, 4, 5} and R is a relation defined on set A as (a, b) © R if and only if b – a = 1.

7. Set A = {1, 2, 3, 4, 5, 6} and R is a relation defined on set A as (a, b) © R if and only if a + b = 9.

8. Set A = {3, 6, 9, 12, 15} and R is a relation defined on set A as (a, b) © R if and only if a + 3 = b.

9. Set A = {–2, –1, 0, 1, 3} and R is a relation defined on set A as (a, b) © R if and only if | a | O | b |.

10. Set A = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (3, 1), (2, 2), (3, 3), (4, 1), (4,4)}.

8.10 Closure of relation

8.10.1 Closure of relation: Let R be a relation on set A. The smallest relation R^* on the set A that contains R as a subset and which possesses the desired property is called the **Closure** of relation R with respect to the property under consideration.

There are different closures of R as below:

(1) Reflexive Closure: Let R be a relation on set A. The reflexive Closure of R is the smallest reflexive relation on A that contains R as a subset.

Examples:

‹ Set A = {1, 2, 3, 4} and relation R defined on A is, R = { (1, 1), (2, 2), (3, 3) }, Then its reflexive closure R* = { (1, 1), (2, 2), (3, 3), (4, 4) }.

‹ Set A = {1, 2, 3, 4} and R = {(1, 1), (3, 1), (4, 1), (4,4)}. Then its reflexive closure R* {(1, 1), (3, 1), (2, 2), (3, 3), (4, 1), (4,4)}.

(2) Symmetric Closure: Let R be a relation on set A. The symmetric Closure of R is the smallest Symmetric relation on A that contains R as a subset.

Examples:

‹ Set A = {1, 2, 3, 4} and relation R defined on A is, R = { (1, 1), (2, 2), (3, 3) }, Then its symmetric closure R* = R ={ (1, 1), (2, 2), (3, 3) }.

‹ Set A = {1, 2, 3, 4} and R = { (1, 1), (1, 3), (2, 2), (4, 1), (4,4)} Then its symmetric closure R* { (1, 1), (1, 3), (3, 1), (2, 2), (4, 1), (4, 1), (4,4)}.

(3) Transitive Closure: Let R be a relation on set A. The transitive Closure of R is the smallest Transitive relation on A that contains R as a subset.

Examples:

‹ Set A = {1, 2, 3, 4} and relation R defined on A is, R = { (1, 1), (2, 2), (3, 3) }, Then its transitive closure R* = R ={ (1, 1), (2, 2), (3, 3) }.

‹ Set A = {1, 2, 3, 4} and R = { (1, 1), (1, 3), (2, 4), (4, 2), (4,4) } Then its reflexive closure R* = { (1, 1), (1, 3), (2, 2), (4, 2), (2, 4), (4,4) }.

8.11 Self-Test 4

Exercise 1 : Select the correct alternatives from the given.

1. If R is a relation on set A. Then what is the smallest reflexive relation on A that contains R as a subset called?
a) Transitive closure of R **b)** Symmetric closure of R
c) Reflexive closure of R **d)** Partial closure of R

2. If R is a relation on set A. Then what is the smallest symmetric relation on A that contains R as a subset called?
a) Transitive closure of R **b)** Symmetric closure of R
c) Reflexive closure of R **d)** Partial closure of R

3. If R is a relation on set A. Then what is the smallest transitive relation on A that contains R as a subset called?
a) Transitive closure of R **b)** Symmetric closure of R
c) Reflexive closure of R **d)** Partial closure of R

Exercise 2: Find the reflexive, symmetric and transitive closures by observation.

1. Let A = {1, 2, 5} and B = {1, 5, 9, 16} and relation R from A to B is, R = {(1,5), (1,9), (1, 16), (2,5), (2,9), (2,16), (5, 9), (5, 16) }.

2. Let A = {1, 2, 3, 4} and R is a relation on set A, R = {(1, 1), (2, 2), (3, 3),(4, 4)}.

3. Let A = {1, 2, 3, 4, 5} and B = {1, 4, 9, 10, 16}. R is a relation from set A to set B defined as (a, b) © R if and only if $b = a^2$.

4. Let A = {1, 2, 3, 4} and B = {1, 4, 9, 16}. R is a relation from set A to set B defined as (a, b) © R if and only if b is a multiple of a.

5. Let A = {2, 4, 6, 8, 10} and B = {2, 3, 4, 5}. R is a relation from set A to set B defined as (a, b) © R if and only if a 2 b.

8.12 Summary

In this unit learners studied the following topics in details:

1. Revised the concept of the Cartesian product of the two sets.
2. Definition and idea of a relation or a binary relation R from set A to set B
3. Different types of relations such as one to one, many to one and many to many relations.
4. Different types of relations such as reflexive, symmetric and transitive relations and equivalence relations.
5. How to represent a relation by a matrix.
6. The concept of the Closure of relation R and different types of closures.

Unit 9: Functions

9.0 Objectives

By the end of this Unit, learners should be able to:

- Understand concept of function.
- Describe Injective or one to one function.
- Describe surjective or onto function.
- Describe bijective function.
- Find inverse function of a bijective function.
- Find composition of functions.

9.1Introduction

The concept of function is of fundamental importance in mathematics. Function is a special type of relations, which plays an important role in mathematics, physics, computer science and many other fields.

Functions are used in a variety of situations. While studying Geometry in school, we learn that the area of a circle of radius r units equals to dr^2 square units. We say that the area of a circle is a function of radius r of that circle. In physics we learn about velocity of a vehicle at time t, velocity is a function of time t. Functions can be of one, or two or more variables. In this unit we study functions of one variable, its basic properties and different types.

9.2 Functions

9.2.1 Function: If A and B are nonempty sets then, a function f from A to B denoted by x : *A* , *B* is a relation from A to B such that every element of set A is related to a unique element of set B. Function is also referred to as a mapping.

If x© A is related to y © B by the function f i. e. (x, y) © f, then we express it as y = f(x). In this case y is called as a image of x and x is called as a preimage of y.

If x : *A* , *B* is a function then A is called as domain of the function f and B is called as co-domain of the function f. And the set of images, which is a subset of B, is called as the range of the function f.

Examples:

- If A = {3, 5, 8, 15} and B= {5, 10, 15}.

  Let f = {(3, 5), (5,5), (8,15), (15,15)}, then f is a relation from set A to set B as f § A ˆ B. And relation f is a function from set A to Set B, because every element of set A is related to a unique element of set B. i.e. here x : *A* , *B* is a function and we can define it as f(3) = 5, f(5) = 5, f(8) = 15 and f(15) = 15.

  Note that two elements 3 and 5 of set A, are having the image 5, but for each of them it is unique image. Also the two elements 8 and 15 of set A, are having the same image 15, but for each of them it is unique image. So f is a function.

For this function f, the domain of f = {3, 5, 8, 15}, the co -domain of f = {5, 10, 15} and the range of f = {5, 15}.

‹ If A = {3, 5, 8, 15} and B= {5, 10, 15}.

Let g = {(3, 5), (3,10), (5,10), (8,15), (15,15) }, then g is a relation from set A to set B as g § A ˆ B. But the relation g is not a function from set A to Set B, because every element of set A is not related to a unique element of set B. We observe that the element 3 of set A is having two images 5 and 10, and hence by the definition of function g is not a function.

‹ If A = {3, 5, 8, 15} and B= {5, 10, 15} and k = {(3, 5), (8,10), (15,15)}, then k is a relation from set A to set B as k § A ˆ B. But the relation k is not a function from set A to Set B, because every element of set A is not related to an element of set B. We observe that the element 5 of set A, is not related to any of the elements from set B, and hence by the definition of function k is not a function.

‹ On set A = {1, 3, 5, 7} define a relation f as, for a © A and b© B, (a, b) © f if a O b. i.e. f(a) = b if a O b.

We observe that by this relation 1 is related to 3, 5 and 7. So 1 in set A has three images but by the definition of function each element in the domain set must have a unique image.

Also 7 in A is not related to any of the other elements and by the definition of function each element in the domain set must have image which must be unique. Hence this relation f is not a function.

‹ Let A = { –2, –1, 1, 2, 3 } and B = {1, 4, 9, 25 }. Define a relation g from set A to set B as (a, b) © R i.e. g(a) = b if and only if b = a^2.

Then g as a set of ordered pairs can be written as g = { (–2, 4), (–1,1), (1,1), (2, 4), (3, 9) }. We observe that –1 and 1 from A are related to unique element 1, –2 and 2 from A are related to unique element 4 and also 3 is related to unique element 9 of set B. So this relation satisfies the definition of a function. In this case we write that " x : *A* , *B* is a function defined as f(x) = x^2 ".

9.3 Types of functions

9.3.1 Injective (or one to one) function: A function x : *A* , *B* is said to be an injective function if distinct elements of A have distinct images in B under f. An injective function is also called as a one to one function.

Note that: The function x : *A* , *B* is an injective function if for x, y©A, x (x) = x (y) implies that x = y.

Examples:

‹ Let A = {1, 3, 5, 7, 9} and B = {1, 9, 25, 49, 81, 100}. Define a function x : *A* , *B* as x (x) = x^2.

Then x (1) = 1, x (3) = 9, x (5) = 25, x (7) = 49 and f(9) = 81.

We observe that every element of A is related to a different element of set B, so this function is an injective function from the set A to the set B.

‹ Let A = {1, 3, 5, 7} and B = {1, 2, 3, ..., 10 }. Define a function x : *A* , *B* as x (x) = x.

Then x (1) = 1, x (3) = 3, x (5) = 5, 5 and x (7) = 7.

We observe that every element of A is related to a different element of set B, so this function is an injective function from the set A to the set B.

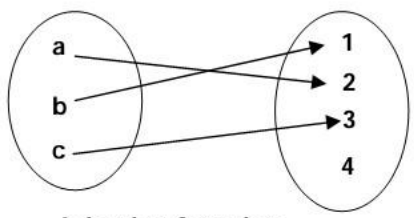
‹ Let A = {–2, –1, 1, 2, 3} and B = {1, 4, 9, 25}. Define a function x : *A* ,   *B* as x (x) = x^2.

Then x (–2) = 4 , x (–1) = 1, x (1) = 1, x (2) = 4 and x (3) = 9.

We observe that every element of A is not related to a different element of set B. Here –1 and 1 from A are related to 1, so x (–1) = 1 = x (1) but it does not imply that –1 = 1.

Also x (–2) = 4 = x (2) but –2 ‘ 2 so this function is not injective function .

‹ The Venn diagram below represents an injective function from set A = {a, b, c} to the set {1, 2, 3, 4}



Injective function

9.3.2 Surjective (or onto) function: A function x : *A* ,   *B* is said to be an surjective function if every element of set B (i.e. codomain), has at least one preimage in set A (i.e. domain). A surjective function is also called as an onto function.

Examples:

‹ Let A = {–2, –1, 1, 2, 3} and B = { 1, 4, 9}. Define a function x : *A* ,   *B* as x (x) = x^2.

Then x (–2) = 4 , x (–1) = 1, x (1) = 1, x (2) = 4 and x (3) = 9.

We observe that every element of set B, has at least one preimage in set A. So this function is a surjective function from the set A to the set B.

‹ If A = {3, 5, 8} and B= {5, 10, 15}. Let f = {(3, 5), (5, 15), (8,10)}, then f is a function from the set A to the set B such that every element of the set B is related to a unique element of the set A. So by definition this function is a surjective function from the set A to the set B.

‹ Let A = {1, 3, 5, 7, 9} and B = {1, 9, 25, 49, 81, 100}. Define a function x : *A* ,   *B* as f(x) = x^2.

Then x (1) = 1,   x (3) = 9, x (5) = 25 , x(7) = 49 and x (9) = 81.

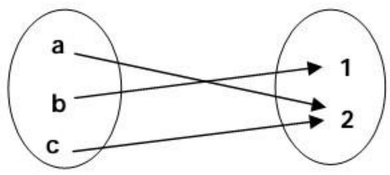
We observe that every element of B is not related with an element of set A, in particular the number 100 in set B has no preimage in set A. So this function is not surjective function from the set A to the set B.

‹ Let A = {1, 3, 5, 7} and B = {1, 2, 3, ..., 10}. Define a function x : *A* ,   *B* as f(x) = x.

Then f(1) = 1,   f(3) = 3, f(5) = 5, 5 and f(7) = 7.

We observe that every element of B is not related with an element of set A. So this function is not surjective function from set A to set B.

- The Venn diagram below represents a surjective function from set A = { a, b, c} to the set { 1, 2 }



Surjective function

9.3.3 Bijective function: A function x : *A* , *B* is said to be a bijective function if it is injective and surjective both. A bijective function is also called as a one to one and onto function.

Examples:

- If A = {3, 5, 8} and B= {5, 10, 15}. Let x = {(3, 5), (5, 15), (8,10)}, then f is a function from the set A to the set B which is injective as all images are distinct. Also every element of set B has a unique preimage under x in set A. So by definition this function is a surjective function.

  Therefore it is a bijective function from the set A to the set B.

- Let A = {1, 3, 5, 7, 9} and B = {1, 9, 25, 49, 81}. Define a function x : *A* , *B* as x (x) = x^2.

  Then x (1) = 1, x (3) = 9, x (5) = 25, x (7) = 49 and x (9) = 81.

  As discussed above this function is injective function from the set A to the set B. and it is surjective function as every element of set B is an image of some element from set A. Therefore it is a bijective function.

- Let A = {1, 3, 5, 7} and B = {1, 2, 3, ..., 10}. Define a function x : *A* , *B* as x (x) = x.

  Then x (1) = 1, x (3) = 3, x (5) = 5 and x (7) = 7.

  As discussed above this function is injective function from set A to set B. But it is not surjective function, because the numbers 2, 4, 6, 8, 9, 10 from set B have no preimages in set A under the function f. Therefore it is not a bijective function.

- Let A = { –2, –1, 1, 2, 3 } and B = { 1, 4, 9, 25 }. Define a function x : *A* , *B* as x (x) = x^2.

  Then x (–2) = 4 , x (–1) = 1, x (1) = 1, x (2) = 4 and x (3) = 9.

  As discussed above this function is not injective function from set A to set B, because x (–1) = 1 = x (1) but –1 ' 1 and x (–2) = 4 = x (2) but –2 ' 2. Therefore it cannot be a bijective function. But observe that it is a surjective function, as all the numbers from set B have preimages in set A under the function f.

Note that: for a function x : *A* , *B*, being injective (or surjective) function does not depend only on the definition of the function or only on the sets A and B. So if a function x : *A* , *B* is injective (or surjective) function from a set A to a set B then the same function(i.e. rule), may not be injective (or surjective) function from some other set C to some other set D.

e.g. from above examples $f : A \to B$ as $f(x) = x^2$ is a bijective function for one pair of sets A and B while it is not a bijective function for another pair of sets A and B.

- If $f : \mathbb{R} \to \mathbb{R}$ is a function defined as $f(x) = 3x + 2$, where $\mathbb{R}$ is a set of real numbers then is f a bijective function?

  Solution: To determine whether this function is a bijective function or not we should determine whether it is an injective function and a surjective function.

  i) Injective: Let $x, y \in \mathbb{R}$ are such that $f(x) = f(y)$.

  $$\Rightarrow \quad 3x + 2 = 3y + 2$$
  $$\Rightarrow \quad 3x = 3y$$
  $$\Rightarrow \quad x = y.$$

  Hence for $x, y \in \mathbb{R}$, we know, $f(x) = f(y)$ implies that $x = y$. So this function $f : \mathbb{R} \to \mathbb{R}$ is an injective function.

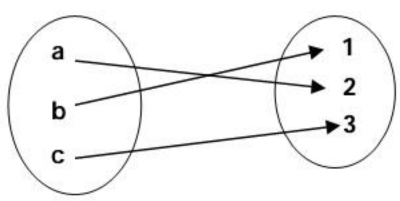
  ii) Surjective: Let $a, b \in \mathbb{R}$ are such that $f(a) = b$.

  $$\Rightarrow \quad 3a + 2 = b$$
  $$\Rightarrow \quad 3a = b - 2$$
  $$\Rightarrow \quad a = (b - 2) / 3 \in \mathbb{R}$$

  Hence for every $b \in \mathbb{R}$ (i.e in codomain), we can find preimage $a \in \mathbb{R}$ (i.e in domain). This implies that this function $f : \mathbb{R} \to \mathbb{R}$ is a surjective function.

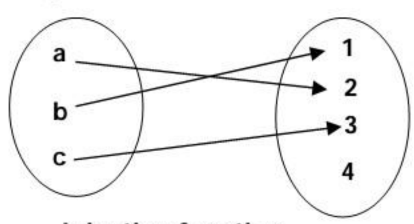
  From above explanation we can say that this function $f : \mathbb{R} \to \mathbb{R}$ is a bijective function.

- The Venn diagram below represents a bijective function from set A = {a, b, c} to the set {1, 2, 3}



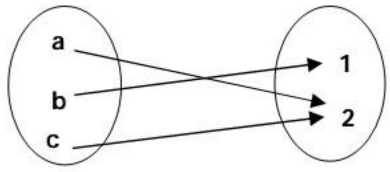
Bijective function

- The Venn diagram below represents an injective function from set A = {a, b, c} to the set {1, 2, 3, 4} but it is not a surjective function as 4 from set B has no preimage in A. Hence this function is not a bijective function.



Injective function

‹ The Venn diagram here represents a surjective function from set A = {a, b, c} to the set {1, 2}, but it is not an injective function as the images are not distinct. Here x (a) = 2 = x (c) but a and c are different elements from A. Hence this function is not a bijective function.

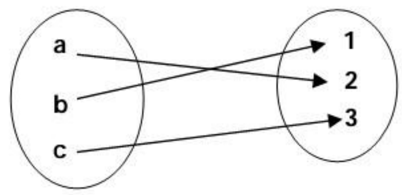


Surjective function

9.3.4 Inverse function: If x : *A* , *B* is a bijective function then its inverse function denoted by x^{-1} exists. And the inverse function x^{-1} : *B* , *A* is defined as, if x (x) = y then x^{-1}(y) = x.

Note that: Inverse of function f exists if and only if f is bijective function.

Consider the bijective function represented by a Venn diagram given here. This is a function x : *A* , *B*, where set A = {a, b, c} and B = {1, 2, 3}



Bijective function

Under this function x (a) = 2, x (b) = 1 and x (c) = 3. Hence preimage of 2 is a, preimage of 1 is b and preimage of 3 is c. If we consider the preimages under function x we obtain one function from the set B to the set A. This function is called inverse function of x.

Examples:

‹ If A = {3, 5, 8} and B= {5, 10, 15} and f is a function from set A to set B, where as a relation f can be represented as a set of ordered pairs x = {(3, 5), (5, 15), (8,10)}. As discussed earlier this function is a bijective function from set A to set B.

  Therefore inverse function of f exists. The inverse function, denoted by x^{-1} is a function from the set B to the set A and it can be represented as a set of ordered pairs,

  x^{-1} = {(5, 3), (15, 5), (10,8)}.

‹ Let A = {1, 3, 5, 7, 9} and B = {1, 9, 25, 49, 81}. Define a function x : *A* , *B* as x (x) = x^2.

  Then x (1) = 1, x (3) = 9, x (5) = 25, x (7) = 49 and x (9) = 81.

  As discussed above this function is a bijective function. So there exists inverse function x^{-1} from set B to set A, which is such x^{-1}(1) = 1, x^{-1}(9) = 3, x^{-1}(25) = 5, x^{-1} (49) = 7 and x^{-1} (81) = 9.

  In this case we can describe this inverse function as, x^{-1} : *B* , *A* is defined as, $x^{-1}(x) = \sqrt{X}$.

If x : š , š is a function defined as x (x) = 3x + 2, then this function is a bijective function as verified earlier.

For x, y © š , we know, if x (x) = y, then 3x + 2 = y

P 3x = y – 2

P x = (y - 2) / 3 © š

So the inverse function is described as, x^{-1}: š , š and f^{-1}(y) = *x* =(y - 2) / 3.

As it is common practice to write all function definitions using variable x, so this definition of inverse function is generally written as $x^{-1}(x)$ =(x - 2) / 3.

9.4 Composition of functions

While studying functions in details we need to perform different operations on functions such as addition or difference etc. Addition and difference of functions are very simple operations to understand. So here we will study the composition of functions, which is the most important operation of functions.

9.4.1 Composition of functions: If x : *A* , *B* and *g* : *B* , *C* are any two functions , then the Composition of functions f and g denoted by **g o f** is a function from A to C i.e.

g o x : *A* , *C*, it is a function defined by ,

g o x (x) = g[x (x)], for all x © A.

Note that:

(i) The composition of two functions f and g is defined only when the codomain of the first function is identical to the domain of the second function.
(ii) The composition of functions is not a commutative operation.
(iii) The composition of functions is an associative operation.
(iv) The composition f o f (if it exists) is also denoted by f^2.

Examples:

‹ Let A = {1, 2, 3}, B = {5, 15, 25, 35} and C = {7, 17, 27, 37}. Let x : *A* , *B* be defined as x (1) = 15, x (2) = 25 , x (3) = 35 and g : *B* , *C* be defined as g(5) = 7, g(15) = 17, g(25) = 27 and g(35) = 37.

Now the Composition of functions f and g denoted by *g* o x is a function, *g* o x :*A* , *C* defined by , *g* o x(x) = g(x (x)), for all x © A.

P *g* o x(1) = g(x (1)) = g (15) = 1 7 ; *g* o x (2) = g(x (2)) = g (25) = 2 7 and *g* o x(3) = g(x (3)) = g (35) = 3 7.

Observe that in this case the composition x o *g* does not exist.

‹ Let A = { –1, –2, –3 } , B = { 1, 4, 9, 10 } and C = {1,2,3,...,15 } are three sets. Define a functionx : *A* , *B* as x (x) = x^2 and another function *g* : B , *C* as g(x) = x + 3. Then x (–1) = 1, x (–2) = 4 and x (–3) = 9. Also g(1) = 4, g(4) = 7 , g(9) = 12 and g(10) = 13.

Now the Composition, *g* o x : *A* , *C* is a function defined by , g o x (x) = g(x (x)), for all x © A.

P g o x (–1) = g(x (–1)) = g (1) = 4 ; g o x (–2) = g(x (–2)) = g (4) = 7 and g o x (–3) = g(x (–3)) = g (9) = 12.

Observe that in this case the composition f o g does not exist.

‹ Let N denotes the set of natural numbers. Define functions x : *N* , *N* and *g* : *N* , *N* as x (x) = 2x + 1 and g(x) = x^2. Then Composition, *g* o x : *N* , *N* is a function defined by, g o x (x) = g(x (x)), for all x © A.

P g o x (x) = g(x (x)) = g (2x + 1) = $(2x + 1)^2$ = 4 x^2 + 4x + 1.

Observe that in this case the composition x o *g* also exists. And the composition, x o *g* : *N* , *N* is a function defined by, x o g (x) = x (g(x)), for all x © A.

P x o g (x) = x (g (x)) = x (x^2) = $2x^2 + 1$.

Obviously g o x (x) ' x o g (x) in this case.

P The composition of functions is not a commutative operation.

Further note that in this case the compositions x o x and g o g also exist.

The composition x o x : *N* , *N* is defined as

x o x (x) = x (x (x)) = x (2x + 1) = 2 (2x + 1) +1 = 4x +3.

The composition *g* o *g* : *N* , *N* is defined as

g o g (x) = g(g (x)) = g (x^2) = $(x^2)^2 = x^4$.

‹ Let R denotes the set of natural numbers. Define functions x : *R* , *R* and *g* : *R* , *R* as f(x) = x^2+ x and g(x) = 5x + 7.

Then Composition, *g* o x : *R* , *R* is a function defined by,

g o x (x) = g(f(x)) = g (x^2+ x) = 5 (x^2+ x) + 7= 5 x^2 + 5x + 7.

And the composition, x o *g* : *R* , *R* is a function defined by,

x o g (x) = x (g (x)) = x (5x + 7) = $(5x + 7)^2$ + (5x + 7)

= 25 x^2 + 70 x + 49 + 5x +7 = 25 x^2 + 75 x + 56.

The composition x o x : *R* , *R* is defined as

x o x (x) = x (x (x)) = x (x^2+ x) = $(x^2+ x)^2$+ (x^2+ x) = $(x^2)^2$ + $2x^3$ + x^2 + (x^2+ x)

= x^4+ $2x^3$+ $2x^2$+ x .

The composition *g* o *g* : *R* , *R* is defined as

g o g (x) = g(g (x)) = g (5x + 7) = 5(5x + 7) + 7= 25 x + 35 + 7 = 25 x + 42 .

9.5 Summary

In this unit learners studied the following topics in details:

1. The concept of function, which is a special type of relation.
2. Definition and examples of an Injective or one to one function
3. Definition and examples of a surjective or onto function.
4. Definition and examples of a Bijective function.
5. Inverse function of a bijective function
6. The concept of the composition of functions.

UNIT 10 Vectors

10.0 Objectives

By the end of this Unit, learners should be able to:

- Understand concept of vectors.
- Describe and discuss different types of vectors.
- Perform different operations on vectors.
- Compute dot product, cross product, scalar triple product of vectors.
- Identify Collinear and coplanar vectors.

10.1Introduction

Vectors are fundamental objects in the study of physical sciences. They can be used to represent any quantity that has both magnitude and direction. For example, velocity, the magnitude of which is speed. Another quantity represented by a vector is force since it has a magnitude and direction. Vectors are also used to describe many other physical quantities, such as displacements, acceleration, momentum, angular momentum etc. Electric field and magnetic field are also represented as a system of vectors at each point of a physical space. In this unit we will study the definition, examples, different types and properties of vectors.

10.2 Vectors

The physical quantities, which have only magnitude and are independent of direction, are called scalar quantities. But there are some quantities, which need direction along with the magnitude to describe them completely. The physical quantities, which are described by, both magnitude and direction are called as vector quantities. Real or any numbers are scalars and displacement, velocity, acceleration etc. are vector quantities.

10.2.1 Vector: A vector is a geometric object that has both, the magnitude or length and the direction. A vector is frequently represented by a directed line segment i.e. by an arrow. A directed line segment AB with an initial point *A* and with a terminal point *B*, represents the vector $\overrightarrow{AB}$.

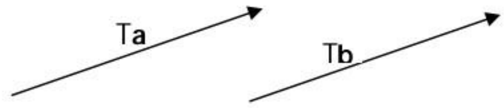
10.2.2 The magnitude (or modulus) of the Vector: The magnitude (or modulus) of the vector $\overrightarrow{AB}$ is the length of the segment AB and it is denoted as $|\overrightarrow{AB}|$.

Note that: Instead of denoting vectors by the initial and terminal vertices as $\overrightarrow{AB}$, $\overrightarrow{PQ}$ etc, vectors are also denoted as Ta, Tb, Tc, Tp, Tq etc.

10.3 Types of vectors

1. **Equal vectors:** The vectors having the same magnitude and the same direction are said to be equal vectors.

The two vectors Ta andTb are said to be equal i.e. Ta = Tb if |Ta | = |Tb| and Ta, Tb have same direction vectors. In the diagramTa andTb are equal vectors.



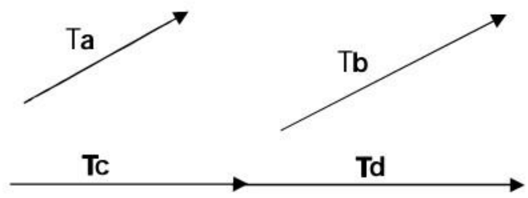
2. **Unit Vector:** A vector whose magnitude is one unit is said to be unit vector. A unit vector in the direction of the vector Ta is denoted by $\hat{a}$ (It is read as 'a cap').

   Obviously $\hat{a}$ 1 $\dfrac{\bar{a}}{|\ a\ |}$.

   In geometry the unit vectors along x, y and z-axis are denoted by $\hat{i}$, $\hat{j}$ *and* $\hat{k}$ respectively.

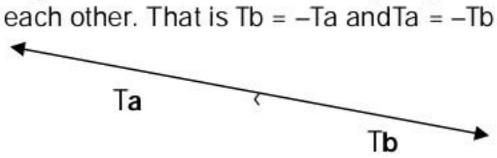
3. **Zero Vector or null vector:** A vector whose magnitude is zero unit is said to be unit vector (or null vector). Obviously for a zero vector the initial point and the terminal point coincides.

4. **Collinear vectors:** Vectors, which are parallel to the same line, are called collinear vectors. In the diagram vectors Ta andTb are collinear vectors and also vectors Tc andTd are collinear vectors.



5. **Coplanar vectors:** Vectors which are lying in the same plane or in the parallel planes are called coplanar vectors.

6. **Negative of a vector:** A vector having the same magnitude but opposite direction is the negative of the given vector. In the following diagram vectorsTa andTb are negative vectors of each other. That is Tb = –Ta andTa = –Tb.



10.4 Algebra of vectors

As a vector is considered as a geometrical object, there are some algebraic operations defined on the set of vectors. These operations are addition, subtraction and scalar multiplication of vectors. After studying how to perform these operations, we will study the concept of position vector and how to represent a point in plane or in space in terms of vectors. This is needed because it is easy to deal with vectors in geometry using position vectors.

10.4.1 Addition of vectors

1. **Triangle law of vector addition:** If two vectors are such that the initial point of the second vector is the terminal point of the first vector then triangle law of vector addition is used for their addition.

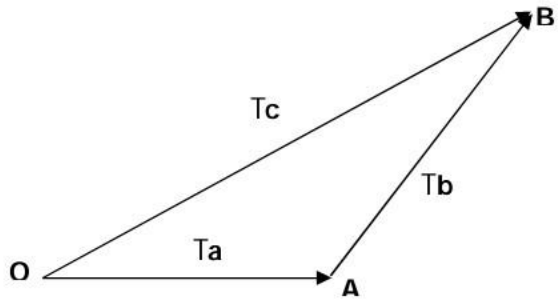
   The law is stated as: If O, A, B are three points such that $\overrightarrow{OA}$ 1 $\bar{a}$ and $\overrightarrow{AB}$ 1 $\bar{b}$ then the vector $\overrightarrow{OB}$ 1 $\bar{c}$ is called the addition of the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

   i.e. Tc = Ta +Tb or $\overrightarrow{OB}$ 1 $\overrightarrow{OA}$ + $\overrightarrow{AB}$

Note that:

(i) If A, B and C are any three non-collinear points, then using triangle law of vector addition, we have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

(ii) Using the above result for any two points A and B we can say that ,

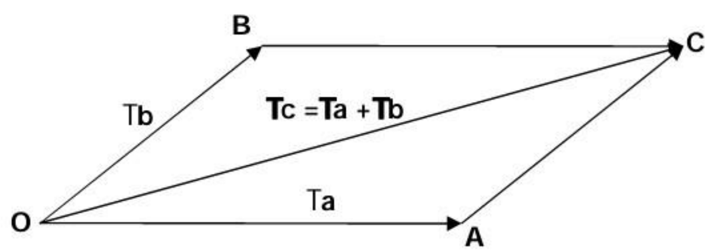


$\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \vec{0}$. $\therefore$ $\overrightarrow{AB} = -\overrightarrow{BA}$.

2. Parallelogram law of vector addition: If two vectors are such that the initial point of the first vector is same as the initial point of the second vector then parallelogram law of vector addition is used for their addition.

The law is stated as: If O, A, B, C are four points such that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are adjacent sides of a parallelogram, then the diagonal of the parallelogram passing through common initial point represents the addition of theses two vectors i.e. $\overrightarrow{OA} + \overrightarrow{OB}$.

If $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$ then the vector $\overrightarrow{OC} = \bar{a} + \bar{b}$ the addition of the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ as shown in the diagram here:



Properties of addition of vectors:

(i) Addition of vectors is commutative

i.e. for any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(ii) Addition of vectors is associative

i.e. for any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

(iii) For any vector $\vec{a}$, $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.

(iv) For any vector $\vec{a}$, there exists the negative vector $-\vec{a}$ such that

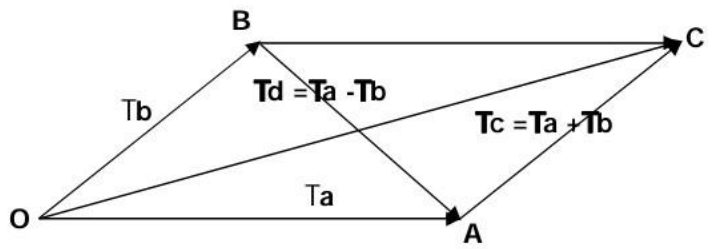
$$\vec{a} + (-\vec{a}) = \vec{0}.$$

10.4.2 Subtraction of vectors For any two vectors $\vec{a}$ and $\vec{b}$ their difference $\vec{a} - \vec{b}$ is defined as, $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

If O, A, B, C are four points such that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are adjacent sides of a parallelogram, then by the parallelogram law of vector addition, the addition of these two vectors i.e. $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ 1 $\bar{a}$ Ž $\bar{b}$ where $\overrightarrow{OA}$ 1 $\bar{a}$ and $\overrightarrow{OB}$ 1 $\bar{b}$. Then we also observe in the parallelogram drawn below that,

$\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$ ....(1) by triangle law of vector addition.

Now $\overrightarrow{BC}$ = $\overrightarrow{OA}$ 1 $\bar{a}$, as these vectors have the same magnitude and the same direction; and $\overrightarrow{CA} = \overrightarrow{BO} = -\overrightarrow{OB}$ =.–Tb. **P** $\overrightarrow{BA}$ = Td =Ta –Tb ... From (1) above.

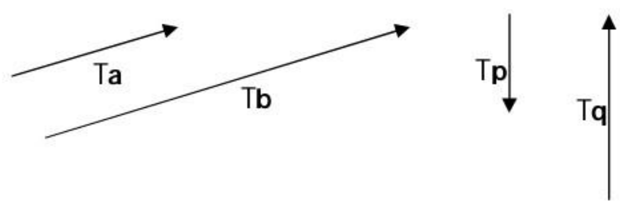


10.4.3 Scalar Multiplication of vectors For any vectorTa and any scalar (i.e a real number) k, scalar multiplication is defined as, the vector kTa.

Note that:

I. The vectors Ta and kTa are always collinear vectors.
II. Also two collinear vectors are scalar multiples of each other.

In the following diagram Ta and Tb are scalar multiples of each other andTp and Tq are scalar multiples of each other.



Clearly |Tb |= 2|Ta | and Ta and Tb have the same direction.
Pb = 2Ta or equivalently

Ta = $\frac{1}{2}$Tb.

And |Tq |= 2 |Tp | and Tp and Tq have the opposite directions.

P Tq = –2Tp or equivalently Tp = $-\frac{1}{2}$Tq.

Properties of scalar multiplication: If Ta and Tb are any two vectors and m, n are any real numbers then

(i) Scalar multiplication of a vector is associative i.e. m (nTa) = m n (Ta) and also m (nTa) = mn (Ta) = n(mTa).

(ii) Scalar multiplication is distributive over vector addition i.e.

m (Ta + Tb) = mTa + mTb.

(iii) Addition is distributive over scalar multiplication of a vector i.e.
(m + n)Ta = mTa + nTa .

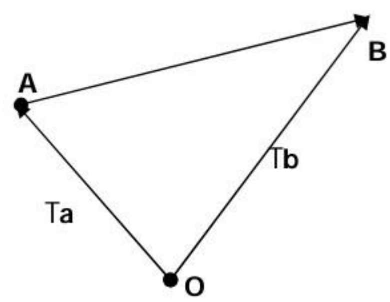
10.4.4 Representation of a point in plane using vectors

1. **Position vector of a point:** If P is any point and O is some fixed point, then the vector $\overrightarrow{OP}$ is called position vector of point P with respect to O. In particular if P is a

point in plane then the Cartesian coordinates (x, y) of point P determine the position of point P uniquely. Hence the directed line segment $\overrightarrow{OP}$ is called position vector of point P, with reference to origin O. If $\overrightarrow{OP}$ is denoted by Tp then the notation P(Tp)is used to denote that the point P is with position vector Tp.

2. Representation of any vector in terms of position vectors

Let A and B be two points with position vectors Ta and Tb respectively that means for some fixed point O, $\overrightarrow{OA}$ =Ta and $\overrightarrow{OB}$ =Tb as shown in above diagram. Thenby triangle law of vector addition, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

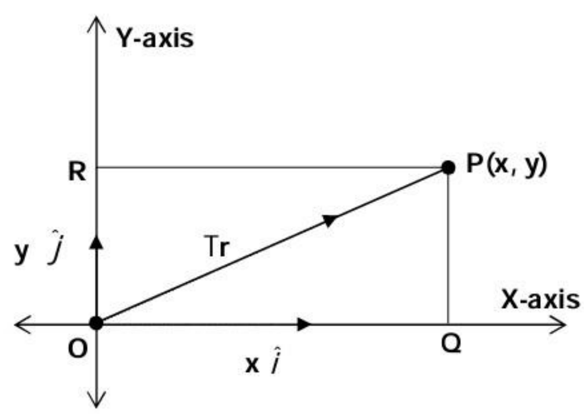


P $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

P $\overrightarrow{AB}$ = Tb –Ta .

Thus any vector $\overrightarrow{AB}$ can be written as the difference of position vector of point B and position vector of point A.

3. Representation of a vector in plane: Let P be a point in plane with the Cartesian coordinates (x, y). Let O denotes the origin in the Cartesian plane and $\hat{i}$, $\hat{j}$ be unit vectors along X -axis and Y- axis respectively. Then as per the diagram given, $\overrightarrow{OQ} = x\,\hat{i}$ and $\overrightarrow{OR} = y\,\hat{j}$ as these two vectors are collinear with the unit vectors $\hat{i}$ *and* $\hat{j}$ respectively. Now by parallelogram law of vector addition



$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{OR} = x\,\hat{i} + y\,\hat{j}$

P Tr $= x\,\hat{i} + y\,\hat{j}$.

Thus for every point P(x, y) in the Cartesian plane, there exists a unique vector representing it, which is $\overrightarrow{OP}$ = Tr $= x\,\hat{i} + y\,\hat{j}$.

The magnitude of this vector is, $|\overrightarrow{OP}|$ = | Tr |= $\sqrt{x^2 \text{ Ž } y^2}$, by distance formula.

Similarly it can be explained that, for every point P(x, y, z) in the 3-D space , there exists a unique vector representing it, which is $\overrightarrow{OP}$ = Tr $= x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$.

The magnitude of this vector is, $|\overrightarrow{OP}|$ = |Tr | = $\sqrt{x^2 \text{ Ž } y^2 \text{ Ž } z^2}$, by distance formula.

10.4.5 Product of vectors Earlier in this unit we studied the operations of vector additions, subtraction and scalar multiplication. Product of vectors is one more important concept we need to understand. There are two types of products of vectors, which are dot product and cross product. $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along x, y and z axis respectively.

1. **Dot product of vectors:** If $\vec{a}$ and $\vec{b}$ are any two vectors and θ, is angle between them (for $0 \le \theta \le \pi$), then their dot product denoted by $\vec{a} \cdot \vec{b}$ is defined as,
$\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$.

   Using the different properties of dot product which are given below, we can verify that, if two position vectors in space are $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

   Properties of dot product:

   (i) For any two vectors $\vec{a}$ and $\vec{b}$ their dot product $\vec{a} \cdot \vec{b}$ is a scalar i.e. a real number, hence this product is also referred to as scalar product of vectors.

   (ii) Dot product of vectors is commutative i.e. for any two vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta = |\vec{b}|\ |\vec{a}| \cos\theta = \vec{b} \cdot \vec{a}$.

   (iii) Any two nonzero vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other i.e. $\theta = \pi/2$ if and only if $\vec{a} \cdot \vec{b} = 0$.

   (iv) If $\vec{a}$ and $\vec{b}$ are two collinear vectors then in this case $\theta = 0$ or π.

   $\therefore \vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}|$ or $\vec{a} \cdot \vec{b} = -\ |\vec{a}|\ |\vec{b}|$ because $\cos 0 = 1$ and $\cos\pi = -1$.

   (v) $\vec{a} \cdot \vec{a} = |\vec{a}|\ |\vec{a}| \cos 0 = |\vec{a}|\ |\vec{a}| = |\vec{a}|^2$.

   (vi) If unit vectors along x, y and z axis are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively. Then these are perpendicular to each other and their magnitudes are 1 unit.

   $\therefore \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 1.1. \cos(\pi/2) = 1 \times 1 \times 0 = 0$

   and $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1. 1. \cos 0 = 1 \times 1 \times 1 = 1$.

   (vii) Dot product is distributive over vector addition i.e.

   $$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.$$

2. **Cross product of vectors** If $\vec{a}$ and $\vec{b}$ are any two vectors and θ, is angle between them (for $0 \le \theta \le \pi$), then their cross product denoted by $\vec{a} \times \vec{b}$ is a vector whose magnitude is $|\vec{a}|\ |\vec{b}| \sin\theta$ and which has the direction perpendicular to both $\vec{a}$ and $\vec{b}$. If $\hat{n}$ denotes a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, it is such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right hand triplet. Then $\vec{a} \times \vec{b}$ is along $\hat{n}$.
Hence, $\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\ \hat{n}$.

   If two position vectors in space are $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then their cross product can be obtained using determinant as,

   $$\vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$1 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}\,(a_2 b_3 - b_2 a_3) - \hat{j}\,(a_1 b_3 - b_1 a_3) + \hat{k}\,(a_1 b_2 - b_1 a_2)$$

Using determinant it becomes easy to solve problems related with cross product.

Properties of cross product:

(i) For any two vectors Ta and Tb their cross product Ta ˆ Tb is a vector hence this product is also referred to as vector product of vectors.

(ii) Cross product of vectors is not commutative. For any two vectors Ta and Tb, we know the product Ta ˆ Tb is along $\hat{n}$ but Tb ˆ Ta is along $-\hat{n}$.

P Ta ˆ Tb = |Ta| |Tb| sine $\hat{n}$ and Tb ˆ Ta = |Tb| |Ta| sine ($-\hat{n}$).

P Ta ˆ Tb ' Tb ˆ Ta.

But $\hat{n}$ is unit vector, hence $|\hat{n}| = |-\hat{n}|$, **P** |Ta ˆ Tb| = |Tb ˆ Ta|.

(iii) If Ta and Tb are any two vectors which are perpendicular to each other i.e. e = $d/2$ then Ta ˆ Tb = |Ta| |Tb| $\hat{n}$; because sin $d/2$ = 1.

(iv) IfTa and Tb are two collinear vectors then in this case e = 0 or d and Ta ˆ Tb = 0.

P Any two nonzero vectorsTa andTb are collinear vectors if and only ifTa ˆ Tb = 0, because sin0 = sind = 0.

(v) Ta ˆ Ta = |Ta| |Ta| sin 0 $\hat{n}$ = 0.

(vi) If unit vectors along x, y and z axis are denoted by $\hat{i}$, $\hat{j}$ *and* $\hat{k}$ respectively. Then these are perpendicular to each other and their magnitudes are 1unit. Here $\hat{i}$ ˆ $\hat{j}$ is a unit vector perpendicular to bothT$\hat{i}$ *and* $\hat{j}$, it is such that $\hat{i}$, $\hat{j}$ *and* $\hat{i}$ ˆ $\hat{j}$ form a right hand triplet.

P T$\hat{i}$ ˆ $\hat{j}$ = $\hat{k}$ and T$\hat{j}$ ˆ $\hat{i}$ = $-\hat{k}$,

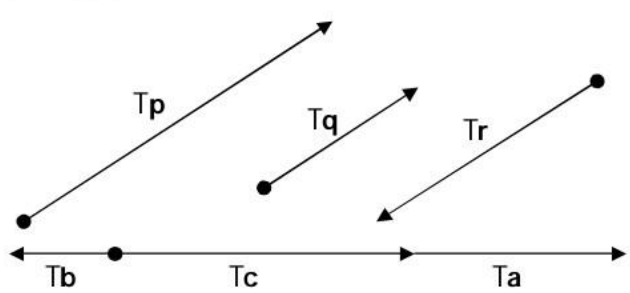
Also $\hat{j}$ ˆ $\hat{k}$ 1 $\hat{i}$ and $\hat{k}$ ˆ $\hat{j}$ 1 $-\hat{i}$; similarly $\hat{k}$ ˆ $\hat{i}$ = $\hat{j}$ and $\hat{i}$ ˆ $\hat{k}$ = $-\hat{j}$.

(vii) Cross product is distributive over vector addition i.e.
Ta ˆ (Tb + Tc) = Ta ˆ Tb + Ta ˆ Tc.

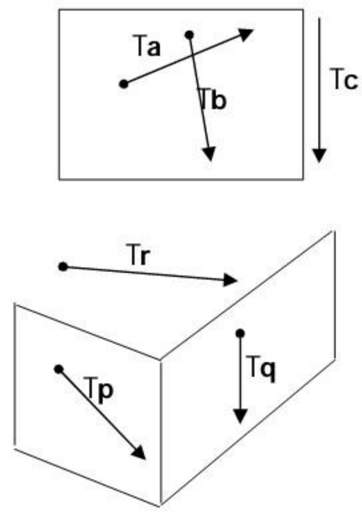
10.5 Collinear and coplanar vectors

Earlier in this unit we studied the definitions of collinear and coplanar vectors. We will get some more information of these types of vectors.

Collinear vectors: If two or more vectors are parallel to the same line or are along the same line are called collinear vectors. The collinear vectors need not lie on the same line always. In the diagram vectors Tp, Tq andTr are collinear vectors and also vectors Ta, Tb andTc are collinear vectors.



Coplanar vectors: If more than two vectors are parallel to the same plane or are on the same plane are called coplanar vectors. The coplanar vectors need not lie on the same plane always. In the diagram vectors Ta, Tb andTc are coplanar vectors as they are parallel to the same plane. But the vectors Tp, Tq andTr are not coplanar vectors.



Some important properties of collinear and coplanar vectors:

1. Two collinear vectors are scalar multiples of each other and conversely the vectors, which are scalar multiples of each other, are always collinear.

2. Two vectors Ta and Tb are collinear vectors if and only if Ta ˆ Tb = 0

3. Two nonzero vectors Ta and Tb are collinear vectors if and only if there exist two nonzero real numbers k_1 and k_2 such that k_1 a + k_2Tb = T0.

4. Any two parallel or intersecting vectors form a plane hence they are always coplanar.

5. Three vectors Ta, Tb andTc are coplanar vectors if and only if their scalar triple product is equal to 0. i.e. Ta •Tb ˆ Tc = 0.

6. Three nonzero vectors a, Tb andTc are coplanar if and only if there exist real numbers k_1 , k_2 and k_3 not all zero such that k_1Ta + k_2Tb + k_3Tc = T0.

10.6 Summary

In this unit learners studied the following topics in details:

1. The concept of vector quantities and scalar quantities.
2. Different types of vectors such as, equal vectors, unit vector, zero vector, collinear vectors and coplanar vectors etc.
3. Vector operations such addition, subtraction and scalar multiplication, dot product of vectors and cross product of vectors.
4. Representation of a point in plane or in space using vectors.

UNIT 11 Matrices and Determinants

11.0 Objectives

By the end of this Unit, learners should be able to:

- Understand the concept of matrix.
- Describe and discuss different types of matrices.
- Perform different operations on matrices such as addition, subtraction, scalar multiplication, multiplication etc.
- Find inverse of a matrix.
- Understand concept of determinant.
- Describe and discuss different properties of determinant.
- Evaluate determinant of a matrix.

11.1 Introduction

The theory of matrices (single matrix) is of great importance in many branches of higher mathematics such as astronomy, mechanics, nuclear physics and aerodynamics etc. Matrices are used to describe linear equations and to record data that depend on multiple parameters. Hence matrices are widely used in computer science. They can be added, multiplied, and decomposed in various ways, which also makes them a key concept in the field of linear algebra.

In elementary algebra we mainly deal with single numbers. These numbers are combined by various operations to obtain other numbers. But in some branches of algebra we need to consider a set of numbers. For example in plane geometry the Cartesian coordinates of a point are given by a pair of two numbers these numbers can be represented as a single entity using matrix. A matrix is only arrangements of numbers but it is treated as a single entity rather than a collection of numbers. In this unit we will get introduced with elementary concepts about matrices.

11.2 Matrices

In mathematics, a matrix (plural matrices) is an arrangement of numbers into rows and columns. The horizontal lines in a matrix are called rows and the vertical lines are called columns.

11.2.1 Matrix: A matrix is a set of numbers, which are arranged into rows and columns.

A matrix with m rows and n columns is called an $m \times n$ matrix. It is commonly said that an $m \times n$ matrix is of order $m \times n$. The order of a matrix is always given with the number of rows first, then the number of columns.

The entry that lies in the i-th row and the j-th column of a matrix is typically referred to as the (i, j)th entry of the matrix. The row is always noted first and then the column. Generally capital letters are used to denote matrices.

We often write $A = \left[a_{ij}\right]_{m \times n}$ to define an $m \times n$ matrix A. In this case, the entries $a_{i,j}$ are defined separately for all integers $1 \le i \le m$ and $1 \le j \le n$.
We can denote a general 3×4 matrix as $A = \left[a_{ij}\right]_{3 \times 4}$, then this matrix can also be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4}$$

Examples:

- $A = [1 \quad 2 \quad 3]$ is a matrix with 1 row and 3 columns. ⇒ A has the order 1×3.
- $B = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ is a matrix with 3 row and 3 columns. ⇒ A has the order 3×3.
- $P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a 2×4 matrix.
- $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a 4×4 matrix.

11.3 Types of Matrices

1. **Row matrix:** A matrix having only one row is called a row matrix. If P is a row matrix then its order is of the form $1 \times n$, where n is the number of columns in this matrix.

   Example:

   - $P = [1 \quad 5 \quad 7 \quad -4]$ is a row matrix as it has 1 row and 4 columns.

   ⇒ P has order 1×4.

2. **Column matrix:** A matrix having only one column is called a column matrix. If A is a column matrix then its order is of the form $m \times 1$, where m is the number of rows in this matrix.

   Example:

   - $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a column matrix as it has 3 rows and 1 column.

   ⇒ A has an order 3×1.

3. **Square matrix:** If the number of rows in a matrix is equal to the number of columns in it, then that matrix is called a square matrix.

If matrix $A_{m \times n}$ is a square matrix then m = n. So its order can also be written as n × n.

For a general square matrix $A = \left[a_{ij}\right]_{n \times n}$, the entries a_{11}, a_{22}, a_{33}, ..., a_{nn} are called the diagonal elements.

Examples:

- $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ is a matrix with 2 rows and 2 columns. ∴ A is a square matrix. In this matrix the diagonal elements are $a_{11} = 2$ and $a_{22} = 2$.
- $B = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 9 & 9 \\ 2 & 11 & -4 \end{bmatrix}$ is a 3 × 3 matrix, so it is a square matrix. In this matrix the diagonal elements are $b_{11} = 1$, $a_{22} = 9$ and $a_{33} = -4$.
- $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a square matrix with all the diagonal elements equal to 1.

4. **Zero or null matrix:** If every element in a matrix is equal to zero, then that matrix is called a zero matrix or a null matrix. It is denoted by 0.

Examples:

- $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ∴ A is a square matrix. In this matrix all elements are 0, and so it is a null matrix of order 2 × 2. It is also denoted as $0_{2\times2}$.
- $N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a null matrix, of order 3 × 5.

5. **Unit or Identity matrix:** A square matrix in which all diagonal elements are equal to 1 and all non- diagonal elements are equal to 0, is called an identity matrix. An identity matrix of order n × n is denoted by I_n.

Examples:

- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix of order 2 × 2.
- $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 4 × 4.

6. **Diagonal matrix:** A square matrix, in which all non-diagonal elements are equal to 0, is called a diagonal matrix.

Note that, in a diagonal matrix all non-diagonal elements are equal to 0 and some or all of the diagonal elements may be equal to 0.

Examples:

- An identity matrix I_n is always a diagonal matrix.
- A square matrix which is a null matrix is a diagonal matrix.
- $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ is a diagonal matrix of order 4×4.
- $A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ is a diagonal matrix of order 2×2.

7. Triangular matrix:

(i) Upper triangular matrix: A square matrix, in which all the elements below the main diagonal are equal to 0, is called an upper triangular matrix.

(ii) Lower triangular matrix: A square matrix, in which all the elements above the main diagonal are equal to 0, is called a lower triangular matrix.

Note that:

1. A triangular matrix is either a lower triangular or an upper triangular matrix.
2. A diagonal matrix and an identity matrix are both lower triangular and upper triangular matrix.

Examples:

- $P = \begin{pmatrix} 1 & 5 & 0 & -2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ is an upper triangular matrix.
- $Q = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 5 & 30 \end{pmatrix}$ is a lower triangular matrix.
- $A = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ is both lower triangular and upper triangular matrix.

8. Transpose of a matrix: If A is a matrix of order $m \times n$, then the transpose of matrix A is denoted by A^t or A', is the matrix obtained by interchanging the rows and the columns of matrix A. Order of A^t is $n \times m$.

For a general square matrix $A = \left(a_{ij}\right)_{m \times n}$, its transpose is $A^t = \left(a_{ji}\right)_{n \times m}$.

Note that,

1. Transpose of a row matrix is a column matrix.
2. Transpose of a column matrix is a row matrix.
3. Transpose of an identity matrix is itself.

Examples:

- $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ is a row matrix with order 1×3.

  $\Rightarrow$ A^t is a column matrix with order 3×1 and $A^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- $P = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 6 & 4 & 0 & 5 \end{bmatrix}$ is a matrix with order 2×4.

  $\Rightarrow$ P^t is a matrix with order 4×2 and $P^t = \begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 2 & 0 \\ 1 & 5 \end{bmatrix}$

- $B = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ is a matrix with order 3×3.

$\Rightarrow$ B^t is a matrix with order 3×3 and $B^t = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$

9. **Symmetric matrix:** A square matrix $A = [a_{ij}]_{n \times n}$ is a symmetric matrix, if $a_{ij} = a_{ji}$ for all i and j.

   Note that,

   1. A square matrix A is symmetric matrix, if and only if $A = A^t$.
   2. Every identity matrix is a symmetric matrix.
   3. Every null matrix of order $n \times n$ is a square matrix for natural number $n>1$.
   4. Every diagonal matrix is a symmetric matrix.

Examples:

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ then $A^t = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $\Rightarrow$ A is a symmetric matrix.
- If $B = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ then $B^t = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ $\Rightarrow$ B is a symmetric matrix.
- If $P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 7 & 3 \\ 1 & 3 & 8 \end{bmatrix}$ then $P^t = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 7 & 3 \\ 1 & 3 & 8 \end{bmatrix}$. $\Rightarrow$ P is a symmetric matrix.

11.4 Algebra of matrices

11.4.1 Equality of matrices: Two matrices are equal if they are of the same order and if the corresponding elements are equal. So, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices then, A = B if and only if $a_{ij} = b_{ij}$ for all i and j. Note that, matrices of different orders cannot be equal.

11.4.2 Addition of matrices: If two matrices A and B are of the same order then they can be added and their addition denoted by A + B is the matrix of the same order which is obtained by adding the corresponding elements of A and B.

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{m \times n}$ then, $A + B = \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}_{m \times n}$

Note that, matrices of different orders cannot be added.

Examples:

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A + B = \begin{bmatrix} 2+1 & 2+2 \\ 2+3 & 2+4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

- If $C = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 7 & 3 \\ 1 & 3 & 8 \end{bmatrix}$, then $C + D = \begin{bmatrix} 3 & 5 & 3 \\ 5 & 9 & 12 \\ 3 & 12 & 16 \end{bmatrix}$

If $P = \begin{bmatrix} -1 & 3 & 2 & 1 \\ 6 & 4 & 0 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 3 & 5 & -2 & 1 \\ 7 & 9 & 4 & 5 \end{bmatrix}$, then

$$P + Q = \begin{bmatrix} -1+3 & 3+5 & 2+(-2) & 1+1 \\ 6+7 & 4+9 & 0+4 & 5+5 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 0 & 2 \\ 13 & 13 & 4 & 10 \end{bmatrix}$$

Properties of addition of matrices:

1. Addition of matrices is commutative i.e. A + B = B + A.
2. Addition of matrices is associative i.e. A + (B + C) = (A + B)+ C.
3. A + 0 = 0 +A = A, where 0 is a null matrix of appropriate order.

11.4.3 Subtraction of matrices: If two matrices A and B are of the same order then the subtraction can be performed, the subtraction denoted by A – B is the matrix of the same order which is obtained by subtracting the corresponding elements of B from the elements of A.

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{m \times n}$ then, $A - B = \begin{bmatrix} a_{ij} - b_{ij} \end{bmatrix}_{m \times n}$.

Note that, matrices of different orders cannot be subtracted.

Examples:

- If $A = \begin{bmatrix} 4 & 5 \\ 1 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A - B = \begin{bmatrix} 4-1 & 5-2 \\ 1-3 & 9-4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -2 & 5 \end{bmatrix}$

- If $C = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 7 & 3 \\ 1 & 3 & 8 \end{bmatrix}$, then

$$C - D = \begin{bmatrix} 1-2 & 5-0 & 2-1 \\ 5-0 & 2-7 & 9-3 \\ 2-1 & 9-3 & 8-8 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 1 \\ 5 & -5 & 6 \\ 1 & 6 & 0 \end{bmatrix}.$$

If $P = \begin{bmatrix} -1 & 3 & 2 & 1 \\ 6 & 4 & 0 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 3 & 5 & -2 & 1 \\ 7 & 9 & 4 & 5 \end{bmatrix}$, then

$$P - Q = \begin{bmatrix} -1-3 & 3-5 & 2-(-2) & 1-1 \\ 6-7 & 4-9 & 0-4 & 5-5 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 4 & 0 \\ -1 & -5 & -4 & 0 \end{bmatrix}.$$

11.4.4 Scalar Multiplication: If $A = \left[a_{ij}\right]_{m \times n}$ is any matrix and k is a scalar i.e. a real number then ka , the scalar multiple of A is defined as $kA = \left[ka_{ij}\right]_{m \times n}$.

Examples:

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ and k =1/3 then $kA = \begin{bmatrix} 2/3 & 2/3 \\ 2/3 & 2/3 \end{bmatrix}$
- If $A = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{bmatrix}$ and k = 2, then $kA = \begin{bmatrix} 2 & 10 & 4 \\ 10 & 4 & 18 \\ 4 & 18 & 16 \end{bmatrix}$

If $Q = \begin{bmatrix} 3 & 5 & -2 & 1 \\ 7 & 9 & 4 & 5 \end{bmatrix}$ and k = 5, then $kQ = \begin{bmatrix} 15 & 25 & -10 & 5 \\ 35 & 45 & 20 & 25 \end{bmatrix}$

Properties of scalar multiplication:

1. Scalar multiplication is distributive over addition and subtraction of matrices i.e. $k(A + B) = kA + kB$.
2. Addition and subtraction of scalars is distributive over scalar multiplication i.e. $(k_1 + k_2)A = k_1A + k_2A$.
3. If k = –1, then kA = –A where $-A = -\left[a_{ij}\right]_{m \times n} = \left[-a_{ij}\right]_{m \times n}$.
4. If k = 0 then kA = 0, where 0 is a null matrix of appropriate order.

11.4.5 Multiplication of Matrices: If two matrices A and B are such that the number of columns of A is equal to the number of rows of B then their multiplication denoted by AB can be performed. If the order of A is $m \times n$ and the order of B is $n \times p$ then the product matrix AB is of order $m \times p$.

If $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{n \times p}$ are two matrices then,

$$AB = \left[a_{ij}\right]_{m \times n} \times \left[b_{ij}\right]_{n \times p} = \left[\sum_{k=1}^{n} (a_{ik} \times b_{kj})\right]_{m \times p}$$

Examples:

- If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2\times 2}$, then

$$AB = \begin{bmatrix} 1\times 1+1\times(-1) & -1\times 1+1\times 1 \\ 1\times 1+1\times(-1) & -1\times 1+1\times 1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- If $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$, then

$$AB = \begin{bmatrix} 2\times 1+2\times 3 & 2\times 2+2\times 4 \\ 2\times 1+2\times 3 & 2\times 2+2\times 4 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 8 & 12 \\ 8 & 12 \end{bmatrix}$$

- If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} 7 & 1 & 5 & 2 \\ 3 & 8 & 4 & 7 \\ 6 & 3 & 9 & 5 \end{bmatrix}_{3\times 4}$ then AB is a matrix of order 2×4.

$$AB = \begin{bmatrix} 1\times 7+3\times 3+5\times 6 & 1\times 1+3\times 8+5\times 3 & 1\times 5+3\times 4+5\times 9 & 1\times 2+3\times 7+5\times 5 \\ 2\times 7+4\times 3+6\times 6 & 2\times 1+4\times 8+6\times 3 & 2\times 5+4\times 4+6\times 9 & 2\times 2+4\times 7+6\times 5 \end{bmatrix}_{2\times 4}$$

$$= \begin{bmatrix} 46 & 40 & 62 & 48 \\ 62 & 52 & 80 & 62 \end{bmatrix}$$

Properties of Multiplication of Matrices: If A,B,C are matrices such that their multiplications are possible, 0 denotes null matrix of appropriate order and I denotes identity matrix of appropriate order, then

1. Multiplication of matrices is not commutative. i.e. $AB \neq BA$, for all matrices A and B.
2. Multiplication of matrices is associative, i.e. $A(BC) = (AB)C$.
3. $A + 0 = 0 + A = A$, where 0 is a null matrix of appropriate order
4. Multiplication of matrices is distributive over addition and subtraction of matrices i.e. $A(B \pm C) = AB \pm AC$.
   and $(B \pm C)A = BA \pm CA$.
5. Identity matrix is multiplicative identity i.e. $IA = AI = A$.
6. $0A = A0 = 0$, where 0 is a null matrix of appropriate order.
7. It is possible that $AB = 0$, but A and B are such that $A \neq 0$ as well as $B \neq 0$.

11.5 Determinant

If A is any square matrix of real numbers, then the determinant of A is a certain real number assigned to the matrix A. It is denoted by | A | or det(A).

The formula to find the determinant changes according to the order of the matrix.

11.5.1 Determinant of a Matrix of order 2×2 : If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix, then its determinant is defined as ,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

Example:

- If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$ then its determinant is, $|A| = 1 \times 4 - 3 \times 2 = 4 - 6 = -2.$

11.5.2 Determinant of a Matrix of order 3 × 3:

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a 3 × 3 matrix, then its determinant is defined as,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Example:

- If $A = \begin{pmatrix} 1 & 5 & 2 \\ 5 & 2 & 9 \\ 2 & 9 & 8 \end{pmatrix}$ is a 3 × 3 matrix then ,its determinant is,

$$\begin{aligned} |A| &= 1 \times (2\times 8 - 9\times 9) - 5 \times (5\times 8 - 2\times 9) + 2 \times (5\times 9 - 2\times 2) \\ &= 1 \times (16 - 81) - 5 \times (40 - 18) + 2 \times (45 - 4) \\ &= 1 \times (-65) - 5 \times (22) + 2 \times (41) = -65 - 110 + 82 = -93. \end{aligned}$$

11.5.3 Singular Matrix: A square matrix A is said to be a singular matrix if its determinant is equal to zero i.e. $|A| = 0$. Otherwise it is called as a non-singular matrix.

Examples:

- If $A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}_{2\times 2}$ then $|A| = 2 \times 2 - 2 \times 2 = 4 - 4 = 0$. ∴ A is a singular matrix.
- and $B = \begin{pmatrix} 4 & 5 \\ 3 & 6 \end{pmatrix}_{2\times 2}$, then $|B| = 4 \times 6 - 3 \times 5 = 24 - 15 = 9 \neq 0$.

  ∴ B is a non-singular matrix.

Properties of determinants:

(i) If the rows and the columns of a matrix are interchanged then the value of the determinant remains the same. Hence for every square matrix and its transpose the value of the determinant is same, i.e. for any square matrix A , we have $|A| = |A^t|$.

(ii) If any two rows (or columns) of a matrix are interchanged then the value of the determinant changes in sign only.

(iii) If any two rows (or columns) of a matrix are identical then the value of the determinant is zero.

(iv) If every element of any one row (or column) of a matrix is multiplied by a constant k, then the value of the determinant is multiplied by k.

11.6 Inverse of a matrix

11.6.1 Inverse of a Matrix: If two square matrices A and B are such that AB = BA = I, where I denotes identity matrix of the same order, then B is known as inverse of A, and it is denoted by A^{-1}.If B is inverse of A then A is inverse of B. i.e. if $B = A^{-1}$ then $A = B^{-1}$.

Inverse of every square matrix does not exist. If A is a matrix such that its inverse exists then it is called as invertible matrix.

It is known that a matrix is invertible if and only if its determinant is nonzero i.e. if and only if it is a non-singular matrix.

It is possible to find inverse of an invertible matrix using various methods. We will now study one of the methods to find inverse of a matrix, which is using adjoint of the matrix. To understand what is adjoint we need to understand few more terms such as minor and cofactor.

11.6.2 Minor of an element: If $A = \left[a_{ij}\right]_{n \times n}$ is any square matrix, then the minor of the $(i, j)^{th}$ element a_{ij} is the determinant of the submatrix obtained by deleting i^{th} row and j^{th} column from A. It is denoted by m_{ij}.

Example:

- If $A = \begin{bmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{bmatrix}$ is a square matrix, then we can find minor m_{ij} of every element a_{ij} of A . By the above definition m_{ij} =determinant of the submatrix obtained by deleting i^{th} row and j^{th} column of A.

  $\therefore$ m_{11} = determinant of the submatrix obtained by deleting 1^{st} row and 1^{st} column of A

  $\therefore m_{11} = \begin{vmatrix} 6 & 2 \\ 0 & -1 \end{vmatrix} = 6 \times (-1) - 0 \times 2 = -6 - 0 = -6.$

  Similarly $m_{12} = \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} = 2 \times (-1) - 1 \times 2 = -2 - 2 = -4.$

  $m_{13} = \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix} = 2 \times 0 - 1 \times 6 = 0 - 6 = -6.$

  $m_{21} = \begin{vmatrix} 1 & -4 \\ 0 & -1 \end{vmatrix} = 1 \times (-1) - 0 \times (-4) = -1 - 0 = -1.$

  $m_{22} = \begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix} = 1 \times (-1) - 1 \times (-4) = -1 + 4 = 3.$

$$m_{23} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \times 0 - 1 \times 1 = 0 - 1 = -1.$$

$$m_{31} = \begin{vmatrix} 1 & -4 \\ 6 & 2 \end{vmatrix} = 1 \times 2 - 6 \times (-4) = 2 + 24 = 26.$$

$$m_{32} = \begin{vmatrix} 1 & -4 \\ 2 & 2 \end{vmatrix} = 1 \times 2 - 2 \times (-4) = 2 + 8 = 10.$$

$$m_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} = 1 \times 6 - 1 \times 2 = 6 - 2 = 4.$$

11.6.3 Cofactor of an element: If $A = \left(a_{ij}\right)_{n \times n}$ is any square matrix, and the minor of the (i, j)th element a_{ij} is m_{ij}, then the cofactor of the element a_{ij} is denoted by c_{ij} is defined as $c_{ij} = (-1)^{i+j} \times m_{ij}$.

Example:

- For a square matrix $A = \begin{pmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{pmatrix}$, we have minor m_{ij} of every element a_{ij} of A , as computed above. Using these minors we can find the corresponding cofactors $c_{ij} = (-1)^{i+j} \times m_{ij}$,as below,

$m_{11} = -6$ $\Rightarrow$ $c_{11} = (-1)^{1+1} \times m_{11} = (-1)^2 \times (-6) = 1 \times (-6) = -6.$

$m_{12} = -4$ $\Rightarrow$ $c_{12} = (-1)^{1+2} \times m_{12} = (-1)^3 \times (-4) = (-1) \times (-4) = 4.$

$m_{13} = -6$ $\Rightarrow$ $c_{13} = (-1)^{1+3} \times m_{13} = (-1)^4 \times (-6) = 1 \times (-6) = -6.$

$m_{21} = -1$ $\Rightarrow$ $c_{21} = (-1)^{2+1} \times m_{21} = (-1)^3 \times (-1) = (-1) \times (-1) = 1.$

$m_{22} = 3.$ $\Rightarrow$ $c_{22} = (-1)^{2+2} \times m_{22} = (-1)^4 \times 3 = 1 \times 3 = 3.$

$m_{23} = -1$ $\Rightarrow$ $c_{23} = (-1)^{2+3} \times m_{23} = (-1)^5 \times (-1) = (-1) \times (-1) = 1.$

$m_{31} = 26$ $\Rightarrow$ $c_{31} = (-1)^{3+1} \times m_{31} = (-1)^4 \times 26 = 1 \times 26 = 26.$

$m_{32} = 10$ $\Rightarrow$ $c_{32} = (-1)^{3+2} \times m_{32} = (-1)^5 \times 10 = (-1) \times 10 = -10.$

$m_{33} = 4$ $\Rightarrow$ $c_{33} = (-1)^{3+3} \times m_{33} = (-1)^6 \times 4 = 1 \times 4 = 4.$

11.6.4 Cofactor matrix: If $A = \left(a_{ij}\right)_{n \times n}$ is any square matrix then cofactor matrix of A is the matrix $C = \left(c_{ij}\right)_{n \times n}$ in which the (i, j)th element c_{ij} is the cofactor of the element a_{ij} of A.

Example:

- For a square matrix $A = \begin{bmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{bmatrix}$, we have cofactor c_{ij} of every element a_{ij} of A, as computed above. Using these cofactors we can write the cofactor matrix C of A as below,

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -6 & 4 & -6 \\ 1 & 3 & 1 \\ 26 & -10 & 4 \end{bmatrix}$$

11.6.5 Adjoint of a Matrix: Adjoint of a matrix A is the transpose of the matrix of cofactors of A. This is denoted by Adj(A).

Using these definitions and matrix algebra, some theorems can be proved, which give a formula to find inverse of any square nonsingular matrix.

This formula is stated as , where | A | is the determinant of A.

Examples:

- For a square matrix $A = \begin{bmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{bmatrix}$, we have cofactor matrix C of A as

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} -6 & 4 & -6 \\ 1 & 3 & 1 \\ 26 & -10 & 4 \end{bmatrix}$$

$$\therefore \text{Adj}(A) = C^t = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -6 & 1 & 26 \\ 4 & 3 & -10 \\ -6 & 1 & 4 \end{bmatrix}$$

The inverse of A is computed by $A^{-1} = \frac{1}{|A|} \times Adj(A)$.

$$\text{Now } |A| = \begin{vmatrix} 1 & 1 & -4 \\ 2 & 6 & 2 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= 1 \times [6 \times (-1) - 0 \times 2] - 1 \times [2 \times (-1) - 1 \times 2] + (-4) \times [2 \times 0 - 1 \times 6]$$

$$= [1 \times (-6)] - [1 \times (-4)] + [(-4) \times (-6)]$$

$$= -6 + 4 + 24$$

$= 22 \neq 0$. $\therefore$ A is invertible matrix.

$$\therefore A^{-1} = \frac{1}{22} \times \begin{pmatrix} -6 & 1 & 26 \\ 4 & 3 & -10 \\ -6 & 1 & 4 \end{pmatrix} = \begin{pmatrix} -6/22 & 1/22 & 26/22 \\ 4/22 & 3/22 & -10/22 \\ -6/22 & 1/22 & 4/22 \end{pmatrix}$$

‹ If $B = \begin{pmatrix} 4 & 5 \\ 3 & 6 \end{pmatrix}_{2 \times 2}$, then $|B| = 4 \times 6 - 3 \times 5 = 24 - 15 = 9 \neq 0$.

$\therefore$ B is a non-singular matrix. $\therefore$ B is invertible matrix, hence by definition of minor we have,

m_{11} = determinant of the submatrix obtained by deleting 1st row and 1st column of B

$\therefore m_{11} = |\ 6\ | = 6.$

And similarly $m_{12} = |\ 3\ | = 3$, $m_{21} = |\ 5\ | = 5$ and $m_{22} = |\ 4\ | = 4$.

Using these minors we can find the corresponding cofactors $c_{ij} = (-1)^{i+j} \times m_{ij}$, as $c_{11} = (-1)^{1+1} \times m_{11} = (-1)^2 \times 6 = 1 \times 6 = 6$.

$c_{12} = (-1)^{1+2} \times m_{12} = (-1)^3 \times 3 = (-1) \times 3 = -3.$

$c_{21} = (-1)^{2+1} \times m_{21} = (-1)^3 \times 5 = (-1) \times 5 = -5.$

and $c_{22} = (-1)^{2+2} \times m_{22} = (-1)^4 \times 4 = 1 \times 4 = 4.$

Hence cofactor matrix $C = \begin{pmatrix} 6 & -3 \\ -5 & 4 \end{pmatrix}$ $\therefore$ $Adj(B) = C^t = \begin{pmatrix} 6 & -5 \\ -3 & 4 \end{pmatrix}$

By formula to find inverse, $B^{-1} = \frac{1}{|B|} \times Adj(B) = \frac{1}{9} \times \begin{pmatrix} 6 & -5 \\ -3 & 4 \end{pmatrix}$

11.7 Summary

In this unit learners studied the following topics in details:

1. The concept of matrix.
2. Different types of matrices such as, equal matrices, identity matrix, null matrix, diagonal matrices and singular matrices etc.
3. Operations on matrices such addition, subtraction and scalar multiplication and multiplication of matrices.
4. Concept of determinant. Evaluation of determinant of a matrix.
5. Concept of inverse of a matrix and adjoint method to find the inverse of a matrix.

UNIT12 Mensuration

12.0 Objectives

By the end of this Unit, learners should be able to:

- Compute the areas of plane figures such as triangle, rectangle and circle.
- Understand how to find Perimeter and circumference of above figures.
- Find surface areas of cube, cuboids, spheres and right circular cylinders.
- Compute Volumes of cube, cuboids, spheres and right circular cylinders.

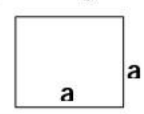
12.1 Introduction

Mensuration in its literal meaning is the science of measurement. This word is generally used where geometrical figures are concerned and where one has to determine various physical quantities such as length, area and volume. Measurement is fundamental in science; it is one of the things that distinguish science from pseudoscience. Measurement is the process of estimating the magnitude of some attribute of an object, such as its length, weight, or depth relative to some standard unit of measurement. Measurement is also essential in industry, commerce, engineering, construction, manufacturing, pharmaceutical production, and electronics.

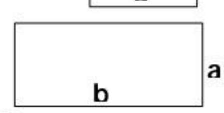
12.2 Areas of plane figures

Area is a quantity expressing the size of a figure typically a region bounded by a closed curve in the Euclidean plane or on a two-dimensional surface. Points and lines have zero area. A figure may have infinite area, for example the entire Euclidean plane. Although area seems to be one of the basic notions in geometry, it is not easy to define even in the Euclidean plane. The expressions for the areas of some simple plane figures are as given below:

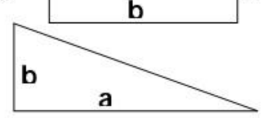
1. **Square:** The area A of a square of side length a units is given by, $A = a^2$ square units.



2. **Rectangle:** The area A of a rectangle of side lengths a and b i.e. height a and breadth b is given by $A = a \times b$ square units.



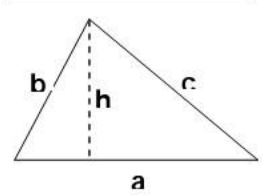
3. **Right-angled triangle:** The area A of a triangle having sides enclosing the right angle, of lengths a and b is given by, $A = (a \times b)/2$ square units.



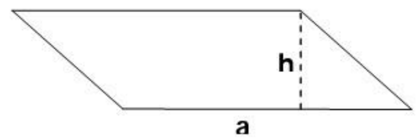
4. **Triangle:** The area A of a triangle with length of base a, and height h from the opposite angle is given by, $A = (\text{base} \times \text{height}) / 2 = (a \times h) / 2$ square units. **OR**

   The area A of a triangle with lengths of the three sides a units, b units and c units respectively is given by ,

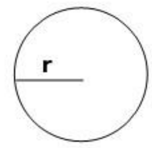
   $A = \sqrt{s \times (s - a) \times (s - b) \times (s - c)}$ square units where $s = (a + b + c) / 2$.



5. **Parallelogram:** The area A of a parallelogram with the two opposite sides a and perpendicular distance between them h is given by, $A = h \times a$ square units.



6. **Circle:** The mensuration of the circle is founded on the property that the areas of different circles are proportional to the squares on their diameters. The area A of a circle is given by, $A = \pi \times r^2$ square units, where r is the radius, and π=3.14159 approximately.



Examples:

- Area of a square of side 5 cm. is equal to 25 cm^2.
- If ABCD is a rectangle with the length of side AB = 12cms. and the length of side AD = 21cm,

  Then the area A = length of side AB $\times$ length of side AD

  $\therefore A = 12 \times 21 = 252$ cm^2.
- The area A of a triangle having height 32 mm. and length of base 65 mm

  $$A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 32 \times 65 = 1040\ mm^2.$$
- The area A of a triangle with sides 5 cm, 12 cm and 13 cm is,

  $$A = \sqrt{s \times (s-5) \times (s-12) \times (s-13)}.$$

  $$\text{where } s = \frac{a+b+c}{2} = \frac{5+12+13}{2} = 15$$

$$\therefore A = \sqrt{15 \times (15-5) \times (15-12) \times (15-13)} = \sqrt{15 \times 10 \times 3 \times 2} = \sqrt{900} = 30\ cm^2.$$

- The area A of a parallelogram with the length of two opposite sides 6cm and perpendicular distance between them 5cm is given by,

  $A = 6 \times 5 = 30$ square cm.
- The area A of a circle with radius 2 cm is,

  $A = \pi \times 2^2 = 4\pi = 12.5663\ cm^2$.

12.3 Perimeters of plane figures

The perimeter is the length of the line that bounds an area. The word may also be used to refer to the boundary line itself, but we will use perimeter as a measure. In the special case where the area is circular or elliptical the perimeter is called as the circumference.The perimeter of a polygon is the addition of the lengths of its sides. The expressions for the perimeters of some simple plane figures are given below:

1. Square: The perimeter P of a square of side length a is, $P = 4 \times a$ units.
2. Rectangle: The perimeter P of a rectangle of side lengths a and b i.e. say height a and breadth b is given by, $P = 2 \times (a + b)$ units.

3. Parallelogram: The perimeter P of a parallelogram with the two opposite sides a and b is given by, P = 2 ˆ (a + b) units.

4. Circle: The perimeter P of a circle is called the circumference of the circle. It is given by P =2ˆdˆr units, where r is the radius of the circle, and d=3.14159 approximately.

Examples:

‹ The perimeter of a square of side 5 cm. is equal to 4 ˆ5 = 20 cm.

‹ If ABCD is a rectangle with adjacent sides 12 cm and 21 cm in length length, then the perimeter = 2 ˆ(12+ 21) = 66 cm.

‹ The perimeter of a parallelogram with the length of adjacent sides 32 mm. and 65 mm is,

P = 2 ˆ(32+ 65) = 194 mm.

‹ The perimeter of a triangle with sides 5 cm, 12 cm and 13 cm is,

P = a + b+c = 5+ 12 +13 = 30 cm.

‹ The perimeter i.e. circumference of a circle with radius 4 cm is,

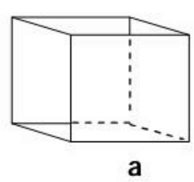
P =2ˆdˆr = 8 d = 25.1327 cm.

12.4 Volumes of solid objects

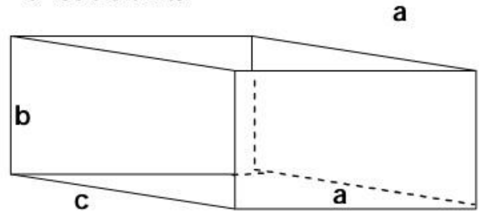
The 3-dimensional analog of area is the volume. The volume of any solid, plasma, vacuum or theoretical object is how much three dimensional space it occupies, often quantified numerically. One-dimensional figures and two-dimensional shapes) are assigned zero volume in the three-dimensional space. Volumes of straight-edged and circular shapes are calculated using arithmetic formulae.

Volume and capacity are sometimes distinguished, with capacity being used for how much a container can hold and volume being how much space an object displaces .The volume of a dispersed gas is the capacity of its container. Volume and capacity are also distinguished in a capacity management setting, where capacity is defined as volume over a specified time period.

The expressions for the volumes of some simple figures in three dimensional space are as given below:

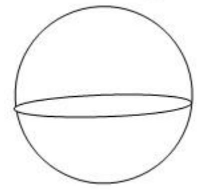


1. **Cube:** A cube is a three dimensional solid object bounded by six square faces or sides, with three meeting at each vertex. All the faces are perpendicular to each other. Volume V of a cube of side length a units of a square face is given by, V= a^3 cubic units.



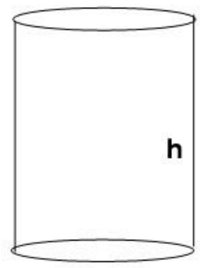
2. **Cuboids:** A cuboid or a rectangular paralleopiped is a three dimensional solid object bounded by six rectangular faces or sides, with three meeting at each vertex. All the faces are perpendicular to each other. Volume V of a cuboid of side lengths a , b and c units respectively is given by ,
   V = length ˆ height ˆ width = a ˆ b ˆ c cubic units.

3. Sphere: A sphere is a symmetrical geometrical object. In non-mathematical usage, the term is used to refer either to a round ball or globe. In mathematics, a sphere is the set of all points in three-dimensional space which are at distance *r* from a fixed point of that space, where *r* is a positive real number called the radius of the sphere and the fixed point is the centre of the sphere. The volume Vof a sphere of radius r is given by, $V = \frac{4}{3}$ ˆ d ˆ r^3 cubic units.

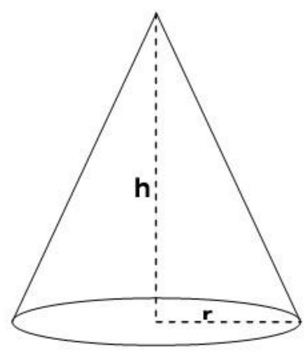


4. **Right circular cylinder:** A cylinder is bounded by two parallel planes and by the surface generated by a line segment rotating parallel to itself.

The parallel planes are called the bases. A right circular cylinder is generated by revolving a rectangle about one of its sides as an axis. For a right circular cylinder the bases are circles. The height of a cylinder is the perpendicular distance between the bases. If the right circular cylinder has a radius r and height h, then its volume V is given by, V= d ˆ r^2 ˆ h cubic units.



5. **Right circular cone:** A right circular cone is generated by revolving a right-angled triangle about one of its sides as an axis. For a right circular cone its base is a circle. The height of a right circular cone is the perpendicular from the vertex to the centre of the circular base. If the right circular cone has height h and the radius of the circular base r, then its volume V is given by , V= $\frac{1}{3}$ ˆ d ˆ r^2 ˆ h cubic units.



Examples:

‹ Volume of a cube of side 5 cm. is equal to 125 cm^3.

‹ Volume of a rectangular parallelepiped with length 30cm, height 5 cm and width 20 cm, is V = length ˆ height ˆ width

P V = 30 ˆ 5 ˆ 20 = 3000 cm^3.

‹ Volume of a sphere with radius 5 cm is V = $\frac{4}{3}$ ˆ d ˆ r^3

P V = $\frac{4}{3}$ ˆ d ˆ 5^3 = $\frac{4}{3}$ ˆ d ˆ 125 = 523.5987 cm^3.

‹ Volume of a right circular cylinder with radius 4.6 cm and height 8.5cm, is given by, V= d ˆ $(4.6)^2$ ˆ 8.5 = 565.0468 cm^3.

‹ Volume of right circular cone that has height 20 cm and the radius of the circular base 15 cm, then its volume V = $\frac{1}{3}$ ˆ d ˆ $(15)^2$ ˆ 20 = 4712.3889 cm^3.

12.5 Surface areas of solid objects

In general, the surface area is the sum of all the areas of all the shapes that cover the surface of the object. The term surface area refers to the total area of the exposed surface of a 3-dimensional solid, such as the sum of the areas of the exposed sides of a polyhedron. Surface area is the measure of how much exposed area an object has. It is expressed in square units. If an object has flat faces, its surface area can be calculated by adding together the areas of its faces. Even objects with smooth surfaces, such as spheres, have surface area.

The expressions for the surface areas of some simple figures are as given below:

1. **Cube:** As a cube is a solid object bounded by six square faces, with three of them meeting at each vertex. All square faces have equal area which is given by a^2 square units.

   P The surface area S of a cube of side length a units is, $S = 6 \times a^2$ square units.

2. **Cuboids:** A cuboid is a three dimensional solid object bounded by six rectangular faces with three meeting at each vertex. The faces apposite to each other have equal area, which can be obtained by length $\times$ height formula for area of a rectangle.

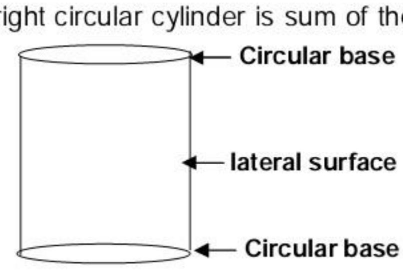
   P The surface area S of a cuboid of side lengths a, b and c units respectively is given by,

   $$S = 2 \times \text{length} \times \text{height} + 2 \times \text{height} \times \text{width} + 2 \times \text{length} \times \text{width}$$
   $$= (2 \times a \times b + 2 \times b \times c + 2 \times a \times c) \text{ square units.}$$

3. **Sphere:** The surface area S of a sphere of radius r is four times the surface area of a great circle of the sphere.
   P The surface area S of a sphere of radius r is given by,
   $S = 4 \times \pi \times r^2$ square units.

4. **Right circular cylinder:** The surface area S of right circular cylinder is sum of the lateral surface area and the area of two bases. For a right circular cylinder if the circular base has radius r. Then the area of the two bases

   is given by $2 \times \pi \times r^2$.

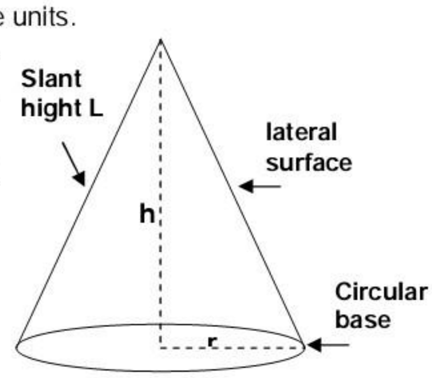
   

   If the right circular cylinder has and height h, then the lateral surface area is given by $2 \times \pi \times r \times h$.

   P The surface area $S = 2 \times \pi \times r^2 + 2 \times \pi \times r \times h$
   $$= 2 \times \pi \times r \times (r + h) \text{ square units.}$$

5. **Right circular cone:** The surface area S of right circular cone is sum of the lateral surface area and the area of the circular base. For a right circular cone if the circular base has radius r, the area of base is given by $\pi \times r^2$. If the right circular cone has slant height L, then the lateral surface area is given by

   $\frac{1}{2} \times 2 \times \pi \times r \times L = \pi \times r \times L$.

$\therefore$ The surface area $S = \pi \times r^2 + \pi \times r \times L$

$$= \pi \times r \times (r + L) \text{ square units.}$$

Note that, if the right circular cone has slant height L, the circular base has radius r and the vertical height is h then by Pythagoras theorem, we have

$L^2 = r^2 + h^2$ or $L = \sqrt{r^2 + h^2}$

Examples:

- Surface area of a cube of side 5 cm. is equal to $6 \times 25 = 150$ cm^2.
- Surface area S, of a rectangular parallelepiped with length 30cm, height 5 cm and width 20 cm, is

  $S = 2 \times$ (length $\times$ height + height $\times$ width + length $\times$ width).

  $\therefore S = 2 \times (30 \times 5 + 5 \times 20 + 30 \times 20) = 1700$ cm^2.
- Surface area S of a sphere with radius 5 cm is

  $S = 4 \times \pi \times r^2 = 4 \times \pi \times 5^2$

  $\therefore S = 314.1592$ cm^2.
- Surface area S of a right circular cylinder with radius 4.6 cm and height 8.5 cm, is given by,

  $S = 2 \times \pi \times r \times (r + h) = 2 \times \pi \times 4.6 \times (4.6 + 8.5) = 378.6247$ cm^2
- Surface area S of right circular cone that has height 20 cm and the radius of the circular base 15 cm, then its

  $S = \pi \times r^2 + \pi \times r \times L$

  where $L = \sqrt{r^2 + h^2} = \sqrt{15^2 + 20^2} = \sqrt{225 + 400} = \sqrt{625} = 25$ cm

  $\therefore S = \pi \times 15 \times (15 + 25) = 1884.9555$ cm^2.

12.6 Summary

In this unit learners studied the following topics in details:

1. The areas of plane figures such as triangle, rectangle and circle.
2. Perimeter and circumference of plane figures.
3. Volumes of cube, cuboids, spheres and right circular cylinders.
4. Surface areas of cube, cuboids, spheres and right circular cylinders.

UNIT 13 System of Linear Equations

13.0 Objectives

By the end of this Unit, learners should be able to:

- Define Linear equation.
- Understand and explain system of linear equations
- Find Solution of System of linear equations
- Apply Cramer's Rule to find solution of a system containing upto 3 variables and 3 equations.

13.1Introduction

The equations, which can be used to represent a straight line in a plane or in three-dimensional space, are linear equations. Besides geometry, there are other fields, which involve a linear relationship between different unknown quantities or variables. The study of linear equations is important and very useful for solving problems related with speed and time, age, profit and loss etc.

We will study system of linear equations containing upto 3 variables.

13.2 Linear equations

Consider the following problem: If the cost of 2 pens and 3 pencils is Rs. 26 and the cost of 3 pens and 2 pencils is Rs. 34, then what is the cost of one pen and one pencil respectively?

To solve such problems we use linear equations. For the values we want to find we assume some unknowns or variables. Using these variables we convert the given information in equations and then solve them. Before solving this problem let us define linear equation formally.

13.2.1Linear equation: A linear equation in variables $x_1, x_2, x_3, \ldots, x_n$ is defined as

$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$, where $a_1, a_2, a_3, \ldots, a_n$ and b are real numbers. $a_1, a_2, a_3, \ldots, a_n$ are called coefficients and b is called the constant term in this linear equation.

Note that all variables in a linear equation occur with power 1 only and no multiplication or division of variables occurs in it.

Examples:

- $3x_1 + 7x_2 - x_3 = -3$, is a linear equation in 3 variables.
- Equation of any line in plane is a linear equation.
- $8x - y + 4z + \sqrt{6}\,w = 4$, is a linear equation in 4 variables.

- $3 x_1 \times x_2 - 3x_3 = 14$, is not a linear equation as the first term involves multiplication of variables.
- $7x = 12y$ is a linear equation in 2 variables.
- $4x + 77y - 4\sqrt{z} = -5/21$, is not a linear equation.

13.2.2 Solution of a linear equation: A solution of linear equation $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$, is a sequence of n numbers $s_1, s_2, s_3, \ldots, s_n$ such that the equation is satisfied when we substitute $x_1 = s_1$, $x_2 = s_2$, $x_3 = s_3$, $\ldots$, $x_n = s_n$, That is $a_1s_1 + a_2s_2 + a_3s_3 + \ldots + a_ns_n = b$.

The set of all solutions is called the solution set of the linear equation.

Examples:

- $3x_1 + 7x_2 - x_3 = -3$, is a linear equation, it is satisfied by the values $x_1 = 1$, $x2 = -1$, $x_3 = -1$ because $3 \times 1 + 7 \times (-1) - (-1) = 3 - 7 + 1 = -3$.

  Hence 1, –1, –1 is a solution to the linear equation $3x_1 + 7x_2 - x_3 = -3$.

  Note that this is not the only possible solution of this equation.

- $8x - y - 4z + \sqrt{6}\,w = 4$, is a linear equation which is satisfied when we substitute $x = 2$, $y = \sqrt{6}$, $z = 3$ and $w = 1$, because

$$(8 \times 2) - (\sqrt{6}) - (4 \times 3) + (\sqrt{6} \times 1) = 4$$

  So the equation $8x - y - 4z + \sqrt{6}\,w = 4$ has a solution 2, $\sqrt{6}$, 3, 1.

  Note that this is not the only possible solution of this equation. There exist other solutions also.

13.3 System of linear equations

13.3.1 System of linear equation: A finite set of linear equations in variables $x_1, x_2, x_3, \ldots, x_n$ is called a system of linear equations, in these variables.

In general a system of m equations in n variables $x_1, x_2, x_3, \ldots, x_n$ is written as:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2\\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3\\
&\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m
\end{aligned}$$

13.3.2 Homogeneous/non-homogeneous system of linear equation: A system of linear equations in variables $x_1, x_2, x_3, \ldots, x_n$ is called a homogeneous system of linear equations if the constant term of each equation of the system is zero. A system is called as a non-homogeneous system if it is not a homogeneous system.

In general a homogeneous system of m equations in n variables $x_1, x_2, \ldots, x_n$ is written as:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = 0\\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = 0\\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = 0\\
&\ldots\\
&a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = 0.
\end{aligned}$$

Examples:

‹ $2x_1 - 3x_2 = -1$; $x_1 + x_2 = 2$; $x_1 - x_2 = 0$

is a system of linear equations in 2 variables x_1 and x_2. This system is containing 3 equations. It is non-homogeneous system.

‹ $x + 3y + 5z = 0$; $3x + 2y + 7z = 0$; $4x - 7y - 3z = 0$

It is a homogeneous system of linear equations in 3 variables x, y and z. This system is containing 3 equations.

‹ $x + y + 2z = 7$; $-x - 2y + 3z = 6$; $3x - 7y + 6z = 1$

It is a non-homogeneous system of linear equation in 3 variables x, y and z, it contains 3 equations.

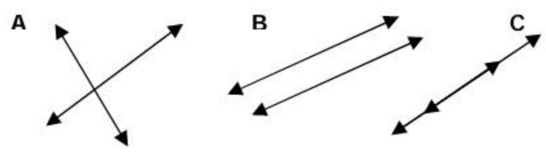
13.3.3 Solution of system of linear equation: A sequence of n numbers $s_1, s_2, s_3, \ldots, s_n$ is a solution of the system of linear equation if $x_1 = s_1$, $x_2 = s_2$, $x_3 = s_2$, $\ldots$, $x_n = s_n$, is a solution of each equation in the system.

The set of all solutions is called the solution set of the system of linear equation.

Note that:

A system of linear equations has either a unique solution, or infinitely many solutions or no solution.

It is easy to explain this geometrically, if we consider a set of 2 straight lines on plane (or in space), then one of the following situations will occur.



A) In this situation the two lines are intersecting at a point. And the corresponding linear equations form a system, which has an unique solution.

B) In this situation the two lines are parallel they are not intersecting each other. And the corresponding linear equations form a system, which has no solution.

C) In this situation the two lines are coincident, i.e. they are intersecting at every point. And the corresponding linear equations form a system, which has infinitely many solutions.

Examples:

‹ Consider the system of equations

$x + 2y = 3$; $3x + 2y = 5$; $x - y = 0$

Each equation of this system is satisfied by the values $x = 1$, $y = 1$

P 1, 1 is a solution of this system. It is a unique solution to this system.

‹ Consider the system of equations

$x + y = 2$; $5x + 5y = 10$

Each equation of this system is satisfied by the values $x = t$, $y = 2 - t$ where t is a real number.

e.g. $x = 1$, $y = 1$ is a solution of this system; also $x = 2$, $y = 0$ is a solution of this system; $x = 3$, $y = -1$ is a solution of this system; etc .

P This system has infinitely many solutions.

‹ Consider the system of equations

$$x + y = 2; \qquad 5x + 5y = 11$$

In this case from first equation we get $y = 2 - x$, substituting it in the other equation we get

$$5x + 5(2 - x) = 11$$

P 5x + 5 ˆ 2 – 5 x = 11

P 10 = 11 which is not possible hence it is not possible that $y = 2 - x$

P This system has no solution.

13.3.4 Consistent / inconsistent system: A system of linear equations is called as a consistent system if it has at least one solution otherwise i.e. if it has no solution then it is called as an inconsistent system.

Examples:

‹ $x + 3y + 5z = 0$; $\quad 3x + 2y + 7z = 0$; $\quad 4x - 7y - 3z = 0$

Each equation of this system is satisfied by the values $x = 0$, $y = 0$ and $z = 0$.

P 0, 0, 0 is a solution of this system. Hence this system is a consistent system.

Note that in general a homogeneous system is always consistent system, because 0,0,0,....0 is always solution of it.

‹ $2x_1 - 3x_2 = -1$; $\quad x_1 + x_2 = 2$; $\quad x_1 - x_2 = 0$

is a system of linear equation. Each equation of this system is satisfied by the values $x_1 = 1$, $x_2 = 1$.

P 1, 1 is a solution of this system. Hence this system is a consistent system.

‹ $x + y + 2z = 7$; $\quad -x - 2y + 3z = 6$; $\quad 3x - 7y + 6z = 1$

Each equation of this system is satisfied by the values $x = -1$, $y = 2$ and $z = 3$.

P –1, 2, 3 is a solution of this system. Hence this system is a consistent system.

13.4 Representation of system of equations in matrix form

Consider the general system of m equations in n variables $x_1, x_2, x_3, \ldots, x_n$ written as:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3$$

. . .

:

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m$$

Using matrix we can write this system as,

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \cdots \\ \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \\ \\ b_m \end{pmatrix}$$

This is equivalent to the matrix multiplication,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or A X = B where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

So the general system of m equations in n variables has the matrix form

A X = B where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is called the matrix of coefficients, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is called the matrix of variables and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is called the matrix of constant terms.

Note that:

1. If A is invertible then the system AX = B has unique solution.
2. There are different methods to solve the system AX = B.
3. If A is invertible matrix then solution of AX = B is $X = A^{-1}B$.
4. One method to find solution of system of linear equations is Cramer's rule, which we will study later in this unit.

Examples:

- The system of linear equations

  $x + 3y + 5z = 0$; $3x + 2y + 7z = 0$; $4x - 7y - 3z = 0$

  has the matrix form A X = B where,

the matrix of coefficients, $A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & 7 \\ 4 & -7 & -3 \end{pmatrix}$, the matrix of variables $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the matrix of constant terms, $B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

- The system of linear equations

  $2x_1 - 3x_2 = -1$; $\quad x_1 + x_2 = 2$; $\quad x_1 - x_2 = 0$,

  has the matrix form $AX = B$ where , the matrix of coefficients (A), the matrix of variables (X) and the matrix of constant terms (B) are:

  $A = \begin{pmatrix} 2 & -3 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\quad B = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$.

13.5 Cramer's Rule

If a system of n linear equations and n unknowns is such that the matrix of coefficients is invertible then this system has unique solution. One method of solving such system is called the method of Cramer's Rule, it is based on the use of determinants. To solve any system of linear equation using Cramer's rule we need to write the system in its matrix form.

13.5.1 Cramer's Rule: If $AX = B$ is a matrix form of the system of n linear equations in n variables, such that the determinant $|A| \neq 0$, then the system has a unique solution given by the formula, $x_i = \dfrac{|A_i|}{|A|}, 1 \le i \le n$, where A_i is the matrix obtained from A by replacing the elements in i^{th} column by the entries in the matrix of constants B.

Note that the system in n variables and n equations only can be solved using this rule. A system for which coefficient matrix is not a square matrix cannot be solved using Cramer's rule.

1. **Cramer's rule for a system of 2 equation in 2 variables is as below:**

   If a system is $a_1 x + b_1 y = c_1$ and $a_2 x + b_2 y = c_2$ then in its matrix form, $AX = B$, the matrix of coefficients is , $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, the matrix of variables is $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and the matrix of constant terms is, $B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

   In this case the solution by the Cramer's rule is , $x = \dfrac{|A_1|}{|A|}$, $y = \dfrac{|A_2|}{|A|}$.

   where $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $|A_1| = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $|A_2| = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

Example:

- Consider the problem with which we started our discussion: If the cost of 2 pens and 3 pencils is Rs. 26 and the cost of 3 pens and 2 pencils is Rs. 34, the what is the cost of one pen and one pencil respectively?

  Solution: To solve this problem assume that the cost of a single pen is Rs. x and the cost of a single pencil is Rs. y.

  The information that "the cost of 2 pens and 3 pencils is Rs. 26" is now converted to the equation $2x + 3y = 26$.

  And the information that "the cost of 3 pens and 2 pencils is Rs. 34" is now converted to the equation $3x + 2y = 34$.

  We want to find x and y satisfying above two equations. That means we want to find solution of the following system in two variables x and y.

  $2x + 3y = 26$

  $3x + 2y = 34$

  In matrix form the system is A X = B where the matrix of coefficients (A), the matrix of variables (X) and the matrix of constant terms (B) are:

  $$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 26 \\ 34 \end{pmatrix}$$

  In this case the solution by the Cramer's rule is , $x = \dfrac{|A_1|}{|A|}$, $y = \dfrac{|A_2|}{|A|}$ ....(1)

  Where

  $$|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 2 \times 2 - 3 \times 3 = 4 - 9 = -5.$$

  $$|A_1| = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 26 & 3 \\ 34 & 2 \end{vmatrix} = 26 \times 2 - 34 \times 3 = 52 - 102 = -50.$$

  and $$|A_2| = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 26 \\ 3 & 34 \end{vmatrix} = 2 \times 34 - 3 \times 26 = 68 - 78 = -10.$$

  $\therefore$ by the Cramer's rule $x = \dfrac{|A_1|}{|A|} = (-50) / (-5) = 10$ and $y = \dfrac{|A_2|}{|A|} = (-10) / (-5) = 2$

  $\therefore$ Solution is x = the cost of a pen is Rs. 10 and y = the cost of a pencil is Rs .2.

2. **Cramer's rule for a system of 3 equation in 3 variables is as below:**

   If a system is $a_1 x + b_1 y + c_1 z = d_1$

   $a_2 x + b_2 y + c_2 z = d_2$

   $a_3 x + b_3 y + c_3 z = d_3$

   then its matrix form is A X = B where, the matrix of coefficients (A), the matrix of variables (X) and the matrix of constant terms (B) are:

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

In this case the solution by the Cramer's rule is, $x = \dfrac{|A_1|}{|A|}$, $y = \dfrac{|A_2|}{|A|}$, $z = \dfrac{|A_3|}{|A|}$.

where $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $|A_1| = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $|A_2| = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

and $|A_3| = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$.

Examples:

- Solve the following system of 3 equation in 3 variables, by Cramer's rule

$$\begin{aligned} x + y + 2z &= 7 \\ -x - 2y + 3z &= 6 \\ 3x - 7y + 6z &= 1 \end{aligned}$$

Solution: The given system has matrix form, A X = B where, the matrix of coefficients (A), the matrix of variables (X) and the matrix of constant terms (B) are:

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 6 \end{pmatrix}; \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 1 \end{pmatrix}$$

In this case the solution by the Cramer's rule is, $x = \dfrac{|A_1|}{|A|}$, $y = \dfrac{|A_2|}{|A|}$, $z = \dfrac{|A_3|}{|A|}$.

where $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 6 \end{vmatrix}$

$= 1 \times [(-2 \times 6) - (-7 \times 3)] - 1 \times [(-1 \times 6) - (3 \times 3)] + 2 \times [(-1 \times -7) - (3 \times -2)]$

$= 1 \times [(-12) - (-21)] - 1 \times [(-6) - (9)] + 2 \times [7 + 6]$

$= 1 \times [9] - 1 \times [-15] + 2 \times [13]$

$= 9 + 15 + 26 = 50$

$|A_1| = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 7 & 1 & 2 \\ 6 & -2 & 3 \\ 1 & -7 & 6 \end{vmatrix}$

= 7ˆ [(–2 ˆ 6) – (–7 ˆ 3)] –1ˆ [(6 ˆ 6) –(1 ˆ 3)] +2ˆ [(6 ˆ –7) –(1 ˆ –2)]

=7ˆ [(–12) – (–21)] –1ˆ [(36) –(3)] +2ˆ [– 42 +2]

= 7ˆ [9] –1ˆ [33] +2ˆ [– 40]

= 63 – 33 – 80 = –50

$$|A_2| = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 2 \\ !1 & 6 & 3 \\ 3 & 1 & 6 \end{vmatrix}$$

= 1ˆ [(6 ˆ 6) – (1 ˆ 3)] –7ˆ [(–1 ˆ 6) –(3 ˆ 3)] +2ˆ [(–1 ˆ 1) –(3 ˆ 6)]

=1ˆ [(36) – (3)] –7ˆ [(–6) –(9)] +2ˆ [–1 –18]

= 1ˆ [33] –7ˆ [–1 5] +2ˆ [–19]

= 33+ 105 –38 = 100

$$\text{and } |A_3| = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 7 \\ !1 & !2 & 6 \\ 3 & !7 & 1 \end{vmatrix}$$

= 1ˆ [(–2 ˆ 1) – (–7 ˆ 6)] –1ˆ [(–1 ˆ 1) –(3 ˆ 6)] +7ˆ [(–1 ˆ –7) –(3 ˆ –2)]

=1ˆ [(–2) – (–42)] –1ˆ [(–1) –(18)] +7ˆ [7 +6]

= 1ˆ [40] –1ˆ [–1 9] +7ˆ [13]

= 40 + 19 + 91 = 150

P By the Cramer's rule $x = \dfrac{|A_1|}{|A|}$ = (-50) / 50 = –1

$$y = \frac{|A_2|}{|A|} = 100 / 50 = 2$$

$$\text{and } z = \frac{|A_3|}{|A|} = 150 / 50 = 3$$

P Solution of the system is x = –1, y = 2 , z = 3

P Solution set of the system = { –1, 2 , 3 }.

13.5 Summary

In this unit learners studied the following topics in details:

1. The concept of Linear equation, solution of linear equation.
2. System of linear equations and solution of system of linear equations.
3. Matrix representation of system of linear equations.
4. Cramer's Rule to find solution of a system containing upto 3 variables and 3 equations.

Unit 14 Polynomials

14.0 Objectives

By the end of this Unit, learners should be able to:

- Define polynomials in one variable
- Find the degree of polynomial
- Understand Equality of Polynomials
- Perform addition, subtraction, multiplication and division of polynomials
- Determine factors and roots of polynomial equation
- Quadratic equations and to find the roots of Quadratic equations

14.1Introduction

Polynomials are one of the most important concepts in algebra and throughout mathematics, science and engineering. They are used to form polynomial equations which are used to solve a wide range of problems which appear in basic chemistry, physics and even economics.. Polynomials are also used to approximate other functions. Polynomials are widely used because they are flexible and can take many forms of data. The generating functions are special type of polynomials, which are used as a powerful technique for solving complicated problems from combinatorics an important branch of applied mathematics. In this unit we will study about polynomials in one variable and the related terminology. We will also study about different operations on polynomials and about the roots of polynomials.

14.2 Polynomials

In mathematics a polynomial is an expression constructed from variables and constants using the operations of addition, subtraction, multiplication and raising to constant non negative powers. For example $5x^2 + 9x + 4$ is a polynomial but $6x^2 + 9x + 4x^{½}$ is not a polynomial because the later involves an exponent, which is not an integer. We will start the discussion with the formal definition of polynomial.

14.2.1 Polynomial: An algebraic expression of the form $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is called a "polynomial in variable x over real numbers", where n is either 0 or a positive integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers which are called coefficients.

Polynomials are generally denoted as function of x i.e. as f(x), p(x), q(x) etc.

14.2.2 Degree of Polynomial: The degree of the polynomial is the exponent in the term with the highest power.

If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is a polynomial such that $a_n \neq 0$, then p(x) is a polynomial of degree n.

A polynomial of degree 1 is also called a linear polynomial, a polynomial of degree 2 is called a quadratic polynomial, and a polynomial of degree 3 is called a cubic polynomial.

14.2.3 Constant polynomial: If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is a polynomial such that $a_0 \neq 0$ and $a_1 = 0$, $a_2 = 0$, … , $a_n = 0$, then $p(x) = a_0$ and it is called a constant polynomial. Clearly, a constant polynomial is of degree 0. As the highest power term is x^0.

14.2.4 Zero polynomial: If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is a polynomial such that all coefficients are 0 i.e. $a_0 = 0$, $a_1 = 0$, … , $a_n = 0$, then $p(x)$ is called a zero polynomial. For a zero polynomial the degree is not defined.

Examples:

- $f(x) = 3x - 2$, is a polynomial of degree 1, in this polynomial coefficient of x is 3 and the coefficient of x^0 or the constant term is -2 .
- $g(x) = 9x^2 - \frac{2}{x}$, it is not a polynomial. (Why?)
- $p(x) = -6x^2 + 9x + \frac{1}{2}$, is a polynomial of degree 2 i.e. a quadratic polynomial. In this polynomial coefficient of x^2 is -6, coefficient of x is 9 and the constant term is $\frac{1}{2}$.
- $q(x) = 4x^3 + \frac{1}{4}x - 4$, is a polynomial of degree 3 i.e. a cubic polynomial. In this polynomial coefficient of x^3 is 4, coefficient of x^2 is 0 , coefficient of x is $\frac{1}{4}$ and the constant term is -4 .
- $r(x) = 5x^7$ is a polynomial of degree 7.. In this polynomial coefficient of x^7 is 5 and all other coefficients are zero.
- $g(x) = 8$, is a constant polynomial as it is of degree 0.
- $z(x) = 0 = 0x + 0 = 0x^3 + 0x^2 + 0x + 0$, is a zero polynomial.

14.2.5 Equal polynomials: Two polynomials are said two be equal if the coefficients of like powers of x are equal. i.e. the polynomials $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$ are equal if $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, …, $a_n = b_n$.

14.3 Operations on polynomials

14.3.1 Multiplication of a polynomial by a real number: If k is a real number and $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is any polynomial then $k.p(x)$ is the polynomial in which the coefficients are k^{th} multiples of the coefficients of $p(x)$. i.e.

$$k\,p(x) = ka_0 + ka_1x + ka_2x^2 + \ldots + ka_nx^n .$$

Examples:

- $f(x) = 3x + 2$, $\Rightarrow -2f(x) = -2(3x + 2) = -6x - 4$.
- $p(x) = 6x^2 + 9x + 2$, $\Rightarrow (1/3)p(x) = (1/3)(6x^2 + 9x + 2) = 2x^2 + 3x + (2/3)$.

14.3.2 Addition of two polynomials: The addition of two polynomials is obtained by adding the terms of like powers.

If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$ are two polynomials and m is greater than n then,

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n + b_{n+1}x^{n+1} + \ldots + b_mx^m \quad b_mx^m$$

If $p(x)$ is a polynomial of degree n and $q(x)$ is a polynomial of degree m, then the degree of the addition polynomial $p(x) + q(x)$ is equal to the greater of the two integers m and n.

Examples:

- $f(x) = 3x - 2$ and $g(x) = 8x^2 + 9x + 3$ are two polynomials, then their addition is

  $f(x) + g(x) = (3x - 2) + (8x^2 + 9x + 3) = (0+8)x^2 + (3+9)x + (-2+3)$

  $= 8x^2 + 12x + 1.$

- $p(x) = 6x^3 + 9x^2 + ½$ and $q(x) = 4x^3 + ¼ x - 4$, are two polynomials, then their addition is $p(x) + q(x) = (6x^3 + 9x^2 + ½) + (4x^3 + ¼ x - 4)$

  $= (6 + 4)x^3 + (9 + 0)x^2 + (0+¼)x + (½ - 4)$

  $= 10x^3 + 9x^2 + ¼x - (7/2).$

14.3.3 Difference of two polynomials: The difference of two polynomials is obtained by subtracting the terms of like powers.

If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_n x^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_m x^m$ are two polynomials and m is greater than n then,

$p(x) - q(x) = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \ldots + (a_n - b_n)x^n - b_{n+1}x^{n+1} - \ldots - b_m x^m \quad b_m x^m.$

If $p(x)$ is a polynomial of degree n and $q(x)$ is a polynomial of degree m, then the degree of the difference polynomial $p(x) - q(x)$ is equal to the greater of the two integers m and n.

Examples:

- $f(x) = 3x - 2$ and $g(x) = 8x^2 + 9x + 3$ are two polynomials, then their difference is $f(x) - g(x) = (0x^2 + 3x - 2) - (8x^2 + 9x + 3)$

  $= (0 - 8)x^2 + (3 - 9)x + (-2 - 3) = -8x^2 - 6x - 5.$

- $p(x) = 6x^3 + 9x^2 + ½$ and $q(x) = 4x^3 + ¼ x - 4$, are two polynomials, then their difference is $p(x) - q(x) = (6x^3 + 9x^2 + ½) - (4x^3 + ¼ x - 4)$

  $= (6 - 4)x^3 + (9 - 0)x^2 + (0 - ¼)x + (½ - (-4))$

  $= 2x^3 + 9x^2 - ¼x + (9/2).$

14.3.4 Multiplication of two polynomials: If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_n x^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_m x^m$ are two polynomials having degrees n and m respectively then, their multiplication is the polynomial, obtained by term by term multiplication.

$\Rightarrow$ $p(x) \times q(x) = c_0 + c_1x + c_2x^2 + \ldots + c_k x^k$, where $c_k = \sum_{i=0}^{k} a_i \times b_{k-i}$.

i.e. $c_0 = (a_0 \times b_0)$, $c_1 = (a_1 \times b_0 + a_0 \times b_1)x$ and $c_2 = (a_2 \times b_0 + a_1 \times b_1 + a_0 \times b_2)$ etc.

If $p(x)$ is a polynomial of degree n and $q(x)$ is a polynomial of degree m, then the degree of the multiplication polynomial $p(x) \times q(x)$ is equal to $m \times n$.

Examples:

- $f(x) = 3x - 2$ and $g(x) = 8x^2 + 9x + 3$ are two polynomials, then their product or multiplication is $f(x) \times g(x) = (3x - 2) \times (8x^2 + 9x + 3)$

$= 3x \times (8x^2 + 9x + 3) - 2 \times (8x^2 + 9x + 3)$

$= 24x^3 + 27x^2 + 9x - 16x^2 - 18x - 6$

$= 24x^3 + 11x^2 - 9x - 6$

- $p(x) = 6x^3 + 9x^2 + ½$ and $q(x) = 4x^3 + ¼x - 4$, are two polynomials , then their multiplication is $p(x) \times q(x) = (6x^3 + 9x^2 + ½) \times (4x^3 + ¼x - 4)$

  $= 6x^3 \times (4x^3 + ¼x - 4) + 9x^2 \times (4x^3 + ¼x - 4) + ½ \times (4x^3 + ¼x - 4)$

  $= (24x^6 + (6/4)x^4 - 24x^3) + (36x^5 + (9/4)x^3 - 36x^2) + (2x^3 + (1/8)x - 2)$

  $= 24x^6 + 36x^5 + (6/4)x^4 - (79/4)x^3 - 36x^2 + (1/8)x - 2.$

14.3.5 Division of two polynomials: Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$ are two polynomials having degrees n and m respectively with m u n. If there exists a polynomial t(x) such that, $p(x) = t(x) \times q(x)$,then we say that "p(x) is divisible by q(x)" or "q(x) is a factor of p(x)." The division or the quotient t(x) is a polynomial of degree m – m and it is obtained by step by step procedure explained below.

Examples:

- $f(x) = 3x - 2$ and $g(x) = 6x^2 + 9x + 3$ are two polynomials , then their division $g(x) / f(x) = (6x^2 + 9x + 3) / (3x - 2)$ is obtained as follows.

  First the highest degree term in g (x) is divided by the highest degree term in f(x), and by that division every term in g(x) is multiplied. This product is then subtracted from g (x). Then highest degree term in the subtraction obtained is divided by the highest degree term in f(x) and the above procedure is repeated again and again till it is possible.

	$2x + 4$
$3x - 2$	$6x^2 + 8x - 8$ $6x^2 - 4x$ $-$ $+$
	$12x - 8$ $12x - 8$ $-$ $+$
	$0x + 0$

$$\Rightarrow \frac{6x^2 + 8x - 8}{3x - 2} = 2x + 4$$

- The division $(x^3 - 6x^2 + 11x - 6) / (x - 2)$ is obtained as follows.

	$x^2 - 4x + 3$
$x - 2$	$x^3 - 6x^2 + 11x - 6$ $x^3 - 2x^2$ $-$ $+$
	$-4x^2 + 11x - 6$ $-4x^2 + 8x$ $+$ $-$
	$3x - 6$ $3x - 6$ $-$ $+$
	0

$$\Rightarrow \frac{x^3 - 6x^2 + 11x - 6}{x - 2} = x^2 - 4x + 3$$

- $p(x) = 24x^6 + 36x^5 + (6/4)\,x^4 - (79/4)\,x^3 - 36\,x^2 + (1/8)\,x - 2$ and $q(x) = 4x^3 + ¼\,x - 4$, are two polynomials , then their division p(x) / q(x) is obtained as follows.

	$6x^3 + 9x^2 + ½$
$4x^3 + ¼\,x - 4$	$24\,x^6 + 36\,x^5 + (6/4)x^4 - (79/4)x^3 - 36\,x^2 + (1/8)x - 2$
	$24\,x^6 + 0 + (6/4)x^4 - 24\,x^3$
	$- \quad - \quad - \quad +$
	$36\,x^5 + 0 + (17/4)x^3 - 36\,x^2$
	$36\,x^5 + 0 + (9/4)x^3 - 36\,x^2$
	$- \quad - \quad +$
	$2x^3 + 0 + (1/8)x - 2$
	$2x^3 + 0 + (1/8)x - 2$
	$- \quad - \quad +$
	0

P $p(x) / q(x) = = 6\,x^3 + 9x^2 + ½$.

14.3.6 Synthetic division: If for polynomials p(x) and q(x), there exists a polynomial t(x) such that, p(x) = t(x) ˆ q(x) ,then we say that "q(x) is a factor of p(x)." If this factor q(x) is linear i.e. if the polynomial q(x) is of degree one and has the form q(x) = x – a, then the polynomial t(x) can be found using synthetic division which is easier that usual polynomial division.

This method is explained with examples below:

Examples:

- Find $(x^3 - 6x^2 + 11x - 6)$ / (x –2) using the synthetic division.

  <u>Solution:</u> Here $p(x) = x^3 - 6x^2 + 11x - 6$ and $q(x) = x - 2$. We will write a table in which the first row of second column contains the coefficients 1, –6, 11 and –6 of p(x) in the decreasing order of powers of x. We also write +2 in the first column of the table as the factor of p(x) is x – 2. Write the first coefficient 1 in third row of the table. Multiply the coefficients –6, 11 and –6 by 2 and write each product below each of the coefficients. In the next row perform the addition of the numbers previously written. This addition gives the coefficients of the required quotient polynomial. The remainder is written in a rectangle. If the remainder is not zero then q(x) is not a factor of p(x).

	1	–6	11	–6
2	ƒ	2	– 8	6
	1	–4	3	$\boxed{0}$

From the third row the coefficients of the required quotient polynomial are 1, –4 and 3. So the quotient is the polynomial $x^2 - 4\,x + 3$. And the factorization of p(x) is, $p(x) = x^3 - 6x^2 + 11x - 6 = (x - 2)$ ˆ $(x^2 - 4\,x + 3)$.

- Find $(4x^3 - 20x^2 + 17x - 4)$ / (x – 4) using the synthetic division.

  <u>Solution:</u> Here $p(x) = 4x^3 - 20x^2 + 17x - 4$ and $q(x) = x - 4$. We will write a table in which the first row of second column contains the coefficients 4, –20, 17 and – 4 of p(x) in the decreasing order of powers of x. We also write +4 in the first column of the table as (x – 4) is the factor of p(x). Write the first coefficient 4 in third row of the table. Multiply the coefficients –20, 17 and – 4 of p(x) by 4 and write each product below each of these coefficients. In the next row perform the addition of the numbers previously written. This addition gives the coefficients of the required quotient polynomial. The last number is remainder, which is written in the rectangle.

	4	– 20	17	– 4
4	ƒ	16	– 16	4
	4	– 4	1	$\boxed{0}$

From the third row the coefficients of the required quotient polynomial are 4, – 4 and 1. So the quotient is the polynomial $4x^2 - 4x + 1$. And the factorization of p(x) is, $p(x) = 4x^3 - 20x^2 + 17x - 4 = (x - 4)$ ˆ $(4x^2 - 4x + 1)$.

14.4 Roots of Polynomial equation

14.4.1 Polynomial equation: If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is a polynomial, then the equation p(x) = 0 i.e. $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$ is called a polynomial equation.

14.4.2 Roots of Polynomial equation: If $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$ is a polynomial equation then the values of the variable x satisfying this equation are called roots of this equation.

For a polynomial equation $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$, x = %ois a root, if $a_0 + a_1$‰+ a_2 ‰^{2} + …+ a_n ‰^{n} = 0. If polynomial is of degree n, then that polynomial equation can have maximum n distinct roots.

Examples:

‹ For the polynomial equation $x^3 - 6x^2 + 11x - 6 = 0$, its root is x = 2, because 2 satisfies this equation, i.e. $2^3 - 6$ ˆ $2^2 + 11$ˆ$2 - 6 = 8 - 24 + 22 - 6 = 30 - 30 = 0$.

‹ For the polynomial equation $4x^3 - 20x^2 + 17x - 4 = 0$, its root is x = 4 because 4 satisfies this equation, i.e. (4 ˆ 4^3)– (20 ˆ 4^2)+ (17 ˆ 4) – 4 = 256 – 320 + 68 – 4 = 324 – 324 = 0.

All roots of a polynomial equation p(x) = 0 can be found by different methods, one of the methods is of synthetic division. We studied this method earlier, but to find the roots by this method we need one factor of the polynomial p(x). If (x –U) is a factor of polynomial p(x) then x = U is a root of polynomial equation p(x)= 0, then the remaining roots can be found by synthetic division method.

To find a factor of the given polynomial tests of divisibility are useful. If these tests are not applicable then an integer root of p(x)= 0 can be from the divisors of the constant term in p(x), this we can find by trial and error method.

14.5 Tests of divisibility

1. **Test for (x – 1):** If the sum of all the coefficients of a polynomial p(x) in x is zero, then (x – 1) is a factor of that polynomial. And x = 1 is a root of the polynomial equation p(x) = 0.

If $x = 1$ is a root of the polynomial equation then remaining roots can be found by synthetic division method.

Examples:

‹ Find a factor of $x^3 - 6x^2 + 9x - 4 = 0$

Let $p(x) = x^3 - 6x^2 + 9x - 4$. Sum of all coefficients in $p(x) = 1 - 6 + 9 - 4 = 0$

P $(x - 1)$ is a factor of $p(x)$.

‹ Find a factor of $x^3 + x^2 - x - 1 = 0$.

Here the sum of all coefficients $= 1 + 1 - 1 - 1 = 0$.

P $(x - 1)$ is a factor of $x^3 - x^2 + x - 1$.

2. **Test for $(x + 1)$:** If for a polynomial $p(x)$, the sum of all the coefficients of even powers of x is equal to the sum of all the coefficients of odd powers of x then $(x + 1)$ is a factor of the polynomial $p(x)$. And $x = -1$ is a root of the polynomial equation $p(x) = 0$.

If $x = -1$ is a root of the polynomial equation then remaining roots can be found by synthetic division method.

Examples:

‹ Find a factor of $x^3 + 6x^2 + 9x + 4 = 0$

The sum of all the coefficients of even powers of $x = 6 + 4 = 10$ and the sum of all the coefficients of odd powers of $x = 1 + 9 = 10$

P $(x + 1)$ is a factor of $p(x)$.

‹ Find a factor of $x^3 + x^2 - x - 1 = 0$.

The sum of all the coefficients of even powers of $x = 1 + (-1) = 0$ and the sum of all the coefficients of odd powers of $x = 1 + (-1) = 0$

P $(x + 1)$ is a factor of $p(x)$.

‹ Find all roots of $x^3 - 6x^2 + 9x - 4 = 0$

__Solution:__ Let $p(x) = x^3 - 6x^2 + 9x - 4$.

Sum of all coefficients in $p(x) = 1 - 6 + 9 - 4 = 0$

P By test for $(x - 1)$, it is a factor of $p(x)$. Now to obtain the other factors use synthetic division. $(x - 1)$ is a factor so $x = 1$ is a root of $p(x) = 0$.

1	1	-6	9	-4
	ƒ	1	-5	4
	1	! 5	4	0

P the other factor of $p(x) = x^2 - 5x + 4$.

So $p(x) = (x - 1)$ ˆ $(x^2 - 5x + 4)$

Let $x^2 - 5x + 4 = q(x)$.

Sum of all coefficients in $q(x) = 1 - 5 + 4 = 0$. **P** By test for $(x - 1)$, it is a factor of $q(x)$. Now to obtain the other factor use synthetic division again,

1	1	-5	4
	↓	1	-4
	1	−4	0

$\Rightarrow$ $q(x) = x^2 - 5x + 4 = (x - 1) \times (x - 4)$

Hence $p(x) = (x - 1) \times (x - 1) \times (x - 4)$

$\Rightarrow$ The equation $p(x) = 0$ has the roots $x = 1$, $x = 1$ and $x = 4$.

- Find all roots of $x^3 + x^2 - x - 1 = 0$

Solution: Let $p(x) = x^3 + x^2 - x - 1 = 0$.

Sum of all coefficients in $p(x) = 1 + 1 - 1 - 1 = 0$ $\Rightarrow$ By test for $(x - 1)$, it is a factor of $p(x)$. Now to obtain the other factors use synthetic division. $(x - 1)$ is a factor so $x = 1$ is a root of $p(x) = 0$.

1	1	1	-1	-1
	↓	1	2	1
	1	2	1	0

$\Rightarrow$ the other factor of $p(x) = x^2 + 2x + 1$.

So $p(x) = (x - 1) \times (x^2 + 2x + 1)$

Let $q(x) = (x^2 + 2x + 1)$.

Sum of all coefficients in $q(x) = 1 + 2 + 1 = 4 \neq 0$. $\Rightarrow$ The test for $(x - 1)$ fails.

The sum of all the coefficients of even powers of $x = 1 + 1 = 2$

and the sum of all the coefficients of odd powers of $x = 2$

$\Rightarrow$ $(x + 1)$ is a factor of $q(x)$ i.e. $x = -1$ is a root of $q(x) = 0$. Now to obtain the other factor use synthetic division again,

−1	1	2	1
	↓	-1	-1
	1	1	0

$\Rightarrow$ $q(x) = x^2 + 2x + 1 = (x + 1) \times (x + 1)$

Hence $p(x) = (x - 1) \times (x + 1) \times (x + 1)$

$\Rightarrow$ All the roots of $x^3 + x^2 - x - 1 = 0$, are $x = 1$, $x = -1$ and $x = -1$.

14.6 Quadratic equations and their roots

14.6.1 Quadratic equation: If $p(x) = 0$ is a polynomial equation, where $p(x)$ is a polynomial of degree 2, then such equation is called a quadratic equation.

In general an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$, is a quadratic equation. It has two roots. These two roots of a quadratic equation can be found using above synthetic division method or by using following formulae.

If $ax^2+bx+c=0$, where a, b, c are real numbers and $a \neq 0$, is a quadratic equation, and α and β are its roots, then $\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

<u>Properties of roots of quadratic equation $ax^2+bx+c=0$:</u>

(i) If $\sqrt{b^2-4ac}=0$, then the roots are real and equal.

(ii) If $\sqrt{b^2-4ac}$ is positive and a perfect square, then the roots are rational numbers and unequal.

(iii) If $\sqrt{b^2-4ac}$ is positive and not a perfect square, then the roots are irrational numbers and unequal.

(iv) If $\sqrt{b^2-4ac}$ is negative, then the roots are complex numbers and not real numbers.

Examples:

- Find the roots of the quadratic equation $x^2-6x+9=0$,

  <u>Solution</u>: In this equation the coefficients are $a=1$, $b=-6$ and $c=9$.

  Let α and β be its roots,

  then $\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

  i.e. $\alpha = \frac{-(-6) \pm \sqrt{6^2-4\times1\times9}}{2\times1} = \frac{6\pm\sqrt{36-36}}{2} = \frac{6\pm0}{2} = \frac{6}{2} = 3$

  and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$

  i.e. $\beta = \frac{-(-6)-\sqrt{6^2-4\times1\times9}}{2\times1} = \frac{6-\sqrt{36-36}}{2} = \frac{6-0}{2} = \frac{6}{2} = 3$.

  $\therefore$ The roots of equation $x^2-6x+9=0$, are $x=3$, $x=3$ i.e. the roots are multiple roots.

- Find the roots of the quadratic equation $x^2-7x+10=0$,

  <u>Solution</u>: In this equation the coefficients are $a=1$, $b=-7$ and $c=10$.

  Let α and β be its roots, then

  $$\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \text{ so,}$$

  $$\alpha = \frac{-(-7) \pm \sqrt{7^2-4\times1\times10}}{2\times1} = \frac{7\pm\sqrt{49-40}}{2} = \frac{7\pm\sqrt{9}}{2} = \frac{7\pm3}{2} = \frac{10}{2} = 5$$

  and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$

  i.e. $\beta = \frac{-(-7)-\sqrt{7^2-4\times1\times10}}{2\times1} = \frac{7-\sqrt{49-40}}{2} = \frac{7-\sqrt{9}}{2} = \frac{7-3}{2} = \frac{4}{2} = 2$.

P The roots of equation $x^2 - 7x + 10$, are $x = 5$, $x = 2$.

‹ Find the roots of $3x^2 - x - 10 = 0$,

Solution: In this equation the coefficients are $a = 3$, $b = -1$ and $c = -10$.

Let U and V be its roots, then

$$U = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

i.e. $U = \dfrac{-(-1) + \sqrt{1^2 - 4 \times 3 \times (-10)}}{2 \times 3} = \dfrac{1 + \sqrt{121}}{6} = \dfrac{1 + 11}{6} = \dfrac{12}{6} = 2$

and $V = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

i.e.

$V = \dfrac{-(-1) - \sqrt{1^2 - 4 \times 3 \times (-10)}}{2 \times 3} = \dfrac{1 - \sqrt{121}}{6} = \dfrac{1 - 11}{6} = \dfrac{-10}{6} = \dfrac{-5}{6}$.

P All the roots of $3x^2 - x - 10 = 0$, are $x = 2$, $x = -(5/6)$.

14.6 Summary

In this unit learners studied the following topics in details:

1. The concept of polynomial in variable x over real numbers, its degree.
2. Different polynomials such as zero polynomial and constant polynomials.
3. Different operations on polynomials such as addition, difference, and multiplication, division of polynomials and synthetic division.
4. Roots of Polynomial equation and tests of divisibility.
5. Quadratic equations and the formula to find its roots.

UNIT 15 Introduction to Graph Theory

15.0 Objectives

By the end of this Unit, learners should be able to:

- Understand the concept of a graph in discrete mathematics.
- Understand common terminology in Graph theory.
- Use different representation of graphs.
- Explain different types of graphs.
- Describe Eulerian and Hamiltonian graphs.
- Describe Planar graphs and colouring problem.
- Define and use trees.

15.1Introduction

Graph theory is one of the widely used branches of mathematics. Although the first paper in graph theory was published in 1736 A.D. there has been widespread and intense interest in this subject since 1920s. One of the reasons for the recent interest in graph theory is its applicability in many diverse fields including computer science, chemistry, geography, electronics, electrical engineering etc. It is because one can represent a highway map of some region using a graph or even an electronics network can also be represented by a graph.

It is known that The first problem solved in graph theory was the "Seven bridges problem of Königsberg." The paper about solution to this problem was presented by well-known mathematician Leonhard Euler in 1736 A.D. So he is also referred as the" father of graph theory."

15.2 Graph

In the city of Königsberg (now called Kaliningrad in Russia), two islands in the Pregel river were connected to each other and the river banks by seven bridges. See a crude map drawn of this geographical topology in the figure1.

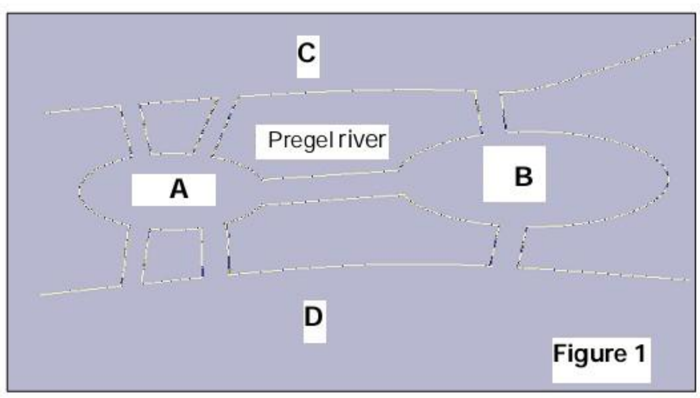


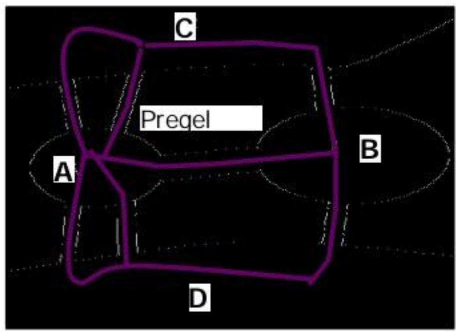
Figure 1

The puzzle related with this geographical topology was that: "Starting from any one of the four land portions A, B, C and D is it possible to walk over each bridge exactly once and return to the starting point?" Leonard Euler in his paper explained that it is not possible to do this. He represented the map of this geographical topology in a diagrammatic way, which is now termed as a graph.

The graph representation of above map of Königsberg is as in Figure 2 on the next page. In this diagram or a graph the land portions A, B, C, D are denoted by dots or small circles called vertices and the bridges were denoted by arcs named e_1, e_2, e_3, e_4, e_5, e_6 and e_7 called as the edges in the graph.

Observe that how in the graph the land portion are represented by vertices i.e. the points and the bridges are represented by edges or arcs.

We will now study the formal definition of a graph.



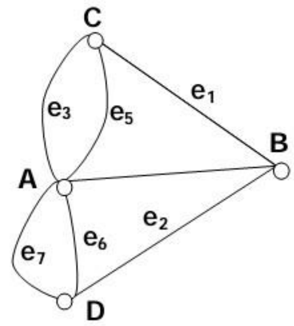


Figure 2: Representation of Seven bridges problem

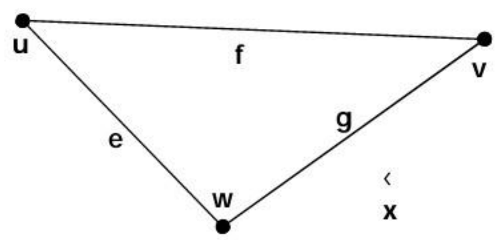
15.2.1 Graph: A graph G consists of a non empty set V of vertices and a set E of edges such that each edge is associated with a pair of vertices. We denote such graph as G = (V, E).

Every graph can be represented diagrammatically in which vertices are represented by points or dots and the edged are represented by lines or arcs joining the corresponding points i.e. vertices.

Examples:

- Let G = (V,E)be a graph where the set of vertices, V = { u, v, w, x } and the set of edges, E = { e, f, g }. Where the edge e represents the pair (u, w) or (w, u), f represents the pair (u, v) or (v, u), g represents the pair (v, w) or (w, v). Note that the pairs in set E are unordered.

  This graph G can also be represented by the diagram as shown here.



- The graph representing seven bridges problem in Figure3 above is a graph G=(V, E); where the set of vertices, V = {A, B, C, D} and the set of edges, E = {(A, B), (A, C), (A, C),(A, D), (A, D), (B, C),(B, D)}.

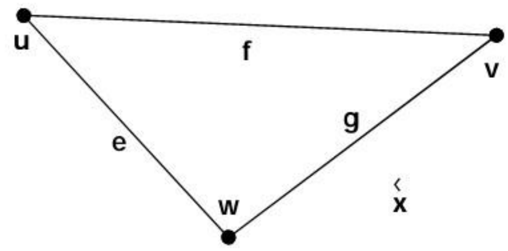
  We can write this edge set also as $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ where

  e_1 = (B, C) or (C, B) , e_2 = (B, D) or (D, B), e_3 = (A, C) or (C, A), e_4 = (A, B) or (B, A), e_5 = (A, C) or (C, A), e_6 = (A, D) or (D, A) and e_7= (A, D) or (D, A).

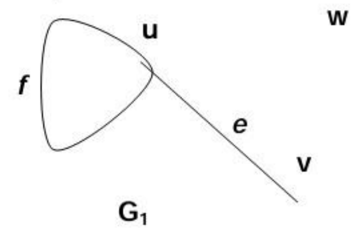
15.3 Commonly used terminology in graph theory

1. **Incidence and adjacency:** If in a graph an edge e is associated with the unordered pair of vertices *u* and *v* then, the edge e is said to be incident on *u* and *v*, and the vertices *u* and *v* are also said to be adjacent vertices.

Examples:



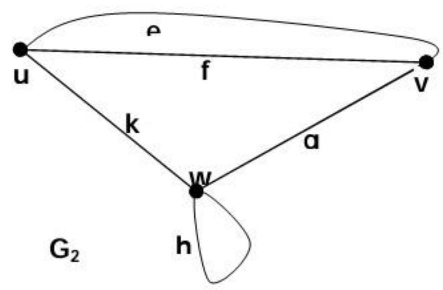
- In the graph given here the edge f is incident on the vertices u and v, so u and v are adjacent vertices, the edge e is incident on the vertices u and w so u and w are adjacent vertices and the edge g is incident on the vertices v and w so v and w are adjacent vertices. So all vertices are adjacent to each other except the vertex X.



G$_1$

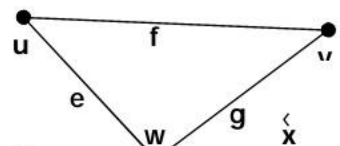
- In the graph G_1 the edge e is incident on the vertices u and v, so u and v are adjacent vertices and the edge f is incident on the vertex u only. Such edge is called a loop. But in this graph u and w are not adjacent vertices as there is no edge from vertex u to the vertex w. In fact w is not adjacent to any other vertices.

2. **Degree of a vertex:** The degree (or valency) of a vertex is the number of edges incident at that vertex. Degree of a vertex v is denoted as d(v). A loop contributes 2 to the degree of that vertex on which it is incident.



Examples:

- In the graph G_2 on the right side, the degrees of vertices are d(u) = 3 , d(v) = 3 and d(w) = 4



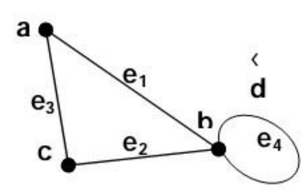
G$_3$

- In the graph G_3 on the right side, the degrees of vertices are: d(u) = 2 , d(v) = 2 , d(w) = 2 and d(x) = 0.

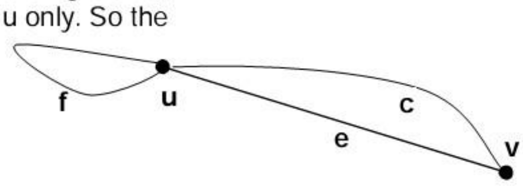
3. **Self Loop and parallel edges:**

If in a graph an edge is incident on a single vertex then it is called as a loop. The edges, which are incident on the same pair of vertices, are called parallel edges.

Examples:

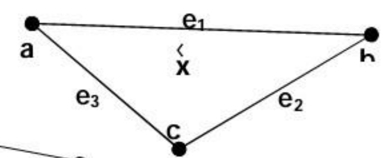


- In the graph drawn in figure on the right side, the edge e_4 is incident on the vertex c only. So the edge e_4 is a loop.
- In the graph drawn in the figure on the right side, the edge f is incident on the vertex u only. So the edge f is a loop. In this graph the edges c and e are incident on the same pair of vertices i.e. u and v so these are parallel edges.

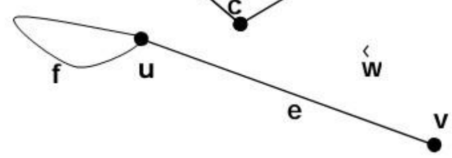


4. **Isolated vertex**: A vertex, which is not incident on any edge i.e., a vertex, which is not adjacent to any other vertex, is called an isolated vertex in a graph.

Examples:



- In the graph drawn in figure on the right side, the vertex x is an isolated vertex.

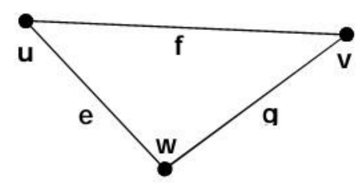


- In the graph drawn in figure on the right side, the vertex w is an isolated vertex.

5. **Path in a graph:** Let u_0 and u_n are vertices in a graph G, a path P from vertex u_0 to vertex u_n is an alternating sequence of edges and vertices beginning with vertex u_0 and ending with vertex u_n i.e. P: $u_0\ e_1\ u_1\ e_2\ u_2\ e_3\ u_3\ e_4 \ldots\ e_n\ u_n$ such that every edge in this sequence is incident on preceding and succeeding vertices.

Note:
1. A path is of length n if it contains n edges.
2. If in a path no vertices are repeated then it is called a simple path.

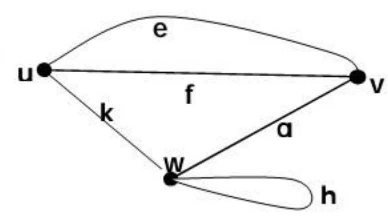
Examples:



- Two of the paths in the graph drawn in figure on the right side are:

  P: u e w g v, it is u to v path of length 2

  Q: u f v g w e u, it is path of length 3.
  Both are simple paths



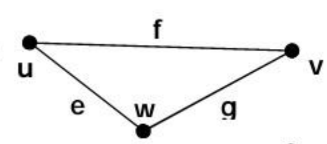
- Following are some paths in the graph drawn in figure on the right side,

  P: u k w h w g v, it is u to v path of length 3

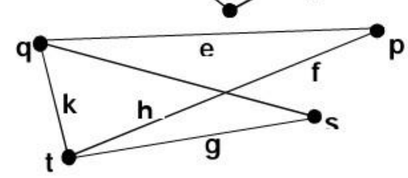
  Q: u f v e u k w g v, it is path of length 4.
  Both are not simple paths.

6. **Cycle in a graph:** A cycle (or circuit) in a graph is a path of non-zero length from a vertex u to u with no repeated edges.

Examples:



- A cycle in the graph drawn in figure on the right side is, C: u f v g w e u

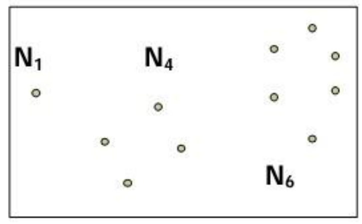


- Some cycles in the graph drawn in figure on the right side are,
  C_1: p e q k t f p; C_2: q k t g s h q

  C_3: p e q h s g t f p

15.4 Some important types of graphs

15.4.1 Null graph: The graph containing no edges is called a null graph. A null graph containing n vertices is denoted as N_n.



Examples:

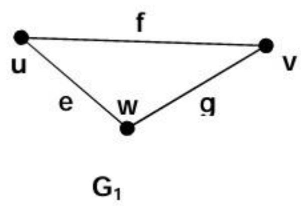
- The null graphs N_1, N_4, and N_6 are null graphs containing 1, 4 and 6 vertices respectively; these null graphs can be drawn as in following figure.

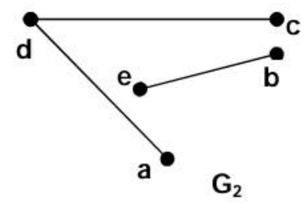
15.4.2 Simple graph: A graph, which contains neither loops nor parallel edges, is called a simple graph.

Examples:

- The graphs G_1 and G_2 figures here are simple graphs but the graph representing seven bridges problem mentioned earlier is not a simple graph.

  Note that a null graph is a simple graph.

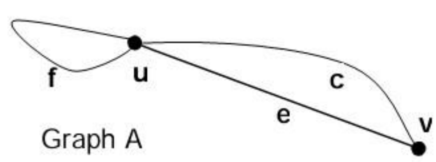




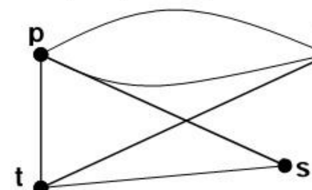
15.4.3 Multigraph or multiple graph: A multigraph is a graph containing multiple or parallel edges and /or loops between the same vertices.

Examples:

- The following graphs, the graph A and the graph B below are multiple graphs. In the graph A the edge f is a loop and the edges e and g are parallel edges. And in the graph B there are 3 parallel edges incident on the pair p and q of vertices.



Graph A

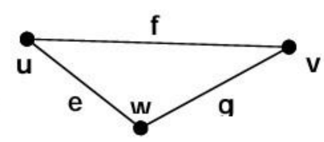


Graph B

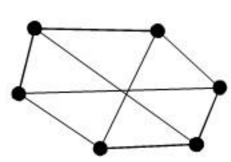
15.4.4 Regular graph: A regular graph is a graph, in which all of the vertices have the same degree. i.e. in a regular graph the number of edges incident at every vertex is the same. If each vertex has degree r then that regular graph is called as r-regular graph.

Examples:

- In the graph drawn in figure on the right side d(u) = 2, d(v) = 2, d(w) = 2 , So it is a regular graph of degree two or a 2-regular graph.



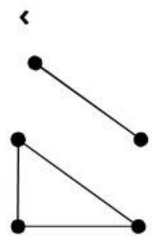
- In the graph drawn in figure on the right side is a regular graph of degree 3 or a 3- regular graph, as each vertex of this graph has degree three.



- Null graph is a regular graph of degree 0.

15.4.5 Complete graph: The **complete graph** on p vertices denoted by K_p, is a simple graph with p vertices in which there is an edge between every pair of distinct vertices.

Examples:



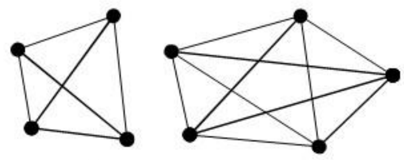
K_1

K_2

K_3

- The complete graph K_1 containing 1 vertex is just a vertex without any edge.
- K_2 is a complete graph on 2 vertices and K_3 is a complete graph on3 vertices,

K_4 is a complete graph on 4 vertices and K_5 is a complete graph on 5 vertices. These are as shown in the figure on right side.



K_4 K_5

Note that all complete graphs are regular graphs. In fact a complete graphs K_p is a (p–1) regular graph because every vertex in this graph is adjacent to remaining p – 1 vertices.

15.5 Representation of Graph using Matrix

15.5.1 Adjacency matrix: Let G be a graph with n vertices which are ordered as $v_1, v_2, v_3, v_4, \ldots, v_n$. Then the Adjacency matrix of the graph G is a matrix $A = [a_{ij}]$ of order $n \times n$, defined by

a_{ij} = the number of edges incident on the adjacent vertices v_i and v_j; and

$a_{ij} = 0$, when vertex v_i and the vertex v_j are not adjacent.

Note that if the graph contains loops incident on a vertex v_i then

a_{ii} = twice the number of loops incident on the vertex v_i.

Examples:

- The adjacency matrix of the graph in the figure 1 is,

$$A = \begin{array}{c} u \\ v \\ w \end{array} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

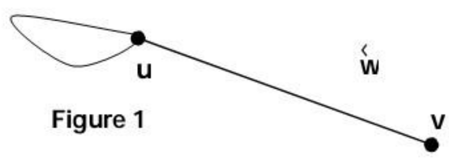


Figure 1

This is because there are 2 edges i.e. a loop from vertex u to u and 1 edge from vertex u to vertex v and vice versa.

- The adjacency matrix of the graph in the figure 2 on the right side is,

$$A = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

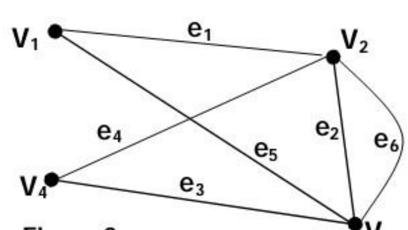


Figure 2

15.5.2 Incidence matrix: Let G be a graph with n vertices which are ordered as v_1, v_2, v_3, v_4,..., v_n and m edges which are ordered as e_1, e_2, e_3, e_4,..., e_m.

Then the Incidence matrix of the graph G is a matrix $A = [a_{ij}]$ of order $n \times m$, defined by

$a_{ij} = 1$, if vertex v_i is incident on the edge e_j and

$a_{ij} = 0$, otherwise.

Examples:

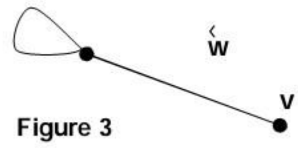


Figure 3

- The incidence matrix of the graph in the figure 3 is,

$$A = \begin{array}{c} u \\ v \\ w \end{array} \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

In this matrix the first column corresponds to the edge (u, u) and there are 2 edges i.e. a loop from vertex u to u. And the second column corresponds to the edge (u, v), so we write 1 in the rows corresponding to vertex u and vertex v.

- The incidence matrix of the graph in the figure 4 is,

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

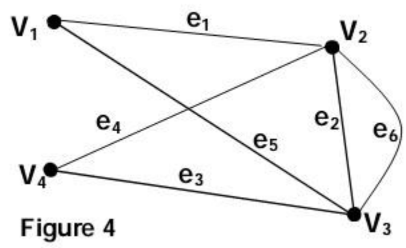


Figure 4

15.6 Eulerian and Hamiltonian graphs

15.6.1 Connected graph: A graph G is connected if for every pair of vertices u and v in G, there is a path from vertex u to vertex v. Other wise the graph G is disconnected.

Examples:

- The graph shown in the figure 5 is a connected graph because for every pair of vertices in this graph, there is a path from first vertex to the other vertex.

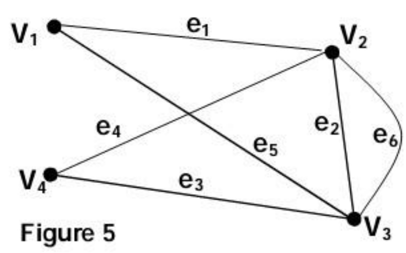


Figure 5

- The graph shown in the figure 6 is a disconnected graph because for the pair of vertices u and w in this graph, there is no path from vertex u to the other vertex w .In fact there is no path from the vertex w to any other vertex.

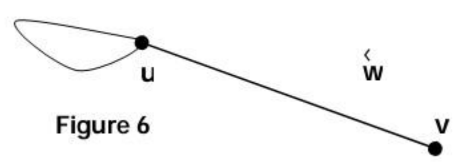


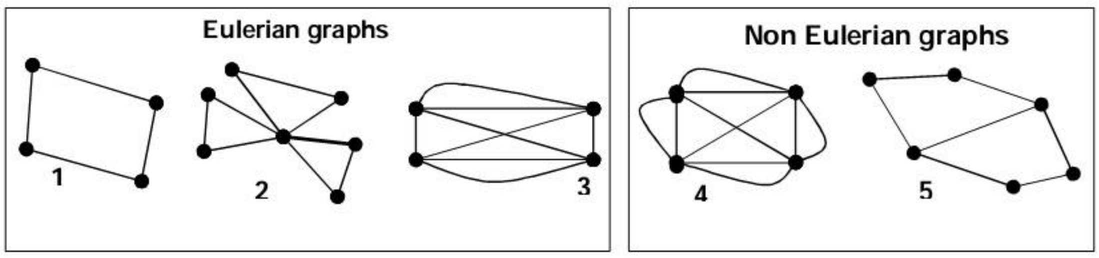
Figure 6

15.6.2 Eulerian graph: A graph is called as Eulerian graph if it contains a cycle which includes all of the edges and all of the vertices.

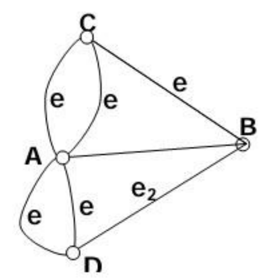
Note that, a Eulerian graph is always a connected graph in which all vertices have even degree.

Examples:

- The complete graphs K_3, K_5, K_7 etc are Eulerian graphs. In general complete graph K_n is a Eulerian graph when n is odd.
- The graphs 1, 2 and 3 shown in the figure on the next page are all Eulerian graphs. The graphs 4 and 5 are not Eulerian graphs.



- The important example of non Eulerian graphs is the graph which represents seven bridges problem, drawn below.

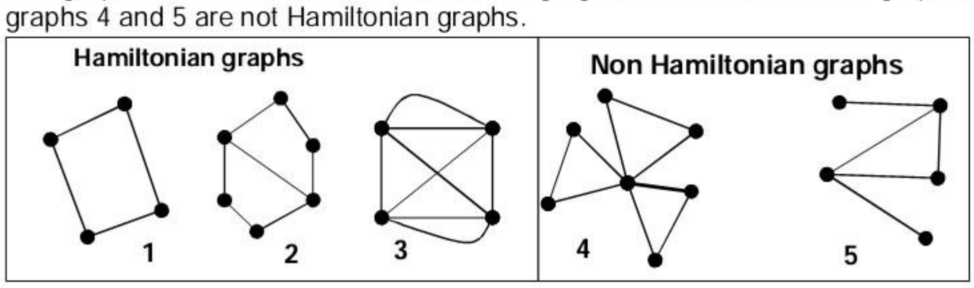


15.6.3 Hamiltonian graph: A graph G is called as Hamiltonian graph if it contains a cycle which includes every vertex of G exactly once, except that for the starting and ending vertex that appears twice.

Note that, a Hamiltonian graph is always a connected graph but any necessary and sufficient condition for a graph to be Hamiltonian is not known.

Examples:

- All complete graphs K_n are Hamiltonian graphs, for n ‡ 3. As each complete graph includes at least one cycle in which every vertex other than initial vertex is appearing only once.
- The graphs 1,2 and 3 shown in the following figure are all Hamiltonian graphs. The graphs 4 and 5 are not Hamiltonian graphs.

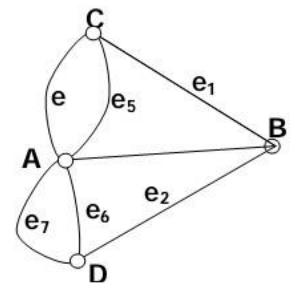


- The graph, which represents seven bridges problem, drawn here is an example of, a Hamiltonian graphs.

  Some Hamiltonian cycles in this graph are:

  C_1: A e_6 D e_2 B e_1 C e_5 A

  C_2: A e_7 D e_2 B e_1 C e_3 A

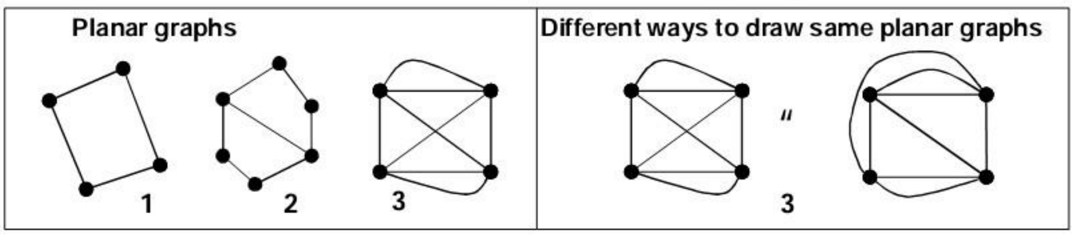


15.7 Planar graphs and colouring problem

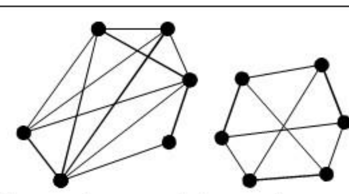
15.7.1 Planar graph: A graph is called as a planar graph if it can be drawn on a plane in such a way that no edges crass each other (except , of course at common vertices).

Examples:

- The complete graphs K_1, K_2, K_3 and K_4 are planar graphs. But other complete graphs K_n are not planar graphs, for n ‡ 4.
- The 3 graphs shown in the following figure are all planar graphs. The 3rd graph in this diagram can be drawn in a different way and the crossing of edges can be avoided.



- The 2 graphs shown on the right hand side are non-planar graphs. These graphs in this diagram cannot be drawn in a different way such that the crossing of edges can be avoided.



15.7.2 Colouring of a graph: Colouring of a graph means to assign one or more distinct colours to the vertices of a graph in such a way that no two adjacent vertices are assigned the same colour.

The idea of colouring of planar graphs is useful in solving problems in day to day life such as timetable problem.

A planar graph can be used to represent a map also, hence colouring a map is equivalent to colouring a planar graph. Lots of research is done about colouring in graph theory. Now it has been proved by the four colour theorem that every simple planar graph can be coloured with four colours.

15.8 Trees

Trees form one of the most widely used subclass of graphs. There are many applications of trees. Trees are useful in computer science to organize data in a database. There are many equivalent definitions of a tree. We will see the simplest form of it.

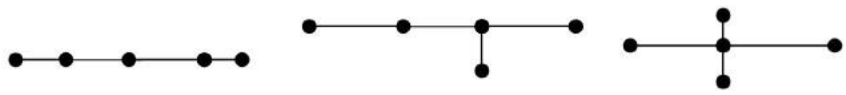
15.8.1 Tree: A tree is a simple, connected graph in which no cycle exists.

Examples:

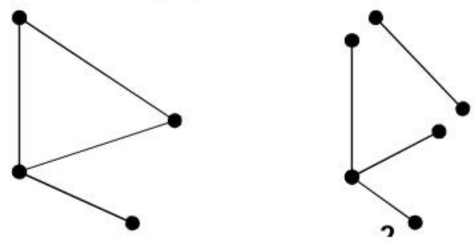
- The complete graphs K_1 and K_2 drawn below are trees, because these are connected graphs without any cycles. But other complete graphs K_n are not trees, for n ‡ 3.

K_1 K_2

- The following 3 graphs are non isomorphic trees on 5 vertices



- The following 2 graphs are not trees because the first graph includes a cycle and the second graph is not a connected graph.



15.8.2 Properties of trees

1. A tree containing n vertices has n – 1 edges.
2. There exists unique path between any two vertices of a tree.
3. If any two nonadjacent vertices in a tree are joined by an edge, then the resultant graph contains exactly one cycle.

15.9 Summary

In this unit learners studied the following topics in details:

1. The "Seven bridges problem of Königsberg."
2. The concept of a graph and its different representations.
3. Different terms used in the graph theory.
4. Different types of graphs such as, null graph, simple graph, multigraph, regular graph and complete graph etc.
5. Adjacency matrix and incidence matrix.
6. Connectivity, Eulerian and Hamiltonian graphs.
7. Planar graphs.
8. Trees.