MATH 1B FINAL (PRACTICE 1)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \int \sec(x)\ dx = \ln|\sec(x)+\tan(x)| + C$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_{Mid_n}| \leq \frac{K(b-a)^3}{24n^2} \qquad |E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. Compute the following integrals:

   (a) (10 points)

$$\int x^2 \ln(x^3) dx$$

Solution:

$$x^2 \ln(x^3) = 3x^2 \ln(x)$$

$$f(x) = \ln(x) \quad g'(x) = 3x^2$$

$$f'(x) = \frac{1}{x} \quad g(x) = x^3$$

$$\Rightarrow \int x^2 \ln(x^3)\, dx = x^3 \ln(x) - \int x^2\, dx$$

$$= x^3 \ln(x) - \frac{1}{3}x^3 + C$$

(b) (15 points)

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx$$

Solution:

$$x = 3\sec\theta \implies \frac{dx}{d\theta} = 3\tan\theta\sec\theta$$

$$\implies dx = 3\tan\theta\sec\theta\,d\theta$$

$$\implies \int \frac{\sqrt{x^2-9}}{x^4}\,dx = \int \frac{3\tan\theta \cdot 3\tan\theta\sec\theta}{81\sec^4\theta}\,d\theta$$

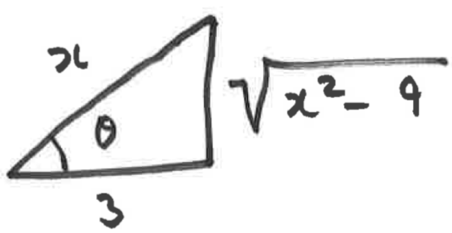
$$= \frac{9}{81}\int \sin^2\theta\cos\theta\,d\theta$$

Let $u = \sin\theta \implies \frac{du}{d\theta} = \cos\theta \implies d\theta = \frac{du}{\cos\theta}$

$$\implies \frac{9}{81}\int \sin^2\theta\cos\theta\,d\theta = \frac{9}{81}\int u^2\,du = \frac{3u^3}{81} + C$$

$$= \frac{3\sin^3\theta}{81} + C$$

$\frac{x}{3} = \sec\theta$



$\implies \sin\theta = \frac{\sqrt{x^2-9}}{x}$

$$\implies \int \frac{\sqrt{x^2-9}}{x^4}\,dx = \frac{3}{81}\cdot\frac{(x^2-9)^{3/2}}{x^3} + C$$

2. (25 points) Calculate the area of the surface of revolution (around the x-axis) of the curve

$$y = (x-1)^3,$$

between $x = 1$ to $x = 2$.

Solution:

$f(x) = (x-1)^3 \Rightarrow f'(x) = 3(x-1)^2$

$\Rightarrow$

$$2\pi f(x)\sqrt{1 + (f'(x))^2} = 2\pi (x-1)^3\sqrt{1 + 9(x-1)^4}$$

Let $u = 1 + 9(x-1)^4 \Rightarrow \dfrac{du}{dx} = 36(x-1)^3$

$\Rightarrow$ Surface Area $= \displaystyle\int_1^2 2\pi (x-1)^3\sqrt{1 + 9(x-1)^4}\, dx$

$$= \int_1^{10} \frac{\pi}{18}\sqrt{u}\, du = \frac{\pi}{18}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10}$$

$$= \frac{\pi}{27}\left(10^{3/2} - 1\right).$$

3. (25 points) Determine if the following series are convergent or divergent. You do not need to show your working.

   (a)
$$\sum_{n=2}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n}\right)$$

   Solution: Convergent

   (b)
$$\sum_{n=1}^{\infty}(-1)^n\cos(1/n^3)$$

   Solution: Divergent

   (c)
$$\sum_{n=1}^{\infty}\frac{n+1}{n^4-10}$$

   Solution: Convergent

   (d)
$$\sum_{n=1}^{\infty}\frac{5^n}{3^n+4^n}$$

   Solution: Divergent

   (e)
$$\sum_{n=1}^{\infty}\frac{n^n}{n!}$$

   Solution: Divergent

4. (a) (20 points) Using the integral test, determine whether

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}e^{\sqrt{n}}}$$

is convergent or divergent. Be sure to check all the hypotheses of the integral test.

Solution:

$$f(x) = \frac{1}{\sqrt{x}\, e^{\sqrt{x}}}$$

1/ $f(x) > 0$ on $[1, \infty)$

2/ f continuous on $[1, \infty)$

3/ $\sqrt{x}$, $e^{\sqrt{x}} > 0$ and increasing on $[1, \infty)$ $\Rightarrow$ $f(x)$ decreasing on $[1, \infty)$

Let $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\, du$

$$\Rightarrow \int \frac{1}{\sqrt{x}\, e^{\sqrt{x}}}\, dx = 2\int \frac{1}{e^u}\, du = -2e^{-u} + C$$

$$= \frac{-2}{e^{\sqrt{x}}} + C$$

$$\Rightarrow \int_1^{\infty} \frac{1}{\sqrt{x}\, e^{\sqrt{x}}}\, dx = \lim_{t \to \infty} \left. \frac{-2}{e^{\sqrt{x}}} \right|_1^t = \lim_{t \to \infty} \frac{2}{e} - \frac{2}{e^{\sqrt{t}}}$$

$$= \frac{2}{e} \Rightarrow \text{convergent}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, e^{\sqrt{n}}} \ \underline{\text{convergent}}.$$

Integral Test

(b) (5 points) Using this, or otherwise, determine if the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{ne^n}$$

is convergent or divergent.

Solution:

$$0 < \frac{1}{n e^n} \leq \frac{1}{\sqrt{n} e^{\sqrt{n}}} \quad \text{for all} \quad n \geq 1$$

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} e^{\sqrt{n}}}$ <u>convergent</u> $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n e^n}$ <u>convergent</u>

Comparison Test

5. Let
$$f(x) = \frac{(x^2+1)}{e^{x^2}}.$$

(a) (20 points) Calculate the Maclaurin series of $f(x)$. Be sure to include a general term.

Solution:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$\Rightarrow e^{(-x^2)} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} \dots + (-1)^n \frac{x^{2n}}{n!} + \dots$$

$$\Rightarrow x^2 e^{(-x^2)} = x^2 - x^4 + \dots \frac{x^6}{2!} + (-1)^{n-1} \frac{x^{2n}}{(n-1)!} + \dots$$

$$\Rightarrow \frac{x^2+1}{e^{x^2}} = 1 + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} \right) x^{2n}$$

For all x in $(-\infty, \infty)$

Maclaurin series of $\frac{x^2+1}{e^{x^2}}$

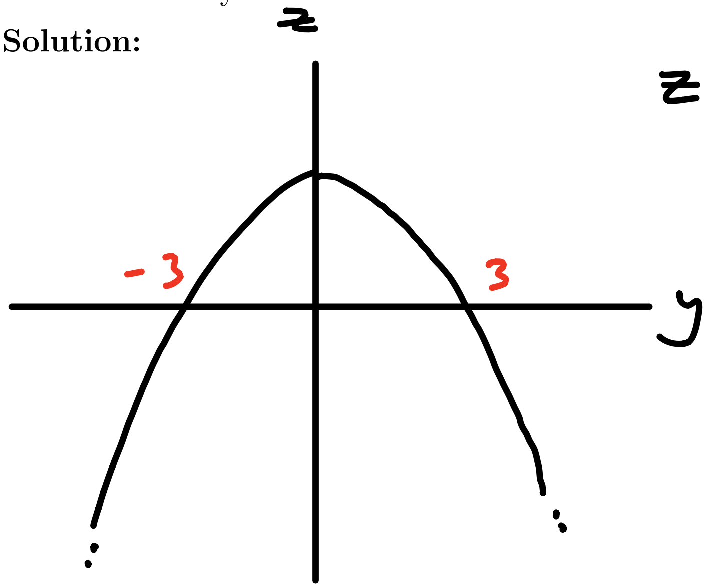
(b) (5 points) Using this, or otherwise, determine the value of $f^{(2n)}(0)$.

Solution:

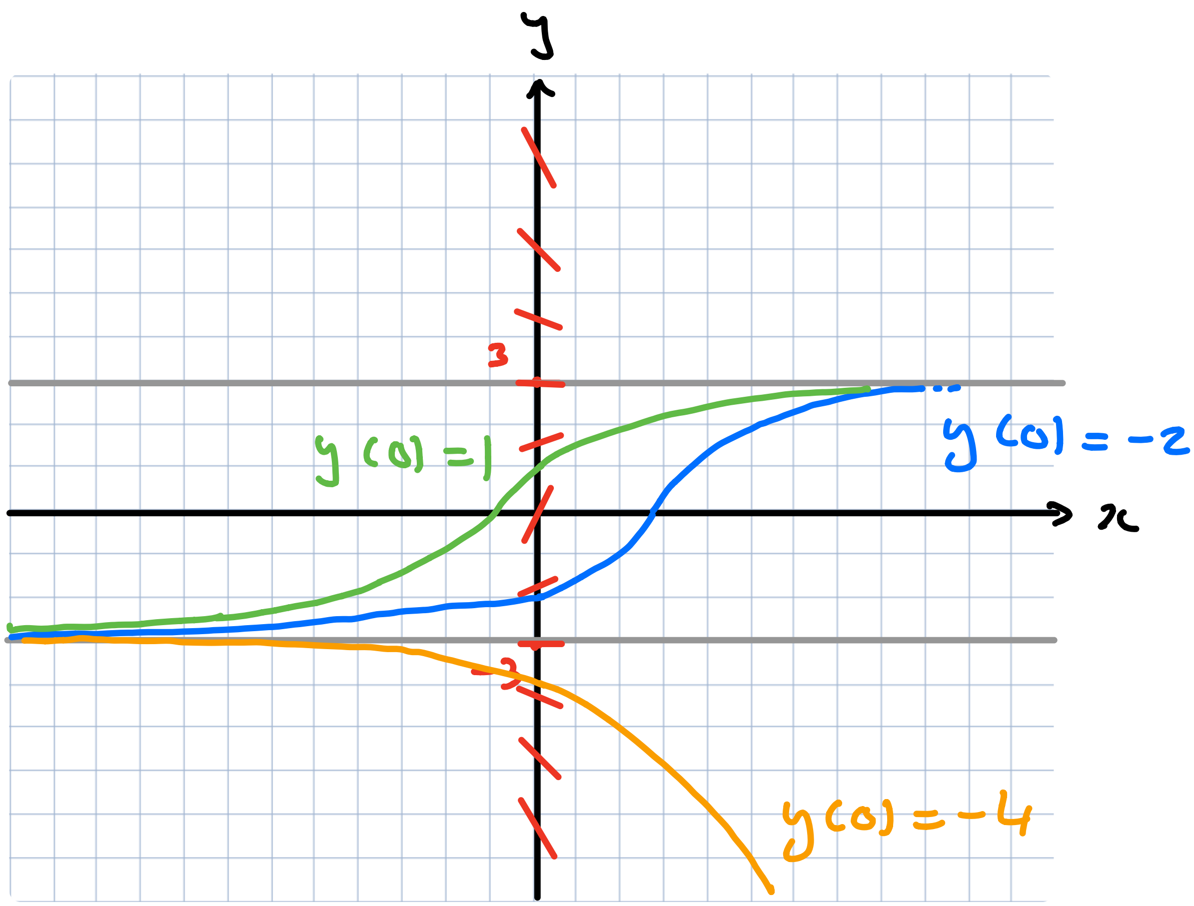
$$f^{(2n)}(0) = c_{2n} \cdot 2n! = \left(\frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} \right) \cdot 2n!$$

6. (25 points) Consider the differential equation $y' = 9 - y^2$. Graph the solutions with the following initial conditions: $y(0) = -2$, $y(0) = 1$ and $y(0) = -4$. Be sure to label each solution carefully.

Solution:



$z = 9 - y^2$



7. (25 points) Find an equation for the orthogonal trajectory to the family of curves

$$y = \frac{1}{k-x} + 1 \qquad (k \text{ any constant})$$

which contains the point $(1, 2)$.

Solution:

$$\frac{dy}{dx} = \frac{1}{(k-x)^2} = (y-1)^2$$

Must solve $\frac{dy}{dx} = \frac{-1}{(y-1)^2} \Rightarrow \int (y-1)^2 \, dy = \int -1 \cdot dx$

$$\Rightarrow \frac{1}{3}(y-1)^3 = -x + c$$

$$\Rightarrow (y-1)^3 = -3x + 3c$$

$$\Rightarrow y = \left(\sqrt[3]{-3x + 3c}\right) + 1$$

$$(2-1)^3 = -3 \cdot 1 + 3c \Rightarrow 3c = 4$$

$$\Rightarrow y = \left(\sqrt[3]{-3x + 4}\right) + 1$$

8. (25 points) Solve the following initial value problem:

$$t^3y' + 2y = 1 \qquad y(1) = e + 1/2.$$

Solution:

$$y' + \frac{2}{t^3} y = \frac{1}{t^3} . \quad \int \frac{2}{t^3} dt = \frac{-1}{t^2} + c$$

$$\Rightarrow \quad I(t) = e^{-\frac{1}{t^2}} \quad \Rightarrow$$

$$y = \frac{1}{e^{-\frac{1}{t^2}}} \cdot \int e^{-1/t^2} \cdot \frac{1}{t^3} \, dt$$

Let $u = -\frac{1}{t^2} \Rightarrow \frac{du}{dt} = \frac{2}{t^3} \Rightarrow$

$$\int e^{-1/t^2} \cdot \frac{1}{t^3} dt = \frac{1}{2} \int e^u \, du = \frac{1}{2} e^{-1/t^2} + c$$

$$\Rightarrow y = \frac{1}{2} + C e^{+1/t^2}$$

$$y(1) = \frac{1}{2} + Ce = \frac{1}{2} + e \Rightarrow C = 1$$

$$\Rightarrow y = \frac{1}{2} + e^{1/t^2}.$$

9. (25 points) Find a general solution to the differential equation:

$$y'' + y = x\sin(x).$$

Solution:

$$y'' + y = 0 \rightsquigarrow r^2 + 1 = 0 \iff r = \pm i$$

$$\Rightarrow y_h = c_1\cos(x) + c_2\sin(x) \leftarrow$$ general homogeneous solution

$$\Rightarrow y_p = (A_0 x + A_1 x^2)\cos(x) + (B_0 x + B_1 x^2)\sin(x)$$

$$\Rightarrow y_p' = (A_0 + 2A_1 x)\cos(x) - (A_0 x + A_1 x^2)\sin(x) + (B_0 + 2B_1 x)\sin(x) + (B_0 x + B_1 x^2)\cos(x)$$

$$\Rightarrow y_p'' = 2A_1\cos(x) - (A_0 + 2A_1 x)\sin(x) - (A_0 + 2A_1 x)\sin(x) - (A_0 x + A_1 x^2)\cos(x) + 2B_1\sin(x) + (B_0 + 2B_1 x)\cos(x) + (B_0 + 2B_1 x)\cos(x) - (B_0 x + B_1 x^2)\sin(x)$$

$$\Rightarrow y_p'' + y_p = ((2A_1 + 2B_0) + 4B_1 x)\cos(x) + ((-2A_0 + 2B_1) - 4A_1 x)\sin(x) = x\sin(x)$$

$$\Rightarrow \begin{cases} 2A_1 + 2B_0 = 0 \\ 4B_1 = 0 \\ -2A_0 + 2B_1 = 0 \\ -4A_1 = 1 \end{cases} \Rightarrow \begin{cases} A_1 = \frac{-1}{4} \\ B_0 = \frac{1}{4} \\ B_1 = 0 \\ A_0 = 0 \end{cases}$$

general solution

$$\Rightarrow y = c_1\cos(x) + c_2\sin(x) + \frac{-1}{4}x^2\cos(x) + \frac{1}{4}x\sin(x)$$

10. (25 points) Find a power series solution to the following initial value problem:

$$y'' + xy' + 2y = 0, \qquad y(0) = 0,\ y'(0) = 1 \Rightarrow \begin{matrix} c_0 = 0 \\ c_1 = 1 \end{matrix}$$

Solution:

$$2y = \sum_{n=0}^{\infty} 2c_n x^n, \quad xy' = \sum_{n=1}^{\infty} n c_n x^n,$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$\Rightarrow \quad y'' + xy' + 2y = (2c_0 + 2c_2) + \sum_{n=1}^{\infty} \left((n+2)(n+1) c_{n+2} + (n+2) c_n\right) x^n$$

$$\Rightarrow \quad 2c_0 + 2c_2 = 0 \Rightarrow c_2 = -c_0 = 0;$$

$$c_{n+2} = \frac{-c_n}{n+1} \quad (n \geq 1)$$

This actually also works for $n=0$, but thats a fluke

$\underline{n=1}$ $\quad c_3 = \frac{-c_1}{2} = \frac{-1}{2}$ $\qquad \underline{n=3}$ $\quad c_5 = \frac{-c_3}{4} = \frac{(-1)^2}{4 \cdot 2}$

$\underline{n=2}$ $\quad c_4 = \frac{-c_2}{3} = 0$ $\qquad \underline{n=4}$ $\quad c_6 = 0$ $\qquad \frac{(-1)^2}{2^2 \cdot 2!}$

<u>Pattern</u> $\quad c_{2k+1} = \frac{(-1)^k}{2^k \cdot k!} \quad k \geq 0 \quad (0! = 1)$

$c_{2k} = 0$

$$\Rightarrow \quad y = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k \cdot k!}$$

END OF EXAM