Magnetic field due to Straight Cylindrical wire

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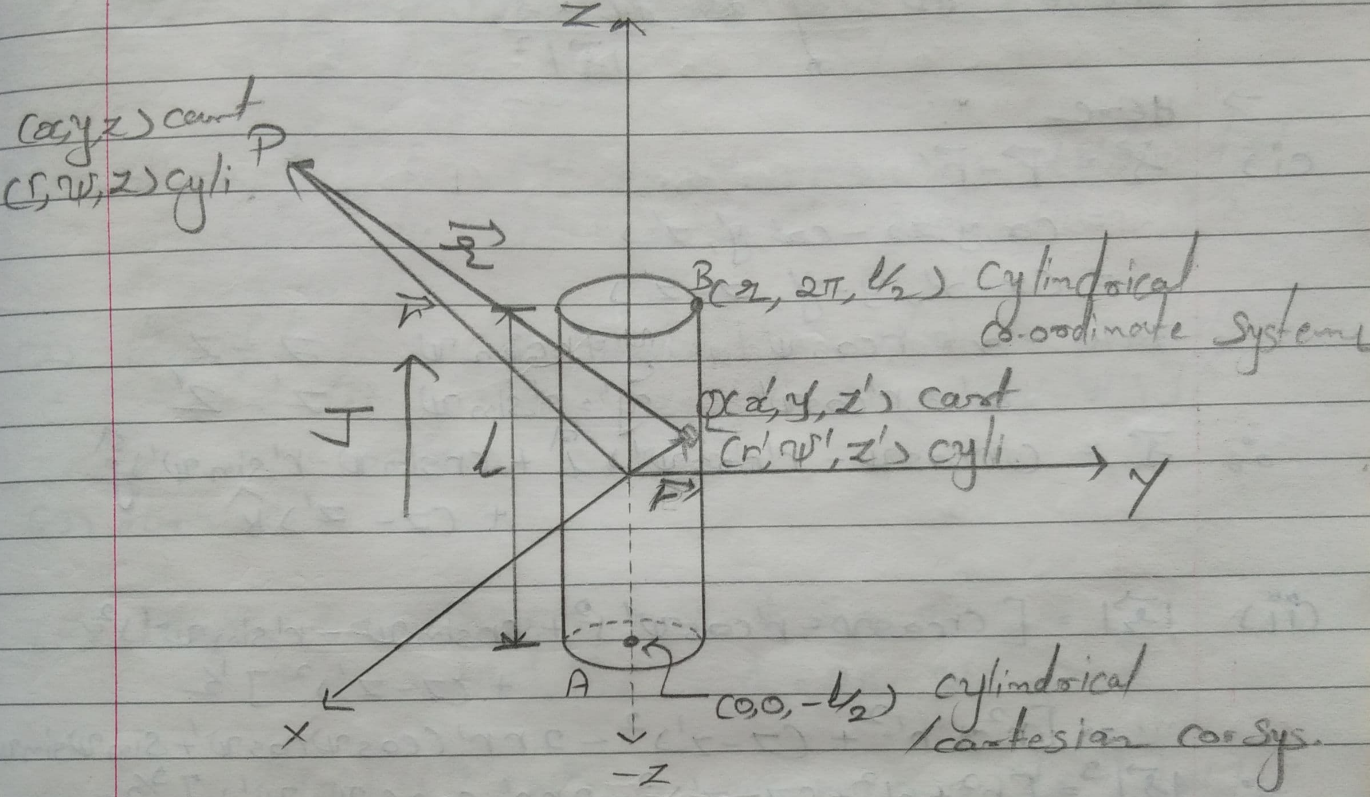

→ A straight conducting cylindrical wire is along z-axis and its centre is at origin

→ Here,

(1) Source point, $Q(r', \psi', z')$

(2) Observation Point, $P(r, \psi, z)$

(3) $A(0, 0, -l/2)$ and $B(r, 2\pi, l/2)$ are two diametrically opposite points.

(4) The current density is flowing along z-axis.

→ The magnetic field at point P can be written as,

$$\vec{B} = \frac{\mu_0 J}{4\pi} \int_A^B ds' \frac{d\vec{l} \times \vec{r}}{|\vec{r}|^3} = \frac{\mu_0 J}{4\pi} \vec{H}$$

→ where, $\vec{H} = \int_{A}^{B} ds' \frac{d\vec{I} \times \vec{\imath}}{|\vec{\imath}|^3}$

→ Here

(i) $\vec{\imath} = \vec{r} - \vec{r}'$

$= (x, y, z) - (x', y', z')$

$= (x - x', y - y', z - z')$

take $x = r\cos\psi$ $\quad y = r\sin\psi$ $\quad z = z$

$x' = r'\cos\psi'$ $\quad y' = r'\sin\psi$ $\quad z' = z'$

$\therefore \vec{\imath} = (r\cos\psi - r'\cos\psi')\hat{i} + (r\sin\psi - r'\sin\psi')\hat{j} + (z - z')\hat{k}$

(ii) $|\vec{\imath}| = [(r\cos\psi - r'\cos\psi')^2 + (r\sin\psi - r'\sin\psi')^2 + (z - z')^2]^{1/2}$

$= [r^2 + r'^2 + (z - z')^2 - 2rr'(\cos\psi\cos\psi' + \sin\psi\sin\psi')]^{1/2}$

$\therefore |\vec{\imath}|^3 = [r^2 + r'^2 + (z - z')^2 - 2rr'\cos(\psi - \psi')]^{3/2}$

(iii) current element

$d\vec{I} = dz'\hat{k}$

(iv) area element

$ds' = r'dr'd\psi'$

(v) cross-product

$$d\vec{I} \times \vec{\imath} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & dz' \\ r\cos\psi - r'\cos\psi' & r\sin\psi - r'\sin\psi' & z - z' \end{vmatrix}$$

$= \hat{i}[0 - (r\sin\psi - r'\sin\psi')dz'] - \hat{j}[0 - (r\cos\psi - r'\cos\psi')dz']$

$= \hat{i}(r'\sin\psi' - r\sin\psi)dz' + \hat{j}(r\cos\psi - r'\cos\psi')dz'$

$$\rightarrow \vec{H} = \int_{r'=0}^{2}\int_{\psi'=0}^{2\pi}\int_{z'=-l/2}^{l/2} r'\,dr'\,d\psi'\,dz' \frac{[\hat{i}(r'\sin\psi' - r\sin\psi) + \hat{j}(r\cos\psi - r'\cos\psi')]}{|\vec{R}|^3}$$

(1) $$H_x = \int_0^2\int_0^{2\pi}\int_{-l/2}^{l/2} \frac{(r'^2\sin\psi' - rr'\sin\psi)\,dr'\,d\psi'\,dz'}{[r^2+r'^2+(z-z')^2 - 2rr'\cos(\psi-\psi')]^{3/2}}$$

(2) $$H_y = \int_0^2\int_0^{2\pi}\int_{-l/2}^{l/2} \frac{(rr'\cos\psi - r'^2\cos\psi')\,dr'\,d\psi'\,dz'}{[r^2+r'^2+(z-z')^2 - 2rr'\cos(\psi-\psi')]^{3/2}}$$

(3) $H_z = 0.0$

$\rightarrow$ take $\phi = \psi' - \psi \quad \Rightarrow \phi_1 = 0 - \psi$

$\Rightarrow d\phi = d\psi' \qquad \phi_2 = 2\pi - \psi$

(1) $$H_x = \int_0^2\int_{-l/2}^{l/2}\left[\int_{\phi_1}^{\phi_2} \frac{(r'^2\sin(\phi+\psi) - rr'\sin\psi)\,d\phi}{[r^2+r'^2+(z-z')^2 - 2rr'\cos\phi]^{3/2}}\right]dr'\,dz'$$

(2) $$H_y = \int_0^2\int_{-l/2}^{l/2}\left[\int_{\phi_1}^{\phi_2} \frac{(rr'\cos\psi - r'^2\cos(\phi+\psi))\,d\phi}{[r^2+r'^2+(z-z')^2 - 2rr'\cos\phi]^{3/2}}\right]dr'\,dz'$$

(3) $H_z = 0.0$

$\rightarrow$ take $b^2 = r^2 + r'^2 + (z-z')^2$

(1) $$H_x = \int_0^2\int_{-l/2}^{l/2}\left[\int_{\phi_1}^{\phi_2} \frac{[(r'^2\cos\psi)\sin\phi + (r'^2\sin\psi)\cos\phi - rr'\sin\psi]\,d\phi}{[b^2 - 2rr'\cos\phi]^{3/2}}\right]dr'dz'$$

$$= \int_0^2\int_{-l/2}^{l/2}[(r'^2\cos\psi)I_2 + (r'^2\sin\psi)I_1 - (rr'\sin\psi)I_3]\,dr'dz'$$

$$\rightarrow H_y = \int_0^{r}\int_{-l/2}^{l/2}\left[\int_{\phi_1}^{\phi_2}\frac{[rr'\cos\psi-(r'^2\cos\psi)\cos\phi+(r'^2\sin\psi)\sin\phi]\,d\phi}{[b^2-2rr'\cos\phi]^{3/2}}\right]dr'\,dz'$$

$$= \int_0^{r} dr'\int_{-l/2}^{l/2} dz'\,[(rr'\cos\psi)I_3-(r'^2\cos\psi)I_1+(r'^2\sin\psi)I_2]$$

→ So our Eq^{n}s reduces to,

(4) $$H_x = \int_0^{r} dr'\int_{-l/2}^{l/2} dz'\,[(r'^2\cos\psi)I_2+(r'^2\sin\psi)I_1-(rr'\sin\psi)I_3]$$

(5) $$H_y = \int_0^{r} dr'\int_{-l/2}^{l/2} dz'\,[(rr'\cos\psi)I_3-(r'^2\cos\psi)I_1+(r'^2\sin\psi)I_2]$$

(6) $H_z = 0.0$

where, $I_1 = \int_{\phi_1}^{\phi_2}\frac{\cos\phi\,d\phi}{[b^2-2rr'\cos\phi]^{3/2}}$, $I_2 = \int_{\phi_1}^{\phi_2}\frac{\sin\phi}{[b^2-2rr'\cos\phi]^{3/2}}$

$I_3 = \int_{\phi_1}^{\phi_2}\frac{d\phi}{[b^2-2rr'\cos\phi]^{3/2}}$ and $\phi_k = \psi_k - \psi$

→ The solution of I_2 is given by,

Case-1 If $r \neq 0$ then, $I_2 = \frac{-1}{rr'}\sum_{j=1}^{2}(-1)^j\frac{1}{\sqrt{b^2-2rr'\cos\phi_j}}$

Case-2 If $r=0$ then, $I_2 = \frac{-1}{(r'^2+(z-z')^2)^{3/2}}\sum_{j=1}^{2}(-1)^j\cos\phi_j$

→ Now Let's solve these eq^n with respect to z'

(5) $$H_y = \int_0^{r} dr'\,[(rr'\cos\psi)I_{z3}-(r'^2\cos\psi)I_{z1}+(r'^2\sin\psi)I_{z2}]$$

(4) $$H_x = \int_0^{r} dr'\,[(r'^2\cos\psi)I_{z2}+(r'^2\sin\psi)I_{z1}-(rr'\sin\psi)I_{z3}]$$

(6) $H_z = 0.0$

$\rightarrow$ $$I_{z1} = \int_{\omega_1}^{\omega_2}\int_{\phi_1}^{\phi_2} \frac{\cos\phi \; d\phi \; d\omega}{[r^2+r'^2+\omega^2-2rr'\cos\phi]^{3/2}}$$

$\left\{ \because \text{by taking} \quad \omega = z'-z \quad \omega_k = z_k - z \right.$
$\left. \Rightarrow d\omega = dz' \quad k=1,2 \right\}$

take, $P^2 = r^2+r'^2-2rr'\cos\phi$

must note that $P^2 > 0$ as $\cos\phi \in [-1,1]$

(Proof: $(r+r')^2 \geq r^2+r'^2-2rr'\cos\phi \geq (r-r')^2$)

$$\therefore I_{z1} = \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi \cdot \int_{\omega_1}^{\omega_2} \frac{d\omega}{[P^2+\omega^2]^{3/2}}$$

takin $\omega = P\tan\theta$ we can solve the integral

$$\therefore I_{z1} = \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi \cdot \frac{1}{P^2} \sum_{k=1}^{2} (-1)^k \frac{\omega_k}{\sqrt{\omega_k^2+P^2}}$$

$$= \cancel{\frac{1}{P^2}} \sum_{k=1}^{2} (-1)^k \int_{\phi_1}^{\phi_2} \frac{\cos\phi \cdot \omega_k}{P^2\sqrt{\omega_k^2+P^2}} \, d\phi$$

$$= \sum_{k=1}^{2} (-1)^k \omega_k \int_{\phi_1}^{\phi_2} \frac{\cos\phi \; d\phi}{(r^2+r'^2-2rr'\cos\phi)\sqrt{r^2+r'^2+\omega^2-2rr'\cos\phi}}$$

$\rightarrow$ Similarly for I_{z3}

$$I_{z3} = \int_{\omega_1}^{\omega_2}\int_{\phi_1}^{\phi_2} \frac{d\phi}{[r^2+r'^2+\omega^2-2rr'\cos\phi]^{3/2}} \, d\omega$$

$$= \int_{\phi_1}^{\phi_2} d\phi \int_{\omega_1}^{\omega_2} \frac{d\omega}{[P^2+\omega^2]^{3/2}}$$

$$= \int_{\phi_1}^{\phi_2} d\phi \; \frac{1}{P^2} \sum_{k=1}^{2} (-1)^k \frac{\omega_k}{\sqrt{\omega_k^2+P^2}}$$

$$\therefore I_{z3} = \sum_{k=1}^{2} (-1)^k \omega_k \int_{\phi_1}^{\phi_2} \frac{d\phi}{(r^2+r'^2-2rr'\cos\phi)\sqrt{r^2+r'^2+\omega^2-2rr'\cos\phi}}$$

$\rightarrow$ for I_{z2},

Case-1 $r \neq 0$ then $I_2 = \frac{-1}{rr'} \sum_{j=1}^{2} (-1)^j \frac{1}{\sqrt{r^2 + r'^2 + (z-z')^2 - 2rr'\cos\phi_j}}$

$$\therefore I_{z2} = \frac{-1}{rr'} \sum_{j=1}^{2} (-1)^j \int_{\omega_1}^{\omega_2} \frac{d\omega}{\sqrt{r^2 + r'^2 + \omega^2 - 2rr'\cos\phi_j}}$$

$(\because \omega = z' - z)$

take $P^2 = r^2 + r'^2 - 2rr'\cos\phi_j$

$$\therefore I_{z2} = \frac{-1}{rr'} \sum_{j=1}^{2} (-1)^j \int_{\omega_1}^{\omega_2} \frac{d\omega}{\sqrt{P^2 + \omega^2}}$$

$$= \frac{-1}{rr'} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \ln\left|\omega_k + \sqrt{P^2 + \omega_k^2}\right|$$

$$\therefore I_{z2} = \frac{-1}{rr'} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \ln\left|\omega_k + \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}\right|$$

Case-2 $r = 0$ then $I_2 = \frac{-1}{(r'^2 + (z-z')^2)^{3/2}} \sum_{j=1}^{2} (-1)^j \cos\phi_j$

$$\therefore I_{z2} = -\sum_{j=1}^{2} (-1)^j \cos\phi_j \int_{\omega_1}^{\omega_2} \frac{d\omega}{(r'^2 + \omega^2)^{3/2}}$$

$(\because \omega = z' - z \Rightarrow d\omega = dz')$

$\rightarrow$ take, $\omega = r' \tan\theta$

$$\therefore \int_{\omega_1}^{\omega_2} \frac{d\omega}{(r'^2 + \omega^2)^{3/2}} = \frac{1}{r'^2} \sum_{k=1}^{2} (-1)^k \frac{\omega_k}{\sqrt{\omega_k^2 + r'^2}}$$

$$\therefore I_{z2} = -\sum_{j=1}^{2} (-1)^j \cos\phi_j \cdot \frac{1}{r'^2} \sum_{k=1}^{2} (-1)^k \frac{\omega_k}{\sqrt{\omega_k^2 + r'^2}}$$

$$\therefore\ I_{Z_2} = \frac{-1}{r'^2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{j+k}\cos\phi_j\frac{\omega_k}{\sqrt{\omega_k^2+r'^2}}$$

→ So now our eq^n becomes

$$(7)\quad H_x = \int_0^{r} dr'\left[(r'^2\cos\psi)I_{Z_2} + (r'^2\sin\psi)I_{Z_1} - (rr'\sin\psi)I_{Z_3}\right]$$

$$(8)\quad H_y = \int_0^{r} dr'\left[(rr'\cos\psi)I_{Z_3} - (r'^2\cos\psi)I_{Z_1} + (r'^2\sin\psi)I_{Z_2}\right]$$

$$(9)\quad H_z = 0.0$$

where, $$I_{Z_1} = \sum_{k=1}^{2}(-1)^k\omega_k\int_{\phi_1}^{\phi_2}\frac{\cos\phi\, d\phi}{(r^2+r'^2-2rr'\cos\phi)\sqrt{r^2+r'^2+\omega_k^2-2rr'\cos\phi}}$$

Case-1 ($r \neq 0$): $$I_{Z_2} = \frac{-1}{rr'}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{j+k}\ln\left(\omega_k+\sqrt{r^2+r'^2+\omega_k^2-2rr'\cos\phi_j}\right)$$

Case-2 ($r=0$): $$I_{Z_2} = \frac{-1}{r'^2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{j+k}\cos\phi_j\frac{\omega_k}{\sqrt{\omega_k^2+r'^2}}$$

$$I_{Z_3} = \sum_{k=1}^{2}(-1)^k\omega_k\int_{\phi_1}^{\phi_2}\frac{d\phi}{(r^2+r'^2-2rr'\cos\phi)\sqrt{r^2+r'^2+\omega_k^2-2rr'\cos\phi}}$$

$$(7)\quad H_x = \cos\psi\int_0^{r} r'^2 I_{Z_2}\,dr' + \sin\psi\left[\int_0^{r}(r'^2 I_{Z_1} - rr' I_{Z_3})\,dr'\right]$$

$$= \cos\psi\, I_{r2} - \sin\psi\, I_{r13}$$

$$(8)\quad H_y = \sin\psi\int_0^{r} r'^2 I_{Z_2}\,dr' + \cos\psi\left[\int_0^{r}(rr' I_{Z_3} - r'^2 I_{Z_1})\,dr'\right]$$

$$= \sin\psi\, I_{r2} + \cos\psi\, I_{r13}$$

$$(9)\quad H_z = 0.0$$

(10) $H_x = \cos\psi \, I_{22} - \sin\psi \, I_{213}$

(11) $H_y = \sin\psi \, I_{22} + \cos\psi \, I_{213}$

(12) $H_z = 0.0$

→ where, $I_{22} = \int_0^2 r'^2 I_{z2} \, dr'$

Case-1 $r \neq 0$ $\quad I_{z2} = \dfrac{-1}{rr'} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \ln\left(\omega_k + \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}\right)$

Case-2 $r = 0$ $\quad I_{z2} = \dfrac{-1}{r'^2} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \cos\phi_j \dfrac{\omega_k}{\sqrt{\omega_k^2 + r'^2}}$

→ $I_{213} = \int_0^2 (rr' I_{z3} - r'^2 I_{z1}) \, dr'$

$$I_{z1} = \sum_{k=1}^{2} (-1)^k \omega_k \int_{\phi_1}^{\phi_2} \frac{\cos\phi \, d\phi}{(r^2 + r'^2 - 2rr'\cos\phi)\sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi}}$$

$$I_{z3} = \sum_{k=1}^{2} (-1)^k \omega_k \int_{\phi_1}^{\phi_2} \frac{d\phi}{(r^2 + r'^2 - 2rr'\cos\phi)\sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi}}$$

$$\therefore I_{213} = \sum_{k=1}^{2} (-1)^k \cdot \omega_k \int_0^2 \int_{\phi_1}^{\phi_2} \frac{(rr' - \cos\phi \, r'^2) \, dr' \, d\phi}{(r^2 + r'^2 - 2rr'\cos\phi)\sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi}}$$

→ Now,

$$I_{22} = \int_0^2 r'^2 I_{z2} \, dr'$$

Case-2 $r = 0$ $\quad I_{z2} = \dfrac{-1}{r'^2} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \cos\phi_j \dfrac{\omega_k}{\sqrt{\omega_k^2 + r'^2}}$

$$\therefore I_{22} = -\sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \cos\phi_j \, \omega_k \int_0^2 \frac{dr'}{\sqrt{\omega_k^2 + r'^2}}$$

$$= -\sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \cos\phi_j \, \omega_k \left[\ln\left(r' + \sqrt{\omega_k^2 + r'^2}\right)\right]_0^2$$

$$\therefore\ I_{z2} = -\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{j+k}\cos\phi_j\,\omega_k \ln\left(\frac{r + \sqrt{\omega_k^2 + r^2}}{\omega_k}\right)$$

→ <u>Case-1</u> $r \neq 0$, $I_{z2} = \frac{-1}{rr'}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{j+k}\ln\left(\omega_k + \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}\right)$

$$I_{z2} = \int_0^{r} r'^2 I_{z2}\,dr'$$

$$= \frac{-1}{r}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{j+k}\int_0^{r} r' \ln\left(\omega_k + \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}\right)dr'$$

→ $\phi = \psi' - \psi$ $\quad \phi_1 = 0 - \psi \ \| \ \phi_1 = -\pi - \psi$

$\quad \phi_2 = 2\pi - \psi \ \| \ \phi_2 = \pi - \psi$

→ $\omega = z' - z$ $\quad \omega_1 = -l/2 - z$

$\quad \omega_2 = l/2 - z$