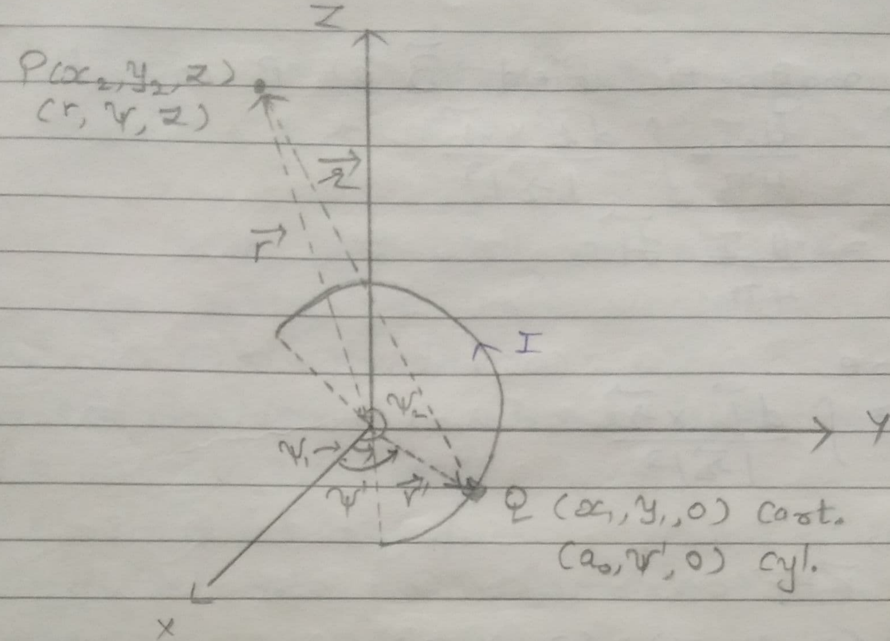
Magnetic field due to a conducting filament in the form of a circular arc



→ A current carrying circular conducting filament of approximately zero thickness placed in the XY-Plane so that its center is at the origin and axis is Z-axis of a right handed rectangular co-ordinate system. The magnetic field ~~and~~ ~~vector~~ p will be calculated in (in) Cartesian co-ordinate system.

→ Here,

(1) Source point, $Q(a_0, \psi', 0)$

(2) Observation point, $P(r, \psi, z)$

(3) a_0 is the radius of the filament

(4) ψ_1 is the angular distance of the nearest end from x-axis measured along the direction of current

(5) ψ_2 is the angular distance of the farthest end from x-axis measured along the direction of current.

(6) Current I is flowing from ψ_1 to ψ_2

→ The magnetic field $\vec{B}$ at P can be written as

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3}$$

$$= \frac{\mu_0 I}{4\pi} \vec{H}$$

where,

$$\vec{H} = \int \frac{d\vec{l}' \times \vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3}$$

Here,

(i) $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$= (x_2, y_2, z) - (x_1, y_1, 0)$

$= (x_2 - x_1, y_2 - y_1, z)$ (in Cartesian coordinates)

take, $x_2 = r\cos\psi$, $x_1 = a_0 \cos\psi'$

$y_2 = r\sin\psi$, $y_1 = a_0 \sin\psi'$

$\therefore \vec{\mathscr{r}} = (r\cos\psi - a_0\cos\psi')\hat{i} + (r\sin\psi - a_0 \sin\psi')\hat{j} + z\hat{k}$

(ii) Current Element, $d\vec{l}' = a_0 \psi' \hat{\psi}$

(In cylindrical coord. sys.)

→ Must note that our current element is in cylidrical co-ordinate system and $\vec{\mathscr{r}}$ is in Cartesian co-ordinate system

→ Now, substitude these values in eq^n of $\vec{H}$

$$\therefore \vec{H} = \int_{\psi_1}^{\psi_2} \frac{a_0 \psi' \hat{\psi} \times [(r\cos\psi - a_0\cos\psi')\hat{i} + (r\sin\psi - a_0\sin\psi')\hat{j} + z\hat{k}]}{[(r\cos\psi - a_0\cos\psi')^2 + (r\sin\psi - a_0\sin\psi')^2 + z^2]^{3/2}}$$

→ Let's try to simplify $\vec{H}$.

(iii) $|\vec{\mathscr{r}}|^3 = [(r\cos\psi - a_0\cos\psi')^2 + (r\sin\psi - a_0\sin\psi')^2 + z^2]^{3/2}$

$= [r^2\cos^2\psi + a_0^2\cos^2\psi'^2 - 2ra_0\cos\psi'\cos\psi$
$+ r^2\sin^2\psi + a_0^2\sin^2\psi'^2 - 2ra_0\sin\psi'\sin\psi + z^2]^{3/2}$

$$= [r^2 + a_0^2 + z^2 - 2ra_0(\cos\psi'\cos\psi + \sin\psi'\sin\psi)]^{3/2}$$

$$\therefore |\vec{r}|^3 = [r^2 + a_0^2 + z^2 - 2ra_0\cos(\psi - \psi')]^{3/2}$$

(iv) In this Part we will convert the unit vector "$\hat{\psi}$" in cartesian unit vector, so that we can do cross product.

-> transformation Equations (Cylidrical -> Cartesian)

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

-> Using Second transformation Eq^n we can write,

$$\hat{\psi} = -\sin\psi'\,\hat{i} + \cos\psi'\,\hat{j}$$

-> So our current Element becomes,

$$a_0\,d\psi'\,\hat{\psi} = a_0\,d\psi'(-\sin\psi'\,\hat{i} + \cos\psi'\,\hat{j})$$

(v) Now, we are ready to do cross product

$$\rightarrow d\vec{l}' \times \vec{r} = a_0\,d\psi'(-\sin\psi'\hat{i} + \cos\psi'\hat{j}) \times [(r\cos\psi - a_0\cos\psi')\hat{i} + (r\sin\psi - a_0\sin\psi')\hat{j} + z\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\psi' & \cos\psi' & 0 \\ r\cos\psi - a_0\cos\psi' & r\sin\psi - a_0\sin\psi' & z \end{vmatrix} a_0\,d\psi'$$

$$= (a_0 z\cos\psi'\,d\psi')\hat{i} + (a_0 z\sin\psi'\,d\psi')\hat{j} + \hat{k}\,a_0\,d\psi' \begin{vmatrix} \sin\psi' & \cos\psi' \\ r\cos\psi - a_0\cos\psi' & r\sin\psi - a_0\sin\psi' \end{vmatrix}$$

-> Let's solve the determinante,

$$a_0\,d\psi' \begin{vmatrix} -\sin\psi' & \cos\psi' \\ r\cos\psi - a_0\cos\psi' & r\sin\psi - a_0\sin\psi' \end{vmatrix}$$

$$= a_0 d\psi' [(-r\sin\psi\sin\psi' + a_0\sin^2\psi') - (r\cos\psi\cos\psi' - a_0\cos^2\psi')]$$

$$= a_0 d\psi' [a_0 - r[\cos\psi\cos\psi' + \sin\psi\sin\psi']]$$

$$= a_0 (a_0 - r\cos(\psi-\psi'))\, d\psi'$$

$$\therefore d\vec{l}' \times \vec{r} = (a_0 z\cos\psi' d\psi')\hat{i} + (a_0 z\sin\psi' d\psi')\hat{j} + a_0(a_0 - r\cos(\psi-\psi'))d\psi'\hat{k}$$

→ finally we can substitude all values of $\vec{H}$

$$\therefore \vec{H} = \int_{\psi_1}^{\psi_2} \frac{(a_0 z\cos\psi' d\psi')\hat{i} + (a_0 z\sin\psi' d\psi')\hat{j} + (a_0(a_0 - r\cos(\psi-\psi'))d\psi')\hat{k}}{[r^2 + a_0^2 + z^2 - 2ra_0\cos(\psi-\psi')]^{3/2}}$$

→ The components of $\vec{H}$,

$$(1)\quad \vec{H}_x = a_0 z \int_{\psi_1}^{\psi_2} \frac{\cos\psi'\, d\psi'}{[r^2+a_0^2+z^2-2ra_0\cos(\psi-\psi')]^{3/2}} \quad \hat{i}$$

$$(2)\quad \vec{H}_y = a_0 z \int_{\psi_1}^{\psi_2} \frac{\sin\psi'\, d\psi'}{[r^2+a_0^2+z^2-2ra_0\cos(\psi-\psi')]^{3/2}} \quad \hat{j}$$

$$(3)\quad \vec{H}_z = a_0 \int_{\psi_1}^{\psi_2} \frac{(a_0 - r\cos(\psi-\psi'))}{[r^2+a_0^2+z^2-2ra_0\cos(\psi-\psi')]^{3/2}} \quad \hat{k}$$

→ To solve Eq^n 1, 2 and 3,

Let, $b^2 = a_0^2 + r^2 + z^2$

$\phi = \psi' - \psi$ also, $\phi_1 = \psi_1 - \psi$

$\Rightarrow d\phi = d\psi'$ $\quad \phi_2 = \psi_2 - \psi$

→ So our Eq^n reduce to,

$$(4)\quad H_x = a_0 z \int_{\phi_1}^{\phi_2} \frac{\cos(\phi+\psi)\, d\phi}{[b^2 - 2ra_0\cos\phi]^{3/2}}$$

(5) $$H_y = a_o z \int_{\phi_1}^{\phi_2} \frac{\sin(\phi+\psi)\,d\phi}{[b^2-2ra_o\cos\phi]^{3/2}}$$

(7) $$H_z = a_o \int_{\phi_1}^{\phi_2} \frac{(a_o - r\cos\phi)\,d\phi}{[b^2-2ra_o\cos\phi]^{3/2}}$$

→ Now,

(4) $$H_x = a_o z \int_{\phi_1}^{\phi_2} \frac{\cos(\phi+\psi)\,d\phi}{[b^2-2ra_o\cos\phi]^{3/2}}$$

$$= a_o z \int_{\phi_1}^{\phi_2} \frac{(\cos\phi\cos\psi - \sin\phi\sin\psi)\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}}$$

$$= a_o z \left(\cos\psi \int_{\phi_1}^{\phi_2} \frac{\cos\phi\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}} - \sin\psi \int_{\phi_1}^{\phi_2} \frac{\sin\phi\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}} \right)$$

(5) $$H_y = a_o z \int_{\phi_1}^{\phi_2} \frac{(\sin\phi\cos\psi + \sin\psi\cos\phi)\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}}$$

$$= a_o z \left(\cos\psi \int_{\phi_1}^{\phi_2} \frac{\sin\phi\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}} + \sin\psi \int_{\phi_1}^{\phi_2} \frac{\cos\phi\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}} \right)$$

(6) $$H_z = a_o \int_{\phi_1}^{\phi_2} \frac{(a_o - r\cos\phi)\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}}$$

$$= a_o^2 \int_{\phi_1}^{\phi_2} \frac{d\phi}{(b^2-2ra_o\cos\phi)^{3/2}} - a_o r \int_{\phi_1}^{\phi_2} \frac{\cos\phi\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}}$$

→ Now take,

$$I_1 = \int_{\phi_1}^{\phi_2} \frac{\cos\phi\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}}$$

$$I_2 = \int_{\phi_1}^{\phi_2} \frac{\sin\phi\,d\phi}{(b^2-2ra_o\cos\phi)^{3/2}}, \quad I_3 = \int_{\phi_1}^{\phi_2} \frac{d\phi}{(b^2-2ra_o\cos\phi)^{3/2}}$$

→ So, Eq^n (4), (5) and (6) can be written as

(4) $H_x = a_0 z(\cos\psi I_1 - \sin\psi I_2)$

(5) $H_y = a_0 z(\cos\psi I_2 + \sin\psi I_1)$

(6) $H_z = a_0(a_0 I_3 - r I_1)$

→ $I_2 = \int_{\phi_1}^{\phi_2} \frac{\sin\phi \, d\phi}{[b^2 - 2ra_0\cos\phi]^{3/2}}$ Case-1

Let $t = \cos\phi$

$\Rightarrow dt = -\sin\phi \, d\phi$

$\therefore I_2 = \int \frac{-dt}{[b^2 - 2ra_0 t]^{3/2}}$

$= -\int (b^2 - 2ra_0 t)^{-3/2} dt$

$= -\frac{(b^2 - 2ra_0 t)^{-1/2}}{(-1/2)(-2ra_0)}$

$= \frac{-1}{ra_0}\left[\frac{1}{\sqrt{b^2 - 2ra_0\cos\phi}}\right]_{\phi_1}^{\phi_2}$

$\therefore I_2 = \frac{-1}{ra_0} \sum_{i=1}^{2} (-1)^i \frac{1}{\sqrt{b^2 - 2ra_0\cos\phi_i}}$ iff $r \neq 0$

Case-2 if $r = 0$

→ $I_2 = \int_{\phi_1}^{\phi_2} \frac{\sin\phi}{b^3} d\phi$

$= \frac{1}{(a_0^2 + z^2)^{3/2}} [-\cos\phi]_{\phi_1}^{\phi_2}$

$\therefore I_2 = \frac{-1}{(a_0^2 + z^2)^{3/2}} \sum_{i=1}^{2} (-1)^i \cos\phi_i$

→ Sub. the values of $\vec{H}$ components in $\vec{B}$ we get,

(7) $B_x = \frac{\mu_0 I}{4\pi} a_0 z (\cos\psi I_1 - \sin\psi I_2)$

(8) $B_y = \frac{\mu_0 I}{4\pi} a_0 z (\cos\psi I_2 + \sin\psi I_1)$

(9) $B_z = \frac{\mu_0 I}{4\pi} a_0 (a_0 I_3 - r I_1)$

where → $I_1 = \int_{\phi_1}^{\phi_2} \frac{\cos\phi \, d\phi}{(b^2 - 2ra_0\cos\phi)^{3/2}}$

→ If $r=0$, $I_2 = \frac{-1}{(a_0^2+z^2)^{3/2}} \sum_{i=1}^{2} (-1)^i \cos\phi_i$

If $r \neq 0$, $I_2 = \frac{-1}{ra_0} \sum_{i=1}^{2} (-1)^i \frac{1}{\sqrt{b^2 - 2ra_0\cos\phi_i}}$

→ $I_3 = \int_{\phi_1}^{\phi_2} \frac{d\phi}{(b^2 - 2ra_0\cos\phi)^{3/2}}$

and → $\phi = \psi' - \psi \Rightarrow \phi_k = \psi_k - \psi$, $k = 1,2$

→ $b^2 = a_0^2 + r^2 + z^2$

Note: We can not solve I_1 and I_3 analyticaly. We can Express I_1 and I_3 in term of Elliptic Integrals. Please see the book in references

* References:

1 I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series and Products. Page 180, 182, 179

2 Analytical derivation of magnetic field and vector potential due to different current carrying conducting geometries, IPR

3 Special thanks to Grattu Ramesh Sir.