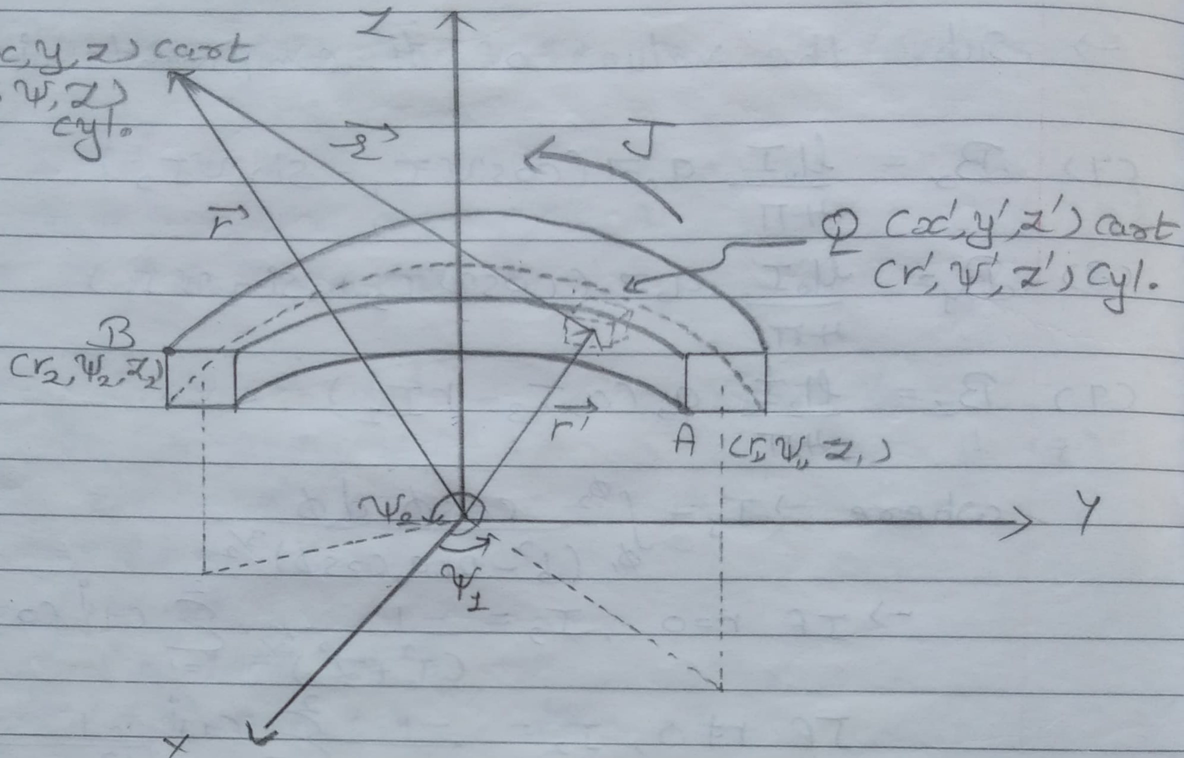
Magnetic Field due to a conducting circular bar of rectangular cross-section



→ A current carrying circular conductor of rectangular cross-section placed parallel to the XY-plane.

→ Here,

(1) Source point, $Q(r', \psi', z')$

(2) observation point, $P(r, \psi, z)$

(3) $A(r_1, \psi_1, z_1)$ and $B(r_2, \psi_2, z_2)$ are two dimetrically opposite points

(4) ψ_1 is the angular distance of the nearest end from X-axis

(5) ψ_2 is the angular distance of the farthes end from X-axis

(6) Current density J is flowing from ψ_1 to ψ_2

→ The magnetic field $\vec{B}$ at P can be written as.

$$\vec{B} = \frac{\mu_0 J}{4\pi} \int_A^B ds' \frac{d\vec{l}' \times \vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3}$$

$$= \frac{\mu_0 J}{4\pi} \vec{H}$$

where,

$$\vec{H} = \int_A^B ds' \frac{d\vec{l}' \times \vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3}$$

Here,

(i) $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$= (x, y, z) - (x', y', z')$

$= (x - x', y - y', z - z')$

take $x' = r'\cos\psi'$ $\quad y' = r'\sin\psi'$ $\quad z' = z'$

$x = r\cos\psi$ $\quad y = r\sin\psi$ $\quad z = z$

$\therefore \vec{\mathscr{r}} = (r\cos\psi - r'\cos\psi')\hat{i} + (r\sin\psi - r'\sin\psi')\hat{j} + (z - z')\hat{k}$

(ii) current Element or elementary length of arc

$d\vec{l}' = r' \, d\psi' \, \hat{\psi}$

(iii) area Element, $ds' = dr' dz'$

→ Now, sub. these values in eq^n of $\vec{H}$ we get

$$\therefore \vec{H} = \int_{\psi_1}^{\psi_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} \frac{dr' dz' \cdot r' d\psi' \hat{\psi} \times \vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3}$$

$$\therefore \vec{H} = \int_{\psi_1}^{\psi_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} \frac{r' \hat{\psi} \times \vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3} dr' dz' d\psi'$$

$$= \int_{\psi_1}^{\psi_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} r' \frac{(-\sin\psi' \hat{i} + \cos\psi' \hat{j}) \times \vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3} dr' dz' d\psi'$$

using $\hat{r} = \cos\theta \, \hat{i} + \sin\theta \, \hat{j}$

$\hat{\theta} = -\sin\theta \, \hat{i} + \cos\theta \, \hat{j}$

(iv) Cross Product $= (-\sin\psi'\,\hat{i} + \cos\psi'\,\hat{j}) \times ((r\cos\psi - r'\cos\psi')\hat{i}$
$+ (r\sin\psi - r'\sin\psi')\hat{j} + (z-z')\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\psi' & \cos\psi' & 0 \\ r\cos\psi - r'\cos\psi' & r\sin\psi - r'\sin\psi' & z-z' \end{vmatrix}$$

$$= ((z-z')\cos\psi')\hat{i} + ((z-z')\sin\psi')\hat{j} + \begin{vmatrix} -\sin\psi' & \cos\psi' \\ r\cos\psi - r'\cos\psi' & r\sin\psi - r'\sin\psi' \end{vmatrix}\hat{k}$$

→ Let's solve the determinant

$$\begin{vmatrix} -\sin\psi' & \cos\psi' \\ r\cos\psi - r'\cos\psi' & r\sin\psi - r'\sin\psi' \end{vmatrix}$$

$$= -r\sin\psi\sin\psi' + r'\sin^2\psi' - r\cos\psi\cos\psi' + r'\cos^2\psi'$$

$$= r' - r(\cos\psi\cos\psi' + \sin\psi\sin\psi')$$

$$= r' - r\cos(\psi-\psi')$$

∴ Cross product $= ((z-z')\cos\psi')\hat{i} + ((z-z')\sin\psi')\hat{j}$
$+ (r' - r\cos(\psi-\psi'))\hat{k}$

→ Substitude this value in eq^n of H we get

(1) $$H_x = \int_{\psi_1}^{\psi_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} \frac{r'(z-z')\cos\psi'\,dr'\,dz'\,d\psi'}{(r^2+r'^2+(z-z')^2-2rr'\cos(\psi-\psi'))^{3/2}}$$

(2) $$H_y = \int_{\psi_1}^{\psi_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} \frac{r'(z-z')\sin\psi'\,dr'\,dz'\,d\psi'}{(r^2+r'^2+(z-z')^2-2rr'\cos(\psi-\psi'))^{3/2}}$$

(3) $$H_z = \int_{\psi_1}^{\psi_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} \frac{r'(r'-r\cos(\psi-\psi'))\,dr'\,dz'\,d\psi'}{(r^2+r'^2+(z-z')^2-2rr'\cos(\psi-\psi'))^{3/2}}$$

→ Let's try to simplify H_x, H_y and H_z

(1) $$H_x = \int_{r_1}^{r_2} dr' \int_{z_1}^{z_2} dz' \int_{\psi_1}^{\psi_2} \frac{r'(z-z')\cos\psi' \, d\psi'}{(r^2+r'^2+(z-z')^2-2rr'\cos(\psi-\psi'))^{3/2}}$$

take $b^2 = r^2+r'^2+(z-z')^2$

$\phi = \psi' - \psi$, $\phi_j = \psi_j - \psi$, $j = 1, 2$

$$\therefore H_x = \int_{r_1}^{r_2} r'dr' \int_{z_1}^{z_2} (z-z')dz' \int_{\phi_1}^{\phi_2} \frac{\cos(\phi+\psi)\, d\phi}{[b^2-2rr'\cos\phi]^{3/2}}$$

$$= \int_{r_1}^{r_2} r'dr' \int_{z_1}^{z_2} (z-z')dz' \left(\int_{\phi_1}^{\phi_2} \frac{\cos\phi\cos\psi - \sin\phi\sin\psi \; d\phi}{(b^2-2rr'\cos\phi)^{3/2}} \right)$$

$$= \int_{r_1}^{r_2} r'dr' \int_{z_1}^{z_2} (z-z')dz' \left(\cos\psi \, I_1 - \sin\psi \, I_2\right)$$

where, $$I_1 = \int_{\phi_1}^{\phi_2} \frac{\cos\phi \; d\phi}{(b^2-2rr'\cos\phi)^{3/2}}$$

$$I_2 = \int_{\phi_1}^{\phi_2} \frac{\sin\phi \; d\phi}{(b^2-2rr'\cos\phi)^{3/2}}$$

$$\therefore H_x = \int_{r_1}^{r_2} r'dr' \left(\cos\psi \int_{z_1}^{z_2} (z-z') I_1 \, dz' - \sin\psi \int_{z_1}^{z_2} (z-z') I_2 \, dz' \right)$$

$$\therefore H_x = \int_{r_1}^{r_2} r'dr' \left(\cos\psi \, I_{R1} - \sin\psi \, I_{R2} \right) \qquad \text{------ (4)}$$

where, $$I_{R1} = \int_{z_1}^{z_2} (z-z') I_1 \, dz'$$

$$I_{R2} = \int_{z_1}^{z_2} (z-z') I_2 \, dz'$$

(2) $$H_y = \int_{r_1}^{r_2} r'dr' \int_{z_1}^{z_2} (z-z')dz' \int_{\psi_1}^{\psi_2} \frac{\sin\psi' \; d\psi'}{(r^2+r'^2+(z-z')^2-2rr'\cos(\psi-\psi'))^{3/2}}$$

take, $b^2 = r^2+r'^2+(z-z')^2$

$\phi = \psi' - \psi$, $\phi_j = \psi_j - \psi$, $j = 1, 2$

$$\therefore \mathcal{H}_y = \int_{r_1}^{r_2} r' dr' \int_{z_1}^{z_2} (z-z')dz' \int_{\phi_1}^{\phi_2} \frac{\sin(\phi+\psi)\, d\phi}{(b^2-2rr'\cos\phi)^{3/2}}$$

$$= \int_{r_1}^{r_2} r' dr' \int_{z_1}^{z_2} (z-z')dz' \int_{\phi_1}^{\phi_2} \frac{\sin\phi\cos\psi + \sin\psi\cos\phi}{(b^2-2rr'\cos\phi)^{3/2}}\, d\phi$$

$$= \int_{r_1}^{r_2} r' dr' \int_{z_1}^{z_2} (z-z')dz' \left(\cos\psi\, I_2 + \sin\psi\, I_1\right)$$

$$= \int_{r_1}^{r_2} r' dr' \left(\cos\psi \int_{z_1}^{z_2} (z-z')I_2\, dz' + \sin\psi \int_{z_1}^{z_2} (z-z')I_1\, dz'\right)$$

$$\therefore \mathcal{H}_y = \int_{r_1}^{r_2} r' dr' \left(\cos\psi\, I_{R2} + \sin\psi\, I_{R1}\right) \quad \text{-----} (5)$$

(3) $$\mathcal{H}_z = \int_{r_1}^{r_2} dr' \int_{z_1}^{z_2} dz' \int_{\psi_1}^{\psi_2} \frac{r'(r'-r\cos(\psi-\psi'))\, d\psi'}{(r^2+r'^2+(z-z')^2-2rr'\cos(\psi-\psi'))^{3/2}}$$

$$= \int_{r_1}^{r_2} dr' \int_{z_1}^{z_2} dz' \int_{\phi_1}^{\phi_2} \frac{r'(r'-r\cos\phi)}{[b^2-2rr'\cos\phi]^{3/2}}\, d\phi$$

$$= \int_{r_1}^{r_2} dr' \int_{z_1}^{z_2} dz' \left(r'[r' I_3 - r I_1]\right)$$

where, $$I_3 = \int_{\phi_1}^{\phi_2} \frac{d\phi}{[b^2-2rr'\cos\phi]^{3/2}} \quad \text{-----} (6)$$

-> Let's collect the eq^n (4), (5) and (6)

(4) $$\mathcal{H}_x = \int_{r_1}^{r_2} r' dr' \left(\cos\psi\, I_{R1} - \sin\psi\, I_{R2}\right)$$

(5) $$\mathcal{H}_y = \int_{r_1}^{r_2} r' dr' \left(\cos\psi\, I_{R2} + \sin\psi\, I_{R1}\right)$$

(6) $$\mathcal{H}_z = \int_{r_1}^{r_2} dr' \int_{z_1}^{z_2} dz' \left(r'[r' I_3 - r I_1]\right)$$

where, $b^2 = r^2 + r'^2 + (z-z')^2$

$\phi_j = \psi_j - \psi, \quad j = 1, 2$

$$I_{R1} = \int_{z_1}^{z_2} (z-z') I_1 \, dz' \quad , \quad I_{R2} = \int_{z_1}^{z_2} (z-z') I_2 \, dz'$$

$$I_1 = \int_{\phi_1}^{\phi_2} \frac{\cos\phi \, d\phi}{(b^2 - 2rr'\cos\phi)^{3/2}} \quad , \quad I_2 = \int_{\phi_1}^{\phi_2} \frac{\sin\phi \, d\phi}{(b^2 - 2rr'\cos\phi)^{3/2}}$$

$$I_3 = \int_{\phi_1}^{\phi_2} \frac{d\phi}{(b^2 - 2rr'\cos\phi)^{3/2}}$$

→ The solution of I_2 is given by

Case-1 If $r \neq 0$ then, $I_2 = \dfrac{-1}{rr'} \displaystyle\sum_{j=1}^{2} (-1)^j \frac{1}{\sqrt{b^2 - 2rr'\cos\phi_j}}$

Case-2 If $r = 0$ then, $I_2 = \dfrac{-1}{(r'^2 + (z-z')^2)^{3/2}} \displaystyle\sum_{j=1}^{2} (-1)^j \cos\phi_j$

(i) $I_{R1} = \int_{z_1}^{z_2} (z-z') I_1 \, dz'$

where, $I_1 = \int_{\phi_1}^{\phi_2} \dfrac{\cos\phi \, d\phi}{(b^2 - 2rr'\cos\phi)^{3/2}}$

$$b^2 = r^2 + r'^2 + (z-z')^2$$

$$\therefore \; I_{R1} = \int_{z_1}^{z_2} \int_{\phi_1}^{\phi_2} \frac{(z-z')\cos\phi \, d\phi \, dz'}{(r^2 + r'^2 + (z-z')^2 - 2rr'\cos\phi)^{3/2}}$$

take $q = r^2 + r'^2 - 2rr'\cos\phi$

$$\therefore \; I_{R1} = \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi \int_{z_1}^{z_2} \frac{(z-z') \, dz'}{(q + (z-z')^2)^{3/2}}$$

$$= \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi \; I_d$$

→ where

$$I_d = \int_{z_1}^{z_2} \frac{(z-z') \, dz'}{(q + (z-z')^2)^{3/2}}$$

take $\omega = z - z'$ $\quad$ $\omega_1 = z - z_1$

$\Rightarrow dz' = -d\omega$ $\quad$ $\omega_2 = z - z_2$

$$\therefore \cancel{I_{d_1}}\ I_d = -\int_{\omega_1}^{\omega_2} \frac{\omega\, d\omega}{(q+\omega^2)^{3/2}}$$

$$= \frac{-1}{2}\int_{\omega_1}^{\omega_2} \frac{2\omega\, d\omega}{(q+\omega^2)^{3/2}}$$

take $\omega^2 = t \Rightarrow dt = 2\omega\, d\omega$

$$\therefore I_d = \frac{-1}{2}\int \frac{dt}{(q+t)^{3/2}}$$

$$= \frac{-1}{2} \frac{(q+t)^{-1/2}}{-1/2}$$

$$= \left[\frac{1}{\sqrt{q+\omega^2}}\right]_{\omega_1}^{\omega_2}$$

$$\therefore I_d = \sum_{K=1}^{2} (-1)^K \frac{1}{\sqrt{q+\omega_K^2}}$$

$$\therefore I_{R1} = \int_{\phi_1}^{\phi_2} \cos\phi\, d\phi \sum_{K=1}^{2} (-1)^K \frac{1}{\sqrt{q+\omega_K^2}}$$

$$= \sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \frac{\cos\phi}{\sqrt{q+\omega_K^2}}\, d\phi$$

$$= \sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \frac{\cos\phi\ \ d\phi}{\sqrt{r^2+r'^2-2rr'\cos\phi+\omega_K^2}}$$

$$\therefore \boxed{I_{R1} = \sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \frac{\cos\phi\, d\phi}{[r^2+r'^2+\omega_K^2-2rr'\cos\phi]^{1/2}}}$$

→ Let's solve I_{R2} now. For I_{R2} we will consider two cases when $r \neq 0$ and $r = 0$

(ii) → $I_{R2} = \int_{z_1}^{z_2} (z-z') I_2 \, dz'$

where $I_2 = \int_{\phi_1}^{\phi_2} \frac{\sin\phi \, d\phi}{(b^2 - 2rr'\cos\phi)^{3/2}}$

Case-1, If $r \neq 0$, $I_2 = \frac{-1}{rr'} \sum_{j=1}^{2} (-1)^j \frac{1}{[b^2 - 2rr'\cos\phi_j]^{1/2}}$

Case-2, If $r = 0$, $I_2 = \frac{-1}{[r'^2 + (z-z')^2]^{3/2}} \sum_{j=1}^{2} (-1)^j \cos\phi_j$

and $b^2 = r^2 + r'^2 + (z-z')^2$

→ Case-1 If $r \neq 0$

$$I_{R2} = \int_{z_1}^{z_2} (z-z') I_2 \, dz'$$

$$= \int_{z_1}^{z_2} (z-z') \frac{(-1)}{rr'} \sum_{j=1}^{2} (-1)^j \frac{1}{(b^2 - 2rr'\cos\phi_j)^{1/2}} dz'$$

$$= \frac{-1}{rr'} \sum_{j=1}^{2} (-1)^j \int_{z_1}^{z_2} \frac{(z-z') \, dz'}{[r^2 + r'^2 + (z-z')^2 - 2rr'\cos\phi_j]^{1/2}}$$

take, $q = r^2 + r'^2 - 2rr'\cos\phi_j$

$\omega = z - z'$ $\qquad \omega_1 = z - z_1$

$\Rightarrow dz' = -d\omega$ $\qquad \omega_2 = z - z_2$

$$\therefore I_{R2} = \frac{-1}{rr'} \sum_{j=1}^{2} (-1)^j \int_{\omega_1}^{\omega_2} \frac{-\omega \, d\omega}{[q + \omega^2]^{1/2}}$$

$$= \frac{1}{2rr'} \sum_{j=1}^{2} (-1)^j \int_{\omega_1}^{\omega_2} \frac{2\omega \, d\omega}{[q + \omega^2]^{1/2}}$$

$$= \frac{1}{rr'} \sum_{j=1}^{2} (-1)^j I_d$$

$\rightarrow$ where, $I_d = \int_{\omega_1}^{\omega_2} \frac{\omega\, d\omega}{[q+\omega^2]^{1/2}}$

take $t = \omega^2$

$\Rightarrow dt = 2\omega\, d\omega$

$\therefore I_d = \frac{1}{2} \int \frac{dt}{[q+t]^{1/2}}$

$= \frac{1}{2} \int [q+t]^{-1/2}\, dt$

$= \frac{1}{2} \frac{[q+t]^{1/2}}{1/2}$

$= [q+\omega^2]^{1/2} \Big|_{\omega_1}^{\omega_2}$

$\therefore I_d = \sum_{k=1}^{2} (-1)^k \sqrt{q+\omega_k^2}$

$\therefore I_{R2} = \frac{1}{rr'} \sum_{j=1}^{2} (-1)^j \sum_{k=1}^{2} (-1)^k \sqrt{q+\omega_k^2}$

$$\boxed{I_{R2} = \frac{1}{rr'} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \sqrt{r^2+r'^2-2rr'\cos\phi_j + \omega_k^2}}$$

$\rightarrow$ Case-2 , $r=0$

$I_{R2} = \int_{z_1}^{z_2} (z-z') I_2\, dz'$

$= \int_{z_1}^{z_2} (z-z') \frac{(-1)}{[r'^2+(z-z')^2]^{3/2}} \sum_{j=1}^{2} (-1)^j \cos\phi_j\, dz'$

$\therefore I_{R2} = -\sum_{j=1}^{2} (-1)^j \cos\phi_j \int_{z_1}^{z_2} \frac{(z-z')\, dz'}{(r'^2+(z-z')^2)^{3/2}}$

$\rightarrow$ take $\omega = z-z' \Rightarrow dz' = -d\omega$

$\omega_1 = z - z_1 \quad , \omega_2 = z - z_2$

$$\therefore I_{R_2} = -\sum_{j=1}^{2} (-1)^j \cos\phi_j \int_{\omega_1}^{\omega_2} \frac{-\omega\, d\omega}{(r'^2 + \omega^2)^{3/2}}$$

$$= -\sum_{j=1}^{2} (-1)^j \cos\phi_j \, I_d$$

$\rightarrow$ where $I_d = \dfrac{-1}{2} \displaystyle\int_{\omega_1}^{\omega_2} \frac{2\omega\, d\omega}{(r'^2 + \omega^2)^{3/2}}$

take $t = \omega^2$

$\Rightarrow dt = 2\omega\, d\omega$

$$\therefore I_d = \frac{-1}{2} \int (r'^2 + t)^{-3/2}\, dt$$

$$= \frac{-1}{2} \frac{(r'^2 + t)^{-1/2}}{-1/2}$$

$$= \left[\frac{1}{\sqrt{r'^2 + \omega^2}} \right]_{\omega_1}^{\omega_2}$$

$$\therefore I_d = \sum_{K=1}^{2} (-1)^K \frac{1}{\sqrt{r'^2 + \omega_K^2}}$$

$$\therefore I_{R_2} = -\sum_{j=1}^{2} (-1)^j \cos\phi_j \sum_{K=1}^{2} (-1)^K \frac{1}{\sqrt{r'^2 + \omega_K^2}}$$

$$\boxed{I_{R_2} = -\sum_{j=1}^{2} \sum_{K=1}^{2} (-1)^{j+K} \frac{\cos\phi_j}{\sqrt{r'^2 + \omega_K^2}}}$$

$\rightarrow$ Now, from eq^n (6)

$$H_z = \int_{r_1}^{r_2} dr' \int_{z_1}^{z_2} dz' \left(r'^2 I_3 - r r' I_1 \right)$$

$$= \int_{r_1}^{r_2} dr' \left(r'^2 \int_{z_1}^{z_2} I_3\, dz' - r r' \int_{z_1}^{z_2} I_1\, dz' \right)$$

$$\therefore \; H_z = \int_{r_1}^{r_2} dr' \left(r'^2 I_{z3} - r r' I_{z1} \right)$$

where, $I_{z3} = \int_{z_1}^{z_2} I_3 \, dz'$

$$I_{z1} = \int_{z_1}^{z_2} I_1 \, dz'$$

$$I_3 = \int_{\phi_1}^{\phi_2} \frac{d\phi}{(b^2 - 2rr'\cos\phi)^{3/2}} \quad \text{and} \quad I_1 = \int_{\phi_1}^{\phi_2} \frac{\cos\phi \, d\phi}{(b^2 - 2rr'\cos\phi)^{3/2}}$$

$$b^2 = r^2 + r'^2 + (z - z')^2$$

(iii) $$I_{z3} = \int_{z_1}^{z_2} \int_{\phi_1}^{\phi_2} \frac{d\phi}{(b^2 - 2rr'\cos\phi)^{3/2}} \, dz'$$

$$= \int_{\phi_1}^{\phi_2} d\phi \int_{z_1}^{z_2} \frac{dz'}{(r^2 + r'^2 + (z-z')^2 - 2rr'\cos\phi)^{3/2}}$$

take, $q = r^2 + r'^2 - 2rr'\cos\phi$

$$\therefore \; I_{z3} = \int_{\phi_1}^{\phi_2} d\phi \int_{z_1}^{z_2} \frac{dz'}{(q + (z-z')^2)^{3/2}}$$

take $\omega = z - z'$ $\qquad \omega_1 = z - z_1$

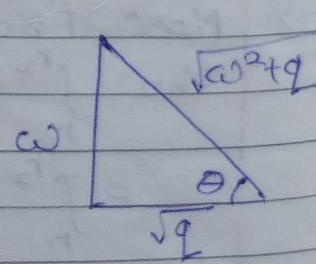
$\Rightarrow dz' = -d\omega$ $\qquad \omega_2 = z - z_2$

$$\therefore \; I_{z3} = \int_{\phi_1}^{\phi_2} d\phi \int_{\omega_1}^{\omega_2} \frac{-d\omega}{(q + \omega^2)^{3/2}}$$

$$= -\int_{\phi_1}^{\phi_2} d\phi \; I_d$$

$\rightarrow$ where,

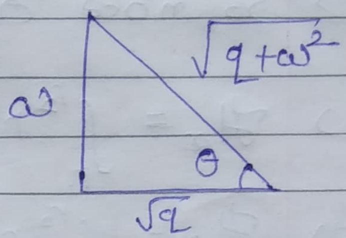
$$I_d = \int_{\omega_1}^{\omega_2} \frac{d\omega}{(q + \omega^2)^{3/2}}$$



take $\omega = \sqrt{q} \tan\theta$

$\Rightarrow d\omega = \sqrt{q} \sec^2\theta \, d\theta$

$$\therefore I_d = \int \frac{q^{1/2}\sec^2\theta \, d\theta}{(q + q\tan^2\theta)^{3/2}}$$



$$= \int \frac{q^{1/2}}{q^{3/2}} \frac{\sec^2\theta}{\sec^3\theta} \, d\theta$$

$$= \frac{1}{q} \int \cos\theta \, d\theta$$

$$= \frac{1}{q} \sin\theta = \frac{1}{q} \frac{\omega}{\sqrt{q+\omega^2}} \Bigg|_{\omega_1}^{\omega_2}$$

$$= \frac{1}{q} \sum_{k=1}^{2} (-1)^k \frac{\omega_k}{\sqrt{q+\omega_k^2}}$$

$$= \frac{1}{(r^2 + r'^2 - 2rr'\cos\phi)} \sum_{k=1}^{2} (-1)^k \frac{\omega_k}{\sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi}}$$

$$\therefore I_{z3} = -\int_{\phi_1}^{\phi_2} d\phi \, \frac{1}{q} \sum_{k=1}^{2} (-1)^k \frac{\omega_k}{\sqrt{q+\omega_k^2}}$$

$$= -\sum_{k=1}^{2} (-1)^k \int_{\phi_1}^{\phi_2} \frac{1}{q} \frac{\omega_k}{\sqrt{q+\omega_k^2}} \, d\phi$$

$$\therefore \boxed{I_{z3} = -\sum_{k=1}^{2} (-1)^k \int_{\phi_1}^{\phi_2} \frac{\omega_k \, d\phi}{(r^2 + r'^2 - 2rr'\cos\phi)\sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi}}}$$

(iv) $$I_{z1} = \int_{z_1}^{z_2} \int_{\phi_1}^{\phi_2} \frac{\cos\phi \, d\phi \, dz'}{(b^2 - 2rr'\cos\phi)^{3/2}}$$

$$= \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi \int_{z_1}^{z_2} \frac{dz'}{\left(r^2 + r'^2 + (z-z')^2 - 2rr'\cos\phi\right)^{3/2}}$$

$$\therefore \boxed{I_{z1} = -\sum_{k=1}^{2} (-1)^k \int_{\phi_1}^{\phi_2} \frac{\omega_k \cos\phi \, d\phi}{(r^2 + r'^2 - 2rr'\cos\phi)\sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi}}}$$

→ So, now our eq^n (4), (5) and (6) becomes

(4) $$H_x = \int_{r_1}^{r_2} r'\,dr' \left(\cos\psi\, I_{R1} - \sin\psi\, I_{R2} \right)$$

(5) $$H_y = \int_{r_1}^{r_2} r'\,dr' \left(\cos\psi\, I_{R2} + \sin\psi\, I_{R1} \right)$$

(6) $$H_z = \int_{r_1}^{r_2} dr' \left(r'^2 I_{Z3} - r r' I_{Z1} \right)$$

→ where,

(i) $$I_{R1} = \sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \frac{\cos\phi\, d\phi}{(r^2+r'^2+\omega_K^2-2rr'\cos\phi)^{1/2}}$$

(ii) → for I_{R2},

→ case-1, $r \neq 0$ $$I_{R2} = \frac{1}{rr'} \sum_{j=1}^{2}\sum_{K=1}^{2} (-1)^{j+K} \sqrt{r^2+r'^2+\omega_K^2-2rr'\cos\phi_j}$$

→ case-2, $r = 0$ $$I_{R2} = -\sum_{j=1}^{2}\sum_{K=1}^{2} (-1)^{j+K} \frac{\cos\phi_j}{\sqrt{r'^2+\omega_K^2}}$$

(iii) $$I_{Z3} = -\sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \frac{\omega_K\, d\phi}{(r^2+r'^2-2rr'\cos\phi)\sqrt{r^2+r'^2+\omega_K^2-2rr'\cos\phi}}$$

(iv) $$I_{Z1} = -\sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \frac{\omega_K \cos\phi\, d\phi}{(r^2+r'^2-2rr'\cos\phi)\sqrt{r^2+r'^2+\omega_K^2-2rr'\cos\phi}}$$

where, $\omega_K = z - z_K$, $K = 1, 2$

$\phi_j = \psi_j - \psi$, $j = 1, 2$

→ To solve the eq^n (4), (5) and (6) with respect to r' consider following

→ Let's Integrate with respect to r'

(10) $H_x = \cos\psi \, I_{R21} - \sin\psi \, I_{R22}$

(11) $H_y = \cos\psi \, I_{R22} + \sin\psi \, I_{R21}$

(12) $H_z = I_{Z23} - r \, I_{Z21}$

where, $I_{R21} = \int_{r_1}^{r_2} r' I_{R1} \, dr'$, $I_{R22} = \int_{r_1}^{r_2} r' I_{R2} \, dr'$

$I_{Z23} = \int_{r_1}^{r_2} r'^2 I_{Z3} \, dr'$, $I_{Z21} = \int_{r_1}^{r_2} r' I_{Z1} \, dr'$

(i) $I_{R21} = \int_{r_1}^{r_2} r' I_{R1} \, dr'$

$$= \int_{r_1}^{r_2} r' \cdot \sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \frac{\cos\phi \, d\phi}{[r^2 + r'^2 + \omega_K^2 - 2rr'\cos\phi]^{1/2}} \, dr'$$

$$= \sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi \int_{r_1}^{r_2} \frac{r' \, dr'}{[r^2 + r'^2 + \omega_K^2 - 2rr'\cos\phi]^{1/2}}$$

$$= \sum_{K=1}^{2} (-1)^K \int_{\phi_1}^{\phi_2} \cos\phi \, d\phi \, I_d$$

→ where,

$$I_d = \int_{r_1}^{r_2} \frac{r' \, dr'}{[r^2 + r'^2 + \omega_K^2 - 2rr'\cos\phi]^{1/2}}$$

tak $m^2 = r^2 + \omega_K^2$

$n = r\cos\phi$

$\therefore \quad I_d = \int_{r_1}^{r_2} \frac{r' \, dr'}{[r'^2 - 2nr' + m^2]^{1/2}}$

$\rightarrow \quad r'^2 - 2nr' + m^2 = r'^2 - 2nr' + n^2 + m^2 - n^2$

$= (r' - n)^2 + K$

where $K = m^2 - n^2$

$= r^2 + \omega_k^2 - r^2 \cos^2\phi$

$= \omega_k^2 + r^2 \sin^2\phi > 0$

$$\therefore \quad I_d = \int_{r_1}^{r_2} \frac{r' \, dr'}{[(r'-n)^2 + K]^{1/2}}$$

$$= \int_{r_1}^{r_2} \frac{r-n}{[(r'-n)^2+K]^{1/2}} dr' + n \int_{r_1}^{r_2} \frac{dr'}{[(r'-n)^2+K]^{1/2}}$$

$$= I_{d1} + n \, I_{d2}$$

$\rightarrow \quad I_{d_1} = \int_{r_1}^{r_2} \frac{r-n}{[(r'-n)^2+K]^{1/2}} dr'$

$t = (r'-n)^2 + K$

$\Rightarrow \quad dt = 2(r'-n) \, dr'$

$$\therefore \quad I_{d_1} = \frac{1}{2} \int \frac{2(r'-n) \, dr'}{[(r'-n)^2 + K]^{1/2}} = \frac{1}{2} \int \frac{dt}{t^{1/2}}$$

$$= \frac{1}{2} \frac{t^{1/2}}{1/2} = \left[\sqrt{(r'-n)^2 + K} \right]_{r_1}^{r_2}$$

$$= \sum_{i=1}^{2} (-1)^i \sqrt{(r_i - n)^2 + K}$$

$$= \sum_{i=1}^{2} (-1)^i \sqrt{r_i^2 - 2n r_i + m^2}$$

$$= \sum_{i=1}^{2} (-1)^i \sqrt{r^2 + r_i^2 + \omega_k^2 - 2 r r_i \cos\phi}$$

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$\rightarrow I_{d2} = \int_{r_1}^{r_2} \frac{dr'}{[(r'-r_1)^2 + K]^{1/2}}$

take $\quad t = r' - r_1$

$\Rightarrow dt = dr'$

$\therefore I_{d_2} = \int \frac{dt}{[t^2+K]^{1/2}} = \int \frac{dt}{\sqrt{\sqrt{K}^2 + t^2}}$

$= \ln | t + \sqrt{K + t^2} |$

$= \ln | r' - r_1 + \sqrt{K + (r'-r_1)^2} \Big|_{r_1}^{r_2}$

$= \sum_{i=1}^{2} (-1)^i \ln | r_i - r\cos\phi + \sqrt{r_i^2 + r^2 + \omega_k^2 - 2rr_i\cos\phi} |$

$\therefore I_d = \sum_{i=1}^{2} (-1)^i \Big[\sqrt{r^2 + r_i^2 + \omega_k^2 - 2rr_i\cos\phi}$

$+ r\cos\phi \ln (r_i - r\cos\phi + \sqrt{r^2 + r_i^2 + \omega_k^2 - 2rr_i\cos\phi}) \Big]$

$$\boxed{\therefore I_{R21} = \sum_{i=1}^{2} \sum_{K=1}^{2} (-1)^{i+K} \Big[\int_{\phi_1}^{\phi_2} \cos\phi \sqrt{r^2 + r_i^2 + \omega_k^2 - 2rr_i\cos\phi}\, d\phi + \int_{\phi_1}^{\phi_2} r\cos^2\phi \ln \Big(r_i - r\cos\phi + \sqrt{r^2 + r_i^2 + \omega_k^2 - 2rr_i\cos\phi} \Big) d\phi \Big]}$$

(iii) $I_{R22} = \int_{r_1}^{r_2} r' I_{R2}\, dr'$

for I_{R2},

case-1, $r = 0$: $\quad I_{R2} = -\sum_{j=1}^{2} \sum_{K=1}^{2} (-1)^{j+K} \frac{\cos\phi_j}{\sqrt{r'^2 + \omega_k^2}}$

case-2, $r \neq 0$: $\quad I_{R2} = \frac{1}{rr'} \sum_{j=1}^{2} \sum_{K=1}^{2} (-1)^{j+K} \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}$

→ Case - I $r=0$

$$I_{Rzz} = \int_{r_1}^{r_2} r' I_{Rz} dr'$$

$$= \int_{r_1}^{r_2} r' \left(- \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \frac{\cos\phi_j}{\sqrt{r'^2 + \omega_k^2}} \right) dr'$$

$$= - \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \cos\phi_j \int_{r_1}^{r_2} \frac{r'}{\sqrt{\omega_k^2 + r'^2}} dr'$$

Here $\displaystyle\int_{r_1}^{r_2} \frac{r'\, dr'}{\sqrt{\omega_k^2 + r'^2}} = \sum_{i=1}^{2} (-1)^i \sqrt{r_i^2 + \omega_k^2}$

$$\therefore \boxed{I_{Rzz} = - \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{i+j+k} \cos\phi_j \sqrt{r_i^2 + \omega_k^2}}$$

→ Case - II $r \neq 0$

$$I_{Rzz} = \int_{r_1}^{r_2} r' I_{Rz} dr'$$

Where $\displaystyle I_{Rz} = \frac{1}{rr'} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}$

$$\therefore I_{Rzz} = \int_{r_1}^{r_2} r' \frac{1}{rr'} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}\, dr'$$

$$= \frac{1}{r} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} \int_{r_1}^{r_2} \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}\, dr'$$

$$= \frac{1}{r} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{j+k} I_d$$

→ $\displaystyle I_d = \int_{r_1}^{r_2} \sqrt{r^2 + r'^2 + \omega_k^2 - 2rr'\cos\phi_j}\, dr'$

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take, $m^2 = r^2 + \omega_k^2$

$n = r\cos\phi_j$

$$\therefore I_d = \int_{r_1}^{r_2} \sqrt{r'^2 + m^2 - 2r'n}\, dr'$$

$$\rightarrow r'^2 - 2r'n + m^2 = r'^2 - 2nr' + n^2 + m^2 - n^2$$
$$= (r'-n)^2 + K$$

Where $K = m^2 - n^2$
$$= r^2 + \omega_k^2 - r^2\cos^2\phi_j$$
$$= \omega_k^2 + r^2\sin^2\phi_j > 0$$

$$\therefore I_d = \int_{r_1}^{r_2} \sqrt{K + (r'-n)^2}\, dr'$$

From, $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

$$\therefore I_d = \left[\frac{(r'-n)}{2}\sqrt{K+(r'-n)^2} + \frac{K}{2}\ln\left|r'-n+\sqrt{K+(r'-n)^2}\right|\right]_{r_1}^{r_2}$$

$$= \sum_{i=1}^{2} (-1)^i \left[\frac{(r_i - r\cos\phi_j)}{2}\sqrt{r^2+r_i^2+\omega_k^2-2rr_i\cos\phi_j} + \frac{(\omega_k^2+r^2\sin^2\phi_j)}{2}\ln\left(r_i - r\cos\phi_j + \sqrt{r^2+r_i^2+\omega_k^2-2rr_i\cos\phi_j}\right)\right]$$

$$\boxed{\therefore I_{R22} = \frac{1}{2r}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k}\left[(r_i - r\cos\phi_j)\sqrt{r^2+r_i^2+\omega_k^2-2rr_i\cos\phi_j} + (\omega_k^2+r^2\sin^2\phi_j)\ln\left(r_i - r\cos\phi_j+\sqrt{r^2+r_i^2+\omega_k^2-2rr_i\cos\phi_j}\right)\right]}$$

→ Calculations for H_z follow the same pattern.

→ Next page involves final eq^{n}s for H_x, H_y, and H_z.

→ These eq^{n}s are used in code.

(7) $H_x = \cos\psi \; I_{Rz1} - \sin\psi \; I_{Rz2}$

(8) $H_y = \cos\psi \; I_{Rz2} + \sin\psi \; \cancel{I_{Rz2}} \; I_{Rz1}$

$$\rightarrow \; I_{Rz1} = \sum_{i=1}^{2} \sum_{k=1}^{2} (-1)^{i+k} \left[\int_{\phi_1}^{\phi_2} \cos\phi \sqrt{r^2 + r_i^2 + \omega_k^2 - 2 r r_i \cos\phi} \, d\phi \right.$$

$$\left. + \int_{\phi_1}^{\phi_2} r \cos^2\phi \, \ln\left(r_i - r\cos\phi + \sqrt{r^2 + r_i^2 + \omega_k^2 - 2 r r_i \cos\phi} \right) d\phi \right]$$

$\rightarrow$ for I_{Rz2}

Case - 1 $\quad r = 0$

$$I_{Rz2} = - \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{i+j+k} \cos\phi_j \sqrt{r_i^2 + \omega_k^2}$$

Case - 2 $\quad r \neq 0$

$$I_{Rz2} = \frac{1}{2r} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \left[(r_i - r\cos\phi_j) \sqrt{r^2 + r_i^2 + \omega_k^2 - 2 r r_i \cos\phi_j} \right.$$

$$\left. + (\omega_k^2 + r^2 \sin^2\phi_j) \ln\left(r_i - r\cos\phi_j + \sqrt{r^2 + r_i^2 + \omega_k^2 - 2 r r_i \cos\phi_j} \right) \right]$$

(9) $H_z = -\omega_k I_{zz1} + \omega_k I_{zz2}$

$\rightarrow$ $$I_{zz1} = \sum_{i=1}^{2} \sum_{k=1}^{2} (-1)^{i+k} \omega_k \int_{\phi_1}^{\phi_2} \ln\left(r_i - r\cos\phi + \sqrt{r^2 + r_i^2 + \omega_k^2 - 2 r r_i \cos\phi}\right) d\phi$$

$\rightarrow$ $$I_{zz2} = \sum_{k=1}^{2} (-1)^{k} \int_{r_1}^{r_2} \int_{\phi_1}^{\phi_2} \frac{\omega_k (r^2 - r r' \cos\phi)}{(r^2 + r'^2 - 2 r r' \cos\phi)\sqrt{r^2 + r'^2 + \omega_k^2 - 2 r r' \cos\phi}} \, d\phi \, dr'$$

also, $\phi_j = \psi_j - \psi$, $j = 1, 2$

$\omega_k = z - z_k$, $k = 1, 2$