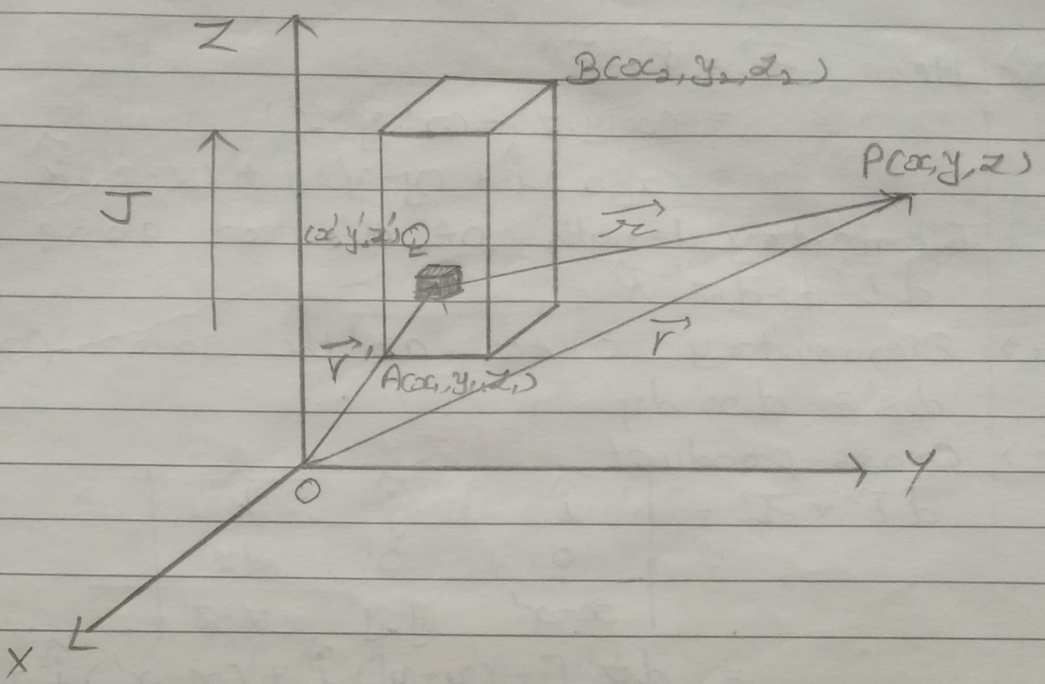
Magnetic field due to a conducting bar of rectangular cross section



→ A current carrying conductor of rectangular cross section is placed parallel to z-axis.

→ Here,

(1) Source point , $Q(x', y', z')$

(2) Observation Point , $P(x, y, z)$

(3) $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two dimetrically opposite points.

(4) The electric current density J is flowing along z direction.

→ The magnetic field $\vec{B}$ at point P can be written as,

$$\vec{B} = \frac{\mu_0 J}{4\pi} \int_A^B ds' \frac{d\vec{l}' \times \vec{r}}{|\vec{r}|^3}$$

$$= \frac{\mu_0 J}{4\pi} \vec{H}$$

→ where,

$$\vec{H} = \int_A^B ds' \frac{d\vec{l}' \times \vec{r}}{|\vec{r}|^3} \qquad (1)$$

→ Here, $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$= (x, y, z) - (x', y', z')$

$= (x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}$

→ Elementry length of the rod along z-axis,

$d\vec{l}' = dz'\,\hat{k}$

→ Elementry area at the source point

$ds' = dx'\,dy'$

→ Cross-product,

$$d\vec{l}' \times \vec{\mathscr{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & dz' \\ x - x' & y - y' & z - z' \end{vmatrix}$$

$$= dz'\left[-(y - y')\hat{i} + (x - x')\hat{j} + 0\hat{k}\right]$$

→ $|\vec{\mathscr{r}}| = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{1/2}$

→ Now, Sub. these all values in eq^n ① we get,

$$\vec{H} = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \frac{(-(y - y')\hat{i} + (x - x')\hat{j} + 0\hat{k})\,dx'\,dy'\,dz'}{\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{3/2}}$$

→ The component of $\vec{H}$ are,

(1) $$H_x = -\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \frac{(y - y')\,dx'\,dy'\,dz'}{\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{3/2}}$$

(2) $$H_y = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \frac{(x - x')\,dx'\,dy'\,dz'}{\left[(x - x')^2 + (y - y') + (z - z')^2\right]^{3/2}}$$

(3) $H_z = 0.0$

→ Let, $u = x - x'$, $v = y - y'$, $\omega = z - z'$

$\Rightarrow u_1 = x - x_1$, $\Rightarrow v_1 = y - y_1$, $\Rightarrow \omega_1 = z - z_1$

$\Rightarrow u_2 = x - x_2$, $\Rightarrow v_2 = y - y_2$, $\Rightarrow \omega_2 = z - z_2$

and $du = -dx'$, $dv = -dy'$, $d\omega = -dz'$

→ Now, Substitude all these values in (1),(2),(3) we get,

$$(4)\quad H_x = \int_{u_1}^{u_2}\int_{v_1}^{v_2}\int_{\omega_1}^{\omega_2} \frac{v\, du\, dv\, d\omega}{(u^2+v^2+\omega^2)^{3/2}}$$

$$(5)\quad H_y = -\int_{u_1}^{u_2}\int_{v_1}^{v_2}\int_{\omega_1}^{\omega_2} \frac{u\, du\, dv\, d\omega}{(u^2+v^2+\omega^2)^{3/2}}$$

$$(6)\quad H_z = 0.0$$

→ Let us solve H_x first

$$H_x = \int_{u_1}^{u_2} du \int_{\omega_1}^{\omega_2} d\omega \int_{v_1}^{v_2} \frac{v\, dv}{[u^2+v^2+\omega^2]^{3/2}}$$

$$= \int_{u_1}^{u_2} du \int_{\omega_1}^{\omega_2} d\omega\; I_1$$

$$I_1 = \int \frac{v\, dv}{[u^2+v^2+\omega^2]^{3/2}}$$

$$= \frac{1}{2}\int \frac{2v\, dv}{[a^2+v^2]^{3/2}} \quad , \quad a^2 = u^2+\omega^2$$

take $t = v^2 \Rightarrow dt = 2v\,dv$

$$\therefore I_1 = \frac{1}{2}\int [a^2+t]^{-3/2}\, dt$$

$\therefore I_1 = \frac{1}{2} \frac{(a^2+t)^{-1/2}}{-1/2}$

$= \left[\frac{-1}{\sqrt{u^2+\omega^2+v^2}} \right]_{v_1}^{v_2}$

$= -\sum_{j=1}^{2} (-1)^j \frac{1}{\sqrt{u^2+v_j^2+\omega^2}}$

→ Now, again,

$\therefore H_x = -\int_{\omega_1}^{\omega_2} d\omega \int_{u_1}^{u_2} \sum_{j=1}^{2} (-1)^j \frac{du}{\sqrt{u^2+v_j^2+\omega^2}}$

$= -\sum_{j=1}^{2} (-1)^j \int_{\omega_1}^{\omega_2} d\omega \int_{u_1}^{u_2} \frac{du}{\sqrt{u^2+v_j^2+\omega^2}}$

We know that, $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} \equiv \log\left|x+\sqrt{x^2+a^2}\right|$

$\therefore H_x = -\sum_{j=1}^{2} (-1)^j \int_{\omega_1}^{\omega_2} d\omega \left[\sinh^{-1}\left(\frac{u}{\sqrt{\omega^2+v_j^2}} \right) \right]_{u_1}^{u_2}$

$= -\sum_{j=1}^{2} (-1)^j \int_{\omega_1}^{\omega_2} \sum_{i=1}^{2} (-1)^i \sinh^{-1}\left(\frac{u_i}{\sqrt{\omega^2+v_j^2}} \right) d\omega$

$= -\sum_{j=1}^{2} \sum_{i=1}^{2} (-1)^{i+j} \int_{\omega_1}^{\omega_2} \sinh^{-1}\left(\frac{u_i}{\sqrt{\omega^2+v_j^2}} \right) d\omega$

$= -\sum_{j=1}^{2} \sum_{i=1}^{2} (-1)^{i+j} I$ ---------(7)

→ where,

$I = \int \sinh^{-1}\left(\frac{u_i}{\sqrt{\omega^2+v_j^2}} \right) d\omega$

$= \omega \sinh^{-1}\left(\frac{u_i}{\sqrt{\omega^2+v_j^2}} \right) - \int \omega \frac{d}{d\omega}\left(\sinh^{-1}\left(u_i [\omega^2+v_j^2]^{-1/2} \right) \right) d\omega$

Here, $\frac{d}{d\omega} \sinh^{-1}\left(\frac{u_i (\omega^2+v_j^2)^{-1/2}}{1} \right)$

$= \frac{1}{\sqrt{1+u_i^2((\omega^2+v_j^2)^{-1/2})^2}} \frac{d}{d\omega}\left(u_i (\omega^2+v_j^2)^{-1/2} \right)$

$$= \frac{u_i}{\sqrt{1+u_i^2(\omega^2+v_j^2)^{-1}}}\left(\frac{-1}{2}\right)(\omega^2+v_j^2)^{-3/2}\,2\omega$$

$$= \frac{-u_i\,(\omega^2+v_j^2)^{1/2}}{\sqrt{u_i^2+v_j^2+\omega^2}}\;\frac{\omega}{(\omega^2+v_j^2)^{3/2}}$$

$$= \frac{-u_i\,\omega}{(\omega^2+v_j^2)\sqrt{\omega^2+u_i^2+v_j^2}}$$

$$\therefore\; I = \omega\sinh^{-1}\left(\frac{u_i}{\sqrt{\omega^2+v_j^2}}\right) + u_i\int\frac{\omega^2\,d\omega}{(\omega^2+v_j^2)\sqrt{\omega^2+u_i^2+v_j^2}}$$

$$= \omega\sinh^{-1}\left(\frac{u_i}{\sqrt{\omega^2+v_j^2}}\right) + u_i I_1$$

$\rightarrow$ where,

$$I_1 = \int\frac{\omega^2\,d\omega}{(\omega^2+v_j^2)\sqrt{\omega^2+u_i^2+v_j^2}}$$

$$= \int\frac{\omega^2+v_j^2-v_j^2}{(\omega^2+v_j^2)\sqrt{\omega^2+u_i^2+v_j^2}}\,d\omega$$

$$= \int\frac{d\omega}{\sqrt{\omega^2+u_i^2+v_j^2}} - v_j^2\int\frac{d\omega}{(\omega^2+v_j^2)\sqrt{\omega^2+u_i^2+v_j^2}}$$

$$= \sinh^{-1}\left(\frac{\omega}{\sqrt{u_i^2+v_j^2}}\right) - v_j^2 I_2$$

$$\therefore\; I = \omega\sinh^{-1}\left(\frac{u_i}{\sqrt{\omega^2+v_j^2}}\right) + u_i\sinh^{-1}\left(\frac{\omega}{\sqrt{u_i^2+v_j^2}}\right) - u_i v_j^2 I_2$$

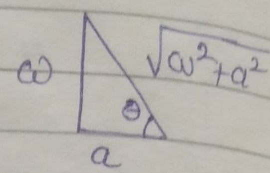
$\rightarrow$ where,

$$I_2 = \int\frac{d\omega}{(\omega^2+v_j^2)\sqrt{\omega^2+u_i^2+v_j^2}}$$

$$\therefore\; I_2 = \int\frac{d\omega}{(\omega^2+d^2)\sqrt{\omega^2+a^2}}\,,\qquad a^2 = u_i^2+v_j^2,\quad d = v_j$$

$\rightarrow$ Put $\omega = a\tan\theta$

$\Rightarrow d\omega = a\sec^2\theta \, d\theta$



$$\therefore I_2 = \int \frac{a\sec^2\theta \, d\theta}{(a^2\tan^2\theta + d^2)\sqrt{a^2\tan^2\theta + a^2}}$$

$$= \int \frac{\sec\theta \, d\theta}{(d^2 + a^2\tan^2\theta)}$$

$$= \int \frac{\cos\theta \, d\theta}{d^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \int \frac{\cos\theta \, d\theta}{d^2 + (a^2 - d^2)\sin^2\theta}$$

$\rightarrow$ take $t = \sin\theta$

$\Rightarrow dt = \cos\theta \, d\theta$

$$\therefore I_2 = \int \frac{dt}{d^2 + (a^2 - d^2)t^2}$$

$$= \frac{1}{a^2 - d^2} \int \frac{dt}{\left(\frac{d^2}{a^2 - d^2}\right) + t^2}$$

$$= \frac{1}{a^2 - d^2} \frac{1}{\sqrt{\frac{d^2}{a^2 - d^2}}} \tan^{-1}\left(\frac{t}{\sqrt{\frac{d^2}{a^2 - d^2}}}\right)$$

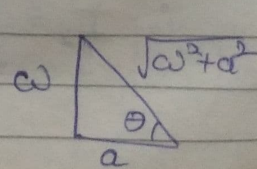
$$\left(\because \text{from} \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$\therefore I_2 = \frac{1}{d\sqrt{a^2 - d^2}} \tan^{-1}\left(\frac{\sqrt{a^2 - d^2}}{d} t\right)$$

$$= \frac{1}{v_j\sqrt{u_i^2 + v_j^2 - v_j^2}} \tan^{-1}\left(\frac{\sqrt{u_i^2 + v_j^2 - v_j^2}}{v_j} \sin\theta\right)$$

$$= \frac{1}{v_j u_i} \tan^{-1}\left(\frac{u_i}{v_j} \sin\theta\right)$$

$$= \frac{1}{v_j u_i} \tan^{-1}\left(\frac{u_i}{v_j} \frac{\omega}{\sqrt{\omega^2 + a^2}}\right)$$



$$\therefore \; I = \omega \sinh^{-1}\left(\frac{u_i}{\sqrt{\omega^2+v_j^2}}\right) + u_i \sinh^{-1}\left(\frac{\omega}{\sqrt{u_i^2+v_j^2}}\right) - v_j \tan^{-1}\left(\frac{u_i}{v_j}\frac{\omega}{\sqrt{\omega^2+u_i^2+v_j^2}}\right)$$

→ Sub. this value in eq^n (7) we get

$$\therefore \; H_x = -\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k}\left[\omega_k \sinh^{-1}\left(\frac{u_i}{\sqrt{\omega_k^2+v_j^2}}\right) + u_i \sinh^{-1}\left(\frac{\omega_k}{\sqrt{u_i^2+v_j^2}}\right) - v_j \tan^{-1}\left(\frac{u_i}{v_j}\frac{\omega_k}{\sqrt{\omega_k^2+u_i^2+v_j^2}}\right)\right]$$

→ Similarly we can prove that

$$H_y = -\int_{u_1}^{u_2}\int_{v_1}^{v_2}\int_{\omega_1}^{\omega_2} \frac{u\,du\,dv\,d\omega}{(u^2+v^2+\omega^2)^{3/2}}$$

$$= -\int_{v_1}^{v_2} dv \int_{\omega_1}^{\omega_2} d\omega \left(-\sum_{i=1}^{2} \frac{(-1)^i}{\sqrt{u_i^2+v^2+\omega^2}}\right)$$

$$= \sum_{i=1}^{2}(-1)^i \int_{\omega_1}^{\omega_2} d\omega \int_{v_1}^{v_2} \frac{dv}{\sqrt{u_i^2+v^2+\omega^2}}$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\int_{\omega_1}^{\omega_2} \sinh^{-1}\left(\frac{v_j}{\sqrt{u_i^2+\omega^2}}\right) d\omega$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k}\left[\omega_k \sinh^{-1}\left(\frac{v_j}{\sqrt{\omega_k^2+u_i^2}}\right) + v_j \sinh^{-1}\left(\frac{\omega_k}{\sqrt{u_i^2+v_j^2}}\right) - u_i \tan^{-1}\left(\frac{v_j}{u_i}\frac{\omega_k}{\sqrt{\omega_k^2+u_i^2+v_j^2}}\right)\right]$$

and, $H_z = 0.0$

→ Let,

$$J(u_i, v_j, \omega_k) = -\left[\omega_k \sinh^{-1}\left(\frac{u_i}{\sqrt{\omega_k^2+v_j^2}}\right) + u_i \sinh^{-1}\left(\frac{\omega_k}{\sqrt{u_i^2+v_j^2}}\right) - v_j \tan^{-1}\left(\frac{u_i}{v_j}\frac{\omega_k}{\sqrt{u_i^2+v_j^2+\omega_k^2}}\right)\right]$$

$$J(v_j, u_i, \omega_k) = -\left[\omega_k \sinh^{-1}\left(\frac{v_j}{\sqrt{\omega_k^2+u_i^2}}\right) + v_j \sinh^{-1}\left(\frac{\omega_k}{\sqrt{u_i^2+v_j^2}}\right) - u_i \tan^{-1}\left(\frac{v_j}{u_i}\frac{\omega_k}{\sqrt{u_i^2+v_j^2+\omega_k^2}}\right)\right]$$

$$\therefore H_x = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k} J(u_i, v_j, \omega_k)$$

$$H_y = -\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k} J(v_j, u_i, \omega_k)$$

$$H_z = 0.0$$

→ Hence, The magnetic field components are,

1. $B_x = \frac{\mu_0 J}{4\pi}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k} J(u_i, v_j, \omega_k)$
2. $B_y = -\frac{\mu_0 J}{4\pi}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k} J(v_j, u_i, \omega_k)$
3. $B_z = 0.0$

→ where, $u_i = x - x_i$, $v_j = y - y_j$, $\omega_k = z - z_k$

$i = 1, 2$ $j = 1, 2$ $k = 1, 2$

* References:-

1 Analytical derivation of magnetic field and vector potential due to different current carrying conducting geometries, IPR.

2 Special thanks to Gattu Ramesh sir