Differential Equations (MA 1150)

Neeraj Kumar

Department of Mathematics, IIT Hyderabad

Lecture 1

March 26, 2019

Information about this course

- Instructor: Neeraj Kumar
- Office: Academic Block C, 208 -E
- Email: neeraj@iith.ac.in

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- Reference Textbooks
  - Advanced Engineering Mathematics by Erwin Kreyszig, 8th Edition, John Wiley and Sons (1999).
  - Elementary Differential Equations by William Trench, available at ramanujan.math.trinity.edu/wtrench/texts/index.shtml.
- Office hour for meeting: 10:00 am to 11:30 am (Monday and Wednesday)

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- 2 Assignments 5 marks each. Main Quiz 5 marks.

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- Minimum passing marks is 20.

Attendance policy

Attendance 100% This is compulsory.

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Surprise quiz may happen in any lecture.

Absent in lectures/quizzes due to health reason. You need to produce medical report slip (of same day) from OPD, IITH Hospital and drop an email reporting absence.

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- In April: 2nd, 5th, 9th, 12th, 16th, –, 23rd, and 26th.

Examination

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Examination

- 1st Assignment April 5, 2019 (5 marks)
- Main Quiz April 12, 2019 (5 marks)
- 2nd Assignment April 16, 2019 (5 marks)

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- Final examination Saturday April 27, 2019 from 8:30 am to 10:30 am.
- About final exam evaluation !!!
  - Solution will be uploaded at 11:30 am, April 27, 2019.
  - Comments on solution till 2:00 pm, April 27, 2019. Updated solution and marking scheme will be finalized.
- Will update about when to show your answer sheet !!!

Course content

Advanced Engineering Mathematics by Erwin Kreyszig.

- Chapter 1: First-Order Differential Equations,
- Chapter 2: Linear Differential Equations of Second and Higher Order.

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Timeline.

- Chapter 1 in Lectures $1-4$,
- Chapter 2 in Lectures $5-8$,
- Revision in Lecture 9.

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Assignment: A good strategy is to try each problem yourself first, then get together with others to discuss your solutions and questions, and finally write up the solutions yourself neatly.

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Assignment: A good strategy is to try each problem yourself first, then get together with others to discuss your solutions and questions, and finally write up the solutions yourself neatly.

Goal is to sharpen your mathematical writing skills, and homework and assignments are a place to practice.

Ordinary Differential Equation

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- Ordinary differential equation by ODE.
- Derivative of y by y', $\dfrac{dy}{dx}$ or $y^{(1)}$.

The order of an ODE is the highest order of derivative of y occurring in the ODE.

Examples.

- $y' = x^3y^4 + y$ is a 1st order ODE.
- $y'' + x^5y' + y = \cos x$ is a 2nd order ODE.
- $y^{(4)} + xy^{(1)}y^{(2)} + 2xy = \sin x$ is a 4th order ODE.

Open Intervals

Given any $a, b \in \mathbb{R}$, we define the open interval from a to b to be the set

$$(a, b) = \{x \in \mathbb{R} \ : \ a < x < b\},$$

and the closed interval from a to b to be the set $[a, b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$.

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It is often useful to consider the symbols ∞ (called infinity) and $-\infty$ (called minus infinity), which may be thought as the fictional (right and left) endpoints of the number line.

We use the symbols ∞ and $-\infty$ to define, for any $a \in \mathbb{R}$, the following semi-infinite (open) intervals:

$$(-\infty, b) = \{x \in \mathbb{R} \,:\, x < b\} \text{ and } (a, \infty) = \{x \in \mathbb{R} \,:\, a < x\}.$$

The set $\mathbb{R}$ can also be thought of as the doubly infinite (open) interval $(-\infty, \infty)$.

Solution of an ODE

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Note that $y(x) = \sin x + c$ is a solution to this ODE on $\mathbb{R}$, where c is arbitrary constant. (Why)

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Question Do ODE always have solutions?

Home work: Revise the definition of explicit and implicit function!!

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General and Particular solution $y = \sin x + c$ is a general solution and if we choose specific c, e.g. say $c = 1$, then $y = \sin x + 1$ is a particular solution.

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Question Can we recover all possible solutions of a given ODE from general solution? (Think! you may refer to Textbook).

Many scientific and in particular physical laws appear in the form of differential equations.

Applications. Modeling. Initial Value Problems

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Examples Radioactive substance decay, Growth of bacteria, Falling stone from certain height, Pendulum movement, Population growth etc. (For more details with explanations, please refer to textbook).

Initial Value Problem (IVP)

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$$y' = f(x, y), \text{ where } y(x_0) = y_0.$$

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Solution to IVP. A function $y = y(x)$ defined on some open interval (a, b) containing x_0 is a solution of the IVP if y satisfies the ODE on open interval (a, b) and $y(x_0) = y_0$.

Separation of Variables to solve ODE: Example

Solve. $y' + 2xy = 0$?

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We can solve this equation by separating the variables.

$$\begin{aligned}
& \frac{1}{y}dy = -2x\ dx \\
\Longrightarrow \quad & \int \frac{1}{y}dy = \int -2x\ dx, \\
\Longrightarrow \quad & \ln|y| = -x^2 + c, \\
\Longrightarrow \quad & y = c_1 e^{-x^2}.
\end{aligned}$$

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Question. Is this y solution to given ODE?

Separation of Variables: General Method

Let $M(x, y) + N(x, y)\dfrac{dy}{dx} = 0$ be a differential equation.

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Assume that ODE is separable.

Let H_1 and H_2 be anti derivatives of M and N respectively. This means ${H_1}'(x) = M(x)$ and ${H_2}'(y) = N(y)$. Then our ODE is

$$
{H_1}'(x) + {H_2}'(y)\frac{dy}{dx} = 0
$$

Separation of Variables: General Method (continued...)

${H_1}'(x) + {H_2}'(y)\dfrac{dy}{dx} = 0$ can be re-written as

$$\frac{d}{dx}\left[H_1(x) + H_2(y(x))\right] = 0.$$

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In general, the separable variables method only gives us the implicit solution to the given ODE.

Random attendance

Roll no.	Name	
CE18BTECH11001	ANURAAG CHANDRA SHUKLA	Present
CE18BTECH11002	ARUPULA BHAVIKA	Present
CE18BTECH11024	MANISH YADAV	Present
CH18BTECH11005	ANISH KUMAR PURBEY	Present
CH18BTECH11006	BHUKYA NANDINI	Present
CS17BTECH11023	M SAI SUBODH	Present
CS17BTECH11033	SYAMALA VENU PRIYA	Absent
EE17BTECH11007	AMRIT PARIMI	Absent
EE17BTECH11031	PRANJAL SINGH	Absent
EE18BTECH11006	C SHRUTI	Present
EE18BTECH11013	DIVYANSH MADURIYA	Present
EE18BTECH11036	PIYUSH KUMAR UTTAM	Present
EP18BTECH11002	AVANEESH SINGH	Present

Random attendance

Roll no.	Name	
EP18BTECH11012	RAJDEEP MUKESH AGRAWAL	Present
ES17BTECH11020	SIDDHANT SINGH CHAUHAN	Absent
ES17BTECH11028	SOURADEEP CHATTERJEE	Absent
ES18BTECH11010	ROHIT REDDY KOMATIREDDI	Present
ES18BTECH11023	VINOD CHOUPAL	Present
MA18BTECH11001	ANURAG REDDY KARRI	Present
MA18BTECH11009	RITESH PARMAR	Present
ME18BTECH11001	ABHIJEET UMESH GUNJAL	Present
ME18BTECH11012	GOVIND SEHRAWAT	Present
ME18BTECH11035	SAHAJ GUPTA	Present
MS17BTECH11001	ABHINAV GOYAL	Present
MS17BTECH11011	NIKHIL SACHAN	Present

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Remark Note that $y \equiv 0$ is also a solution. However this solution cannot be obtained for any choice of c.

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Remark Note that $y \equiv 0$ is also a solution. However this solution cannot be obtained for any choice of c. (Does this remind you of something!!!)

Converting to Separable Equation

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We can make this ODE separable by changing our variables.

Let $v = \dfrac{y}{x}$ or $y = vx$. Then $y' = v'x + v$.

Substituting we get $v'x = 1$.

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Question. Is this y solution to given ODE? Under what conditions, y will be solution to given ODE?

- Any non linear ODE of the form $y' = q\left(\frac{y}{x}\right)$ can be converted into a separable ODE by substituting $y = vx$.

Example

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$$v'x + v = v^2 + v - 1.$$

Re-write it as

$$\frac{v'}{v^2 - 1} = \frac{1}{x}$$

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Re-write it as

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1.$$

Substituting $y = vx$, we get

$$v'x + v = v^2 + v - 1.$$

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Integrating, we get

$$\frac{1}{2}\left(\ln|v-1| - \ln|v+1|\right) = \ln|x| + c_1.$$

Example (Continued ...)

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Both $y = x$ and $y = -x$ are also solutions, but only $y = x$ can be obtained by choosing a particular value of $c = 0$.

Converting to Separable Equation

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Use substitution $X = x + 2$, and $Y = y - 3$.

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Question. Is this y an explicit solution of ODE $y' + ay = 0$?

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Consider the simplest example of an ordinary differential equation (ODE)

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Clearly, this solution is defined on any open interval on which $f(x)$ is defined.

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is a solution of ODE on $\mathbb{R}$.

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Let us assume that $a(x)$ is a function defined on the interval $I = (x_0 - \epsilon, x_0 + \epsilon)$. Define a function

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Example

Example Solve the ODE: $y' - 2xy = 0.$

The function $a(x) = 2x$ is defined on all of $\mathbb{R}$.

Thus, the solution is given by

$$y(x) = c \exp\left(\int_0^x 2s ds\right) = c\ e^{x^2}.$$

Existence and Uniqueness Theorem

Theorem Let $D = (a, b) \times (c, d)$ be an open rectangle containing the point (x_0, y_0) and consider the IVP

$$y' = f(x, y), \text{ where } y(x_0) = y_0.$$

(a) (Existence) Assume $f(x, y)$ is continuous on D. Then IVP has at least one solution on some interval $(a_1, b_1) \subset (a, b)$ containing x_0.

(b) (Uniqueness) If both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on D, then IVP has a unique solution on some interval $(a', b') \subset (a, b)$ containing x_0.