Linear Algebra, Numerical and Complex Analysis (MA11004)
Department of Mathematics
Indian Institute of Technology Kharagpur
Tutorial Sheet 10, Spring 2025

Topics: Complex line integrals, Cauchy's integral theorem, Cauchy's integral formula.

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1. (a) Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)dz$, along the paths
   (i) $x = t + 1, y = 2t^2 - 1$,
   (ii) the straight line joining $1 - i$ and $2 + i$.

   (b) Evaluate $\int_{0}^{1+i} (x - y + ix^2)dz$ along
   (i) the straight line joining $z = 0$ and $z = 1 + i$,
   (ii) real axis from $z = 0$ to $z = 1$ and then a line parallel to imaginary axis from $z = 1$ to $z = 1+i$.

   (c) Find the value of the integral $\int_C (z + 1)^2 dz$, where C is the boundary of the rectangle with vertices at the points $1 + i$, $-1 + i$,$-1 - i$ and $1 - i$.

   (d) Compute $\int_\Gamma |z|dz$, where Γ is the left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.

   (e) Find the value of $\int_C (z^2 - iz)dz$ along the curve C which is $y = x^3 - 3x^2 + 4x - 1$ joining points $(1, 1)$ and $(2, 3)$.

2. (a) Verify that the value of the integral $\int_C z^2 dz$ is same for the following cases:
   (i) C is the straight line joining the points $A = (0, 0)$ and $B = (1, 2)$.
   (ii) C is the straight line path from $A = (0, 0)$ to $P = (1, 0)$ followed by the straight line path from $P = (1, 0)$ to $B = (1, 2)$.
   (iii) C is the parabolic path $y = 2x^2$ joining the points $A = (0, 0)$ and $B = (1, 2)$.

   (b) Integrate the function $f(z) = xz$ along the straight line from $A = (1, 1)$ to $B = (2, 4)$ in the complex plane. Is the value same if the path of integration from A to B is along the curve $x = t$, $y = t^2$?

   (c) Evaluate the function f defined by the integral $f(z) = \oint_{|w|=1} \frac{e^{w^2}-1}{w-z} dw$.

3. Suppose $F(a)=\oint_C \dfrac{(4z^2 + z + 5)}{(z - a)} dz$, where C is the ellipse $(\frac{x}{2})^2+(\frac{y}{3})^2 = 1$ taken in the counter clockwise direction. Find $F(3.5)$, $F(i)$, $F'(-1)$ and $F''(-i)$.

4. (a) Evaluate $\oint_C \dfrac{3z^2 + z + 1}{(z^2 - 1)(z + 3)} dz$, where C is the circle $|z| = 2$.

   (b) Compute $\oint_C \dfrac{e^{z^2}}{(z - 2)} dz$ over the contour C, where $C : |z - (2 + i)| = 3$.

   (c) Compute $\oint_C \dfrac{1}{(z^2 + 4)^2} dz$ over the contour $C : |z - i| = \frac{3}{2}$.

(d) Evaluate $\oint_C \frac{1}{(z-a)^n}dz$, where n is any integer and C is any closed curve containing the point a.

5. (a) Show that $\left|\oint_C \frac{e^z}{z^2+1}dz\right| \le \frac{4}{3}\pi e^2$ where $C : |z| = 2$.

   (b) Estimate an upper bound for $\left|\oint_{|z|=3} \frac{\text{Log}(z)}{z-4i}dz\right|$.

6. (a) Find the value of $\oint_{|z|=1} \frac{4z^2-4z+1}{(z-2)(z^2+4)}dz$.

   (b) Compute $\oint_{|z+1-i|=2} \frac{z+4}{z^2+2z+5}dz$.

   (c) Compute $\oint_{|z|=6} (\frac{e^{2iz}}{z^4} - \frac{z^4}{(z-i)^3})dz$.

   (d) Evaluate the integral $\oint_{|z|=1} \frac{dz}{2-\bar{z}}$.

7. Evaluate $\oint_C \frac{z^2+1}{z^2-1}$, where C is the circle

   (a) $|z| = \frac{3}{2}$.
   (b) $|z-1| = 1$.
   (c) $|z| = \frac{1}{2}$.

8. Verify the Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2i$ if C is

   (a) the circle $|z| = 1$.
   (b) the circle $|z-1| = 2$.
   (c) the ellipse $|z-3i| + |z+3i| = 20$.

9. If $0 < r < R$, evaluate the integral $I = \oint_C \frac{R+z}{z(R-z)}dz$, where C:$|z| = r$.
   Further using this result deduce that

   (a) $\int_0^{2\pi} \frac{d\theta}{R^2 - 2rR\cos\theta + r^2} = \frac{2\pi}{R^2-r^2}$.

   (b) $\int_0^{2\pi} \frac{\sin\theta d\theta}{R^2 - 2rR\cos\theta + r^2} = 0$.

10. By integrating $f(z) = \frac{1}{(R-z)}$ over C:$|z| = r$, $0 < r < R$ and using Problem 10, show that $\int_0^{2\pi} \frac{R\cos\theta}{R^2 - 2rR\cos\theta + r^2}d\theta = \frac{2\pi r}{R^2-r^2}$.

11. Evaluate $I = \int_{|z|=r} \frac{|dz|}{|z-\alpha|^2}$, where α is constant with $|\alpha| \ne r$.