Linear Algebra, Numerical and Complex Analysis (MA11004)
Department of Mathematics
Indian Institute of Technology Kharagpur
Tutorial Sheet 9, Spring 2025

Topics: Complex Analysis: Limits, Continuity, Differentiability, Cauchy-Riemann equations, Analytic functions.

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1. Find the following limits (if exist):

   (a) $\lim_{z\to 0} \dfrac{\bar{z}}{z}$.

   (b) $\lim_{z\to 0} \dfrac{\mathrm{Im}(z)}{|z|}$.

   (c) $\lim_{z\to 1+i} (z^2 - 5z + 10)$.

   (d) $\lim_{z\to 0} \left[\dfrac{1}{1-e^{\frac{1}{y}}} + iz^2\right]$.

2. Test the continuity of the following functions at $z = 0$:

   (a) $f(z) = \begin{cases} \frac{\mathrm{Re}(z)\,\mathrm{Im}(z)}{|z|^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

   (b) $f(z) = \begin{cases} \frac{\mathrm{Re}(z)}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

   (c) $f(z) = \begin{cases} \frac{\mathrm{Im}(z^2)}{z^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

3. Test the differentiability of the following functions at $z = 0$:

   (a) $f(z) = |z|$.

   (b) $f(z) = \mathrm{Re}(z)$.

   (c) $f(z) = |z|^2$.

4. Let
$$f(z) = \begin{cases} \frac{\bar{z}^3}{z^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$
   Show that

   (a) $f(z)$ is continuous everywhere on $\mathbb{C}$.

   (b) $f'(0)$ does not exist.

5. Show that the function $f(z) = |\,\mathrm{Re}(z)\,\mathrm{Im}(z)|^{1/2}$ satisfies the Cauchy-Riemann equations at $z = 0$, but $f'(0)$ does not exist.

6. Show that following functions are harmonic and find their harmonic conjugates:

(a) $u(x, y) = 4xy - x^3 + 3xy^2$.

(b) $u(x, y) = e^{-x}(x \sin y - y \cos y)$.

(c) $u(x, y) = x^3 - 3xy^2$.

(d) $u(r, \theta) = r^2 \sin 2\theta$.

7. Using Cauchy-Riemann equations, show that $f(z) = (1 + 2i)x^2y^2$ is nowhere analytic.

8. Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{|z|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Show that $f(z)$ is continuous everywhere, but nowhere analytic on $\mathbb{C}$.

9. Let $f(z) = u + iv$ be analytic in a domain D. Prove that f is constant in D if any one of the followings hold:

(a) $f'(z)$ vanishes in D.

(b) $\operatorname{Re} f(z) = u =$ constant.

(c) $\operatorname{Im} f(z) = v =$ constant.

(d) $|f(z)| =$ constant (non zero).

10. Show that there exist no analytic function f such that $\operatorname{Re} f(z) = y^2 - 2x$.

11. Show that the function $f(z) = (\bar{z} + 1)^3 - 3\bar{z}$ is nowhere analytic.

12. If $f(z)$ is analytic and $Re(f(z)) = x^2 - y^2$ then find $f(z)$. Also, If $f(0) = 0$, then find $f(3 + 2i)$.