Context Free Grammars.

$$G = (N, \Sigma, P, S).$$

N - finite set (non-terminal symbols).

Σ - finite set (terminal symbols)

Assumption : $N \cap \Sigma = \phi$.

P - finite subset of $N \times (N \cup \Sigma)^*$ [productions]

Language generated by G.

$$L(G) = \{ x \in \Sigma^* \mid S \xrightarrow[G]{*} x \}$$

$B \subseteq \Sigma^*$ is a context free language (CFL) if

$B = L(G)$ for some CFG G.

Balanced Parenthesis.

$G = (N, \Sigma, P, S)$ where

$\Sigma = \{[,]\}$ $\quad N = \{S\}$

$P: \quad S \to [S] \mid SS \mid \epsilon$

Claim. $L(G) = \{x \in \{[,]\}^* \mid x$ is balanced$\}$.

$L(x) = \#[(x)$: Number of $[$ in string x.

$R(x) = \#](x)$: Number of $]$ in string x.

$x \in \Sigma^*$ is balanced iff

1. $L(x) = R(x)$

2. For all prefix y of x; $L(y) \geq R(y)$

$G: S \longrightarrow [S] \mid SS \mid \epsilon.$

x is balanced iff
(1) $L(x) = R(x)$
(2) $L(y) \geq R(y)$ $\forall$ prefix y of x.

Theorem. $L(G) = \{x \in \{[,]\}^* \mid x$ satisfies (1) and (2)$\}$

Claim. if $S \xrightarrow[G]{*} x$ then x satisfies (1) and (2)

Proof.
Induction on the length of the derivation

For any $\alpha \in (N \cup \Sigma)^*$, if $S \xrightarrow[G]{*} \alpha$ then α satisfies (1.) and (2.)

Base Case: if $S \xrightarrow[G]{0} \alpha$ then $\alpha = S$

Induction step. Suppose $S \xrightarrow[G]{n+1} \alpha$ | $S \xrightarrow[G]{n} \beta \xrightarrow[G]{1} \alpha$

By induction hypothesis, β satisfies (1) and (2).

$S \to \epsilon$, $S \to SS$ | $\exists\, \beta_1, \beta_2 \in (N \cup \Sigma)^*$ s.t.

$\beta = \beta_1 S \beta_2$

$$\alpha = \begin{cases} \beta_1 \beta_2 & \text{if } S \to \epsilon \\ \beta_1 SS \beta_2 & \text{if } S \to SS \end{cases}$$

if $S \to [S]$ $\quad \exists\, \beta_1, \beta_2 \in (N \cup \Sigma)^*$, $\beta = \beta_1 S \beta_2$

$\alpha = \beta_1 [S] \beta_2$.

By IH, β satisfies (1) & (2)

$$L(\alpha) = L(\beta) + 1 = R(\beta) + 1 = R(\alpha)$$

Thus α satisfies (1.)

$$S \xrightarrow[G]{n} \overbrace{\beta_1 S \beta_2}^{\beta} \xrightarrow[G]{1} \beta_1 [S] \beta_2 = \alpha$$

To show: α satisfies 2.

Let γ be an arbitrary prefix of α.

To show: $L(\gamma) \geq R(\gamma)$

— γ is a prefix of $\beta_1 \Rightarrow \gamma$ is a prefix of β

— γ is a prefix of $\beta_1[S$ but not of β_1

$$L(\gamma) = L(\beta_1) + 1 \geq R(\beta_1) + 1 \quad \text{[IH, Since } \beta_1 \text{ is a prefix of } \beta\text{]}$$

$$> R(\beta_1) = R(\gamma)$$

— $\gamma = \beta_1[S]\delta$; where δ is a prefix of β_2.

$$L(\gamma) = L(\beta_1 S \delta) + 1$$

$$\geq R(\beta_1 S \delta) + 1 \quad \text{[Induction hypothesis]}$$

$$= R(\gamma).$$

Thus, in all the cases, we have shown that

$$L(\gamma) \geq R(\gamma).$$

Therefore, α satisfies (2.)

Claim. if x satisfies (1) and (2) then $S \xrightarrow[G]{*} x$

Proof. Induction on $|x|$.

Base Case. $|x| = 0$ - Trivial since $S \to \epsilon \rightsquigarrow P$

Induction step. if $|x| > 0$ – Consider 2 cases.

(a) $\exists$ a proper prefix y of x satisfying (1) & (2).

$(0 < |y| < |x|)$.

(b) no such prefix exists.

Case (a). $x = yz$ for some z, $0 < |z| < |x|$.

Claim. z satisfies (1) and (2)

By Definition

$$L(z) = \underbrace{L(x)}_{=} - \underbrace{L(y)}_{=} = R(x) - R(y) = R(z)$$

For any prefix w of z

$$L(w) = L(yw) - L(y)$$

$$\geq R(yw) - R(y) \quad \text{(Since } yw \text{ is a prefix of } x \text{ and } L(y) = R(y)\text{)}$$

$$= R(w). \quad \text{(By definition)}$$

By induction hypothesis $S \xrightarrow[G]{*} y$, $S \xrightarrow[G]{*} z$

$$S \xrightarrow[G]{1} SS \xrightarrow[G]{*} yS \xrightarrow[G]{*} yz = x.$$

Case b. $x = [z]$ for some z.

Claim: z satisfies (1) and (2).

$L(z) = L(x) - 1 = R(x) - 1 = R(z)$.

$\forall$ non-null prefixes u of z,

$L(u) - R(u) = L([u) - 1 - R([u) \geq 0$

Since $L([u) - R([u) \geq 1$.

By induction hypothesis $S \xrightarrow[G]{*} z$

$S \xrightarrow[G]{1} [S] \xrightarrow[G]{*} [z] = x$.