To prove $L(M_3) = A \cap B$

Theorem. $L(M_3) = L(M_1) \cap L(M_2)$

Lemma 1. For all $x \in \Sigma^*$,

$$\hat{\delta}_3(\underbrace{(q_1, q_2)}_{\in Q_3}, x) = \underbrace{(\overbrace{\hat{\delta}_1(q_1, x)}^{\in Q_1}, \overbrace{\hat{\delta}_2(q_2, x)}^{\in Q_2})}_{\in Q_3}$$

Proof. Induction on $|x|$.

Base Case $x = \epsilon$; $\hat{\delta}_3((q_1, q_2), \epsilon) = (q_1, q_2) = (\hat{\delta}_1(q_1, \epsilon), \hat{\delta}_2(q_2, \epsilon))$

Induction step.

$$\hat{\delta}_3((q_1, q_2), xa) \underbrace{= (\hat{\delta}_1(q_1, xa), \hat{\delta}_2(q_2, xa))}_{\text{To prove}}$$

$= \delta_3(\hat{\delta}_3((q_1, q_2), x), a)$ - Definition of $\hat{\delta}_3$

$= \delta_3((\hat{\delta}_1(q_1, x), \hat{\delta}_2(q_2, x)), a)$ Induction hypothesis

$= (\delta_1(\hat{\delta}_1(q_1, x), a), \delta_2(\hat{\delta}_2(q_2, x), a))$

Definition of δ_3.

$= (\hat{\delta}_1(q_1, xa), \hat{\delta}_2(q_2, xa))$ Definition of $\hat{\delta}_1$ and $\hat{\delta}_2$.

Theorem. $L(M_3) = L(M_1) \cap L(M_2)$

Proof. For all $x \in \Sigma^*$

$x \in L(M_3) \iff \hat{\delta}_3(s_3, x) \in F_3$ [acceptance]

$\iff \hat{\delta}_3((s_1, s_2), x) \in F_1 \times F_2$ [Def of s_3 & F_3]

$\iff (\hat{\delta}_1(s_1, x), \hat{\delta}_2(s_2, x)) \in F_1 \times F_2$

[Lemma 1]

$\iff \hat{\delta}_1(s_1, x) \in F_1$ and $\hat{\delta}_2(s_2, x) \in F_2$

[definition of set product]

$\iff x \in L(M_1)$ and $x \in L(M_2)$. [Acceptance]

$\iff x \in L(M_1) \cap L(M_2)$ [Defn. of intersection]

$x \in L(M_3)$ iff $x \in L(M_1) \cap L(M_2)$.

Question. Are regular sets closed under Complementation?

Let $A \subseteq \Sigma^*$. if $\underbrace{A \text{ is regular}}_{\exists M \text{ s.t } L(M) = A}$ is $\underbrace{\bar{A} \text{ regular}}_{\exists \bar{M} \text{ s.t } L(\bar{M}) = \bar{A}}$? Yes

Interchange the set of accept states and non-accept states.

Question. if $A, B \subseteq \Sigma^*$ are regular then is $A \cup B$ regular? Yes

$$A \cup B = \overline{(\bar{A} \cap \bar{B})}$$

An explicit Construction

From M_1 $[L(M_1) = A]$ and M_2 $[L(M_2) = B]$ Construct M_3 s.t $L(M_3) = L(M_1) \cup L(M_2)$.

Intersection $F_3 = F_1 \times F_2$

Union $F_3 = (F_1 \times Q_2) \cup (Q_1 \times F_2)$

$$F_3 = \{ (q_1, q_2) \mid q_1 \in F_1 \text{ or } q_2 \in F_2 \}.$$

Non-determinism

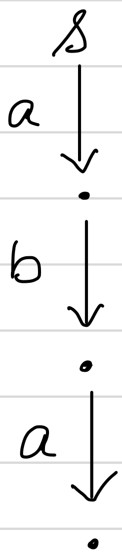
- Next state of a computation is not uniquely determined by current state.
- Important concept in the design of efficient algorithms.
- There are many combinatorial problems with efficient non-deterministic solution but no known efficient deterministic solution.

$$P \stackrel{?}{=} NP$$

DFA-M - $\delta : Q \times \Sigma \to Q$.

$x = aba$

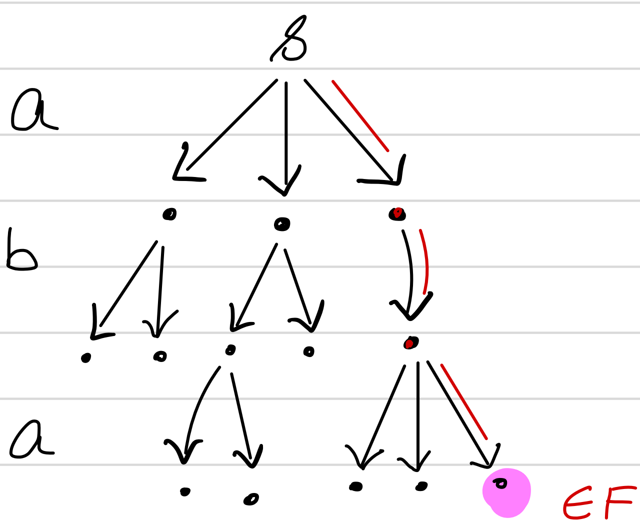
Run of M on x



NFA-N - δ is non-deterministic

$\delta : Q \times \Sigma \to 2^Q$.

Run of N on x



N accepts x if at least one computation path starting from at least one initial state leads to a final state.