A Myhill-Nerode relation for a regular set $R \subseteq \Sigma^*$ is an equivalence relation $\equiv$ satisfying the following.

(P1) It is a right congruence : $\forall x, y \in \Sigma^*, \forall a \in \Sigma$

$$x \equiv y \Rightarrow xa \equiv ya.$$

(P2) It refines R : $\forall x, y \in \Sigma^*$

$$x \equiv y \Rightarrow (x \in R \text{ iff } y \in R)$$

(P3) it is of finite index : there are only finitely many equivalence classes.

Construction 1. $M \mapsto \equiv_M$.

Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA s.t $L(M) = R$

M induces an equivalence relation $\equiv_M$ on Σ^* defined by

$$x \equiv_M y \text{ iff } \hat{\delta}(s, x) = \hat{\delta}(s, y)$$

Observation. $\equiv_M$ is an equivalence relation.

Construction 2. $\equiv \mapsto M_\equiv$

Let $R \subseteq \Sigma^*$ and $\equiv$ be any Myhill-Nerode relation for R. From $\equiv$ we can construct a DFA $M_\equiv$ s.t $L(M_\equiv) = R$.

$M_\equiv = (Q, \Sigma, \delta, s, F)$ where

$Q = \{[x] \mid x \in \Sigma^*\}$; $s = [\epsilon]$; $F = \{[x] \mid x \in R\}$

$\delta([x], a) = [xa]$.

A relation $\equiv_1$ refines another relation $\equiv_2$ if

$$\equiv_1 \subseteq \equiv_2 .$$

That is, $\forall x, y$ if $x \equiv_1 y$ then $x \equiv_2 y$.

For equivalence relation: $\forall x \quad [x]_1 \subseteq [x]_2$

Example $x \equiv y \bmod 6$ on integers refines $x \equiv y \bmod 3$

"Refinement" - relation between equivalence relations is

- reflexive
- transitive (if $\equiv_1$ refines $\equiv_2$ and $\equiv_2$ refines $\equiv_3$ then $\equiv_1$ refines $\equiv_3$).
- antisymmetric (if $\equiv_1$ refines $\equiv_2$ and $\equiv_2$ refines $\equiv_1$ then $\equiv_1$ & $\equiv_2$ are the same)

if $\equiv_1$ refines $\equiv_2$ then $\equiv_1$ is the finer and $\equiv_2$ is the coarser of the two relations.

There is always a "finest" and "coarsest" equivalence rel. on any set $\mathcal{U}$.

$\{(x,x) \mid x \in \mathcal{U}\}$ and $\{(x,y) \mid x, y \in \mathcal{U}\}$

(identity relation) (universal relation)

Let $R \subseteq \Sigma^*$ (not necessarily regular)

Define an equivalence relation $\equiv_R$ on Σ^* as follows:

$x \equiv_R y$ iff $\forall z \in \Sigma^*$ $(xz \in R$ iff $yz \in R)$

Claim 1. For any $R \subseteq \Sigma^*$, $\equiv_R$ satisfies (P1) and (P2) of Myhill-Nerode relation and is the coarsest such relation on Σ^*.

Claim 2. if R is regular then $\equiv_R$ also satisfies (P3)

Corollary. if $R \subseteq \Sigma^*$ is regular then $\equiv_R$ is the coarsest Myhill-Nerode relation for R. Then $M_{\equiv_R}$ corresponds to the unique minimal finite automaton for R.

$x \equiv_R y$ iff $\forall z \in \Sigma^*$ ($xz \in R$ iff $yz \in R$)

Lemma. Let $R \subseteq \Sigma^*$ (R need not be regular)
The relation $\equiv_R$ is a right congruence refining R and it is the coarsest such relation on Σ^*.

Proof.

(1) $\equiv_R$ is a right congruence

In the definition of $\equiv_R$, take $z = aw$, $a \in \Sigma$, $w \in \Sigma^*$

$x \equiv_R y \Rightarrow \forall a \in \Sigma\ \forall w \in \Sigma^*$ ($xaw \in R$ iff $yaw \in R$)

$\Rightarrow \forall a \in \Sigma$ ($xa \equiv_R ya$)

(2). $\equiv_R$ refines R.

In the definition of $\equiv_R$ take $z = \epsilon$.

$x \equiv_R y \Rightarrow$ ($x \in R$ iff $y \in R$)

(3). Any other equivalence relation $\equiv$ that satisfies (P1) and (P2) refines $\equiv_R$.

$x \equiv y$

$\Rightarrow \forall z$ ($xz \equiv yz$) (Using (P1) by induction on $|z|$)

$\Rightarrow \forall z$ ($xz \in R$ iff $yz \in R$) (By (P2))

$\Rightarrow x \equiv_R y$ (By defn. of $\equiv_R$)

Myhill-Nerode Theorem. Let $R \subseteq \Sigma^*$. The following Statements are equivalent.

1. R is regular
2. there exists a Myhill-Nerode relation for R
3. the relation $\equiv_R$ is of finite index.

Proof.

$1 \Rightarrow 2$. Given a DFA M s.t $L(M) = R$, the Construction $M \mapsto \equiv_M$ produces a Myhill-Nerode relation for R.

$2 \Rightarrow 3$. Any Myhill-Nerode relation $\equiv$ for R is of finite index.
By Lemma, $\equiv$ refines $\equiv_R$. So $\equiv_R$ has finite index.

$3 \Rightarrow 1$. if $\equiv_R$ is of finite index then it is a Myhill-Nerode relation for R.

The Construction $\equiv \mapsto M_{\equiv}$ generates a DFA s.t $L(M_{\equiv}) = R$.

Since $\equiv_R$ is the unique coarsest Myhill-Nerode relation for R, $M_{\equiv_R}$ has the fewest states among all DFAs for R.

Suppose $M = (Q, \Sigma, \delta, s, F)$ is a DFA where $L(M) = R$ and $M = M/\approx$ (Collapsed).

That is,

$$p \approx q \text{ iff } \forall x \in \Sigma^* \left(\hat{\delta}(p, x) \in F \text{ iff } \hat{\delta}(q, x) \in F \right)$$

is identity on Q and M has no inaccessible states.

Claim. $\equiv_M$ the Myhill-Nerode relation corresponding to M is same as $\equiv_R$.

Proof.

$$x \equiv_R y \iff \forall z \in \Sigma^* (xz \in R \text{ iff } yz \in R)$$
(Defn of $\equiv_R$.)

$$\iff \forall z \in \Sigma^* (\hat{\delta}(s, xz) \in F \text{ iff } \hat{\delta}(s, yz) \in F)$$
(Defn of acceptance)

$$\iff \forall z \in \Sigma^* (\hat{\delta}(\hat{\delta}(s, x), z) \in F \text{ iff } \hat{\delta}(\hat{\delta}(s, y), z) \in F)$$
(Claim - Exercise)

$$\iff \hat{\delta}(s, x) \approx \hat{\delta}(s, y) \quad \text{(Defn of } \approx)$$

$$\iff \hat{\delta}(s, x) = \hat{\delta}(s, y) \quad \text{(Since } M = M/\approx)$$

$$\iff x \equiv_M y \quad \text{(Defn of } \equiv_M)$$

$$A = \{a^n b^n \mid n \geq 0\}$$

if $k \neq m$ then $a^k \not\equiv_A a^m$, since $a^k b^k \in A$ and $a^m b^k \notin A$

Thus there is a $\equiv_A$-class corresponding to each a^k, $k \geq 0$.

Thus $\equiv_A$ is not of finite index.

By Myhill-Nerode theorem, A is not regular.