Configuration of a Turing machine

Tape of M consists of a string $y\sqcup^{\omega}$ where $y \in \Gamma^*$.

A configuration is $\in Q \times \{\underbrace{y\sqcup^{\omega}} \mid y \in \Gamma^*\} \times \mathbb{N}$

$\underbrace{y\sqcup^{\omega}} \rightarrow$ infinite string with a finite presentation.

(p, z, n)

$p \rightarrow Q$

$z \rightarrow$ Current content of the tape.

$n \rightarrow$ $n \geq 0$ Current position of head on tape

Start configuration - $(s, \vdash x \sqcup^{\omega}, 0)$

$x \rightarrow$ input string

Next configuration relation $\xrightarrow[M]{1}$

For an infinite string z, let z_n be the n^{th} symbol in z

$S_b^n(z)$ - string obtained by substituting b for z_n.

Example $S_b^3(\vdash abaa\ldots) = \vdash abba\ldots$

$$(p, z, n) \xrightarrow[M]{1} \begin{cases} (q, S_b^n(z), n-1) & \text{if } \delta(p, z_n) = (q, b, L) \\ (q, S_b^n(z), n+1) & \text{if } \delta(p, z_n) = (q, b, R) \end{cases}$$

$\alpha \xrightarrow[M]{*} \beta$: Reflexive transitive closure of $\xrightarrow[M]{1}$

M accepts $x \in \Sigma^*$ if $(s, \vdash x \sqcup^\omega, 0) \xrightarrow[M]{*} (t, y, n)$

$L(M) = \{x \in \Sigma^* \mid M \text{ accepts } x\}$

M rejects $x \in \Sigma^*$ if $(s, \vdash x \sqcup^\omega, 0) \xrightarrow[M]{*} (r, y, n)$

M **halts** on input $x \in \Sigma^*$ if it either accepts x or rejects x.

M **loops** on input $x \in \Sigma^*$ if M neither accepts nor rejects x.

Total TM: A Turing machine that halts on all inputs is called a total TM.

$A \subseteq \Sigma^*$ is **recursive** if $A = L(M)$ for a total TM M.

$A \subseteq \Sigma^*$ is **recursively enumerable (r.e)** if $A = L(M)$ for some TM M (need not be a total TM).

Ex. $A = \{a^n b^n c^n \mid n \geq 0\}$ is recursive.

Example. $A = \{ww \mid w \in \{a,b\}^*\}$ — not a CFL

A is recursive since we can define a total TM M s.t $L(M) = A$.

Working of M on input x.

- Scan (left to right) till first $\sqcup$ - symbol) replace $\sqcup$ with $\dashv$ and check that number of symbols in x is even

- In each pass from right to left, M marks the first unmarked a, b with $\acute{a}$, $\acute{b}$

- In each pass from left to right, it marks the first unmarked a, b with $\grave{a}$, $\grave{b}$

Ex. Tape Content $\Gamma = \{a, b, \vdash, \sqcup, \dashv, \grave{a}, \grave{b}, \acute{a}, \acute{b}\}$

$\vdash$ a a b b b a a b b b $\sqcup$ $\sqcup$ $\sqcup$ $\sqcup$

$\vdash$ a a b b b a a b b b $\dashv$ $\sqcup$ $\sqcup$ $\sqcup$

$\vdash$ $\grave{a}$ $\grave{a}$ $\grave{b}$ $\grave{b}$ $\grave{b}$ $\acute{a}$ $\acute{a}$ $\acute{b}$ $\acute{b}$ $\acute{b}$ $\dashv$ $\sqcup$ $\sqcup$ $\sqcup$

$\vdash$ $\sqcup$ $\sqcup$ $\grave{b}$ $\grave{b}$ $\grave{b}$ $\sqcup$ $\sqcup$ $\acute{b}$ $\acute{b}$ $\acute{b}$ $\dashv$ $\sqcup$ $\sqcup$ $\sqcup$

Recursive and recursively enumerable (r.e) Sets.

- Every recursive set is r.e

- Not every TM is equivalent to a total TM.

Recursive sets are closed under complementation.

Suppose $A \subseteq \Sigma^*$ is recursive. Then there exists a total TM M s.t $L(M) = A$.

- Switch the accept and reject states.
  Resulting M': $L(M') = \Sigma^* - A$.

This construction does not work for r.e. sets.

Rejecting and not accepting is not the same in a TM.

M' will still loop on the strings that M loops on.

Such strings are not accepted or rejected by either machines.

Claim. if both A and $\bar{A}$ are r.e then A is recursive