Theorem 1. Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism.

if $A \subseteq \Sigma^*$ is regular then $h(A)$ is regular.

Proof. α is a regular expression s.t $L(\alpha) = A$.

over Σ.

To show: Construct α' s.t $L(\alpha') = h(A)$.

α' - Replace each letter (or symbol) $a \in \Sigma$ in α

with the String $h(a) \in \Gamma^*$.

Formal defn. of α' - By induction on the structure of α.

$$a' = h(a) \quad , a \in \Sigma.$$

$$\phi' = \phi$$

$$(\beta + \gamma)' = \beta' + \gamma'$$

$$(\beta\gamma)' = \beta' . \gamma'$$

$$(\beta^*)' = \beta'^*$$

over Σ.

Claim. For any regular expression β : $L(\beta') = h(L(\beta))$

Corollary : $L(\alpha') = h(A)$

Lemma 2. For $C, D \subseteq \Sigma^*$, $h(CD) = h(C)\,h(D)$.

Lemma 3. For a family of subsets $C_i \subseteq \Sigma^*$, $i \in I$,

we have $h\left[\bigcup_{i\in I} C_i\right] = \bigcup_{i \in I} h(C_i)$

Exercise: Proof of Lemma 2 and Lemma 3.

To prove: For any β, $L(\beta') = h(L(\beta))$

Base Case:

$$L(a') = L(h(a)) = \{h(a)\} = h(\{a\}) = h(L(a))$$

$$L(\phi') = L(\phi) = \phi = h(\phi) = h(L(\phi))$$

Induction Step: Operators $+, \cdot, *$

$$L((\beta+\gamma)') = L(\beta'+\gamma') \quad \text{[Defn of ']}$$

$$= L(\beta') \cup L(\gamma') \quad \text{[Defn. of +]}$$

$$= h(L(\beta)) \cup h(L(\gamma)) \quad \text{[Induction hypothesis]}$$

$$= h(L(\beta) \cup L(\gamma)) \quad \text{[Lemma 3]}$$

$$= h(L(\beta+\gamma)) \quad \text{[Definition of +]}.$$

For concatenation, the proof is similar to +. Use Lemma 2.

$$L(\beta^{*'})$$

$$= L(\beta'^{*}) \quad \text{[Definition of ']}$$

$$= L(\beta')^{*} \quad \text{[Defn. of regular expression operator *].}$$

$$= h(L(\beta))^{*} \quad \text{[Induction hypothesis.]}$$

$$= \bigcup_{n \geq 0} h(L(\beta))^{n} \quad \text{[Defn. of set operator *]}$$

$$= \bigcup_{n \geq 0} h(L(\beta)^{n}) \quad \text{[Lemma 2].}$$

$$= h\left(\bigcup_{n \geq 0} L(\beta)^{n}\right) \quad \text{[Lemma 3]}$$

$$= h(L(\beta)^{*}) \quad \text{[Defn of set operator *]}$$

$$= h(L(\beta^{*})) \quad \text{[Defn of regular expression operator *].}$$

$A \subseteq \Sigma^{*}$

$\hookrightarrow$ Finite .

$B = \{a^n b^n \mid n \geq 0\} = \{\epsilon, ab, aabb, aaabbb, \ldots\}$

Question. Is B regular?

Suppose B is regular. Then there is a DFA M s.t $L(M) = B$

Let k be the number of states of M.

Consider an arbitrary string $a^n b^n$ where $n > k$.

$$\overbrace{aaaaaaaaaa}^{n}\overbrace{bbbbbbbbbbb}^{n}$$

$$\underbrace{aaaaaa}_{u}\ \underbrace{aa}_{v}\ \underbrace{aa\,bbbbbbbbbbb}_{w}$$

$s \qquad p \qquad p \qquad r \in F$

There exists a state p in M such that

$\hat{\delta}(s, u) = p$; $\hat{\delta}(p, v) = p$ where $|v| = j > 0$

and $\hat{\delta}(p, w) = r \in F$

Now consider the run of M on input uw.

$$\hat{\delta}(s, uw) = \hat{\delta}(\hat{\delta}(s, u), w) = \hat{\delta}(p, w) = r \in F$$

$uw \in L(M)$ but $uw = a^{n-|v|} b^n \notin B$

Consider the string $uv^2w = a^{n+|v|} b^n$ and the run of M on uv^2w.

$$\hat{\delta}(s, uvvw) = \hat{\delta}(\hat{\delta}(\hat{\delta}(\hat{\delta}(s,u), v), v), w)$$
$$= \hat{\delta}(\hat{\delta}(\hat{\delta}(p, v), v), w)$$
$$= \hat{\delta}(\hat{\delta}(p, v), w)$$
$$= \hat{\delta}(p, w) = r \in F.$$

Thus $uv^2w \in L(M)$

But $uv^2w = a^{n+|v|} b^n \notin B$.

Contradicts our assumption that $L(M) = B$

Pumping Lemma.

Let A be a regular set. Then the following property (P) holds for A.

(P) There exists $k \geq 0$ such that for any string x, y, z with $xyz \in A$ and $|y| \geq k$ there exists strings u, v, w s.t $y = uvw$, $v \neq \epsilon$ and for all $i \geq 0$, the string $xuv^iwz \in A$.

Contrapositive of (P): Suppose A satisfies the following:

($\neg$P) For all $k \geq 0$ there exists strings x, y, z such that $xyz \in A$, $|y| \geq k$ and for all u, v, w with $y = uvw$ and $v \neq \epsilon$, there exists $i \geq 0$ s.t $xuv^iwz \notin A$.

Then A is not regular.

Suppose A satisfies the following:

(¬P) For all $k \geq 0$ there exists strings x, y, z such that $xyz \in A$, $|y| \geq k$ and for all u, v, w with $y = uvw$ and $v \neq \epsilon$, there exists $i \geq 0$ s.t. $xuv^iwz \notin A$.

Then A is not regular.

Example. $A = \{a^n b^m \mid n \geq m\}$.

Claim. A is not regular.

By pumping lemma, suffices to show that A satisfies ¬P

Consider any $k \geq 0$, let $x = a^k$ $y = b^k$ & $z = \epsilon$.

Then $xyz \in A$. Consider any split of $y = uvw$ with $v \neq \epsilon$. Say $y = \underbrace{b^j}_{u} \underbrace{b^m}_{v} \underbrace{b^n}_{w}$. $\therefore k = j + m + n$

Let $i = 2$. $xuv^2wz = a^k b^j b^m b^m b^n$

$= a^k b^{j+2m+n}$

$= a^k b^{k+m} \notin A$

Suppose A satisfies the following:

(¬P) For all $k \geq 0$ there exists strings x, y, z such that
$xyz \in A$, $|y| \geq k$ and for all u, v, w
with $y = uvw$ and $v \neq \epsilon$, there exists
$i \geq 0$ s.t $xuv^i wz \notin A$.

Then A is not regular.

$$A = \{ww \mid w \in \{a,b\}^*\}.$$

Claim. A is not regular.

Proof. Consider any $k \geq 0$ and let

$x = \epsilon$, $y = a^k$, $z = ba^k b$. Then $xyz \in A$.

Consider any split of $y = uvw$ s.t $v \neq \epsilon$.

Say $y = a^j a^m a^n$ where $k = j+m+n$; $m > 0$.

Let $i = 2$. Then $xuv^2wz = a^j a^m a^m a^n b a^k b$

$$= a^{k+m} b a^k b$$

Since $m > 0$, $a^{k+m} b a^k b \notin A$.

Thus A is not regular.

Use of closure properties.

$$A = \{ x \in \{a,b\}^* \mid \#a(x) = \#b(x) \}.$$

$\#a(x)$: number of a's in the string x.

Claim: A is not regular.

Proof. Suppose A is regular.

$$A' = A \cap L(a^* b^*).$$

1) $L(a^* b^*)$ is regular.

2) if A is regular and $L(a^*b^*)$ is regular then A' is regular. Since regular sets are closed under intersection.

$$A' = A \cap L(a^* b^*) = \underbrace{\{a^n b^n \mid n \geq 0\}}_{\text{is not regular.}}$$

This results in a contradiction

Thus A is not regular.

$A = \{a^n b^m \mid n \geq m\}$

Claim. A is not regular.

Proof.

Suppose A is regular. Then rev(A) is regular.

$rev(A) = \{b^m a^n \mid n \geq m\}$.

Consider the homomorphism h where $h(a) = b$ & $h(b) = a$

Let $A' = h(rev(A))$. Since rev(A) is regular, $h(rev(A))$ is also regular.

$A' = \{a^m b^n \mid n \geq m\}$.

$$A \cap A' = \{\underbrace{a^n b^n \mid n \geq 0}_{\text{is not regular}}\}.$$

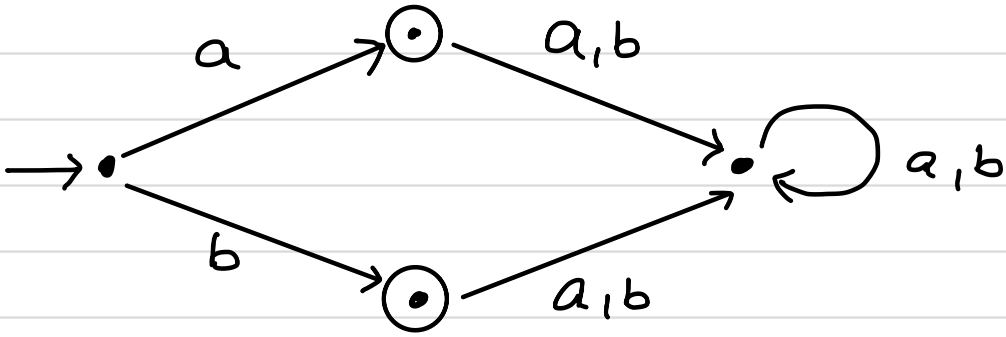
is not regular - Contradiction

Thus A is not regular.

State Minimization – DFA.

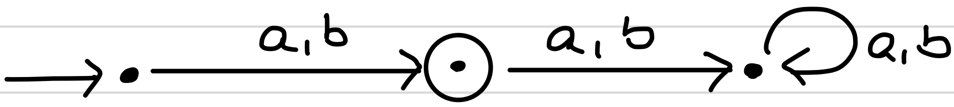
Example 1.

M_1 :



$L(M_1) = \{a, b\}$

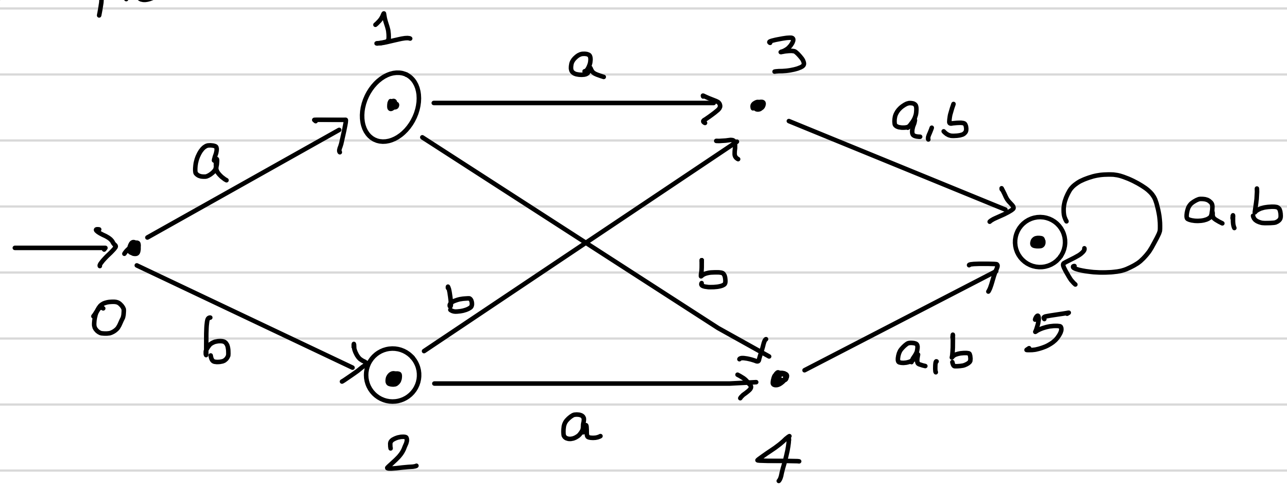
M_2 :



$L(M_1) = L(M_2)$

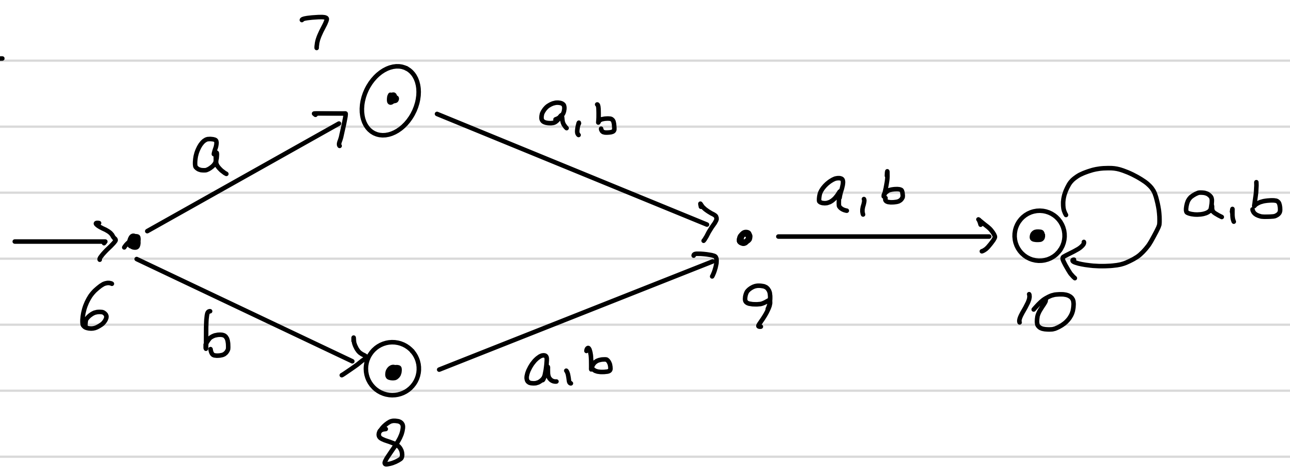
Exemple2.

M_1

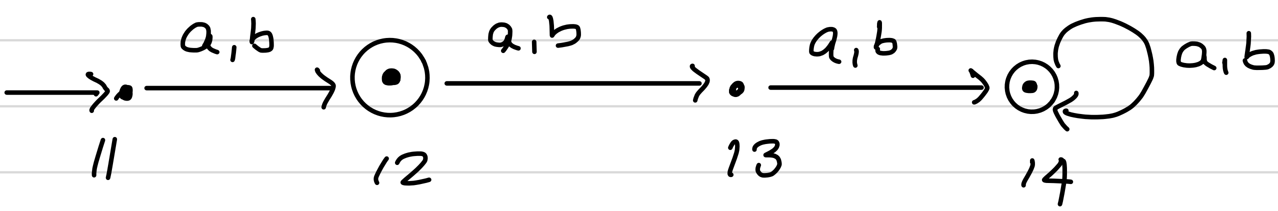


$L(M_1) = \{a, b\} \cup \{\text{strings of length at least } 3\}$

M_2

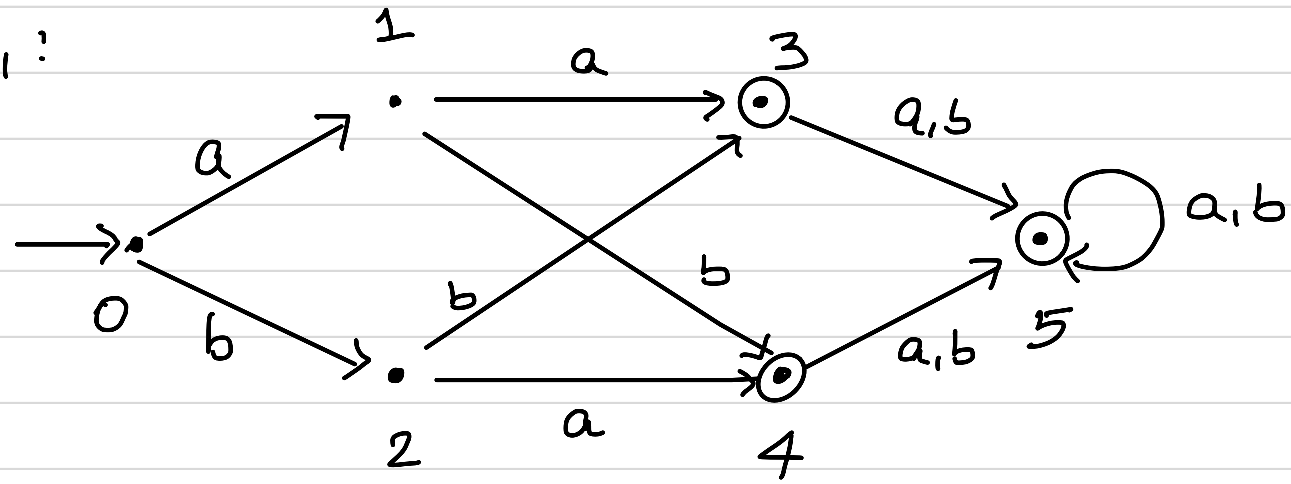


M_3:



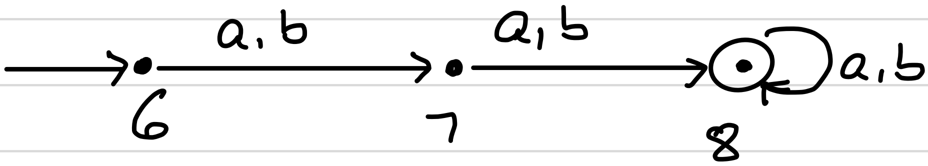
Example 3.

M_1:



$L(M_1) = \{$strings of length atleast 2$\}$

M_2



For a DFA $M = (Q, \Sigma, \delta, s, F)$ the state minimization process consists of:

1. Removing inaccessible states.
2. Collapsing "equivalent" states.

- We cannot collapse a final state p and a non-final state q.

$$\text{if } p = \hat{\delta}(s, x) \text{ and } q = \hat{\delta}(s, y) \text{ then}$$

$$x \in L(M) \text{ and } y \notin L(M).$$

- if we collapse states p & q then we should also collapse $\delta(p, a)$ and $\delta(q, a)$. Otherwise the resulting automata will not be deterministic.

Definition of an equivalence relation on Q.

$p \approx q$ iff $\forall x \in \Sigma^* \, (\hat{\delta}(p,x) \in F$ iff $\hat{\delta}(q,x) \in F)$

Claim. $\approx$ is an equivalence relation.

$\approx$ partitions Q into a set of equivalence classes.

$[p] = \{q \mid q \approx p\}$.

Easy to verify that $p \approx q$ iff $[p] = [q]$.