HP is not recursive, HP is r.e

MP is not recursive, MP is r.e

$\overline{HP}$ is not r.e

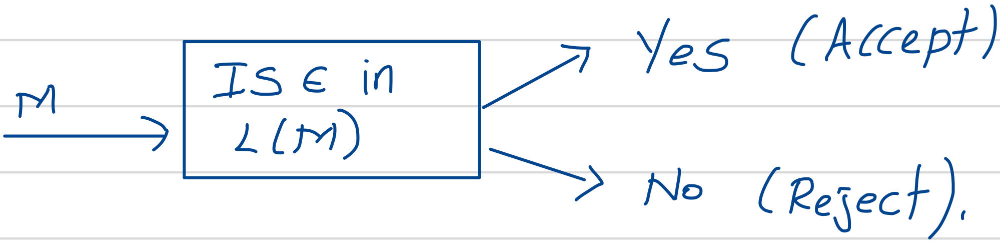
$\overline{MP}$ is not r.e.

Properties of TMs.

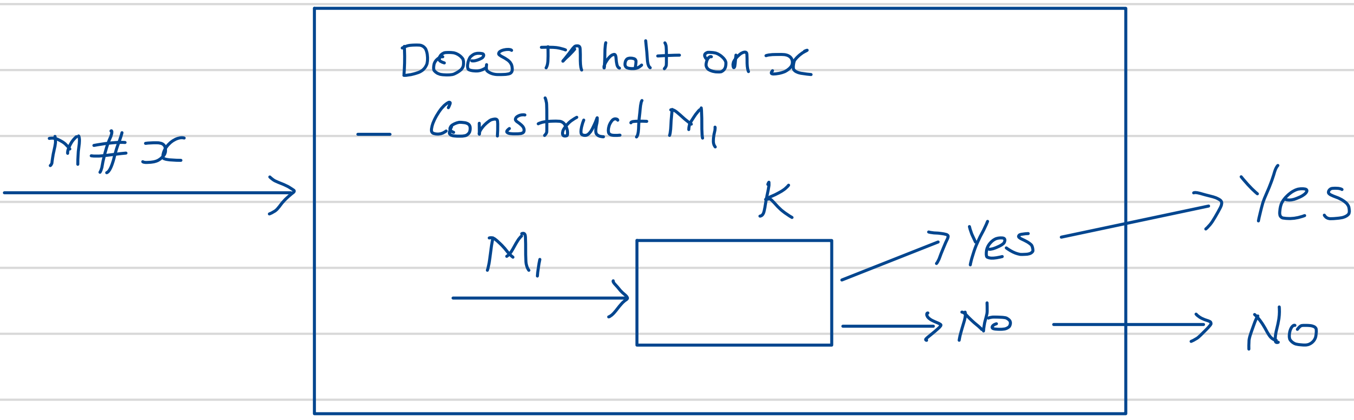
- Is it decidable if a TM M accepts ϵ ?
- Is it decidable if for a TM M, $L(M) = \phi$?
- Is it decidable if for a TM M, $L(M) = \Sigma^*$?

- Is it decidable if for a TM M, L(M) is regular ?
- Is it decidable if for a TM M, L(M) is a CFL ?
- Is it decidable if for a TM M, L(M) is recursive ?

A total TM K



A total TM K'



Is it decidable if a TM M accepts ϵ ?

Is $\{M \mid \epsilon \text{ is in } L(M)\}$ recursive ?

Suppose $\exists$ a total TM K that can decide if a given TM M accepts ϵ. i.e. $L(K) = \{M \mid \epsilon \text{ is in } L(M)\}$

Then we can construct a total TM K' s.t $L(K') = HP$

- Given a TM M and x, to determine if M halts on x.

Construct M_1 that on input y works as follows.

1. Erases its input y.
2. writes x on the tape.
3. Runs M on input x

M and x are hard coded in M'.

4. Accept if M halts on x.

if M halts on x, M_1 accepts y for all input y.

Thus $L(M_1) = \begin{cases} \Sigma^* & \text{if } M \text{ halts on } x \\ \phi & \text{if } M \text{ does not halt on } x. \end{cases}$

Run K with input M_1

if K accepts $\Rightarrow \epsilon \in L(M_1) \Rightarrow L(M_1) = \Sigma^* \Rightarrow M$ halts on x

if K rejects $\Rightarrow \epsilon \notin L(M_1) \Rightarrow L(M_1) = \phi \Rightarrow M$ does not halt on x.

- Given a TM M and x, to determine if M halts on x.

Construct M_1 that on input y works as follows.

1. Erases its input y.
2. writes x on the tape.
3. Runs M on input x

} M and x are hard coded in M'.

4. Accept if M halts on x.

if M halts on x, M_1 accepts y for all input y.

Thus $L(M_1) = \begin{cases} \Sigma^* & \text{if } M \text{ halts on } x \\ \phi & \text{if } M \text{ does not halt on } x. \end{cases}$

Run K with input M_1

if K accepts $\Rightarrow \epsilon \in L(M_1) \Rightarrow L(M_1) = \Sigma^* \Rightarrow M$ halts on x

if K rejects $\Rightarrow \epsilon \notin L(M_1) \Rightarrow L(M_1) = \phi \Rightarrow M$ does not halt on x.

Questions.

- Is it decidable if for a TM M, $L(M) = \phi$?
  Is $\{M \mid L(M) = \phi\}$ recursive?

- Is it decidable if for a TM M, $L(M) = \Sigma^*$?
  Is $\{M \mid L(M) = \Sigma^*\}$ recursive?

Same construction as above.

Question. Is it decidable, for a TM M, if $L(M)$ is regular.

Is $\{M \mid L(M) \text{ is regular}\}$ recursive?

Choose a set A that is r.e. but not recursive.
Eg. $A = HP$ or $A = MP$. Let N be a TM s.t $L(N) = A$.

Suppose $\exists$ a total TM K that can decide, given an arbitrary TM M if $L(M)$ is regular.
Then using K, we can construct a total TM K' s.t $L(K') = HP$

Given M and x to determine if M halts on x.

Construct a TM M_2 which on input y does the following:
↳ with multiple tracks.

1. Writes y on one of the tracks

2. Writes x on a separate track. } M and x are
3. Runs M on input x. } hard coded in M_2

4. if M halts on x then M_2 runs N on y.
(y is M_2 original input)

M_2 accepts if N accepts y.

if M does not halt on x then M_2 does not accept any string

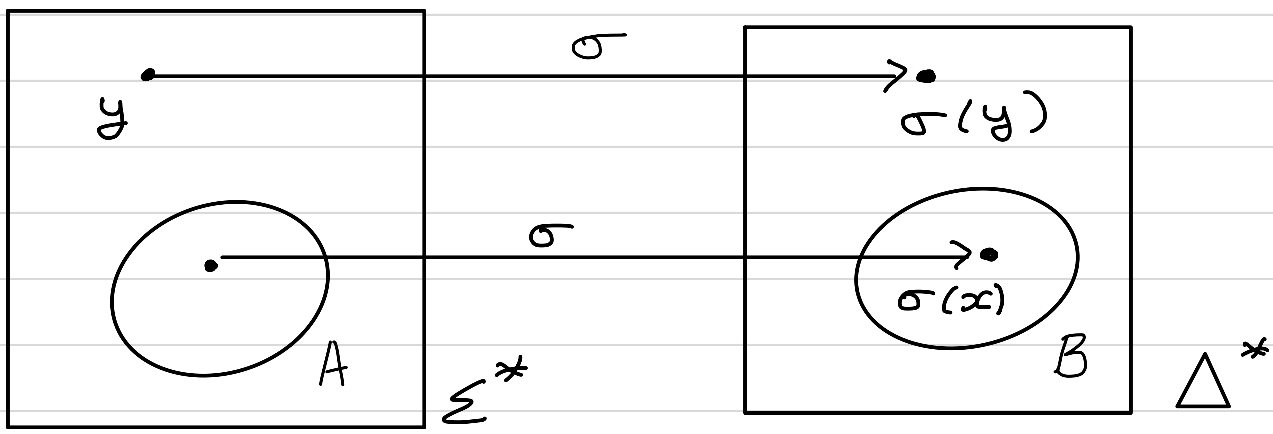
$$L(M_2) = \begin{cases} A & \text{if } M \text{ halts on } x. \\ \phi & \text{if } M \text{ does not halt on } x. \end{cases}$$

A is not recursive, so A is not regular; not a CFL
ϕ is regular, CFL and recursive.

Reductiòn.

Given $A \subseteq \Sigma^*$, $B \subseteq \Delta^*$, a (many-one) reduction of A to B is a computable function

$\sigma : \Sigma^* \to \Delta^*$ s.t $\forall x \in \Sigma^*$, $x \in A$ iff $\sigma(x) \in B$



σ should be computable by a total TM.
A total TM that on any input x writes σ(x) on the tape and halts.

σ need not be one-to-one or onto.

$A \leq_m B$ - A reduces to B via a map σ.

Observation. $\leq_m$ between sets is transitive.

if $\underset{\sigma}{A \leq_m B}$ and $\underset{\tau}{B \leq_m C}$ then $\underset{\tau \circ \sigma}{A \leq_m C}$

Example 1. Given M is ϵ in $L(M)$?

$A = \{ M\#x \mid M \text{ halts on } x \} = HP$

$B = \{ M \mid \epsilon \in L(M) \}$.

σ is the computable function $M\#x \mapsto M_1$.

Theorem.

1. if $A \leq_m B$ and B is r.e then A is r.e.

↳ if $A \leq_m B$ and A is not r.e then B is not r.e.

2. if $A \leq_m B$ and B is recursive then A is recursive

↳ if $A \leq_m B$ and A is not recursive then B is not recursive.