$M = (Q, \Sigma, \Gamma, \delta, s, \perp, F)$

Configurations: A configuration of M is an element of $Q \times \Sigma^* \times \Gamma^*$.

Q: current state

Σ^*: part of the input that is unread.

Γ^*: current contents of the stack

Start configuration: $(s, x, \perp)$

Define: 1 step next configuration relation $\xrightarrow[M]{1}$

if $((p, a, A), (q, \gamma)) \in \delta$ then for any $y \in \Sigma^*$ and $\beta \in \Gamma^*$

$$(p, ay, A\beta) \xrightarrow[M]{1} (q, y, \gamma\beta)$$

if $((p, \epsilon, A), (q, \gamma)) \in \delta$ then for any $y \in \Sigma^*$ and $\beta \in \Gamma^*$

$$(p, y, A\beta) \xrightarrow[M]{1} (q, y, \gamma\beta)$$

Let $\xrightarrow[M]{*}$ denote the reflexive transitive closure of $\xrightarrow[M]{1}$

Acceptance. Two types: By final state and empty stack.

By final state: M accepts x by final state if

$$(s, x, \perp) \xrightarrow[M]{*} (q, \epsilon, \gamma) \text{ for some } q \in F, \gamma \in \Gamma^*$$

↳ can be any string.

By empty stack. M accepts x by empty stack if

$$(s, x, \perp) \xrightarrow[M]{*} (q, \epsilon, \epsilon) \text{ for some } q \in Q.$$

↳ can be any state

$L(M)$ - set of all strings $x \in \Sigma^*$ accepted by M.

Example: Set of balanced parentheses.
NPDA - accepting by empty stack.

$Q = \{q\}$, $\Sigma = \{[,]\}$, $\Gamma = \{\perp, [\}$ $s = q$.

$((q, [, \perp), (q, [\perp)) \in \delta$.

$((q, [, [), (q, [[)) \in \delta$

$((q,], [), (q, \epsilon)) \in \delta$.

$((q, \epsilon, \perp), (q, \epsilon)) \in \delta$.

$L = \{ w w^R \mid w \in \{a,b\}^* \}$

$M = (\{q_0, q_1, q_2\}, \{a,b\}, \{a,b,\perp\}, \delta, q_0, \perp, F)$

$F = \{q_2\}$ - Accepting by final state.

- $((q_0, a, \perp), (q_0, a\perp)) \in \delta$; $((q_0, b, \perp), (q_0, b\perp)) \in \delta$

- $$\left. \begin{array}{l} ((q_0, a, a), (q_0, aa) \; ; ((q_0, a, b), (q_0, ab)) \\ ((q_0, b, a), (q_0, ba) \; ; \; ((q_0, b, b), (q_0, bb)) \end{array} \right\} \in \delta$$

- $$\left. \begin{array}{r} ((q_0, \epsilon, a), (q_1, a)) \; ; ((q_0, \epsilon, b), (q_1, b)) \\ ((q_0, \epsilon, \perp), (q_1, \perp)) \end{array} \right\} \in \delta$$

- $((q_1, a, a) \; (q_1, \epsilon))$; $((q_1, b, b), (q, \epsilon))$

- $((q_1, \epsilon, \perp), (q_2, \epsilon))$

NPDAs and CFGs have equivalent expressive power

Theorem 1. Given a CFG G, we can construct an NPDA M s.t $L(G) = L(M)$.

Theorem 2. Given an NPDA M, we can construct a CFG G s.t $L(G) = L(M)$.

Closure properties of CFLs.

Union. Suppose A and B are CFLs where $L(G_1)=A$, $L(G_2)=B$ and the start symbols are S_1 for G_1 and S_2 for G_2.

Construct a grammar G s.t $L(G)=A\cup B$ as follows:

Ensure that G_1 and G_2 have disjoint set
[rename the nonterminals if required]

Combine the productions of G_1 & G_2.

Add a new start symbol S and the productions: $S\to S_1$, $S\to S_2$

Concatenation if A and B are CFLs with $L(G_1)=A$ and $L(G_2)=B$ with start symbols S_1 & S_2. Construct G such that $L(G)=AB=\{xy \mid x\in A, y\in B\}$ as follows:

- Combine the grammars G_1 & G_2.
- Add a new start symbol S with production $S\to S_1S_2$

Kleene Star. if A is a CFL with $L(G_1)=A$ and start symbol S_1, Construct G s.t $L(G)=A^*$ as follows:

- Take G_1 along with a new start symbol S along with the production

$$S\to S_1S \mid \epsilon.$$

Intersection. CFLs are not closed under intersection.

$$\{a^n b^m c^n \mid m,n \geq 0\} \cap \{a^m b^n c^n \mid m,n \geq 0\}$$

$$= \underbrace{\{a^n b^n c^n \mid n \geq 0\}}_{\text{Not a CFL.}}$$

Intersection of a CFL and a regular set

Theorem. CFLs are closed under intersection with regular sets.

if $A \subseteq \Sigma^*$ is a CFL and $B \subseteq \Sigma^*$ is a regular set then $A \cap B$ is a CFL.

Proof idea. Consider an NPDA M_1 and a DFA M_2 s.t $L(M_1) = A$ and $L(M_2) = B$

NPDA N: Apply the product construction on M_1 and M_2

- States of N are product of states of M_1 and M_2
- Stack of N simulates stack of M_1