Working of the universal TM $\mathcal{U}$.

$\mathcal{U}$ takes as input an encoding of a TM M and a string x and simulates M on x.

- halts and accepts if M halts and accepts x
- halts and rejects if M halts and rejects x.
- loops if M loops on x.

$\mathcal{U}$ simulates M step by step.

Question. Can we do better than blind simulation?

Eg. If M halts on x then simulate M on x
if M does not halt on x then terminate the simulation and reject.

That is: Build $\mathcal{U}'$ that takes as input $M\#x$ and

- halts and accepts if M halts and accepts x
- halts and rejects if M halts and rejects x.
- halts and rejects if M loops on x.

Thus $L(\mathcal{U}') = L(\mathcal{U}) = \underbrace{\{M\#x \mid x \in L(M)\}}_{\text{recursive ?}}$

$HP = \{M\#x \mid M \text{ halts on } x\}$

Question. Is HP recursive?

Does there exists a total TM M s.t $L(M) = HP$

Answer. No

Question. Is HP recursively enumerable (r.e.)?

Does there exists a TM M s.t $L(M) = HP$?

Answer. Yes.

$MP = \{M\#x \mid x \in L(M)\}$

Question. Is MP recursively enumerable (r.e.)?

Does there exists a TM M s.t $L(M) = MP$?

Answer. Yes

Question. Is MP recursive?

Does there exists a total TM M s.t $L(M) = MP$

Answer. No.

Cantor's Diagonalization.

$$2^{\mathbb{N}} = \{A \mid A \subseteq \mathbb{N}\}$$

Claim. There does not exist a function $f : \mathbb{N} \to 2^{\mathbb{N}}$ that is onto. (Surjective)

Proof. Suppose there exist such an onto function f.

	0	1	2	3	4	5	6	7	8	...
$f(0)$	1	0	1	1	0	0	0	1	1	...
$f(1)$	0	0	1	1	0	1	0	0	0	...
$f(2)$	1	1	0	0	0	1	1	0	0	...
$f(3)$	0	0	1	0	1	0	0	1	1	...
$f(4)$	0	1	0	1	1	1	1	1	0	...
$\vdots$										

i^{th} row describes the set $f(i)$

$$(\overline{1\,0\,0\,0\,1\cdots}) = (0\,1\,1\,1\,0\cdots) = \{1, 2, 3, \dots\}$$

The general argument for any set A.

$f: A \to 2^A$; Let $B = \{x \in A \mid x \notin f(x)\}$

Then $B \subseteq A$

Since f is onto, $\exists y \in A$ s.t $f(y) = B$.

Question. Is $y \in f(y)$?

$y \in f(y)$ iff $y \in B$ (Since $B = f(y)$)

iff $y \notin f(y)$ (Definition of B)

So no such f exists.

Theorem. HP is not recursive.

For $x \in \{0,1\}^*$, let M_x denote the TM with input alphabet $\{0,1\}$ whose encoding is x.

	ϵ	0	1	00	01	11	000	001	--
M_ϵ	H	L	H	H	L	L	L	H	--
M_0	L	H	L	L	H	L	H	H	
M_1	H	L	H	H	L	H	L	H	
M_{00}	L	L	L	H	L	H	H	H	
M_{01}	H	H	L	H	H	L	L	L	
M_{11}	L	L	H	H	H	L	H	L	
M_{000}	L	H	L	H	L	H	H	L	
M_{001}	H	L	H	H	L	H	H	H	
⋮									

– x^{th} row describes for input y if M_x halts on y.

	ϵ	0	1	00	01	11	000	001 ---
M_ϵ	H	L	H	H	L	L	L	H ..
M_0	L	H	L	L	H	L	H	H
M_1	H	L	H	H	L	H	L	H
M_{00}	L	L	L	H	L	H	H	H
M_{01}	H	H	L	H	H	L	L	L
M_{11}	L	L	H	H	H	L	H	L
M_{000}	L	H	L	H	L	H	H	L
M_{001}	H	L	H	H	L	H	H	H

Suppose $\exists$ a total TM K such that $L(K) = HP$

For any x and y, K can determine the entry in the $(x,y)^{th}$ cell in the above table.

On input $M\#x$,
- K halts and accepts if M halts on x
- K halts and rejects if M loops on x

Consider a TM N that on input $x \in \{0,1\}^*$ does the following:

1) Constructs M_x from x and writes $M_x\#x$ on the tape

2) Runs K on input $M_x\#x$, accepting if K rejects and going into a trivial loop if K accepts.

For any $x \in \{0,1\}^*$, N halts on x iff K rejects $M\#x$ iff M_x loops on x.

That is, N is different from every M_x on at least one string - the string x.
This gives a contradiction.

Theorem. MP is not recursive.

Suppose $\exists$ a total TM K s.t $L(K) = MP$.

Given a TM M and input x. To check if M halts on x

Build a new TM N_M that does the following.

- Similar to M, N_M accepts if M accepts or rejects.

For all $x \in \Sigma^*$, N_m accepts x iff M halts on x.

For any M & x, to check if M halts on x

Construct N_M and run K on input $N_M \# x$

By assumption K is a total TM. But then we can construct a total TM K' s.t $L(K') = HP$

This is a contradiction.