DFA $M = (Q, \Sigma, \delta, s, F)$ where. $\delta : Q \times \Sigma \to Q$

We can "lift" δ as a transition function over strings.

$\hat{\delta} : Q \times \Sigma^* \to Q$.

Defining $\hat{\delta}$ inductively.

$$\hat{\delta}(q, \epsilon) = q$$

$$\hat{\delta}(q, xa) = \delta(\underbrace{\hat{\delta}(q, x)}_{\text{well defined by induction}}, a)$$

Claim 1. We can show that $\hat{\delta}(q,a) = \delta(q,a)$

$$\hat{\delta}(q,a) = \hat{\delta}(q, \epsilon a) \quad [\text{Since } a = \epsilon a]$$

$$= \delta(\hat{\delta}(q, \epsilon), a)$$

$$= \delta(q, a)$$

x is accepted by M if $\hat{\delta}(s, x) \in F$

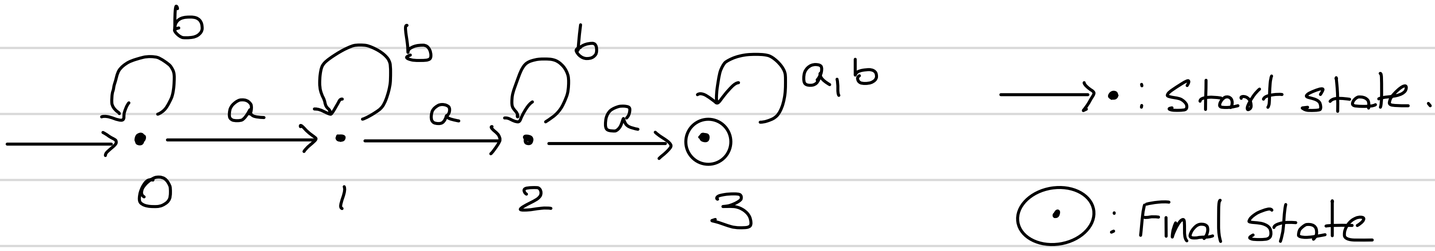
x is rejected by M if $\hat{\delta}(s, x) \notin F$

Language of the DFA M.

$$L(M) = \{ x \in \Sigma^* \mid \hat{\delta}(s, x) \in F \}$$

$A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M.

Example:



Input $x = a\,a\,b\,b\,a$

M accept x since $\hat{\delta}(0, x) = 3$ and $3 \in F$.

Input $y = a\,b\,b\,a\,b$

M rejects y since $\hat{\delta}(0, y) = 2$ and $2 \notin F$

$L(M) = \{ x \in \{a, b\}^* \mid x$ contains at least three a's $\}$.

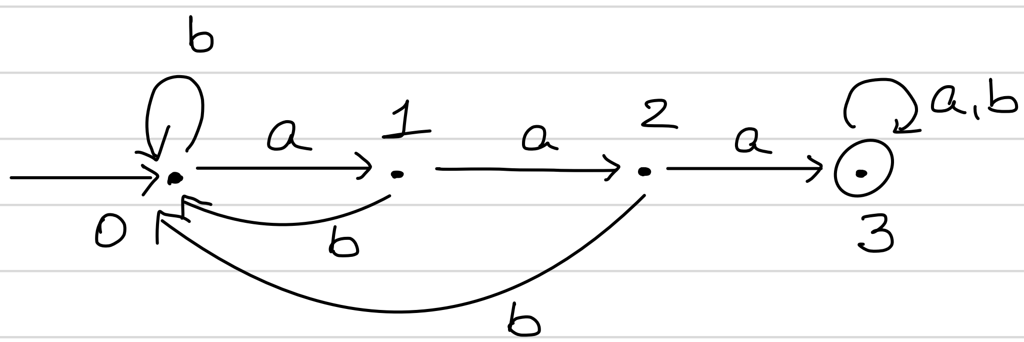
$A = \{x \in \{a,b\}^* \mid x$ contains a substring of three consecutive a's$\}$.

b a a b b a a a b b $\in A$

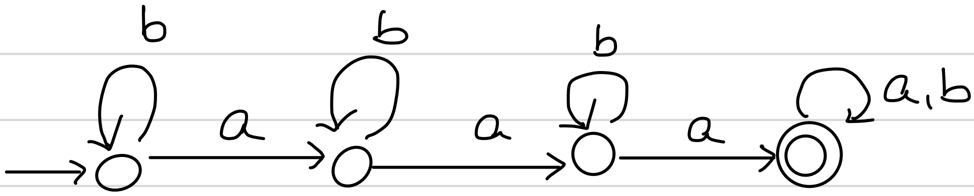
b a a b b a b b $\notin A$

Question. Is A regular ?

To show that A is regular it suffices to construct a DFA M such that $L(M) = A$.

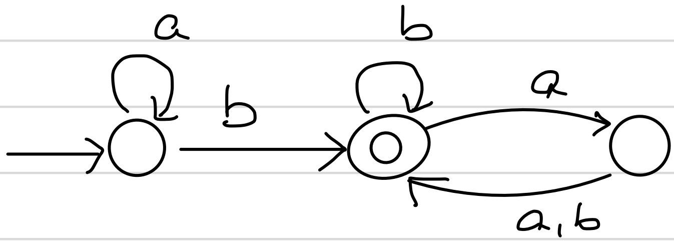


Example



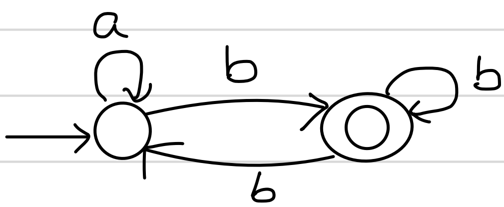
x: baabbaab $\in L(M)$
y: babbbab $\notin L(M)$.

$L(M) = \{x \in \Sigma^* \mid x$ contains 3 or more a's$\}$.



aabbb $\in L(M)$
aab $\in L(M)$
abaa $\in L(M)$
abaaa $\notin L(M)$

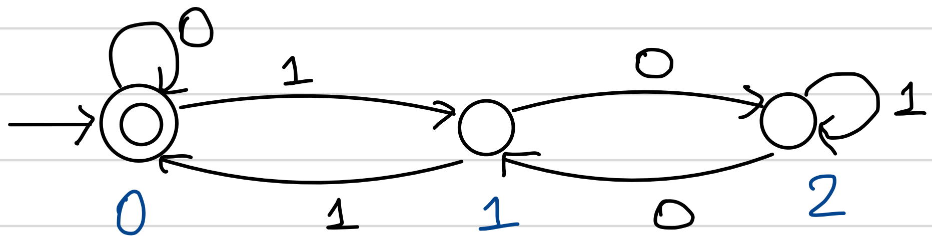
$L(M) = \{x \in \Sigma^* \mid x$ contains atleast one b and an even number of a's follow the last b$\}$.



$L(M) = \{x \in \Sigma^* \mid x$ ends in b$\}$

$A = \{x \in \{0,1\}^* \mid x \text{ represents a multiple of three in binary}\}$.

M:



For $x \in \{0,1\}^*$

$\hat{\delta}(0, x) = 0$ iff $\#x \equiv 0 \bmod 3$

$\hat{\delta}(0, x) = 1$ iff $\#x \equiv 1 \bmod 3$

$\hat{\delta}(0, x) = 2$ iff $\#x \equiv 2 \bmod 3$

To prove: $\hat{\delta}(0, x) = \#x \bmod 3$.

$$\left.\begin{aligned} \#(x0) &= 2(\#x) + 0 \\ \#(x1) &= 2(\#x) + 1 \end{aligned}\right\} \overbrace{\#(xc) = 2(\#x) + c \quad \forall c \in \{0,1\}.}^{B}$$

For all $q \in \{0,1,2\}$ and symbol $c \in \{0,1\}$.

$\delta(q, c) = (2q + c) \bmod 3$. —— A

Induction on $|x|$.

Base Case $x = \epsilon$.

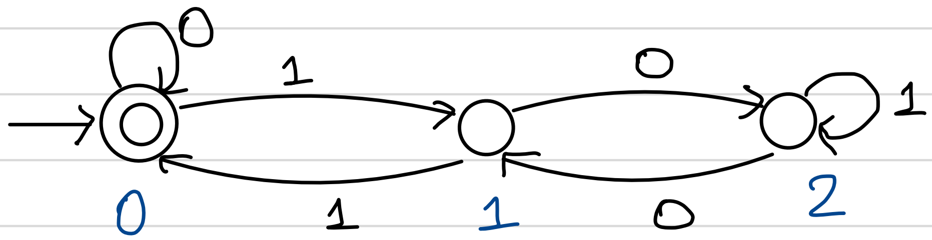
$\hat{\delta}(0, \epsilon) = 0$ [def. of $\hat{\delta}$]

$= \#\epsilon$

$= \#\epsilon \bmod 3$.

$A = \{x \in \{0,1\}^* \mid x \text{ represents a multiple of three in binary}\}$.

M:



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$$\hat{\delta}(0,x) = 2 \text{ iff } \#x \equiv 2 \bmod 3$$

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For all $q \in \{0,1,2\}$ and symbol $c \in \{0,1\}$.

$$\delta(q,c) = (2q + c) \bmod 3. \quad \text{———} \quad A$$

Induction on $|x|$.

$$\hat{\delta}(0,xc) = \delta(\hat{\delta}(0,x),c) \quad [\text{Definition of } \hat{\delta}]$$
$$= \delta(\#x \bmod 3, c) \quad [\text{Induction Hypothesis}]$$
$$= (2(\#x \bmod 3) + c) \bmod 3 \quad [\text{From } A]$$
$$= (2(\#x) + c) \bmod 3.$$
$$= \#xc \bmod 3. \quad [\text{From } B]$$

Closure properties of regular sets.

Let $A, B \subseteq \Sigma^*$

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} \qquad \text{Union}$$

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} \qquad \text{Intersection}$$

$$\overline{A} = \{x \in \Sigma^* \mid x \notin A\} \qquad \text{Complement}$$

Question. Are regular set closed under union, intersection, complementation?

Intersection:

if A and B are regular then is $A \cap B$ regular?

Let $A, B \subseteq \Sigma^*$

Statement: if A and B are regular then $A \cap B$ is regular.

$\exists M_1$ s.t $L(M_1) = A$; $\exists M_2$ s.t $L(M_2) = B$

To construct M_3 s.t $L(M_3) = A \cap B$.

M_3 runs over strings in Σ^* ; so $L(M_3) \subseteq \Sigma^*$

$M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ $\quad M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

$M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$.

$Q_3 = Q_1 \times Q_2 = \{ (q_1, q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2 \}$.

$s_3 = (s_1, s_2)$

$F_3 = F_1 \times F_2 = \{ (q_1, q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2 \}$.

$\delta_3 : Q_3 \times \Sigma \to Q_3$

$$\delta_3(\underbrace{(q_1, q_2)}_{\in Q_3}, a) = \underbrace{(\overbrace{\delta_1(q_1, a)}^{\in Q_1}, \overbrace{\delta_2(q_2, a)}^{\in Q_2})}_{\in Q_3}$$

M_3 - product of M_1 & M_2.