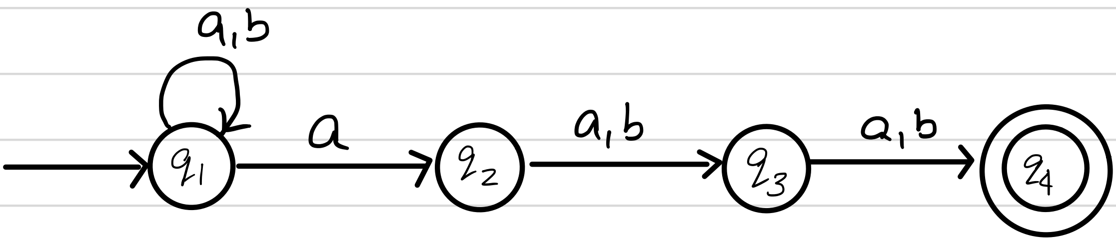
Example

$A = \{x \in \{a,b\}^* \mid \text{the third symbol from right is } a\}$

ababab b $\in A$

ababbab $\notin A$.

NFA N:



Claim: $L(N) = A$.

Nondeterministic Finite Automata (NFA)

$N = (Q, \Sigma, \Delta, S, F)$

Q - Finite set of states.

$S \subseteq Q$ - set of start states $\quad F \subseteq Q$ set of final states.

$\Delta : Q \times \Sigma \to 2^Q \quad$ where $2^Q = \{A \mid A \subseteq Q\}$.

$\Delta(q,a)$ - set of all states that N is allowed to move to from q in one step under the symbol a.

$q \xrightarrow{a} p$ if $p \in \underbrace{\Delta(q,a)}_{\text{can be empty.}}$

Extending Δ over strings.

$$\hat{\Delta} : 2^Q \times \Sigma^* \to 2^Q$$

$$\hat{\Delta}(A, \epsilon) = A.$$

$$\hat{\Delta}(A, xa) = \bigcup_{q \in \hat{\Delta}(A,x)} \Delta(q, a)$$

For $A \subseteq Q$ and $x \in \Sigma^*$, $\hat{\Delta}(A, x)$ is the set of all states reachable under string x from some state in A.

$r \in \hat{\Delta}(A, xa)$ if $\exists q \in \hat{\Delta}(A, x)$ and $r \in \Delta(q, a)$

$$p \overset{x}{\dashrightarrow} q \xrightarrow{a} r$$

$$\hat{\Delta}(A, a) = \bigcup_{p \in \hat{\Delta}(A, \epsilon)} \Delta(p, a) = \bigcup_{p \in A} \Delta(p, a)$$

N accepts $x \in \Sigma^*$ if $\hat{\Delta}(S, x) \cap F \neq \phi$.

$$L(N) = \{x \in \Sigma^* \mid N \text{ accepts } x\}.$$

DFA $M=(Q,\Sigma,\delta,s,F)$ and NFA $N=(Q,\Sigma,\Delta,S,F)$

Statement. NFAs and DFAs have the same expressive power

Note. A DFA M can be easily "converted" to an equivalent NFA.
Take $S=\{s\}$ and $\Delta(q,a)=\{\delta(q,a)\}$

Subset Construction – Converts an NFA to an equivalent DFA.

Let $N=(Q_N,\Sigma,\Delta_N,S_N,F_N)$ to construct

$M=(Q_M,\Sigma,\delta_M,s_M,F_M)$ s.t $L(M)=L(N)$.

$$Q_M = 2^{Q_N} \quad \text{– the power set of } Q_N.$$

$$\delta_M(A,a)=\hat{\Delta}_N(A,a)$$

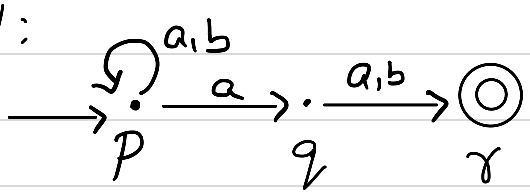
$$s_M = S_N$$

$$F_M=\{A\subseteq Q_N \mid A\cap F_N\neq\emptyset\}.$$

Example.

$A=\{x\in\{a,b\}^* \mid \text{the second symbol from right is } a\}.$

N:



$Q_M=\{\emptyset,\{p\},\{q\},\{r\},$
$\{p,q\},\{p,r\},\{q,r\},$
$\{p,q,r\}\}$

Lemma 1. For any $x, y \in \Sigma^*$, and $A \subseteq Q$.

$$\hat{\Delta}(A, xy) = \hat{\Delta}(\hat{\Delta}(A, x), y).$$

[Proof by induction on $|y|$]

Lemma 2. For any $A \subseteq Q_N$, and $x \in \Sigma^*$

$$\hat{\delta}_M(A, x) = \hat{\Delta}_N(A, x)$$

Proof. By induction on $|x|$

Base Case: $x = \epsilon$ $\quad \hat{\delta}_M(A, \epsilon) = A \quad \hat{\Delta}_N(A, \epsilon) = A.$

$=$

Induction Step.

$$\hat{\delta}_M(A, xa) = \delta_M(\hat{\delta}_M(A, x), a) \quad [\text{defn. of } \hat{\delta}_M]$$

$$= \delta_M(\hat{\Delta}_N(A, x), a) \quad [\text{Induction Hypothesis}]$$

$$= \hat{\Delta}_N(\hat{\Delta}_N(A, x), a) \quad [\text{Defn of } \delta_M]$$

$$= \hat{\Delta}_N(A, xa) \quad [\text{Lemma 1.}]$$

Theorem. $L(M) = L(N)$

For $x \in \Sigma^*$ $\quad x \in L(M) \iff \hat{\delta}_M(s_M, x) \in F_M \quad [\text{acceptance}]$

$$\iff \hat{\Delta}_N(s_N, x) \cap F_N \neq \phi \quad [\text{Lemma 2, } s_M, F_M]$$

$$\iff x \in L(N) \quad [\text{Defn of acceptance of } N]$$