Quotient Automata.

DFA $M = (Q, \Sigma, \delta, s, F)$ $\quad M/\approx = (Q', \Sigma, \delta', s', F')$

$Q' = \{[p] \mid p \in Q\}$ - states of $M/\approx$ are the $\approx$-equivalence classes.

$\delta'([p], a) = [\delta(p, a)]$

$s' = [s]$, $F' = \{[p] \mid p \in F\}$

To show. δ' is well defined.

Lemma 1. if $p \approx q$ then $\delta(p,a) \approx \delta(q,a)$

That is, if $[p] = [q]$ then $[\delta(p,a)] = [\delta(q,a)]$

Proof. Suppose $p \approx q$. For $a \in \Sigma$ and $y \in \Sigma^*$

$\hat{\delta}(\delta(p,a), y) \in F$ iff $\hat{\delta}(p, ay) \in F$

iff $\hat{\delta}(q, ay) \in F$ [Since $p \approx q$]

iff $\hat{\delta}(\delta(q,a), y) \in F$.

Since the above holds for all $y \in \Sigma^*$, $\delta(p,a) \approx \delta(q,a)$

[By defn of $\approx$].

DFA $M = (Q, \Sigma, \delta, s, F)$ $\quad M/\approx = (Q', \Sigma, \delta', s', F')$

Lemma 2. $p \in F$ iff $[p] \in F'$.

Proof. $\Rightarrow$ Follows from the definition of F'

$\Leftarrow$ if $p \approx q$ and $p \in F$ then $q \in F$. That is,

every $\approx$-equivalence class is either a subset of F or disjoint from F.

Follows by taking $x = \epsilon$ in the definition of $p \approx q$.

Lemma 3. For all $x \in \Sigma^*$, $\hat{\delta}'([p], x) = [\hat{\delta}(p, x)]$

Proof. By induction on $|x|$.

Base case: $x = \epsilon$. $\hat{\delta}'([p], \epsilon) = [p] = [\hat{\delta}(p, \epsilon)]$

Induction Step.

$$\hat{\delta}'([p], xa) = \delta'(\hat{\delta}'([p], x), a) \quad [\text{Defn of } \hat{\delta}']$$

$$= \delta'([\hat{\delta}(p, x)], a) \quad [\text{Induction hypothesis}]$$

$$= [\delta(\hat{\delta}(p, x), a)] \quad [\text{Defn. of } \delta']$$

$$= [\hat{\delta}(p, xa)] \quad [\text{Defn of } \hat{\delta}]$$

Lemma 3. For all $x \in \Sigma^*$, $\hat{\delta}'([p], x) = [\hat{\delta}(p, x)]$

Lemma 2. $p \in F$ iff $[p] \in F'$.

Theorem. $L(M/\approx) = L(M)$.

Proof. For $x \in \Sigma^*$,

$x \in L(M/\approx)$ iff $\hat{\delta}'(s', x) \in F'$

iff $\hat{\delta}'([s], x) \in F'$ [Defn. of s']

iff $[\hat{\delta}(s, x)] \in F'$ [Lemma 3]

iff $\hat{\delta}(s, x) \in F$ [Lemma 2]

iff $x \in L(M)$ [Defn. of acceptance]

What if you do the quotient construction again on $M/\approx$?

$$[p] \sim [q] \text{ iff } \forall x \in \Sigma^* \left(\hat{\delta}'([p], x) \in F' \text{ iff } \hat{\delta}'([q], x) \in F' \right)$$

Use $\sim$ to denote the equivalence relation on Q' to distinguist it from the relation $\approx$ on Q.

$$[p] \sim [q] \Rightarrow \forall x \left(\hat{\delta}'([p], x) \in F' \text{ iff } \hat{\delta}'([q], x) \in F' \right)$$

[Defn. of $\sim$]

$$\Rightarrow \forall x \left([\hat{\delta}(p, x)] \in F' \text{ iff } [\hat{\delta}(q, x)] \in F' \right)$$

[Lemma 3]

$$\Rightarrow \forall x \left(\hat{\delta}(p, x) \in F \text{ iff } \hat{\delta}(q, x) \in F \right)$$

[Lemma 2]

$$\Rightarrow p \approx q \Rightarrow [p] = [q]$$

$\therefore$ Two equivalent states of $M/\approx$ are equal and $\sim \subseteq Q' \times Q'$ is the identity relation.

Algorithm for DFA minimization

Algorithm takes as input a DFA $M = (Q, \Sigma, \delta, s, F)$ and marks pairs of states (unordered)

A pair $\{p, q\}$ will be marked when the procedure detects that p and q are not equivalent.

Algorithm.

1. All pairs of states $\{p, q\}$ are initially unmarked.
2. if $p \in F$ and $q \notin F$ or vice versa, mark $\{p, q\}$.
3. Repeat until no change in marking
4. if $\exists \{p, q\}$-unmarked such that $\{\delta(p, a), \delta(q, a)\}$ is marked for some $a \in \Sigma$ then mark $\{p, q\}$.
5. $p \approx q$ iff $\{p, q\}$ is not marked.

$$\boxed{p \approx q \text{ iff } \forall x \in \Sigma^* \left(\hat{\delta}(p, x) \in F \text{ iff } \hat{\delta}(q, x) \in F\right)}$$

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Termination. There are atmost $\binom{n}{2}$ pairs.

Step 4 marks at least one pair in each iteration

$\therefore$ Repeat step [line 3] can have atmost $\binom{n}{2}$ iterations.

Algorithm.

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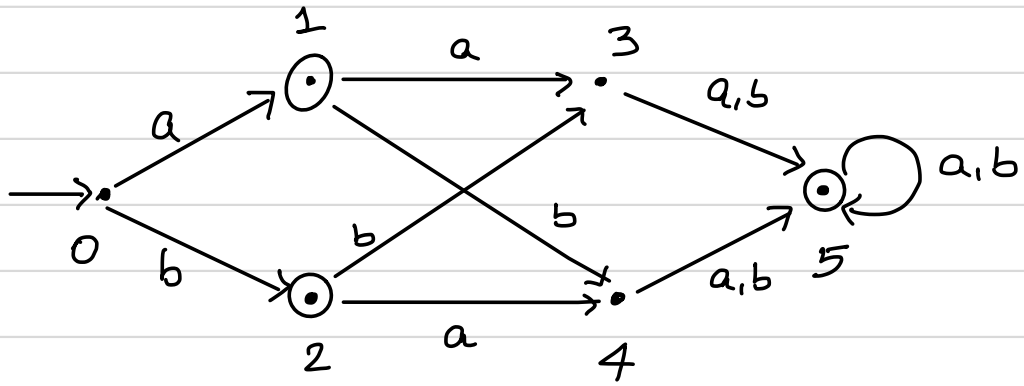
Example.



0					
✓	1				
✓	—	2			
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✓	✓	✓	✓	✓	5

$\{0,3\} \xrightarrow{a} \{1,5\}$.

$\{0,3\} \xrightarrow{b} \{2,5\}$

$\{1,5\} \xrightarrow{a} \{3,5\}$.

$\{2,5\} \xrightarrow{a} \{4,5\}$.

Theorem. The pair $\{p, q\}$ is marked by Ito Algorithm iff $\exists x \in \Sigma^*$, s.t $\hat{\delta}(p, x) \in F$ and $\hat{\delta}(q, x) \notin F$ or vice versa. That is, iff $p \not\approx q$.

Proof. By induction.

Two DFAs $M = (Q_M, \Sigma, \delta_M, s^M, F_M)$

$N = (Q_N, \Sigma, \delta_N, s^N, F_N)$

M is isomorphic to N if $\exists$ a bijection $F : Q_M \to Q_N$

s.t $F(s^M) = s^N$; $p \in F_M$ iff $F(p) \in F_N$

$F(\delta_M(p, a)) = \delta_N(F(p), a) \quad \forall p \in Q_M, \ \forall a \in \Sigma$

Myhill - Nerode Theorem.

if M and N are any two DFAs with no inaccessible states s.t $L(M) = L(N)$, then $M/\approx$ and $N/\approx$ are isomorphic.

Corollary.

The output of the collapsing algorithm is the minimal DFA for the set and it is unique upto isomorphism.

Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA s.t $L(M) = R$

M induces an equivalence relation $\equiv_M$ on Σ^* defined by

$$x \equiv_M y \text{ iff } \hat{\delta}(s, x) = \hat{\delta}(s, y)$$

Observation. $\equiv_M$ is an equivalence relation.

$\equiv_M$ also satisfies the following properties

(P1) It is a right congruence: $\forall x, y \in \Sigma^*, \forall a \in \Sigma$

$$x \equiv_M y \Rightarrow xa \equiv_M ya.$$

Suppose $x \equiv_M y$. Then $\delta(s, xa) = \delta(\hat{\delta}(s, x), a)$
$= \delta(\hat{\delta}(s, y), a)$
$= \hat{\delta}(s, ya)$

(P2) It refines R: $\forall x, y \in \Sigma^*$

$$x \equiv y \Rightarrow (x \in R \text{ iff } y \in R)$$

Since $\hat{\delta}(s, x) = \hat{\delta}(s, y)$

(P3) it is of finite index: there are only finitely many equivalence classes.

exactly one equivalence class corresponding to each $q \in Q$.

$\{x \in \Sigma^* \mid \hat{\delta}(s, x) = q\}$

Myhill-Nerode Relation

Let $R \subseteq \Sigma^*$ be a regular set.

An equivalence relation $\equiv$ on Σ^* is a Myhill-Nerode relation for R if it satisfies the following 3 properties

(P1) It is a right congruence: $\forall x, y \in \Sigma^*, \forall a \in \Sigma$

$$x \equiv y \Rightarrow xa \equiv ya.$$

(P2) It refines R: $\forall x, y \in \Sigma^*$

$$x \equiv y \Rightarrow (x \in R \text{ iff } y \in R)$$

(P3) it is of finite index: there are only finitely many equivalence classes.

Observation.

Any DFA $M = (Q, \Sigma, S, \delta, F)$ with $L(M) = R$ defines a Myhill-Nerode relation $\equiv_M$:

Let $R \subseteq \Sigma^*$ and $\equiv$ be any Myhill-Nerode relation for R. From $\equiv$ we can construct a DFA $M_\equiv$ s.t $L(M_\equiv) = R$.

For $x \in \Sigma^*$ let $[x] = \{y \mid y \equiv x\}$ be the equivalence classes of string x.

Note: By (P3) there are only finitely many equivalence classes.

$M_\equiv = (Q, \Sigma, \delta, s, F)$ where

$Q = \{[x] \mid x \in \Sigma^*\}$; $s = [\epsilon]$; $F = \{[x] \mid x \in R\}$

$\delta([x], a) = [xa]$.

Question. if you choose another $y \in [x]$, can it be that $[xa] \neq [ya]$? No by right congruence.

Proposition 1. $x \in R$ iff $[x] \in F$.

$\Rightarrow$ Definition of F.

$\Leftarrow$ Definition of F and (P2).

Lemma 1. $\hat{\delta}([x], y) = [xy]$.

Proof. Induction on $|y|$

Base Case: $\hat{\delta}([x], \epsilon) = [x] = [x\epsilon]$

Induction Step:

$$\hat{\delta}([x], ya) = \delta(\hat{\delta}([x], y), a)$$

$$= \delta([xy], a) \quad \text{(Induction hypothesis)}$$

$$= [xya] \quad \text{(Definition of } \delta\text{)}$$

Theorem. $L(M_{\equiv}) = R$

Proof

$$x \in L(M_{\equiv}) \Leftrightarrow \hat{\delta}([\epsilon], x) \in F$$

$$\Leftrightarrow [x] \in F \quad \text{(Lemma 1)}$$

$$\Leftrightarrow x \in R \quad \text{(Proposition 1)}$$

Lemma 2. Let $\equiv$ be a Myhill-Nerode relation for $R \subseteq \Sigma^*$. if we apply the construction $\equiv \mapsto M_\equiv$ and then apply the construction $M \mapsto \equiv_M$ to the result, the resulting relation $\equiv_{M_\equiv}$ is identical to $\equiv$

Proof.

Let $M_\equiv = (Q, \Sigma, S, s, F)$

For any $x, y \in \Sigma^*$

$x \equiv_{M_\equiv} y$ iff $\hat{\delta}(s, x) = \hat{\delta}(s, y)$ (Defn of $\equiv_{M_\equiv}$)

iff $\hat{\delta}([\epsilon], x) = \hat{\delta}([\epsilon], y)$ (Defn of s)

iff $[x] = [y]$ (By Lemma 1)

iff $x \equiv y$

Lemma 3. Let M be a DFA for R (with no inaccessible states).
If we apply the construction $M \mapsto \equiv_M$ and then apply $\equiv \mapsto M_{\equiv}$, the resulting DFA $M_{\equiv_M}$ is isomorphic to M

Proof. Let $M = (Q, \Sigma, \delta, s, F)$ and $M_{\equiv_M} = (Q', \Sigma, \delta', s', F')$

$$[x] = \{y \mid y \equiv_M x\} = \{y \mid \hat{\delta}(s,y) = \hat{\delta}(s,x)\}$$

$$Q' = \{[x] \mid x \in \Sigma^*\} \qquad s' = [\epsilon] \qquad F' = \{[x] \mid x \in R\}$$

$$\delta'([x], a) = [xa]$$

Consider the function $f: Q' \to Q$ defined as

$$f([x]) = \hat{\delta}(s,x)$$

$[x] = [y]$ iff $\hat{\delta}(s,x) = \hat{\delta}(s,y)$ (From defn of $\equiv_M$)
So f is well-defined and one-to-one.

f is onto since M has no inaccessible states.

claim. $M_{\equiv_M}$ and M are isomorphic under f.

Consider the function $f: Q' \to Q$ defined as
$$f([x]) = \hat{\delta}(s, x)$$

claim. $M_{\equiv_M}$ and M are isomorphic under f.
To show.

(1) $f(s') = s$

(2) $f(\delta'([x], a)) = \delta(f([x]), a)$

(3) $[x] \in F'$ iff $f([x]) \in F$

(1). $f(s') = f([\epsilon])$ (Defn of s')
$= \hat{\delta}(s, \epsilon)$ (defn of f)
$= s$ (Defn of $\hat{\delta}$)

(2). $f(\delta'([x], a)) = f([xa])$ (Defn of δ')
$= \hat{\delta}(s, xa)$ (Defn of f)
$= \delta(\hat{\delta}(s, x), a)$ (Defn of $\hat{\delta}$)
$= \delta(f([x]), a)$ (Defn of f)

(3). $[x] \in F'$ iff $x \in R$ (Defn of F and prop 2)
iff $\hat{\delta}(s, x) \in F$ (since $L(M) = R$)
iff $f([x]) \in F$ (Defn of f).

Theorem. Let Σ be a finite alphabet. Upto isomorphism of automata, there is a one-to-one correspondence between deterministic finite automata over Σ (with no inaccessible states) accepting $R \subseteq \Sigma^*$ and Myhill-Nerode relations for R on Σ^*.