Introduction to the Theory of Computation - Sipser

Automata and Computability - Kozen

Introduction to Automata Theory, Languages and Computation - Hopcroft, Ullman, Motwani

Decision Problem. Function with a one-bit output: "Yes" or "No".

How to specify a decision problem?

- Set A of possible inputs.
- Subset $B \subseteq A$ of "Yes" instances.

An abstraction:

Set of possible inputs to a decision problem

consists of

Set of finite length strings over some fixed finite alphabet.

Alphabet - finite set - denoted by Σ

Ex. $\{0, 1, 2, \ldots, 9\}$ - decimal numbers.

$\{0, 1\}$ - bit strings.

Notation: $a, b \in \Sigma$.

Strings over Σ.
finite length sequence of elements of Σ.

Ex. $\Sigma = \{a, b\}$ abbaa - string of length 5.

Notation:

x, y, z - denote strings.

$|x|$ - number of symbols in x.

length of the string x.

Unique string of length 0 - null string.

ϵ [epsilon].

$|\epsilon| = 0$.

$a \in \Sigma$, a^n - string of a's of length n.

$a^0 = \epsilon$ $\quad$ $a^{n+1} = a^n a$

Set of all strings over an alphabet Σ - denoted by Σ^*.

Ex.

$\{a, b\}^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$.

$\{a\}^* = \{\epsilon, a, aa, aaa, \ldots\}$.

$= \{a^n \mid n \geq 0\}$

$\phi^* = \{\epsilon\}$

String and sets are not the same.

$\{a, b\} = \{b, a\}$ / $ab \neq ba$

$\{a, a, b\} = \{a, b\}$ / $aab \neq ab$

ϕ - empty set

ϵ - null string.

$\{\epsilon\}$ - Set with one element - null string.

Operations on Strings

Concatenation. 2 string x & y and creates a new string xy

Note. $xy \neq yx$ are in general different

- Concatenation is associative $(xy)z = x(yz)$
- null string ϵ is identity for concatenation

$$\epsilon x = x\epsilon = x$$

- $|xy| = |x| + |y|$

Set Σ^* with concatenation as a binary operator and ϵ as identity is a monoid.

Let $x \in \Sigma^*$

x^n - String consisting of n copies of x.

Example: $(abb)^3 = abb\,abb\,abb$.

Formally x^n can be defined inductively.

- $x^0 = \epsilon$.
- $x^{n+1} = x^n x$

Prefix of a String.

Prefix of x is a string y s.t there exists z with $x = yz$.

- Null string is a prefix of every string.
- Every string is a prefix of itself.

Operations on sets. : subsets of Σ^*

A, B - sets of strings.

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Complement $\bar{A} = \{x \in \Sigma^* \mid x \notin A\}$.

Set Concatenation

$AB = \{xy \mid x \in A \text{ and } y \in B\}$.

$\{a, ab\}\{b, ba\} = \{ab, aba, abb, abba\}$

Note: In general AB and BA are different.

A^n - inductively

$A^0 = \{\epsilon\}$ $A^{n+1} = AA^n$

$\{ab, aab\}^0 = \{\epsilon\}$
$\{ab, aab\}^1 = \{ab, aab\}$.
$\{ab, aab\}^2 = \{abab, abaab, aabab, aabaab\}$.

$\{a, b\}^n = \{x \in \{a,b\}^* \mid |x| = n\}$.

A^* - Union of all finite powers of the set A

Σ^* - Set of all finite strings over Σ.

$$A^* = \bigcup_{n\geq 0} A^n = A^0 \cup A^1 \cup A^2 \cup \cdots$$

$$A^* = \{ x_1 x_2 \ldots x_n \mid n \geq 0 \text{ and } x_i \in A,\ 1 \leq i \leq n \}$$

ϵ is in A^* for any A - Since n can be 0.

$$A^+ = AA^* = \bigcup_{n\geq 1} A^n$$

Properties of Set operations

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

$(AB)C = A(BC)$

Commutative

Associative.

Concatenation is not Commutative

$\{\epsilon\}$ - Identity for concatenation.

$\{\epsilon\} A = A \{\epsilon\} = A$

$A\phi = \phi A = \phi$

ϕ is an annihilator for set concatenation.

$*$ satisfies the following properties:

$A^* A^* = A^*$

$A^{**} = A^*$

$\phi^* = \{\epsilon\}$