$A = \{a^n b^n \mid n \geq 0\}$. - not regular.

Recursive definition

- $\epsilon \in A$
- if $w \in A$ then $awb \in A$.

Context Free Languages.

Context Free Grammars.

$$G = (N, \Sigma, P, S).$$

N - finite set (non-terminal symbols).

Σ - finite set (terminal symbols)

Assumption : $N \cap \Sigma = \phi$.

P - finite subset of $N \times (N \cup \Sigma)^*$ [productions]
$A \rightarrow \alpha$

$S \in N$ (Start symbol).

Notation. A, B, C - non terminal symbols
a, b, c - terminal symbols

α, β, γ - strings over $(N \cup \Sigma)^*$

$L = \{a^n b^n \mid n \geq 0\}$ - is a CFL.

CFG. $G = (N, \Sigma, P, S)$ $N = \{S\}$, $\Sigma = \{a, b\}$.

$P = \{S \to aSb,\ S \to \epsilon\}$.

$S \to aSb \mid \epsilon$

$a^3 b^3$: $S \xrightarrow[G]{1} aSb \xrightarrow[G]{1} a\,aSbb$

$\xrightarrow[G]{1} aaaSbbb \xrightarrow[G]{1} aaabbb$.

Context Free Grammars.

$$G = (N, \Sigma, P, S).$$

N - finite set (non-terminal symbols).

Σ - finite set (terminal symbols). $\Sigma \cap N = \phi$

P - finite subset of $N \times (N \cup \Sigma)^*$ [productions]

$P \subseteq N \times (N \cup \Sigma)^*$ $\quad \{(A, \alpha_1), (A, \alpha_2), (A, \alpha_3)\} \subseteq P$

$A \rightarrow \alpha_1 \mid \alpha_2 \mid \alpha_3$.

$S \in N$ (Start Symbol).

Suppose $\alpha, \beta \in (N \cup \Sigma)^*$. β is derivable from α in one step $\alpha \xrightarrow{1} \beta$ if β can be obtained from α by replacing some occurence of a non-terminal A in α with γ where $A \rightarrow \gamma \in P$.

if $\exists\ \alpha_1, \alpha_2 \in (N \cup \Sigma)^*$ s.t $\alpha = \alpha_1 A \alpha_2$ and $\exists\ A \rightarrow \gamma \in P$ then $\alpha \underset{G}{\xrightarrow{1}} \beta = \alpha_1 \gamma \alpha_2$

$\alpha \xrightarrow[G]{1} \beta$ - one step derivation

$\xrightarrow[G]{*}$: reflexive transitive closure of the relation $\xrightarrow[G]{1}$.

$\alpha \xrightarrow[G]{0} \alpha$ for all α

$\alpha \xrightarrow[G]{n+1} \beta$ if $\exists \gamma$ s.t $\alpha \xrightarrow[G]{n} \gamma$ and $\gamma \xrightarrow[G]{1} \beta$.

$\alpha \xrightarrow[G]{*} \beta$ if $\alpha \xrightarrow[G]{n} \beta$ for some $n \geq 0$.

Language generated by G.

$$L(G) = \{x \in \Sigma^* \mid S \xrightarrow[G]{*} x\}$$

$B \subseteq \Sigma^*$ is a Context free language (CFL) if

$B = L(G)$ for some CFG G.

$Z = \{a^n b^n \mid n \geq 0\}$ - is a CFL.

CFG. $G = (N, \Sigma, P, S)$ $\quad N = \{S\}$, $\Sigma = \{a, b\}$.

$P = \{S \to aSb,\ S \to \epsilon\}$.

$L(G) = Z$.

$a^3 b^3$: $\quad S \xrightarrow[G]{1} aSb \xrightarrow[G]{1} a\,aSbb$

$\xrightarrow[G]{1} aaaSbbb \xrightarrow[G]{1} aaabbb.$

By induction on n we can show: $S \xrightarrow[G]{n+1} a^n b^n$

$\Rightarrow$ all strings of the form $\quad a^n b^n \in L(G)$.

Conversly, the only strings in $L(G)$ are of the form $a^n b^n$ - induction on the length of the derivation.

Example

$X = \{ w \in \{0,1\}^* \mid w$ has equal number of 0's and 1's$\}$

$S \rightarrow 0S1S \mid 1S0S \mid \epsilon$

$X = \{ w \in \{0,1\}^* \mid w = rev(w) \}$

$S \rightarrow \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$

w is even

$S \rightarrow \epsilon \mid 0S0 \mid 1S1$

$X = \{ w \in \{0,1\}^* \mid w$ is odd and middle symbol is $0 \}$

$S \rightarrow 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid 0$

$X = \{ w \in \{0,1\}^* \mid w$ contains at least three 1s$\}$

$S \rightarrow A1A1A1A$

$A \rightarrow 0A \mid 1A \mid \epsilon$

$$X = \{ a^i b^j c^k \mid i, j, k \geq 0 ; \ i + j = k \}$$

$$S \rightarrow aSc \mid Z$$

$$Z \rightarrow bZc \mid \epsilon$$