EE2100: Matrix Theory

Quiz - 4

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1. (3 points) Compute the value if x such that $\mathbf{Det}(\mathbf{A}) = c$ where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & x \\ 2 & 3 & 4 \end{bmatrix}$

Solution: The determinant of the given matrix is $\mathbf{Det}(\mathbf{A}) = x - 6 \implies x = c + 6$

2. (3 points) Let x be such that $\mathbf{Det}(\mathbf{A}) = c$ where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & x \end{bmatrix}$. Compute the determinant of matrix $\mathbf{B}$ given by $\mathbf{B} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 13 & 18 \\ 4 & 5 & x \end{bmatrix}$

Solution: Let $\mathbf{a}_1^T, \mathbf{a}_2^T$ and $\mathbf{a}_3^T$ denote the row vectors of $\mathbf{A}$. The row vectors of $\mathbf{B}$ are $2\mathbf{a}_1^T, 3\mathbf{a}_2^T + 2\mathbf{a}_1^T, \mathbf{a}_3^T$. Thus,

$$\begin{aligned} \mathbf{Det}(\mathbf{B}) &= \mathbf{Det}\left(2\mathbf{a}_1^T, 3\mathbf{a}_2^T + 2\mathbf{a}_1^T, \mathbf{a}_3^T\right) \\ &= 2\mathbf{Det}\left(\mathbf{a}_1^T, 3\mathbf{a}_2^T + 2\mathbf{a}_1^T, \mathbf{a}_3^T\right) \\ &= 6\mathbf{Det}\left(\mathbf{a}_1^T, \mathbf{a}_2^T, \mathbf{a}_3^T\right) + 4\underbrace{\mathbf{Det}\left(\mathbf{a}_1^T, \mathbf{a}_1^T, \mathbf{a}_3^T\right)}_{0} \\ &= 6c \end{aligned} \tag{1}$$

3. (3 points) Let x and y be such that the maximum eigen value of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & y & x \end{bmatrix}$ is c. Compute the maximum eigen value of matrix $\mathbf{B}$ given by $\mathbf{B} = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & y & x-2 \end{bmatrix}$.

Solution: Observe that $\mathbf{B} = \mathbf{A} - 2\mathbf{I}$. If λ is the eigen value of $\mathbf{A}$, then $\lambda - 2$ is the eigen value of $\mathbf{B}$. Thus, the maximum eigen value of $\mathbf{B}$ is $c - 2$.

4. (3 points) Let $\mathbf{A} \in \mathcal{R}^{3\times 3}$ and $\mathbf{B} \in \mathcal{R}^{3\times 3}$ be full rank matrices such that the eigen values of $\mathbf{AB}$ are $1, 2$ and 3. If $\mathbf{Tr}(\mathbf{B}^T\mathbf{B}) = c$ and $\mathbf{Tr}(\mathbf{A}^T\mathbf{A}) = \frac{14}{c}$, the minimum eigen value of $\mathbf{BA}$ is

Solution: If λ is an eigen value of $\mathbf{AB}$, then λ is also an eigen value of $\mathbf{BA}$.

Proof: Since λ is an eigen value of $\mathbf{AB}$, there exists a vector $\mathbf{v}$ such that

$$\mathbf{ABv} = \lambda\mathbf{v} \tag{2}$$

Pre-multiplying (2) with $\mathbf{B}$ gives

$$(\mathbf{BA})\mathbf{Bv} = \lambda\mathbf{Bv} \tag{3}$$

Thus, λ is an eigen value of $\mathbf{BA}$ with the corresponding vector $\mathbf{Bv}$.

The minimum eigen value of $\mathbf{BA}$ is 1.

5. (3 points) Let $\mathbf{A}^{3\times 3}$ be a skew symmetric matrix whose eigen values are 0, $-cj$. The imaginary part of the other eigen value of $\mathbf{A}$ is

Solution: Since $\mathbf{A}$ is a skew symmetric matrix, the sum of its eigen values is 0. Thus,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \implies \lambda_3 = \lambda_2 + \lambda_1 \implies \lambda_3 = cj \tag{4}$$