EE2100: Matrix Theory

Quiz - 1

Please Note :

1. For numerical questions, answers within the accuracy of $\pm 2\%$ are considered good enough to receive full points.

2. If there are any inconsistencies in grading, please indicate it as a comment in the corresponding page on Canvas.

Group 1

(One among the following set of questions)

1. (2 points) Consider $\mathbf{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathcal{R}^3$. Compute the angle that $\mathbf{a}$ makes with the standard basis vector $\mathbf{e}_3$.

Solution: If θ denoted the angle between $\mathbf{a}$ and $\mathbf{e}_3$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{e}_3}{\|\mathbf{a}\|\|\mathbf{e}_3\|} = \frac{z}{\sqrt{x^2+y^2+z^2}} \implies \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \tag{1}$$

The final answer varies depending on the value of x, y and z given in the question.

2. (2 points) Consider $\mathbf{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathcal{R}^3$. Compute the angle that $\mathbf{a}$ makes with the standard basis vector $\mathbf{e}_2$.

Solution: If θ denoted the angle between $\mathbf{a}$ and $\mathbf{e}_2$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{e}_2}{\|\mathbf{a}\|\|\mathbf{e}_3\|} = \frac{y}{\sqrt{x^2+y^2+z^2}} \implies \theta = \cos^{-1}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) \tag{2}$$

The final answer varies depending on the value of x, y and z given in the question.

Group 2

(Understanding the role of vectors in operations related to signals)

1. Let $\mathbf{x} \in \mathcal{R}^N$ and $\mathbf{y} \in \mathcal{R}^N$ denote a finite discrete time signal of length N. (Note that the discrete-time signals are usually indexed from 0). The auto-correlation function of a signal $x[n]$ (typically denoted as $R_{\mathbf{xx}}$) is defined as

$$R_{\mathbf{xx}}[k] = \sum_n x[n]x[n-k] \tag{3}$$

On the other hand, the cross-correlation between two signals $x[n]$ and $y[n]$ is defined as

$$R_{\mathbf{xy}}[k] \;=\; \sum_n x[n]y[n-k] \tag{4}$$

Finally, the correlation coefficient $C_{\mathbf{xy}}$ (which can be considered as a measure of similarity between two signals) is defined as

$$C_{\mathbf{xy}} \;=\; \frac{R_{\mathbf{xy}}[0]}{\sqrt{R_{\mathbf{xx}}[0]R_{\mathbf{yy}}[0]}} \tag{5}$$

(a) (2 points) Compute the auto-correlation coefficient $R_{\mathbf{xx}}[0]$ for a discrete-time signal of length 8, which, when represented as a vector $\mathbf{x} \in \mathcal{R}^8$, is given by

$$\mathbf{x} = \sum_{i=1}^{8} \alpha_i \mathbf{e}_i \tag{6}$$

(Note: To calculate the exact numerical answer, substitute the value of α_i given in the question).

Solution: The auto-correlation coefficient $R_{\mathbf{xx}}[0]$ is given by

$$R_{\mathbf{xx}}[0] \;=\; \sum_n x[n]x[n] \tag{7}$$

Equation (7) indicates that the auto-correlation is related to norm of $\mathbf{x}$ i.e., $R_{\mathbf{xx}}[0] = \|\mathbf{x}\|^2$. Accordingly,

$$R_{\mathbf{xx}}[0] = \sum_{i=1}^{8} \alpha_i^2 \tag{8}$$

(b) (2 points) Compute the cross-correlation coefficient $R_{\mathbf{xy}}[0]$ for discrete-time signals (of length 8), which, when represented as vectors, are given by

$$\mathbf{x} = \sum_{i=1}^{8} \alpha_i \mathbf{e}_i \text{ and } \mathbf{y} = \sum_{i=1}^{8} \beta_i \mathbf{e}_i \tag{9}$$

(Note: To calculate the exact numerical answer, substitute the value of $\alpha_i,\ \beta_i$ given in the question).

Solution: The cross-correlation coefficient $R_{\mathbf{xy}}[0]$ is given by

$$R_{\mathbf{xy}}[0] \;=\; \sum_n x[n]y[n] \tag{10}$$

Equation (10) indicates that the auto-correlation is related to inner product of $\mathbf{x}$ and $\mathbf{y}$ i.e., $R_{\mathbf{xy}}[0] = \mathbf{x} \cdot \mathbf{y}$. Accordingly,

$$R_{\mathbf{xy}}[0] = \sum_{i=1}^{8} \alpha_i \beta_i \tag{11}$$

(c) (2 points) Compute the correlation coefficient $C_{\mathbf{xy}}$ for discrete-time signals (of length 8), which, when represented as vectors, are given by

$$\mathbf{x} = \sum_{i=1}^{8} \alpha_i \mathbf{e}_i \text{ and } \mathbf{y} = \sum_{i=1}^{8} \beta_i \mathbf{e}_i \tag{12}$$

(Note: To calculate the exact numerical answer, substitute the value of $\alpha_i,\ \beta_i$ given in the question).

Solution: The correlation coefficient $C_{\mathbf{xy}}$ is given by

$$C_{\mathbf{xy}} = \frac{R_{\mathbf{xy}}[0]}{\sqrt{R_{\mathbf{xx}}[0] R_{\mathbf{yy}}[0]}} \tag{13}$$

Using (7) and (10), the correlation coefficient can be expressed as

$$C_{\mathbf{xy}} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} = \frac{\sum_{i=1}^{8} \alpha_i \beta_i}{\sum_{i=1}^{8} \alpha_i^2 \sum_{i=1}^{8} \beta_i^2} \tag{14}$$

(d) (2 points) What is the maximum and minimum limit of the correlation coefficient $C_{\mathbf{xy}}$.

Solution: Using Cauchy Schwarz inequality and (14), it can be inferred that $-1 \leq C_{\mathbf{xy}} \leq 1$. Accordingly, the maximum limit is 1 and the minimum limit is -1.

Group 3

(The basic idea behind Grahm-Schmidt Algorithm)

1. (2 points) Let $\mathbf{x} = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ \alpha + 1 \end{bmatrix}$. Compute $\mathbf{x} \cdot (\mathbf{y} - \mathbf{Proj_x y})$.

Solution: The projection of $\mathbf{y}$ on $\mathbf{x}$ is given by

$$\mathbf{Proj_x y} = (\mathbf{y} \cdot \mathbf{e_x})\, \mathbf{e_x} \tag{15}$$

Accordingly,

$$\begin{aligned} \mathbf{x} \cdot (\mathbf{y} - \mathbf{Proj_x y}) &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{e_x})\, \mathbf{e_x} \\ &= \mathbf{x} \cdot \mathbf{y} - (\mathbf{y} \cdot \mathbf{e_x}) \underbrace{\mathbf{x} \cdot \mathbf{e_x}}_{\|\mathbf{x}\|} \\ &= \mathbf{x} \cdot \mathbf{y} - (\mathbf{y} \cdot \mathbf{x}) = 0 \end{aligned} \tag{16}$$

So $\mathbf{x} \perp (\mathbf{y} - \mathbf{Proj_x y})$ (The basic idea behind Grahm-Schmidt Algorithm).