EE2100: Matrix Theory

Quiz - 2

Please Note :

1. If there are any inconsistencies in grading, please use the google form shared with you.

Group 1

(One among the following set of questions)

1. (3 points) Consider the problem of computing the orthonormal basis of the subspace using the Grahm Schmidt Approach. Let $\mathbb{W}$ denote the subspace spanned by the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \in \mathcal{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \text{ and, } \mathbf{v}_3 = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The first entry of the third orthonormal basis vector for the subspace $\mathbb{W}$ is (Please Note: Apply the Grahm Schmit algorithm by considering vectors in the following order, i.e., $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$)

Solution: Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ denote the orthogonal basis of $\mathbb{W}$ obtained using Grahm Schmidt Approach. Accordingly,

$$\mathbf{b}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}, \mathbf{b}_2 = \mathbf{v}_2 - \underbrace{\mathbf{Proj}_{\mathbf{b}_1}\mathbf{v}_2}_{\mathbf{0} \text{ since } \mathbf{v}_2 \perp \mathbf{b}_1} = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}, \mathbf{b}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{b}_1}\mathbf{v}_3 - \mathbf{Proj}_{\mathbf{b}_2}\mathbf{v}_3 \text{ where}$$
$$\mathbf{Proj}_{\mathbf{b}_1}\mathbf{v}_3 = \frac{c}{1+a^2}\begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix} \text{ and } \mathbf{Proj}_{\mathbf{b}_2}\mathbf{v}_3 = \mathbf{0}. \text{ Accordingly, } \mathbf{b}_3 = \begin{bmatrix} \frac{a^2c}{1+a^2} \\ -\frac{ac}{1+a^2} \\ 0 \end{bmatrix} \tag{2}$$

The third orthonormal basis vector of the subspace is thus

$$\mathbf{e}_{\mathbf{b}_3} = \frac{\mathbf{b}_3}{\|\mathbf{b}_3\|} = \begin{bmatrix} \frac{a}{1+a^2} \\ -\frac{1}{1+a^2} \\ 0 \end{bmatrix} \tag{3}$$

The final answer to the question is $\frac{a}{1+a^2}$ and the exact value depends on the value of a given in the question.

2. (3 points) Consider the problem of computing the orthonormal basis of the subspace using the Grahm Schmidt Approach. Let $\mathbb{W}$ denote the subspace spanned by the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \in \mathcal{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \text{ and, } \mathbf{v}_3 = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

The second entry of the third orthonormal basis vector for the subspace $\mathbb{W}$ is (Please Note: Apply the Grahm Schmit algorithm by considering vectors in the following order, i.e., $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$)

Solution: Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ denote the orthogonal basis of $\mathbb{W}$ obtained using Grahm Schmidt Approach. Accordingly,

$$\mathbf{b}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix}, \mathbf{b}_2 = \mathbf{v}_2 - \underbrace{\mathbf{Proj}_{\mathbf{b}_1}\mathbf{v}_2}_{\mathbf{0} \text{ since } \mathbf{v}_2 \perp \mathbf{b}_1} = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}, \mathbf{b}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{b}_1}\mathbf{v}_3 - \mathbf{Proj}_{\mathbf{b}_2}\mathbf{v}_3 \text{ where}$$
$$\mathbf{Proj}_{\mathbf{b}_1}\mathbf{v}_3 = \frac{c}{1+a^2}\begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix} \text{ and } \mathbf{Proj}_{\mathbf{b}_2}\mathbf{v}_3 = \mathbf{0}. \text{ Accordingly, } \mathbf{b}_3 = \begin{bmatrix} \frac{a^2c}{1+a^2} \\ -\frac{ac}{1+a^2} \\ 0 \end{bmatrix} \tag{5}$$

The third orthonormal basis vector of the subspace is thus

$$\mathbf{e}_{\mathbf{b}_3} = \frac{\mathbf{b}_3}{\|\mathbf{b}_3\|} = \begin{bmatrix} \frac{a}{1+a^2} \\ -\frac{1}{1+a^2} \\ 0 \end{bmatrix} \tag{6}$$

The final answer to the question is $-\frac{1}{1+a^2}$ and the exact value depends on the value of a given in the question.

Group 2

(One among the following set of questions)

1. (3 points) Consider the problem of expressing the vector $\mathbf{v} \in \mathcal{R}^3$ as a linear combination of combination of the basis vectors $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_1}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2}, \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_3} \right\}$ i.e., $\mathbf{v} = \sum_{i=1}^{3} \alpha_i \mathbf{b}_i$. Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Compute the coefficient of linear combination α_3.

Solution: The given basis vectors are orthogonal. Thus,

$$\alpha_3 = \frac{\mathbf{v} \cdot \mathbf{b}_3}{\|\mathbf{b}_3\|^2} = \frac{y+z-2x}{6} \tag{7}$$

The final answer depends on the values of x, y and z given in the question.

2. (3 points) Consider the problem of expressing the vector $\mathbf{v} \in \mathcal{R}^3$ as a linear combination of combination of the basis vectors $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_1}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2}, \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_3} \right\}$ i.e., $\mathbf{v} = \sum_{i=1}^{3} \alpha_i \mathbf{b}_i$. Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Compute the coefficient of linear combination α_1.

Solution: The given basis vectors are orthogonal. Thus,

$$\alpha_1 = \frac{\mathbf{v} \cdot \mathbf{b}_1}{\|\mathbf{b}_1\|^2} = \frac{x+y+z}{3} \tag{8}$$

The final answer depends on the values of x, y and z given in the question.

3. (3 points) Consider the problem of expressing the vector $\mathbf{v} \in \mathcal{R}^3$ as a linear combination of combination of the basis vectors $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_1}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2}, \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_3} \right\}$ i.e., $\mathbf{v} = \sum_{i=1}^{3} \alpha_i \mathbf{b}_i$. Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Compute the coefficient of linear combination α_2.

Solution: The given basis vectors are orthogonal. Thus,

$$\alpha_2 = \frac{\mathbf{v} \cdot \mathbf{b}_2}{\|\mathbf{b}_2\|^2} = \frac{z-y}{2} \tag{9}$$

The final answer depends on the values of y and z given in the question.

Group 3

(One among the following set of questions)

1. (1 point) For some $\mathbf{a}$,$\mathbf{b} \in \mathcal{R}^n$, that satisfy $\mathbf{b} \neq \alpha\mathbf{a}$, $\mathbf{Proj_a b} = \mathbf{0}$

A. True

B. False

Solution: For any pair of orthogonal vectors $\mathbf{a}, \mathbf{b} \in \mathcal{R}^n$, $\mathbf{Proj_a b} = \mathbf{0}$.

2. (1 point) For all $\mathbf{a}$,$\mathbf{b} \in \mathcal{R}^n$, that satisfy $\mathbf{b} \neq \alpha\mathbf{a}$, $\mathbf{Proj_a b} = \mathbf{0}$

   A. True

   B. False

Solution: Only if $\mathbf{a}, \mathbf{b} \in \mathcal{R}^n$ orthogonal (or one of them is a zero-vector), $\mathbf{Proj_a b} = \mathbf{0}$.

Group 4

(One among the following set of questions)

1. (1 point) Consider all possible sets of 3 linearly independent vectors (denoted by say, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{R}^3$) such that $\mathbf{b} \perp \mathbf{a}$. Let $\mathbf{d} = \mathbf{c} - \mathbf{Proj_b c} - \mathbf{Proj_a c}$. Then, $\mathbf{d} \perp \mathbf{b}$ and $\mathbf{d} \perp \mathbf{a}$.

   A. True

   B. False

Solution: A simple way to prove the given statement is as follows.

$$
\begin{aligned}
\mathbf{d} \cdot \mathbf{a} &= (\mathbf{c} - \mathbf{Proj_b c} - \mathbf{Proj_a c}) \cdot \mathbf{a} \\
&= (\mathbf{c} - (\mathbf{c} \cdot \mathbf{e_b})\mathbf{e_b} - (\mathbf{c} \cdot \mathbf{e_a})\mathbf{e_a}) \cdot \mathbf{a} \\
&= \mathbf{c} \cdot \mathbf{a} - (\mathbf{c} \cdot \mathbf{e_b}) \underbrace{\mathbf{e_b} \cdot \mathbf{a}}_{0} - \underbrace{(\mathbf{c} \cdot \mathbf{e_a})\mathbf{e_a} \cdot \mathbf{a}}_{\mathbf{c} \cdot \mathbf{a}} = 0 \implies \mathbf{d} \perp \mathbf{a} \\
\text{Similarly, } \mathbf{d} \cdot \mathbf{b} &= (\mathbf{c} - \mathbf{Proj_b c} - \mathbf{Proj_a c}) \cdot \mathbf{b} \\
&= (\mathbf{c} - (\mathbf{c} \cdot \mathbf{e_b})\mathbf{e_b} - (\mathbf{c} \cdot \mathbf{e_a})\mathbf{e_a}) \cdot \mathbf{b} \\
&= \mathbf{c} \cdot \mathbf{b} - \underbrace{(\mathbf{c} \cdot \mathbf{e_b})\mathbf{e_b} \cdot \mathbf{b}}_{\mathbf{c} \cdot \mathbf{b}} - (\mathbf{c} \cdot \mathbf{e_a}) \underbrace{\mathbf{e_a} \cdot \mathbf{b}}_{0} = 0 \implies \mathbf{d} \perp \mathbf{b}
\end{aligned} \tag{10}
$$

2. (1 point) Consider all possible sets of 3 linearly independent vectors (denoted by say, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{R}^3$) . Let $\mathbf{d} = \mathbf{c} - \mathbf{Proj_b c} - \mathbf{Proj_a c}$. Then, $\mathbf{d} \perp \mathbf{b}$ and $\mathbf{d} \perp \mathbf{a}$.

   A. True

   B. False

Solution: A simple way to disprove the given statement is as follows.

$$\begin{aligned}
\mathbf{d}\cdot\mathbf{a} &= (\mathbf{c} - \mathbf{Proj_b c} - \mathbf{Proj_a c})\cdot\mathbf{a} \\
&= (\mathbf{c} - (\mathbf{c}\cdot\mathbf{e_b})\mathbf{e_b} - (\mathbf{c}\cdot\mathbf{e_a})\mathbf{e_a})\cdot\mathbf{a} \\
&= \mathbf{c}\cdot\mathbf{a} - (\mathbf{c}\cdot\mathbf{e_b})\underbrace{\mathbf{e_b}\cdot\mathbf{a}}_{\neq 0} - \underbrace{(\mathbf{c}\cdot\mathbf{e_a})\mathbf{e_a}\cdot\mathbf{a}}_{\mathbf{c}\cdot\mathbf{a}} \neq 0 \implies \mathbf{d} \text{ is not orthogonal to } \mathbf{a}
\end{aligned} \tag{11}$$

Group 5

1. Consider all possible $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{R}^3$ such that $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{b} \perp \mathbf{c}$. Then $\mathbf{a} \perp \mathbf{c}$.

   A. True

   B. False

Solution: A counter example to the given statement is the set of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ given by

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ and, } \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

For the chosen set of vectors $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{b} \perp \mathbf{c}$. But, $\mathbf{a}$ is not orthogonal to $\mathbf{c}$.

Group 6

(One among the following set of questions)

1. (3 points) The coordinate vector of $\mathbf{x} \in \mathcal{R}^3$ in basis $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_1}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2}, \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_3} \right\}$ is $\mathbf{x}_{\mathbb{B}}$. If $\mathbf{x}_{\mathbb{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the first entry of $\mathbf{x}$ in the standard basis is

Solution: The vector x in the standard basis is given by

$$\mathbf{x} = \mathbf{x}_{\mathbb{B}}[1]\mathbf{b}_1 + \mathbf{x}_{\mathbb{B}}[2]\mathbf{b}_2 + \mathbf{x}_{\mathbb{B}}[3]\mathbf{b}_3 = \begin{bmatrix} a - 2c \\ a - b - c \\ a + b + c \end{bmatrix} \tag{13}$$

The final answer to the question is $a - 2c$ and the exact value depends on the values of a, b and c given in the question.

2. (3 points) The coordinate vector of $\mathbf{x} \in \mathcal{R}^3$ in basis $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_1}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2}, \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_3} \right\}$ is $\mathbf{x}_{\mathbb{B}}$. If $\mathbf{x}_{\mathbb{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the second entry of $\mathbf{x}$ in the standard basis is

Solution: The vector x in the standard basis is given by

$$\mathbf{x} = \mathbf{x}_{\mathbb{B}}[1]\mathbf{b}_1 + \mathbf{x}_{\mathbb{B}}[2]\mathbf{b}_2 + \mathbf{x}_{\mathbb{B}}[3]\mathbf{b}_3 \;=\; \begin{bmatrix} a - 2c \\ a - b + c \\ a + b + c \end{bmatrix} \tag{14}$$

The final answer to the question is $a - b + c$ and the exact value depends on the values of a, b and c given in the question.

3. (3 points) The coordinate vector of $\mathbf{x} \in \mathcal{R}^3$ in basis $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_1}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{\mathbf{b}_2}, \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{b}_3} \right\}$ is $\mathbf{x}_{\mathbb{B}}$. If $\mathbf{x}_{\mathbb{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the third entry of $\mathbf{x}$ in the standard basis is

Solution: The vector x in the standard basis is given by

$$\mathbf{x} = \mathbf{x}_{\mathbb{B}}[1]\mathbf{b}_1 + \mathbf{x}_{\mathbb{B}}[2]\mathbf{b}_2 + \mathbf{x}_{\mathbb{B}}[3]\mathbf{b}_3 \;=\; \begin{bmatrix} a - 2c \\ a - b + c \\ a + b + c \end{bmatrix} \tag{15}$$

The final answer to the question is $a + b + c$ and the exact value depends on the values of a, b and c given in the question.