EE2100: Matrix Theory

Quiz - 3

Group 1

(One among the following set of questions)

1. (3 points) Consider a matrix $\mathbf{A} \in \mathcal{R}^{4\times 4}$ given by

$$\mathbf{A} = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Let $\mathbf{B} \in \mathcal{R}^{4\times 4}$ denote the inverse of a matrix. Compute the entry B_{13} [Hint: use the idea of inverse linear transformation].

Solution: The matrix $\mathbf{A}$ represents a linear transformation (say T) whose output (when applied to $\mathbf{x}$) is given by

$$T(\mathbf{x}) = \mathbf{A}\mathbf{x} = \underbrace{\begin{bmatrix} x_1 + ax_2 \\ x_2 + bx_3 \\ x_3 + cx_4 \\ x_4 \end{bmatrix}}_{\text{say } \mathbf{y}} \tag{2}$$

The inverse of $\mathbf{A}$ is a matrix corresponding to the inverse transformation of T i.e., the matrix corresponding to the transformation $T_1(\mathbf{y}) = \mathbf{x}$. The relation between the entries of $\mathbf{y}$ and $\mathbf{x}$ can be shown to be

$$\begin{aligned} x_4 &= y_4 \\ x_3 &= y_3 - c\, y_4 \\ x_2 &= y_2 - b\, y_3 + bc\, y_4 \\ x_1 &= y_1 - a\, y_2 + ab\, y_3 - abc\, y_4 \end{aligned} \tag{3}$$

The inverse of $\mathbf{A}$ is thus

$$\mathbf{B} = \begin{bmatrix} 1 & -a & ab & -abc \\ 0 & 1 & -b & -bc \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

The final answer to the question is ab and the exact value depends on the values of a and b given in the question.

2. (3 points) Consider a matrix $\mathbf{A} \in \mathcal{R}^{4\times 4}$ given by

$$\mathbf{A} = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Let $\mathbf{B} \in \mathcal{R}^{4\times 4}$ denote the inverse of a matrix. Compute the entry B_{14} [Hint: use the idea of inverse linear transformation].

Solution: The matrix $\mathbf{A}$ represents a linear transformation (say T) whose output (when applied to $\mathbf{x}$) is given by

$$T(\mathbf{x}) = \mathbf{Ax} = \underbrace{\begin{bmatrix} x_1 + ax_2 \\ x_2 + bx_3 \\ x_3 + cx_4 \\ x_4 \end{bmatrix}}_{\text{say } \mathbf{y}} \tag{6}$$

The inverse of $\mathbf{A}$ is a matrix corresponding to the inverse transformation of T i.e., the matrix corresponding to the transformation $T_1(\mathbf{y}) = \mathbf{x}$. The relation between the entries of $\mathbf{y}$ and $\mathbf{x}$ can be shown to be

$$\begin{aligned} x_4 &= y_4 \\ x_3 &= y_3 - c\, y_4 \\ x_2 &= y_2 - b\, y_3 + bc\, y_4 \\ x_1 &= y_1 - a\, y_2 + ab\, y_3 - abc\, y_4 \end{aligned} \tag{7}$$

The inverse of $\mathbf{A}$ is thus

$$\mathbf{B} = \begin{bmatrix} 1 & -a & ab & -abc \\ 0 & 1 & -b & -bc \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The final answer to the question is $-abc$ and the exact value depends on the values of a, b and c given in the question.

Group 2

(One among the following set of questions)

1. (1 point) If $\forall \mathbf{v} \in \mathcal{R}^n$ $\mathbf{Av} = \mathbf{Bv}$ (where $\mathbf{A} \in \mathcal{R}^{m\times n}$, $\mathbf{B} \in \mathcal{R}^{m\times n}$), then $\mathbf{A} = \mathbf{B}$.

   A. True

   B. False

2. (1 point) If $\exists \mathbf{v} \in \mathcal{R}^n$ such that $\mathbf{Av} = \mathbf{Bv}$ (where $\mathbf{A} \in \mathcal{R}^{m\times n}$, $\mathbf{B} \in \mathcal{R}^{m\times n}$), then $\mathbf{A} = \mathbf{B}$.

   A. True

   B. False

Group 3 (All the questions)

1. (2 points) Consider a matrix $\mathbf{A} \in \mathcal{R}^{4\times 4}$ whose column vectors are given by

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{e}_1 \\ \mathbf{a}_2 &= \mathbf{e}_3 + x\mathbf{e}_2 \\ \mathbf{a}_3 &= x\mathbf{e}_3 - \mathbf{e}_2 \\ \mathbf{a}_4 &= \mathbf{e}_4 \end{aligned} \tag{9}$$

   The rank of the matrix is

   Solution: The column vectors of $\mathbf{A}$ are orthogonal (and hence, linearly independent). Thus, $\mathbf{Rank}(\mathbf{A}) = 4$.

2. (2 points) Consider a matrix $\mathbf{A} \in \mathcal{R}^{4\times 4}$ whose column vectors are given by

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{e}_1 \\ \mathbf{a}_2 &= \mathbf{e}_3 + x\mathbf{e}_2 \\ \mathbf{a}_3 &= x\mathbf{e}_3 - \mathbf{e}_2 \\ \mathbf{a}_4 &= \mathbf{e}_4 \end{aligned} \tag{10}$$

   The nullity of the matrix is

   Solution: The columns of $\mathbf{A}$ are orthogonal (linearly independent). Thus, (by Rank-Nullity Theorem) $\mathbf{Nullity}(\mathbf{A}) = 0$.

3. (4 points) Consider a matrix $\mathbf{A} \in \mathcal{R}^{4\times 4}$ whose column vectors are given by

$$\begin{aligned}\mathbf{a}_1 &= \mathbf{e}_1\\ \mathbf{a}_2 &= \mathbf{e}_3 + x\mathbf{e}_2\\ \mathbf{a}_3 &= x\mathbf{e}_3 - \mathbf{e}_2\\ \mathbf{a}_4 &= \mathbf{e}_4\end{aligned} \tag{11}$$

The second entry of the vector $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$ (where $\mathbf{b} = [p, q, r, s]^t$ is

Solution: The second entry of $\mathbf{x}$ is the coefficient of linear combination α_2 such that

$$\mathbf{b} = \sum_{i=1}^{4} \alpha_i \mathbf{a}_i \tag{12}$$

Since the column vectors of $\mathbf{A}$ are orthogonal,

$$\alpha_2 = \frac{\mathbf{b}\cdot\mathbf{a}_2}{\|\mathbf{a}_2\|^2} = \frac{qx + r}{x^2 + 1} \tag{13}$$

The exact answer depends on the values of x, q and r.

4. (4 points) Consider a matrix $\mathbf{A} \in \mathcal{R}^{4\times 4}$ whose column vectors are given by

$$\begin{aligned}\mathbf{a}_1 &= \mathbf{e}_1\\ \mathbf{a}_2 &= \mathbf{e}_3 + x\mathbf{e}_2\\ \mathbf{a}_3 &= x\mathbf{e}_3 - \mathbf{e}_2\\ \mathbf{a}_4 &= \mathbf{e}_4\end{aligned} \tag{14}$$

Let $\mathbf{C} = \mathbf{A}^T\mathbf{A}$. The entry $C_{2,2}$ is

Solution: The column vectors of $\mathbf{A}$ are orthogonal. Thus

$$\mathbf{C} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{a}_1\cdot\mathbf{a}_1 & \mathbf{a}_1\cdot\mathbf{a}_2 & \mathbf{a}_1\cdot\mathbf{a}_3 & \mathbf{a}_1\cdot\mathbf{a}_4\\ \mathbf{a}_2\cdot\mathbf{a}_1 & \mathbf{a}_2\cdot\mathbf{a}_2 & \mathbf{a}_2\cdot\mathbf{a}_3 & \mathbf{a}_2\cdot\mathbf{a}_4\\ \mathbf{a}_3\cdot\mathbf{a}_1 & \mathbf{a}_3\cdot\mathbf{a}_2 & \mathbf{a}_3\cdot\mathbf{a}_3 & \mathbf{a}_3\cdot\mathbf{a}_4\\ \mathbf{a}_4\cdot\mathbf{a}_1 & \mathbf{a}_4\cdot\mathbf{a}_2 & \mathbf{a}_4\cdot\mathbf{a}_3 & \mathbf{a}_4\cdot\mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & x^2+1 & 0 & 0\\ 0 & 0 & x^2+1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

Thus, $C_{2,2} = x^2 + 1$. The exact answer depends on the value of x given in the question.

5. (4 points) Consider a matrix $\mathbf{A} \in \mathcal{R}^{4\times 4}$ whose column vectors are given by

$$\begin{aligned}
\mathbf{a}_1 =& \mathbf{e}_1 \\
\mathbf{a}_2 =& \mathbf{e}_3 + x\mathbf{e}_2 \\
\mathbf{a}_3 =& x\mathbf{e}_3 - \mathbf{e}_2 \\
\mathbf{a}_4 =& \mathbf{e}_4
\end{aligned} \tag{16}$$

Let $\mathbf{C}$ be the inverse of $\mathbf{A}$. The entry $C_{2,2}$ is

Solution: $C_{2,2} = \frac{x}{x^2+1}$ (See Tutorial 6) The exact answer depends on the value of x given in the question.