MSO202A COMPLEX VARIABLES
Soluition–5

Problems for Discussion:

1. Integrate the following functions counterclockwise around the unit circle $|z| = 1$:
$(a)\ \dfrac{\sinh 2z}{z^4} \quad (b)\ \dfrac{z^2}{(2z-1)^3} \quad (c)\ \dfrac{e^{3z}}{(4z-\pi i)^3}$.
Solution: Use the Cauchy integral formula.

2. Evaluate the integral $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{ze^{zt}}{(z+1)^3}\,dz$ where C is a positively oriented simple closed enclosing $z = -1$.
Solution: Using the Cauchy integral formula : $\frac{1}{2}(2t - t^2)e^{-t}$.

3. Find the Taylor series of the function $(a) f(z) = \dfrac{1}{z^2}$ at $z = a \neq 0, (b) f(z) = \dfrac{6z+8}{(2z+3)(4z+5)}$ at $z = 1$ (c) $f(z) = \dfrac{e^z}{z+1}$ at $z = 1$.
Solution: 2(a). Let $t = z - a$. $\dfrac{1}{z^2} = \dfrac{1}{(t+a)^2} = \dfrac{1}{a^2}\displaystyle\sum_0^\infty \frac{(-1)^n(n+1)(z-a)^n}{a^n}$.
(b) Let $t = z - 1$. $f(z) = \dfrac{1}{2z+3} + \dfrac{1}{4z+5} = \dfrac{1}{2t+3} + \dfrac{1}{4t+5}$. This is equal to $\displaystyle\sum_0^\infty \frac{(-2)^n(z-1)^n}{5^{n+1}} + \sum_0^\infty \frac{(-4)^n(z-1)^n}{9^{n+1}}$.
(c) $f(z) = \dfrac{e^z}{z+1} = \dfrac{e}{2}\left[\displaystyle\sum_0^\infty \frac{(z-1)^n}{n!}\right]\left[\sum_0^\infty \frac{(-1)^n(z-1)^n}{2^n}\right]$. the coefficient of $(z-1)^n$ is $\dfrac{e}{2}\displaystyle\sum_{j=0}^n \frac{(-1)^{n-j}}{j!2^{n-j}}$.

4. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and let $f : \mathbb{D} \to \mathbb{C}$ be analytic such that $|f(z) - f(w)| \leq K, \forall z, w \in \mathbb{D}$. Show that $2|f'(0)| \leq K$.
Solution: By Cauchy's integral formula, $f'(0) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z^2}\,dz$ where C : $re^{i\theta}, 0 \leq \ \leq \pi, r < 1$. Let $g(z) = f(z)$. Then,g is analytic on $\mathbb{D}$ and so by Cauchy's integral formula, $g'(0) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{g(z)}{z^2}\,dz$. As $g'(0) = -f'(0)$, it follows that
$$2f'(0) = \frac{1}{2\pi i}\int_C \frac{f(z) - f(-z)}{z^2}\,dz \Rightarrow 2f'(0) \leq \frac{1}{2\pi}\frac{K}{r^2}2\pi r \leq \frac{K}{r}.$$
Take limit as $r \to 1$, to get $2f'(0) \leq K$.

Problems for Tutorial

1. Integrate the following functions counterclockwise
   (a) $f(z) = z^{-2}\tan \pi z$, C any contour enclosing 0.
   (b) $f(z) = \frac{\cosh 4z}{(z-4)^3}$, C consists of $|z| = 6$ counterclockwise and $|z - 3| = 2$ counterclockwise.

Solution: Use cauchy's integral formula: (a) $2\pi^2 i$ (b) $16\pi i \sinh 16$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\{z \in \mathbb{C} : z \neq 0\}$ and bounded in some neighborhood of 0, say $\{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. Prove that $\int_{|z|=R} f(z)dz = 0$ for every $R > 0$.

   Solution: Let $r < \frac{1}{2}$. By Cauchy's theorem for multiply connected domains, $\displaystyle\int_{C_R} f(z)\,dz = \int_{C_r} f(z)\,dz$ where C_r is the circle $|z| = r$. Let $|f(z)| \leq M \forall z$ with $|z| \leq 1/2$. Then $\displaystyle\left|\int_C f(z)\,dz\right| = \left|\int_{C_r} f(z)\,dz\right| \leq 2\pi r M$. Take limit as $r \to 0$, to get $\displaystyle\left|\int_C f(z)\,dz\right| \to 0$. Hence proved.

3. (a) Let f be an entire function bounded by M on $|z| = R$. Show that the coefficients a_k in its power series expansion about 0 satisfy $|a_k| \leq \frac{M}{R^k}$.

   (b) If a polynomial is bounded by 1 on a unit disc, show that each of its coefficients is also bounded by 1.

   Solution: For the first part use Cauchy's integral formula and then the ML inequality. The second part follows from the fist part.

4. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that the image of the function has to necessarily meet the real axis and imaginary axis.

   Solution: Let $f = u + iv$. Assume that the image of f does not meet the real axis i.e., $v(x, y) \neq 0, \forall (x, y)$ or $\mathrm{Im} f(z) \neq 0, \forall z$. Thus we have $\mathrm{Im} f(z) > 0$ or $\mathrm{Im} f(z) < 0, \forall z$.

   (Note: In the case of continuous functions of one variable it is a consequence of intermediate value property. In two (or higher variables) it is a consequence of the fact that $\mathbb{R}^2$ is connected and connected sets in $\mathbb{R}$ are intervals together with the fact that continuous functions map connected sets to connected sets. You may ask students to assume these facts, which are in any case intutive.)

   If, $\mathrm{Im} f(z) > 0$, consider $g(z) = e^{if(z)}$. Then g is entire and $|g(z)| \leq 1$, and thus bu Liouville's theorem g is a constant function which implies f is a constant function. Similarly, in case $\mathrm{Im} f(z) < 0$, consider $h(z) = e^{-if(z)}$ and proceed as before to conclude that f is a constant function. Hence the image of f function has to necessarily meet the real axis, and like wise the image of f function has to necessarily meet the imaginary axis as well.

5. Let f be entire and $|f(z)| \leq a + b|z|^n$ for some positive real numbers a and b and $n \in \mathbb{N}$. Show that f is a polynomial of degree at most n.

   Solution: $\displaystyle f^{n+1}(0) = \frac{(n+1)!}{2\pi i} \int_{C_R} \frac{f(z)}{z^{n+2}}\,dz$ where $C_R : Re^{i\theta}, 0 \leq \ \leq \pi, R > 0$. So $\displaystyle |f^{n+1}(0)| \leq \frac{(n+1)!}{2\pi} \frac{a + bR^n}{R^{n+2}} 2\pi R \to 0$ as $R \to \infty$. Hence f is a polynomial of degree at most n.