Problem Set 7

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ satisfying $T(1,1,1) = (1,2,3,0), T(1,2,-1) = (2,1,3,0)$ and $T(1,5,-7) = (0,0,0,1)$? Give reasons for your answer.

2. Can we ever find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ which is onto?

3. Find out $[\mathbf{v}]_{\mathcal{B}}$, where $\mathcal{B}$ is an ordered basis:

   (a) $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}, \mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$ (b) $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}, \mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$

4. Find out $\mathbf{v}$ given $[\mathbf{v}]_{\mathcal{B}}$, where $\mathcal{B}$ is an ordered basis:

   (a) $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}, [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$ (b) $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}, [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$

5. Give three linear transformations from $\mathbb{R}^3$ to $\mathbb{W} = \{\mathbf{w} \in \mathbb{R}^5 : w_1 - w_2 + w_3 - w_4 + w_5 = 0\}$. Give their coordinate matrix w.r.t the ordered bases $\mathcal{B}_1 = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$ on $\mathbb{R}^3$ and some ordered basis of $\mathbb{W}$.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ as $T \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x - y + z\\x + 2z\end{bmatrix}$. Find

   (a) a basis of Range (T),
   (b) rank (T),
   (c) a basis for $\mathcal{N}(T)$, and
   (d) $\dim(\mathcal{N}(T))$.

7. **(T)** Find all linear transformations from $\mathbb{R}^n \longrightarrow \mathbb{R}$.

8. Let $\mathbf{v} \in \mathbb{R}^n$ and $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis of $\mathbb{R}^n$. Form a matrix $B = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$. Is $B[\mathbf{e}_1]_{\mathcal{B}} = \mathbf{e}_1$? What is $B\,[[\mathbf{e}_1]_{\mathcal{B}}, \dots, [\mathbf{e}_n]_{\mathcal{B}}]$? Show that B is invertible and $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$.

9. **(T)** Show that a linear transformation is one-one if and only if null-space of $\mathcal{N}(T)$ is $\{0\}$.

10. Describe all 2×2 orthogonal matrices. Prove that action of any orthogonal matrix on a vector $\mathbf{v} \in \mathbb{R}^2$, is either a rotation or a reflection about a line.

11. **(T)** Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $n \geq 2$, with $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$. Prove that there exist an orthogonal matrix A such that $A(\mathbf{v}) = \mathbf{w}$. Prove also that A can be chosen such that $\det(A) = 1$.
    (*This is why orthogonal matrices with determinant one are called rotations.*))