Causality Condition

- For a causal LTI digital system, the output sequence $y[n]$ at time index $n = N_o$ depends only on the samples of the input sequence $x[n]$ for all values of $n \le N_o$
- Output $y[n]$ does not depend on future values of $x[n]$ for $n > N_o$

Causality Condition

- This condition can only be satisfied if the impulse response $h[n]$ of the LTI digital system is a causal sequence, that is,

$$\boxed{h[n] = 0 \quad \text{for} \quad n < 0}$$

- Thus, for a causal LTI digital system, the input-output relation is given by

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

Stability Condition

- A much simpler BIBO stability condition for an LTI digital system is in terms of its impulse response
- It can be shown that an LTI digital system is BIBO stable, if its impulse response $h[n]$ is an absolutely summable sequence, that is

Stability Condition

$$\boxed{S = \sum_{-\infty}^{+\infty} |h[n]| < \infty}$$

Example – Let $h[n] = \alpha^n \mu[n]$

- To test BIBO stability we compute

$$S = \sum_{-\infty}^{+\infty} |\alpha^n \mu[n]| = \sum_{0}^{+\infty} |\alpha^n|$$

- If $|\alpha| < 1$, then $\sum_{n=0}^{\infty} |\alpha|^n = \dfrac{1}{1-|\alpha|} < \infty$

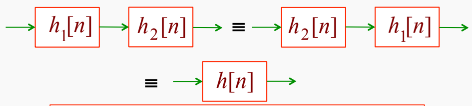
Stability Condition

- Since $h[n]$ is absolutely summable for $|\alpha| < 1$, the LTI digital system $h[n] = \alpha^n \mu[n]$ is BIBO stable for $|\alpha| < 1$
- On the other hand, if $|\alpha| \ge 1$, the infinite series $\sum_{n=0}^{+\infty} |\alpha|^n$ is not convergent and then the digital system is not BIBO stable

Interconnected LTI Digital Systems

- Most LTI digital systems are usually designed by an interconnection of simple LTI digital systems
- There are three basic interconnection schemes

Cascade Connection



$$\boxed{h[n] = h_1[n] \circledast h_2[n] = h_2[n] \circledast h_1[n]}$$

- If the individual systems in a cascade are stable, then the overall cascaded system is also stable

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Cascade Connection

Example – Let $h_1[n] = \alpha^n \mu[n]$ and $h_2[n] = \beta^n \mu[n]$ with $|\alpha| < 1$ and $|\beta| < 1$

- The impulse response $h[n]$ of their cascade is given by

$$h[n] = \sum_{k=0}^{n} h_1[k] h_2[n-k] = \sum_{k=0}^{n} \alpha^k \beta^{n-k}$$

$$= \frac{1}{\alpha - \beta}\left(\alpha^{n+1} - \beta^{n+1}\right), \quad n \geq 0$$

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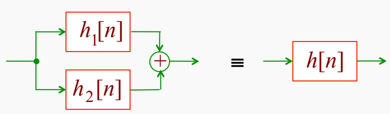
Parallel Connection



$$\boxed{h[n] = h_1[n] + h_2[n]}$$

- If the individual systems connected in parallel are stable, then the overall system is also stable

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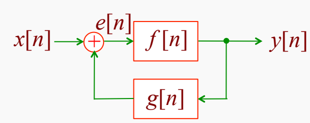
Feedback Connection



- Analysis of the feedback system yields

$$e[n] = x[n] + g[n] \circledast y[n]$$

$$y[n] = e[n] \circledast f[n]$$

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Feedback Connection

- Combining the two equations we get

$$y[n] = e[n] \circledast f[n] = \Big(x[n] - g[n] \circledast y[n]\Big) \circledast f[n]$$

- From the above we arrive at the input-output relation of the feedback system as

$$\boxed{\Big(\delta[n] + g[n] \circledast f[n]\Big) \circledast y[n] = f[n] \circledast x[n]}$$

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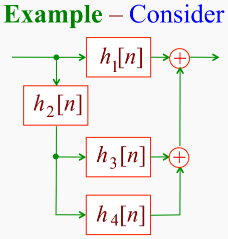
Interconnected LTI Digital Systems

Example – Consider the system shown below



$$h_1[n] = \delta[n] + 0.5\delta[n-1]$$

$$h_2[n] = 0.5\delta[n] - 0.25\delta[n-1]$$

$$h_3[n] = 2\delta[n]$$

$$h_4[n] = -2\mu[n]$$

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Interconnected LTI Digital Systems

- The overall impulse response $h[n]$ of the interconnected LTI digital system is given by

$$h[n] = h_1[n] + h_2[n] \circledast (h_3[n] + h_4[n])$$
$$= h_1[n] + h_2[n] \circledast h_3[n] + h_2[n] \circledast h_4[n]$$

making use of the distributive property of the convolution sum

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Interconnected LTI Digital Systems

- Substituting the values of the individual impulse responses given in Slide No. 12 in the last equation we arrive at

$$h[n] = \delta[n] - 0.5\mu[n-1]$$

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Inverse Digital System

$$x[n] \rightarrow \boxed{g[n]} \rightarrow \boxed{h[n]} \rightarrow x[n]$$

- The LTI digital system with an impulse response $h[n]$ is called the inverse of the LTI digital system with an impulse response $g[n]$, and vice-versa, if

$$\boxed{g[n] \circledast h[n] = \delta[n]}$$

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Inverse Digital System

Example – Consider the accumulator with an impulse response $h[n] = \mu[n]$

- The impulse response $h^{-1}[n]$ of its inverse system is given by

$$h^{-1}[n] = \delta[n] - \delta[n-1]$$

known as the backward difference system

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Inverse Digital System

Example – Consider the LTI digital system with an impulse response $h[n]$ given by

$$h[n] = A\delta[n] + B\alpha^n \mu[n], \; \alpha < 1$$

- The impulse response $h^{-1}[n]$ of its inverse LTI digital system is given by

$$h^{-1}[n] = \frac{1}{A}\delta[n] - \frac{B}{A(A+B)}\left(\frac{\alpha A}{A+B}\right)^n \mu[n]$$

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Inverse Digital System

Example – Let $h[n] = 0.25\delta[n] - 0.05(0.4)^n \mu[n]$

- Let the impulse response $g[n]$ of its inverse system be given by $h^{-1}[n] = C\delta[n] + D\beta^n \mu[n]$
- Here $A = 0.25, \; B = -0.05, \; \alpha = 0.4$
- From the previous slide we have

$$C = \frac{1}{A} = \frac{1}{0.25} = 4$$
$$D = -\frac{B}{A(A+B)} = \frac{-0.05}{0.25(0.25-0.05)} = 1$$

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Inverse Digital System

$$\beta = \frac{\alpha A}{A+B} = \frac{0.4\times 0.25}{0.25+0.05} = 0.5$$

- Hence, $h^{-1}[n] = 4\delta[n] + (0.5)^n \mu[n]$
- It can be shown that

$$\left(0.25\delta[n] - (0.04)^n \mu[n]\right) \circledast \left(4\delta[n] + (0.5)^n \mu[n]\right) = \delta[n]$$

LTI Digital System in the Time-Domain

- The LTI digital systems of interest to us in this course are characterized in the time-domain by linear constant-coefficient difference equations of the form

$$y[n] + \sum_{\ell=1}^{N} q_\ell y[n-\ell] = \sum_{\ell=0}^{M} p_\ell x[n-\ell]$$

where $y[n]$ and $x[n]$ are, respectively, the output and input signals

LTI Digital System in the Time-Domain

and the coefficients $\{q_\ell\}$ and $\{p_\ell\}$ are constants

- The constants N and M are positive integers with max(N,M) denoting the order of the difference equation

LTI Digital System in the Time-Domain

Example – Discrete-time representation of the trapezoidal method of integration

$$y[n] = y[n-1] + \tfrac{T}{2}(x[n] + x[n-1])$$

Example - The factor-of-2 linear interpolator

$$y[n] = x_u[n] + \tfrac{1}{2}(x_u[n+1] - x_u[n-1])$$

Solution of Difference Equation

- A direct method of computing the sample values of the output sequence $\{y[n]\}$ of an LTI digital system characterized by a difference equation is by recursion

Solution of Difference Equation

- If the system is assumed to be causal, then the difference equation given in Slide No. 20 can be rewritten to determine the n-th sample of the output sequence $y[n]$ as a function of the n-th sample and all past sample values of the input sequence $\{x[n]\}$:

$$y[n] = -\sum_{\ell=1}^{N} q_\ell y[n-\ell] + \sum_{\ell=0}^{M} p_\ell x[n-\ell]$$

Solution of Difference Equation

- Using the previous equation we can then determine $y[n]$ for all values of the time index $n \geq n_o$, knowing all values of $x[n]$ for $n < n_o$ and the set of past output samples
$$y[n_o - 1],\ y[n_o - 2], \cdots, y[n_o - N]$$
known as the initial conditions

Solution of Difference Equation

- If all initial conditions are equal to zero at the time index n_o when the causal input sequence is applied to a causal digital system, the system is said to be at rest
- For the case $N = 0$, the difference equation reduces to
$$y[n] = \sum_{\ell=0}^{M} p_\ell x[n-\ell]$$

Solution of First-Order Difference Equation

- Consider $y[n] + q_0 y[n-1] = x[n]$
- We determine $y[n]$ for $x[n] = A\gamma^n \mu[n]$ and $y[-1] = y_o$
- The complementary solution $y_c[n]$ satisfies the homogeneous equation
$$y[n] + q_0 y[n-1] = 0$$
- We assume $y_c[n] = B\xi^n$

Solution of First-Order Difference Equation

- Substituting $y_c[n] = B\xi^n$ in the homogeneous equation we get
$$B\xi^n - \alpha B\xi^{n-1} = (\xi + q_0)B\xi^{n-1} = 0$$
- Solution of the characteristic polynomial $\xi + q_0 = 0$ is given by $\xi = -q_0$
- Thus, $y_c[n] = B(-q_0)^n$ where the constant B is to be determined later

Solution of First-Order Difference Equation

- Next, we assume the particular solution to be of the form $y_p[n] = C\gamma^n$, $n \geq 0$
- Substituting the above expression in the first-order difference equation we get
$$C\gamma^n + q_0 C\gamma^{n-1} = A\gamma^n$$
which yields
$$C = \frac{A\gamma}{\gamma + q_0}$$

Solution of First-Order Difference Equation

- Thus, the particular solution is given by
$$y_p[n] = \frac{A}{\gamma + q_0}\gamma^{n+1}, \quad n \geq 0$$
- Therefore, the complete solution is given by
$$y[n] = y_c[n] + y_p[n] = B(-q_0)^n + \frac{A}{\gamma + q_0}\gamma^{n+1}, \ n \geq 0$$
- The constant B is determined by setting $n = 0$ in the first-order difference equation

Solution of First-Order Difference Equation

$$y[0] - \alpha y[-1] = y[0] + q_0 y_o = x[0] = A$$

which yields $y[0] = A - q_0 y_o$

- Setting $n = 0$ in the complete solution given in Slide No. 30 we obtain
$$y[0] = B + \frac{A\gamma}{\gamma + q_0}$$
- Substituting the value of $y[0]$ in the above equation we get $B = \dfrac{Aq_0}{\gamma + q_0} - q_0 y_o$

Solution of First-Order Difference Equation

- Therefore, the complete solution is given by
$$y[n] = y_o(-q_0)^{n+1} + A\frac{\gamma^{n+1} + q_0{}^{n+1}}{\gamma + q_0}, \; n \geq 0$$
- Next, for $n < 0$, $x[n] = 0$
- Then, the difference equation reduces to
$$y[n] + q_0 y[n-1] = 0$$
- It solution is of the form $y[n] = D(-q_0)^n$

Solution of First-Order Difference Equation

- Setting $n = -1$ in the above equation we get
$$y[-1] = y_o = D(-q_0)^{-1}$$
which yields $D = y_o(-q_0)$
- Therefore $y[n] = y_o(-q_0)^{n+1} \quad n < 0$,
- Finally, the complete solution for all values of n in the range $-\infty < n < +\infty$ is given by
$$y[n] = y_o(-q_0)^{n+1} + A\frac{\gamma^{n+1} - (-q_0)^{n+1}}{\gamma + q_0}\mu[n]$$

Step Response of First-Order Difference Equation

- The step response $s[n]$ can be obtained from the solution of $s[n] + q_0 s[n-1] = \mu[n]$
- It can also be obtained from the complete solution given in the previous slide by setting $y_o = 0$, $A = 1$, and $\gamma = 1$ which leads to
$$s[n] = \frac{1 - (-q_0)^{n+1}}{1 + q_0}\mu[n]$$

Impulse Response of a Causal LTI Digital System

- Setting $x[n] = \delta[n]$ in the difference equation representation we obtain the expression for the impulse response $h[n]$
- For a causal LTI digital system, the impulse response $h[n]$ is a causal sequence

Solution of Difference Equation

- Hence, when $N \neq 0$, the impulse response is computed using
$$h[n] = -\sum_{\ell=1}^{N} q_\ell y[n-\ell] + \sum_{\ell=0}^{M} p_\ell h[n-\ell]$$
with the system at rest

Solution of Difference Equation

Example – Consider the causal LTI digital system with an input-output relation given by $y[n] = -q_0 y[n-1] + x[n]$

- Setting $x[n] = \delta[n]$ we get

$$h[n] = -q_0 h[n-1] + \delta[n]$$

- We compute the impulse response samples starting at $n = 0$

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Solution of Difference Equation

$$h[0] = \delta[0] = 1$$
$$h[1] = -q_0 h[0] + \delta[1] = -q_0$$
$$h[2] = q_0 h[1] + \delta[2] = (-q_0) \times (-q_0) = (-q_0)^2$$
$$h[3] = -q_0 h[2] + \delta[3] = (-q_0)^2 \times (-q_0) = (-q_0)^3$$
$$h[4] = -q_0 h[3] + \delta[4] = (-q_0)^3 \times (-q_0) = (-q_0)^4$$
$$\vdots$$

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Solution of Difference Equation

- It follows from the previous set of equations that the expression for the impulse response $h[n]$ can be written in compact form as

$$h[n] = \begin{cases} (-q_0)^n, & 0 \le n < \infty \\ 0, & n < 0 \end{cases}$$

- In general, it may not be possible to determine the impulse response in compact form of a causal LTI digital system from the constant coefficient difference equation

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Solution of Difference Equation

- Except for $N = 0$ in which case the impulse response is given by

$$h[n] = \sum_{\ell=0}^{M} p_\ell \delta[n-\ell] = \begin{cases} p_\ell, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

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Impulse Response Using MATLAB

- The function `impz` can be used to compute the output samples of a causal stable LTI digital system characterized by a difference equation
- The basic form is

```
h = impz(num,den,N)
```

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Impulse Response Using MATLAB

- where `num` and `den` are the vectors containing the coefficients $\{p_k\}$ and $\{q_k\}$, respectively, in increasing values of `k`, `N` denoting the total number of impulse response samples to be computed, and `h` is the vector containing the computed impulse response samples

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Impulse Response Using MATLAB

Example – Consider a causal LTI digital system given by

$$0.5y[n]+0.1y[n-1]-0.04y[n-2]$$
$$=2x[n]+4x[n-1]+6x[n-2]$$

- We compute the first 8 impulse response samples of the above system
- The code fragments used are given in the next slide

Impulse Response Using MATLAB

```
N = 8;
num = [2  4  6];
den = [0.5  0.1  -0.04];
H = impz(num,den,N);
```

which yield

Impulse Response Using MATLAB

- H =

```
     4.0000
     7.2000
    10.8800
    -1.6000
     1.1904
    -0.3661
     0.1684
    -0.0630
```


Impulse Response Using MATLAB

- Hence, the impulse response is given by

$$\{h[n]\}=\{\underset{\uparrow}{4},\ 7.2,\ 10.88,\ -1.6,\ 1.1904$$
$$-0.3661,\ 0.1684,\ -0.0630,\ \ldots\}$$

Output Response Using MATLAB

- The function `filter` can be used to compute the output samples of a causal stable LTI system characterized by a difference equation

Output Response Using MATLAB

- Basic form of this function is `y = filter(num,den,x)` where `x` is the vector containing the input samples, num and den are the vectors containing the coefficients $\{p_k\}$ and $\{q_k\}$, respectively, in increasing values of `k`, and `y` is the vector containing the computed output samples of same length as `x`

Output Response Using MATLAB

Example – $\{h[n]\} = \{-3,\ 7,\ -5,\ 4\},\ 0 \le n \le 3$

- We compute the first 7 samples of its output for a length-7 causal input $x[n] = 1,\ 0 \le n \le 6$
- Code fragments used are

```
x = ones(1,7);
h = [-3  7  -5  4];
y = filter(h,1,x);
```


Output Response Using MATLAB

- The computed output samples are

```
y =
  -3   4  -1   3   3   3   3
```


Finite Impulse Response System

- For a finite impulse response (FIR) LTI digital system, the length of its impulse response $h[n]$ is finite, that is

$$h[n] = 0 \text{ for } n < N_a \text{ and } n > N_b$$

with $N_b > N_a$

Finite Impulse Response System

- For the above FIR LTI digital system the input-output relation is given by

$$y[n] = \sum_{k=N_a}^{N_b} h[k]x[n-k]$$

- The amplitude of the output sample at all value of the time index n can be computed using the above equation as it involves a finite sum of products

Infinite Impulse Response System

- The impulse response of an infinite impulse response (IIR) is of infinite length
- The convolution sum description of an IIR system given by

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

cannot be employed to compute the output sample for all values of the time index n

Infinite Impulse Response System

- Fortunately, the class of IIR systems of interest to us in this course is characterized by the constant coefficient difference equation of the form

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- The above involves two finite sum of products and thus can be employed to compute the output sample at all values of n

Infinite Impulse Response System

Example – The discrete-time representation of the trapezoidal method of integration

$$y[n] = y[n-1] + \frac{T}{2}(x[n] + x[n-1])$$

Nonrecursive LTI Digital Systems

- Here, output is computed for increasing values of n using only the present and past values of input

Example – Factor-of-2 interpolator given by

$$y[n] = x_u[n] + 0.5(x_u[n-1] + x_u[n+1])$$

Recursive LTI Digital Systems

Example – Moving-average filter given by

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

Recursive LTI digital system

- Here, the output is computed using the past computed values of the output, and the present and past values of the input for increasing values of n

Recursive LTI Digital Systems

Example - The recursive accumulator given by

$$y[n] = y[n-1] + x[n]$$

Example – Recursive implementation of the moving-average filter given by

$$y[n] - y[n-1] + \frac{1}{M}(x[n] - x[n-M])$$

Block Diagram Representation

- A convenient form of an LTI digital system
- Provides additional information about its operation and also its hardware or software implementation
- Makes use of the schematic representations of the basic operations described earlier

Block Diagram Development

Example - Consider the first-order LTI digital system

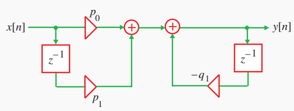
$$y[n] + q_1 y[n-1] = p_0 x[n] + p_1 x[n-1]$$

- We solve the above equation for $y[n]$ resulting in

$$y[n] = -q_1 y[n-1] + p_0 x[n] + p_1 x[n-1]$$

Block Diagram Development

- Hence, to compute $y[n]$ we multiply $y[n-1]$ with a constant $-q_1$, multiply $x[n]$ with a constant p_0, multiply $x[n-1]$ with a constant p_1, and add these products



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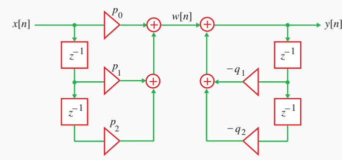
Block Diagram Analysis

Example – Consider the block diagram given below



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Block Diagram Analysis

- We first express the outputs of each adder as given below:

$$w[n] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2]$$

$$y[n] = w[n] - q_1 y[n-1] - q_2 y[n-2]$$

- Rearranging the last equation we obtain

$$y[n] + q_1 y[n-1] + q_2 y[n-2] = w[n]$$

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Block Diagram Analysis

- Substituting the expression for $w[n]$ in the last equation we arrive at the difference equation representation of the LTI digital system:

$$y[n] + q_1 y[n-1] + q_2 y[n-2]$$
$$= p_0 x[n] + p_1 x[n-1] + p_2 x[n-2]$$

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Equivalent Structures

- Different block diagram representations having identical differential equation representation are called equivalent structures

Block-Diagram Manipulation

- An equivalent structure can be easily developed by simple block diagram manipulations

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Equivalent Structures

Example - This approach is illustrated next by developing equivalent structures of the LTI digital system in Slide No. 62

- Now, the structure in Slide No. 62 can be considered as a cascade of two sections, one with an input sequence $x[n]$ and an output sequence $w[n]$, and the other with an input sequence $w[n]$ and an output sequence $y[n]$

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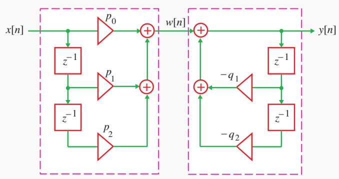
Equivalent Structures



Section 1 Section 2

- The above structure is equivalent to the one shown in the next slide

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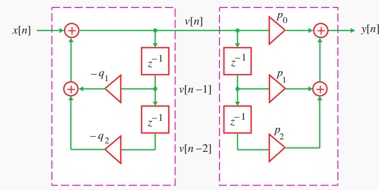
Equivalent Structures



Section 2 Section 1

- The output of the two top unit delay blocks are both $v[n-1]$

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Equivalent Structures

- Likewise, the output of the two bottom unit delay blocks are both $v[n-2]$
- Hence, we can eliminate two of the unit delay blocks resulting in the structure shown in the next slide without changing the input-output relation

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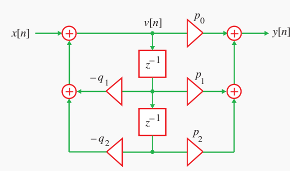
Equivalent Structures



- Analysis yields

$$v[n] = x[n] - q_1 v[n-1] - q_2 v[n-2]$$
$$y[n] = p_0 v[n] + p_1 v[n-1] + p_2 v[n-2]$$

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Equivalent Structures

Transpose Form

- A simple method of generating an equivalent structure of a block-diagram realization is obtained by applying the transpose operation
- The three steps of the transpose operation are:

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Equivalent Structures

- Step 1: Reverse all directed paths
- Step 2: Replace adders with pick-off nodes and vice-versa
- Step 3: Interchange input and output nodes
- Figure in the next slide shows the equivalent structure obtained by applying the transpose operation to the block-diagram in Slide No. 62

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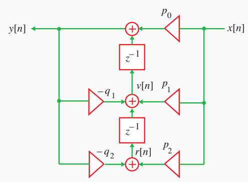
Equivalent Structures



- Analysis of the above structure yields

Equivalent Structures

$$r[n] = p_2 x[n] - q_2 y[n]$$

$$v[n] = p_1 x[n] - q_1 y[n] + r[n-1]$$

$$y[n] = p_0 x[n] + v[n-1]$$

- From the first equation we get

$$r[n-1] = p_2 x[n-1] - q_2 y[n-1]$$

- Substituting the above equation in the second equation at the top we get

Equivalent Structures

$$v[n] = p_1 x[n] + p_2 x[n-1] - q_1 y[n] - q_2 y[n-1]$$

- From the above equation we get

$$v[n-1] = p_1 x[n-1] + p_2 x[n-2] \\ -q_1 y[n-1] - q_2 y[n-2]$$

- Substituting the above in the third equation at the top in Slide No. 74 we arrive at

$$y[n] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2] \\ -q_1 y[n-1] - q_2 y[n-2]$$