Problem Set 4

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Determine whether the following sets of vectors are linearly independent or not

   (a) $\{(1,0,0),(1,1,0),(1,1,1)\}$ of $\mathbb{R}^3$

   Solution: Yes. Look at the null space, $N(A)$ of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. One can show that $N(A) = \{0\}$ by computing the RREF.

   (b) $\{(1,0,0,0),(1,1,0,0),(1,2,0,0),(1,1,1,1)\}$ of $\mathbb{R}^4$

   Solution: No. The null space $N(A)$ of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ contains $\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$.

   (c) $\{(1,0,2,1),(1,3,2,1),(4,1,2,2)\}$ in $\mathbb{R}^4$.

   Solution: Yes. Similar to (a).

2. Find a maximal linearly independent subset of

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   Find another. And another. Do they have the same cardinality?

   Solution: Compute the RREF. Verify that $\mathrm{RREF}([S]) = [I_5,\ \mathbf{u}_1,\ \mathbf{u_2},\ \mathbf{u}_3]$. Since all the entries of $\mathbf{u}_i$, for $i = 1,2,3$, are non-zero and hence any five vectors form a maximal independent set.

$$\mathrm{RREF}([S]) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{11}{21} & \frac{5}{7} & -\frac{1}{21} \\ 0 & 1 & 0 & 0 & 0 & -\frac{11}{7} & -\frac{6}{7} & \frac{6}{7} \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & \frac{20}{21} & \frac{3}{7} & -\frac{2}{21} \\ 0 & 0 & 0 & 0 & 1 & \frac{17}{14} & \frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

3. Give 2 bases for the trace 0 real symmetric matrices of size 3×3. Extend these bases to bases of the real matrices of size 3×3.

   Solution: $\{\mathbf{e}_{12}+\mathbf{e}_{21}, \mathbf{e}_{13}+\mathbf{e}_{31}, \mathbf{e}_{32}+\mathbf{e}_{23}, \mathbf{e}_{11}-\mathbf{e}_{22}, \mathbf{e}_{11}-\mathbf{e}_{33}\}$,
   $\{\mathbf{e}_{12}+\mathbf{e}_{21}+\mathbf{e}_{13}+\mathbf{e}_{31}, \mathbf{e}_{13}+\mathbf{e}_{31}-(\mathbf{e}_{32}+\mathbf{e}_{23}), \mathbf{e}_{12}+\mathbf{e}_{21}+2(\mathbf{e}_{13}+\mathbf{e}_{31})-3(\mathbf{e}_{32}+\mathbf{e}_{23}), \mathbf{e}_{11}-\mathbf{e}_{22}, \mathbf{e}_{11}-\mathbf{e}_{33}\}$.
   Just add the vectors $\{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}\}$

4. Consider $\mathbb{W} = \{\mathbf{v} \in \mathbb{R}^6 : v_1+v_2+v_3 = 0, v_2+v_3+v_4 = 0, v_5+v_6 = 0\}$. Supply a basis for $\mathbb{W}$ and extend it to a basis of $\mathbb{R}^6$.

5. Let M be the vector space of all 2×2 matrices and let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

   (a) Give a basis of M.

   Solution: One basis would be $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

   (b) Describe a subspace of M which contains A and does not contain B.

   Solution: The subspace consisting of all multiples of A is a subspace which contains A but not B.

   (c) Prove that if a subspace of M contains A and B, it must contain the identity matrix.

   Solution: True : If a subspace contains A and B then it also contains $A - B = I$.

6. **[T]** Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be a basis of the finite dimensional vector space $\mathbb{V}$. Let $\mathbf{v} \neq \mathbf{0}$ be any vector in $\mathbb{V}$. Show that there exists $\mathbf{w}_i$ such that if we replace $\mathbf{w}_i$ by $\mathbf{v}$ then we still have a basis.

   Solution: Since $\mathbf{v} \neq \mathbf{0}$, let $\mathbf{v} = \sum\limits_{i=1}^{n} \alpha_i \mathbf{w}_i$ with $\alpha_1 \neq 0$. So, $\mathbf{w}_1 = \frac{1}{\alpha_1}\mathbf{v} - \frac{1}{\alpha_1} \sum\limits_{i=2}^{n} \alpha_i \mathbf{w}_i$. Thus $\{\mathbf{v}, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ also spans $\mathbb{V}$. Now, $\beta_1 \mathbf{v} + \beta_2 \mathbf{w}_2 + \cdots + \beta_n \mathbf{w}_n = \mathbf{0} \Rightarrow \beta_1 \alpha_1 \mathbf{w}_1 + \sum\limits_{i=2}^{n} (\beta_1 \alpha_i + \beta_i) \mathbf{w}_i = \mathbf{0}$
   As $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis, $\Rightarrow \beta_1 \alpha_1 = 0 \Rightarrow \beta_1 = 0 \ (\alpha_1 \neq 0) \Rightarrow \beta_i = 0, \ i \geq 2$.

7. Show that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent if and only if $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

8. **(T)** Show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{R}^n$ is linearly independent if and only if $\{A\mathbf{u}_1, \ldots, A\mathbf{u}_k\}$ is linearly independent for any invertible matrix $A \in \mathbb{M}_n(\mathbb{R})$, *i.e.*, suppose we have an $n \times n$ invertible matrix A and consider the map $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(\mathbf{x}) = A\mathbf{x}$. Then, '$\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent if and only if the set consisting of their images is also linearly independent'.

   Solution: Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly dependent. Then $\exists \alpha_i$'s, not all 0 s.t. $\sum \alpha_i \mathbf{u}_i = \mathbf{0}$. So $\mathbf{0} = A\mathbf{0} = A \sum \alpha_i \mathbf{u}_i = \sum \alpha_i (A\mathbf{u}_i)$. Hence, $\{A\mathbf{u}_1, \ldots, A\mathbf{u}_n\}$ is linearly dependent.

   Now, suppose that $\{A\mathbf{u}_1, \ldots, A\mathbf{u}_n\}$ is linearly dependent. Then $\exists \alpha_i$'s not all 0 s.t. $\sum \alpha_i (A\mathbf{u}_i) = \mathbf{0}$. So $\mathbf{0} = \sum \alpha_i (A\mathbf{u}_i) = A \sum \alpha_i \mathbf{u}_i$. Hence, $A^{-1}\mathbf{0} = A^{-1} A \sum \alpha_i \mathbf{u}_i = \sum \alpha_i \mathbf{u}_i$. Thus, $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly dependent.

9. Show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{V}$ is linearly independent if and only if $\left\{ \sum\limits_{i=1}^{k} a_{i1} \mathbf{u}_i, \ldots, \sum\limits_{i=1}^{k} a_{ik} \mathbf{u}_i \right\}$ is linearly independent for any invertible matrix $A \in \mathbb{M}_k(\mathbb{R})$. This means: In $\mathrm{LS}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis if and only if the vectors $\mathbf{w}_j = \sum\limits_{i=1}^{k} a_{ij} \mathbf{u}_i$ (which are nothing but some linear combinations of $\mathbf{u}_i$'s given by the matrix A) is a basis.

   Solution: Put $\mathbf{w}_r = \sum\limits_{i=1}^{k} a_{ir} \mathbf{u}_i$. Then $\begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_r \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \end{bmatrix} = A^T \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \end{bmatrix}$.

Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be linearly dependent. Then $\exists \alpha_i$'s, not all 0 s.t. $[\alpha_1 \quad \cdots \quad \alpha_k] \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_k \end{bmatrix} = \mathbf{0}$. So

$$\mathbf{0} = [\alpha_1 \quad \cdots \quad \alpha_k] \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \end{bmatrix} = [\beta_1 \quad \cdots \quad \beta_k] \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \end{bmatrix},$$

where $[\alpha_1 \quad \cdots \quad \alpha_k]A^T = [\beta_1 \quad \cdots \quad \beta_k] \neq \mathbf{0}$. Thus $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly dependent.

Converse: Similar.

10. **(T)** If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ is a basis for a vector space $\mathbb{V}$, then show that any set of n vectors in $\mathbb{V}$ with $n > d$, say $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, is linearly dependent.

    Solution: Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ is a basis for $\mathbb{V}$ and $\mathbf{w}_j \in \mathbb{V}$ for $j = 1, \ldots, n$, there exist constants a_{ij}, $1 \le i \le d, 1 \le j \le n$ such that

    $$\mathbf{w}_j = \sum_{i=1}^{d} a_{ij}\mathbf{v}_i.$$

    Consider a linear combination of $\mathbf{w}_j$'s that equals zero, that is, $\sum\limits_{j=1}^{n} c_j\mathbf{w}_j = 0$. Then,

    $$\sum_{j=1}^{n} c_j\mathbf{w}_j = 0 \iff \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{d} a_{ij}\mathbf{v}_i\right) = 0 \iff \sum_{i=1}^{d} \left(\sum_{j=1}^{n} a_{ij}c_j\right)\mathbf{v}_i = 0.$$

    As $\mathbf{v}_i$'s are linearly independent, we have $A\mathbf{c} = \mathbf{0}$ where the matrix A is a d by n matrix and $\mathbf{c}$ is a column vector of size n with $[A]_{ij} = a_{ij}$. As A is a rectangular matrix with more columns than rows, its null space is non-trivial. We therefore have non-zero c_j's with $\sum\limits_{j=1}^{n} c_j\mathbf{w}_j = \mathbf{0}$. Thus, vectors $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is linearly dependent.

11. Suppose $\mathbb{V}$ is a vector space of dimension d. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be a set of vectors from $\mathbb{V}$. Then show that S does not span $\mathbb{V}$ if $n < d$.

    Solution: Let B be a basis of V . Since $\dim(\mathbb{V}) = d$, the definition implies that B is a linearly independent set of d vectors that spans $\mathbb{V}$.

    Now, suppose on the contrary that S does span $\mathbb{V}$. Then B is a larger set of vectors which is linearly independent. This contradicts the result in the previous problem.

12. **(T)** Let $T = \{1, x^2 - x + 5, 4x^3 - x^2 + 5x, 3x^4 + 2\}$. Is $\mathrm{LS}(T) = \mathbb{R}[x; 4]$?

    Solution: The vector space $\mathbb{R}[x; 4]$ has dimension 5. Since T contains only 4 vectors, T does not span $\mathbb{R}[x; 4]$.

13. Let $\mathbb{W}$ be a proper subspace of a finite dimensional vector space $\mathbb{V}$.

(a) Show that there is a subspace $\mathbb{U}$ of $\mathbb{V}$ such that $\mathbb{W} \cap \mathbb{U} = \{\mathbf{0}\}$ and $\mathbb{U} + \mathbb{W} = \mathbb{V}$.

Solution: Extend the basis of $\mathbb{W}$ to a basis of $\mathbb{V}$ and define $\mathbb{U}$ to be the span of new basis elements.

(b) Show that there is no subspace $\mathbb{U}$ such that $\mathbb{U} \cap \mathbb{W} = \{\mathbf{0}\}$ and $\dim \mathbb{U} + \dim \mathbb{W} > \dim \mathbb{V}$.

Solution: Follows from $\dim(\mathbb{U} + \mathbb{W}) = \dim(\mathbb{U}) + \dim(\mathbb{W}) - \dim(\mathbb{U} \cap \mathbb{W})$ (just ask the students to assume this result) and the fact that $\mathbb{U} + \mathbb{W}$ is a subspace of $\mathbb{V}$.

14. **(T)** Describe all possible ways in which two planes (passing through origin) in $\mathbb{R}^3$ could intersect.

Solution: Let $\mathbb{U}$ and $\mathbb{V}$ be planes. Then, $dim(\mathbb{U}+\mathbb{V}) = dim(\mathbb{U}) + dim(\mathbb{V}) - dim(\mathbb{U}\cap\mathbb{V})$ implies that $dim(\mathbb{U}+\mathbb{V}) = 4 - dim(\mathbb{U}\cap\mathbb{V})$. Clearly, $2 \leq dim(\mathbb{U}+\mathbb{V}) \leq 3$. If $dim(\mathbb{U}+\mathbb{V}) = 2$, then $dim(\mathbb{U}\cap\mathbb{V}) = 2$ which implies $\mathbb{U}+\mathbb{V} = \mathbb{U} = \mathbb{V} = \mathbb{U}\cap\mathbb{V}$, i.e., $\mathbb{U} = \mathbb{V}$. If $dim(\mathbb{U}+\mathbb{V}) = 3$, then $dim(\mathbb{U}\cap\mathbb{V}) = 1$ which implies that $\mathbb{U}$ and $\mathbb{V}$ intersect on a line.

15. Construct a matrix with the required property or explain why this is impossible:

(a) Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}$.

Solution: $\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix}$.

(b) Column space has basis $\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}\right\}$, null-space has basis $\left\{\begin{bmatrix}3\\1\\1\end{bmatrix}\right\}$. What if $\begin{bmatrix}3\\1\\1\end{bmatrix}$ belongs to the null space (but not necessarily forms a basis)?

Solution: Not possible; dimension of the column space and the dimension of the null-space must add to 3. For the second part, take $A = \begin{bmatrix}1 & 1 & -4\\1 & 1 & -4\\3 & 3 & -12\end{bmatrix}$.

(c) The dimension of null-space is one more than the dimension of left null-space.

Solution: $\begin{bmatrix}1 & 1\end{bmatrix}$ or $\begin{bmatrix}1 & 1 & 2\\1 & 1 & 2\end{bmatrix}$ or $\begin{bmatrix}1 & 0 & 2\\0 & 1 & 5\end{bmatrix}$

(d) Left null-space contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space contains $\begin{bmatrix}3\\1\end{bmatrix}$.

Solution: $\begin{bmatrix}-9 & -3\\3 & 1\end{bmatrix}$.

16. Suppose A is a 3 by 4 matrix and B is a 4 by 5 matrix with $AB = \mathbf{0}$. Show that

$$\text{rank}(A) + \text{rank}(B) \leq 4.$$

Solution: As $AB = \mathbf{0}$, $\text{col}(B) \subseteq N(A)$. Therefore, the $\dim(\text{col}(B)) \leq \dim(N(A))$. This implies $\text{rank}(B) \leq 4 - \text{rank}(A)$.

17. **(T)** Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ and $B \in \mathbb{M}_{n,p}(\mathbb{R})$ with $\operatorname{rank}(A) = \operatorname{rank}(B) = n$. Show that $\operatorname{rank}(AB) = n$.

    Solution: Since $\operatorname{rank}(A) = n$, $m \geq n$. Thus, there exists an invertible matrix P such that $PA = \begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$. As P is invertible

$$\operatorname{rank}(AB) = \operatorname{rank}(PAB) = \operatorname{rank}\left(\begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix} B\right) = \operatorname{rank}\left(\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}\right) = \operatorname{rank}(B).$$