Problem Set 4

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Determine whether the following sets of vectors are linearly independent or not
   (a) $\{(1,0,0),(1,1,0),(1,1,1)\}$ of $\mathbb{R}^3$
   (b) $\{(1,0,0,0),(1,1,0,0),(1,2,0,0),(1,1,1,1)\}$ of $\mathbb{R}^4$
   (c) $\{(1,0,2,1),(1,3,2,1),(4,1,2,2)\}$ in $\mathbb{R}^4$.

2. Find a maximal linearly independent subset of
$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$
   Find another. And another. Do they have the same cardinality?

3. Give 2 bases for the trace 0 real symmetric matrices of size 3×3. Extend these bases to bases of the real matrices of size 3×3.

4. Consider $\mathbb{W} = \{\mathbf{v} \in \mathbb{R}^6 : v_1 + v_2 + v_3 = 0, v_2 + v_3 + v_4 = 0, v_5 + v_6 = 0\}$. Supply a basis for $\mathbb{W}$ and extend it to a basis of $\mathbb{R}^6$.

5. Let M be the vector space of all 2×2 matrices and let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
   (a) Give a basis of M.
   (b) Describe a subspace of M which contains A and does not contain B.
   (c) Prove that if a subspace of M contains A and B, it must contain the identity matrix.

6. **[T]** Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be a basis of the finite dimensional vector space $\mathbb{V}$. Let $\mathbf{v} \neq \mathbf{0}$ be any vector in $\mathbb{V}$. Show that there exists $\mathbf{w}_i$ such that if we replace $\mathbf{w}_i$ by $\mathbf{v}$ then we still have a basis.

7. Show that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent if and only if $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$ is linearly independent.

8. **(T)** Show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{R}^n$ is linearly independent if and only if $\{A\mathbf{u}_1, \ldots, A\mathbf{u}_k\}$ is linearly independent for any invertible matrix $A \in \mathbb{M}_n(\mathbb{R})$, *i.e.*, suppose we have an $n \times n$ invertible matrix A and consider the map $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(\mathbf{x}) = A\mathbf{x}$. Then, '$\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent if and only if the set consisting of their images is also linearly independent'.

9. Show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{V}$ is linearly independent if and only if $\left\{ \sum\limits_{i=1}^{k} a_{i1}\mathbf{u}_i, \ldots, \sum\limits_{i=1}^{k} a_{ik}\mathbf{u}_i \right\}$ is linearly independent for any invertible matrix $A \in \mathbb{M}_k(\mathbb{R})$. This means: In $\mathrm{LS}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis if and only if the vectors $\mathbf{w}_j = \sum\limits_{i=1}^{k} a_{ij}\mathbf{u}_i$ (which are nothing but some linear combinations of $\mathbf{u}_i$'s given by the matrix A) is a basis.

10. **(T)** If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ is a basis for a vector space $\mathbb{V}$, then show that any set of n vectors in $\mathbb{V}$ with $n > d$, say $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, is linearly dependent.

11. Suppose $\mathbb{V}$ is a vector space of dimension d. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be a set of vectors from $\mathbb{V}$. Then show that S does not span $\mathbb{V}$ if $n < d$.

12. **(T)** Let $T = \{1, x^2 - x + 5, 4x^3 - x^2 + 5x, 3x^4 + 2\}$. Is $\mathrm{LS}(T) = \mathbb{R}[x; 4]$?

13. Let $\mathbb{W}$ be a proper subspace of a finite dimensional vector space $\mathbb{V}$.

    (a) Show that there is a subspace $\mathbb{U}$ of $\mathbb{V}$ such that $\mathbb{W} \cap \mathbb{U} = \{\mathbf{0}\}$ and $\mathbb{U} + \mathbb{W} = \mathbb{V}$.

    (b) Show that there is no subspace $\mathbb{U}$ such that $\mathbb{U} \cap \mathbb{W} = \{\mathbf{0}\}$ and $\dim \mathbb{U} + \dim \mathbb{W} > \dim \mathbb{V}$.

14. **(T)** Describe all possible ways in which two planes (passing through origin) in $\mathbb{R}^3$ could intersect.

15. Construct a matrix with the required property or explain why this is impossible:

    (a) Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}$.

    (b) Column space has basis $\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}\right\}$, null-space has basis $\left\{\begin{bmatrix}3\\1\\1\end{bmatrix}\right\}$. What if $\begin{bmatrix}3\\1\\1\end{bmatrix}$ belongs to the null space (but not necessarily forms a basis)?

    (c) The dimension of null-space is one more than the dimension of left null-space.

    (d) Left null-space contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space contains $\begin{bmatrix}3\\1\end{bmatrix}$.

16. Suppose A is a 3 by 4 matrix and B is a 4 by 5 matrix with $AB = \mathbf{0}$. Show that

$$\mathrm{rank}(A) + \mathrm{rank}(B) \leq 4.$$

17. **(T)** Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ and $B \in \mathbb{M}_{n,p}(\mathbb{R})$ with $\mathrm{rank}(A) = \mathrm{rank}(B) = n$. Show that $\mathrm{rank}(AB) = n$.