LECTURE-9

CAUCHY'S THEOREM.

Lecture 9: Cauchy's theorem

We saw in the last lecture that if f has an antiderivative in a domain D, then for any closed contour C in D we have

$$\int_C f(z)\,dz = 0.$$

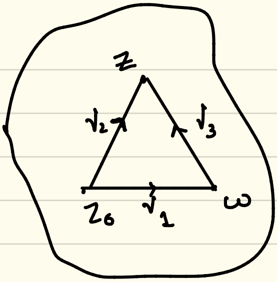
It turns out that $\int_C f(z)\,dz = 0$ is equivalent to f having an antiderivative.

(Indeed, if $\int_C f(z)\,dz = 0$ for any closed contour C in D then $\int_{z_1}^{z_2} f(z)\,dz$ is independent of the contour. Now, define for $z \in D$

$$F(z) := \int_{z_0}^{z} f(\zeta)\,d\zeta.$$

Consider $\dfrac{F(z)-F(\omega)}{z-\omega} - f(\omega)$

$$= \frac{\int_{z_0}^{z} f(\zeta)\,d\zeta - \int_{z_0}^{\omega} f(\zeta)\,d\zeta}{z-\omega} - f(\omega) \qquad (*)$$



$$\int_{\gamma_1+\gamma_3-\gamma_2} f(z)\,dz = 0 \quad \text{(given)}.$$

$$\Rightarrow \int_{\gamma_3} f(z)\,dz = \int_{\gamma_2} f(z)\,dz - \int_{\gamma_1} f(z)\,dz$$

So, (*) becomes

$$\left(\frac{\int_{\omega}^{z} f(\zeta)\,d\zeta}{z-\omega} \right) - f(\omega)$$

$$\therefore \left| \frac{F(z)-F(\omega)}{z-\omega} - f(\omega) \right| = \frac{1}{|z-\omega|} \left| \int_{\omega}^{z} \big(f(\zeta)-f(\omega)\big)\,d\zeta \right|$$

$$\leq \frac{1}{|z-\omega|} \cdot |z-\omega| \sup_{\zeta \in \overline{z\omega}} |f(\zeta)-f(\omega)|$$

as $z \to \omega$, $|f(\zeta)-f(\omega)| \to 0$ $(\because |\zeta-\omega| \leq |z-\omega|)$

<u>Question</u>: Under what condition on f, do we have $\int_C f(z)\,dz = 0$ for any closed contour C.

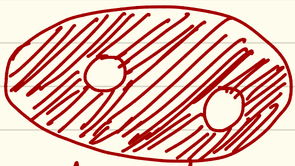
For simplicity of our discussion we always consider <u>simple</u> closed contours.

The answer to the above question is a centre-piece in complex analysis: CAUCHY'S THEOREM

<u>THEOREM</u>: (Cauchy's theorem): Let f be an analytic function on a <u>simply connected</u> domain D and C be a simple closed contour lying in D then $\int_C f(z)\,dz = 0$

Defn: every simple closed contour in D contains points of D alone.

simple connected



not simply connected

(We make use of Green's theorem so we assume that f' is also continuous. The proof of the general statement involves topological arguments. So the proof will not be discussed).

C is assumed to be positively oriented.

Pf: Let $f(z) = u(x,y) + i\,v(x,y)$.

Let $\gamma(t) = x(t) + iy(t)$, $a \le t \le b$, be the contour C.

Then $$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_a^b \Big[u(x(t),y(t)) + i\,v(x(t),y(t))\Big]\Big[x'(t) + iy'(t)\Big]dt$$

$$= \int_a^b (ux' - vy')(t)\,dt + i\int_a^b (vx' + uy')(t)\,dt.$$

$$= \int_a^b (u\,dx - v\,dy) + i\int_a^b (v\,dx + u\,dy)$$

Green's theorem:
$$\int_{\partial D} M\,dx + N\,dy = \iint_D (N_x - M_y)\,dx\,dy.$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \iint_R (-v_x - u_y)\,dx\,dy + i\iint_R (u_x - v_y)\,dx\,dy$$

where R is the region enclosed in C.

$= 0$ (by CR-equations)

Consequences of Cauchy's theorem:

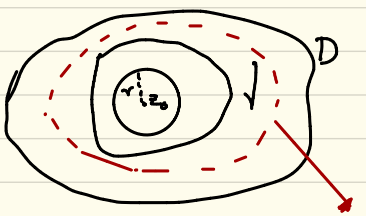
1. Existence of anti-derivative: (already seen)

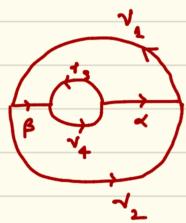
2. Independence of path: $\int_{z_1}^{z_2} f(z)\,dz$ is independent of path chosen from z_1 to z_2.

3. **Deformation theorem**:

Let D be a s.c. domain.
Let f be holomorphic on D except possibly at a point $\{z_0\}$. Let γ be a closed contour in D containing z_0 be a point in the region enclosed by γ.

Then $$\int_{\gamma} f(z)\,dz = \int_{C_{z_0,r}} f(z)\,dz$$





$C_1 = \alpha + \gamma_1 + \beta - \gamma_3$ is a closed contour

$C_2 = \alpha - \gamma_2 + \beta + \gamma_4$, , , . . .

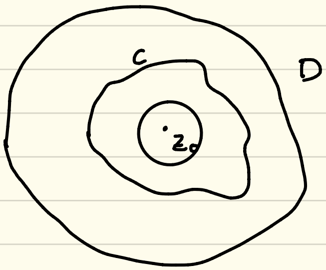
$$\int_{C_1} f(z)\,dz = 0 = \int_{C_2} f(z)\,dz \Rightarrow \int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_3+\gamma_4} f(z)\,dz$$

Cauchy Integral Formula:

Let f be an analytic fn. on a simply connected domain D. Suppose $z_0 \in D$ and C be a simple closed curve in D enclosing z_0.

Then $$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0).$$

(oriented anti-clockwise) * this is a very important convention



Pf: By deformation theorem $$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_{z_0,r}} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_0^{2\pi} \frac{f(z_0+re^{it})}{re^{it}}\, rie^{it}\,dt$$

$$= i\int_0^{2\pi} f(z_0+re^{it})\,dt$$

Hence, $\left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz - f(z_0) \right|$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} \left(f(z_0 + re^{it}) - f(z_0) \right) dt \right|$$

$$\leq \frac{1}{2\pi} \times 2\pi \times \sup_{t \in [0, 2\pi]} \left| f(z_0 + re^{it}) - f(z_0) \right| \quad \text{— } (*)$$

Since f is continuous on D, in particular at z_0,

given $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $|f(\zeta) - f(z_0)| < \varepsilon$ $\forall$ $|\zeta - z_0| < \delta$

if $r < \delta$, $(*)$ above becomes $\leq \varepsilon$.

Thus, $\boxed{\frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz = f(z_0)}$.

Example: $\displaystyle\int_{C_{0,5}} \frac{\cos z}{z}\,dz = 2\pi i(\cos 0) = 2\pi i$

CAUCHY INTEGRAL FORMULA II:

Theorem 3: If f is analytic on a simply connected domain D then f has derivatives of all orders in D; for any $z_0 \in D$,

$$f^{n}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

where C is a simple closed contour (oriented counterclockwise) around z_0 in D.

Proof: Using CIF,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

$$= \lim_{h\to 0} \left[\frac{\displaystyle\int_C \frac{f(z)}{z-(z_0+h)}\, dz - \int_C \frac{f(z)}{z-z_0}\, dz}{2\pi i h} \right]$$

$$= \lim_{h \to 0} \left[\frac{1}{2\pi i h} \int_C \frac{f(z)\left[(z-z_0) - [z-(z_0+h)]\right]}{(z-z_0)\,(z-(z_0+h))} \right]$$

$$= \lim_{h \to 0} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)(z-(z_0+h))} \, dz$$

We wish to show that $f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$

Consider
$$\left| \int_C \left[\frac{f(z)}{(z-z_0)(z-z_0+h)} - \frac{f(z)}{(z-z_0)^2} \right] dz \right|$$

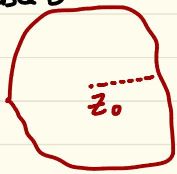
$$\left| \int_C \frac{f(z)\,h}{(z-z_0)^2(z-z_0+h)} \, dz \right|$$

We have to show that $\downarrow \; < \varepsilon$ when $|h| < \delta$ for a suitable $\delta > 0$.

To use ML inequality we have to estimate

Let $\alpha = \min_{z \in C} |z - z_0| > 0$

$(\because z_0 \notin C$ & C is closed) and bounded



Then $\alpha \leq |z - z_0| = |z - z_0 - h + h|$

$$\leq |z - (z_0+h)| + |h|$$

For $|h| < \frac{\alpha}{2}$, $\quad |z - (z_0+h)| > \alpha/2$

$$\left|\frac{f(z)}{(z-z_0)^2\,(z-(z_0+h))}\right| \leq \frac{|f(z)|}{\alpha^2 \cdot \alpha/2}$$

$$\therefore \sup_{z \in C} \frac{2|f(z)|}{\alpha^3} = M$$

By ML-inequality, we get

$$\left|\frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)^2}\,dz\right|$$

$$= \left|\frac{1}{2\pi i}\int_C \frac{f(z)\,h}{(z-z_0)^2\,(z-(z_0+h))}\,dz\right| \leq \frac{M|h|\,l}{2\pi}$$

where l = length of C.

The RHS $\to 0$ as $|h| \to 0$

$$\therefore\ f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

A similar argument as above for arbitrary $n>0$, gives

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\, f(z_0) \quad \text{if } n=0$$

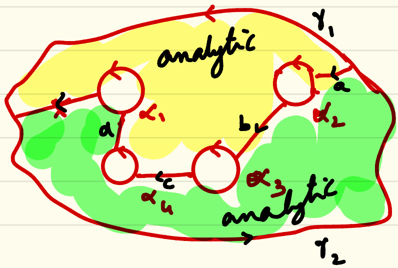
$$= \frac{2\pi i}{n!} f^{(n)}(z_0) \quad \text{if } n\geq 1.$$

REMARK: If z_0 is not contained in the region enclosed by C then the above integral is 0 (by Cauchy's theorem)

APPENDIX

Cauchy's theorem for multiply connected domains ie if f is analytic in D except at finitely many pts $z_1,\dots,z_n$ then

$$\int_\gamma f(z)\,dz = \int_{\gamma_1+\gamma_2} \dots$$



$$0 = \int_{-\gamma_1+\alpha_2^u+\alpha_3^u+\alpha_4^u+\alpha_1^u \cancel{+a+b+c+d}} \dots + \int_{-\gamma_2 \cancel{-a-b-c-d}+\alpha_1^l+\alpha_4^l+\alpha_3^l+\alpha_2^l} \dots$$

$$\therefore \int_{\gamma_1+\gamma_2} = \int_{\alpha_1} + \int_{\alpha_2} + \int_{\alpha_3} + \int_{\alpha_4}.$$