MSO202A COMPLEX ANALYSIS
Assignment 3

Exercise Problems:

1. (a) The hyperbolic functions $\cosh z$ and $\sinh z$ are defined as $\cos iz$ and $-i\sin iz$, respectively. Show that $\cosh^2 z - \sinh^2 z = 1$.

   (b) Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$. Conclude that $\cos z$ is not bounded in $\mathbb{C}$.

   (c) Show that $\cos z = 0 \iff z = (2n+1)\pi/2$ for $n \in \mathbb{Z}$.

   Proof: (a) Direct checking using the relations $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.
   (b) The first part follows from : $\cos z = \cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y$. For the second part, we may assume $y > 0$. Then $\cos^2 x + \sinh^2 y \geq \sinh^2(y) = \frac{e^{2y}+e^{-2y}-2}{4} \geq \frac{e^{2y}-2}{4}$ (the last inequality follows from noticing that $y > 0 \Rightarrow e^{-2y} > 0$). Hence $\cos z$ is unbounded on $\mathbb{C}$.
   (c) From (b) we get $\cos z = 0 \iff \cos x = 0 = \sinh y$. Thus, $x = (2n+1)\pi/2$ and $y = 0$. Therefore $\cos z$ has the same set of zeros as $\cos x$, $x \in \mathbb{R}$.

2. Find the roots of the equation $\sin z = 2$.

   Proof: $\sin z = 2 \Leftrightarrow \dfrac{e^{iz}-e^{-iz}}{2i} = 2 \Leftrightarrow e^{iz} - e^{-iz} = 4i$. Set $w = e^{iz}$, to get $w^2 - 4iw - 1 = 0$. So $w = i(2 \pm \sqrt{3})$ and $e^{i(x+iy)} = i(2 \pm \sqrt{3})$. So, $e^{-y} = |e^{iz}| = 2 \pm \sqrt{3}$ and $\cos x = 0$. Thus, we get $z = (2k+1)\pi/2 - i\ln(2 \pm \sqrt{3})$.

3. Express the following complex numbers in the standard form $x + iy$ and find their principal value. (a) i^{-i} (b) $(-1+i\sqrt{3})^i$. (*Note: For* $c \in \mathbb{C}$, $z^c = e^{c\log z}$, *and for principal value of* z^c *we take* $z^c = e^{c\,\mathrm{Log}\, z}$, *where* $\mathrm{Log}\,(z) = \ln|z| + i\mathrm{Arg}(z)$, *with* $\mathrm{Arg}(z) \in (-\pi, \pi]$ *and* $\log(z) = \mathrm{Log}\,(z) + i2\pi k$.)

   Proof:

   (a) $i = \ln 1 + i(\dfrac{\pi}{2} + 2k\pi) \Rightarrow i^{-i} = e^{-i(\ln 1 + i(\frac{\pi}{2}+2k\pi))} = e^{\frac{\pi}{2}+2k\pi}$, k an integer. For the principal value, take $k = 0$.

   (b) As $z = -1+i\sqrt{3} = 2e^{i(2\pi/3+2\pi k)}, k \in \mathbb{Z} \Rightarrow \ln z = \ln 2 + i(\dfrac{2\pi}{3} + 2k\pi), k \in \mathbb{Z}$ and so $(-1+i\sqrt{3})^i = e^{i\ln z} = e^{i(\ln 2 + i(\frac{2\pi}{3}+2k\pi))} = e^{i\ln 2}e^{-(\frac{2\pi}{3}+2k\pi)}$. For the principal value, take $k = 0$.

4. Using the method of parametric representation, evaluate $\oint_C f(z)\,dz$ for (a) $f(z) = \overline{z}$, (b) $f(z) = z + \frac{1}{z}$, (c) $f(z) = \mathrm{Re}\ z$ (d) $f(z) = \sin z/z$ and C is the unit circle centered at origin oriented counterclockwise.

Proof: Let $z = e^{i\theta}$, $-\pi < \theta \leq \pi$. Then

(a) $\oint \overline{z}\, dz = \int_{-\pi}^{\pi} e^{-i\theta}\, ie^{i\theta}\, d\theta = 2\pi i$.

(b)

$$\oint (z + \frac{1}{z})\, dz = \int_{-\pi}^{\pi} (e^{i\theta} + e^{-i\theta})\, ie^{i\theta}\, d\theta = i\int_{-\pi}^{\pi} (e^{2i\theta} + 1)\, d\theta. = i(\frac{e^{2i\theta}}{2i} + \theta)|_{-\pi}^{\pi} = 2\pi i.$$

(c)

$$\oint \operatorname{Re} z\, dz = \int_{-\pi}^{\pi} \cos\theta\, ie^{i\theta}\, d\theta = i\int_{-\pi}^{\pi} (\cos^2\theta + i\cos\theta\sin\theta)\ d\theta$$
$$= i\int_{-\pi}^{\pi} \cos^2\theta\ d\theta - \int_{-\pi}^{\pi} \cos\theta\sin\theta\ d\theta = i\pi.$$

(d)We have the power series

$$\sin z = z\left(\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}\right).$$

The series within brackets on the right hand side also has infinite radius of convergence (use ratio test), hence $\frac{\sin z}{z}$ is analytic everywhere. Now apply Cauchy's theorem.

5. Evaluate the integral $\int_\Gamma ze^{z^2}dz$ where Γ is the curve from 0 to $1+i$ along the parabola $y = x^2$.

Proof: Let $g(z) = \frac{e^{z^2}}{2}$. Then $g'(z) = ze^{z^2}$. Hence $\int_\Gamma g'(z)dz = g(1+i) - g(0) = \frac{1}{2}(e^{(1+i)^2} - 1)$.

6. (a) Assign an appropriate meaning to the integral $\displaystyle\int_{-i}^{i} \frac{1}{z}\, dz$ and find its value.

   (b) $\int_C \sin^2 z\, dz$, C is the curve from $-\pi i$ to πi along $|z| = \pi$ taken counter -clockwise.

Proof:

(a) The integral is defined as line integral of the function $\frac{1}{z}$, along any path from $-i$ to i contained in simply connected domain $\mathbb{C} \setminus \{$ the negative real axis$\}$. This definition is independent of the chosen path as $\frac{1}{z}$ being analytic on simply connected domain $\mathbb{C} \setminus \{$ the negative real axis$\}$ has a primitive $Ln(z)$ in this domain. Further, $\displaystyle\int_{-i}^{i} \frac{1}{z}\, dz = Ln(i) - Ln(-i) = i\pi$.

(b) Since $\sin^2 z = \frac{1-\cos 2z}{2}$ has primitive $F(z) = \frac{z}{2} - \frac{\sin 2z}{4}$, $\int_C \sin^2 z\, dz = F(\pi i) - F(-\pi i) = \pi i + 2\sin(2\pi i)$.

Problem for Tutorial:

1. A function $u : U \to \mathbb{R}$ is said to be *harmonic* on an open subset $U \subset \mathbb{R}^2$ if its 1st and 2nd order partial derivatives w.r.t x and y exist, are continuous and satisfy the equation $u_{xx}+u_{yy} = 0$ on U. A harmonic function $v : U \to \mathbb{R}$ is said to be a *harmonic conjugate* of u if the function $f(z) := u(x, y) + iv(x, y)$ is analytic (equivalently, if the CR equations hold for u and v).

   (a) Let $f : D \subset \mathbb{C} \to \mathbb{C}$ be a twice* continuously differentiable function on a domain D. Then show that
   (i) u, v are harmonic functions and v is a harmonic conjugate of u;
   (ii) v is unique upto a constant, *i*.e., if v' is another harmonic conjugate of u then $v' = v + c$ for some $c \in \mathbb{R}$;
   (iii) further, if u is a harmomic conjugate of v as well, then u and v are constants.

   (b) Find a harmonic conjugate of $u(x, y) = 3xy^2 - x^3$ on $\mathbb{C}$.

   Proof: (a) (i) Since f satisfies CR equations, $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Since the partial derivative are continuous we also have, $v_{xy} = v_{yx}$. Hence, $u_{xx} = -u_{yy}$ as required. Similarly for v.
   (ii)Let v and v' be harmonic conjugates of u. Then $u + iv$ and u_iv' are analytic on U. Thus their difference $i(v - v')$ is also analytic. This function has real part is 0 on U, so $i(v - v')$ is a constant. Thus $v - v'$ is a constant $c \in \mathbb{R}$.
   (iii) Given v is harmonic conjugate of u, so $u_x = v_y; u_y = -v_x$. Further, if u is a harmonic conjugate of v then $v_x = u_y; v_y = -u_x$. So, we get $2u_x = 0 = 2u_y$,*i*.e., u is a constant. Similarly v is a constant.
   (b) We have $v_y = u_x = 3y^2 - 3x^2$; $v_x = -u_y = -6xy$. From these relations we get $v(x, y) = -3x^2y + \phi(y)$, and $\phi'(y) = 3y^2$. Thus, $v(x, y) = -3x^2y + y^3 + 1$.

2. Show that $u(x, y) := \log(|\sqrt{x^2 + y^2}|)$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$ (*i*.e., $\mathbb{C} \setminus \{0\}$, also denoted as $\mathbb{C}^*$) but it does not have any harmonic conjugates there.

   Proof: The first part is a direct checking. For the second part, suppose that v is a harmonic conjugate of u. Then $f(z) = u(x, y) + iv(x, y)$ is analytic on $\mathbb{C}^*$, so is $g(z) = ze^{-f(z)}$. Then $|g(z)| = |z||e^{-\operatorname{Log}|z|}| = |z|\frac{1}{|z|}$. This is possible only if $g(z)$ is constant. Hence $g'(z) = 0$, which implies that $e^{-f(z)}(1-zf'(z)) = 0$. Since $e^{-f(z)} \neq 0$ for any z, we have $f'(z) = 1/z \forall z \neq 0$. Since f' has an anti-derivative we have by Cauchy's theorem $\int_{C(0,2)} \frac{1}{z} = 0$ where $C(0, 2)$ denotes the circle of radius 2 around 0. However, the fundamental integral is $2\pi i \neq 0$, a contradiction. We conclude therefore that such a v does not exist. *Note: If we consider u on $\mathbb{C} \setminus \{$negative real axis$\}$ then u has a harmonic conjugate.*

*A function that is analytic in a domain is infinitely differentiable.

3. Express i^i in the standard form $x+iy$ and find its principal value.

Proof: (*Recall: For* $c \in \mathbb{C}$, $z^c = e^{c \ln z}$, *and for principal value of* z^c *we take* $z^c = e^{c\,\mathrm{Log}\,z}$, *where the definition of is* $\mathrm{Log}\,(z) = \ln|z| + i\mathrm{Arg}(z)$, *with* $\mathrm{Arg}(z) \in (-\pi, \pi]$ *and* $\ln z = \ln|z| + i\ \arg(z)$.) We have $i = e^{i(\pi/2+2k\pi)} \Rightarrow i^i = e^{i[\ln 1 + i(\pi/2+2k\pi)]} = e^{-\pi/2-2k\pi}$. For $k=0$ we get the principal value.

4. Evaluate the following integrals by parametrizing the contour

   (a) $\int_{\mathcal{C}} Re\, z\ dz$ where $\mathcal{C}$ is the line segment joining 1 to i.

   (b) $\int_{\mathcal{C}} (z-1) dz$ where $\mathcal{C}$ is the semicircle (in the lower half plane) joining 0 to 2.

Proof: (a) Let $\gamma(t) = (1-t) + it$ with t goes from 0 to 1. Then $\int_{\mathcal{C}} x dz = \int_0^1 (1-t)(i-1)\,dt = \frac{i-1}{2}$.

(b) Use the parametrisation $z = 1 + e^{i\theta}$, θ goes from $-\pi$ to 0, and $dz = ie^{i\theta}\,d\theta$. $\int_{\mathcal{C}} (z-1)\,dz = \int_{-\pi}^0 e^{i\theta}(ie^{i\theta})\,d\theta = 0$.

5. Let $\bar{\mathbb{D}} = \{z \in \mathbb{C} \ :\ |z| \leq 1\}$ and f be analytic on $\mathbb{D}$. Let $a,\ b \in \mathbb{D}$ and $\gamma(t) = a + t(b-a)$, $t \in [0,1]$ be the straight line joining a and b.

   (a) Prove that $\frac{f(b)-f(a)}{b-a} = \int_0^1 f'(\gamma(t))dt$.

   (b) Using the above, if required, show that if $Re\ f'(z) > 0$ for all $z \in \mathbb{D}$ then f is injective.

Proof: (a) Since f is analytic on $\mathbb{D}$, we know that the integral $\int_\gamma f'(z)dz$ is independent of the path. It is equal to $f(b) - f(a)$. On the otherhand, $\int_\gamma f'(z)dz = \int_0^1 f'(\gamma(t))\gamma'(t)dt$. As $\gamma'(t) = b - a$, we get (a).

(b) Since $Re(f'(\gamma(t)) : [0,1] \to \mathbb{R}$ is a real-valued function taking values > 0, the Riemann integral $\int_0^1 f'(\gamma(t))dt \neq 0$. We have,

$$\frac{f(b)-f(a)}{b-a} = \int_0^1 Ref'(\gamma(t))dt + i\int_0^1 Imf'(\gamma(t))dt \neq 0.$$

Thus, we get $f(b) \neq f(a)$ for arbitrary $a, b \in \mathbb{D}$.