ESC201T : Introduction to Electronics

Lecture 13: Sinusoidal Steady State Analysis

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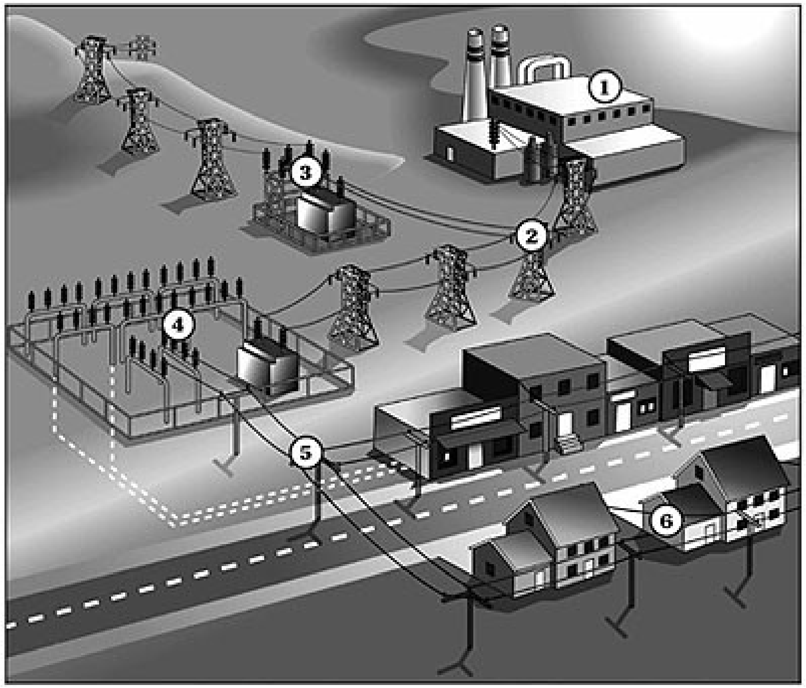
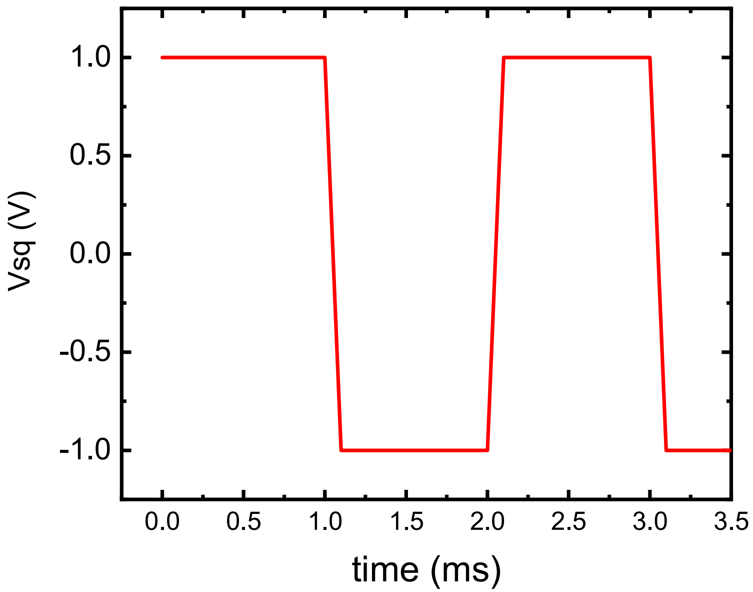
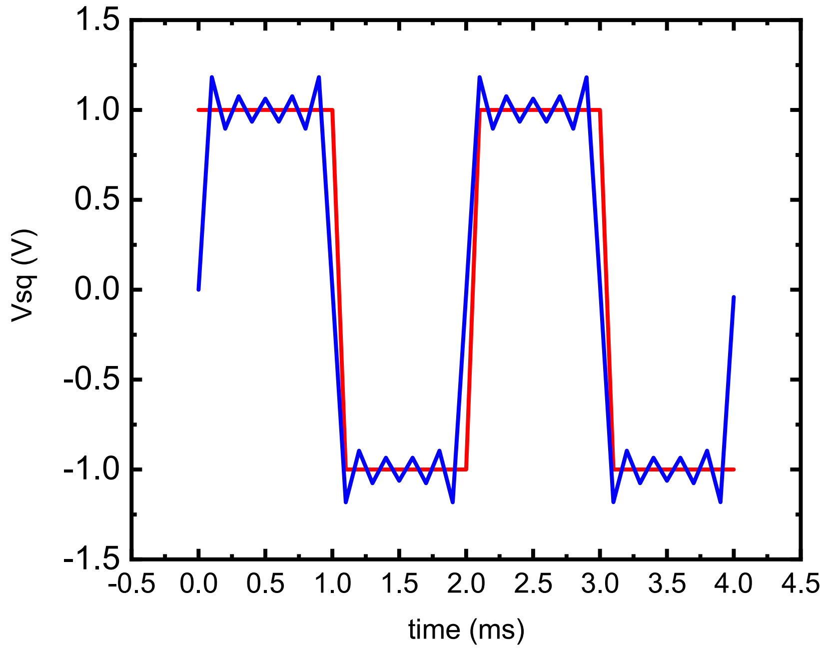
Sinusoidal Signals are widely used In Electrical Systems

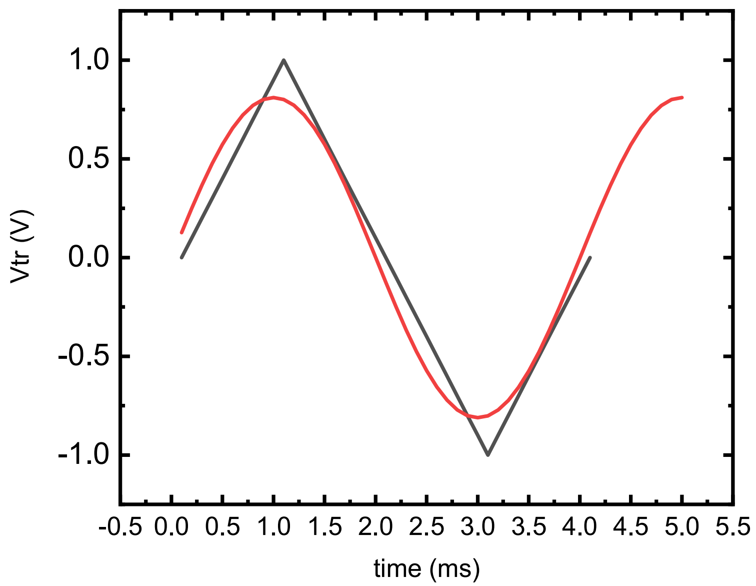


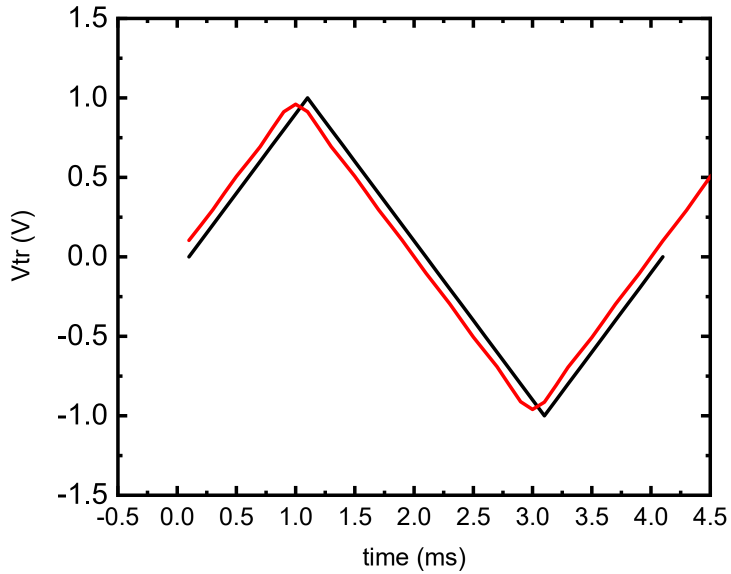
Any periodic signal can be expressed as sum of sinusoids





$$f(t) = \frac{4}{\pi} \sum_{1,3,5}^{\infty} \frac{1}{n} \sin(n \frac{2\pi t}{T})$$





Sum of 5 sinusoids

Sinusoids have interesting mathematical property that their derivatives and integrals are sinusoids too

$$\frac{d\,(Sin\ x)}{dx} = Cos\ x = \ Sin(90 - x)$$

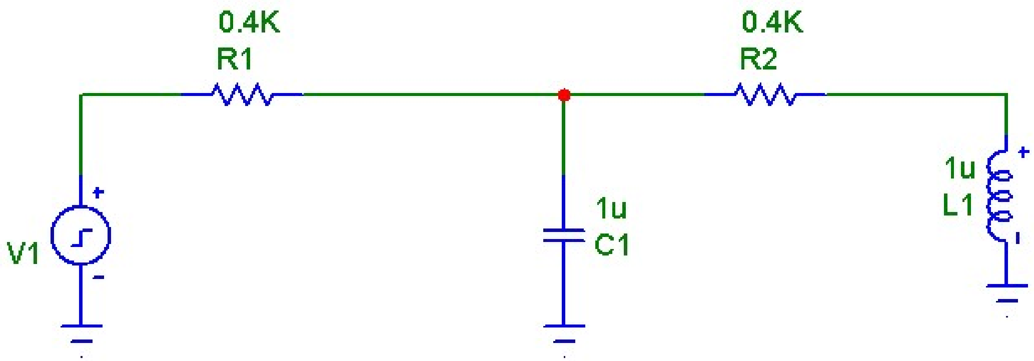
$$i_c = C\,\frac{dv_c}{dt}$$

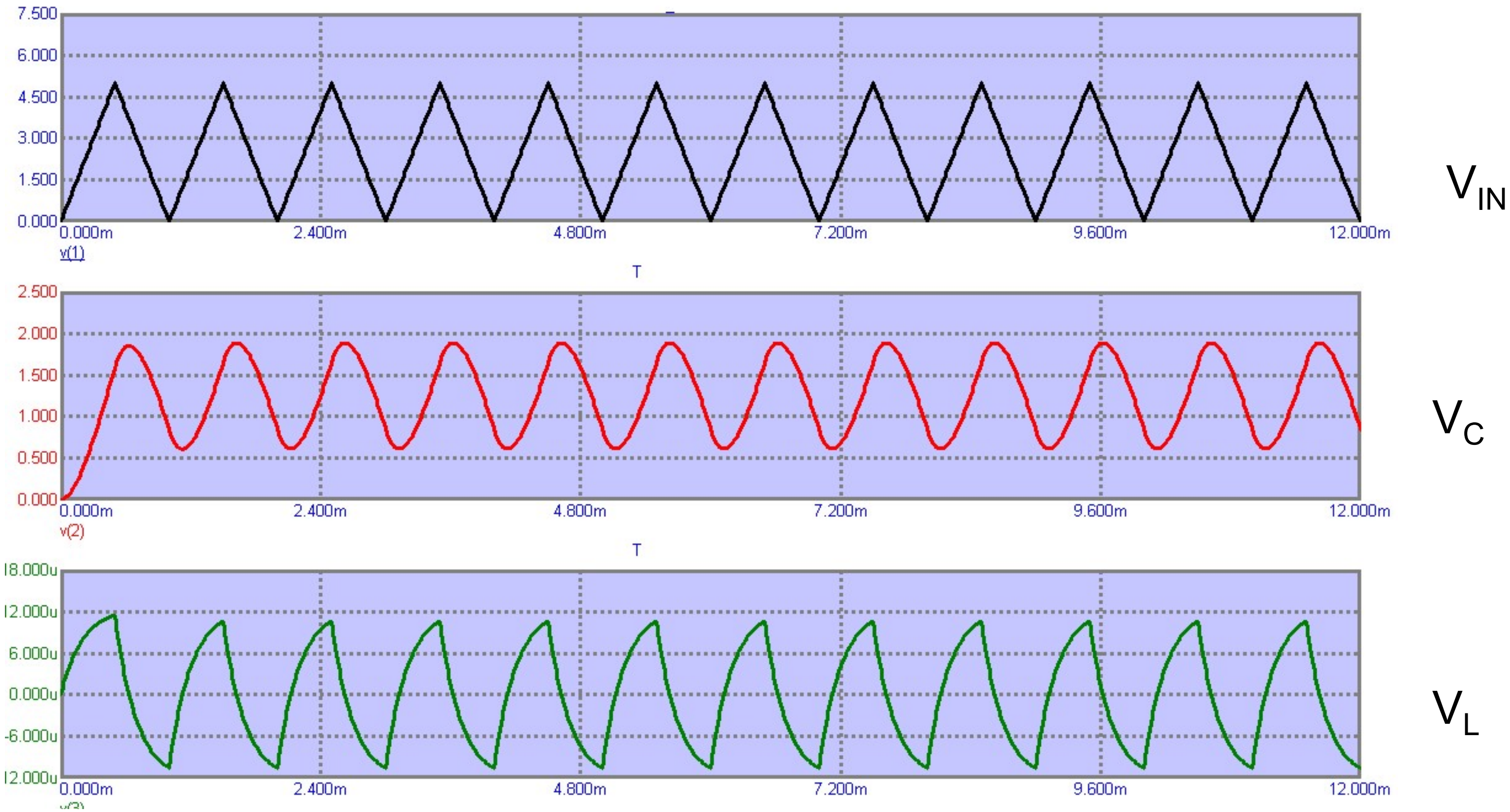
$$\int Sin\ x\, dx = -\,Cos\ x = \ Sin(x - 90)$$

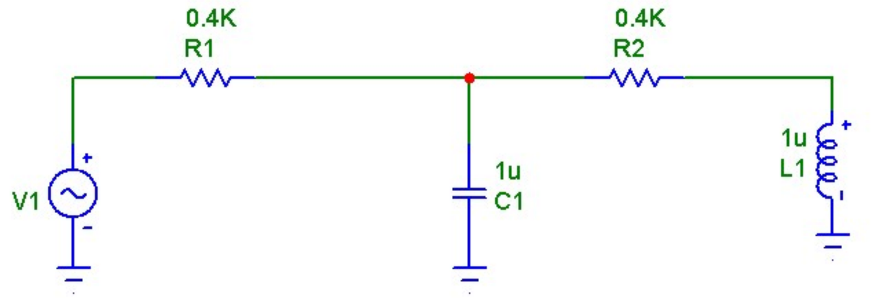
$$v = L\,\frac{di}{dt}$$

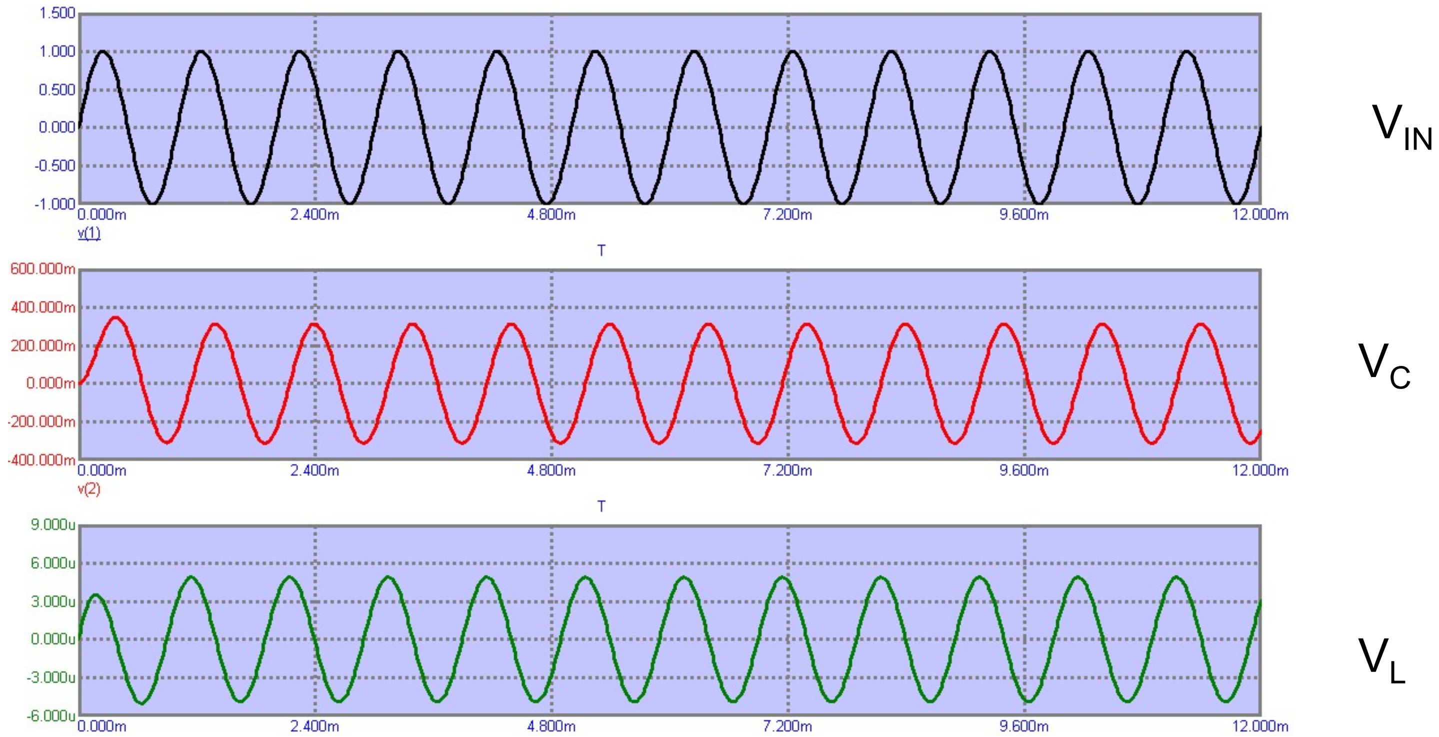
So as a sinusoidal signal goes through a linear circuit, it remains a sinusoid

This makes analysis easier


0.4K
R1
0.4K
R2
V1
1u
C1
1u
L1

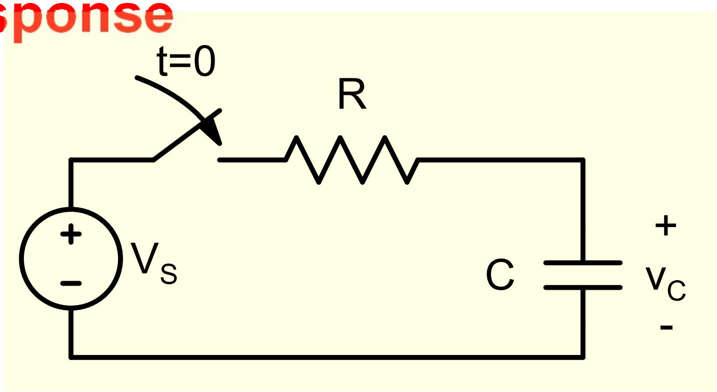

7.500
6.000
4.500
3.000
1.500
0.000
0.000m
2.400m
4.800m
7.200m
9.600m
12.000m
v(1)
T
V_{IN}
2.500
2.000
1.500
1.000
0.500
0.000
0.000m
2.400m
4.800m
7.200m
9.600m
12.000m
v(2)
T
V_C
18.000u
12.000u
6.000u
0.000u
-6.000u
12.000u
0.000m
2.400m
4.800m
7.200m
9.600m
12.000m
V_L





Voltage everywhere in the circuit is sinusoidal

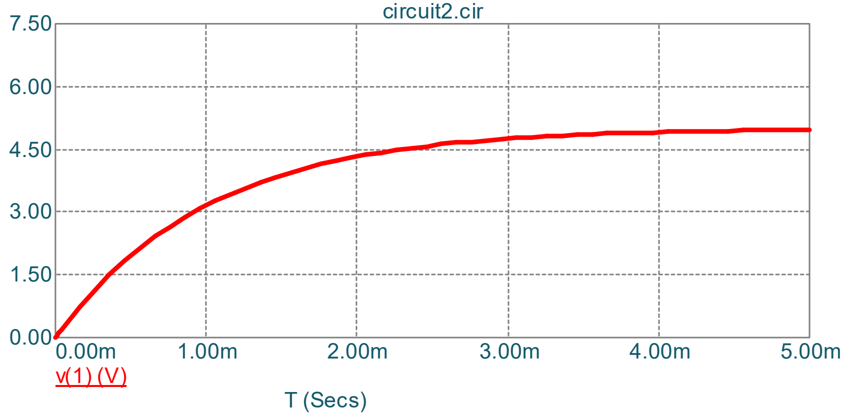
Transient and Forced Response



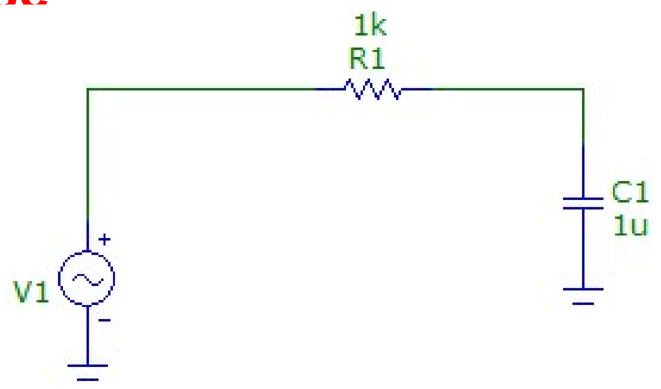
$$v_C(t) = V_S - V_S \times e^{-\frac{t}{RC}}$$

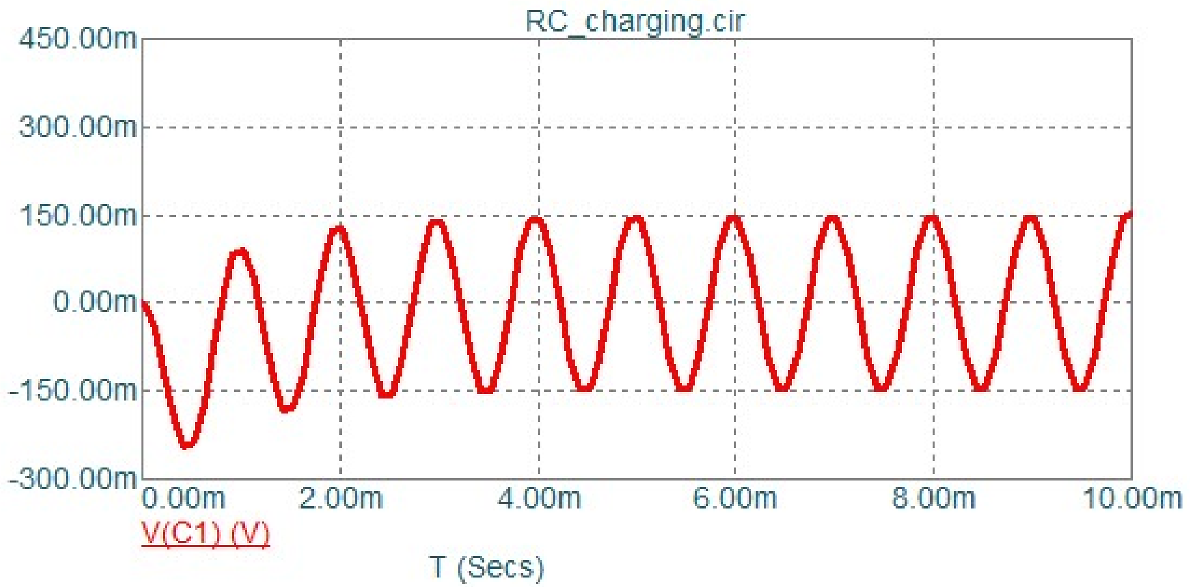
Steady-state/forced response

Transient response

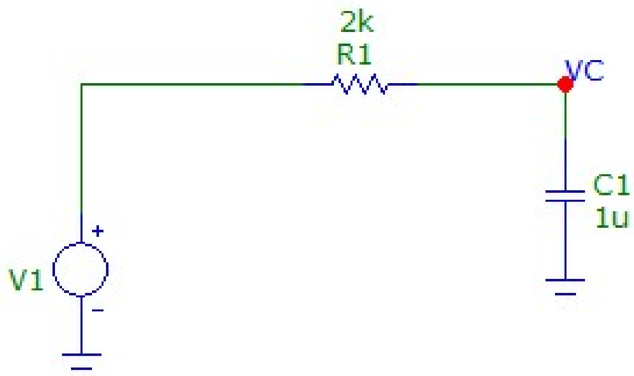


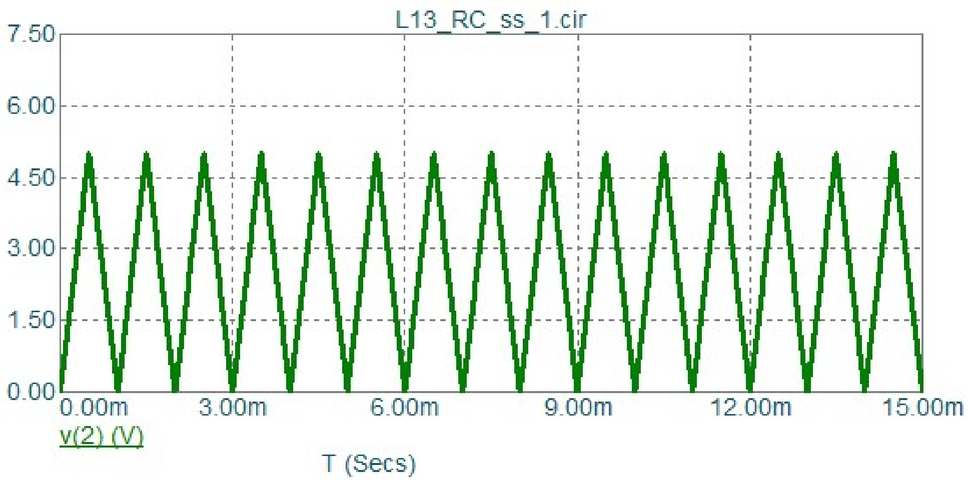
Transient and Forced Response

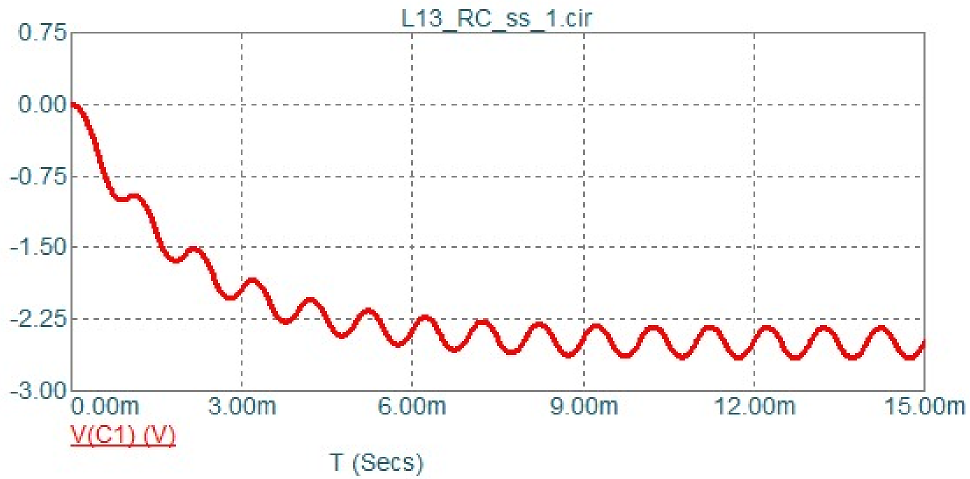




Transient and Forced Response

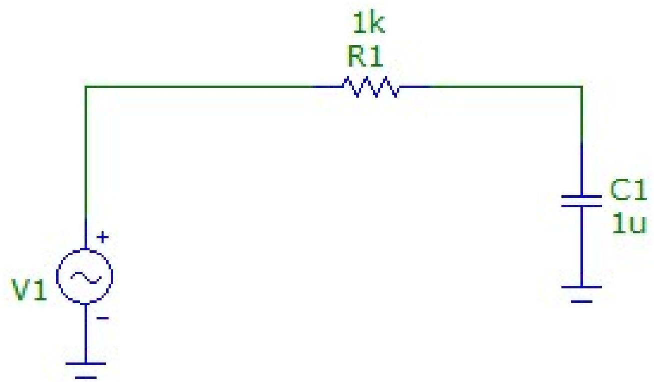


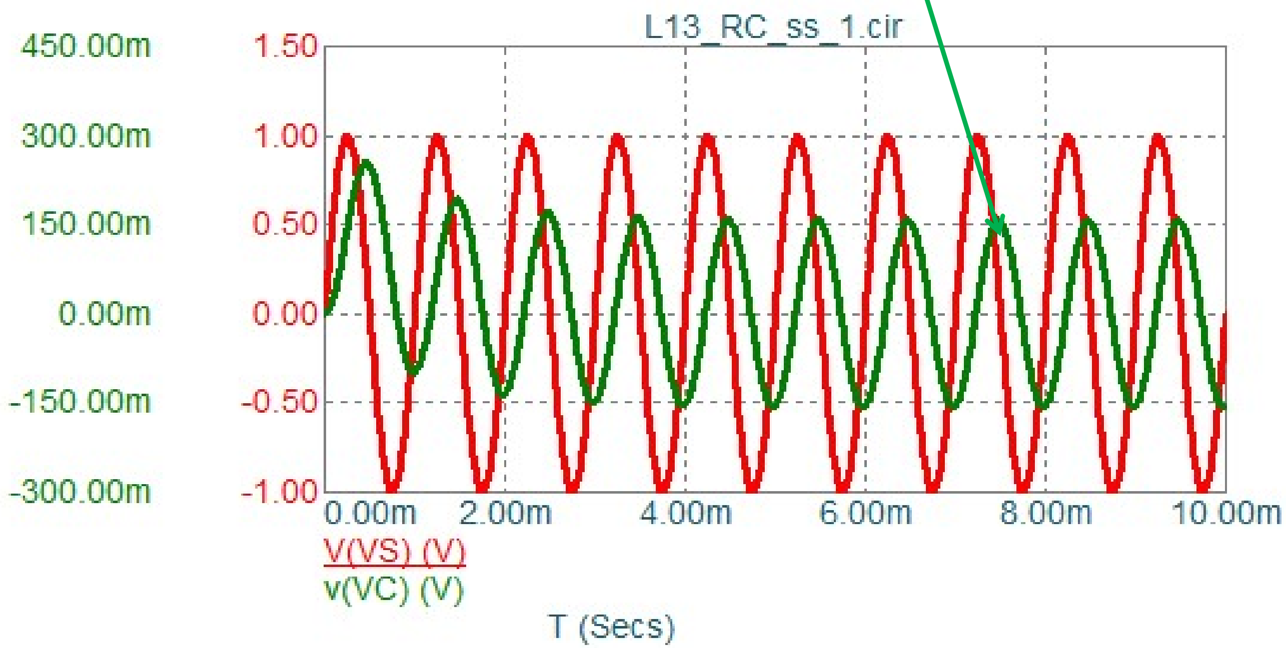


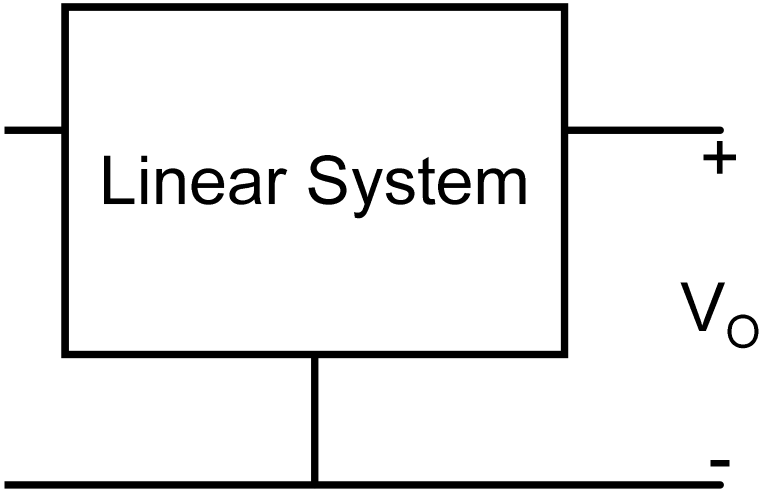


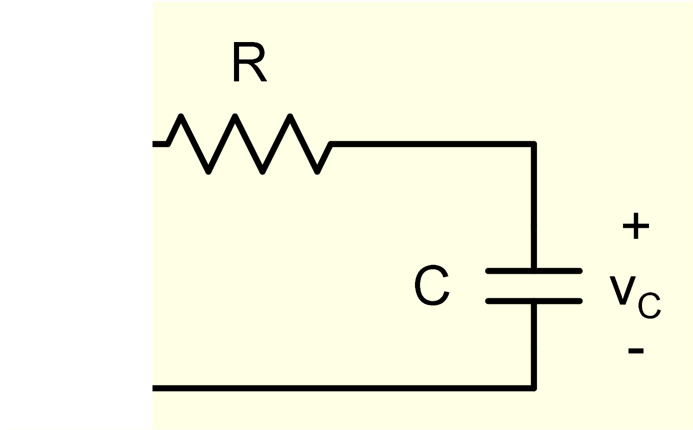
Sinusoidal Steady-State

When the excitation is sinusoidal, the response (voltage or currents) in linear circuit will be sinusoidal as well. If input persists, the response persist and is called steady state response









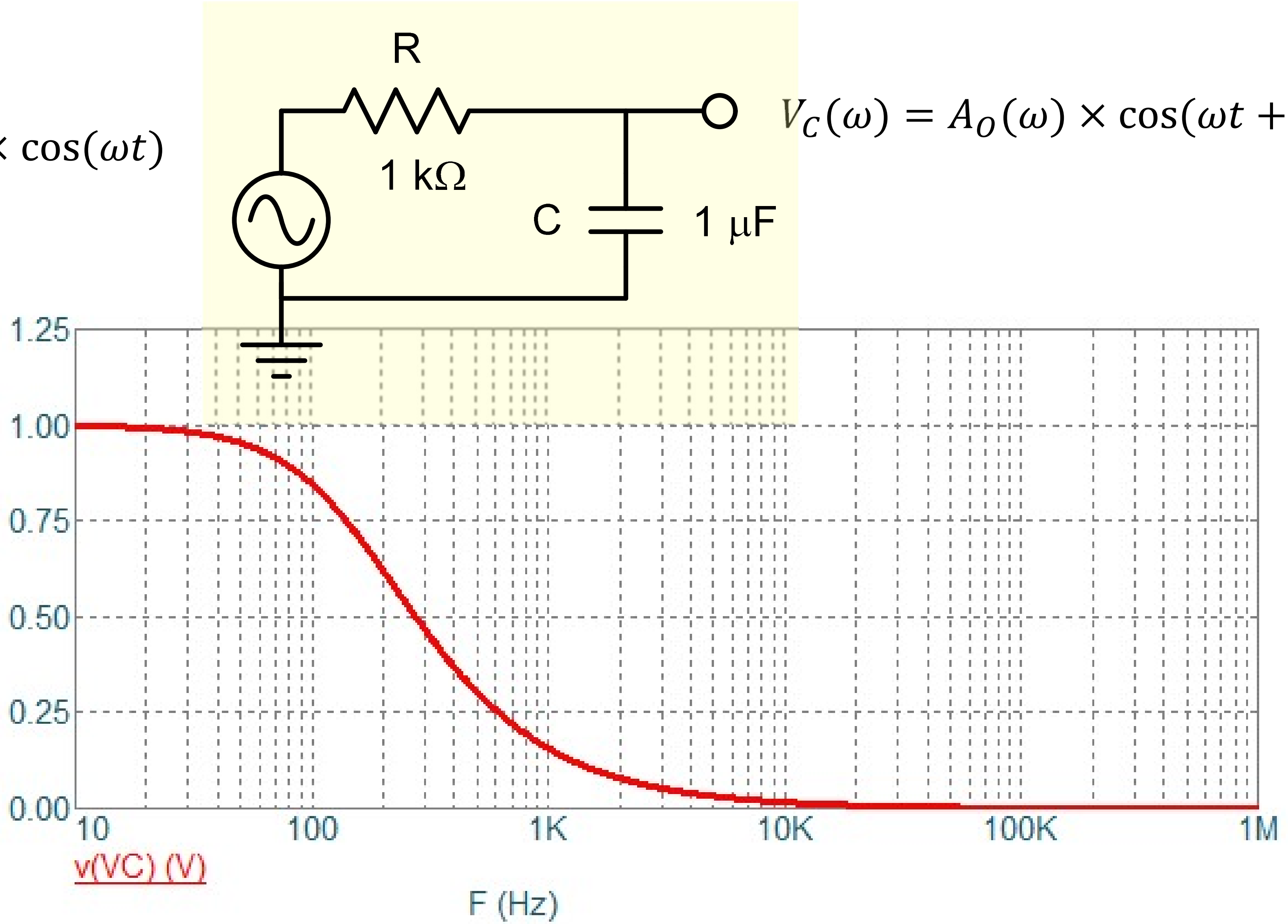
$$V_S = 1 \times \cos(\omega t) \qquad V_O = A_O(\omega) \times \cos(\omega t + \theta(\omega))$$

$A_O(\omega)$ and $\theta(\omega)$ determine the complete characteristics of the system

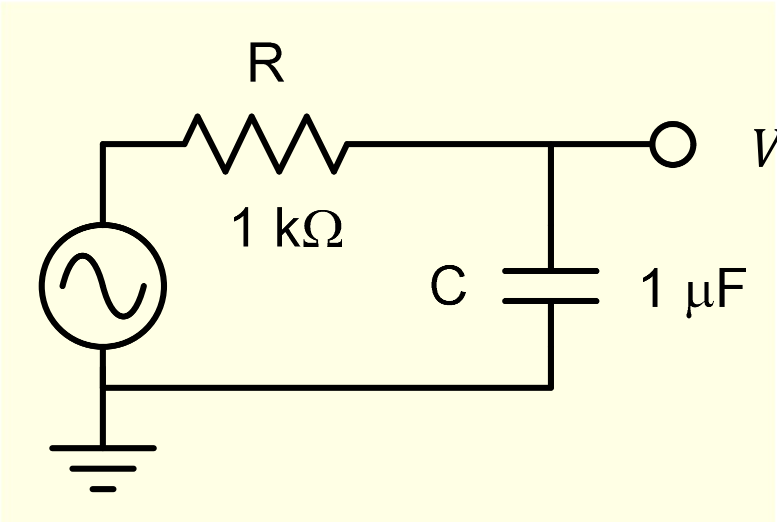
$V_S = 1 \times \cos(\omega t)$

$V_C(\omega) = A_O(\omega) \times \cos(\omega t + \theta(\omega))$

$A_O(\omega$

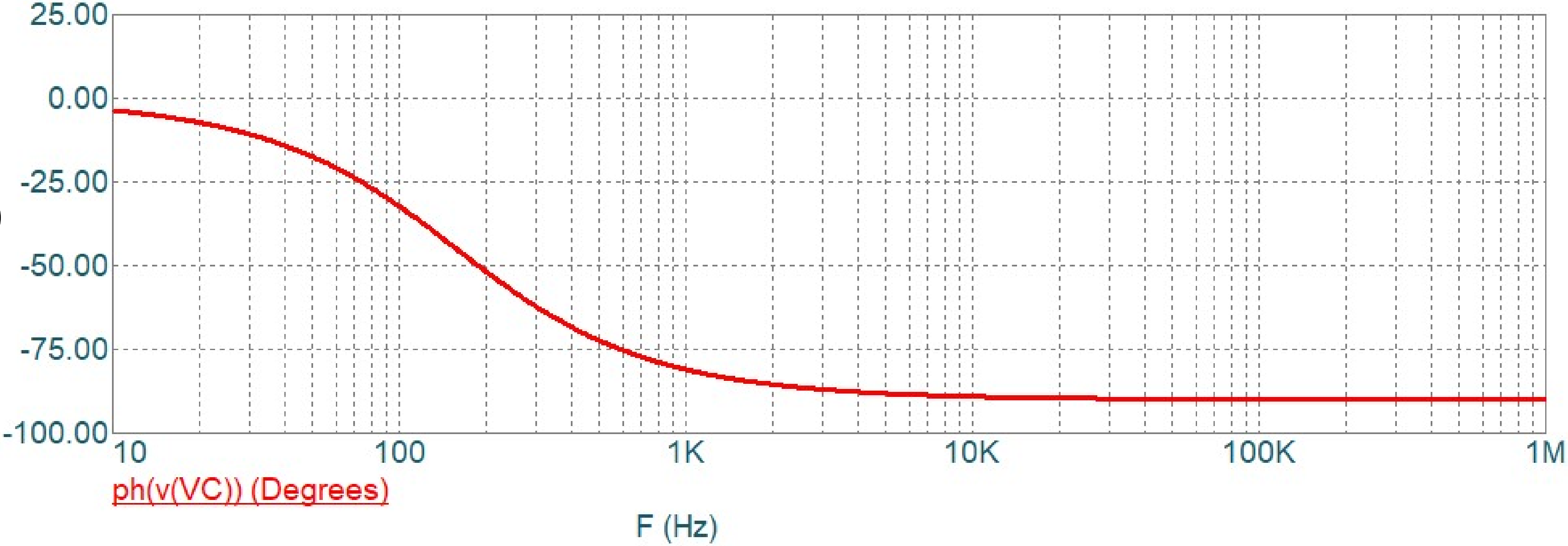


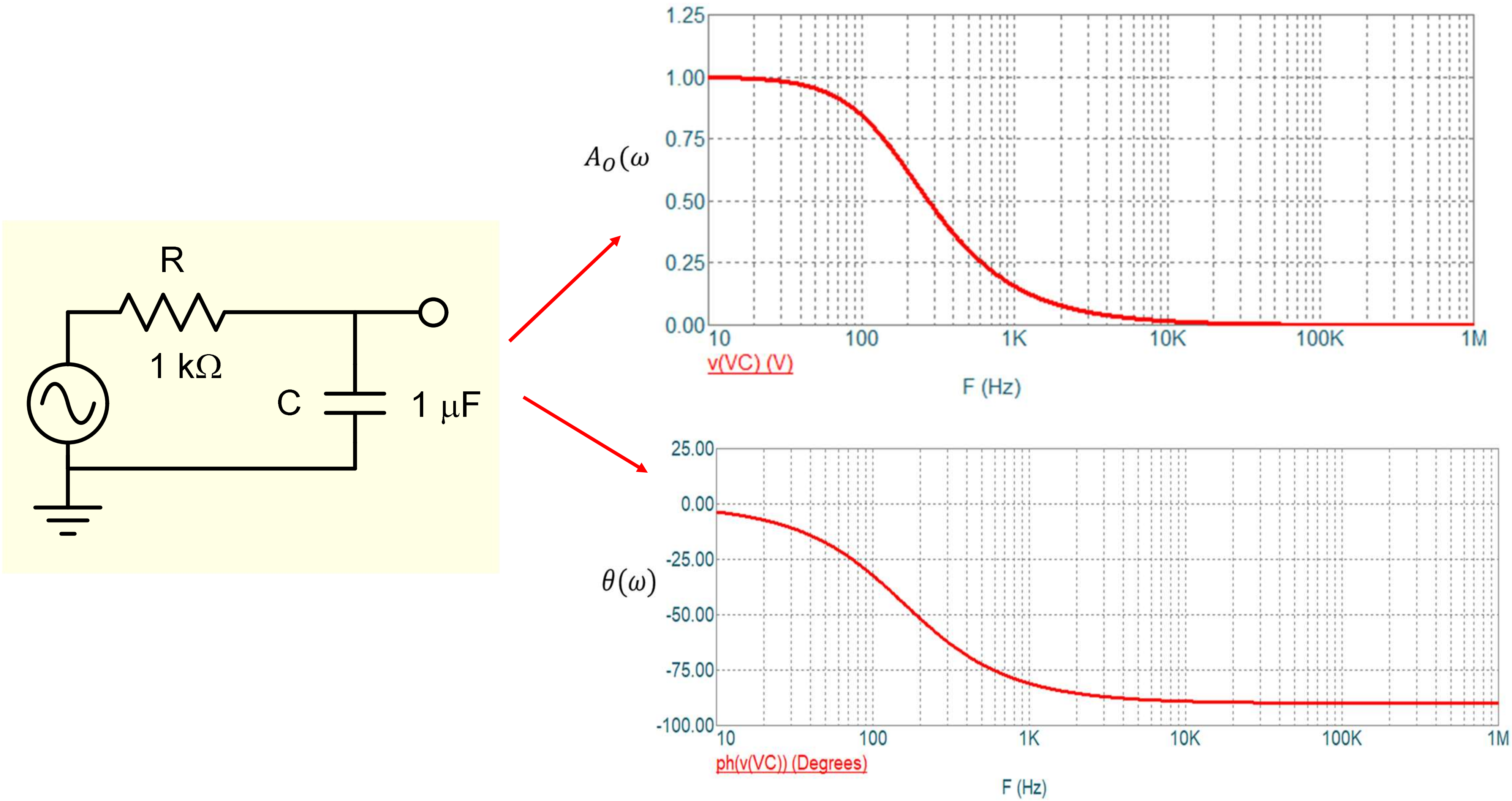
$$V_S = 1 \times \cos(\omega t)$$



$$V_C(\omega) = A_O(\omega) \times \cos(\omega t + \theta(\omega))$$

$\theta(\omega)$



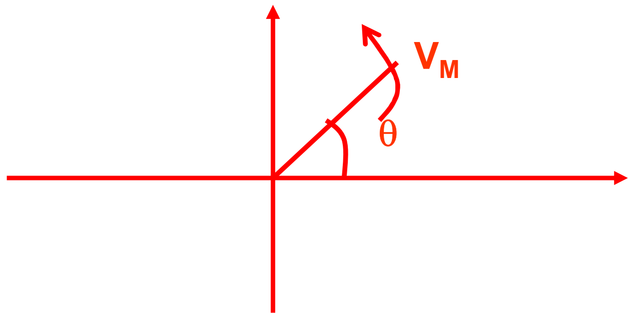

R
1 kΩ
C
1 μF
1.25
1.00
0.75
0.50
0.25
0.00
$A_O(\omega$
10
100
1K
10K
100K
1M
v(VC) (V)
F (Hz)
25.00
0.00
-25.00
-50.00
-75.00
-100.00
$\theta(\omega)$
10
100
1K
10K
100K
1M
ph(v(VC)) (Degrees)
F (Hz)

Sinusoidal Voltages and Currents

$$v(t) = V_M \times \cos(\omega t + \theta)$$

V_m is the peak value

ω is the angular frequency in radians per second



T is the period, where $f = \frac{1}{T}$ is the frequency

$\omega = \frac{2\pi}{T}, \omega = 2\pi f; \theta$ is the phase angle

Example-1

$$5\sin(4\pi t - 60^o)$$

What is the amplitude, phase, angular frequency, time period, frequency?

$$v(t) = V_m \cos(\omega t + \theta)$$

$$\sin(z) = \cos(z - 90^\circ)$$

$$v(t) = 5\cos(4\pi t - 60^o - 90^o)$$

Amplitude = 5 ; Phase = -150°

Phase in radians: 360° = 2 π

$$\theta = \frac{-150}{360} \times 2\pi = -2.618\ Radians$$

$$\omega = 4\pi \quad r/s$$

$$\omega = \frac{2\pi}{T} = 4\pi \Rightarrow T = 0.5s$$

$$f = \frac{1}{T} = 2Hz$$

Example-2

Find the phase difference between the two currents

$$i_1 = 4\sin(377t + 25^o) \qquad i_2 = -5\cos(377t - 40^o)$$

$$x(t) = x_m \cos(\omega t + \theta)$$

$$i_1 = 4\cos(377t + 25^o - 90^o) \qquad \theta_1 = -65^o$$

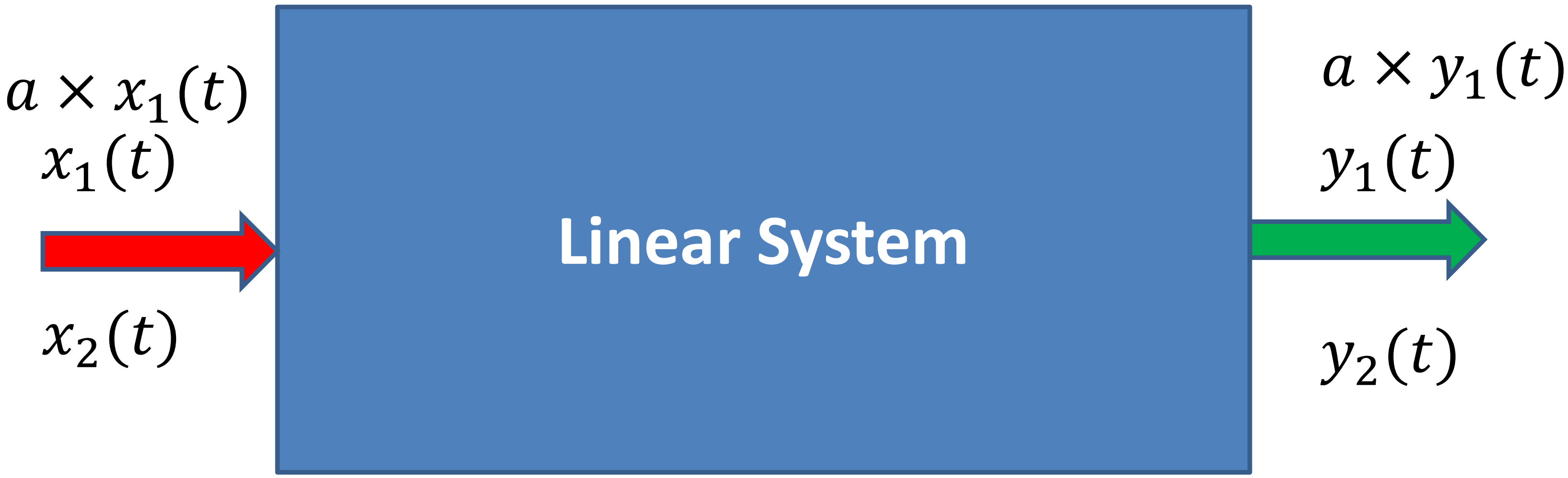
$$i_2 = 5\cos(377t - 40^o + 180^o) \qquad \theta_2 = 140^o$$

$$\sin(\omega t \pm 180^o) = -\sin \omega t$$
$$\cos(\omega t \pm 180^o) = -\cos \omega t$$
$$\sin(\omega t \pm 90^o) = \pm\cos \omega t$$
$$\cos(\omega t \pm 90^o) = \mp\sin \omega t$$

$$\theta_1 - \theta_2 = -205^o$$

$$\theta_1 - \theta_2 = -205 + 180 = -25°$$

Linear Systems



$$x(t) = a_1\, x_1(t) + a_2 x_2(t) \qquad\qquad y(t) = a_1\, y_1(t) + a_2 y_2(t)$$

$$x_1(t) = A\cos(\omega t + \theta)$$

$$x_2(t) = A\sin(\omega t + \theta)$$

$$y_1(t)$$

$$y_2(t)$$

$$x(t) = x_1(t) + jx_2(t)$$

$$y(t) = y_1(t) + jy_2(t)$$

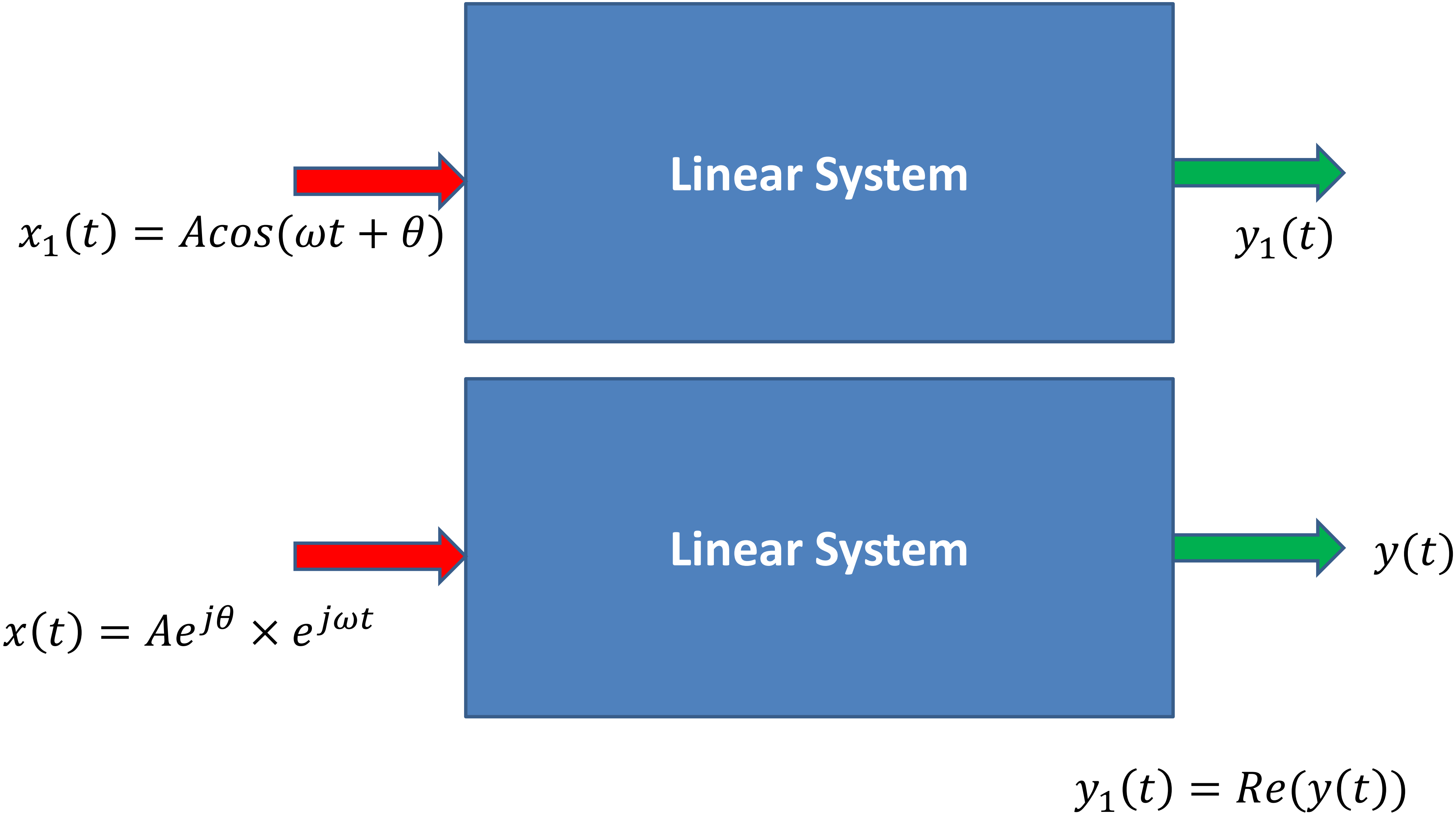
$$j = \sqrt{-1}$$

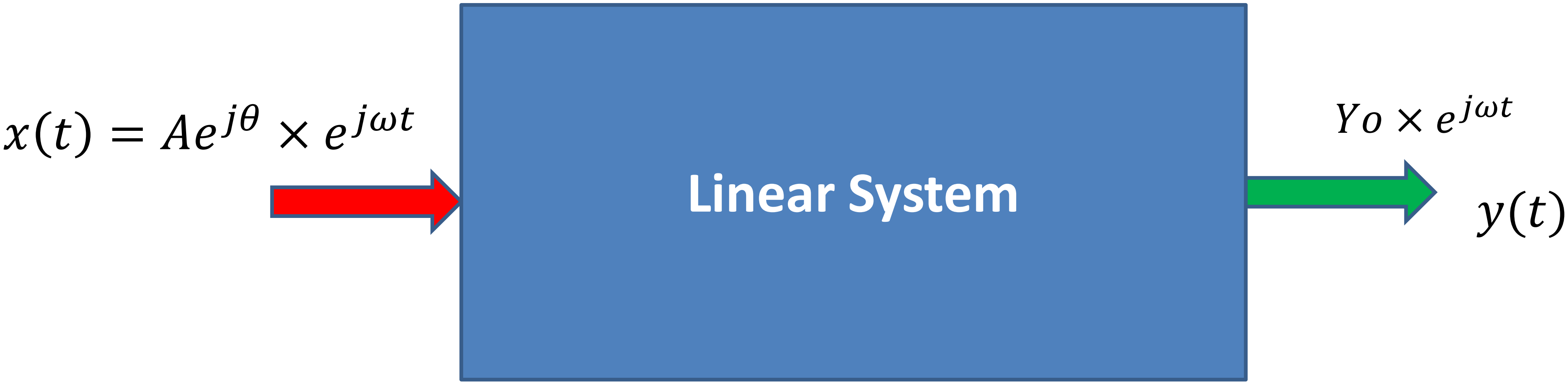
$$x(t) = Ae^{j\theta} \times e^{j\omega t}$$

$$y_1(t) = Re(y(t))$$

$$y_2(t) = Im(y(t))$$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$


Linear System
$x_1(t) = Acos(\omega t + \theta)$
$y_1(t)$
Linear System
$x(t) = Ae^{j\theta} \times e^{j\omega t}$
$y(t)$
$y_1(t) = Re(y(t))$



$$a_n \frac{d^n y}{\mathrm{dt^n}} + a_{n-1} \frac{d^{n-1} y}{\mathrm{dt^{n-1}}} + \cdots + a_o y = Ae^{j\theta} \times e^{j\omega t}$$

$$\text{Solution : } \mathrm{y(t)} = Yo \times e^{j\omega t}$$

$$a_n (\mathrm{j}\omega)^n Y_O + a_{n-1} (\mathrm{j}\omega)^{n-1} Y_O + \cdots + a_o Y_O = A \times e^{j\theta}$$

Solve an algebraic equation in complex variables

$A\angle\theta$

Different Representations of a complex Number

Rectangular form: $Z = 4 + j3$

Polar form: $Z = \sqrt{4^2 + 3^2} \angle \tan^{-1}\left(\frac{3}{4}\right) = 5\angle 36.87°$

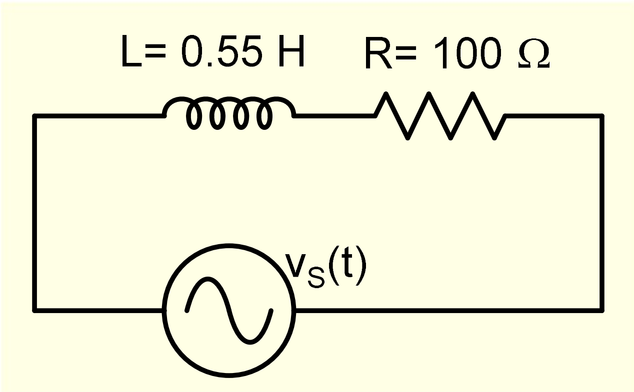
Exponential form: $Z = 5 \times e^{j36.87}$

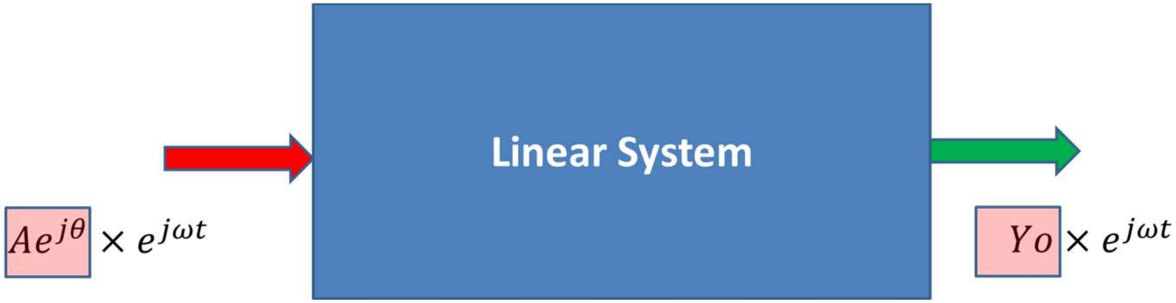
$Z1 = 4\angle 30°$ $\qquad$ $Z2 = 6\angle 60°$

$Z2 \times Z1 = 6 \times 4 \angle 60° + 30° = 24\angle 90°$

$$\frac{Z2}{Z1} = \frac{6}{4} \angle 60° - 30° = 1.5\angle 30°$$

Example-1





$$5 \times e^{j(314.16t+30°)}$$

$$v_S(t) = 5 \times \cos(314.16t + 30°)$$

$$V_S = 5 \times e^{j30} \qquad V_R = 2.5 \times e^{-j3}$$

$$V_L = V_S - V_R$$

$$v_R(t) = 2.5 \times \cos(314.16t - 30°)$$

$$V_S - V_R = 2.165 + j3.75 = 4.33 \times e^{j60}$$

$$v_L(t) = Re(4.33 \times e^{j60} \times e^{j\omega t})$$

$$v_L(t) = v_S(t) - v_R(t)$$

$$v_L(t) = 4.33 \times \cos(314.16t + 60°)$$

$$v_S(t) = 5 \times \cos(314.16t + 30°)$$
$$v_R(t) = 2.5 \times \cos(314.16t - 30°)$$

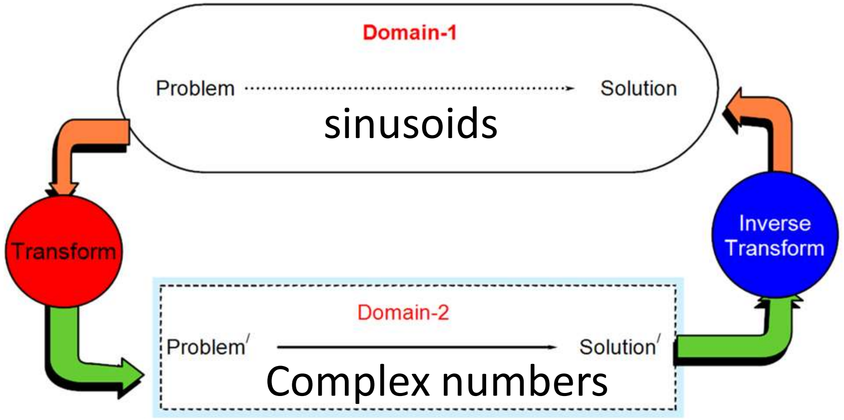
$$v_S(t) - v_R(t) = ?$$

Strategy

$$V_S = 2.16 - j1.25$$
$$V_R = 4.33 + j2.5$$

$$V_S - V_R = 4.33 + j2.5 - 2.16 + j1.25$$
$$= 2.165 + j3.75$$

Sinusoidal variables → Complex variables → Perform Algebra on Complex variables → Transform complex variable Back to sinusoid



$$v_L(t) = Re\left(4.33 \times e^{j60} \times e^{j\omega t}\right)$$
$$v_L(t) = 4.33 \times \cos(314.16t + 60°)$$

$$v(t) = V_m \cos(\omega t + \theta)$$

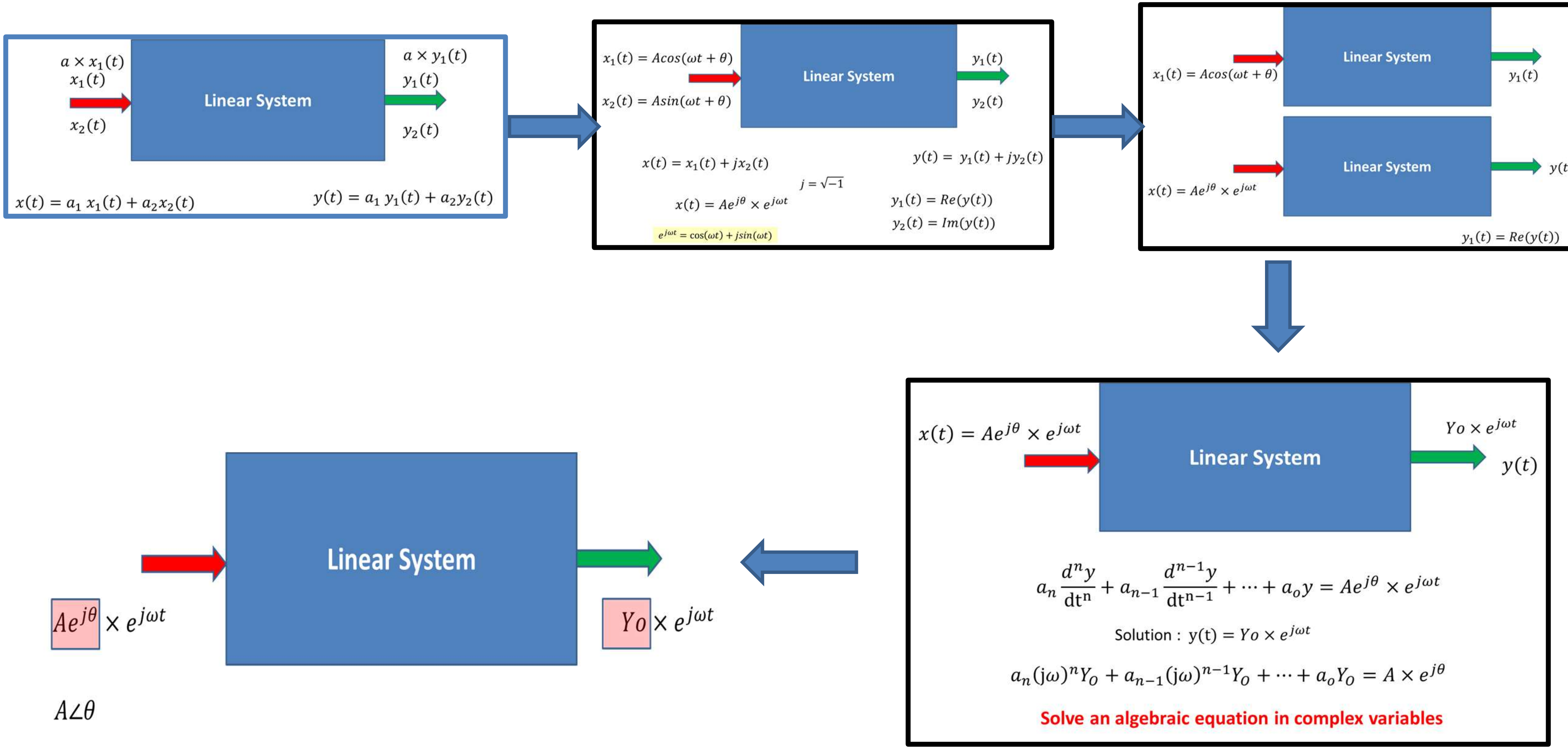
$$v(t) = \mathrm{Re}(V_m \times e^{j(\omega t + \theta)}) \Rightarrow \mathrm{Re}(V_m \angle \omega t + \theta)$$

$$v(t) = V_m \cos(\omega t + \theta) \Leftrightarrow V_m \angle \theta$$

Phasor

$$V_M \angle \theta \rightarrow V_m \times e^{j\theta}$$

Phasor Analysis



$$v(t) = 3\ \cos(\omega t + 45) \Longrightarrow 3\ \angle 45 \Rightarrow 3\ \cos(45) + j3\sin(45)$$

$$v(t) = 5\ \cos(\omega t - 60) \Longleftarrow 5\ \angle -60$$