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ISOLATED SINGULARITIES

Poles and residue.

Types of isolated singularities

When f has an isolated singularity at "a", Laurent theorem says that, in an annulus, $0 < |z-a| < r$ f has a Laurent expansion, $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$.

The sum $\sum_{n=-\infty}^{-1} c_n (z-a)^n$ is called the principal part of the Laurent expansion.

We classify singularities depending on C_{-n}, $n \in \mathbb{N} = \{1, 2, \cdots\}$

If (i) $C_{-n} = 0$ $\forall n \in \mathbb{N}$ then the singularity is said to "removable".

(ii) $C_{-m} = 0$ $\forall m > n$ then the singularity is called a "pole"; If $c_{-n} \neq 0$ then the pole is said to be of order n.

(iii) if none of the above happens, that is if there are infinitely many non-zero terms in the principal part then the singularity is said to be an "isolated essential singularity".

If "a" is a removable singularity, then $f(z)$ is given by the series $\sum_{n=0}^{\infty} c_n (z-a)^n$ in the annulus $0 < |z-a| < r$. So, if we define $f(a) = c_0$, then $f(z)$ agrees with $\sum_{n=0}^{\infty} c_n (z-a)^n$ $\forall z \ni |z-a| < r$ and so is analytic.

So a removable singularity is not a singularity in reality.

Eg: $f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}\right)$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}$$

$z=0$ is a removable singularity, if $f(0) := 1$.

Essential singularity is on the other end of the spectrum. Infact, there are interesting phenomena observed even for fns with essential singularities.

However, the kind of 'singularity that we will study mostly is "poles".

§: Characterization of a pole of order m.

Let f have a pole of order m at 'a'.

Then $\lim\limits_{z \to a} (z-a)^n f(a) = 0 \quad \forall\, n > m$

and $\lim\limits_{z \to a} (z-a)^m f(a) \neq 0$. (Pf: easy).

§ Relation between poles and zeroes

Suppose f is holomorphic in an open disc $B_r(a)$. We have seen that its zeros are isolated; infact $f(z) = (z-a)^m \underbrace{\sum_{n=m+1}^{\infty} a_n \cdot (z-a)^n}_{g(z) \,\ni\, g(a) \neq 0}$

Theorem: With notations as above
f has a zero of order m at 'a' if and only if $1/f$ has a pole of order m at 'a'.

Pf: $f(z) = (z-a)^m g(z)$, g holo on $B_r(a)$ and $g(a) \neq 0$

$\therefore g(z) \neq 0$ in a small disk around a.

$$\Rightarrow \frac{1}{f(z)} = \frac{1/g(z)}{(z-a)^m}.$$

By above characterization, 'a' is a pole of order m for $1/f$ if $\underset{z\to a}{\text{Lt}}\,(z-a)^n \dfrac{1}{f(z)} = 0 \;\forall\, n > m$

and $\underset{z\to a}{\text{Lt}}\,(z-a)^m \dfrac{1}{f(z)} \neq 0$

$$\underset{z\to a}{\text{Lt}}\,(z-a)^{n-m} \cdot 1/g(z) = 0 \qquad (\because n-m>0)$$

$\qquad\qquad \& \; 1/g(a) \neq 0$

$$\underset{z\to a}{\text{Lt}}\,(z-a)^m \cdot \frac{(1/g(z))}{(z-a)^m} = \frac{1}{g(a)} \neq 0$$

Conversely, $\dfrac{1}{f(z)} = \dfrac{C_{-m}}{(z-a)^m} + \dfrac{C_{-m+1}}{(z-a)^{m-1}} + \cdots$

$$= \frac{1}{(z-a)^m}\left[C_{-m} + C_{-m+1}(z-a) + \cdots\right]$$

$$= \frac{1}{(z-a)^m}\Big[\underbrace{\sum_{n=-m}^{\infty} C_n (z-a)^{n+m}}_{\text{analytic} = h(z)\ \&\ h(a)\neq 0}\Big]$$

$$\therefore f(z) = (z-a)^m \cdot \frac{1}{h(z)}$$

$\Rightarrow f$ has a zero of order m.

§ Counting order of a pole

Let f have a pole of order m at 'a'.

(a) Let g be holomorphic at 'a'. Then

(i) if g has a zero of order 'n' at 'a',

fg has a pole of order $m-n$ (if $m>n$)

fg has a zero of order $n-m$ (if $n>m$)

fg has a removable singularity (if $n=m$)

(ii) g has a pole of order 'n'. at 'a'.

Then fg has a pole of order $n+m$ at a

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RESIDUE

Let f be holomorphic in a punctured disk around 'a'. Let 'a' be a pole of order 'm'.

Then $$\int_\gamma f(z)\,dz = 2\pi i\, C_{-1}$$

where γ is a positively oriented contour.

$$\left(\text{Pf: } C_{-1} := \frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega^{-1+1}}\,.d\omega\right)$$

So "c_{-1}" has a very special status in contour integration. So much so that it is given a special name "residue at a".

Defn: Residue of f at a is coeff of $\frac{1}{(z-a)}$ in the Laurent series expansion of f "around a". Denote it as $\mathrm{Res}(f;a)$

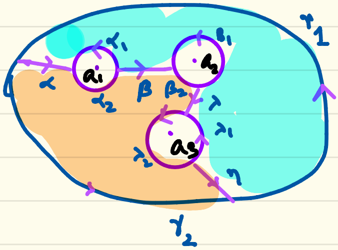
CAUCHY RESIDUE THEOREM

Let f be holomorphic inside and on a positively oriented contour γ except for finitely many isolated singularities $a_1, \dots, a_N$

Then,
$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \mathrm{Res}(f, a_j).$$

Pf: f has a Laurent expansion about $a_1, \dots, a_N$

$$\int_{\gamma_1+\gamma_2} = \int_{\substack{C_{r_1}(a_1) \\ = \\ \alpha_1+\alpha_2}} + \int_{\substack{C_{r_2}(a_2) \\ \beta_1+\beta_2}} + \int_{\substack{C_{r_3}(a_3) \\ \lambda_1+\lambda_2}} + \cdots$$



$$= 2\pi i \, \mathrm{Res}(f;a_1) + 2\pi i \, \mathrm{Res}(f;a_2) + \cdots + 2\pi i \, \mathrm{Res}(f;a_k)$$

$$\Rightarrow \int_{\gamma} f(z)\,dz = 2\pi i \sum_{i=1}^{N} \mathrm{Res}(f;a_k). \quad \boxtimes$$

<u>Strategy to calculate residues</u> (when 'a' is a simple pole)

① $\mathrm{Res}(f(z);a) = \lim_{z\to a} (z-a) f(z).$

if 'a' is a simple pole (ie order = 1).

② If $f(z) = \dfrac{g(z)}{z-a}$ (where g is analytic at a) then $\mathrm{Res}(f(z);a) = g(a).$

③ If $f = \dfrac{h}{k}$, $h(a) \neq 0$, $k(a)=0$, $k'(a) \neq 0$

Then $\lim_{z\to a} (z-a)\dfrac{h(z)}{k(z)} = \lim_{z\to a} h(z)\cdot\dfrac{z-a}{k(z)-k(a)}$

$$= \frac{h(a)}{k'(a)}.$$

§ When 'a' is a multiple pole (ie order >1)

(a) $f(z) = \dfrac{g(z)}{(z-a)^m}$ & $g(a) \neq 0$

Then $\dfrac{g^{(m-1)}(a)}{(m-1)!} = \operatorname{Res}(f; a)$

$$\left(\because \; g^{(m-1)}(a) = \frac{(m-1)!}{2\pi i} \int_{C_{r/2}(a)} \frac{g(z)}{(z-a)^m}\, dz \right.$$

$$= \frac{(m-1)!}{2\pi i} \int_{C_{r/2}(a)} f(z)\, dz \;= \frac{(m-1)!}{2\pi i}\, C_{-1} \cdot 2\pi i$$

$$\left. = (m-1)!\, \operatorname{Res}(f; a) \right)$$

REMARK: In calculating residue of $\dfrac{h(z)}{k(z)}$ it is good practice to put all factors of $k(z)$ which do not contribute to a zero at 'a' into numerator. ☐

4 Using Laurent's series expansion.

Eg: $e^{iz} z^{-4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \quad \forall\ 0 < |z|$

$$= \frac{1}{z^4} + \frac{i}{z^3} - \frac{1}{2!\,z^2} - \frac{i}{3!\,z} + \frac{1}{4!} + \cdots$$

$$\therefore \ \mathrm{Res}\left(e^{iz} z^{-4}; 0\right) = \frac{-i}{3!}$$