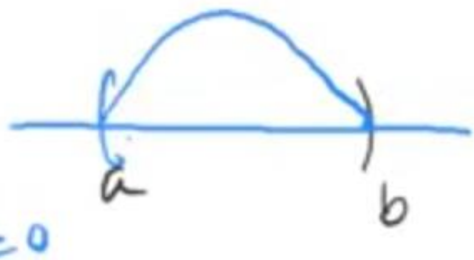
=:: Strum comparison theorem ::=

Theorem Let I be an open interval in $\mathbb{R}$ and $q_1, q_2 \in C(I)$ with the property that $q_2 \geq q_1$ on I. but $q_1 \not\equiv q_2$.. Let u_i be a non-trivial solution of the problem.

$$\rightarrow u_i'' + q_i u_i = 0 \quad \text{on } I, \quad i = 1, 2$$

and suppose $a < b$ are two distinct zeros of "u_1", then "u_2" has a zero in (a, b).

Proof



$u_1(a) = u_1(b) = 0$

$u_1 > 0$ on (a,b)

$\Rightarrow \underline{u_1'(a) > 0}$, $\underline{u'(b) \leq 0} \rightarrow$ (A)

claim 'u_2' vanishes on (a,b)

$u_2 > 0$ on (a,b)

$u_1'' u_2 - u_2'' u_1 = (q_2 - q_1) u_1 u_2$

$\Leftrightarrow \int_a^b (u_1' u_2 - u_2' u_1)' = \int_a^b (q_2 - q_1) u_1 u_2$

$\Rightarrow u_1'(b) u_2(b) - u_1'(a) u_2(a) = \int_a^b (q_2 - q_1) u_1 u_2$

Laplace Equation 1.

Laplace operator, $\Omega \subseteq \mathbb{R}^n$ open.

$$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x).$$

$\Delta u = 0$ in Ω.

Poisson Equation

$-\Delta u = f$

given fn

Harmonic functions

A $C^2(\Omega)$ is said to be Harmonic if $\Delta u = 0$ is satisfied in Ω.

Examples :-

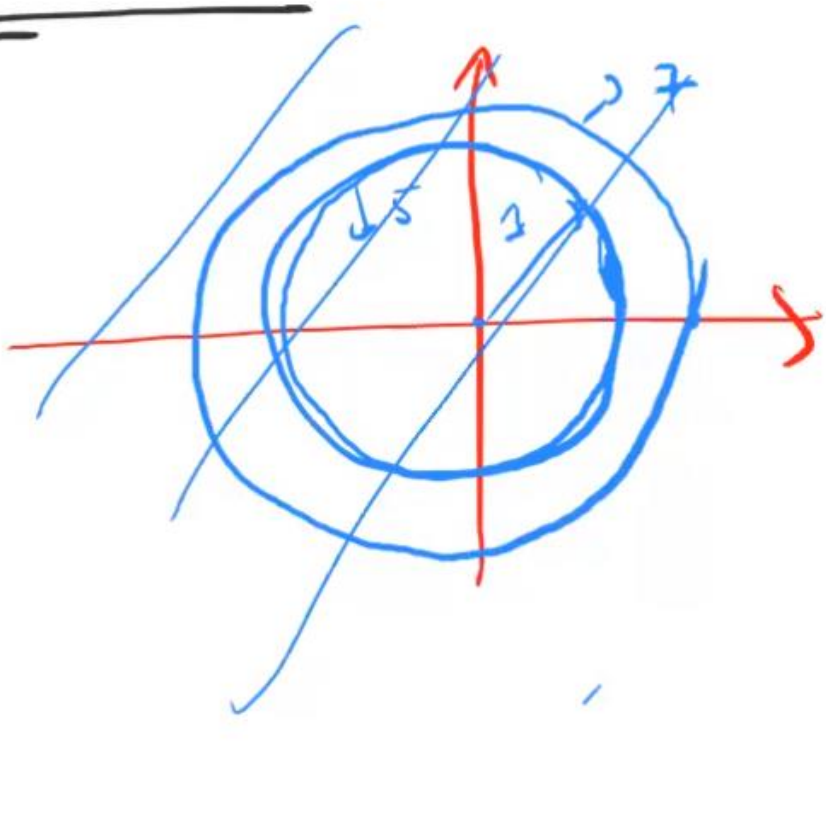
1. $u(x,y) = 5$
2. $u(x,y) = ax + by$
3. $u(x,y) = x^2 - y^2$

$a, b \in \mathbb{R}$

$\Delta u = 0$

"Radial" Harmonic fn

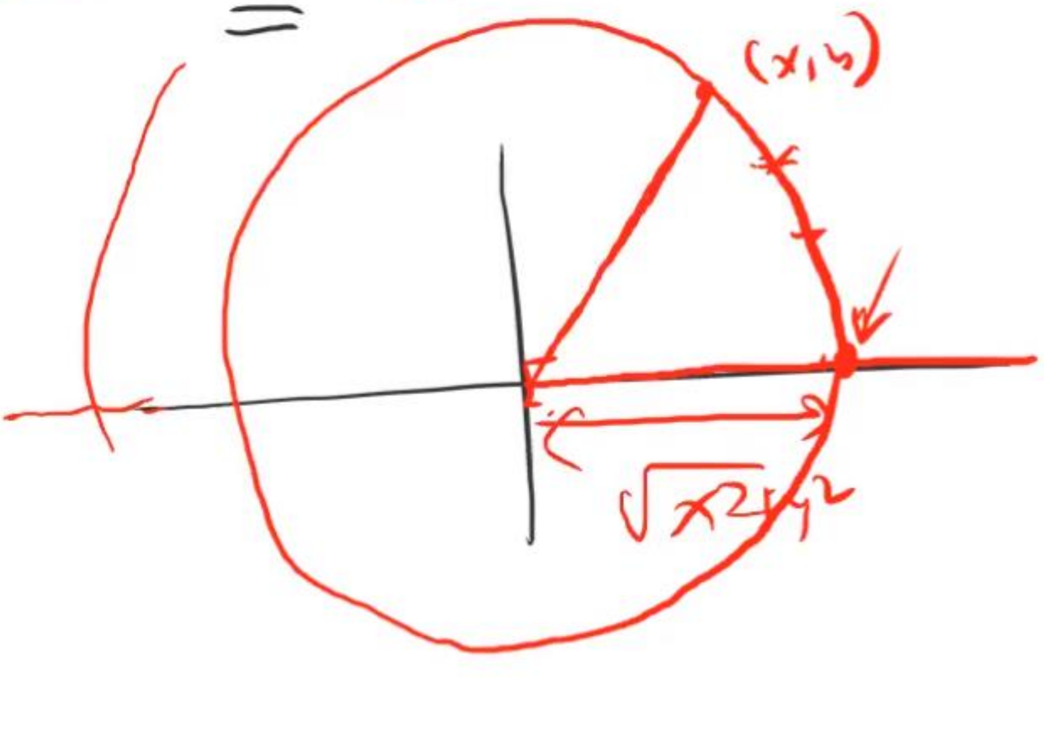
$u(x,y) = 5$



Radial f^n.

A function $u : \mathbb{R}^2 \to \mathbb{R}$ is said to be radial if $\exists$ $f : [0, \infty) \to \mathbb{R}$ such that

$$u(x, y) = f\left(\sqrt{x^2 + y^2}\right)$$



Non trivial Harmonic f^n **4.**

$\underline{\Delta u = 0}$ $\qquad u(x,y) = f\left(\sqrt{x^2+y^2}\right)$

$\underline{x^2+y^2=r^2} \longrightarrow \dfrac{\partial r}{\partial x} = x/r$

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)\, x/r$$

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left(\frac{r - x^2/r}{r^2}\right) + \left(x/r\right)^2 f''(r)$$

$$= \frac{1}{r}f'(r) - x^2/r^3 f'(r) + x^2/r^2 f''(r)$$

$$u_{yy} = 1/r\, f'(r) - y^2/r^3 f'(r) + y^2/r^2 f''(r)$$

$$\underline{\Delta u} = 2/r\, f'(r) - 1/r\, f'(r) + f''(r) = 0$$

$$f''(r) + 1/r\, f'(r) = 0.$$

$$\frac{f''(r)}{f'(r)} = -1/r.$$

$$\left(\log(f'(r))\right)' = -1/r.$$

$$\Rightarrow \quad \log(f'(r)) = -\log r + \log c.$$

$$\Rightarrow \quad f'(r) = c/r.$$

$$\Rightarrow \quad \boxed{f(r) = c\log r + c_2}$$

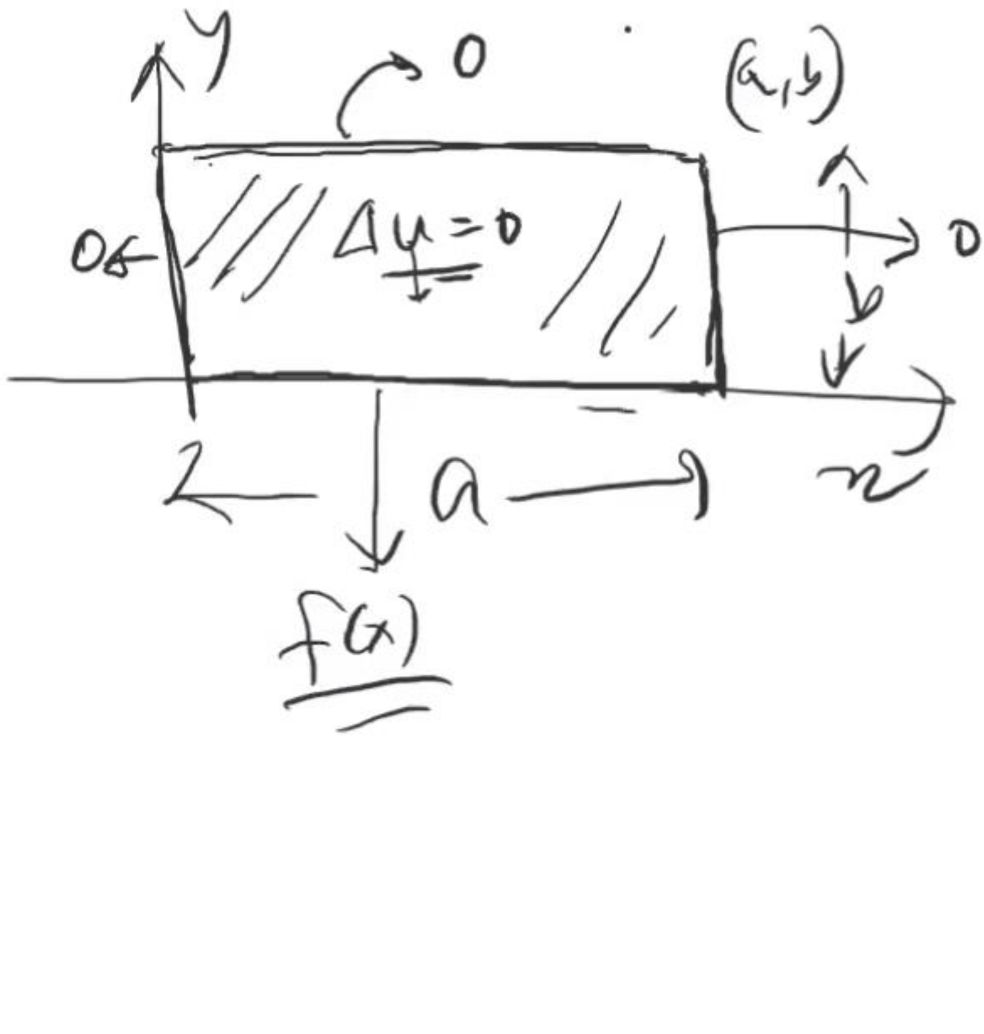
$\Delta u = 0$ on $\mathbb{R}^2 \setminus (0,0)$

$$u(x,y) = f(r).$$
$$= c \log\left(\sqrt{x^2+y^2}\right) + c_2$$

$\{\mathbb{R}^2 \setminus (0,0)\}$ $\quad c_1, c_2 \in \mathbb{R}$.

SOLVING LAPLACE EQ ON RECTANGLES

$$\begin{cases} \Delta u(x,y) = 0 & \text{on } (0,a) \times (0,b). \\ u(x,b) = 0, \quad u(0,y) = 0 \\ u(a,y) = 0, \quad u(x,0) = \underline{\underline{f(x)}} \end{cases}$$



Idea (seperation of variables) 2.

$$\boxed{u(x,y) = F(x)\,G(y)}$$

$\Delta u = 0$

$u_{xx} = F''(x)\,G(y)$

$u_{yy} = G''(y)\,F(x)$

$$F''(x)\,G(y) + G''(y)\,F(x) = 0$$

$$\Leftrightarrow \boxed{\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}, \quad \forall x \in (0,a),\ y \in (0,b)}$$

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda.$$

$$\underbrace{u(0,y)}_{\parallel \atop F(0)\,G(y)} = 0 \qquad G(y) \neq 0$$

$$\Rightarrow \underline{F(0) = 0}.$$

Similarly

$$u(a,y) = 0 \Rightarrow \underline{F(a) = 0}$$

$$\begin{cases} F''(x) + \lambda F(x) = 0 \quad \text{on } (0,a) \\ F(0) = 0 = F(a) \end{cases}$$

$$\underline{\lambda_n} = \left(\frac{n\pi}{a}\right)^2, \quad n \in \mathbb{N}.$$

$$F_n(x) = \sin\left(\frac{n\pi}{a}x\right)$$

$$G_n''(y) = \left(\frac{n\pi}{a}\right)^2 G_n(y)$$ 4.

General Solution for the above $\underline{Eqn}$ is given by:

$$\boxed{G_n(y) = A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}}$$

Eqation in y - Laplace

Law of Superposition

$$u_n(x,y) = \underline{F_n(x)}\ \underline{G_n(y)}$$

$$u(x,y) = \sum_{n=1}^{\infty} F_n(x)\, G_n(y).$$

$$u(x,b) = 0 = \sum_{n=1}^{\infty} F_n(x)\, G_n(b)$$

$$= \sum_{n=1}^{\infty} F_n(x) \Big[\underbrace{A_n e^{n\pi b/a} + B_n e^{-n\pi b/a}}_{=0} \Big]$$

(5)

$$u(x,b) = 0$$

$$\Rightarrow A_n e^{n\pi b/a} + B_n e^{-n\pi b/a} = 0$$

$$\Rightarrow \boxed{\underline{\underline{B_n}} = -A_n \frac{e^{n\pi b/a}}{e^{-n\pi b/a}}}$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2A_n \sin\left(\frac{n\pi x}{a}\right)}{e^{-n\pi b/a}} \left[\underline{\underline{\frac{e^{\frac{n\pi(y-b)}{a}} - e^{-\frac{n\pi(y-b)}{a}}}{2}}} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2A_n \sin\left(\frac{n\pi x}{a}\right)}{e^{-n\pi b/a}} \underline{\sinh} \underline{\frac{n\pi(y-b)}{a}}$$

$$\underline{u(x,0)} = f(x) = \sum_{n=1}^{\infty} \frac{2A_n \sin\left(\frac{n\pi x}{a}\right)}{e^{-\frac{\pi n b}{a}}} \sinh\left(\frac{-n\pi b}{a}\right)$$

6.

$$f(x) = \sum_{n=1}^{\infty} \frac{2 A_n}{e^{-n\pi b/a}} \sinh\left(-\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$\Rightarrow$ Appeal to the Knowledge of Fourier Series

f - differentiable fn then if we choose

$$\boxed{\frac{2 A_n}{e^{-bn\pi/a}} \sinh\left(-\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx}$$

Final - Laplace

Properties of Harmonic functions

$B(x,r)$ $\quad r > 0$

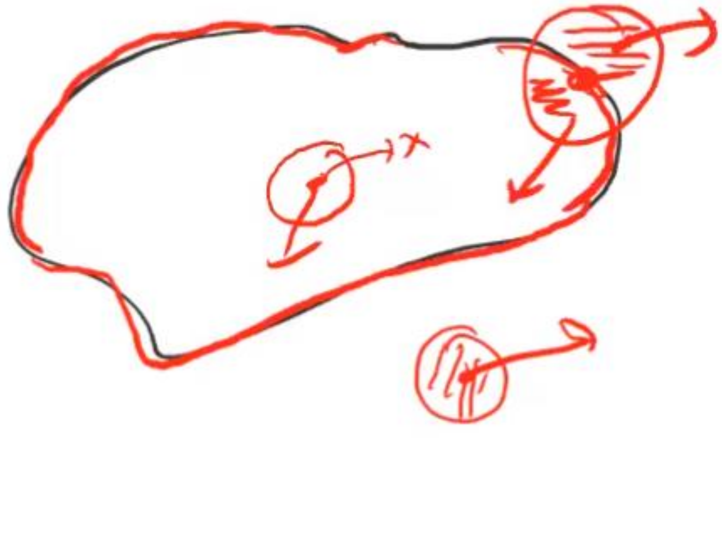
= open Ball of radius 'r' around 'x'.

$A \subseteq \mathbb{R}^2$. Boundary of A, denoted by ∂A is

$$\partial A = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} B(x,r) \cap A \neq \phi \\ B(x,r) \cap A^c \neq \phi \\ \forall\, r > 0 \end{array} \right\}$$

→ Ex

A . open



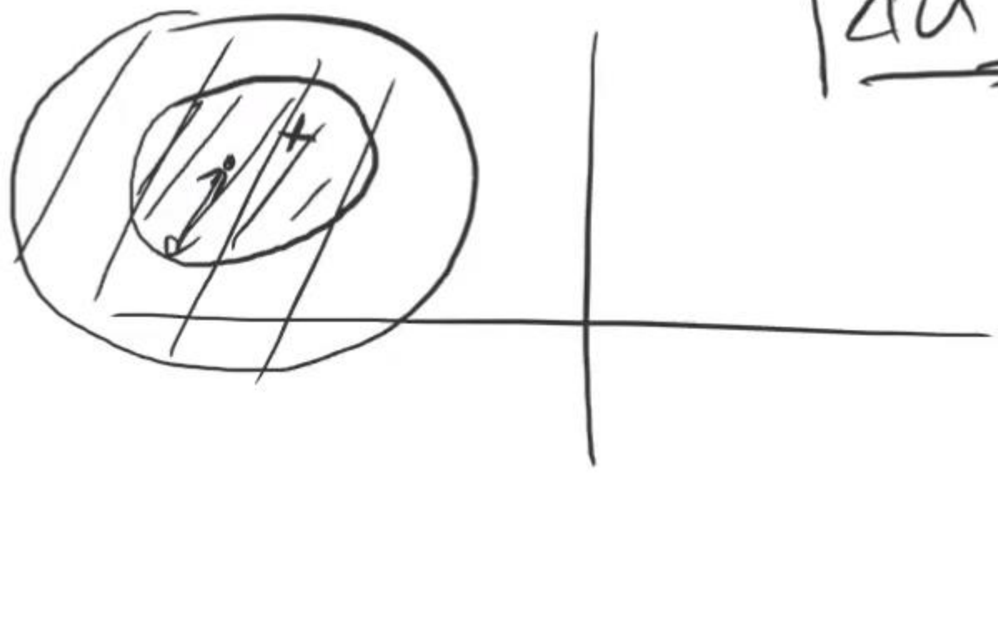
Mean value property ②.

Let u is a harmonic function in $\mathbb{R}^2$. Then

$$u(x) = \frac{1}{\pi r^2} \int_{B(x,r)} u(y)\,dy$$

$\forall r>0,\ x \in \mathbb{R}^2.$

$\Delta u = 0$



Maximum principal

open $\boxed{3}$

Let $\boxed{u \in C^2(\bar{\Omega})}$ and $\Omega \subseteq \mathbb{R}^2$

Harmonic.

Then (1) $\boxed{\max_{\bar{\Omega}} u} = \max_{\partial\Omega} u.$

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u.$$

Furthermore, if $\Omega =$ Ball.
and there exist a point $x_0 \in \Omega$
such that

$$\Longrightarrow \quad u(x_0) = \max_{\bar{\Omega}} u$$

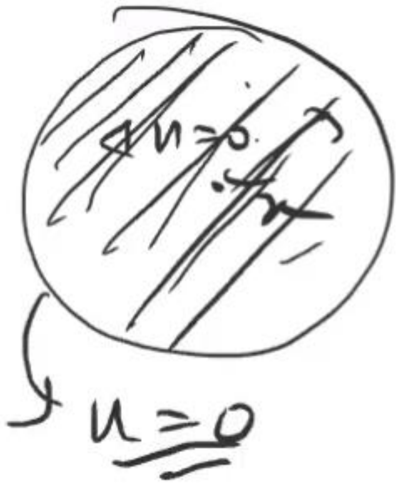
then u is constant in Ω.

$$\bar{\Omega} = \Omega \cup \partial\Omega$$

Let $u \in C^2(\overline{\Omega})$ (uniqueness) u

$$\begin{cases} \Delta u = 0 & \text{in } B(x,r) \\ u = 0 & \text{on } \partial B(x,r) \end{cases} \quad \text{(Dirichlet)}$$

The above problem has unique solution.



→ Max principal.

$$\Rightarrow \; u(y) \le 0 \quad \forall y \in B(x,r)$$

min is taken on the Boundary ⇒

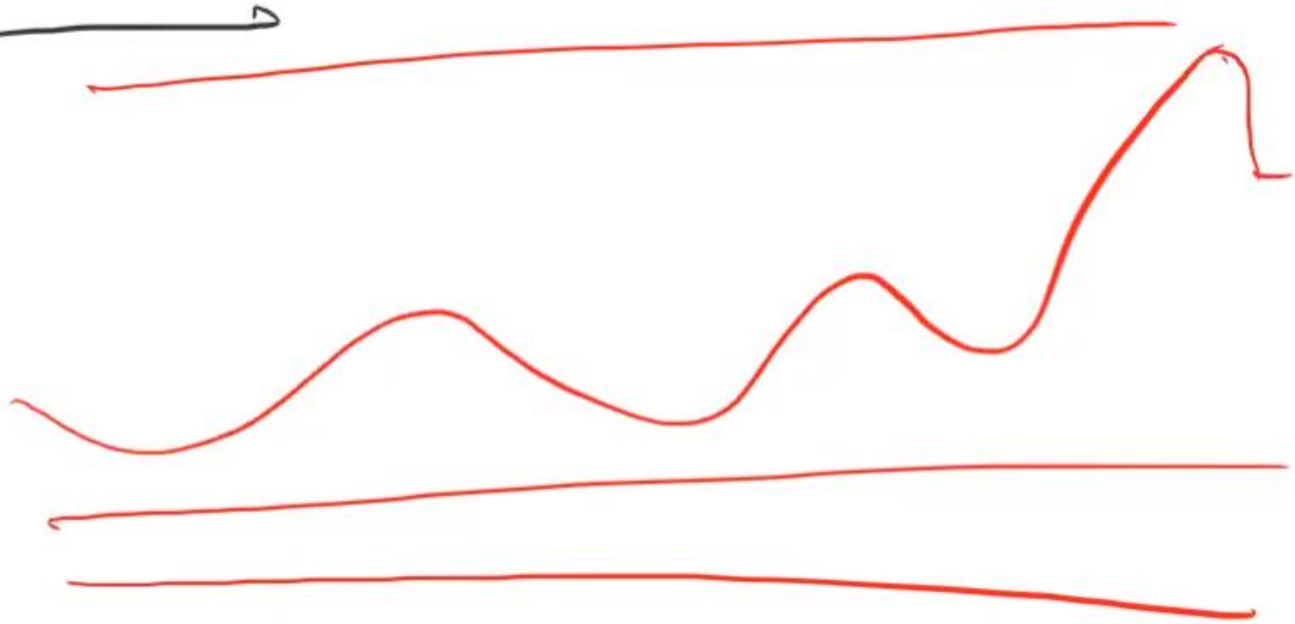
$$u(y) \ge 0 \quad \forall y \in B(x,r)$$

$$\Rightarrow u(y) \equiv 0 \; //$$

5.

Liouville theorem

Let 'u' be a Bounded harmonic function on $\mathbb{R}^n$; Then 'u' is constant.



SOLVING LAPLACE EIGEN VALUE PROBLEM ON RECTANGLES

$$\begin{cases} u'' + \lambda u = 0 & \text{on } (0,\pi) \\ u(0) = u(\pi) = 0 \end{cases}$$

$$\lambda_n = n^2 .$$

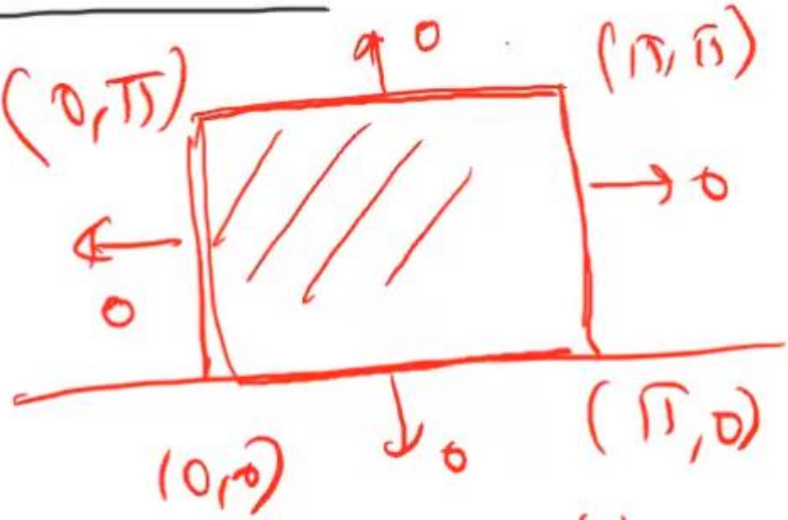
Laplace Eigen value problem, $\Omega \subseteq \mathbb{R}^2$

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega . \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} u'' + \lambda u = 0 & \text{on } (a,b) \\ u(a) = 0 = u(b) \end{cases}$$

$\Delta u + \lambda u = 0$ on $(0,\pi)\times(0,\pi)$ $\quad$ 2.

$u(x,0) = u(x,\pi) = u(0,y) = u(\pi,y) = 0$



Idea - Seperation of variables

$$u(x,y) = F(x)G(y) \longrightarrow (1)$$

$$\Delta u(x,y) = F''(x)G(y) + G''(y)F(x)$$

$$0 = \Delta u + \lambda u = F''(x)G(y) + G''(y)F(x) + \lambda G(y)F(x)$$

$$\frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} = -\lambda . \quad \underline{\underline{3}}$$

$-\mu_n$ $\longrightarrow$ (*)

$\forall x, y \in (0, \pi)$

Let us assume

$$\frac{F''(x)}{F(x)} = -\mu \Rightarrow \begin{cases} F''(x) + \mu F(x) = 0 \\ F(0) = 0 \\ F(\pi) = 0 \end{cases}$$

For Boundary condition

$$u(0,y) = 0 = F(0)\,G(y)$$

But $G(y) \neq 0 \Rightarrow F(0) = 0$

Similarly $\quad u(\pi, y) = 0$

$$F(\pi) = 0$$

$$\boxed{\mu_n = n^2} \qquad F_n(x) = \sin(nx)$$

from (*)

$$\begin{cases} G''(y) = (-\lambda + n^2) G(y) \\ \Rightarrow G''(y) + (\lambda - n^2) G(y) = 0 \\ G(0) = 0 = G(\pi) \end{cases}$$

again Eigen Value problem

$$\begin{cases} u(x,0) = 0 \\ F(x) G(0) = 0 \\ \Rightarrow G(0) = 0 \end{cases}$$

$\lambda - n^2 = m^2 \quad m \in \mathbb{N}$

$\lambda_{n,m} = n^2 + m^2 \quad n, m \in \mathbb{N}$