MTH 102A ASSIGNMENT-12 (ODE) **Apr 07 & 09, 2020.**

1. Locate singular points, if any, and classify them for the following ODE:
   (a) $(x^3 + x^2)y'' + (x^2 - 2x)y' + 4y = 0$.
   (b) $(x^5 + x^4 - 6x^3)y'' + x^2y' + (x - 2)y = 0$.
   (c) $x^3(x - 1)y'' - 2(x - 1)y' + 3xy = 0$.
   (d) $(3x + 1)xy'' - xy' + 2y = 0$.

2. (a) Find the general solution of the ODE $y'' + y' - xy = 0$ using the method of power series centred at the origin. Discuss about the radius of convergence of the solution.
   (b) Solve the IVP of the ODE corresponding to the following initial conditions:
      i. $y(0) = 1, y'(0) = 0$.
      ii. $y(0) = 0, y'(0) = 1$.

3. Find the general solution of $(1 + x^2)y'' - 4xy' + 6y = 0$ using the power series method centred at origin. Also, discuss and compare the radius of convergences of the solution and the coefficients in the normalised form.

4. For any given constant p, the ODE

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

   is called the *Legendre* equation.
   (a) Find the singular points and classify them.
   (b) Show that the Legendre equation has the power series solution about the ordinary point origin as follows: $y_1(x) = \sum_{k=0}^{\infty} c_{2k}x^{2k}$ and $y_2(x) = \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1}$, where $c_0 = c_1 = 1$, for $k \geq 0$,

$$c_{2k+2} = -\frac{(p - 2k)(p + 2k + 1)}{(2k + 1)(2k + 2)}c_{2k}$$

   and, for $k \geq 1$,

$$c_{2k+1} = -\frac{(p - 2k + 1)(p + 2k)}{2k(2k + 1)}c_{2k-1}.$$

   (c) For the case $p = 0$, verify that $y_1(x) = 1$ and $y_2(x) = \dfrac{1}{2}\ln\dfrac{1 + x}{1 - x}$. In this case, the polynomial $P_0 := y_1$ is called the Legendre polynomial of degree zero. Similarly, for $p = 1$, verify that $y_2(x) = x$ and $y_1(x) = 1 - \dfrac{x}{2}\ln\dfrac{1 - x}{1 + x}$. In this case, the polynomial $P_1 := y_2$ is called the Legendre polynomial of degree one.

5. For each of the following, verify that the origin is a regular singular point and find two linearly independent solutions:
   (a) $9x^2y'' + (9x^2 + 2)y = 0$.
   (b) $x^2(x^2 - 1)y'' - x(1 + x^2)y' + (1 + x^2)y = 0$.
   (c) $xy'' + (1 - 2x)y' + (x - 1)y = 0$.
   (d) $x(x - 1)y'' + 2(2x - 1)y' + 2y = 0$.