MSO202A COMPLEX ANALYSIS
Solutions–2

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
Problems for discussion:

1. Let $z = x + iy$ and $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$. Write $f(z)$ as a function of z and $\overline{z}$.

   Solution: Using $x = \frac{z+\overline{z}}{2}, y = \frac{z-\overline{z}}{2i}$ we get $f(z) = \overline{z} + 2iz$.

2. Verify Cauchy-Riemann equation for $z^2,\ z^3$.

   Solution: For z^2, $u = x^2 - y^2, v = 2xy \Rightarrow u_x = 2x, u_y = -2y, v_x = 2y, v_y = 2x$. Similarly for z^3.

3. Let $z, w \in \mathbb{C}, |z|, |w| < 1$ and $\overline{z}w \neq 1$. Prove that $\frac{|w-z|}{|1-\overline{w}z|} < 1$. Further, show that the equality holds if either $|z| = 1$ or $|w| = 1$.

   Solution: Sufficient to show that $|w - z|^2 < |1 - \overline{w}z|^2$, i.e. $w\overline{w} + z\overline{z} - w\overline{z} - z\overline{w} < 1 - w\overline{z} - \overline{w}z + w\overline{w}z\overline{z}$. Since $(1 - z\overline{z})(1 - w\overline{w}) > 0$, the above is true.

   In the case of equality, we see that either $(1 - z\overline{z})$ or $(1 - w\overline{w})$ is zero. Hence, in this case either $|z| = 1$ or $|w| = 1$.

4. Using the relations $x = \dfrac{z + \overline{z}}{2}, y = \dfrac{z - \overline{z}}{2i}$ and the chain rule show that $\dfrac{\partial}{\partial z} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y})$; $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y})$. Further show that

$$\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

   Solution: Straight forward.

5. Suppose that $f = u + iv$ is twice continuously differentiable function on an open connected set Ω. Use CR equations to prove that u and v are harmonic functions i.e., they satisfy $\Delta u = 0 = \Delta v$ where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

   (Note: Later we'll see that a holomorphic function is infinitely many times differentiable. So the conclusion holds for any holomorphic function.)

   Solution: As $u_x = v_y, u_y = -v_x \Rightarrow u_{xx} = v_{yx} = v_{xy}, u_{yy} = -v_{xy} \Rightarrow \Delta u = 0$. Similarly $\Delta v = 0$.

6. Show that in polar coordinates (r, θ), the CR equations for the function $f = u + iv$ takes the form $u_r = \frac{1}{r}v_\theta$ and $u_\theta = -\frac{1}{r}v_r$.

   Solution: In polar coordinates, $x = r\cos\theta, y = r\sin\theta$? By chain rule,we see that $u_r = u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta = v_y \cos\theta - v_x \sin\theta$. $v_\theta = v_x x_\theta + v_y y_\theta = -rv_x \sin\theta + rv_y \cos\theta = ru_r$ and so $u_r = \frac{1}{r}v_\theta$ and similarly, $u_\theta = -\frac{1}{r}v_r$.

Problems for tutorial:

1. For a fixed w in the unit disk $\mathbb{D}$, define the mapping $F\colon z \to \frac{w-z}{1-\overline{w}z}$. Show that

   (a) F is a map from $\mathbb{D}$ to $\mathbb{D}$ and is differentiable at every point of $\mathbb{D}$.

   (b) $F(0) = w$ and $F(w) = 0$, i.e. F interchanges 0 and w.

   (c) $|F(z)| = 1$ if $|z| = 1$.

   (d) $F\colon \mathbb{D} \to \mathbb{D}$ is bijective.

   Solution:

   (a) Since both $|w|$ and $|z|$ are less than 1, from by solution (3), $F(z) \in \mathbb{D}$. Since $|1 - \overline{w}z| \geq 1 - |w||z| > 0, F(z)$, being a rational function is differentiable everywhere.

   (b) direct

   (c) Since $|w| < 1$, it follows from (1), that $F(z) < 1$ only if $|z| = 1$.

   (d) Note $F \circ F(z) = z$

2. Suppose that $f = u + iv$ is holomorphic on an open connected set Ω. Prove that in each one of the following cases f is constant. (a) If u is constant (b) v is constant or if (c) $|f|$ is constant.

   Solution: (a) As that $f = u + iv$ is holomorphic u, v satisfy CR equations. As u is constant this implies $u_x = 0 = v_y, u_y = 0 = -v_x \Rightarrow u$ is constant and v is constant using real variable calculus. Hence f is constant.

   (b) is similar.
   (c) $|f|$ is constant $\Rightarrow u^2 + v^2 = \ const$. Thus $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$. Using CR equations $u_x = v_y, u_y = -v_x$, it follows that $(u^2 + v^2)v_x = 0 = (u^2 + v^2)u_x \Rightarrow v_x = 0 = u_x \Rightarrow u_x = 0 = u_y$ and $v_x = 0 = v_y$. Hence u and v are constant $\Rightarrow f$ is constant.

3. Show that $f = u+iv$ satisfy CR-equations if and only if $\partial f/\partial \overline{z} = 0$. Moreover, if f is holomorphic, then $f'(z) = \partial f/\partial z$. Here f is being thought of as a function of z and $\overline{z}$.

   Solution: As $\dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y})$, $\partial f/\partial \overline{z} = \frac{1}{2}\left[(u_x + iv_x) + i(u_y + iv_y)\right] = 0$.
   Further, $f'(z) = u_x + iv_x$ and $\partial f/\partial z = \dfrac{1}{2}\left[(u_x + iv_x) - i(u_y + iv_y)\right] = u_x + iv_x$ and therefore $f'(z) = \partial f/\partial z$.

4. Consider the following functions:

   (a) $f(z) = \begin{cases} \frac{\overline{z}^3}{|z|^2} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases}$

   (b) $f(z) = \sqrt{|xy|}$

In both the above cases, show that f satisfies the CR conditions at 0 but it is not differentiable there.

Solution: (a) $u(x,0) = x, v(x,0) = 0, u(0,y) = 0, v(0,y) = y$. Hence, $u_x(0,0) = 1 = v_y(0,0)$ and $u_y(0,0) = 0 =?v_x(0,0)$. But, $lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim \frac{\bar{z}^2}{z^2}$ which is 1 along the x-axis and -1 along the line $y = x$ and hence does not exists.

(b) $u_x(0,0) = v_y(0,0) = u_y(0,0) = v_x(0,0) = 0$. But, $lim_{z\to 0}\frac{f(z)-f(0)}{z-0}$ is 0 along the x-axis and $\pm\frac{1-i}{2}$ along the line $y = x$ and hence does not exists.