MSO-203 B ASSIGNMENT 2

IIT, KANPUR

23rd October , 2020

Submission Deadline: 31st October, 2020. 6.00 AM.
Submit only problem number 1,2, 3(a), 4(a) and (c), 5,6,7,8. in the order they are mentioned, otherwise your answer script may not be graded.

1. Let P, Q and R are differentiable functions. Consider the following eigenvalue problem for general linear second order equation:

$$P(t)y^{''}(t) + Q(t)y^{'}(t) + R(t)y(t) + \lambda y(t) = 0. \tag{1}$$

Show that the above problem can be reduced to following SLEVP form

$$\big(p(t)y'(t)\big)^{'} + s(t)y(t) + \lambda r(t)y(t) = 0, \tag{2}$$

with the functions

$$p = Pe^{\int \frac{Q-P'}{P}}, s = Re^{\int \frac{Q-P'}{P}}, \text{ and } r = e^{\int \frac{Q-P'}{P}}.$$

HINT: Multiply the first equation by a positive function μ and then equate the two equations to get the expression for the unknown function μ.

2. There can be more than one correct choices, for multiple type questions. Consider the following eigenvalue problem:

$$\begin{cases} y^{''}(t) + \lambda y(t) = 0 \quad \textbf{on } (0, \pi) \\ y(0) = y(\pi) = 0. \end{cases}$$

Then,

a) $\lambda = 0$ is an eigenvalue of the above problem.
b) there exist an eigenvalue which is strictly negative.
c) the function $\sin(2t)$ is an eigenfunction.
d) there are infinitely many eigenvalues of the above problem.

3. Reduce the following eigenvalue problem to SLEVP form:

a) $ty^{''}(t) + 2y^{'}(t) + \lambda y(t) = 0.$
b) $y^{''}(t) + y^{'}(t) + (\lambda + 1)y(t) = 0.$

4. Find the solution of the following eigenvalue problem:

a) $y^{''}(t) + \lambda y(t) = 0$ on $(0, L)$, $y(0) = y(L) = 0$, for fixed $L > 0$.
b) $y^{''}(t) + \lambda y(t) = 0$ on $(0, 1)$, $y(0) = y(1)$, $y'(0) = y'(1)$.
c) $\left(\frac{y'(t)}{t}\right)^{'} + (\lambda + 1)\frac{y(t)}{t^3} = 0$ on $(1, e^{\pi})$, $y(1) = 0$, $y(e^{\pi}) = 0$.
d) $y^{''}(t) + 8y'(t) + (\lambda + 16)y(t) = 0$ on $(0, \pi)$, $y(0) = 0 = y(\pi)$.
e) $y^{''}(t) + \lambda y(t) = 0$ on $(0, 1)$, $y(0) = y'(1) = 0$.

5. It is given to you that the following is an orthogonal family of functions on the interval $(-\pi, \pi)$:

$$\Big\{\sin(nx), \cos(nx))\Big\}_{n\in\mathbb{N}}.$$

Using this fact prove that the following family of functions are also orthogonal on the interval $(-1, 1)$:

$$\Big\{\sin(n\pi x), \cos(n\pi x)\Big\}_{n\in\mathbb{N}}.$$

6. Consider the following problem

$$y''(t) + (e^{t^2} + 1)y(t) = 0 \quad \text{in } \mathbb{R}.$$

Let y_1 denotes a non-trivial solution (not identically zero function) of the above problem. Then show y_1 has infinitely many zeros, that is, y_1 vanishes at infinitely many points on $\mathbb{R}$.

7. Show that the following family of functions

$$\Big\{\sin(n\pi \log(t)\}\Big\}_{n\in\mathbb{N}},$$

are orthogonal. The domain of definition for each function in the above family is assumed to be $(1, e)$. Are they orthonormal ? If not, can they be turned in to an orthonormal family ?

Hint: Try finding eigenvalues and eigenfunctions for the following problem:

$$t^2y''(t) + ty'(t) + \lambda y = 0, \quad \text{on } (1, e), \quad y(1) = y(e) = 0.$$

Further Hint: Change the independent variable $t = \log(x)$ *to reduce the above problem to a more known problem.*

8. Consider the following RSLEVP problem:

$$\big(p(t)y'(t)\big)' + q(t)y(t) + \lambda\sigma(t)y(t) = 0 \quad \text{on } (a, b). \tag{3}$$

Let (μ, ϕ) be any eigenpair for the above problem (Notice that we are not specifying any boundary conditions). Then prove that

$$\mu = \frac{-p\phi\phi'\big|_a^b + \int_a^b \big[p(\phi')^2 - q\phi^2\big]}{\int_a^b \sigma\phi^2}.$$

Now consider $q = 0$ function in (3), and with the boundary condition

$$y(a) = 0, \; y'(b) = 0.$$

Then prove that any eigenvalue of this problem is strictly positive.