Mathematics of Signals and Systems

- Tools of signals and systems are based on some fundamental mathematical concepts and operations such as:
  - Complex Numbers
  - Complex Variables
  - Trigonometric Functions and Identities
  - Arithmetic Operations on Complex Numbers and Variables

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Mathematics of Signals and Systems

- Polynomials
- Series
- Modulo Operation
- Greatest Common Divisor
- Least Common Multiple
- Decibel

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Complex Numbers

- A complex number is defined using two real numbers
- It can be written in one of two different forms
- Both forms make use of the symbol "j" defined by the equation

$$j^2 = -1$$

or, equivalently by $\boxed{j = \sqrt{-1}}$

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Complex Numbers

- It follows from the definition that

$$\frac{1}{j} = -j$$

- It also follows from the definition that

$$j^3 = -j, \quad j^4 = 1$$

- In general

$$j^{2k+1} = -j, \quad j^{2k} = 1$$

where k is a positive integer

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Complex Numbers

Rectangular Form of Representation

- Also known as the Cartesian form, a complex number n_o is written as

$$\boxed{n_o = \alpha_o + j\beta_0}$$

real number

- α_o is called the real part of n_o, denoted as

$$\boxed{\alpha_o = \mathcal{Re}(n_o)}$$

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Complex Numbers

- β_o is called the imaginary part n_o, denoted as

$$\boxed{\beta_o = \mathcal{Im}(n_o)}$$

- Example: $n_o = -2.5 + j4.3301$ is a complex number in rectangular form
- Its real part is -2.5 and its imaginary part is 4.3301

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Complex Numbers

Polar Form of Representation

- Here a complex number n_o is written as

$$\boxed{n_o = r_o e^{j\theta_o}}$$

real number

- where e is the base of the natural logarithm
- Example: $n_o = 5e^{j2\pi/3}$ is a complex number in polar form

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Complex Numbers

- In the polar form of representation,

$$\boxed{r_o = |n_o|}$$

is called the magnitude of the complex number, and

$$\boxed{\theta_o = \arg(n_o) = \angle(n_o)}$$

is called the argument or angle of the complex number

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Complex Numbers

- The magnitude of the complex number $n_o = 5e^{j2\pi/3}$ is 5 and its angle is $2\pi/3$
- As we shall show later

$$\boxed{e^{j2\pi k} = 1}$$

for all integer values of k

- Hence, we can add any multiples of 2π to the angle of a complex number in polar form without changing its value

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Complex Numbers

- Thus,

$$r_o e^{j(\theta_o + 2\pi k)} = r_o e^{j\theta_o} e^{j2\pi k} = r_o e^{j\theta_o} = n_o$$

- Example:

$$5e^{j(14\pi/3)} = 5e^{j(4\pi + \frac{2\pi}{3})} = 5e^{j4\pi} e^{j2\pi/3}$$
$$= 5e^{j2\pi/3}$$

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Complex Numbers

Complex Conjugate of a Complex Number

- Complex conjugate of a complex number n_o is denoted as n_o^*
- Obtained by replacing "j" with "$-j$"
- Complex conjugate of a complex number in rectangular form $n_o = \alpha_o + j\beta_0$ is

$$n_o^* = \alpha_o - j\beta_0$$

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Complex Numbers

- Complex conjugate of a complex number in polar form $n_o = r_o e^{j\theta_o}$ is

$$n_o^* = r_o e^{-j\theta_o}$$

Graphical Representations

- Two forms of representations depending on the mathematical representation of the complex number

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Graphical Representation

Rectangular Form

- Here a complex number is shown as a point in the complex plane, a two-dimensional rectangular coordinate system
- The origin, usually shown as 0, represents the complex number whose real and imaginary parts are zeros, that is

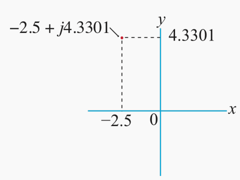
$$n_o = 0 + j0$$

Graphical Representation

- Real part of the complex number is the distance from the origin in the horizontal direction, usually called the x-axis
- Imaginary part of the complex number is the distance from the origin in the vertical direction, usually called the y-axis

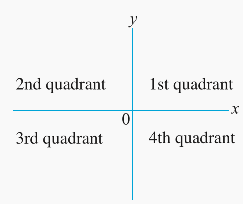
Graphical Representation

- Figure below illustrates the location of the complex number $n_o = -2.5 + j4.3301$ in the complex plane



Graphical Representation

- The complex plane can be divided into 4 parts, called quadrants, as shown below



Graphical Representation

- First quadrant is the region defined by

$$0 \le \mathcal{Re}\{n_o\} < \infty, \quad 0 \le \mathcal{Im}\{n_o\} < \infty$$

- Second quadrant is the region defined by

$$-\infty < \mathcal{Re}\{n_o\} \le 0, \quad 0 \le \mathcal{Im}\{n_o\} < \infty$$

- Third quadrant is the region defined by

$$0 \le \mathcal{Re}\{n_o\} < \infty, \quad -\infty < \mathcal{Im}\{n_o\} \le 0$$

- Fourth quadrant is the region defined by

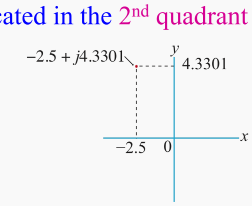
$$-\infty < \mathcal{Re}\{n_o\} \le 0, \quad -\infty < \mathcal{Im}\{n_o\} \le 0$$

Graphical Representation

- Example: The complex number

$$n_o = -2.5 + j4.3301$$

is located in the 2nd quadrant



Graphical Representation

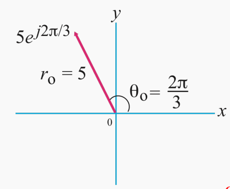
Polar Form

- Here, the complex number $n_o = r_o e^{j\theta_o}$ is represented as a vector of length r_o with θ_o being the angle of the vector with respect to the positive x-axis, usually specified in radians

Graphical Representation

- Figure below illustrates the vector representation of the complex number

$$n_o = 5e^{j2\pi/3}$$



Angle of a Complex Number

Principal Value of the Angle

- Angle θ_o can be either positive or negative
- Usually specified to be in the range

$$-\pi < \theta_o \leq \pi$$

called the principal value

Angle of a Complex Number

- If the specified angle is outside the above range, then $2\pi k$, where k is a positive or negative integer, is added to the angle to bring the result to the range
- Example: The angle of $5e^{j14\pi/3}$ is $\theta_o = 14\pi/3$ which is outside the range
- To bring the angle to the range, we add -4π which results in the new value of the angle $2\pi/3$ that is in the range

Angle of a Complex Number

- Alternate way to bring the angle to the range from $-\pi$ to π, is to rewrite the complex number as shown below

$$5e^{j(\frac{14\pi}{3})} = 5e^{j(\frac{2\pi}{3}+4\pi)} = 5e^{j(\frac{2\pi}{3})}$$

Angle of a Complex Number

- Often, it is more convenient to express the principal value of the angle in degrees
- To convert an angle θ given in radians to degrees, we multiply it with 180 and divide by π, that is

$$\boxed{\text{angle} = \frac{\theta \times 180}{\pi} \text{ degrees}}$$

Angle of a Complex Number

- Example: We determine the angle θ in degrees of the complex number $5e^{j14\pi/3}$
- The principal value of the angle is $2\pi/3$ radians
- Hence, the angle in degrees is given by

$$\text{angle} = \frac{(2\pi/3)\times 180}{\pi} = \frac{2\times 180}{3} = 120^{\text{o}}$$

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Complex Numbers in MATLAB

- In MATLAB an imaginary number is written using either the symbol "`i`" or "`j`"
- Example: The complex number in rectangular form $n_o = -2.5 + j4.3301$ is entered as

```
n = -2.5+4.3301*i
```

or

```
n = -2.5+4.3301*j
```

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Complex Numbers in MATLAB

- Example: The complex number in polar form $n_o = 5e^{j2\pi/3}$ is entered as

```
n = 5*exp(2*pi*i/3)
```

- Real and imaginary parts of a complex number are computed using the MATLAB functions `real` and `imag`
- Complex conjugate of a complex number is computed using the function `conj`

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Complex Numbers in MATLAB

- Example: The code fragments

```
n = - 2.5+4.3301*j;
nreal = real(n);
nimag = imag(n);
nconj(n);
```

yield

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Complex Numbers in MATLAB

```
nreal =
-2.5000
nimag =
4.3301
nconj =
-2.5-4.3301i
```

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Complex Numbers in MATLAB

- In the case of a complex number entered in polar form, the functions `real` and `imag` return, respectively, the real and imaginary parts of the equivalent complex number in rectangular form

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Complex Variables

- The values of a real variable are real numbers
- The complex variable is an extension of the concept of a real variable
- Like the complex number, a complex variable can be mathematically represented in one of two forms

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Complex Variables

Rectangular Form

- A complex variable η is written as

$$\eta = c + jd$$

where c and d are real variables

- Here c is the real part of η, denoted as

$$c = \mathcal{Re}(\eta)$$

and d is the imaginary part of η, denoted as

$$d = \mathcal{Im}(\eta)$$

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Complex Variables

- In the processing of continuous-time signals, the complex variable is denoted as s and is represented in rectangular form as

$$s = \sigma + j\Omega$$

where σ and Ω are real variables

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Complex Variables

PolarForm

- A complex variable η is written as

$$\eta = \gamma e^{j\phi}$$

where γ and ϕ are real variables

- Here, γ is called the magnitude of η, denoted as $\gamma = |\eta|$

and ϕ is the argument or phase of η, denoted as $\phi = \arg(\eta)$

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Complex Variables

- In the processing of discrete-time signals, the complex variable is denoted as z and is represented in polar form as

$$z = re^{j\omega}$$

where r and ω are real variables

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Trigonometric Functions

Basic Trigonometric Functions

- The sine and cosine of an argument θ, are written as $\sin(\theta)$ and $\cos(\theta)$, respectively
- The argument θ, usually called the angle, is a real number and is specified in radians, a dimensionless quantity

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Trigonometric Functions

- If θ is a real variable, then $\sin(\theta)$ and $\cos(\theta)$ are real functions of the real variable θ usually defined for the range

$$-\infty < \theta < +\infty$$

- Plots of $\sin(\theta)$ and $\cos(\theta)$ are shown on the next slide

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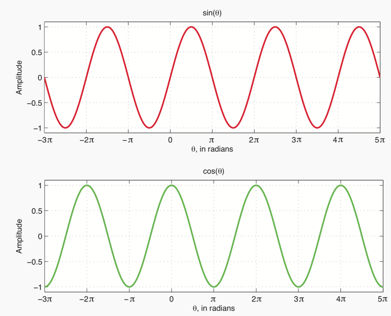
Trigonometric Functions



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Trigonometric Functions

- Commonly used identities in signal processing are as follows:

$$\sin(\theta \pm \tfrac{\pi}{2}) = \pm\cos(\theta)$$

$$\cos(\theta \pm \tfrac{\pi}{2}) = \mp\sin(\theta)$$

- A generalizations of the above identities are:

$$\sin(\theta \pm \psi) = \sin(\theta)\cos(\psi) \pm \cos(\theta)\sin(\psi)$$

$$\cos(\theta \pm \psi) = \cos(\theta)\cos(\psi) \mp \sin(\theta)\sin(\psi)$$

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Trigonometric Functions

- It can be seen from these two plots that

$$-1 \le \sin(\theta) \le +1$$

and

$$-1 \le \cos(\theta) \le +1$$

- In some applications, we may have a scaled trigonometric function, such as, $A\sin(\theta)$, where A is a real positive number and is called the peak amplitude

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Trigonometric Functions

- Some typical values of $\sin(\theta)$ and $\cos(\theta)$ for certain specific values of θ are given below

$$\sin(0) = \sin(\pi) = 0$$

$$\sin(\pi/2) = 1, \quad \sin(3\pi/2) = -1$$

$$\cos(0) = 1, \quad \cos(\pi) = -1$$

$$\cos(\pi/2) = \cos(3\pi/2) = 0$$

$$\sin(\pi/6) = \tfrac{1}{2}, \quad \sin(\pi/4) = \tfrac{1}{\sqrt{2}}, \quad \sin(\pi/3) = \tfrac{\sqrt{3}}{2}$$

$$\cos(\pi/6) = \tfrac{\sqrt{3}}{2}, \quad \cos(\pi/4) = \tfrac{1}{\sqrt{2}}, \quad \cos(\pi/3) = \tfrac{1}{2}$$

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Trigonometric Functions

- Another trigonometric function used in signal processing is the tangent function
- The tangent of an argument θ is written as $\tan(\theta)$
- This trigonometric function is related to the basic functions $\sin(\theta)$ and $\cos(\theta)$ as

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

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Trigonometric Functions

Properties

- All trigonometric functions are periodic functions of their arguments with a period 2π
- Thus, if we replace the argument θ with $\theta + 2\pi k$, where k is an integer, we get back the same trigonometric function
- Example: $\sin(\theta + 2\pi k) = \sin(\theta)$

Trigonometric Functions

- The periodicity properties of $\sin(\theta)$ and $\cos(\theta)$ can be seen from their plots given in Slide No. 38
- Also, it can be seen from these plots that $\sin(\theta)$ is an odd function of θ and $\cos(\theta)$ is an even function of θ, that is,

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$

Trigonometric Identities

- Two commonly used identities are as follows:

$$\sin(\theta \pm \tfrac{\pi}{2}) = \pm\cos(\theta)$$
$$\cos(\theta \pm \tfrac{\pi}{2}) = \mp\sin(\theta)$$

- A generalization of the above are given by

$$\sin(\theta \pm \psi) = \sin(\theta)\cos(\psi) \pm \cos(\theta)\sin(\psi)$$
$$\cos(\theta \pm \psi) = \cos(\theta)\cos(\psi) \mp \sin(\theta)\sin(\psi)$$

Trigonometric Identities

- From $\cos(\theta \pm \psi) = \cos(\theta)\cos(\psi) \mp \sin(\theta)\sin(\psi)$ we obtain after setting $\psi = \theta$,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$$

- From $\sin(\theta \pm \psi) = \sin(\theta)\cos(\psi) \pm \cos(\theta)\sin(\psi)$ we obtain after setting $\psi = \theta$,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Trigonometric Identities

- Additional identities derived from the last two equations of Slide No. 45 are

$$\sin(\theta)\cos(\psi) = \tfrac{1}{2}[\sin(\theta+\psi) + \sin(\theta-\psi)]$$
$$\cos(\theta)\sin(\psi) = \tfrac{1}{2}[\sin(\theta+\psi) - \sin(\theta-\psi)]$$
$$\cos(\theta)\cos(\psi) = \tfrac{1}{2}[\cos(\theta+\psi) + \cos(\theta-\psi)]$$
$$\sin(\theta)\sin(\psi) = \tfrac{1}{2}[\cos(\theta+\psi) - \cos(\theta-\psi)]$$

Inverse of Trigonometric Functions

- Inverse of the sine function is denoted as
  $\sin^{-1}$ or arcsin
- Inverse of the cosine function is denoted as
  $\cos^{-1}$ or arcos
- Inverse of the tangent function is denoted as
  $\tan^{-1}$ or arctan

Inverse of Trigonometric Functions

- Examples: Inverse of $\alpha = \sin(\theta)$ is given by
$$\theta = \arcsin(\alpha) = \sin^{-1}(\alpha)$$
- Inverse of $\beta = \cos(\theta)$ is given by
$$\theta = \arccos(\beta) = \cos^{-1}(\beta)$$
- Inverse of $\gamma = \tan(\theta)$ is given by
$$\theta = \arctan(\gamma) = \tan^{-1}(\gamma)$$

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Trigonometric Functions in MATLAB

- For computing the sine, cosine, and tangent functions, MATLAB functions are `sin`, `cos`, and `tan`, respectively
- For computing the inverses of sine, cosine, and tangent functions, MATLAB functions are `asin`, `acos`, and `atan`, respectively

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Trigonometric Functions in MATLAB

- Example: The plot of the sine function shown in Slide No. 38 has been generated using the code fragments

```
theta = -2*pi:pi/256:8*pi;
x = sin(theta);
```

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Conversion Between Forms

- Makes use of the Euler's formula:
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
- If we represent the complex number as the vertex of a right angle triangle as shown in the next slide

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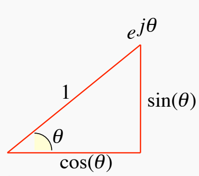
Conversion Between Forms



Then, $\cos(\theta)$ is the length of the base, and $\sin(\theta)$ is the length of the perpendicular

- From Pythagoras' theorem we have
$$\cos^2(\theta) + \sin^2(\theta) = 1$$

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Conversion Between Forms

From Polar to Rectangular Form

- Multiplying both sides of the Euler's formula with r we get
$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$
- If we write $s = re^{j\theta}$ in rectangular form as
$$s = \alpha + j\beta$$
- Then it follows that $\alpha = \mathcal{Re}\{re^{j\theta}\} = r\cos(\theta)$ and $\beta = \mathcal{Im}\{re^{j\theta}\} = r\sin(\theta)$

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Conversion Between Forms

- Example: Consider the complex number $n_o = 5e^{j2\pi/3}$ in polar form
- Its real and imaginary parts are thus given by $\mathcal{Re}\{n_o\} = 5\cos(2\pi/3) = -2.5$
  $\mathcal{Im}\{n_o\} = 5\sin(2\pi/3) = 4.3301$
- Hence, the rectangular form of representation of n_o is given by
  $$n_o = -2.5 + j4.3301$$

Conversion Between Forms

From Rectangular to Polar Form

- From $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ we note that its complex conjugate is given by
  $$(e^{j\theta})^* = e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$
- Adding these two equations we get
  $$2\cos(\theta) = e^{j\theta} + e^{-j\theta}$$
  and hence,
  $$\cos(\theta) = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Conversion Between Forms

- Likewise, subtracting the conjugate of Euler's formula from the Euler's formula we get $2\sin(\theta) = e^{j\theta} - e^{-j\theta}$
  and hence,
  $$\sin(\theta) = \tfrac{1}{2}(e^{j\theta} - e^{-j\theta})$$
- The expressions for $\cos(\theta)$ and $\sin(\theta)$ in terms of the exponential functions are known as the Inverse Euler's formula

Conversion Between Forms

- By squaring both sides of $\alpha = r\cos(\theta)$ and $\beta = r\sin(\theta)$ we get
  $$\alpha^2 + \beta^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta)$$
  $$= r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) = r^2$$
- From the above equation we have
  $$r = \sqrt{\alpha^2 + \beta^2}$$

Conversion Between Forms

- Next dividing $\beta = r\sin(\theta)$ by $\alpha = r\cos(\theta)$ we get
  $$\frac{r\sin(\theta)}{r\cos(\theta)} = \tan(\theta) = \frac{\beta}{\alpha}$$
- Hence, $\theta = \tan^{-1}(\frac{\beta}{\alpha})$ radians

Conversion Between Forms

- Example: We develop the polar form of representation of the complex number $n_o = 2.2 - j6.4$ in rectangular form
- From $r = \sqrt{\alpha^2 + \beta^2}$ we have
  $$r = \sqrt{(2.2)^2 + (6.4)^2} = 6.7676$$
  and from $\theta = \tan^{-1}(\frac{\beta}{\alpha})$ we have
  $$\theta = \tan^{-1}\left(-\frac{6.4}{2.2}\right) = -1.2397 \text{ radians}$$

Ambiguity in the Conversion

- The formula $\theta = \tan^{-1}(\frac{\beta}{\alpha})$ for computing the angle θ_o of a complex number n_o in rectangular form can lead to wrong result if not applied correctly
- Consider for example $n_1 = -2.5 + j4.3301$ and $n_2 = 2.5 - j4.3301$

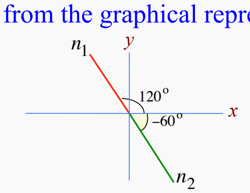
Ambiguity in the Conversion

- The angle θ_1 of the number n_1 is given by
$$\theta_1 = \tan^{-1}\left(\frac{-2.5}{4.3301}\right) = -1.0472$$
radians or -60^o

and the angle θ_2 of the number n_2 is given by
$$\theta_2 = \tan^{-1}\left(\frac{2.5}{-4.3301}\right) = -1.0472$$
radians or -60^o

Ambiguity in the Conversion

- Note from the graphical representations



the vector representing n_1 is in the 2^{nd} quadrant with an angle 120^o and the vector representing n_2 is in the 4^{th} quadrant with an angle -60^o

Ambiguity in the Conversion

Correct Formula for Conversion

$$\theta_o = \begin{cases} \tan^{-1}(\beta_o/\alpha_o), & \alpha_o \geq 0 \\ \tan^{-1}(\beta_o/\alpha_o) \pm \pi, & \alpha_o < 0 \end{cases}$$

- Note from the polar representation of a complex number $n_o = -1$ is $e^{\pm j\pi}$
- Hence, if the real part is negative, then π should be either added or subtracted to bring the computed angle to the range $-\pi$ to π

Conversion Using MATLAB

- The magnitude r and the angle θ of a complex number $n = \alpha + j\beta$ can be determined using the functions `abs` and `angle`
- The code fragments

```
n = -2.5+4.3301*j;
r = abs(n);
theta = angle(n);
```


Conversion Using MATLAB

yield

```
r =
5
theta =
2.0944
```

The angle given above is precisely equal to 2π/3

Conversion Using MATLAB

- The correct value of the argument can also be computed using the MATLAB function atan2
- The pertinent MATLAB code is

```
theta=atan2(imag(n),real(n));
```