LECTURE - 13

Laurent series .

Consider the function $\frac{1}{z^2(1-z)}$.

Since, $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, we immediately see that $\frac{1}{z^2(1-z)} = \sum_{n=-2}^{\infty} z^n$ for $0<|z|<1$

Such series are called "Laurent series".

When functions are analytic in simply connected domains then we have seen many remarkable properties that they satisfy.

Things aren't all that bad if we leave the world of analytic functions, as long as we stay in the world of functions with "isolated singularities". Outside this world all hell is let loose and we won't get there any time soon.

Defns: A pt $z \in \mathbb{C}$ is said to be a <u>singularity</u> of function f defined in a neighbourhood of z if f is not diff'ble at z. The singularity is said to be <u>isolated</u> if f is analytic in a <u>punctured disc</u> around z ie $\exists B_r(z)$ of radius $r>0$ around z $\ni$ f is analytic at every $w \in B_r(z) \setminus \{z\}$.

For functions with isolated singularities a theorem similar to Taylor's theorem is true.

Eg: $\frac{1}{1-z}$ has a Taylor series expansion around '0' with radius of convergence 1.

But, if we inspect the function $\frac{1}{1-z}$, we notice that there is only one pt $z=1$ at which this function is not differentiable. Taylor series have left us with too good an experience to just raise our hands and say, I give up!

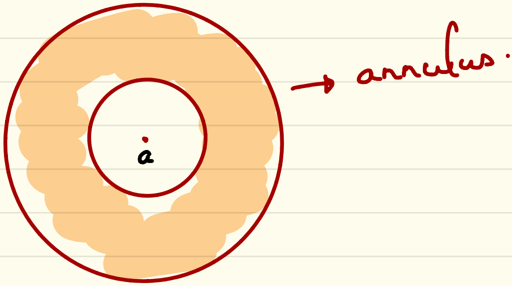
So if $|z|>1$, can we do something to repair the situation:

Notice $|z|>1 \Rightarrow 1/|z|<1$

$$\therefore \frac{1}{1-z} = \frac{-1}{z(1-1/z)} = \frac{-1}{z}\left(\sum_{n=0}^{\infty}(1/z)^n\right)$$

$$= -\sum_{n=1}^{\infty} z^{-n},$$ which is not too bad an expression on the face of it.

This is not special to $\frac{1}{1-z}$. It turns out that if f is a fn which is analytic in a domain like this



then f has an expansion of the form

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

§§ What is the meaning of convergence of such a series?

$$\sum_{n=-1}^{-\infty} c_n (z-a)^n + \sum_{n=0}^{\infty} c_n (z-a)^n$$ converges to $s_1 + s_2$ if $\sum_{n=1}^{\infty} c_{-n} (z-a)^{-n}$ converges to s_1 and $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges to s_2.

Eg: $\sum_{n=-1}^{-\infty} z^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} z^n$ converges

to $\frac{1}{1-z} + \frac{1}{3(1+z/3)}$ for $|z| > 1$ and $\left|\frac{z}{3}\right| < 1$

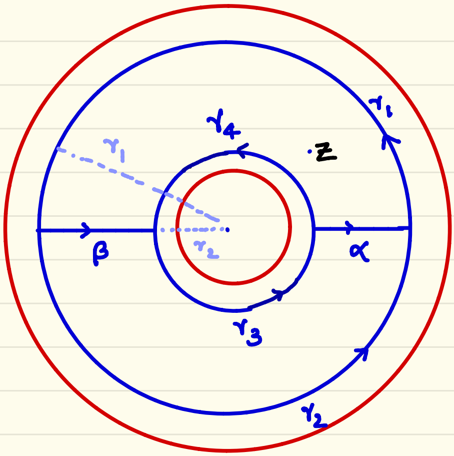
LAURENT'S THEOREM

Let $A = \{z \in \mathbb{C} : R < |z-a| < S\}$ (annulus). Let f be analytic on A. Then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

where $c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$; $\gamma = S_r(a)$.

Pf: w.l.o.g $a = 0$.



$\gamma_1 + \gamma_2 = C_{r_1}(0)$

$\gamma_3 + \gamma_4 = C_{r_2}(0$

$\hat{\gamma} := \gamma_1 + \beta - \gamma_4 + \alpha$ is a closed contour around z

$\therefore$ By CIF $\int_{\hat{\gamma}} \frac{f(w)}{w-z} dw = 2\pi i f(z)$

$\hat{\hat{\gamma}} := \gamma_2 - \alpha - \gamma_3 - \beta$ then $\int_{\hat{\hat{\gamma}}} \frac{f(w)}{w-z} dz = 0$

$$\therefore \int_{\hat{\gamma}+\hat{\hat{\gamma}}} \frac{f(\omega)}{\omega - z}\, d\omega = \int_{\gamma_1+\beta-\gamma_4+\alpha} \cdots + \int_{\gamma_2-\alpha-\gamma_3-\beta} \cdots$$

$$= \int_{\gamma_1+\gamma_2} \cdots - \int_{\gamma_4+\gamma_3} \cdots$$

$$= \int_{C_{r_1}(0)} \cdots - \int_{C_{r_2}(0)} \cdots = 2\pi i f(z)$$

$$\therefore \boxed{f(z) = \frac{1}{2\pi i} \int_{C_{r_1}(0)} \frac{f(\omega)}{\omega - z}\, d\omega - \int_{C_{r_2}(0)} \frac{f(\omega)}{\omega - z}\, d\omega} \quad \text{—}(*)$$

For $\omega \in C_{r_1}(0)$, $|\omega| > |z| \Rightarrow \left|\frac{z}{\omega}\right| < 1$

$$\therefore \frac{1}{\omega - z} = \frac{1}{\omega(1 - z/\omega)} = \frac{1}{\omega}\sum_{n=0}^{\infty} (z/\omega)^n$$

$$= \frac{1}{\omega}\left(\sum_{n=1}^{k} (z/\omega)^n + \frac{(z/\omega)^{k+1}}{1 - z/\omega}\right)$$

For $\omega \in C_{r_2}(0)$, $|\omega| < |z| \Rightarrow \frac{|\omega|}{|z|} < 1$

$$\frac{1}{z(\frac{\omega}{z} - 1)} = \frac{-1}{z} \sum_{n=0}^{\infty} \left(\frac{\omega}{z}\right)^n = \frac{-1}{z}\left(\sum_{n=1}^{l} \left(\frac{\omega}{z}\right)^n + \frac{(\omega/z)^{l+1}}{1 - \omega/z}\right)$$

As in Taylor's theorem, we use the above identities to obtain

$$f(z) = \frac{1}{2\pi i}\int_{C_{r_1}(0)}\left(\sum_{n=1}^{k}\frac{f(w)}{w}\left(z/w\right)^n + \frac{f(w)\left(z/w\right)^{k+1}}{w(1-z/w)}\right)dw$$

$$+\ \frac{1}{2\pi i}\left(\int_{C_{r_2}(0)}\left(\sum_{n=1}^{l}\frac{f(w)}{z}\left(w/z\right)^n + \frac{f(w)}{z(1-w/z)}\left(w/z\right)^{l+1}\right)dw\right.$$

$$= \sum_{n=1}^{k}\left(\frac{1}{2\pi i}\int_{C_{r_1}(0)}\frac{f(w)}{w^{n+1}}\,dw\right)z^n + \frac{1}{2\pi i}\int_{C_{r_1}(0)}\frac{f(w)}{w^{k+1}}\cdot\underbrace{\frac{z^{k+1}}{w-z}dw}_{\rho_k(z)}$$

$$+\ \sum_{n=1}^{l}\left(\frac{1}{2\pi i}\int_{C_{r_2}(0)}\frac{f(w)}{z^{n+1}}w^n\,dw\right) + \underbrace{\frac{1}{2\pi i}\int_{C_{r_2}(0)}\frac{f(w)}{(z-w)}\frac{w^{l+1}}{z^{l+1}}dw}_{R_l(z)}$$

$$\underbrace{\left|\frac{1}{2\pi i}\int_{C_{r_1}(0)}\frac{f(w)}{w-z}dz - \sum_{n=1}^{k}\left(\frac{1}{2\pi i}\int_{C_{r_1}(0)}\frac{f(w}{w^{n+1}}dw\right)z^n\right|}$$

$$\left|\rho_k(z)\right| \le \frac{1}{2\pi}\frac{M_i z^{k+1}}{r_1^{k+1}}\cdot 2\pi r_1 \qquad M_1 = \sup_{w\in C_{r_1}(0)}\left|\frac{f(w)}{w-z}\right|$$

$$\underbrace{\left|\frac{1}{2\pi i}\int_{C_{r_2}(0)}\frac{f(w)}{w-z}dz - \sum_{n=1}^{l}\frac{1}{2\pi i}\int_{C_{r_2}(0)}f(w)w^n\cdot\frac{1}{z^{n+1}}\cdot dw\right|}$$

$$\left|R_l(z)\right| \le \frac{1}{2\pi}\frac{M_2\cdot r_2^{l+1}}{|z|^{l+1}}\cdot 2\pi r_2 \qquad M_2 = \sup_{w\in C_{r_2}(0)}\left|\frac{f(w)}{z-w}\right|$$

Since $r_1 > |z| > r_2$, $\frac{|z|}{r_1} < 1$, $\frac{r_2}{|z|} < 1$

$\therefore$ as $n \to \infty$ & $l \to \infty$

we get $P_n(z) \to 0$ & $R_l(z) \to 0$

Thus,
$$f(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{C_r(0)} \frac{f(\omega)}{\omega^{n+1}} \, d\omega \right) z^n$$

(By Deformation theorem, $\int_{C_{r_1}(0)} h(\omega)\,d\omega = \int_{C_r(0)} h(\omega)\,d\omega$ for h analytic in the annulus containing $C_{r_1}(0)$ & $C_r(0)$.)

$\blacksquare$

REMARK: We do not prove uniqueness of c_n's though it is true! (Follows from Fundamental integral).

Eg: Laurent series of $\dfrac{\sin \pi z}{(z-1)^2}$ around 1.

Taylor series of $\sin \pi z$ around $z=1$

$$= \sum_{n=0}^{\infty} \frac{(\sin \pi z)^{(n)}\big|_{z=1}}{n!} (z-1)^n \qquad \left(\because (\sin \pi z)^{(n)}\big|_{z=1} = 0 \quad n = \text{even}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\pi^{2n+1}}{(2n+1)!} (z-1)^{2n+1} \qquad \text{valid } \forall z$$

$$\left[\text{Alternately, } \sin \pi z \cos(-\pi) + \cos \pi z \sin(-\pi) = \sin \pi(z-1)\right.$$

$$\therefore \; -\sin \pi z = \sin \pi(z-1)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{[\pi(z-1)]^{2n+1}}{(2n+1)!}$$

$$\left.\therefore \; \sin \pi z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\pi^{2n+1}}{(2n+1)!} (z-1)^{2n+1}\right]$$

$\therefore$ Laurent series of $\dfrac{\sin \pi z}{(z-1)^2}$ for $0 < |z-1| < \infty$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\pi^{2n+1}}{(2n+1)!} (z-1)^{2n-1}$$

∎