Problem Set 7

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ satisfying $T(1,1,1) = (1,2,3,0), T(1,2,-1) = (2,1,3,0)$ and $T(1,5,-7) = (0,0,0,1)$? Give reasons for your answer.

   Solution: Verify that the set $\{(1,1,1),(1,2,-1),(1,5,-7)\}$ is linearly dependent. So, their images must be linearly dependent. Which is NOT true in the given example.

2. Can we ever find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ which is onto?

   Solution: No, use the rank-nullity theorem.

3. Find out $[\mathbf{v}]_{\mathcal{B}}$, where $\mathcal{B}$ is an ordered basis:

   (a) $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$, $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. (b) $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$, $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

   Solution: $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}3\\-1\\-1\end{bmatrix}$. $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}1\\0\\2\end{bmatrix}$.

4. Find out $\mathbf{v}$ given $[\mathbf{v}]_{\mathcal{B}}$, where $\mathcal{B}$ is an ordered basis:

   (a) $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$, $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. (b) $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$, $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

   Solution: $\mathbf{v} = \begin{bmatrix}6\\3\\1\end{bmatrix}$. $\mathbf{v} = \begin{bmatrix}3\\5\\4\end{bmatrix}$.

5. Give three linear transformations from $\mathbb{R}^3$ to $\mathbb{W} = \{\mathbf{w} \in \mathbb{R}^5 : w_1 - w_2 + w_3 - w_4 + w_5 = 0\}$. Give their coordinate matrix w.r.t the ordered bases $\mathcal{B}_1 = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$ on $\mathbb{R}^3$ and some ordered basis of $\mathbb{W}$.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ as $T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x - y + z\\ x + 2z\end{bmatrix}$. Find

   (a) a basis of Range (T),
   (b) rank (T),
   (c) a basis for $\mathcal{N}(T)$, and
   (d) $\dim(\mathcal{N}(T))$.

Solution: Verify $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. So, Range $(T) = \mathbb{R}^2$ and rank $(T) = 2$. $\mathcal{N}(T) = \text{LS}\left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$ and $\dim(\mathcal{N}(T)) = 1$.

7. **(T)** Find all linear transformations from $\mathbb{R}^n \longrightarrow \mathbb{R}$.

   Solution: Let $T(\mathbf{e}_i) = \alpha_i, \quad 1 \le i \le n$. So

   $$T(\mathbf{x}) = T\left(\sum_{i=1}^{n} x_i \mathbf{e}_i\right) = \sum_{i=1}^{n} x_i T(\mathbf{e}_i) = \sum_{i=1}^{n} \alpha_i x_i = \langle \mathbf{x}, [\alpha_1, \ldots, \alpha_n]^T \rangle.$$

   Also, given any $T : \mathbb{R}^n \longrightarrow \mathbb{R}$, linear, then we know the images of $\mathbf{e}_i$, for $1 \le i \le n$, the basis vectors. That is, there exists $\beta_i \in \mathbb{R}$ such that $T(\mathbf{e}_i) = \beta_i$, for $1 \le i \le n$. Thus,

   $$T(\mathbf{x}) = \sum_{i=1}^{n} \beta_i x_i = \langle \mathbf{x}, [\beta_1, \ldots, \beta_n]^T \rangle.$$

8. Let $\mathbf{v} \in \mathbb{R}^n$ and $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an ordered basis of $\mathbb{R}^n$. Form a matrix $B = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n]$. Is $B[\mathbf{e}_1]_{\mathcal{B}} = \mathbf{e}_1$? What is $B\left[[\mathbf{e}_1]_{\mathcal{B}}, \ldots, [\mathbf{e}_n]_{\mathcal{B}}\right]$? Show that B is invertible and $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$.

   Solution: Let $\mathbf{e}_1 = \sum\limits_{i=1}^{n} \alpha_i \mathbf{v}_i$.

   Then, $[\mathbf{e}_1]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \Rightarrow B[\mathbf{e}_1]_{\mathcal{B}} = [\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \sum\limits_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{e}_1$.

   Similarly, $B[\mathbf{e}_i]_{\mathcal{B}} = \mathbf{e}_i$, for all $i, 1 \le i \le n$. So,

   $$B\left[[\mathbf{e}_1]_{\mathcal{B}}, \ldots, [\mathbf{e}_n]_{\mathcal{B}}\right] = \left[B[\mathbf{e}_1]_{\mathcal{B}}, \ldots, B[\mathbf{e}_n]_{\mathcal{B}}\right] = [\mathbf{e}_1, \ldots, \mathbf{e}_n] = I.$$

   Thus, $B^{-1} = [[\mathbf{e}_1]_{\mathcal{B}}, \ldots, [\mathbf{e}_n]_{\mathcal{B}}]$. Further, $[\mathbf{e}_i]_{\mathcal{B}} = B^{-1}\mathbf{e}_i$, $\forall i, 1 \le i \le n$. Hence $[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$.

9. **(T)** Show that a linear transformation is one-one if and only if null-space of $\mathcal{N}(T)$ is $\{0\}$.

   Solution: $\mathcal{N}(T) \neq \{\mathbf{0}\} \Rightarrow$ there is an $\mathbf{x} \in \mathcal{N}(T)$, $\mathbf{x} \neq \mathbf{0} \Rightarrow T(\mathbf{x}) = T(\mathbf{0}) \Rightarrow T$ is not one-one. If $\mathcal{N}(T) = \{\mathbf{0}\}$ then $T(\mathbf{x}) = T(\mathbf{y}) \Rightarrow T(\mathbf{x} - \mathbf{y}) = \mathbf{0} \Rightarrow \mathbf{x} - \mathbf{y} \in \mathcal{N}(T) \Rightarrow \mathbf{x} = \mathbf{y}$.

10. Describe all 2×2 orthogonal matrices. Prove that action of any orthogonal matrix on a vector $\mathbf{v} \in \mathbb{R}^2$, is either a rotation or a reflection about a line.

    Solution: As A preserves length, there is a $\theta \in [0, 2\pi)$ such that $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $(\sin^2\theta + \cos^2\theta = 1)$. Since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have, $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \perp A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore, $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$

$\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ or $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$, which further implies that

$$A = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation by an angle } \theta,} \quad \textit{or} \quad A = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}}_{\text{reflection about a line of inclination } \theta/2.}$$

11. **(T)** Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $n \geq 2$, with $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$. Prove that there exist an orthogonal matrix A such that $A(\mathbf{v}) = \mathbf{w}$. Prove also that A can be chosen such that $\det(A) = 1$.
(*This is why orthogonal matrices with determinant one are called rotations.*))

Solution: As composition and inverse of orthogonal matrices are orthogonal, it is enough to choose $\mathbf{v} = \mathbf{e}_1$. So we need an orthogonal matrix A such that $A(\mathbf{e}_1) = \mathbf{w}$. Find an orthonormal basis of $\mathbb{R}^n$, say $\{\mathbf{w}, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ by Gram Schmidt. Let $A = \begin{bmatrix} \mathbf{w} & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{bmatrix}$, that is, first column of A is $\mathbf{w}$ and so on. Then A is orthogonal and $det(A) = \pm 1$. If $det(A) = -1$, then multiply the second column with -1.