Problem Set 3

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Draw and illustrate in $\mathbb{R}^2$.

   (a) $\mathbf{e}_1 + \{n\mathbf{e}_2 | n \in \mathbb{N}\}$. (b) $\mathbf{e}_1 + \{\alpha\mathbf{e}_2 | \alpha \in \mathbb{R}\}$.

2. In $\mathbb{R}^2$, Is $\{\alpha\mathbf{e}_1|\alpha \in \mathbb{R}\} + \{\alpha\mathbf{e}_2|\alpha \in \mathbb{R}\} = \mathbb{R}^2$? What about $\{\alpha\mathbf{e}_1|\alpha \in \mathbb{R}\} + \{\alpha \begin{bmatrix}1\\1\end{bmatrix} |\alpha \in \mathbb{R}\} = \mathbb{R}^2$?

3. In $\mathbb{R}^3$ prove that $\left\{\alpha \begin{bmatrix}2\\1\\1\end{bmatrix} |\alpha \in \mathbb{R}\right\} + \left\{\alpha \begin{bmatrix}1\\1\\0\end{bmatrix} |\alpha \in \mathbb{R}\right\} + \left\{\alpha \begin{bmatrix}0\\1\\1\end{bmatrix} |\alpha \in \mathbb{R}\right\} = \mathbb{R}^3$. Do you use Gauss-Jordan Elimination (GJE) method somewhere?

4. Let L_1 and L_2 be two nonparallel lines passing through origin in $\mathbb{R}^3$. What is $L_1 + L_2$?

5. **(T)** Fix a non-negative integer n and let $\mathbb{R}[x;n] = \left\{\sum\limits_{i=0}^{n} c_i x^i : c_0, c_1, \cdots, c_n \in \mathbb{R}\right\}$. Show that $\mathbb{R}[x;n]$ is a real vector space with respect to the usual addition and scalar multiplication.

6. Recall that $\mathbb{M}_n(\mathbb{R})$ is the real vector space of all $n \times n$ real matrices. Now, prove the following:
   (a) $\mathbb{S} = \{A \in \mathbb{M}_n(\mathbb{R}) : A^T = A\}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.
   (b) Fix $A \in \mathbb{M}_n(\mathbb{R})$. Define $\mathbb{U} = \{B \in \mathbb{M}_n(\mathbb{R}) : AB = BA\}$. Then, $\mathbb{U}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.
   (c) Let $\mathbb{W} = \{a_0 I + a_1 A + \cdots + a_m A^m : m \text{ is a non-negative integer}, a_i \in \mathbb{R}\}$. Then, $\mathbb{W}$ is a subspace of $\mathbb{U}$.

7. In $\mathbb{R}$, define $x \oplus y = x + y - 1$ and $a \odot x = a(x-1)+1$. Show that $\mathbb{R}$ is a real vector space with respect to these operations with additive identity 1 (note that 0 is NOT the additive identity).

8. **(T)** Which of the following are subspaces of $\mathbb{R}^3$:

   (a) $\{(x,y,z) \mid x \geq 0\}$, (b) $\{(x,y,z) \mid x+y=z\}$, (c) $\{(x,y,z) \mid x = y^2\}$.

9. Find the condition on $a, b, c, d \in \mathbb{R}$ so that $S = \{(x,y,z) \mid ax+by+cz = d\}$ is a subspace of $\mathbb{R}^3$.

10. **(T)** Show that $S = \{(x_1,x_2,x_3,x_4) : x_4 - x_3 = x_2 - x_1\} = \mathrm{LS}(\{(1,0,0,-1), (0,1,0,1), (0,0,1,1)\}$.

11. **(T)** Let W_1 and W_2 be subspaces of a vector space V such that $W_1 \cup W_2$ is also a subspace. Prove that one of the spaces $W_i,\ i = 1,2$ is contained in the other.

12. Let $S = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be a subset of a real vector space V. Define **linear span** of S as

$$\mathrm{LS}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} : c_1, c_2, \ldots, c_n \in \mathbb{R}\},$$

i.e., the set of all linear combinations of $\mathbf{v_1}, \ldots, \mathbf{v_n}$. Then $\mathrm{LS}(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\})$ is a subspace of V.

13. Suppose S and T are two subspaces of a vector space V. *Define* the **sum**

$$S + T = \{\mathbf{s} + \mathbf{t} : \mathbf{s} \in S, \mathbf{t} \in T\}.$$

Show that $S + T$ satisfies the requirements for a vector space. Moreover, $LS(S \cup T) = S + T$.

14. **(T)** Find all the subspaces of $\mathbb{R}^2$.

15. **(T)** Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, with $a_{ij} \in \mathbb{C}$. Then the 4 fundamental subspaces are:

(a) The column space of A:

$$\operatorname{col}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{C}^n\} = \operatorname{LS}(A[:,1], \ldots, A[:,n]) = \operatorname{LS}\left(\left\{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}\right\}\right)$$

(b) The column space of A^*:

$$\operatorname{col}(A^*) = \operatorname{LS}(A^*[1,:], \ldots, A^*[m,:]) = \{A^*\mathbf{x} : \mathbf{x} \in \mathbb{C}^m\}.$$

(c) The null space of A:

$$\text{Null Space}(A) = \mathcal{N}(A) = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\}.$$

(d) The null space of A^*:

$$\text{Null Space}(A^*) = \mathcal{N}(A^*) = \{\mathbf{x} \in \mathbb{C}^m : A^*\mathbf{x} = \mathbf{0}\}.$$

Important: In case $A \in \mathbb{M}_{m,n}(\mathbb{R})$, the spaces col($A^*$) and Null Space($A^*$) are called the row-space of A and the left-null space of A, respectively

Now, determine the above 4 mentioned fundamental spaces for the following matrices.

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

(iii) Suppose $B, C \in \mathbb{M}_{m,n}(\mathbb{C})$ and $S = \operatorname{col}(B)$, $T = \operatorname{col}(C)$. Determine $M \in \mathbb{M}_{m,n}(\mathbb{C})$ such that $\operatorname{col}(M) = S + T$.

16. Construct A such that $[\,1\ \ 1\ \ 1\,]^T \in \operatorname{col}(A)$ and $\mathcal{N}(A) = \operatorname{LS}\left([\,1\ \ 1\ \ 1\ \ 1\,]^T\right)$.

17. **(T)** Suppose A is an m by n matrix of rank r.

(a) If $A\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$, what is the column space of A?

(b) In part (a), what are all the relations between the numbers m, n and r?

(c) Give a specific example of a 3 by 2 matrix A of rank 1 with first row $[\,2\;\;5\,]$. Describe the column space, $\mathrm{col}(A)$, and the null space, $N(A)$, completely.

(d) Suppose the right side $\mathbf{b}$ is same as the first column in your example (part c). Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

18. Suppose $R = \mathrm{RREF}(A)$, where $A = \begin{bmatrix} 1 & 2 & 1 & b \\ 2 & a & 1 & 8 \\ & (row & 3) & \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) What can you say immediately about row 3 of A?

(b) What are the numbers a and b?

(c) Describe all solutions of $R\mathbf{x} = \mathbf{0}$. Which among row spaces, column spaces and null spaces are the same for A and for R.

19. Let $A \in \mathbb{M}_n(\mathbb{R})$. Show that $\mathcal{N}(A) \subset \mathcal{N}(A^2) \subset \mathcal{N}(A^3) \cdots$. What if $A = \begin{bmatrix} 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$?

20. **(T)** Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. If $\mathrm{RREF}(A) = \begin{pmatrix} I_r & F \\ 0 & 0 \end{pmatrix}$ then describe $\mathrm{col}(A)$ and $\mathcal{N}(A)$.

21. **(T)** Let $W_1 = \mathrm{span}\left\{ \left[1\ 1\ 0\right]^T, \left[-1\ 1\ 0\right]^T \right\}$ and $W_2 = \mathrm{span}\left\{ \left[1\ 0\ 2\right]^T, \left[-1\ 0\ 4\right]^T \right\}$. Show that $W_1 + W_2 = \mathbb{R}^3$. Give an example of a vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v}$ can be written in two different ways in the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in W_1, \mathbf{v}_2 \in W_2$.