Interconnected LTI Analog Systems

- Most LTI analog systems are designed by interconnecting simple LTI analog systems
- We review next three basic interconnection schemes

Cascade Connection

$$\longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow \;\equiv\; \longrightarrow \boxed{h(t)} \longrightarrow$$

$$\boxed{h(t) = h_1(t) \circledast h_2(t)}$$

1

Cascade Connection

- Because of the commutative property of the convolution integral we have

$$h_1(t) \circledast h_2(t) = h_2(t) \circledast h_1(t)$$

implying

$$\longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_1(t)} \longrightarrow \;\equiv\; \longrightarrow \boxed{h(t)} \longrightarrow$$

- Note: If $h_1(t)$ and $h_2(t)$ are stable, then $h(t)$ is also stable

2

Cascade Connection

Example: We determine the impulse response of the cascade connection of two causal LTI analog systems with impulse responses $f(t) = e^{-\alpha t}\mu(t)$ and $g(t) = e^{-\beta t}\mu(t)$ with $\alpha > 0$ and $\beta > 0$

- We have $h(t) = \int_0^\infty f(\tau) g(t-\tau)\, d\tau$

$$= \int_0^t e^{-\alpha\tau} e^{-\beta(t-\tau)}\, d\tau = \frac{e^{-\alpha\tau} - e^{-\beta\tau}}{\beta - \alpha}\mu(t)$$

3

Cascade Connection

- The identity given earlier for causal signals reduces to

$$\int_0^\infty h(t)\,dt = \left(\int_0^\infty f(t)\,dt\right)\left(\int_0^\infty g(t)\,dt\right)$$

- From $h(t) = \dfrac{e^{-\alpha\tau} - e^{-\beta\tau}}{\beta - \alpha}\mu(t)$

we get $\displaystyle\int_0^\infty h(t)\,dt = \frac{1}{\beta - \alpha}\left(\int_0^\infty e^{-\alpha t}\,dt - \int_0^\infty e^{-\beta t}\,dt\right)$

4

Cascade Connection

- which reduces to

$$\int_0^\infty h(t)\,dt = \frac{1}{\beta - \alpha}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) = \frac{1}{\alpha\beta}$$

- Next we compute

$$\int_0^\infty f(t)\,dt = \int_0^\infty e^{-\alpha t}\,dt = \frac{1}{\alpha}$$

$$\int_0^\infty g(t)\,dt = \int_0^\infty e^{-\beta t}\,dt = \frac{1}{\beta}$$

5

Cascade Connection

- Hence

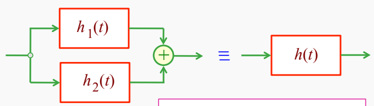
$$\left(\int_0^\infty f(t)\,dt\right)\left(\int_0^\infty g(t)\,dt\right) = \frac{1}{\alpha}\cdot\frac{1}{\beta}$$

$$= \int_0^\infty h(t)\,dt$$

6

Parallel Connection

Parallel Connection

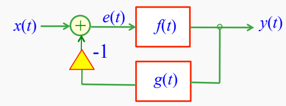


$$\boxed{h(t) = h_1(t) + h_2(t)}$$

- Note: If $h_1(t)$ and $h_2(t)$ are stable, then $h(t)$ is also stable

Feedback Connection

Feedback Connection



- Analysis yields $e(t) = x(t) - g(t) \circledast y(t)$
$$y(t) = e(t) \circledast f(t)$$

Feedback Connection

- Substituting the first equation into the second we arrive at

$$y(t) = [x(t) - g(t) \circledast y(t)] \circledast f(t)$$
$$= x(t) \circledast f(t) - g(t) \circledast f(t) \circledast y(t)$$

using the distributive and commutative properties of the convolution integral

Feedback Connection

- Rewriting the previous equation we get
$$y(t) + g(t) \circledast f(t) \circledast y(t) = x(t) \circledast f(t)$$

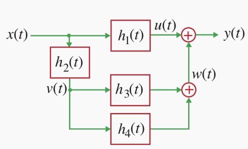
from which we arrive at the input-output relation of the feedback system in the time domain as

$$\boxed{[\delta(t) + g(t) \circledast f(t)] \circledast y(t) = f(t) \circledast x(t)}$$

Output Input

Interconnected LTI Analog Systems

Example: We illustrate the analysis of an analog system composed of 4 simple LTI analog systems



$$h_1(t) = \delta(t) + \tfrac{1}{2}\delta(t-1)$$
$$h_2(t) = \tfrac{1}{2}\delta(t) - \tfrac{1}{4}\delta(t-1)$$
$$h_3(t) = 2\delta(t)$$
$$h_3(t) = -2\mu(t)$$

Interconnected LTI Analog Systems

- The impulse response $h(t)$ of the overall system is given by
$$h(t) = h_1(t) + h_2(t) \circledast [h_3(t) + h_4(t)]$$
$$= h_1(t) + h_2(t) \circledast h_3(t) + h_2(t) \circledast h_4(t)$$
- Substituting the expressions for the individual impulse responses we arrive at after some algebra
$$h(t) = \tfrac{3}{2}\delta(t)$$

Inverse LTI Analog Systems

$x(t) \longrightarrow \boxed{g(t)} \longrightarrow \boxed{h(t)} \longrightarrow x(t)$

$$\boxed{g(t) \circledast h(t) = \delta(t)}$$

- The system with an impulse response $g(t)$ is known as the inverse of the system with an impulse response $h(t)$, and vice-versa
- Notation: $h^{-1}(t)$ will denote the inverse of $h(t)$

Inverse LTI Analog Systems

Example: It can be shown that the inverse of an LTI analog system with an impulse response

$$h(t) = A\delta(t) + Be^{-\alpha t}\mu(t), \quad \alpha > 0$$

has an impulse response given by

$$h^{-1}(t) = C\delta(t) + De^{-\beta t}\mu(t)$$
$$= \tfrac{1}{A}\delta(t) - \tfrac{B}{A^2}e^{-(\alpha + \frac{B}{A})t}\mu(t)$$

Time-Domain Representation of LTI Analog Systems

- The LTI analog systems we shall be concerned with in this course are characterized by an input-output relation in the time-domain by a linear constant-coefficient differential equations of the form

$$\frac{d^N y(t)}{dt^N} + \sum_{k=1}^{N} q_i \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} p_i \frac{d^k x(t)}{dt^k}$$

Time-Domain Representation of LTI Analog Systems

where $y(t)$ and $x(t)$ are, respectively, the output and input signals, and the coefficients $\{q_k\}$ and $\{p_k\}$ are constants

- The constants N and M are positive integers with $\max(N, M)$ denoting the order of the differential equation

Solution of Differential Equation

- The solution $y(t)$ of the differential equation is composed of two parts:

$$y(t) = y_c(t) + y_p(t)$$

where $y_c(t)$, called the complementary solution or homogeneous solution, is the output signal with $x(t) = 0$

Solution of Differential Equation

and $y_p(t)$, called the particular solution, is the output signal due to the prescribed input $x(t)$

- If the input $x(t)$ is a causal signal applied at time instant t_o , $x(t) = 0$ for $t < t_o$, then for a causal stable LTI analog system, we can also assume $y(t) = 0$ for $t < t_o$

Solution of Differential Equation

- To determine the complete solution of the N-th order differential equation, N initial conditions

$$y(t_o),\ \left.\frac{dy(t)}{dt}\right|_{t=t_o},\ \left.\frac{d^2y(t)}{dt^2}\right|_{t=t_o},\ \cdots,\ \left.\frac{d^Ny(t)}{dt^N}\right|_{t=t_o}$$

must be specified

Solution of Differential Equation

- If all initial conditions are equal to zero at the time instant t_o the causal input signal is applied to a causal LTI analog system, the system is said to be at rest
- We shall assume in this course the analog LTI system under consideration is at rest

Solution of First-Order Differential Equation

- Consider $\frac{dy(t)}{dt} + q_0 y(t) = Ae^{-\alpha t}\mu(t)$
- Initial condition $y(0) = y_o$
- We first solve the homogeneous equation

$$\frac{dy_c(t)}{dt} + q_0 y_c(t) = 0$$

to determine the complementary solution $y_c(t)$

Solution of First-Order Differential Equation

- We rewrite the homogeneous equation as

$$y_c(t) = -\frac{1}{q_0}\cdot\frac{dy_c(t)}{dt}$$

- The above form indicates that $y_c(t)$ is proportional to the derivative
- Hence a possible form of $y_c(t)$ is

$$y_c(t) = Ke^{st}$$

Solution of First-Order Differential Equation

- Substituting $y_c(t) = Ke^{st}$ in

$$\frac{dy_c(t)}{dt} + q_0 y_c(t) = 0$$

we get

$$Kse^{st} + q_0 Ke^{st} = (s + q_0)Ke^{st} = 0$$

- Non-trivial solution is obtained with

$$s + q_o = 0 \quad \Rightarrow \quad s = -q_o$$

Solution of First-Order Differential Equation

- Hence $y_c(t) = Ke^{-q_0 t}$
- We next determine the particular solution $y_p(t)$ of

$$\frac{dy(t)}{dt} + q_0 y(t) = Ae^{-\alpha t}\mu(t)$$

- We assume $y_p(t) = Be^{-\alpha t}, \quad t \geq 0$ which results in

$$-\alpha Be^{-\alpha t} + q_0 Be^{-\alpha t} = Ae^{-\alpha t}$$

Solution of First-Order Differential Equation

- Solving for B we get $B = A/(q_0 - \alpha)$
- Thus
$$y_p(t) = \frac{A}{q_0 - \alpha} e^{-\alpha t}, \; t \geq 0$$
- The complete solution is then of the form
$$y(t) = y_c(t) + y_p(t) = Ke^{-q_0 t} + \frac{A}{q_0 - \alpha} e^{-\alpha t}, \; t \geq 0$$

Solution of First-Order Differential Equation

- From the last equation we have
$$y(0) = y_o = K + \frac{A}{q_0 - \alpha}$$
- Hence $K = y_o - \dfrac{A}{q_0 - \alpha}$
- Therefore the complete solution reduces to
$$y(t) = \left(y_o - \frac{A}{q_0 - \alpha} \right) e^{-q_0 t} + \left(\frac{A}{d_0 - \alpha} \right) e^{-\alpha t}, \; t \geq 0$$

Solution of First-Order Differential Equation

- Now, for $t < 0$, $x(t) = 0$, and hence
$$\frac{dy(t)}{dt} + q_0 y(t) = Ae^{-\alpha t} \mu(t)$$
reduces to the homogeneous equation of the form
$$\frac{dy(t)}{dt} + q_0 y(t) = 0$$
- Hence,
$$y(t) = Ke^{-d_0 t}, \; t < 0$$

Solution of First-Order Differential Equation

- Imposing the initial condition $y(0) = y_o$ we get $K = y_o$, and thus
$$y(t) = y_o e^{-q_0 t}, \; t < 0$$
- Hence, the complete solution is given by
$$y(t) = y_o e^{-q_0 t} + \frac{A}{q_0 - \alpha} (e^{-\alpha t} - e^{-q_0 t}) \mu(t)$$

Impulse Response of a Causal First-Order Analog System

- The impulse response $h(t)$ of the causal first-order analog system characterized by
$$\frac{dy(t)}{dt} + q_0 y(t) = Ae^{-\alpha t} \mu(t)$$
is obtained by solving the differential equation
$$\frac{dh(t)}{dt} + q_0 h(t) = \delta(t)$$

Impulse Response of a Causal First-Order Analog System

- Note: The particular solution $h_p(t) = 0$ as $h_p(t)$ cannot include $\delta(t)$
- If it did, then $h_p(t)$ will include a derivative of $\delta(t)$ which is not present in
$$\frac{dh(t)}{dt} + q_0 h(t) = \delta(t)$$
- This implies that $h(t)$ is obtained by solving the homogeneous equation

Impulse Response of a Causal First-Order Analog System

$$\frac{dh(t)}{dt} + q_0 h(t) = 0$$

- Its solution is of the form

$$h(t) = Be^{-q_o t}\mu(t)$$

- Substituting the above in the top equation we get

$$-q_0 Be^{-q_o t}\mu(t) + Be^{-q_o t}\frac{d\mu(t)}{dt} + q_0 Be^{-q_o t}\mu(t) = \delta(t)$$

31

Impulse Response of a Causal First-Order Analog System

which reduces to

$$Be^{-q_o t}\frac{d\mu(t)}{dt} = \delta(t)$$

- Making use of $d\mu(t)/dt = \delta(t)$ and the sampling property of $\delta(t)$ we rewrite the above equation as

$$Be^{-q_o t}\delta(t) = B\delta(t) = \delta(t)$$

32

Impulse Response of a Causal First-Order Analog System

implying $B = 1$

- Hence, the impulse response $h(t)$ of the causal first-order analog system is given by

$$h(t) = e^{-q_o t}\mu(t)$$

33

Step Response of a a Causal First-Order Analog System

- The step response can be computed from the impulse response using the relation

$$s(t) = \int_{-\infty}^{t} h(\psi)d\psi = \int_{0}^{t} h(\psi)d\psi$$

$$= \int_{0}^{t} e^{-q_0\psi}d\psi = \frac{1}{q_0}(1 - e^{-q_0 t})\mu(t)$$

34

Time Constant

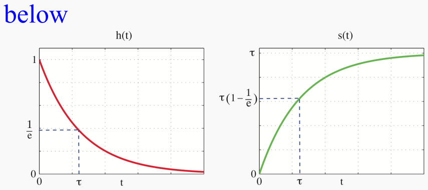
- Note: An important parameter characterizing the impulse response and the step response is the constant q_0
- In practice, this parameter is denoted by its reciprocal $\tau = 1/q_0$, called the time constant
- Thus, $h(t) = e^{-t/\tau}\mu(t)$

$$s(t) = \tau(1 - e^{-t/\tau})\mu(t)$$

35

Time Constant

- Plots of the impulse response and the step response for a typical value of τ are shown below



(a) (b)

36

Time Constant

- Note: The impulse response $h(t)$ of a first-order causal analog system decays exponentially from a value 1 at $t = 0$ to a value 0 as $t \to \infty$
- Note: The step response $s(t)$ of a first-order causal analog system grows exponentially from a value 0 at $t = 0$ to a value to a value τ as $t \to \infty$

Time Constant

- The rate of decay of $h(t)$ and the rate of growth of $s(t)$ is determined by the value of τ
- At $t = \tau$, the impulse response $h(t)$ has decayed to a value that is 1/e times its value at $t = 0$
- At $t = \tau$, the step response $s(t)$ has reached a value 1/e within its final value at $t = \infty$

Block Diagram Representation

- The block-diagram representation of an LTI analog system often provides an additional insight into the operation of the system and also its hardware or software implementation
- The representation is obtained using the schematic representations of the basic operations shown earlier

Block Diagram Development

Example: Consider the first-order LTI analog system characterized by the differential equation

$$\frac{dy(t)}{dt} + q_0 y(t) = p_0 x(t) + p_1 \frac{dx(t)}{dt}$$

- Rewriting the above equation we arrive at

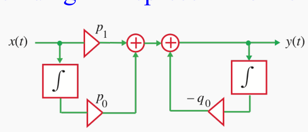
$$\frac{dy(t)}{dt} = p_0 x(t) + p_1 \frac{dx(t)}{dt} - q_0 y(t)$$

Block Diagram Development

- Integrating the last equation we arrive at

$$y(t) = p_0 \int_{-\infty}^{t} x(\tau)d\tau + p_1 x(t) - q_0 \int_{-\infty}^{t} y(\tau)d\tau$$

- The block diagram representation is thus

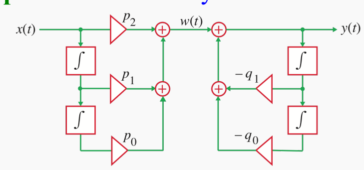


Block Diagram Analysis

- The constant coefficient differential equation representation of the LTI analog system can be determined by analyzing its block-diagram representation as illustrated in the next example

Block Diagram Analysis

Example: Consider the system shown below



- Here we have

$$w(t) = p_2 x(t) + p_1 \int_{-\infty}^{t} x(\tau)d\tau + p_0 \int_{-\infty}^{t} \left[\int_{-\infty}^{\tau} x(\psi)d\psi \right] d\tau$$

43

Block Diagram Analysis

$$y(t) = w(t) - q_1 \int_{-\infty}^{t} y(\tau)d\tau - q_0 \int_{-\infty}^{t} \left[\int_{-\infty}^{\tau} y(\psi)d\psi \right] d\tau$$

- Substituting the expression for $w(t)$ in the above equation we get

$$y(t) + q_1 \int_{-\infty}^{t} y(\tau)d\tau + q_0 \int_{-\infty}^{t} \left[\int_{-\infty}^{\tau} y(\psi)d\psi \right] d\tau$$

$$= p_2 x(t) + p_1 \int_{-\infty}^{t} x(\tau)d\tau + p_0 \int_{-\infty}^{t} \left[\int_{-\infty}^{\tau} x(\psi)d\psi \right] d\tau$$

44

Block Diagram Analysis

- Differentiating both sides of the last equation twice we get the differential equation representation of the LTI analog system of Slide No. 15

$$\frac{d^2 y(t)}{dt^2} + q_1 \frac{dy(t)}{dt} + q_0 y(t)$$

$$= p_2 \frac{d^2 x(t)}{dt^2} + p_1 \frac{dx(t)}{dt} + p_0 x(t)$$

45

Equivalent Structures

- Different block diagram representations of a LTI analog system having the same differential equation representation are called equivalent structures
- A simple way to develop an equivalent structure is by block diagram manipulations as illustrated next

46

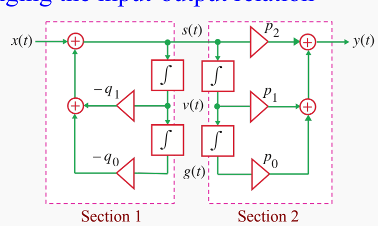
Equivalent Structures

- The structure in Slide 43 can be considered as a cascade of two sections:
- One with an input signal $x(t)$ and an output signal w(t), and the other with an input signal $w(t)$ and an output signal $y(t)$

47

Equivalent Structures

- The order of the these two sections can be interchanged as shown below without changing the input-output relation



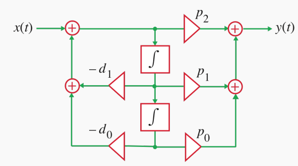
48

Equivalent Structures

- The signal variable at the output of the two top integrators are both $v(t) = \int_{-\infty}^{t} u(\tau)d\tau$
- The signal variable at the output of the two bottom integrators are both $g(t) = \int_{-\infty}^{t} v(\tau)d\tau$
- By eliminating the two of the integrators we arrive at the block diagram shown in the next slide without changing the input-output relation

49

Equivalent Structures



Transpose Operation

- A straight-forward method to generate an equivalent structure from a specified block diagram representation

50

Transpose Operation

- Operation consists of three steps:

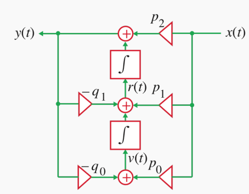
  1) Reverse all directed paths

  2) Replace adders with pick-off nodes and vice-versa

  3) Interchange input and outputs
- Figure in the next slide shows the structure obtained by applying the transpose operation to the structure of Slide 29

51

Equivalent Structures



- To verify the equivalence we analyze the above structure

52

Equivalent Structures

- The outputs of the three adders are given by

$$v(t) = p_0 x(t) - q_0 y(t)$$

$$r(t) = p_1 x(t) - q_1 y(t) - \int_{-\infty}^{t} v(\tau)d\tau$$

$$y(t) = p_2 x(t) + \int_{-\infty}^{t} r(\tau)d\tau$$

- Differentiating both sides of the second equation we get

53

Equivalent Structures

$$\frac{dr(t)}{dt} = p_1 \frac{dx(t)}{dt} - q_1 \frac{dy(t)}{dt} + v(t)$$

- Next, differentiating both sides of the third equation we get

$$\frac{d^2 y(t)}{dt^2} = p_2 \frac{d^2 x(t)}{dt^2} + \frac{dr(t)}{dt}$$

54

Equivalent Structures

- Substituting the top equation into the bottom equation we get

$$\frac{d^2 y(t)}{dt^2} = p_2 \frac{d^2 x(t)}{dt^2} + p_1 \frac{dx(t)}{dt} - q_1 \frac{dy(t)}{dt} + v(t)$$

- Next, we substitute the expression for $v(t)$ from Slide No. 53 resulting in the equation given in the next slide

55

Equivalent Structures

$$\frac{d^2 y(t)}{dt^2} = p_2 \frac{d^2 x(t)}{dt^2} + p_1 \frac{dx(t)}{dt} - q_1 \frac{dy(t)}{dt}$$
$$+ p_0 x(t) - q_0 y(t)$$

- The above equation is precisely the same as the differential equation in Slide 45 after a simple rearrangement

56

Canonic and Noncanonic Structures

- If the number of integrators in a block diagram is the same as the order of its differential equation representation, it is known as a canonic structure
- In a noncanonic structure the number of integrators in the block diagram is greater than the order of its differential equation representation

57