Practice Problems 15 : Integration, Riemann's Criterion for integrability (Part I)

1. Prove the inequality $nr^2\sin(\pi/n)\cos(\pi/n) \leq A \leq r^2\tan(\pi/n)$ given in the lecture notes where A is the area of the circle of radius r.

2. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that there is a partition P of $[a,b]$ such that $L(P,f) = U(P,f)$. Show that f is a constant function.

3. Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $f(x) \geq 0$ for every $x \in [a,b]$. Show that $\underline{\int_a^b} f(x)dx \geq 0$ and $\overline{\int_a^b} f(x)dx \geq 0$. In addition, if f is integrable, show that $\int_a^b f(x)dx \geq 0$.

4. In each of the following cases, evaluate the upper and lower integrals of f and show that f is integrable. Find the integral of f.

   (a) For $\alpha \in \mathbb{R}$, define $f : [a,b] \to \mathbb{R}$ by $f(x) = \alpha$ for every $x \in [a,b]$.

   (b) $f(x) = 0$ for $0 \leq x < \frac{1}{2}$, $f(\frac{1}{2}) = 10$ and $f(x) = 1$ for $\frac{1}{2} < x \leq 1$.

   (c) $f(x) = x$ for all $x \in [0,1]$.

5. Let $f : [a,b] \to \mathbb{R}$ be integrable and P_n be a partition such that $U(P_n,f) - L(P_n,f) \to 0$. Show that $\lim_{n\to\infty} L(P_n,f) = \lim_{n\to\infty} U(P_n,f) = \int_a^b f(x)dx$.

6. In each of the following cases, show that f is integrable using the Riemann criterion.

   (a) $f(x) = x$ on $[0,1]$.

   (b) $f(x) = x^2$ on $[0,1]$.

   (c) $f(x) = \frac{1}{x}$ on $[1,2]$.

7. Let f, f_1 and f_2 be bounded functions on $[0,1]$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in [0,1]$. Suppose that f_1 and f_2 are integrable and $\int_0^1 f_1(x)dx = \int_0^1 f_2(x)dx$, show that f is integrable and find $\int_0^1 f(x)dx$.

8. Let $f : [0,1] \to \mathbb{R}$ be such that $f(x) = x$ for x rational and $f(x) = 0$ for x irrational. Evaluate the upper and lower integrals of f and show that f is not integrable.

9. **(*)** Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p,q \in \mathbb{N} \text{ and } \ p,q \text{ have no common factors} \\ 0 & \text{if } x \text{ is irrational or } x = 0 \end{cases}$$

   (a) For any $N \in \mathbb{N}$ consider the set

$$A_N = \left\{ x \in [0,1] : x = \frac{p}{q} \text{ where } p,q \in \mathbb{N}, q \leq N \text{ and } p,q \text{ have no common factors} \right\}.$$

   Show that the set A_N is finite.

   (b) For given $N \in \mathbb{N}$ and $\epsilon > 0$, show that there are intervals $[x_1,x_2], [x_3,x_4], ..., [x_{m-1},x_m]$ such that $A_N \subseteq (x_1,x_2) \cup (x_3,x_4) \cup ... \cup (x_{m-1},x_m)$ and $|x_1 - x_2| + |x_3 - x_4| + ... + |x_{m-1} - x_m| \leq \frac{\epsilon}{2}$.

   (c) Show that f is integrable.

   (d) Find two integrable functions g and h on $[0,1]$ such that $g \circ h$ (g composition of h) is not integrable.

Practice Problems 15 : Hints/Solutions

1. The area of the inscribed triangle given in Figure 1 in the notes is $2\times\frac{1}{2}r\sin(\pi/n)r\cos(\pi/n)$. The area of the superscribed triangle is $2\times\frac{1}{2}(r\tan(\pi/n))r$.

2. Observe that $U(P,f)-L(P,f)=\sum_{i=1}^{n}(M_i-m_i)\Delta x_i$ and $M_i-m_i\geq 0$ and $\Delta x_i\geq 0$.

3. Follows from the definitions.

4. (a) For any partition $P=\{x_0,x_1,...,x_n\}$ of $[a,b]$, $m_i=M_i=\alpha$ for $i=1,2,...,n$. and hence $U(P,f)=L(P,f)=\alpha(b-a)$. Therefore $\underline{\int_a^b}f(x)dx=\overline{\int_a^b}f(x)dx=\alpha(b-a)$. This implies that f is integrable and $\int_a^b f(x)dx=\overline{\int_a^b}f(x)dx=\alpha(b-a)$.

   (b) Let $P=\{x_0,x_1,...,x_n\}$ be any partition of $[0,1]$ and $\frac{1}{2}\in[x_{i-1},x_i]$. Then $L(P,f)=1-x_i$ and $U(P,f)=10\Delta x_i+(1-x_i)$. Therefore $\underline{\int_a^b}f(x)dx=\overline{\int_a^b}f(x)dx=\frac{1}{2}$. This implies that f is integrable and $\int_a^b f(x)dx=\overline{\int_a^b}f(x)dx=\frac{1}{2}$.

   (c) Let $P_n=\left\{0,\frac{1}{n},\frac{2}{n},...,\frac{n-1}{n},\frac{n}{n}\right\}$. By definition $L(P_n,f)=\frac{(n-1)n}{2n^2}$ and $U(P_n,f)=\frac{n(n+1)}{2n^2}$. Therefore
   $\frac{1}{2}=\sup\{L(P_n,f):n\in\mathbb{N}\}\leq\underline{\int_a^b}f(x)dx\leq\overline{\int_a^b}f(x)dx\leq\inf\{U(P_n,f):n\in\mathbb{N}\}=\frac{1}{2}$.
   Therefore $\underline{\int_a^b}f(x)dx=\overline{\int_a^b}f(x)dx=\frac{1}{2}$ and $\int_a^b f(x)dx=\frac{1}{2}$.

5. Follows from $L(P_n,f)\leq\int_a^b f(x)dx\leq U(P_n,f)$.

6. (a) Let $P_n=\left\{0,\frac{1}{n},\frac{2}{n},...,\frac{n-1}{n},\frac{n}{n}\right\}$. Then $U(P_n,f)-L(P_n,f)=\frac{n}{n^2}-\frac{n-1}{n^2}\to 0$.

   (b) Let $P_n=\left\{0,\frac{1}{n},\frac{2}{n},...,\frac{n-1}{n},\frac{n}{n}\right\}$. Then $U(P_n,f)-L(P_n,f)=\frac{n^2}{n^3}-\frac{(n-1)^2}{n^3}\to 0$.

   (c) Let $P_n=\left\{1,1+\frac{1}{n},1+\frac{2}{n},...,1+\frac{n-1}{n},1+\frac{n}{n}\right\}$. Then $U(P_n,f)-L(P_n,f)=\frac{1}{2n}\to 0$.

7. For any partition P of $[0,1]$, $L(P,f_1)\leq L(P,f)$ and $U(P,f)\leq U(P_2,f)$ which implies that $\int_0^1 f_1(x)dx\leq\underline{\int_0^1}f(x)dx\leq\overline{\int_0^1}f(x)dx\leq\overline{\int_0^1}f_2(x)dx=\int_0^1 f_2(x)dx=\int_0^1 f_1(x)dx$.

8. If $P_n=\left\{0,\frac{1}{n},\frac{2}{n},...,\frac{n-1}{n},\frac{n}{n}\right\}$ then $\overline{\int_a^b}f(x)dx\leq\inf\{U(P_n,f):n\in\mathbb{N}\}=\frac{1}{2}$ (see the solution of Problem 4(c)). If $P=\{x_0,x_1,x_2,...,x_n\}$ be any partition of $[0,1]$, then
   $U(P,f)=\sum_{i=1}^n x_i(x_i-x_{i-1})\geq\sum_{i=1}^n x_i^2-\frac{1}{2}(\sum_{i=1}^n(x_i^2+x_{i-1}^2))=\frac{1}{2}(\sum_{i=1}^n(x_i^2-x_{i-1}^2))=\frac{1}{2}$
   which implies that $\overline{\int_a^b}f(x)dx\geq\frac{1}{2}$. Therefore $\overline{\int_a^b}f(x)dx=\frac{1}{2}$. It is clear that $\underline{\int_a^b}f(x)dx=0$.

9. (a) It is clear that A_N is finite.

   (b) Since the set A_N is finite, this is possible.

   (c) Let $\epsilon>0$. Choose N such that $\frac{1}{N}<\frac{\epsilon}{2}$. Corresponding to this N, choose the partition $P=\{0,x_1,x_2,x_3,...,x_n,1\}$ of $[0,1]$ where $x_i's$ are as given in (b).

   Observe that if $x\in[x_2,x_3]$ or $[x_4,x_5]$ and $f(x)=\frac{1}{q}$ then $q\geq N$ and hence on these intervals $M_j-m_j\leq\frac{1}{N}$.

   Note that
   $U(P,f)-L(P,f)=\sum(M_i-m_i)\Delta x_i=(|x_1-x_2|+|x_3-x_4|+...+|x_{m-1}-x_m|)+\frac{1}{N}<\epsilon$.

   This shows that f is integrable.

   (d) Define $g(0)=0$ and $g(x)=1$ if $x\in(0,1]$. Take $h=f$ where f is defined above.