Problem Set 3

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Draw and illustrate in $\mathbb{R}^2$.

$$(a)\ \mathbf{e}_1 + \{n\mathbf{e}_2 | n \in \mathbb{N}\}. \qquad (b)\ \mathbf{e}_1 + \{\alpha\mathbf{e}_2 | \alpha \in \mathbb{R}\}.$$

2. In $\mathbb{R}^2$, Is $\{\alpha\mathbf{e}_1 | \alpha \in \mathbb{R}\} + \{\alpha\mathbf{e}_2 | \alpha \in \mathbb{R}\} = \mathbb{R}^2$? What about $\{\alpha\mathbf{e}_1 | \alpha \in \mathbb{R}\} + \{\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} | \alpha \in \mathbb{R}\} = \mathbb{R}^2$?

3. In $\mathbb{R}^3$ prove that $\left\{\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} | \alpha \in \mathbb{R}\right\} + \left\{\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} | \alpha \in \mathbb{R}\right\} + \left\{\alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} | \alpha \in \mathbb{R}\right\} = \mathbb{R}^3$. Do you use Gauss-Jordan Elimination (GJE) method somewhere?

   Solution: Put $A = \{\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} | \alpha \in \mathbb{R}\}$, $B = \{\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} | \alpha \in \mathbb{R}\}$, $C = \{\alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} | \alpha \in \mathbb{R}\}$. Then $A + B + C = \{a + b + c | a \in A, b \in B, c \in C\} \subset \mathbb{R}^3$.

   Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$. We want to find α, β, γ s.t. $\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. That is, need to solve $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. We may use GJE to find the values of $\alpha = \frac{x_1 - x_2 + x_3}{2}, \beta = x_2 - x_3, \gamma = \frac{-x_1 + x_2 + x_3}{2}$. But without doing so, we may find the determinant and conclude that the system has a unique solution. But, we will need GJE for higher order vectors.

4. Let L_1 and L_2 be two nonparallel lines passing through origin in $\mathbb{R}^3$. What is $L_1 + L_2$?

5. **(T)** Fix a non-negative integer n and let $\mathbb{R}[x; n] = \left\{\sum\limits_{i=0}^{n} c_i x^i : c_0, c_1, \cdots, c_n \in \mathbb{R}\right\}$. Show that $\mathbb{R}[x; n]$ is a real vector space with respect to the usual addition and scalar multiplication.

   Solution: For $p(x) = \sum\limits_{i=0}^{n} a_i x^i, q(x) = \sum\limits_{i=0}^{n} b_i x^i, r(x) = \sum\limits_{i=0}^{n} c_i x^i$, we define

$$\text{[Vector Addition:]} \qquad (p + q)(x) = \sum_{i=0}^{n} (a_i + b_i) x^i \in \mathbb{R}[x; n]. \tag{1}$$

$$\text{[Scalar Multiplication:]} \qquad (\alpha p)(x) = \sum_{i=0}^{n} (\alpha a_i) x^i \in \mathbb{R}[x; n] \quad \text{for } \alpha \in \mathbb{R}. \tag{2}$$

   Then

   i. $p + q = q + p$ as $(p + q)(x) = \sum\limits_{i=0}^{n} (a_i + b_i) x^i = \sum\limits_{i=0}^{n} (b_i + a_i) x^i = (q + p)(x)$.

ii. $(p+q)+r = p+(q+r)$ as $(a_i+b_i)+c_i = a_i+(b_i+c_i)$ for $0 \le i \le n$.

iii. The *zero* polynomial, $z(x)=0$, satisfies $p+z=p$ as $a_i+0=a_i$ for $0 \le i \le n$.

iv. For all $p(x) \in \mathbb{R}[x;n]$, there is $(-p)(x) := \sum_{i=0}^{n}(-a_i)x^i$ such that $(p+(-p))(x) = 0 = z(x)$.

v. For all $\alpha, \beta \in \mathbb{R}$ and $p(x) \in \mathbb{R}[x;n]$, $(\alpha(\beta p))(x) = \sum_{i=0}^{n} \alpha(\beta a_i)x^i = \sum_{i=0}^{n}(\alpha\beta)a_i x^i = ((\alpha\beta)p)(x)$.

vi. For all $\alpha \in \mathbb{R}$, $\alpha(p+q) = \alpha p + \alpha q$.

vii. For all $\alpha, \beta \in \mathbb{R}$ $(\alpha+\beta)p = \alpha p + \beta p$.

viii. For all $p(x) \in \mathbb{R}[x;n]$, $1(p) = p$ as $(1p)(x) = \sum_{i=0}^{n}(1a_i)x^i = \sum_{i=0}^{n} a_i x^i = p(x)$.

6. Recall that $\mathbb{M}_n(\mathbb{R})$ is the real vector space of all $n \times n$ real matrices. Now, prove the following:

   (a) $\mathbb{S} = \{A \in \mathbb{M}_n(\mathbb{R}) : A^T = A\}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   (b) Fix $A \in \mathbb{M}_n(\mathbb{R})$. Define $\mathbb{U} = \{B \in \mathbb{M}_n(\mathbb{R}) : AB = BA\}$. Then, $\mathbb{U}$ is a subspace of $\mathbb{M}_n(\mathbb{R})$.

   (c) Let $\mathbb{W} = \{a_0 I + a_1 A + \cdots + a_m A^m : m \text{ is a non-negative integer}, a_i \in \mathbb{R}\}$. Then, $\mathbb{W}$ is a subspace of $\mathbb{U}$.

7. In $\mathbb{R}$, define $x \oplus y = x+y-1$ and $a \odot x = a(x-1)+1$. Show that $\mathbb{R}$ is a real vector space with respect to these operations with additive identity 1 (note that 0 is NOT the additive identity).

   Solution: Again, an easy verification of all vector space requirements.

8. **(T)** Which of the following are subspaces of $\mathbb{R}^3$:

   (a) $\{(x,y,z) \mid x \ge 0\}$, (b) $\{(x,y,z) \mid x+y=z\}$, (c) $\{(x,y,z) \mid x=y^2\}$.

   Solution:

   (a) Not a subspace : $-1(1,0,0)$ does not belong to the set.

   (b) Is a subspace.

   (c) Not a subspace : $(1,1,0)+(4,2,0)$ is not in the set. Since the relation is non-linear, closure is a problem.

9. Find the condition on $a,b,c,d \in \mathbb{R}$ so that $S = \{(x,y,z) \mid ax+by+cz=d\}$ is a subspace of $\mathbb{R}^3$.

   Solution: $d=0$. S is subspace $\Rightarrow (0,0,0) \in S \Rightarrow d=0$.

10. **(T)** Show that $S = \{(x_1,x_2,x_3,x_4) : x_4 - x_3 = x_2 - x_1\} = \mathrm{LS}(\{(1,0,0,-1),\ (0,1,0,1),\ (0,0,1,1)\}$.

    Solution: $(x_1,x_2,x_3,x_4) \in S \Rightarrow x_4 = -x_1+x_2+x_3$. Thus, $(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,-x_1+x_2+x_3) = x_1(1,0,0,-1)+x_2(0,1,0,1)+x_3(0,0,1,1)$.

11. **(T)** Let W_1 and W_2 be subspaces of a vector space V such that $W_1 \cup W_2$ is also a subspace. Prove that one of the spaces $W_i,\ i = 1, 2$ is contained in the other.

    Solution: Suppose W_1 is not a subset of W_2. To show: W_2 is a subset of W_1.

    Let $\mathbf{w}_2 \in W_2$. To show that W_2 is contained in W_1, we need to show that $\mathbf{w}_2 \in W_1$. Since $W_1 \not\subset W_2$, we can choose $\mathbf{w}_1 \in W_1$ such that $\mathbf{w}_1 \notin W_2$. Then $\mathbf{w}_2 - \mathbf{w}_1 \in W_1 \cup W_2$ as it is a subspace but $\mathbf{w}_2 - \mathbf{w}_1 \notin W_2$ because then $\mathbf{w}_1 = \mathbf{w}_2 - (\mathbf{w}_2 - \mathbf{w}_1) \in W_2$. So, $\mathbf{w}_2 - \mathbf{w}_1 \in W_1 \Rightarrow \mathbf{w}_2 = (\mathbf{w}_2 - \mathbf{w}_1) + \mathbf{w}_1 \in W_1$.

12. Let $S = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be a subset of a real vector space V. Define **linear span** of S as

    $$\mathrm{LS}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} : c_1, c_2, \ldots, c_n \in \mathbb{R}\},$$

    i.e., the set of all linear combinations of $\mathbf{v_1}, \ldots, \mathbf{v_n}$. Then $\mathrm{LS}(\{\mathbf{v_1}, \ldots, \mathbf{v_n}\})$ is a subspace of V.

    Solution: If $\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n}$ and $\mathbf{w} = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots d_n\mathbf{v_n}$, then

    $$\mathbf{u} + \mathbf{w} = (c_1 + d_1)\mathbf{v_1} + (c_2 + d_2)\mathbf{v_2} + \cdots (c_n + d_n)\mathbf{v_n} \in \mathrm{span}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\})$$

    and

    $$\alpha\mathbf{u} = (\alpha c_1)\mathbf{v_1} + (\alpha c_2)\mathbf{v_2} + \cdots (\alpha c_n)\mathbf{v_n} \in \mathrm{span}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\})$$

    for $\alpha \in \mathbb{R}$. Rest is straightforward.

13. Suppose S and T are two subspaces of a vector space V. *Define* the **sum**

    $$S + T = \{\mathbf{s} + \mathbf{t} : \mathbf{s} \in S, \mathbf{t} \in T\}.$$

    Show that $S + T$ satisfies the requirements for a vector space. Moreover, $LS(S \cup T) = S + T$.

    Solution: Straightforward to check all vector space requirements.

14. **(T)** Find all the subspaces of $\mathbb{R}^2$.

    Solution: $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^2$. Let $W \neq \{\mathbf{0}\}$ be a subspace of $\mathbb{R}^2$. Then there exists $(w_1, w_2) \in W$ with $(w_1, e_2) \neq \mathbf{0}$. If $LS(\{(w_1, w_2)\}) = W$, then W is a line through origin. If $LS(\{(w_1, w_2)\}) \subsetneq W$ then show that $W = \mathbb{R}^2$.
    Let $(u_1, u_2) \in W \setminus LS(\{(w_1, w_2)\})$. So, $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \alpha \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ for all $\alpha \in \mathbb{R}$. So, $A = \begin{bmatrix} w_1 & u_1 \\ w_2 & u_2 \end{bmatrix}$ is invertible. Therefore, the system $A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ in the unknowns α, β has a solution for each $(x, y) \in \mathbb{R}^2$ as A is invertible.

15. **(T)** Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, with $a_{ij} \in \mathbb{C}$. Then the 4 fundamental subspaces are:

(a) The column space of A:

$$\text{col}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{C}^n\} = \text{LS}(A[:,1], \ldots, A[:,n]) = \text{LS}\left(\left\{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}\right\}\right)$$

(b) The column space of A^*:

$$\text{col}(A^*) = \text{LS}(A^*[1,:], \ldots, A^*[m,:]) = \{A^*\mathbf{x} : \mathbf{x} \in \mathbb{C}^m\}.$$

(c) The null space of A:

$$\text{Null Space}(A) = \mathcal{N}(A) = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\}.$$

(d) The null space of A^*:

$$\text{Null Space}(A^*) = \mathcal{N}(A^*) = \{\mathbf{x} \in \mathbb{C}^m : A^*\mathbf{x} = \mathbf{0}\}.$$

Important: In case $A \in \mathbb{M}_{m,n}(\mathbb{R})$, the spaces col$(A^*)$ and Null Space(A^*) are called the row-space of A and the left-null space of A, respectively

Now, determine the above 4 mentioned fundamental spaces for the following matrices.

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

(iii) Suppose $B, C \in \mathbb{M}_{m,n}(\mathbb{C})$ and $S = \text{col}(B)$, $T = \text{col}(C)$. Determine $M \in \mathbb{M}_{m,n}(\mathbb{C})$ such that $\text{col}(M) = S + T$.

Solution: (i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{RREF}(A)$. So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \mathcal{N}(A) = \text{LS}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$.

(iii) Let $M = \begin{bmatrix} B & C \end{bmatrix} \in \mathbb{M}_{m,2n}(\mathbb{C})$. It is easy to see that if $\mathbf{u} \in \text{col}(M)$ then $\mathbf{u} \in S + T$. Similarly, if $\mathbf{u} \in S + T$ then $\mathbf{u} = \mathbf{s} + \mathbf{t}$ where $\mathbf{s} \in \text{col}(B)$ and $\mathbf{t} \in \text{col}(C) \Rightarrow \mathbf{u} \in \text{col}(M)$.

16. Construct A such that $[\ 1\ \ 1\ \ 1\ ]^T \in \text{col}(A)$ and $\mathcal{N}(A) = \text{LS}\left([\ 1\ \ 1\ \ 1\ \ 1\ ]^T\right)$.

Solution: Clearly, the matrix we are looking for is a 3×4 matrix with rank 3. Two such matrices are

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & -3 \\ 1 & 2 & 3 & -6 \\ 1 & 4 & 9 & -14 \end{bmatrix}.$$

17. **(T)** Suppose A is an m by n matrix of rank r.

(a) If $A\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$, what is the column space of A?

Solution: There must be a pivot in every row, so $r = m$ and so $\text{col}(A) = \mathbb{R}^m$.

(b) In part (a), what are all the relations between the numbers m, n and r?

Solution: Using (a), we know that $r = m$. The rank $r \leq n$. Hence $r = m \leq n$.

(c) Give a specific example of a 3 by 2 matrix A of rank 1 with first row $[\,2 \quad 5\,]$. Describe the column space, $\text{col}(A)$, and the null space, $N(A)$, completely.

Solution: Just use multiples of [2 5] for the other rows. For example, $\begin{bmatrix} 2 & 5 \\ 4 & 10 \\ 0 & 0 \end{bmatrix}$. Column space will be the line in $\mathbb{R}^3$ consisting of all multiples of your first column. The null space will be the line in $\mathbb{R}^2$ consisting of all multiples of the null space solution $\begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$.

(d) Suppose the right side $\mathbf{b}$ is same as the first column in your example (part c). Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

Solution: Adding the particular solution $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the null space solution from (c), we get the complete solution $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -5/2 \\ 1 \end{bmatrix}$.

18. Suppose $R = \text{RREF}(A)$, where $A = \begin{bmatrix} 1 & 2 & 1 & b \\ 2 & a & 1 & 8 \\ & (row & 3) & \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) What can you say immediately about row 3 of A?

Solution: As $R[3,:] = [0,0,0,0]$, $A[3,:] = \alpha A[1,:] + \beta A[2,:]$ for some $\alpha, \beta \in \mathbb{C}$.

(b) What are the numbers a and b?

Solution: After one step of elimination, A reduces to $\begin{bmatrix} 1 & 2 & 1 & b \\ 0 & a-4 & -1 & 8-2b \\ & (row & 3) & \end{bmatrix}$. Then comparing with R gives $a = 4$. Now, multiplying the second row by -1 and comparing gives $b = 5$.

(c) Describe all solutions of $R\mathbf{x} = \mathbf{0}$. Which among row spaces, column spaces and null spaces are the same for A and for R.

Solution: Setting x_2 and x_4 as free variables gives the solution of $R\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

The row space and the null space are always the same for A and R whereas column space is different (row operations preserve row space but change column space).

19. Let $A \in \mathbb{M}_n(\mathbb{R})$. Show that $\mathcal{N}(A) \subset \mathcal{N}(A^2) \subset \mathcal{N}(A^3) \cdots$. What if $A = \begin{bmatrix} 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$?

Solution: Let $\mathbf{x} \in \mathcal{N}(A)$. Then $A\mathbf{x} = \mathbf{0} \Rightarrow A^2\mathbf{x} = A(A\mathbf{x}) = \mathbf{0}$. Thus $\mathbf{x} \in \mathcal{N}(A^2)$. In general, it can be shown that $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1}) = \ldots$.

20. **(T)** Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. If $\text{RREF}(A) = \begin{pmatrix} I_r & F \\ 0 & 0 \end{pmatrix}$ then describe $\text{col}(A)$ and $\mathcal{N}(A)$.

Solution: $\text{col}(A)$ is the space of all vectors whose last $m-r$ coordinates are zero. This is clear since $\text{rank}(A) = r$. Further , the first r columns of A are independent as $\text{RREF}(A)$ has I_r in its first block. Denoting by f_{ij} the entry in the the (i,j) position in F. Then $\mathcal{N}(A)$ is the space of all linear combinations of the $n-r$ vectors

$$\begin{bmatrix} -f_{11} \\ -f_{21} \\ \vdots \\ -f_{r1} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -f_{12} \\ -f_{22} \\ \vdots \\ -f_{r2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} -f_{1(n-r)} \\ -f_{2(n-r)} \\ \vdots \\ -f_{r(n-r)} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Clearly, these vectors are linearly independent. Thus $\dim(\mathcal{N}(A)) = n-r$.

21. **(T)** Let $W_1 = \text{span}\left\{[1\ 1\ 0]^T, [-1\ 1\ 0]^T\right\}$ and $W_2 = \text{span}\left\{[1\ 0\ 2]^T, [-1\ 0\ 4]^T\right\}$. Show that $W_1 + W_2 = \mathbb{R}^3$. Give an example of a vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v}$ can be written in two different ways in the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in W_1, \mathbf{v}_2 \in W_2$.

Solution: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \subseteq W_1 + W_2$ and is linearly independent which means $W_1 + W_2 = \mathbb{R}^3$. Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \in W_2$, we have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in W_1 + W_2$. Note that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \in W_1$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{5}{6}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \in W_2$, so

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in W_1 + W_2.$$