MSO202A COMPLEX VARIABLES
ASSIGNMENT–7

Practice Problems: Solutions will not be posted!

1. Find the residue at $z = 0$ of the following functions and indicate the type of singularity they have at 0. (i) $\dfrac{\sin z}{z^2 - \pi^2}$ at $z = \pi$, (ii) $\dfrac{\sin z}{(z-\pi)^2}$ at $z = \pi$ (iii) $\dfrac{z\cos z}{1-\sin z}$ at $z = \pi/2$.

2. Use Cauchy's residue theorem to evaluate the integral of each of the following functions around the circle $|z| = 3$. (a) $\dfrac{e^{-z}}{z^2}$, (b) $\dfrac{e^{-z}}{(z-1)^2}$, (c) $z^2 e^{\frac{1}{z}}$ and $(d)\dfrac{z+1}{z^2-2z}$.

3. Compute the following integrals:

   (a) $\int_{|z|=1} e^z z^{-n} dz;\quad n \in \mathbb{Z};$

   (b) $\int_{|z|=1} \frac{\cos z}{\sin z} dz$

   (c)
$$\int_{|z|=1} (z - \frac{1}{z})^n \frac{dz}{z} = \begin{cases} 2\pi i\binom{n}{n/2}(-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

   Use it to show that

$$\int_0^{2\pi} \sin^n t\; dt = \begin{cases} \frac{\pi}{2^{n-1}}\binom{n}{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

4. Find the isolated singularities and compute the residue of the functions

$$\frac{e^z}{z^2-1}, \qquad \frac{3z}{z^2+iz+2}, \qquad \cot \pi z.$$

5. Let $f(z) = \dfrac{\pi \cot \pi z}{(z+\frac{1}{2})^2}$. Compute the residue of f at isolated singularities.

6. Use Cauchy's residue theorem to evaluate the integral of each of the following functions around the circle $|z| = 3$. (a) $\dfrac{e^{-z}}{z^2}$, (b) $\dfrac{e^{-z}}{(z-1)^2}$, (c) $z^2 e^{\frac{1}{z}}$ and $(d)\dfrac{z+1}{z^2-2z}$.

7. Use the residue integration method to evaluate:
   (i) $\int_0^{\pi} \frac{d\theta}{2+\cos\theta}$ (ii) $\int_0^{2\pi} \frac{d\theta}{8-2\sin\theta}$.

8. Evaluate: (a) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^4+1}\,dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+5)^2}$ (c) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+16}$.

9. Evaluate: (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin 3x}{x^2+a^2}\,dx, a > 0$ (c) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n}, n \geq 1$.

10. Evaluate: (a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)}$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx$ (c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1}\,dx$.