MSO202A COMPLEX ANALYSIS
Assignment 5

Exercise Problems:

1. Evaluate the integral $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{ze^{zt}}{(z+1)^3}\,dz$ where C is a counter-clockwise oriented simple closed contour enclosing $z=-1$.

2. Write down the Taylor series centred at the given point for the following functions and find its disc of convergence:
   (i) $f(z) = \dfrac{1}{z^2}$ at $z_0 \neq 0$ (ii) $f(z) = \dfrac{6z+8}{(2z+3)(4z+5)}$ at $z_0 = 1$
   (iii) $f(z) = \dfrac{e^z}{z+1}$ at $z_0 = 1$.

3. Let $f, g \colon \mathbb{C} \to \mathbb{C}$ be analytic functions such that $f(a_n) = g(a_n), n = 1, 2, \ldots$ for a bounded sequence of distinct complex numbers. Show that $f \equiv g$ on $\mathbb{C}$.

4. Derive the Taylor series representation of $\frac{1}{1-z}$ around i.
   $$\frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \text{ where } |z-i| < \sqrt{2}.$$

5. Let f be analytic in a simply connected domain D and γ be a simple closed curve in D oriented counterclockwise. Suppose z_0 is the only zero of f in the region enclosed by γ. Show that $\int_\gamma \frac{f'(z)}{f(z)}dz = 2\pi i m$ where m is the order of zero of f at z_0. (*If $f(z) = (z-z_0)^m g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$ then f is said to have a zero of order m.*)

6. (Mean Value Theorem) Let D be a simply connected domain and $f : D \to \mathbb{C}$ be an analytic function. Then prove that $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$ for every $r > 0$ such that $B(z_0, r)$ is contained in D.

7. Find the maximum of the function $|f|$ on $\overline{\mathbb{D}}$ if (a) $f(z) = z^2 - z$ (b) $f(z) = \sin z$.

   Problem for Tutorial:

8. Let f be entire and $|f(z)| \leq a + b|z|^n$ for some positive real numbers a and b and $n \in \mathbb{N}$. Show that f is a polynomial of degree at most n.

9. Let $f \colon \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Let $z_0 \in \mathbb{C}$ and $r > 0$ be arbitrary. Show that the image of f intersects the disc $B_r(z_0) = \{z : |z - z_0| < r\}$. (Hence image of a non-constant analytic function intersects every disc in $\mathbb{C}$.)

10. Let f and g be nonzero analytic functions defined on the disc $\mathbb{D}$ with $|f(z)| \leq |g(z)| \forall z$. Assume that z_0 is a zero for $g(z)$ of order n. Show that z_0 is a zero for $f(z)$ of order at least n.