Continuous-Time Fourier Transform

- The frequency-domain representation of a periodic signal composed of a weighted sum of sinusoidal signals with harmonically related frequencies is given by its Continuous-Time Fourier Series
- A generalization of this representation is given by the Continuous-Time Fourier Transform (CTFT)

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Continuous-Time Fourier Transform

- The CTFT represents the frequency-domain representation of a certain class of aperiodic analog signals as a weighted combination of infinite number of complex exponentials whose frequencies are infinitesimally close to each other with the integral replaced by a sum

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Continuous-Time Fourier Transform

Definition

- The continuous-time Fourier transform of an aperiodic signal $x(t)$ is given by

$$\boxed{X(j\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}\,dt}$$

- The signal $x(t)$ can be recovered from its CTFT $X(j\Omega)$ by the Fourier integral

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)e^{j\Omega t}\,d\Omega}$$

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Continuous-Time Fourier Transform

- Note: Ω is a real variable denoting the continuous-time angular frequency in radians per sec.
- The inverse CTFT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)e^{j\Omega t}d\Omega$ can be interpreted as a linear weighted sum with weights $X(j\Omega)$ of complex exponentials of the form

$$\frac{1}{2\pi}e^{j\Omega t}d\Omega$$

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Continuous-Time Fourier Transform

- We shall denote the CTFT and the inverse CTFT pair in compact form as

$$\boxed{x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(j\Omega)}$$

Example – The CTFT $\mathcal{D}(j\Omega)$ of the unit impulse function $\delta(t)$ is given by

$$\mathcal{D}(j\Omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\Omega t}\,dt = 1$$

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Continuous-Time Fourier Transform

Example – The CTFT $X(j\Omega)$ of the time-shifted unit impulse function $x(t) = \delta(t - t_o)$ is given by

$$X(j\Omega) = \int_{-\infty}^{+\infty} \delta(t - t_o)e^{-j\Omega t}\,dt = e^{-j\Omega t_o}$$

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Continuous-Time Fourier Transform

Example – We determine the CTFT $X(j\Omega)$ of the real-valued analog signal

$$x(t) = e^{-\alpha t}\mu(t), \quad 0 < \alpha < \infty$$

$$X(j\Omega) = \int_{-\infty}^{+\infty} e^{-\alpha t}\mu(t)e^{-j\Omega t}dt = \int_{0}^{+\infty} e^{-\alpha t}e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{+\infty} e^{-(\alpha+j\Omega)t}dt = \frac{1}{\alpha + j\Omega}$$

Commonly Used CTFT Pairs

$$\boxed{\delta(t) \overset{\text{CTFT}}{\leftrightarrow} 1}$$

$$\boxed{1 \overset{\text{CTFT}}{\leftrightarrow} 2\pi\delta(\Omega)}$$

$$\boxed{\delta(t - t_o) \overset{\text{CTFT}}{\leftrightarrow} e^{-j\Omega t_o}}$$

Commonly Used CTFT Pairs

$$\boxed{\mu(t) \overset{\text{CTFT}}{\leftrightarrow} \pi\delta(\Omega) + \frac{1}{j\Omega}}$$

$$\boxed{e^{-\alpha t}\mu(t) \overset{\text{CTFT}}{\leftrightarrow} \frac{1}{\alpha + j\Omega}, \; \mathcal{Re}\{\alpha\} > 0}$$

$$\boxed{e^{j\Omega_o t} \overset{\text{CTFT}}{\leftrightarrow} 2\pi\delta(\Omega - \Omega_o)}$$

Continuous-Time Fourier Transform

- The CTFT of an analog signal is sometimes referred to as the Fourier spectrum, or simply, the spectrum
- In general, the CTFT $X(j\Omega)$ of an analog signal $x(t)$ is a complex function of the real variable Ω in the range $-\infty < \Omega < \infty$ and can be expressed either in rectangular or polar form

Parts of CTFT

Magnitude and Phase Spectra

- The complex CTFT in polar form is given by

$$X(j\Omega) = |X(j\Omega)|e^{j\theta(\Omega)}$$

  where $\theta(\Omega) = \arg\{X(\Omega)\}$
- $|X(j\Omega)|$ is called the magnitude spectrum
- $\theta(\Omega)$ is called the phase spectrum

Parts of CTFT

- $|X(j\Omega)|$ and $\theta(\Omega)$ are real functions of Ω
- For a real analog signal

$$|X(-j\Omega)| = |X(j\Omega)|$$

$$\theta(-\Omega) = -\theta(\Omega)$$

- Note: Phase spectra can not be uniquely specified for all values of Ω as demonstrated next

Parts of CTFT

- Define a new CTFT $X_1(j\Omega) = X(j\Omega)e^{j2\pi k}$
- Now, $X_1(j\Omega) = |X(j\Omega)|e^{j\theta(\Omega)}e^{j2\pi k}$

$$= |X(j\Omega)|e^{j\theta(\Omega)} = X(j\Omega)$$

- For uniqueness, $\theta(\Omega)$ is restricted to the range

$$-\pi < \theta(\Omega) \le \pi$$

called the principal value of phase function

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Parts of CTFT

Real and Imaginary Parts

- The complex CTFT in rectangular form is given by

$$X(j\Omega) = X_{re}(j\Omega) + jX_{im}(j\Omega)$$

where $X_{re}(j\Omega)$ and $X_{im}(j\Omega)$ are real and imaginary parts of $X(j\Omega)$, and are real functions of Ω

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Parts of CTFT

- For a real analog signal,

$$X_{re}(-j\Omega) = X_{re}(j\Omega)$$
$$X_{im}(-j\Omega) = -X_{im}(j\Omega)$$

Example – Consider the CTFT $X(j\Omega) = \frac{1}{\alpha + j\Omega}$ of $e^{-\alpha t}\mu(t)$, $\mathcal{R}e(\alpha) > 0$

- Here, $|X(j\Omega)| = \left|\frac{1}{\alpha + j\Omega}\right| = \frac{1}{\sqrt{\alpha^2 + \Omega^2}}$

$$\theta(\Omega) = -\tan^{-1}(\Omega/\alpha)$$

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Parts of CTFT

- Note: $|X(-j\Omega)| = \frac{1}{\sqrt{\alpha^2 + (-\Omega)^2}} = \frac{1}{\sqrt{\alpha^2 + \Omega^2}} = |X(j\Omega)|$

$$\theta(-\Omega) = -\tan^{-1}(-\Omega/\alpha) = \tan^{-1}(\Omega/\alpha) = -\theta(\Omega)$$

- Thus, $|X(j\Omega)|$ is an even function of Ω and $\theta(\Omega)$ is and odd function of Ω, as expected
- Plots of $|X(j\Omega)|$ and $\theta(\Omega)$ shown in the next slide for α = 0.6/sec also verifies the above result

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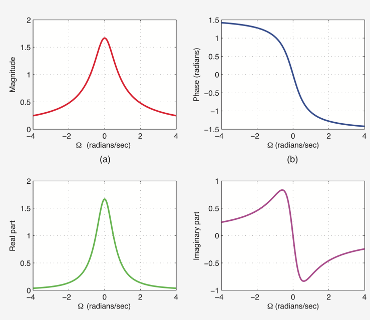
Parts of CTFT



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Parts of CTFT

- Next, we rewrite $X(j\Omega)$ as

$$X(j\Omega) = \frac{1}{\alpha + j\Omega} = \frac{\alpha - j\Omega}{(\alpha + j\Omega)(\alpha - j\Omega)} = \frac{\alpha - j\Omega}{\alpha^2 + \Omega^2}$$
$$= \left(\frac{\alpha}{\alpha^2 + \Omega^2}\right) - j\left(\frac{\Omega}{\alpha^2 + \Omega^2}\right)$$

- Hence, $X_{re}(j\Omega) = \frac{\alpha}{\alpha^2 + \Omega^2}$

$$X_{im}(j\Omega) = -\frac{\Omega}{\alpha^2 + \Omega^2}$$

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Parts of CTFT

- Note:

$$X_{re}(-j\Omega) = \frac{\alpha}{\alpha^2 + (-\Omega)^2} = \frac{\alpha}{\alpha^2 + \Omega^2} = X_{re}(j\Omega)$$

$$X_{im}(-j\Omega) = -\frac{(-\Omega)}{\alpha^2 + (-\Omega)^2} = \frac{\Omega}{\alpha^2 + \Omega^2} = -X_{im}(j\Omega)$$

- The above properties of the real and imaginary parts can also be seen from the plots given in Slide No. 17 for α = 0.6/sec

CTFT Computation Using MATLAB

- The function `freqs` can be used to evaluate a rational CTFT
- The code fragments used to develop the plots shown in Slide No. 17 are given in the next slide

CTFT Computation Using MATLAB

```
w = -4:0.01:4;
h = freqs(1,[1  0.6],w);
mag = abs(h);
phase = angle(h);
Re = real(h);
Im = imag(h);
```

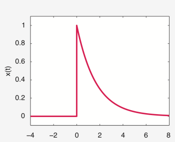

Uniform Convergence Conditions

Given by Dirichlet conditions

- (1) $x(t)$ is absolutely integrable
- (2) $x(t)$ has a finite number of maxima and minima, and a finite number of discontinuities

Uniform Convergence Conditions

Example - $x(t) = e^{-\alpha t}\mu(t)$, $\mathcal{R}e\{\alpha\} > 0$, is absolutely integrable and it has one discontinuity at $t = 0$ and one maximum at $t = 0$



- Thus, its CTFT $X(j\Omega) = \frac{1}{\alpha + j\Omega}$ converges uniformly

Mean-Square Convergence Conditions

- A finite energy $x(t)$ is square integrable but not absolutely integrable
- Its CTFT exists in a mean-square sense
- Let $X(j\Omega)$ denotes its CTFT with $\breve{x}(t)$ denoting its inverse
- Then

$$\int_{-\infty}^{+\infty} \left|\breve{x}(\tau) - x(\tau)\right|^2 d\tau \to 0$$

CTFT Using Delta Functions

- The CTFT can also be defined using ideal delta functions for a number of analog signals that are neither absolutely summable nor square-summable

Example – Consider $x(t)$ with a CTFT given by

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_o)$$

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CTFT Using Delta Functions

- The inverse CTFT of the above CTFT is thus given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\Omega - \Omega_o)e^{j\Omega t}dt = e^{j\Omega_o t}$$

obtained using the sampling property of the delta function

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CTFT Properties

- The CTFT has a number of properties that are often useful in determining the CTFTs of various analog signals that one may encounter in some applications
- We list these properties without any proofs

$$x(t) \overset{\text{CTFT}}{\leftrightarrow} X(j\Omega)$$

$$y(t) \overset{\text{CTFT}}{\leftrightarrow} Y(j\Omega)$$

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CTFT Properties

- Linearity: $\alpha x(t) + \beta y(t) \overset{\text{CTFT}}{\leftrightarrow} \alpha X(j\Omega) + \beta Y(j\Omega)$
- Conjugation: $x^*(t) \overset{\text{CTFT}}{\leftrightarrow} X^*(j\Omega)$
- Time-reversal: $x(-t) \overset{\text{CTFT}}{\leftrightarrow} X(-j\Omega)$
- Time-shifting: $x(t - t_o) \overset{\text{CTFT}}{\leftrightarrow} e^{-j\Omega t_o}X(j\Omega)$

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CTFT Properties

- Frequency-shifting:

$$e^{j\Omega_o t}x(t) \overset{\text{CTFT}}{\leftrightarrow} X(j(\Omega - \Omega_o))$$

- Differentiation-in-frequency:

$$(-jt)x(t) \overset{\text{CTFT}}{\leftrightarrow} \frac{X(j\Omega)}{d\Omega}$$

- Differentiation-in-time: $\frac{dx(t)}{dt} \overset{\text{CTFT}}{\leftrightarrow} j\Omega X(j\Omega)$

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CTFT Properties

- Integration:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{CTFT}}{\leftrightarrow} \frac{dX(j\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$$

- **Example** – Recall

$$\mu(t) = \int_{-\infty}^{t}\delta(\tau)d\tau$$

- Now the CTFT of $\delta(t)$ is given by

$$\mathcal{D}(j\Omega) = 1$$

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CTFT Properties

- Using the integration property of the CTFT we arrive at the CTFT of $\mu(t)$:

$$\mathcal{M}(j\Omega) = \frac{\mathcal{D}(j\Omega)}{j\Omega} + \pi\mathcal{D}(0)\delta(\Omega)$$

$$= \frac{1}{j\Omega} + \pi\delta(\Omega)$$

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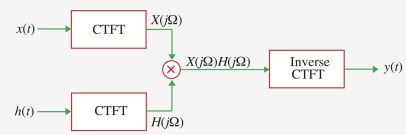
CTFT Properties

- Convolution:

$$x(t) \circledast y(t) \overset{\text{CTFT}}{\longleftrightarrow} X(j\Omega)Y(j\Omega)$$

- Implementation of the convolution integral using the CTFT-based approach



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CTFT Properties

- Modulation:

$$x(t)y(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Psi)Y\big(j(\Omega-\Psi)\big)d\Psi$$

- Parseval's Relation:

$$\int_{-\infty}^{+\infty} |g(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |G(j\Omega)|^2\,d\Omega$$

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CTFT Properties

- **Example** – We determine the inverse CTFT $x(t)$ of

$$X(j\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

- The total energy of $x(t)$ obtained using the Parseval's relation is

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2\,d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} d\Omega = \frac{\Omega_c}{\pi}$$

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CTFT Properties

- Hence, $x(t)$ is a square-integrable analog signal
- The inverse CTFT of $X(j\Omega)$ is

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}\,d\Omega = \frac{\sin(\Omega_c t)}{\pi t}$$

- Note: $x(t)$ is not absolutely integrable

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Band-Limited Analog Signals

- An ideal band-limited analog signal has a CTFT $X(j\Omega)$ that is zero outside a finite frequency range $\Omega_1 \le |\Omega| \le \Omega_2$, that is

$$X(j\Omega) = \begin{cases} 0, & 0 \le |\Omega| < \Omega_1 \\ 0, & \Omega_2 < |\Omega| < \infty \end{cases}$$

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Band-Limited Analog Signals

- An analog signal is said to have a band-limited spectrum if most of its energy is concentrated in a given finite frequency range
- **Lowpass Analog Signal** – has most of its energy in the low frequency range

$$0 \le |\Omega| < \Omega_c < \infty$$

The frequency range from dc to Ω_c is called its bandwidth

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Band-Limited Analog Signals

- **Highpass Analog Signal** – has most of its energy in the high frequency range

$$0 < \Omega_c \le |\Omega| < \infty$$

The frequency range from Ω_c to ∞ is called its bandwidth

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Band-Limited Analog Signals

- **Bandpass Analog Signal** – has most of its energy in the mid frequency range

$$0 < \Omega_{c1} \le |\Omega| \le \Omega_{c2} < \infty$$

The frequency range $\Omega_{c2} - \Omega_{c1}$ is called its bandwidth
- A precise definition of the bandwidth is given by the percent of the total energy in the specified frequency range which depends on applications

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