Department of Mathematics and Statistics Indian Institute of Technology - Kanpur

Problem Set 2

Problems marked **(T)** are for discussions in Tutorial sessions.

1. **(T)** Are the matrices $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ row-equivalent?

2. Supply two examples each and explain their geometrical meaning.

   (a) Two linear equations in two variables with exactly one solution.

   (b) Two linear equations in two variables with infinitely many solutions.

   (c) Two linear equations in two variables with no solutions.

   (d) Three linear equations in two variables with exactly one solution.

   (e) Three linear equations in two variables with no solutions.

3. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are two distinct solutions of the system $A\mathbf{x} = \mathbf{b}$. Prove that there are infinitely many solutions to this system, by showing that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is also a solution, for each $\lambda \in \mathbb{R}$. Do you have a geometric interpretation?

4. Let B be a square invertible matrix. Then, prove that the system $A\mathbf{x} = \mathbf{b}$ and $BA\mathbf{x} = B\mathbf{b}$ are row-equivalent.

5. **[T]** Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have same solutions for every $\mathbf{b}$. Is it true that $A = C$?

6. **[T]** Find matrices A and B with the given property or explain why you can not:

   (a) The only solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (b) The only solution to $B\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

7. Using Gauss Jordan method, find the inverse of $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$.

8. **(T)** Let $B \in \mathbb{M}_n(\mathbb{R})$ be a real skew-symmetric matrix. Show that $I - B$ is non singular.

9. Let A be an $n \times n$ matrix. Prove that

   (a) If $A^2 = \mathbf{0}$ then A is singular.

   (b) If $A^2 = A, A \neq I$ then A is singular.

10. Can $RREF([A|\mathbf{b}]) = \left[\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $\left[\begin{array}{ccc|c} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ or $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$? Explain.
Now, recall the matrices A_j's, for $1 \le j \le 3$ (defined to state the Cramer's rule for solving the linear system $A\mathbf{x} = \mathbf{b}$), that are obtained by replacing the j-th column of A by $\mathbf{b}$. Then, we see that the above system has **NO solution** even though $\det(A) = 0 = \det(A_j)$, for $1 \le j \le 3$.

11. Let A be an $n \times n$ matrix. Prove that the following statements are equivalent:

    (a) $\det(A) \neq 0$.

    (b) A is invertible.

    (c) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

    (d) The row-reduced echelon form of A is I_n.

    (e) A is a product of elementary matrices.

    (f) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every .

    (g) The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$.

12. $A \in \mathbb{M}_n(\mathbb{C})$. Then $\det(A) = 0$ if and only if the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

13. **(T)** Let A be an $n \times n$ matrix. Then, the two statements given below cannot hold together.

    (a) The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

    (b) The system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

14. Suppose the 4×4 matrix M has 4 equal rows all containing a, b, c, d. We know that $det(M) = 0$. The problem is to find by any method

$$det(I + M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}.$$

15. **(T)** If $A \in \mathbb{M}_n(\mathbb{C})$ then $\overline{\det(A)} = \det(A^*)$. Therefore if A is Hermitian, *i.e.*, $A^* = A$) then $\det(A)$ is a real number.

16. The numbers $1375, 1287, 4191$ and 5731 are all divisible by 11. Prove that 11 also divides the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 7 & 8 & 9 & 3 \\ 5 & 7 & 1 & 1 \end{bmatrix}.$$

17. Compute determinant of $\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^4 \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 \end{bmatrix}$.