Problem Set 6

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Find the eigenvalues and corresponding eigenvectors of matrices given below.

   (a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix}$

2. **(T)** Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$. Verify that $\mathbf{x}^T = (c, c, 0)$ is the eigenspace for $\lambda = 1$.

   (a) The above eigenspace is the null space of what matrix constructed from A?

   (b) Find the other two eigenvalues of A and two corresponding eigenvectors.

   (c) The diagonalization $A = S\Lambda S^{-1}$ has a specially nice form because $A = A^t$. Is S orthogonal? If not, can we make it orthogonal?

3. Let A be an $n \times n$ invertible matrix. Show that eigenvalues of A^{-1} are reciprocal of the eigenvalues of A, moreover, A and A^{-1} have the same eigenvectors.

4. Let A be an $n \times n$ matrix and α be a scalar. Find the eigenvalues of $A - \alpha I$ in terms of eigenvalues of A. Further show that A and $A - \alpha I$ have the same eigenvectors.

5. **(T)** Let A be an $n \times n$ matrix. Show that A^T and A have the same eigenvalues. Do they have the same eigenvectors?

6. Let A be an $n \times n$ matrix. Show that:

   (a) If A is idempotent ($A^2 = A$) then eigenvalues of A are either 0 or 1.

   (b) If A is nilpotent ($A^m = \mathbf{0}$ for some $m \geq 1$) then all eigenvalues of A are 0.

   (c) If $A^* = A$ then, the eigenvalues are all real.

   (d) If $A^* = -A$ then, the eigenvalues are either zero or purely imaginary.

   (e) Let A be a unitary matrix ($AA^* = I = A^*A$). Then, the eigenvalues of A have absolute value 1. It follows that if A is real orthogonal then the eigenvalues of A have absolute value 1. Give an example to show that the conclusion may be false if we allow **complex orthogonal**.

7. **(T)** Suppose that $A_{5\times 5}^{15} = \mathbf{0}$. Show that there exists a unitary matrix U such that U^*AU is upper triangular with diagonal entries 0. Further, show that $A^5 = \mathbf{0}$.

8. The matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is NOT diagonalizable, whereas $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable.

9. Show that Hermitian, Skew-Hermitian and unitary matrices are normal.

10. Suppose that $A = A^*$. Show that rank$A$ = number of nonzero eigenvalues of A. Is this true for each square matrix? Is this true for each square symmetric complex matrix?

11. **(T)** Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{R}^n$. If S is linearly independent then show that the matrix $A = \sum\limits_{j=1}^{k} \mathbf{u}_j \mathbf{u}_j^T$ has 0 as an eigenvalue of multiplicity $n - k$. Show that Rank$(A) = k$?

12. **(T)** Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$. Find S such that $S^{-1}AS$ is diagonal. Also, compute A^6.

13. Consider the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ 1 & d & e \\ 0 & 1 & f \end{bmatrix}.$$

Determine the entries a, b, c, d, e, f so that:

- the top left 1×1 block is a matrix with eigenvalue 2;
- the top left 2×2 block is a matrix with eigenvalue 3 and -3;
- the top left 3×3 block is a matrix with eigenvalue 0, 1 and -2.

14. **NOT for mid-sem or end-sem**

(a) Find the eigenvalues and eigenvectors (depending on c) of

$$A = \begin{bmatrix} 0.3 & c \\ 0.7 & 1 - c \end{bmatrix}.$$

For which value of c is the matrix A not diagonallizable (so $A = S\Lambda S^{-1}$ is impossible)?

(b) What is the largest range of values of c (real number) so that A^n approaches a limiting matrix A^∞ as $n \to \infty$?

(c) What is the limit of A^n (still depending on c)? You could work from $A = S\Lambda S^{-1}$ to find A^n.