Digital Signals in the Time Domain

- Most signals encountered in practice are analog signals
- However, increasingly these analog signals are being converted into digital form and are being processed using digital systems
- The processed digital signals are then converted back into analog form

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Digital Signals in the Time Domain

- A digital signal $\{x[n]\}$ is represented in the time-domain as a sequence of numbers called samples where each sample $x[n]$ is a function of the integer-valued variable n with $-\infty < n < +\infty$
- The integer-valued variable n is often called time

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Digital Signals in the Time Domain

$$\{x[n]\} = \{\cdots, -1,\ 2,\ \underset{\uparrow}{5},\ 1,\ -3,\ 7, \cdots\}$$

- The arrow indicates the location of the sample at $n = 0$
- If there is no ambiguity, a digital signal may be shown without the braces
- We shall often refer to a digital signal as a sequence in this course

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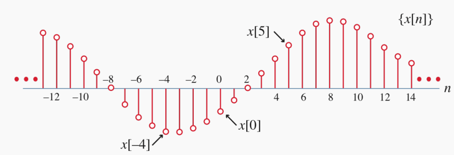
Digital Signals in the Time Domain

- A digital signal also sometimes represented in a graphical form as indicated below



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Digital Signals in the Time Domain

- As mentioned earlier, an analog signal $x_a(t)$ is converted into a digital signal $\{x[n]\}$ by sampling the former periodically at uniform time intervals

$$\boxed{x[n] = x_a(t)\big|_{t=nT} = x_a(nT),\ -\infty < n < +\infty}$$

- The spacing T between two consecutive samples of $\{x[n]\}$ is called the sampling period

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Digital Signals in the Time Domain

- The reciprocal of the sampling period is the sampling frequency:

$$\boxed{F_T = \frac{1}{T}}$$

- Sampling frequency is in Hz if the sampling period is in seconds
- For a real digital signal $\{x[n]\}$, all samples are real-valued and for a complex digital signal $\{x[n]\}$, one or more samples are complex-valued

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Digital Signals in the Time Domain

- A complex digital signal $\{x[n]\}$ is written in the form

$$\boxed{\{x[n]\} = \underbrace{\{x_{re}[n]\}}_{\text{Real part}} + j\underbrace{\{x_{im}[n]\}}_{\text{Imaginary part}}}$$

- $\{x_{re}[n]\}$ and $\{x_{im}[n]\}$ are real sequences
- Complex conjugate of $\{x[n]\}$ is

$$\boxed{\{x^*[n]\} = \{x_{re}[n]\} - j\{x_{im}[n]\}}$$

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Finite- and Infinite-Length Signals

- An infinite-length digital signal is called a right-sided signal if the amplitude of its samples is equal to zero for all values of time index n less than a finite integer N_1
- A right-sided signal is called a causal signal if $N_1 = 0$

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Finite- and Infinite-Length Signals

- An infinite-length digital signal is called a left-sided signal if the amplitude of its samples is equal to zero for all values of time index n greater than a finite integer N_2
- A left-sided signal is called an anti-causal signal if $N_2 = 0$

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Finite- and Infinite-Length Digital Signals

- The amplitudes of the samples of a finite-length signal, also called a finite-duration or finite-extent signal, are equal to zero for all values of time index n less than a finite value N_1 and greater than a finite value N_2 with $N_1 < N_2$
- Length or duration N of the finite-length signal is $N = N_2 - N_1 + 1$

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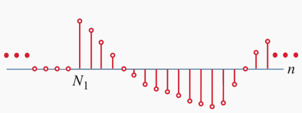
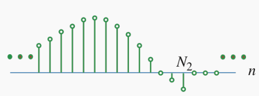
Finite- and Infinite-Length Signals

- Examples of right-sided and left-sided digital signals are shown below



Right-sided signal



Left-sided signal

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Finite- and Infinite-Length Signals

- A two-sided digital signal is defined for both positive and negative values of time index n
- An example of a finite-length two-sided digital signal is shown below

$$\{y[n]\} = \{-3, \quad 0.4, \quad -2, \quad 3.1, \quad 5\} \quad -1 \le n \le 3$$

- Length of the above signal is $3 - (-1) + 1 = 5$

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Finite- and Infinite-Length Signals

- For a right-sided finite-length digital signal defined for $n \geq N_1$ and without the arrow shown, the first sample of the signal is at time index $n = N_1$
- For a causal signal, the first sample is at time index $n = 0$

Periodic and Aperiodic Signals

- An infinite-length digital signal $\tilde{x}[n]$ is called a periodic signal, if for a positive integer N_o, the signal satisfies the condition

$$\boxed{\tilde{x}[n] = \tilde{x}[n + N_o]}$$

for all values of n

Periodic and Aperiodic Signals

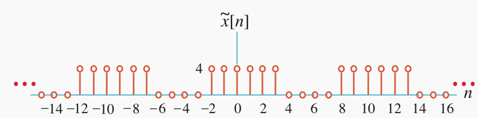
- Note: If the periodicity condition holds for a value of N_o, then

$$\tilde{x}[n + N_o] = \tilde{x}[n + kN_o]$$

where k is a positive integer
- The smallest value of N_o satisfying the periodicity condition is called the fundamental period of $\tilde{x}[n]$, which is the length of one full cycle of the signal

Periodic and Aperiodic Signals

- The square wave signal shown below is an example of a periodic digital signal



Periodic and Aperiodic Signals

- The fundamental frequency, also called cyclic frequency f_o, of the periodic signal $\tilde{x}[n]$ with a fundamental period N_o is defined by

$$\boxed{f_o = \frac{1}{N_o}}$$

which is the number of full cycles in one period

Unit of the fundamental frequency is given by Hz, if the time index n is in seconds

Periodic and Aperiodic Signals

- A periodic digital signal is often characterized by its angular frequency which is given by

$$\boxed{\Omega_o = 2\pi f_o = \frac{2\pi}{N_o}}$$

- The angular frequency Ω_o is in radians per second if N_o is in seconds
- Note: One full cycle of the periodic signal contains 2π radians

Periodic and Aperiodic Signals

- A digital signal is called an aperiodic signal if it is not periodic, that is, there is no value of the constant N_o satisfying the periodicity condition
- A periodic digital signal is denoted with a tilde "~" on top

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Bounded Signals

- A digital signal $x[n]$ is said to be a bounded digital signal if it satisfies the condition

$$\boxed{|x[n]| \le B_x < \infty \,,\, -\infty < n < +\infty}$$

where B_x is a finite real positive number

- The stability of certain types of digital systems is usually defined in terms of the boundedness of signals

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Bounded Signals

- We shall assume in this course that the amplitude of a sample of a digital signal can take any value between $-\infty$ and $+\infty$
- In practice, the signals at the input, output, and anywhere internal to the system are restricted to specific ranges, known as dynamic ranges

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Bounded Signals

- It is thus necessary to ensure that these signals remain as bounded signals lying in the specified dynamic ranges which can be achieved by scaling the signals appropriately
- If the sample amplitude of any of these signals at any time index n exceeds their corresponding dynamic ranges, the pertinent signal is clipped resulting in severe distortion in the processed output signal

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Energy of a Digital Signal

- The total energy of the digital signal $x[n]$ is defined by

$$\boxed{\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2}$$

- The energy of a digital signal over a finite duration $-K \le n \le K$ is defined by

$$\boxed{\mathcal{E}_{x,K} = \sum_{n=-K}^{K} |x[n]|^2}$$

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Power and RMS Value of a Digital Signal

- The average power of the digital signal $x[n]$ is defined by

$$\boxed{\mathcal{P}_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|}$$

- The root-mean square (rms) value of a digital signal $x[n]$ is given by the square-root of its average power $\mathcal{P}_x$, that is

$$x_{rms} = \sqrt{\mathcal{P}_x}$$

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Energy and Power of Digital Signals

- The relation between the average power and the energy of a digital signal is

$$\boxed{\mathcal{P}_x = \lim_{K\to\infty} \frac{1}{2K+1}\mathcal{E}_x}$$

- It follows from the above relation that the average power $\mathcal{P}_x$ of a finite energy digital signal is 0

Energy and Power of Digital Signals

- The average power of a periodic digital signal $\tilde{x}[n]$ with a fundamental period N_o is given by

$$\boxed{\mathcal{P}_{\tilde{x}} = \frac{1}{N_o}\sum_{n=0}^{N_o-1} |\tilde{x}[n]|^2}$$

- A digital signal has infinite energy if the summation $\sum_{n=-\infty}^{\infty} |x[n]|^2$ does not converge to a finite number

Unit Sample Sequence

- The unit sample sequence $\delta[n]$, also known as the unit impulse sequence, is a discrete-time function whose sample is equal to zero for all values of n except $n = 0$ where it has unity value, that is,

$$\boxed{\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}}$$

Unit Sample Sequence

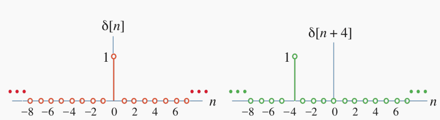
- It follows from the definition,

$$\boxed{\delta[n - N_o] = \begin{cases} 1, & n = N_o \\ 0, & n \neq N_o \end{cases}}$$

- Thus, $\delta[n - N_o]$ is an unit impulse sequence located at $n = N_o$

Unit Sample Sequence

- Figures below show the sequence $\delta[n]$ and the sequence $\delta[n+4]$



Unit Step Sequence

- The unit step sequence denoted by $\mu[n]$ is a causal signal defined by

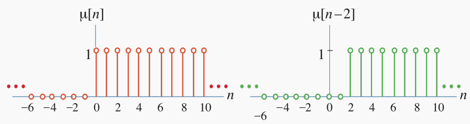
$$\boxed{\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}}$$

- It follows from above

$$\boxed{\mu[n - N_o] = \begin{cases} 1, & n \geq N_o \\ 0, & n < N_o \end{cases}}$$

Unit Step Sequence

- Figures below show the sequence $\mu[n]$ and the sequence $\mu[n-2]$



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Sinusoidal Sequence

- The most general form of the sinusoidal sequence is given by

$$\boxed{\tilde{x}[n] = A_o \sin(2\pi f_o n + \phi_o), \; -\infty < n < \infty}$$

where A_o, f_o, and ϕ_o are real numbers and are called, respectively, the peak amplitude, normalized frequency, and phase

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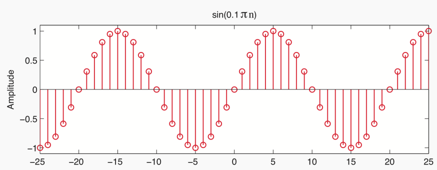
Sinusoidal Sequence

- Figure below show the plot of $\tilde{x}[n] = \sin(0.1\pi n)$



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Sinusoidal Sequence

- An alternate form of the basic sinusoidal digital signal is

$$\tilde{x}[n] = A_o \sin(\omega_o n + \phi_o), \; -\infty < n < \infty$$

where $\boxed{\omega_o = 2\pi f_o}$ is known as the normalized angular frequency

- Unit of ω_o and ϕ_o is radians if n is dimensionless

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Sinusoidal Sequence

- Units of ω_o and ϕ_o is radians per sample, and the unit of f_o is cycles per sample if n is samples
- Units of ω_o and ϕ_o is radians/second, and the unit of f_o is cycles per second or Hz, if the spacing T between two consecutive samples is in seconds

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Sinusoidal Sequence

Periodicity Property

- Unlike the sinusoidal analog signal, the sinusoidal sequence may or may not be a periodic digital signal
- The sinusoidal sequence will be a periodic signal of period N_o if and only if $2\pi f_o N_o$ is an integer multiple of 2π, that is,

$$\boxed{2\pi f_o N_o = 2\pi r}$$

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Sinusoidal Sequence

or equivalently, if

$$\boxed{\frac{2\pi}{\omega_o} = \frac{N_o}{r}}$$

where N_o and r are positive integers

- The smallest value of N_o satisfying the above condition is called the fundamental period

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Sinusoidal Sequence

- If $2\pi/\omega_o$ is a noninteger rational number, then the period N_o of the sinusoidal sequence will be an integer multiple of $2\pi/\omega_o$
- If $2\pi/\omega_o$ is not a rational number, then the sinusoidal sequence is aperiodic but with a sinusoidal envelope

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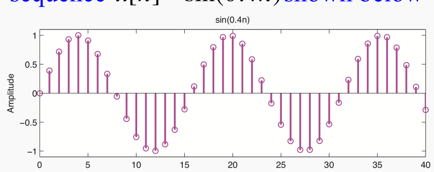
Sinusoidal Sequence

- **Example** – Consider the sinusoidal sequence $x[n] = \sin(0.4n)$ shown below



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Sinusoidal Sequence

- Here the condition $\omega_o N_o = 2\pi r$ leads to $N_o = 2\pi r/0.4 = 5\pi r$
- It can be seen that there is no integer value of r to make N_o an integer
- Hence $x[n] = \sin(0.4n)$ is an aperiodic sinusoidal sequence

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Sinusoidal Sequence

Example – We determine the fundamental period of $\tilde{x}[n] = \sin(0.1\pi n)$

- Here $\omega_o = 0.1\pi$
- Substituting this value in $\omega_o N_o = 2\pi r$ we get $0.1\pi N_o = 2\pi r$ or

$$N_o = \frac{2r}{0.1}$$

which is satisfied for $N_o = 20$ and $r = 1$ which can be verified from the plot

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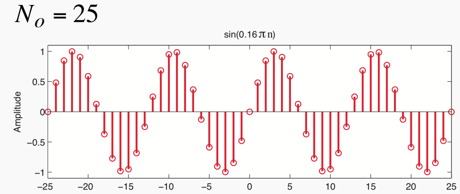
Sinusoidal Sequence

- Example – It can be shown that the fundamental period of $\tilde{x}[n] = \sin(0.16\pi n)$ is $N_o = 25$



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Sinusoidal Sequence

Restriction on the Frequency Range

- Recall that that the angular frequency Ω_o of a sinusoidal analog signal can have any value in the range $0 \le \Omega_o < +\infty$
- Unlike the analog case, the normalized angular frequency ω_o of a sinusoidal digital signal can have a value in a restricted range

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Sinusoidal Sequence

- Consider two sinusoidal sequences given by $\tilde{x}_a[n] = \sin(\omega_a n)$ and $\tilde{x}_b[n] = \sin(\omega_b n)$ with $0 \le \omega_a < 2\pi$ and $\omega_b = \omega_a + 2\pi k$ where k is any positive integer
- We note

$$\tilde{x}_b[n] = \sin(\omega_b n) = \sin(\omega_a n + 2\pi k n)$$
$$= \sin(\omega_a n)\cos(2\pi k n) + \cos(\omega_a n)\sin(2\pi k n)$$

- Now $\cos(2\pi k n) = 1$ and $\sin(2\pi k n) = 0$

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Sinusoidal Sequence

- Hence $\tilde{x}_b[n] = \sin(\omega_a n) = \tilde{x}_a[n]$
- As a result, the two sinusoidal sequences $\tilde{x}_a[n]$ and $\tilde{x}_b[n]$ are indistinguishable
- Another restriction on the frequency range is due to an inherent property of sine and cosine functions which limits the angular frequency to the range $0 \le \omega < \pi$

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Sinusoidal Sequence

- Consider $\tilde{x}_a[n] = \sin(\omega_a n)$ and $\tilde{x}_b[n] = \sin(\omega_b n)$ with $0 \le \omega_a < \pi$ and $\omega_b = 2\pi - \omega_a$
- We note

$$\tilde{x}_b[n] = \sin(\omega_b n) = \sin((2\pi - \omega_a)n)$$
$$= \sin(2\pi n)\cos(\omega_a n) - \cos(2\pi n)\sin(\omega_a n)$$

- Now $\cos(2\pi k n) = 1$ and $\sin(2\pi k n) = 0$
- Hence $\tilde{x}_b[n] = -\sin(\omega_a n) = -\tilde{x}_a[n]$

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Sinusoidal Sequence

- Thus, a sinusoidal sequence with a normalized angular frequency ω_b in the range $\pi \le \omega_b < 2\pi$ becomes a sinusoidal sequence with a normalized angular frequency $\omega_a = \pi - \omega_b$
- The angular frequency π is referred to as the folding frequency

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Sinusoidal Sequence

- Hence, the allowable range of the normalized angular frequency of a sinusoidal digital signal is $0 \le \omega < \pi$
- Frequencies with small values near dc $(\omega = 0)$ are called low frequencies
- Frequencies with high values near π are called the high frequencies

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Sinusoidal Sequence

Example – The sequence $\sin(0.2\pi n)$ is a low-frequency digital signal, whereas, the sequence $\sin(0.8\pi n)$ is a high-frequency digital signal

Complex Exponential Sequence

- Most general form

$$\boxed{x[n] = A\alpha^n, \; -\infty < n < +\infty}$$

where A and α are complex numbers

- Expressing the parameter A in polar form and α in rectangular form:

$$A = |A|e^{j\phi}, \quad \alpha = \sigma_o + j\omega_o$$

Complex Exponential Sequence

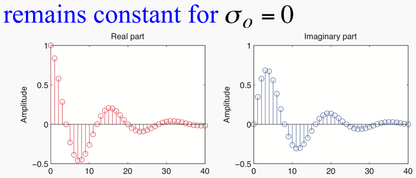
we arrive at

$$x[n] = A\alpha^n = |A|e^{j\phi}e^{(\sigma_o + j\omega_o)n} = |A|e^{j\phi}e^{\sigma_o n}e^{j\omega_o n}$$
$$= |A|e^{j\phi}e^{\sigma_o n}(\cos\omega_o n + j\sin\omega_o n)$$
$$= \underbrace{|A|e^{\sigma_o n}\cos(\omega n + \phi)}_{Re\{x[n]\}} + j\underbrace{|A|e^{\sigma_o n}\sin(\omega n + \phi)}_{Im\{x[n]\}}$$

- The real and imaginary parts of the complex sequence $x[n]$ are real sequences with amplitudes $|A|e^{\sigma_o n}$

Complex Exponential Sequence

- For $n > 0$, the sample amplitudes of the complex exponential sequence are growing for $\sigma_o > 0$, decaying for $\sigma_o < 0$, and remains constant for $\sigma_o = 0$



Complex Exponential Sequence

- A complex exponential sequence $Ae^{j\omega_o n}$ is a periodic sequence with a period N_o, if the condition $\omega_o N_o = 2\pi r$ is satisfied

Real Exponential Sequence

- The sequence $x[n] = A\alpha^n, \; -\infty < n < +\infty$, is a real exponential sequence when A and α are real numbers