Complex Analysis - MSO202A

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Contents

A *complex number* is an ordered pair of real numbers (x, y).
x is called the *real* part.
y is called the *imaginary* part.

$(x, y) = x(1, 0) + y(0, 1)$. (Recall that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$)
Denoting $(0, 1)$ as i we have the representation $(x, y) = x + iy$. This is the representation we use!!

The real and imaginary parts don't interact over $+$.

$x + iy = x' + iy'$ if and only if $x = x'$, $y = y'$.

• $\mathbb{C} = \mathbb{R}^2$

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$$(x+iy)+(x'+iy') = x+x'+i(y+y').$$

$\mathbb{R}^2$ is a vector space $/\mathbb{R}$

$$(x,y)+(x',y') = (x+x', y+y')$$

From now on ....

Definition of a complex number

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The set of all complex numbers is denoted by $\mathbb{C}$.

- $\mathbb{C}$ is a $\mathbb{R}$-vector space.
- Consider the maps
  $Re : \mathbb{C} \to \mathbb{R}$ given by $x + iy \mapsto x$.
  $Im : \mathbb{C} \to \mathbb{R}$ given by $x + iy \mapsto y$.

  They are $\mathbb{R}$-linear.
- $\mathbb{C}$ is a field: $(x_1 + iy_1) + (x_2 + iy_2) := x_1 + x_2 + i(y_1 + y_2)$
  $(x_1 + iy_1).(x_2 + iy_2) := x_1x_2 - (y_1y_2) + i(x_1y_2 + x_2y_1)$

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  $Re(\alpha \cdot x+iy)$
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makes the big difference b/w $\mathbb{R}^2$ & $\mathbb{C}$.

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algebraic

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Geometry of $\mathbb{C}$.

- $z = x + iy$ is the vector $(\mathbf{x}, \mathbf{y})$ on the $\mathbb{R}^2$-plane (or complex plane).
- length of the z is the length of the line segment from $(\mathbf{0}, \mathbf{0})$ to $(\mathbf{x}, \mathbf{y})$. Denoted as $|z|$.

Length of z

For $z = x + iy$, we have $|z| = \sqrt{x^2 + y^2}$.
It is referred also as *modulus of z*.

- Reflection of z about the x-axis is called the conjugate of z, denoted as $\bar{z}$.

Conjugate of z

For $z = x + iy$, we have $\bar{z} = x - iy$

- Note that $|z|^2 = z.\bar{z} = \bar{z}.z$.
  And, $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$.

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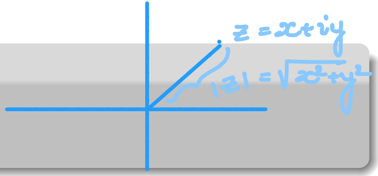

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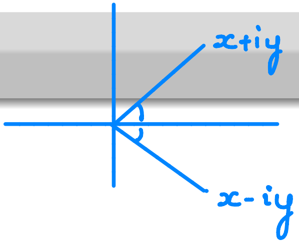

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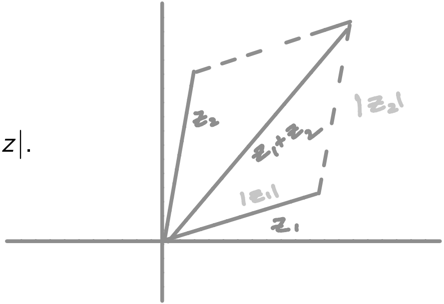
} relating $|z|$ and $\bar{z}$ via the algebraic structures.

Properties of $\bar{z}$ and $|z|$

- $Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2i}$;
  $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$.
- $|z| = |\bar{z}|$.
- $|z| = 0$ if and only if $z = 0$.
- $|z_1 z_2| = |z_1||z_2|$, $\quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- Triangle inequality: $\quad |z_1 + z_2| \leq |z_1| + |z_2|$.
- Parallelogram identity: $\quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

- $\frac{(x+iy)+(x-iy)}{2} = \frac{2x}{2} = Re(x+iy)$
- $|x| \leq \sqrt{x^2+y^2}$

Properties of $\bar{z}$ and $|z|$



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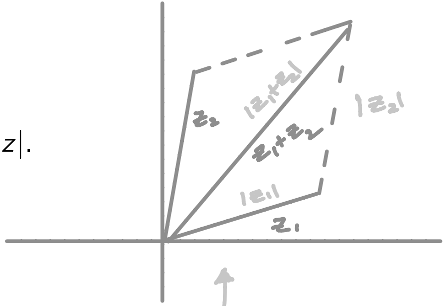

Recall that every point on the real plane (except $(0,0)$) can be written in the form $(r\cos\theta, r\sin\theta)$ for a suitable $r \in \mathbb{R}^{>0}$ and $\theta \in \mathbb{R}$. This gives rise to the polar form of a complex number.

The polar form of a complex number $z \neq 0$ is the representation of z in its polar co-ordinates.

Polar form of $z = x + iy$

If $(x, y) = (r\cos\theta, r\sin\theta)$ in polar co-ordinates then $r(\cos\theta + i\sin\theta)$ is the polar form of z.

Notation

$e^{i\theta} := \cos\theta + i\sin\theta$.
So, polar form of $z = re^{i\theta}$.

Properties of polar form

- $re^{i\theta} = re^{i(\theta+2\pi n)}$ for any integer n.
- $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$.
- $\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$.

- $\cos\theta + i\sin\theta$
$= \cos(\theta+2\pi n) + i\sin(\theta+2\pi n)$

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- $r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 \Big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$

$+ i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\Big]$

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$$= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \right]$$

$$= r_1 r_2 \quad \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \; .$$

de Moivre's Theorem

If m is any integer then

$$(e^{i\theta})^m = e^{im\theta},$$

$$i.e., \ (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta.$$

The roots of $\omega^n = z = re^{i\theta}$ are given by $\sqrt[n]{r}e^{i(\theta+2k\pi)/n}$ where $k = 0, 1, \ldots, n-1$.

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$\omega = se^{i\phi}$

$s^n e^{in\phi} = re^{i\theta}$

$\Rightarrow s^n = r, \quad n\phi = \theta + 2\pi k$

+ve real no: (exists (?))

$(\because e^{i\theta} = 1 \Leftrightarrow \cos\theta = 1 \ \sin\theta = 0$
$\Leftrightarrow \theta = 2k\pi, k \in \mathbb{Z})$

$k_1 = k_2 \bmod n \Rightarrow e^{\frac{i(2\pi(k_1-k_2))}{n}} = e^{i2\pi p} = 1$

$\Rightarrow e^{\frac{i(\theta+2\pi k_1)}{n}} = e^{\frac{i(\theta+2\pi k_2)}{n}}$

$k_1 = k_2 \bmod n$

ie $k_1 - k_2 = pn$, $p \in \mathbb{Z}$

then $2\pi(k_1-k_2) = 2\pi pn$

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$k_1 = k_2 \bmod n$
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if $k_1, k_2 \in \{0, 1, \ldots, n-1\}$

$e^{i\frac{(2\pi k_1+\theta)}{n}} \neq e^{i\frac{(2\pi k_2+\theta)}{n}}$

$\because e^{\frac{i2\pi(k_1-k_2)}{n}} = 1$

$\Rightarrow \frac{2\pi(k_1-k_2)}{n} = 2\pi p$

$\Rightarrow k_1 - k_2 = pn. \Rightarrow k_1 = k_2$

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$\omega = s e^{i\phi}$

$s^n e^{in\phi} = r e^{i\theta}$

$\Rightarrow s^n = r, \quad n\phi = \theta + 2\pi k$

+ve real no: