MTH 102A ASSIGNMENT-07 (ODE) **Feb 25 & 27, 2020.**

1. Classify each of the following differential equations in terms of its linearity and specify its order:
   (a) $y' + x^2 y = xe^x$.
   (b) $y^{(3)} + 4y'' - 5y' + 3y = \sin x$.
   (c) $x^2 dy + y^2 dx = 0$.
   (d) $y^{(4)} + 3(y'')^5 + 5y = 0$.
   (e) $y'' + y \sin x = 0$.
   (f) $y'' + x \sin y = 0$.
   (g) $\frac{d^6x}{dt^6} + \left(\frac{d^4x}{dt^4}\right)\left(\frac{d^3x}{dt^3}\right) + x = t$.
   (h) $\left(\frac{dx}{dt}\right)^3 = \sqrt{\frac{d^2x}{dt^2} + 1}$.

2. (a) Show that the function $(1+x^2)^{-1}$ is a solution of the ODE $(1+x^2)y''+4xy'+2y = 0$ in every open interval $I \subset \mathbb{R}$.
   (b) Show that $x^3 + 3xy^2 = 1$ is an implicit solution of the ODE $2xyy' + x^2 + y^2 = 0$ in the interval $(0, 1)$.

3. (a) Show that the *one* parameter family of functions $(x^3 + c)e^{-3x}$ is a solution of the *first* order ODE $y' + 3y = 3x^2e^{-3x}$.
   (b) Show that the *three* parameter family of functions $c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}$ is a solution of the *third* order ODE $y^{(3)} - 2y'' - 4y' + 8y = 0$.
   (c) Determine the values of m such that $u(x) := e^{mx}$ is a solution of the ODE $y^{(3)} - 3y'' - 4y' + 12y = 0$.

4. (a) Show that $y = (x + c)^{-1}$ is a family of solution for the nonlinear ODE $y' + y^2 = 0$.
   (b) Find the particular solution and the interval on which the particular solution is valid corresponding to the initial conditions:
      i. $y(0) = 5$.
      ii. $y(2) = -1/5$.

5. (a) Show that the BVP

$$\begin{cases} y'' + y &= 0 \\ y(0) &= 0 \\ y(\pi) &= 1 \end{cases}$$

   has no solution of the form $y(x) := \alpha_1 \sin x + \alpha_2 \cos x$. (`The fact that this BVP has no solution will follow from later lectures!`)

   (b) Show that there exists a $h > 0$ such that the IVP

$$\begin{cases} y' &= x^2 \sin y \\ y(1) &= -2 \end{cases}$$

   admits a unique solution in the interval $|x - 1| \leq h$.

6. Analyse the existence and uniqueness of solution for the IVP

$$\begin{cases} (x^2 - 2x)y' &= 2(x-1)y \\ y(x_0) &= y_0. \end{cases}$$

7. Let $I \subset \mathbb{R}$ be an open interval and $Q : I \to \mathbb{R}$ be a continuous function.
   (a) Show that $y \equiv 0$ on I is a solution of the linear homogeneous ODE $y' + Q(x)y = 0$.
   (b) Show that if u is a solution of the ODE such that $u(x_0) = 0$ for some $x_0 \in I$ then $u \equiv 0$ on I.
   (c) If u and v are two solutions of the ODE such that $u(x_0) = v(x_0)$ for some $x_0 \in I$ then $u(x) = v(x)$ for all $x \in I$.
   (d) Show that the set of all solutions of $y' + Q(x)y = 0$ in I form a vector space over $\mathbb{R}$. Can a similar conclusion be made for any k-th order linear homogeneous ODE?
   (e) What is the dimension of the vector space of solutions of $y' + Q(x)y = 0$?

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