MSO202A COMPLEX VARIABLES
ASSIGNMENT–6

Problems for Discussion:

1. Let $f_j\colon \mathbb{C} \to \mathbb{C}, j = 1, 2$ be analytic functions such that $f_1(a_n) = f_2(a_n)$ for a bounded sequence of complex numbers. Show that the two functions are the same.

   Solution: Recall: Bolzano-Weierstrass theorem which says that a bounded sequence of real numbers has a convergent subsequence. Thus, a bounded sequence of complex numbers also has a convergent subsequence.

   Let $z_k = a_{n_k}$ be a convergent subsequence of $\{a_n\}$ converging to $z_0 \in \mathbb{C}$. . Since $(f_1 - f_2)(z_n) = 0$ for all n and since $f_1 - f_2$ is analytic, and hence continuous, we have $(f_1 - f_2)(z_0) = 0$. But then z_0 is a non isolated zero of an analytic function in $\mathbb{C}$, which is not possible unless $f \equiv 0$ in $\mathbb{C}$.

2. Verify the maximum modulus principle in $|z| \leq 1$ for the following functions: (a) $z^2 - 3z + 2$ (b) $z^2 - z$ (c) $\sin z$.

   Solution: In all the cases the function is continuous on $|z| \leq 1$ and analytic inside the unit circle. The maximum will thus occur on the unit circle in all these cases.

   (a) $|f(z)| = |z^2 - 3z + 2| \leq 6$. Let $z = e^{i\theta}$. Then $|f|^2 = 14 - 18\cos\theta + 4\cos 2\theta$. $|f|$ will have a maximum value of 6 at $z = -1$. Otherwise, observe that at $z = -1$ gives the value 6.

   In (b), we get the maximum value of $|f|$ to be 2 at $z = -1$.

   (c) $f(z) = \sin z$. $|\sin z| \leq \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} = \frac{1}{2}(e - \frac{1}{e})$.

   When $z = i, |\sin z| = \frac{1}{2}(e - \frac{1}{e})$.

3. Find the Laurent series of the function $f(z) = \exp(z + \frac{1}{z})$ around 0. Hence show that $\displaystyle \frac{1}{\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta d\theta = \sum_{j=0}^{\infty} \frac{1}{(n+j)!j!}$.

   Solution: $\displaystyle e^{(z+\frac{1}{z})} = \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{j=0}^{\infty} \frac{z^{-j}}{j!} = \sum_{k,j=0}^{\infty} \frac{z^{k-j}}{k!j!}$. Make a change of variable, $k - j = l$, and collect the coefficient of z^l is $\displaystyle \sum_{j=0}^{\infty} \frac{1}{(l+j)!j!}$, $l \geq 0$, and by symmetry, z^{-l} is $\displaystyle \sum_{j=0}^{\infty} \frac{1}{(l+j)!j!}$, $l < 0$. (e.g., power -3 arises as coming from powers -7 and 4, -8 and 5 and so on, the coefficients for which are same as for 7 and -4, 8 and -5 etc.)

   On the other hand, f is analytic on the puncture plane $\mathbb{C} \setminus 0$. By Cauchy's Integral formula $\displaystyle a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}} dz$ where $C : |z| = 1$.

   So, $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos(n\theta)\, d\theta = \sum_{j=0}^{\infty} \frac{1}{(n+j)!j!}$.

4. Expand the given function in a Laurent series that converge for $0 < |z| < R$ and determine the precise region of convergence.
(a) $1/(z^4 - z^5)$ (b) e^{-z}/z^3 (c) $z^{-3}e^{1/z^2}$.

Solution:

$$\frac{1}{z^4 - z^5} = \frac{1}{z^4}\frac{1}{1-z} = \frac{1}{z^4}[1 + z + z^2 + z^3 + \cdots], \quad 0 < |z| < 1.$$

(ii)

$$\frac{e^{-z}}{z^3} = \frac{1}{z^3}\left[1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + - \cdots\right], \quad 0 < |z| < \infty.$$

(iii)

$$z^{-3}e^{1/z^2} = \frac{1}{z^3}\left[1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \frac{1}{3!z^6} + \cdots\right], \quad 0 < |z| < \infty.$$

5. Expand the given function in a Laurent series that converge for $0 < |z - z_0| < R$ and determine the precise region of convergence.
(a) $e^z/(z-1), z_0 = 1$ (b) $1/z^2 + 1, z_0 = i$ (c) $(z^2 - 4)/(z-1), z_0 = 1$.

Solution: (i)

$$\frac{e^z}{z-1} = \frac{ee^{z-1}}{(z-1)} = \frac{e}{(z-1)}\left[1 + z - 1 + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3} + \cdots\right]$$

$$= \frac{e}{(z-1)} + e\left[1 + \frac{z-1}{2!} + \frac{(z-1)^2}{3!} + \cdots\right], \quad |z-1| > 0.$$

(ii)

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{z-i} \times \frac{1}{2i + z - i}$$

$$= \frac{-i}{2(z-i)}\left[1 - \frac{(z-i)}{2i} + \frac{(z-i)^2}{4i^2} - \frac{(z-i)^3}{8i^3} + - \cdots\right], \quad 0 < |z-i| < 2.$$

(iii)

$$\frac{z^2 - 4}{z-1} = \frac{(z-1)^2 - +2(z-1) - 3}{z-1}$$

$$= -\frac{3}{z-1} + +2 + (z-1).$$

6. Find the first three terms of the Laurent expansion in powers of z valid in the region $0 < |z| < \pi$ for the function $f(z) = \dfrac{1}{z^2 \sin z}$. Compute the integral $\displaystyle\int_C \frac{1}{z^2 \sin z}\,dz$ where C is a positively oriented curve in the unit disc enclosing 0.

Solution: $f(z) = \frac{1}{z^2 \sin z} == \frac{1}{z^3(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots)} = \frac{1}{z^3}g(z)$, where g is an analytic function in $|z| < \pi$. Now, g has a Taylor series expansion $\sum_{n=0}^{\infty} d_n z^n, |z| < \pi$. Thus $1 = (1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots)(d_0 + d_1 z + d_2 z^2 + \cdots)$. Solving for d_i we get, $g(z) = 1 + \frac{z^2}{3!} + \frac{7z^4}{360} + \cdots$ and so $f(z) = \frac{1}{z^3} + \frac{1}{6z} + \frac{7z}{360} + \cdots$. Therefore, the integral is $\frac{\pi i}{3}$.

7. Determine and classify as pole/essential singularity of the functions, in case of pole find the order of the pole:
   $(a)\ \dfrac{\sin z}{z^2-\pi^2}\ (b)\ \dfrac{1}{z}+\dfrac{1}{z^3}\ (c)\ \dfrac{\cos z}{z^2}+\sin z.$

   Solution: (a) Since $z = \pm\pi$ is a simple pole.

   (b)$z = 0$ is a pole of order 3.

   (c) $z = 0$ is a pole of order 2.

Problems for Tutorial:

1. Let f and g be nonzero analytic functions defined on Ω with $|f(z)| \leq |g(z)| \forall z \in \Omega$. Assume that z_0 is a zero for $g(z)$ of order n. Show that z_0 is a zero for $f(z)$ of order at least n. Hence conclude that f/g is analytic at z_0. What is the value of $(f/g)(z_0)$?

   Solution: Assume that z_0 is a zero for g of order n. Then $g(z) = (z-z_0)^n h(z)$, where h is analytic on $\mathbb{D}$ and $h(z_0) \neq 0$.

   Since $|f(z_0)| \leq |g(z_0)| = 0$, z_0 is a zero of f. Let m be its order. Then $f(z) = (z-z_0)^m \phi(z)$, for some analytic function ϕ with $\phi(z_0) \neq 0$. As $|f(z)| \leq |g(z)| \Rightarrow |(z-z_0)|^m |\phi(z)| \leq |z-z_0|^n |h(z)|$. If $m < n$, it implies $n - m > 0$, and so $\phi(z_0) = 0$, a contradiction. Therefore, $m \geq n$.

   Again $g(z) = (z-z_0)^n h(z)$, where h is analytic on $\mathbb{D}$ and $h(z_0) \neq 0$.

   As h is a continuous function, and $h(z_0) \neq 0$ there exists $r > 0$ s.t. $h(z)$ is never equal to zero in $B_r(z_0)$.

   Similarly, $f(z) = (z-z_0)^m \phi(z)$, with ϕ analytic on $\mathbb{D}$ and $\phi(z_0) \neq 0$ implies there exists $R > 0$ s.t. $\phi(z)$ is never equal to zero in $B_R(z_0)$.

   Thus $(f/g)(z) = (z-z_0)^{m-n}\psi(z)$, with ψ analytic on $\mathbb{D}$ and $\psi(z) \neq 0$ in some $B_\delta(z_0) \Rightarrow f/g$ is analytic at z_0. If $m > n$, $(f/g)(z_0) = 0$. If $m = n$, then $(f/g)(z_0) = \frac{\phi(z_0)}{h(z_0)} = \frac{a_m}{b_m}$, where $a_m = \frac{1}{m!}\frac{d^m}{dz^m} f(z)|_{z=z_0}$, $b_m = \frac{1}{m!}\frac{d^m}{dz^m} g(z)|_{z=z_0}$.

2. Let $\mathbb{D} = \{z : |z| < 1\}$. Let $f\colon \mathbb{D} \to \mathbb{D}$ be an analytic function with $f(0) = 0$. Show that $|f(z)| \leq |z|, \forall z \in \mathbb{D}$. Further show that if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ in $\mathbb{D}$ or $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $f(z) = cz$ for all $z \in \mathbb{D}$.

   Solution: Set $\phi(z) = \frac{f(z)}{z}, z \in \mathbb{D}$. As $f(0) = 0, f(z) = z^m g(z)$, for some analytic function g with $g(0) \neq 0, m > 0$ integer.

   So $\phi(z) = z^{m-1} g(z)$, and hence analytic on $\mathbb{D}$.

   On $|z| = R < 1, |\phi(z)| \leq 1/R$ because $|f(z)| < 1$. Therefore by maximum modulus principle, $|\phi(z)| \leq 1/R$ in the disc $|z| \leq R$.
   (Note that the bound for $|\phi|$ gets better if we take R near 1 and gets worse if we take R near 0.)

   Taking limit as $R \to 1$, we get $|\phi(z)| \leq 1, \forall z \in \mathbb{D}$. Thus $|f(z)| \leq |z|, \forall z \in \mathbb{D}$.

   If $z_0 \neq 0$ in $\mathbb{D}$ is such that $|f(z_0)| = |z_0|$, then $|\phi|$ has a maximum inside $\mathbb{D}$. Therefore by the maximum modulus principle, $\phi \equiv c \Rightarrow f(z) = cz$ for all $z \in \mathbb{D}$, for some constant c.

Again, if $|f'(0)| = 1$, then $1 = |f'(0)| = \left|\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}\right| = |\lim_{z\to 0} \phi(z)| = |\phi(0)|$. Hence $|\phi|$ has a maximum inside $\mathbb{D}$. Therefore by the maximum modulus principle, $\phi \equiv c \Rightarrow f(z) = cz$ for all $z \in \mathbb{D}$, for some constant c.

3. Expand the following in a Laurent series that converges for $|z| > 0$:

$$\frac{1}{z^2}\int_0^z \frac{e^t - 1}{t}\, dt.$$

Solution: As $f(t) = \frac{e^t-1}{t} = 1+\frac{t}{2!}+\frac{t^2}{3!}+\cdots = F'(t)$, for $F(z) = z+\frac{z^2}{2\cdot 2!}+\frac{z^3}{3\cdot 3!}+\cdots$,

$$\frac{1}{z^2}\int_0^z \frac{e^t - 1}{t}\, dt = \frac{1}{z^2}(F(z) - F(0))$$

$$= \frac{1}{z^2}\left[z + \frac{z^2}{2\cdot 2!} + \frac{z^3}{3\cdot 3!} + \cdots\right]$$

$$= \frac{1}{z} + \frac{1}{2\cdot 2!} + \frac{z}{3\cdot 3!} + \cdots, \quad 0 < |z| < \infty.$$

4. Is there a polynomial $P(z)$ such that $P(z)e^{1/z}$ is an entire function? Justify your answer.

   Solution: As exp is never zero, if $h(z) = P(z)e^{1/z}$ is an entire function, then $z = a$ is a zero of order m of P iff $z = a$ is a zero of order m of h. Hence, $\frac{h}{P}$ is an entire function. But $\frac{h}{P} = e^{1/z}$, for $z \neq 0$, and $e^{1/z}$ has an essential singularity at $z = 0$.

5. Does $\tan(1/z)$ have a Laurent series that converges in a region $0 < |z| < R$.

   Solution: No. Since $\cos(1/z)$ is zero for $z_n = 2/\{(2n+1)\pi\}$. and hence singularities of $\tan(1/z)$. These accumulate at its singularity at $z = 0$. Hence $z = 0$ is non-isolated singularity of $\tan(1/z)$.

6. Determine and classify as pole/essential singularity of the functions, in case of pole find the order of the pole:
   (i) $\frac{\sin 3z}{(z^4-1)^4}$ (ii) $\cosh\left(\frac{1}{z^2+1}\right)$

   Solution: (i)

$$\frac{\sin 3z}{(z^4-1)^4} = \frac{\sin 3z}{(z-1)^4(z+1)^4(z-i)^4(z+i)^4}.$$

   Since $\sin 3z \neq 0$ at $z = \pm 1, \pm i$, the point $z = \pm 1, \pm i$ are poles of order 4.

   (ii) Since

$$\cosh\left(\frac{1}{z^2+1}\right) = \frac{e^{1/(z^2+1)} + e^{-1/(z^2+1)}}{2}$$

   and $z = \pm i$ are essential singularities of $e^{\pm 1/(z^2+1)}$, these are also essential singularities of the given function.