Laplace Transform

- For an absolutely integrable analog signal $x(t)$, the CTFT $X(j\Omega)$ converges uniformly to a finite function of Ω
- In cases where the analog signals of interest are not absolutely integrable, a frequency-domain like representation is obtained by Laplace transform

1

Laplace Transform

- The Laplace transform $X(s)$ of an analog signal $x(t)$ is given by

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt}$$

where s is a complex variable

Compact notation -

$$\boxed{x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)}$$

2

Laplace Transform

- In rectangular form the complex variable s is expressed as

$$s = \sigma + j\Omega$$

where σ and Ω are, respectively, the real and imaginary parts of the complex variable s, and are continuous real variables taking values in the range $-\infty$ to $+\infty$

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Laplace Transform

- We rewrite the Laplace transform $X(s)$ as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{\infty} g(t) e^{-j\Omega t} dt$$

where we have set $g(t) = x(t) e^{-\sigma t}$

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Laplace Transform

- Hence, $X(s)$ can be considered as the CTFT of the analog signal $g(t)$ which exists if $g(t) = x(t)e^{-\sigma t}$ is absolutely integrable
- Now, for an analog signal $x(t)$ that is not absolutely integrable, its CTFT does not converge uniformly

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Region of Convergence

- However, for some values of σ its Laplace transform $X(s)$ may converge uniformly
- The range of values of σ for which the Laplace transform $X(s)$ of an analog signal $x(t)$ exists is known as its region of convergence (ROC)
- A Laplace transform $X(s)$ may have several ROCs

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Region of Convergence

- The exact region defining the ROC of a Laplace transform $X(s)$ depends on the form of the analog signal $x(t)$

Finite-Length Analog Signal

The ROC is the entire s-plane

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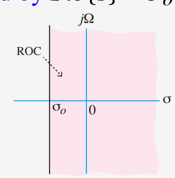
Region of Convergence

Right-Sided Analog Signal

- If the ROC includes the line $\mathcal{Re}\{s\} = \sigma_o$, then the ROC includes the region of the s-plane defined by $\mathcal{Re}\{s\} \geq \sigma_o$



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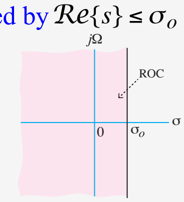
Region of Convergence

Left-Sided Analog Signal

- If the ROC includes the line $\mathcal{Re}\{s\} = \sigma_o$, then the ROC includes the region of the s-plane defined by $\mathcal{Re}\{s\} \leq \sigma_o$



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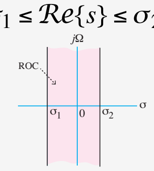
Region of Convergence

Two-Sided Analog Signal

- If the ROC includes the line $\mathcal{Re}\{s\} = \sigma_1$ and $\mathcal{Re}\{s\} = \sigma_2$ with $\sigma_2 > \sigma_1$, then the ROC includes the region of the s-plane defined by $\sigma_1 \leq \mathcal{Re}\{s\} \leq \sigma_2$



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Inverse Laplace Transform

- The analog signal $x(t)$ can be determined from its Laplace transform $X(s)$ by computing the inverse Laplace transform given by

$$x(t) = \int_{c-\infty}^{c+\infty} X(s) e^{st} ds$$

where the constant c is selected to ensure the convergence of the definite integral

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Laplace Transform

- In most practical applications, the analog signal $x(t)$ of interest is absolutely integrable and as a result, its CTFT $X(j\Omega)$ exists
- Hence, its Laplace transform $X(s)$ also exists

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Laplace Transform

- We can thus determine the transform-domain representation $X(s)$ of $x(t)$ from its frequency-domain representation $X(j\Omega)$ by replacing $j\Omega$ with s, and vice-versa
- Also the manipulation of analog signals and systems is much simpler in the transform domain

Rational Laplace Transform

General Rational Form

$$X(s) = \frac{P(s)}{Q(s)} = \frac{\sum_{k=0}^{M} p_k s^k}{s^N + \sum_{k=0}^{N-1} q_k s^k}$$

- M is the degree of the polynomial $P(s)$
- N is the degree of the polynomial $Q(s)$

Factored Form

$$X(s) = \frac{P(s)}{Q(s)} = p_M \left(\frac{\prod_{i=1}^{M} (s + \psi_i)}{\prod_{i=1}^{N} (s + \lambda_i)} \right)$$

- The constant ψ_i is known as the zero as
$$X(s)\big|_{s=-\psi_i} = 0$$
- The constant λ_i is known as the pole as
$$X(s)\big|_{s=-\lambda_i} \to \infty$$

Factored Form

- For a rational Laplace transform $X(s)$ expressed as a ratio of two polynomials in Ω with real coefficients $\{p_i\}$ and $\{q_i\}$, λ_i and ψ_i are real or complex numbers

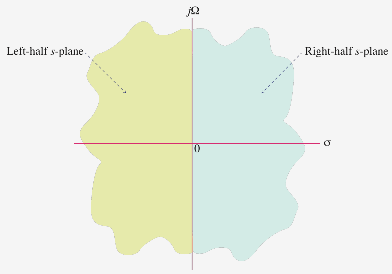
Factored Form

- If ψ_i is a complex number, then $P(j\Omega)$ contains the factors $(j\Omega + \psi_i)$ and $(j\Omega + \psi_i^*)$
- Their product is a second-order polynomial in $(j\Omega)$ given by
$$(j\Omega + \psi_i)(j\Omega + \psi_i^*)$$
$$= (j\Omega)^2 + 2\mathcal{R}e\{\psi_i\}(j\Omega) + |\psi_i|^2$$
which has all real coefficients

Factored Form

- Likewise, if λ_i is a complex number, then $Q(j\Omega)$ contains the factors $(j\Omega + \lambda_i)$ and $(j\Omega + \lambda_i^*)$
- Their product is a second-order polynomial in $(j\Omega)$ given by
$$(j\Omega + \lambda_i)(j\Omega + \lambda_i^*)$$
$$= (j\Omega)^2 + 2\mathcal{R}e\{\lambda_i\}(j\Omega) + |\lambda_i|^2$$
which has all real coefficients

s-Plane

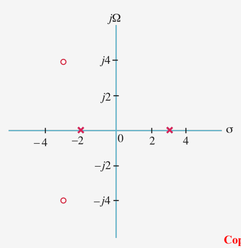


Graphical Representation

- Obtained by showing the locations of the zeros and poles of the rational Laplace transform in the complex s-plane
- The zero is shown with the symbol "o" and the pole is shown with the symbol "×"
- A typical pole-zero plot is shown in the next slide

Graphical Representation

$$X(s) = \frac{3(s^2+6s+25)}{(s+2)(s-3)} = \frac{3(s+3+j4)(s+3-j4)}{(s+2)(s-3)}$$



Partial-Fraction Expansion Form

- In most practical situations, $M \le N$, and the constants $\lambda_i, 1 \le i \le N$, are distinct
- In such a case we can express the rational CTFT in a partial-fraction expansion form given by

$$X(s) = K + \sum_{i=1}^{N} \frac{\rho_i}{s+\lambda_i}$$

Partial-Fraction Expansion Form

where

$$K = \begin{cases} 0, & M < N \\ p_N, & M = N \end{cases}$$

$$\rho_k = X(s)(s+\lambda_k)\big|_{s=-\lambda_k}$$

↑
residue

Partial-Fraction Expansion Form

- **Example** -

$$X(s) = \frac{4s^2+22s+32}{s^2+5s+6} = \frac{4s^2+22s+32}{(s+2)(s+3)}$$

- Its partial-fraction expansion is of the form

$$X(s) = K + \frac{\rho_1}{s+3} + \frac{\rho_2}{s+2}$$

where $K = p_2 = 4$

Partial-Fraction Expansion Form

and $\rho_1 = X(s)\cdot(s+3)\big|_{s=-3}$

$$= \frac{4s^2+22s+32}{s+2}\bigg|_{s=-3} = -2$$

$$\rho_2 = X(s)\cdot(s+2)\big|_{s=-2}$$

$$= \frac{4s^2+22s+32}{s+3}\bigg|_{s=-2} = 4$$

- Thus,

$$X(s) = 4 - \frac{2}{s+3} + \frac{4}{s+2}$$

Partial-Fraction Expansion Form

- For a rational Laplace transform with multiple poles at the same location, the form of the partial-fraction expansion is slightly different
- Without any loss of generality, we consider a rational Laplace transform with $M < N$, and one double pole and remaining $N-2$ poles being simple

Partial-Fraction Expansion Form

- Here the partial-fraction expansion is of the form

$$X(s) = K + \frac{\alpha}{(s+\lambda_1)^2} + \frac{\rho_1}{s+\lambda_1} + \sum_{k=2}^{N} \frac{\rho_k}{s+\lambda_k}$$

- The constants K and the residues ρ_k, $2 \le k \le N$ are determined using the expressions given in Slide No. 22

Partial-Fraction Expansion Form

- Constant α is determined using

$$\alpha = X(s)(s+\lambda_1)^2\big|_{s=-\lambda_1}$$

- To determine ρ_1 we first form

$$X_1(s) = X(s) - \frac{\alpha}{(s+\lambda_1)^2}$$

and then compute

$$\rho_1 = X_1(s)(s+\lambda_1)\big|_{s=-\lambda_1}$$

Partial-Fraction Expansion Form

- **Example** –

$$Y(s) = \frac{3s^2+20s+37}{s^2+6s+9} = \frac{3s^2+20s+37}{(s+3)^2}$$

- The partial-fraction expansion is of the form

$$Y(s) = K + \frac{\alpha}{(s+3)^2} + \frac{\rho_1}{s+3}$$

- The constant K is thus $K = p_2 = 3$

Partial-Fraction Expansion Form

- The constant α is obtained using

$$\alpha = Y(s)(s+3)^2\big|_{s=-3} = (3s^2+20s+37)\big|_{s=-3} = 4$$

- We next form

$$Y_1(s) = Y(s) - \frac{4}{(s+3)^2} = \frac{3s+11}{s+3}$$

- Thus,

$$\rho_1 = Y_1(s)(s+3)\big|_{s=-3} = (3s+11)\big|_{s=-3} = 2$$

Partial-Fraction Expansion Form

- The partial-fraction expansion of $Y(s)$ is hence given by

$$Y(s) = 3 + \frac{4}{(s+3)^2} + \frac{2}{s+3}$$

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Laplace Transform Analysis Using MATLAB

CTFT Computation

- The function `freqs` can be used to compute the CTFT of a rational Laplace transform at specified angular frequencies
- Its application has been demonstrated in Slide 21 of Ch5-2.ppt

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Laplace Transform Analysis Using MATLAB

Factored Form from Rational Form

- Can be determined using the function `tf2zp`
- **Example** – $X(s) = \dfrac{4s^2 + 22s + 32}{s^2 + 5s + 6}$
- Code fragments

```
num = [4  22  32];
den = [1  5  6];
[z, p, k] = tf2zp(num,den)
```

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Laplace Transform Analysis Using MATLAB

which yield

```
z =
   -2.75 + 0.6614i
   -2.75 - 0.6614i
p =
    -3
    -2
k =
     4
```

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Laplace Transform Analysis Using MATLAB

- Hence, the factored form is given by

$$X(s) = \frac{3(s+2.75+j0.6614)(+2.75-j0.6614)}{(s+3)(s+2)}$$

- Zeros are at $s = -2.75 \pm j0.6614$ and the poles are at $s = -2$ and $s = -3$

Rational Form from Factored Form

- Can be determined using the function `zp2tf`

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Laplace Transform Analysis Using MATLAB

- **Example** – We determine the rational form of $X(s)$ from its factored form given in the previous example in Slide 30
- Code fragments used are

```
z=[-2.75+0.6614*i -2.75-06614*i];
p=[-3  -2];
k=4;
[num,den]=zp2tf(r,p,k)
```

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Laplace Transform Analysis Using MATLAB

which yields

```
num =
   4   22   32
den =
   1   5   6
```

- Hence

$$X(s) = \frac{4s^2 + 22s + 32}{s^2 + 5s + 6}$$

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Laplace Transform Analysis Using MATLAB

Partial-Fraction Expansion from Rational Form With Simple Poles

- Can be determined using the function `residue`
- **Example** –

$$X(s) = \frac{4s^2 + 22s + 32}{s^2 + 5s + 6}$$

- Code fragments used

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Laplace Transform Analysis Using MATLAB

```
num = [4  22  32];
den = [1  5  6];
[r,p,k] = residue(num,den)
```

which yield

```
r =
   -2.0000
    4.0000
```

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Laplace Transform Analysis Using MATLAB

```
p =
   -3.0000
   -2.0000
k =
    4
```

- Hence

$$X(s) = 4 - \frac{2}{s+3} + \frac{4}{s+2}$$

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Laplace Transform Analysis Using MATLAB

Partial-Fraction Expansion from Rational Form With Higher Order Poles

- Can be determined using the function `residue`
- **Example** –

$$Y(s) = \frac{3s^2 + 20s + 37}{(s+3)^2} = \frac{3s^2 + 20s + 37}{s^2 + 6s + 9}$$

- Code fragments used

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Laplace Transform Analysis Using MATLAB

```
num = [3  20  37];
den = [1  6  9];
[r, p, k] = residue(num,den)
```

which yield

```
r =
   2
   4
```

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Laplace Transform Analysis Using MATLAB

```
p =
   -3
   -3
k =
   3
```

- Hence,

$$Y(s) = 3 + \frac{2}{s+3} + \frac{4}{(s+3)^2}$$

Laplace Transform Analysis Using MATLAB

Rational Form from Partial-Fraction Expansion

- Can also be determined using the function `residue`
- **Example** – $X(s) = 4 - \frac{2}{s+3} + \frac{4}{s+2}$
- Code fragments used

Laplace Transform Analysis Using MATLAB

```
r = [-2  4];
p = [-3  -2];
k = 4;
[num,den] = residue(r,p,k)
```

which yields

Laplace Transform Analysis Using MATLAB

```
num =
      4   22   32
den =
      1   5   6
```

- Hence

$$X(s) = \frac{4s^2 + 22s + 32}{s^2 + 5s + 6}$$

Laplace Transform Properties

- Properties are listed next without any proofs
- Let

$$x(t) \overset{\mathcal{L}}{\Leftrightarrow} X(s), \ \text{ROC:}\mathcal{R}_x$$

$$h(t) \overset{\mathcal{L}}{\Leftrightarrow} H(s), \ \text{ROC:}\mathcal{R}_h$$

Linearity Property

$$\alpha x(t) + \beta h(t) \overset{\mathcal{L}}{\Leftrightarrow} \alpha X(s) + \beta H(s), \ \text{ROC: } \mathcal{R}_x \cap \mathcal{R}_h$$

Laplace Transform Properties

Conjugation Property

$$x^*(t) \overset{\mathcal{L}}{\Leftrightarrow} X^*(s), \ \text{ROC:}\mathcal{R}_x$$

- Note: If $x(t)$ is a real analog signal, then $X(s) = X^*(s^*)$ ⇨ If $X(s)$ has a zero (pole) at $s = s_k$, then it also has a zero (pole) at $s = s_k^*$

Laplace Transform Properties

Time-Shifting Property

$$x(t-t_o) \overset{\mathcal{L}}{\Leftrightarrow} e^{-t_o s}X(s), \quad \text{ROC:}\, \mathcal{R}_x$$

***s*-Domain Shifting Property**

$$e^{s_o t}x(t) \overset{\mathcal{L}}{\Leftrightarrow} X(s-s_o), \quad \text{ROC:}\, \mathcal{R}_x + \mathcal{R}e\{s_o\}$$

Time-Scaling Property

$$x(\alpha t) \overset{\mathcal{L}}{\Leftrightarrow} \frac{1}{|\alpha|}X(s/\alpha), \quad \text{ROC: } \alpha\mathcal{R}_x$$

Laplace Transform Properties

Time-Reversal Property

$$x(-t) \overset{\mathcal{L}}{\Leftrightarrow} X(-s), \quad \text{ROC:}\, -\mathcal{R}_x$$

Differentiation-in-Time Property

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\Leftrightarrow} sX(s), \quad \text{ROC:}\, \mathcal{R}_x$$

Laplace Transform Properties

Differentiation in the *s*-Domain Property

$$-tx(t) \overset{\mathcal{L}}{\Leftrightarrow} \frac{dX(s)}{ds}, \quad \text{ROC:}\, \mathcal{R}_x$$

Integration in the Time Domain Property

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\Leftrightarrow} \frac{1}{s}X(s), \quad \text{ROC:}\, \mathcal{R}_x \cap \{\mathcal{R}e\{s\} > 0\}$$

Laplace Transform Properties

Convolution Integral Property

$$x(t) \circledast h(t) \overset{\mathcal{L}}{\Leftrightarrow} X(s)H(s), \quad \text{ROC:}\, \mathcal{R}_x \cap \mathcal{R}_h$$

Example

- We prove the identity

$$\int_{-\infty}^{\infty} y(t)dt = \left(\int_{-\infty}^{\infty} x(t)dt\right)\left(\int_{-\infty}^{\infty} h(t)dt\right)$$

where $y(t) = x(t) \circledast h(t)$

Laplace Transform Properties

- From the convolution integral property, we have

$$Y(s_k) = X(s_k)H(s_k)$$

where s_k is a specific value of the complex variable s

- For $s_k = 0$, the identity follows

Laplace Transform Properties

Initial-Value Theorem

- For a right-sided analog signal $x(t)$ with $x(t) = 0$ for $t < 0$, with no impulses or its derivatives at $t = 0$, and with a Laplace transform $X(s)$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Laplace Transform Properties

Final-Value Theorem

- If in addition, the above type of analog signal $x(t)$ has a finite value as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

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Unilateral Laplace Transform

- The Unilateral Laplace Transform $\mathcal{X}(s)$ of an analog signal $x(t)$ is defined by

$$\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

- In the above equation, the lower limit of integration $t = 0^-$ is employed to include impulse function $\delta(t)$ or its derivatives located at $t = 0$

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Unilateral Laplace Transform

Region of Convergence

- As the definition of the unilateral Laplace transform includes only the right-sided part of the signal $x(t)$ from $t = 0^-$ to $t = +\infty$, the ROC of the transform is to the right of the line going through the right-most pole

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Unilateral Laplace Transform

Compact Notation

$$x(t) \overset{\mathcal{L}_u}{\leftrightarrow} \mathcal{X}(s)$$

Properties

- The properties of the unilateral Laplace transform is exactly the same as those of the bilateral Laplace transform except the differentiation in the time domain property

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Unilateral Laplace Transform

- Making use of the integration by parts we obtain from the definition of the unilateral Laplace transform

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st}dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}dt$$
$$= -x(0^-) + s\mathcal{X}(s)$$

- Hence $\quad \dfrac{dx(t)}{dt} \overset{\mathcal{L}_u}{\leftrightarrow} s\mathcal{X}(s) - x(0^-)$

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Unilateral Laplace Transform

- Following the same procedure we obtain

$$\frac{d^2x(t)}{dt^2} \overset{\mathcal{L}_u}{\leftrightarrow} s^2\mathcal{X}(s) - sx(0^-) - \frac{dx(t)}{dt}\Big|_{t=0^-}$$

- An important application of the unilateral Laplace transform is in computing the solution of a constant-coefficient differential equation with prescribed initial conditions

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Solution of First-Order Differential Equation

- We consider the solution of

$$\frac{dy(t)}{dt} + q_0 y(t) = x(t)$$

with nonzero initial condition $y(0^-) = y_o$ for an input $x(t) = Ae^{-\alpha t}\mu(t)$

- Let $\mathcal{Y}(s)$ and $\mathcal{X}(s)$ denote, respectively, the unilateral Laplace transforms of $y(t)$ and $x(t)$

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Solution of First-Order Differential Equation

- Now

$$\mathcal{X}(s) = \frac{A}{s+\alpha}, \ \mathcal{R}e\{s\} > -\alpha$$

- From the first-order differential equation we obtain

$$s\mathcal{Y}(s) - y_o + q_o\mathcal{Y}(s) = \mathcal{X}(s) = \frac{A}{s+\alpha}$$

which after a rearrangement yields

$$(s+q_o)\mathcal{Y}(s) = y_o + \frac{A}{s+\alpha}$$

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Solution of First-Order Differential Equation

- Hence

$$\mathcal{Y}(s) = \frac{y_o}{s+q_o} + \frac{A}{(s+q_o)(s+\alpha)}$$

- Applying partial-fraction expansion we get

$$\mathcal{Y}(s) = \frac{y_o}{s+q_o} + \frac{\frac{A}{\alpha-q_o}}{s+q_o} + \frac{\frac{A}{q_o-\alpha}}{s+\alpha}$$

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Solution of First-Order Differential Equation

$$\mathcal{Y}(s) = \frac{y_o}{s+q_o} + \frac{A}{q_o-\alpha}\left(\frac{1}{s+\alpha} - \frac{1}{s+q_o}\right)$$

- Its inverse transform yields

$$y(t) = \left[y_o e^{-q_o t} + \frac{A}{q_o-\alpha}\left(e^{-\alpha t} - e^{-q_o t}\right)\right]\mu(t)$$

which is exactly the same as that derived before

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