Problem Set 1

Problems marked **(T)** are for discussions in Tutorial sessions.

1. If A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix, then show that $A(BD) = (AB)D$ (Associativity holds).

   Solution: Entry by entry for $1 \leq i \leq m$ and $1 \leq j \leq s$, we have

$$\begin{aligned} [A(BD)]_{ij} &= \sum_{k=1}^{n} [A]_{ik}[BD]_{kj} = \sum_{k=1}^{n} [A]_{ik} \left(\sum_{l=1}^{p} [B]_{kl}[D]_{lj} \right) = \sum_{k=1}^{n} \sum_{l=1}^{p} [A]_{ik}[B]_{kl}[D]_{lj} \\ &= \sum_{l=1}^{p} \sum_{k=1}^{n} [A]_{ik}[B]_{kl}[D]_{lj} = \sum_{l=1}^{p} [D]_{lj} \left(\sum_{k=1}^{n} [A]_{ik}[B]_{kl} \right) \\ &= \sum_{l=1}^{p} [D]_{lj}[AB]_{il} = \sum_{l=1}^{p} [AB]_{il}[D]_{lj} = [(AB)D]_{ij}. \end{aligned}$$

   Hence the result.

2. If A is an $m \times n$ matrix, B and C are $n \times p$ matrices and D is a $p \times s$ matrix, then show that

   (a) $A(B + C) = AB + AC$ (Distributive law holds).

   Solution: Entry by entry for $1 \leq i \leq m$ and $1 \leq j \leq p$, we have

$$\begin{aligned} [A(B + C)]_{ij} &= \sum_{k=1}^{n} [A]_{ik}[B + C]_{kj} = \sum_{k=1}^{n} [A]_{ik} \left([B]_{kj} + [C]_{kj} \right) \\ &= \sum_{k=1}^{n} \left([A]_{ik}[B]_{kj} + [A]_{ik}[C]_{kj} \right) = \sum_{k=1}^{n} [A]_{ik}[B]_{kj} + \sum_{k=1}^{n} [A]_{ik}[C]_{kj} \\ &= [AB]_{ij} + [AC]_{ij} = [AB + AC]_{ij}. \end{aligned}$$

   Hence the result.

   (b) $(B + C)D = BD + CD$ (Distributive law holds).

   Solution: Similar to part (a) with appropriate modifications.

3. **(T)** Let A and B be 2×2 real matrices such that $A \begin{bmatrix} x \\ y \end{bmatrix} = B \begin{bmatrix} x \\ y \end{bmatrix}$ for all $(x, y) \in \mathbb{R}^2$. Prove that $A = B$.

   Solution: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. The given equation imply

$$x \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + y \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = x \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + y \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \tag{1}$$

Now, by substituting $x = 1$ and $y = 0$ in (1), we get

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \tag{2}$$

Similarly, by substituting $x = 0$ and $y = 1$ in (1), we get

$$\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \tag{3}$$

Equations (2) and (3) together imply the result.

4. Let A and B be $m \times n$ real matrices such that $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then, $A = B$

5. $(A + B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$ whenever $A + B$ and AB are defined.

   Solution: Let A and B be $m \times n$ matrices. Then, for $1 \leq i \leq m$ and $1 \leq j \leq n$, we have

   $$[(A + B)^T]_{ij} = [A + B]_{ji} = [A]_{ji} + [B]_{ji} = [A^T]_{ij} + [B^T]_{ij} = [A^T + B^T]_{ij}.$$

   Hence the result.

   Solution: Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then, entry by entry for $1 \leq i \leq p$ and $1 \leq j \leq m$, we have

   $$[(AB)^T]_{ij} = [AB]_{ji} = \sum_{k=1}^{n} [A]_{jk}[B]_{ki} = \sum_{k=1}^{n} [A^T]_{kj}[B^T]_{ik} = \sum_{k=1}^{n} [B^T]_{ik}[A^T]_{kj} = [B^T A^T]_{ij}.$$

   Hence the result.

6. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then $A = S + T$, where $S^* = S$ (Hermitian matrix) and $T^* = -T$ (skew-Hermitian matrix).

7. Give examples of 3×3 non zero matrices A and B such that $A^2 = 0$ and $B^3 = B$.

   $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. Show by an example that if $AB \neq BA$ then $(A + B)^2 = A^2 + 2AB + B^2$ need not hold.

   Solution: Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Clearly, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = BA$. A straightforward calculation shows that

   $$(A + B)^2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} = A^2 + 2AB + B^2.$$

   $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

9. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be invertible matrices. Then $(AB)^{-1} = B^{-1}A^{-1}$.

   Solution: Let $D = B^{-1}A^{-1}$. Then

   $$(AB)D = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

   and

   $$D(AB) = (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

   imply that D is the inverse of AB.

10. Let $A \in \mathbb{M}_n(\mathbb{C})$ be a nilpotent matrix. Then show that $I + A$ is invertible.

Solution: As A is nilpotent, there exists an $N > 0$ such that $A^N = 0$. Define

$$B = \sum_{n=0}^{N-1} (-1)^n A^n.$$

We have

$$\begin{aligned} (I+A)B &= (I+A)\left(\sum_{n=0}^{N-1}(-1)^n A^n\right) = \sum_{n=0}^{N-1}(-1)^n A^n + \sum_{n=0}^{N-1}(-1)^n A^{n+1} \\ &= \sum_{n=0}^{N-1}(-1)^n A^n - \sum_{n=1}^{N-1}(-1)^n A^n = I \end{aligned}$$

and

$$\begin{aligned} B(I+A) &= \left(\sum_{n=0}^{N-1}(-1)^n A^n\right)(I+A) = \sum_{n=0}^{N-1}(-1)^n A^n + \sum_{n=0}^{N-1}(-1)^n A^{n+1} \\ &= \sum_{n=0}^{N-1}(-1)^n A^n - \sum_{n=1}^{N-1}(-1)^n A^n = I \end{aligned}$$

and, therefore, B is the inverse of $I + A$.

11. **(T)** Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Define Tr $(A) = \sum\limits_{i=1}^{n} a_{ii}$. Then show that Tr $(AB) =$ Tr (BA). Hence or otherwise, show that if A is invertible then Tr $(ABA^{-1}) =$ Tr (B). Furthermore, show that there do not exist matrices A and B such that $AB - BA = cI$, for any $c \neq 0$.

Solution: Tr $(AB) =$ Tr (BA) follows from a straightforward calculation shown below:

$$\sum_{i=1}^{n}[AB]_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}[BA]_{jj}.$$

Now let $D = BA^{-1}$. We have,

$$\text{Tr}(ABA^{-1}) = \text{Tr}(AD) = \text{Tr}(DA) = \text{Tr}(BA^{-1}A) = \text{Tr}(B).$$

12. Let $A \in \mathbb{M}_n(\mathbb{C})$. If $AA^* = \mathbf{0}$ then show that $A = \mathbf{0}$.

Solution:

$$AA^* = 0 \Rightarrow \text{Tr}(AA^*) = 0 \Rightarrow \sum_{i=1}^{n}[AA^*]_{ii} = 0 \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}[A]_{ij}[A^*]_{ji} = 0 \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}[A]_{ij}\overline{[A]_{ij}} = 0.$$

We, therefore, have $[A]_{ij} = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq n$ and thus $A = 0$.

13. **(T)** The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$, $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

Solution: As the parabola passes through point $(1, 4)$, we have $4 = a + b \cdot 1 + c \cdot 1^2$ leading to the equation $a + b + c = 4$.

Similarly for points $(2, 8)$ and $(3, 14)$, we get $a + 2b + 4c = 8$ and $a + 3b + 9c = 14$.

We can obtain a, b and c as a solution to

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 14 \end{bmatrix}.$$

Carry out Gauss-elimination as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

We can thus obtain the solution to the given linear system by solving the equivalent system

$$\begin{aligned} a + b + c &= 4 \\ b + 3c &= 4 \\ 2c &= 2 \end{aligned}$$

The solution is $a = 2, b = 1$ and $c = 1$.

14. **(T)** Let $J = \mathbf{1}\mathbf{1}^*$. Then each entry of J equals 1. Determine condition(s) on a and b such that $bJ + (a - b)I_n$ is invertible. Find α and β in terms of a and b such that the inverse has the form $\alpha J + \beta I$.

Solution: Check that each entry of J equals 1 and $J^2 = nJ$. The symmetry of the matrix $bJ + (a - b)I$ motivates us to try to assume that $\alpha J + \beta I$ may be the inverse for some choice of α and β. Verify $\beta = \dfrac{1}{a - b}$ and $\alpha = \dfrac{b}{(b - a)((n - 1)b + a)}$.

15. **(T)** Let $\mathbf{x} \in \mathbb{M}_{3,1}(\mathbb{R})$. Then find $\mathbf{y}, \mathbf{z} \in \mathbb{M}_{3,1}(\mathbb{R})$ such that $\mathbf{x}^T\mathbf{y} = 0$ and $\mathbf{x}^T\mathbf{z} = 0$.

Solution: Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Choose $\mathbf{y}$ and $\mathbf{z}$ such that their dot product with $\mathbf{x}$ is zero.

16. **(T)** Let A be an upper triangular matrix. If $AA^* = AA^*$ then A is a diagonal matrix.

Solution: $(AA^*)_{11} = \sum\limits_{i=1}^{n} a_{1i}\overline{a_{1i}}$ and $(A^*A)_{11} = a_{11}\overline{a_{11}}$. Thus, $\sum\limits_{2=1}^{n} a_{1i}\overline{a_{1i}} = 0$ and hence $a_{1i} = 0$ for all $i \neq 1$. Now, consider the $(2, 2)$-entry of both sides and continue the above argument.