MSO202A COMPLEX ANALYSIS
Assignment 4

Exercise Problems:

1. Verify Cauchy's theorem for $f(z) = z^2$ over the boundary of the square with vertices $1+i$, $-1+i$, $-1-i$ and $1-i$, counterclockwise.

   Proof: Let $C = \cup_{j=1}^{4} C_j$, where C_j, $j = 1, 2, 3, 4$, are the four sides of the square represented as $C_1 : \alpha_1(x) = x - i$, x goes from -1 to 1. $C_2 : \alpha_2(y) = 1 + iy$, y goes from -1 to 1. $C_3 : \alpha_3(x) = x + i$, x goes from 1 to -1 $\quad C_4 : \alpha_4(y) = -1 + iy$, y goes from 1 to -1. Therefore,

$$\oint_C f(z)\, dz = \int_{-1}^{1} (x-i)^2\, dx + \int_{-1}^{1} (1+iy)^2\, idy + \int_{1}^{-1} (x+i)^2\, dx + \int_{1}^{-1} (-1+iy)^2\, idy.$$

$$= \int_{-1}^{1} [(x^2 - 1 - 2ix)dx + (1 - y^2 + 2iy)idy - (x^2 - 1 - 2ix)dx - i(1 - y^2 + 2iy)dy] = 0.$$

2. Use ML-inequality to prove the following:

   (a) $\left| \int_\gamma \frac{1}{1+z^2}\, dz \right| \leq \frac{\pi}{3}$, γ is the arc of $|z| = 2$ from 2 to $2i$.

   (b) $\left| \int_\gamma (1+z^2)\, dz \right| \leq \pi R(R^2+1)$, γ is the semicircular arc of $|z| = R$.

   Proof:

   (a) $\left| \dfrac{1}{1+z^2} \right| \leq \dfrac{1}{|z|^2 - 1} = \dfrac{1}{3}, L = \pi.$

   (b) $|(1+z^2)| \leq |z|^2 + 1 = R^2 + 1, L = \pi R.$

3. By parametrizing the curve or otherwise, evaluate:

   (a) $\int_C \tan z\, dz$, where C is the circle $|z| = 1$ oriented counter -clockwise.

   (b) $\int_C \text{Re } z^2\, dz$, C is the circle $|z| = 1$ oriented counter -clockwise.

   (c) $\int_C e^{4z}\, dz$, C is the shortest path from $8 - 3i$ to $8 - (3+\pi)i$.

   Proof:

   (a) 0, as $\tan z$ is analytic in a disc containing the unit circle $|z| = 1$.

   (b) $\int_C \text{Re } z^2\, dz = \int_0^{2\pi} \cos 2\theta\, d\theta = 0.$

(c) As e^{4z} has primitive $F(z) = \frac{e^{4z}}{4}$, $\int_C e^{4z}\,dz = F(8-(3+\pi)i.) - F(8-3i)$.

4. Use Cauchy's integral formula to find all simple closed curves C for which the following holds:
$(a)\ \displaystyle\int_C \frac{1}{z}\,dz = 0, \quad (b)\ \int_C \frac{e^{1/z}}{z^2+9}\,dz = 0.$

Proof: (a) Any simple closed curve C which does not enclose 0. (b) Any simple closed curve C which does not enclose $0, \pm 3i$.

5. Integrate $\frac{z^2}{z^4-1}$ counter-clockwise around the circle $(a)|z+1| = 1$ (b) $|z+i| = 1$.

Proof: (a) $z^4 - 1 = (z^2+1)(z^2-1)$, $\frac{z^2}{z^4-1} = \frac{z^2-1+1}{z^4-1} = \frac{1}{z^2+1} + \frac{1}{z^4-1}$, $\int_C \frac{z^2}{z^4-1}\,dz = 0 + 2\pi i f(-1) = -\frac{\pi i}{2}$, where $f(z) = \frac{z^2}{(z-1)(z^2+1)}$.
(b) Similar.

6. Integrate the functions counter-clockwise on the unit circle $|z| = 1$:
$(a)\frac{z^3}{2z-i}$ (b) $\frac{\cosh 3z}{2z}$ (c) $\frac{z^3 \sin z}{3z-1}$.

Proof: (a)$2\pi i \frac{z^3}{2}|_{z=i/2}$. (b) $2\pi i \frac{\cosh 3z}{2}|_{z=0}$. (c) $2\pi i \frac{z^3 \sin z}{3}|_{z=1/3}$.

7. Let Γ denote the positively (counter-clockwise) oriented boundary of the square whose sides lie on the lines $x = \pm 2$ and $y = \pm 2$. Using Cauchy's integral formula, evaluate the following integrals:

$$(a) \int_\Gamma \frac{\cos z}{z(z^2+8)}\,dz \quad (b) \int_\Gamma \frac{z}{2z+1}\,dz.$$

Proof: Using the Cauchy integral formula :

(a) $i\pi/4$.

(b) $\displaystyle\int_\gamma \frac{z/2}{z+1/2}\,dz = 2i\pi(-1/4) = -i\pi/2.$

Problem for Tutorial:

8. Let C be the positively oriented circle $|z| = 3$. If $f(w) = \int_C \frac{2z^2-z-2}{z-w}\,dz$, $|w| \neq 3$, then show that $f(2) = 8i\pi$. What is $f(w)$, if $|w| > 3$?

Proof: $f(2) = \displaystyle\int_C \frac{2z^2-z-2}{z-2}\,dz = 2\pi i(2z^2-z-2)|_{z=2} = 8\pi i$. When $|w| > 3$, the integrand is analytic in a open set containing C (since w lies outside C) and is hence 0.

9. Use Cauchy's integral formula to find closed contours C in complex plane satisfying $(a)\quad \int_C \operatorname{Log}(z)\,dz = 0\ (b)\quad \int_C \frac{\cos z}{z^6 - z^2}\,dz = 0$.

Proof: (a) Any closed contours C which is contained in the simply connected domain $\mathbb{C} \setminus$ the negative real axis.

(b) Any closed contours C which does not enclose $0, \pm 1, \pm i$.

10. Using Cauchy's integral formula, integrate counterclockwise:

$$\oint_C \frac{\text{Ln }(z+1)}{z^2+1}\,dz, \quad C:\ |z-2i| = 2.$$

Proof:

$$\oint_C \frac{\text{Ln }(z+1)}{z^2+1}\,dz = \frac{i}{2}\oint_C \text{Ln }(z{+}1)\left[\frac{1}{z-i} - \frac{1}{z+i}\right]\,dz = \frac{i}{2}\oint_C \frac{\text{Ln }(z+1)}{z-i}\,dz = -\pi \text{Ln}(1{+}i).$$

as $z = -i$ lies out side $|z - 2i| = 2$ and hence the integral of that term is zero by Cauchy's integral formula.