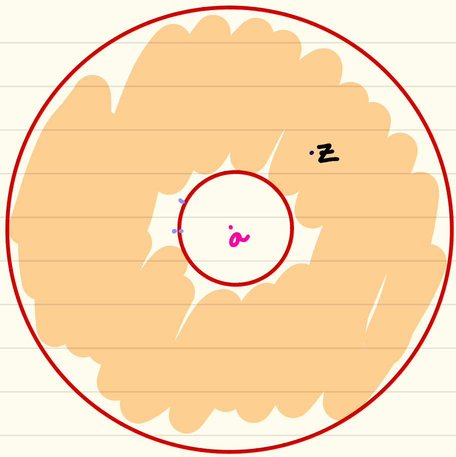
LAURENT'S THEOREM

Let $A = \{ z \in \mathbb{C} : R < |z-a| < S \}$ (annulus). Let f be analytic on A. Then

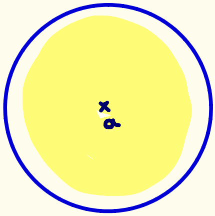
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

where $c_n = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, dw$; $\gamma = S_r(a)$.



TYPES OF ISOLATED SINGULARITIES

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

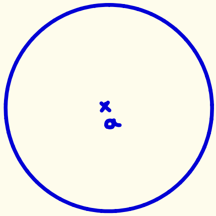


$$\forall\ 0<|z-a|<R$$

$$\underbrace{\sum_{n=-\infty}^{-1} \cdots\cdots}_{\text{principal part}} + \sum_{n=0}^{\infty} \underbrace{\cdots\cdots\cdots}$$

Types of Isolated Singularities

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$



$\forall\ 0<|z-a|<R$

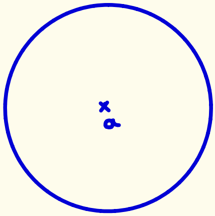
$c_n = 0 \ \forall\, n<0$

$c_n = 0 \ \forall\, n<-N$

$c_n \neq 0$ for infinitely many $n<0$.

TYPES OF ISOLATED SINGULARITIES

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \forall\ 0<|z-a|<R$$



$c_n = 0 \ \forall\ n<0$.

removable singularity

$$\frac{\sin z}{z}.$$

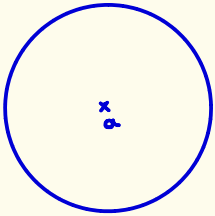
$$\frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z}$$

$c_n = 0 \ \forall\ n<-N$.

$c_n \neq 0$ for infinitely many $n<0$.

Types of Isolated Singularities

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$



$\forall \; 0<|z-a|<R$

$c_n = 0 \;\forall\, n<0$

removable singularity

$$\frac{\sin z}{z}$$

$$\frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z}$$

$c_n = 0 \;\forall\, n<-N$

pole of order N

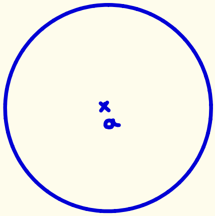
$$\frac{e^z}{z^2}$$

$$\frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \cdots$$

$c_n \neq 0$ for infinitely many $n<0$.

Types of Isolated Singularities

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$



$\forall\ 0<|z-a|<R$

$c_n = 0 \ \forall\, n<0$

removable singularity

$$\frac{\sin z}{z}$$

$$\frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z}$$

$c_n = 0 \ \forall\, n<-N$

pole of order N

$$\frac{e^z}{z^2}$$

$$\frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \cdots$$

$c_n \neq 0$ for infinitely many $n<0$.

isolated essential singularity

$$e^{1/z}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n$$

$\forall\ 0<|z|<\infty$

$$\cdots + c_{-n}\frac{1}{z^n} + \cdots + c_{-1}\frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots .$$

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

$$c_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz .$$

$c_n = 0 \;\forall\, n<0$

removable singularity

$\dfrac{\sin z}{z}$

$c_n = 0 \;\forall\, n<-N$

pole of order N

$\dfrac{e^z}{z^2}$

$c_n \neq 0$ for infinitely many $n<0$.

isolated essential singularity

$$\cdots + c_{-n}\frac{1}{z^n} + \cdots + c_{-1}\frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

$$c_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz.$$

"0"

$c_n = 0 \ \forall n<0$

removable singularity

?

$c_n = 0 \ \forall n<-N$

pole of order N

simple

$N = 1$

multiple

$N > 1$

?

$c_n \neq 0$ for infinitely many $n<0$.

isolated essential singularity

$$\cdots + c_{-n}\frac{1}{z^n} + \cdots + c_{-1}\frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

$$c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

$$c_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz.$$

"0"

$c_n = 0 \ \forall n<0$

removable singularity

?

$c_n = 0 \ \forall n < -N$

pole of order N

simple

$N = 1$

multiple

$N > 1$

?

$c_n \neq 0$ for infinitely many $n<0$.

isolated essential singularity

$\boxed{\text{Laurent series}}$

POLE

at 'a'

SIMPLE

MULTIPLE

$$= f(z) =$$

$$\frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$$

$$\frac{C_{-n}}{(z-a)^n} + \cdots + \frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$$

POLE
at 'a'

SIMPLE

MULTIPLE

$= f(z) =$

$\frac{C_{-1}}{z-a} + \underline{C_0 + C_1(z-a) + \cdots}$

$\frac{C_{-n}}{(z-a)^n} + \cdots + \frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$

$\lim_{z \to a} (z-a) f(z) = C_{-1}$

$f(z) = \frac{C_{-1}}{z-a} + g(z)$

$0 < |z-a| < R$

POLE at 'a'

SIMPLE | MULTIPLE

$= f(z) =$

Simple:

$$\frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$$

$$\lim_{z \to a} (z-a) f(z) = C_{-1}$$

$$f(z) = \frac{C_{-1}}{z-a} + g(z)$$

Multiple:

$$\frac{C_{-n}}{(z-a)^n} + \cdots + \frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$$

$$\frac{1}{(z-a)^n} \underbrace{\left[C_{-n} + \cdots + C_{-1}(z-a)^{n-1} + C_0(z-a)^n + \cdots \right]}_{g(z) \text{ analytic}}$$

POLE at 'a'

SIMPLE

MULTIPLE

$= f(z) =$

$$\frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$$

$$\lim_{z \to a} (z-a) f(z) = C_{-1}$$

$$\boxed{f(z) = \frac{C_{-1}}{z-a} + g(z)}$$

$$\frac{C_{-n}}{(z-a)^n} + \cdots + \frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$$

$$\frac{1}{(z-a)^n} \underbrace{\left[C_{-n} + \cdots + C_{-1}(z-a)^{n-1} + C_0(z-a)^n + \cdots \right]}_{g(z) \text{ analytic}}$$

C_{-1} = coeff of $(z-a)^{n-1}$ in Taylor series of g.

$$= \frac{g^{(n-1)}(a)}{(n-1)!}$$

POLE at 'a'

SIMPLE

MULTIPLE

$$= f(z) =$$

$$\frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$$

$$\frac{C_{-n}}{(z-a)^n} + \cdots + \frac{C_{-1}}{z-a} + C_0 + C_1(z-a) + \cdots$$

$$\lim_{z \to a} (z-a) f(z) = C_{-1}$$

$$f(z) = \frac{C_{-1}}{z-a} + g(z)$$

$$\frac{1}{(z-a)^n} \underbrace{\left[C_{-n} + \cdots + C_{-1}(z-a)^{n-1} + C_0 (z-a)^n + \cdots \right]}_{g(z) \text{ analytic}}$$

C_{-1} = coeff of $(z-a)^{n-1}$ in Taylor series of g.

$$= \frac{g^{(n-1)}(a)}{(n-1)!}$$

$$f(z) = \frac{h(z)}{k(z)}$$

$$f(z) = \frac{h(z)}{k(z)}$$

h, k are analytic in $|z-a| < R$.

& $k(a) = 0$

$f(z) = \dfrac{h(z)}{k(z)}$ h, k are analytic in $|z-a| < R$.

& $k(a) = 0$

$k(z) = (z-a)^m \cdot k_1(z)$. $k_1(a) \neq 0$

$h(a) = 0$ $h(a) \neq 0$

$$f(z) = \frac{h(z)}{k(z)}$$

h, k are analytic in $|z-a| < R$.

& $k(a) = 0$

$$k(z) = (z-a)^m \cdot k_1(z).$$

$h(a) = 0$

$h(a) \neq 0$

$$h(z) = (z-a)^n \cdot h_1(z)$$

$$\frac{h(z)}{k(z)} = (z-a)^{n-m} \frac{h_1(z)}{k_1(z)}$$

$$f(z) = \frac{h(z)}{k(z)}$$ h, k are analytic in $|z-a| < R$.

& $k(a) = 0$

$$k(z) = (z-a)^m \cdot k_1(z).$$

$h(a) = 0$ $h(a) \neq 0$

$$h(z) = (z-a)^n \cdot h_1(z)$$

$$\frac{h(z)}{k(z)} = (z-a)^{n-m} \frac{h_1(z)}{k_1(z)}$$

$n \geq m$, 'a' removable singularity.

$n < m$, 'a' pole of order $m-n$

f is defined in $0<|z-a|<R$

$$f(z) = \frac{h(z)}{k(z)}$$

h, k are analytic in $|z-a|<R$.

& $k(a)=0$

$$k(z) = (z-a)^m \cdot k_1(z).$$

$h(a)=0$

$$h(z) = (z-a)^n \cdot h_1(z)$$

$$\frac{h(z)}{k(z)} = (z-a)^{n-m} \frac{h_1(z)}{k_1(z)}$$

$n \geq m$, 'a' removable singularity.

$n < m$, 'a' pole of order $m-n$

$h(a) \neq 0$

'a' is a pole of order m.

$$f(z) = \frac{h(z)}{k(z)}$$

h, k are analytic in $|z-a| < R$.

& $k(a) = 0$

$$k(z) = (z-a)^m \cdot k_1(z).$$

$h(a) = 0$

$h(a) \neq 0$

$$h(z) = (z-a)^n \cdot h_1(z)$$

$$\frac{h(z)}{k(z)} = (z-a)^{n-m} \frac{h_1(z)}{k_1(z)}$$

'a' is a pole of order m.

$m - n = 1$

$$\lim_{z \to a} (z-a) \frac{h_1(z)}{(z-a) k_1(z)}$$

$$= \frac{h_1(a)}{k_1(a)}$$

$$f(z) = \frac{h(z)}{k(z)}$$

h, k are analytic in $|z-a| < R$.

& $k(a) = 0$

$$k(z) = (z-a)^m \cdot k_1(z).$$

$$k'(z) \Big|_{z=a} = \underline{\underline{m}}\,\cancel{(z-a)^{m-1}} \cdot k_1(z)\Big| + \underline{\underline{\cancel{(z-a)^m}}}\, k_1'(z)\Big|_{z=a}$$

$$k'(a) = k_1(a)$$

$h(a) = 0$

$$h(z) = (z-a)^n \cdot h_1(z)$$

$$\frac{h(z)}{k(z)} = (z-a)^{n-m} \frac{h_1(z)}{k_1(z)}$$

$h(a) \neq 0$

'a' is a pole of order m.

$m - n = 1$ SIMPLE $m = 1$

$$\lim_{z \to a} (z-a) \frac{h_1(z)}{(z-a)\, k_1(z)}$$

$$= \frac{h_1(a)}{k_1(a)}$$

$$\lim_{z \to a} (z-a) \cdot \frac{h(z)}{k(z)}$$

$$= h(a) \cdot \frac{1}{\lim_{z \to a} \frac{k(z) - k(a)}{z-a}}$$

$$= \frac{h(a)}{k'(a)} = \frac{h(a)}{k_1(a)}$$

$$\cdots + C_{-n}\frac{1}{z^n} + \cdots + C_{-1}\frac{1}{z} + C_0 + C_1 z + C_2 z^2 + \cdots$$

$$C_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

$$C_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz\,.$$

CALCULUS OF RESIDUES.

$$\cdots + C_{-n}\frac{1}{z^n} + \cdots + C_{-1}\frac{1}{z} + C_0 + C_1 z + C_2 z^2 + \cdots .$$

$$C_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\,dz$$

$$C_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz .$$

"RESIDUE"

Calculus Of Residues.

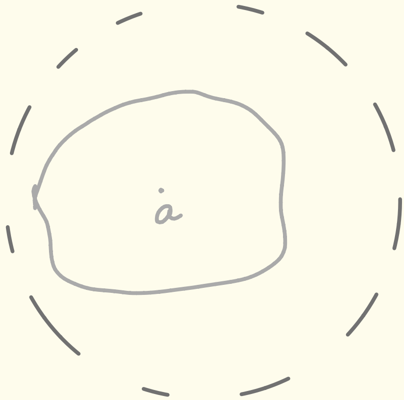
$$\cdots + C_{-n}\frac{1}{(z-a)^n} + \cdots + C_{-1}\frac{1}{z-a} + C_0 + C_1(z-a) + C_2(z-a)^2 + \cdots .$$

$$C_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$C_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz .$$

Residue of f at a.

$0<|z-a|<R$

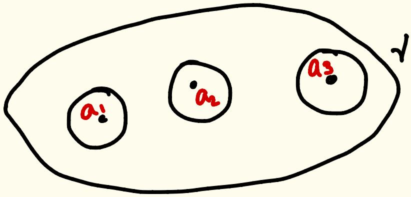


$$\int_{\gamma} f(z)\,dz$$

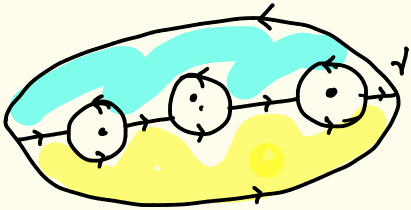
$$= 2\pi i \,\mathrm{Res}(f\,;\,a_1)$$

(via Laurent series)

$$\int_{\gamma} f(z)\,dz$$



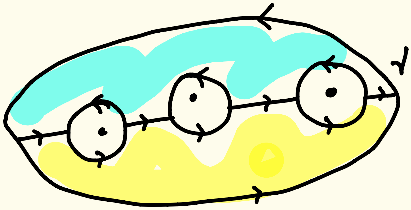
$$\int_\gamma f(z)\,dz$$



$$= \int_{C_{r_1}(a_1)} + \int_{C_{r_2}(a_2)} + \cdots + \int_{C_{r_k}(a_k)}.$$

$$= 2\pi i\left[\operatorname{Res}(f;a_1) + \cdots + \operatorname{Res}(f;a_k)\right]$$

$$\int_\gamma f(z)\,dz$$



$$= \int_{C_{r_1}(a_1)} + \int_{C_{r_2}(a_2)} + \cdots + \int_{C_{r_k}(a_k)}.$$

$$= 2\pi i\left[\operatorname{Res}(f;a_1) + \cdots + \operatorname{Res}(f;a_k)\right]$$

"CAUCHY RESIDUE THEOREM"

Strategy to calculate residues (when 'a' is a simple pole)

$\boxed{0 < |z-a| < R}$

① $\mathrm{Res}(f(z); a) = \lim_{z \to a} (z-a) f(z)$.

if 'a' is a simple pole (ie order = 1).

② → If $f(z) = \dfrac{g(z)}{z-a}$ (where g is analytic at a) then $\mathrm{Res}(f(z); a) = g(a)$.

→ When 'a' is a multiple pole (ie order > 1)

$f(z) = \dfrac{g(z)}{(z-a)^m}$ & $g(a) \neq 0$

Then $\dfrac{g^{(m-1)}(a)}{(m-1)!} = \mathrm{Res}(f; a)$

④ If $f = \dfrac{h}{k}$, $h(a) \neq 0$, $k(a) = 0$, $k'(a) \neq 0$

Then $\lim_{z \to a} (z-a) \dfrac{h(z)}{k(z)} = \lim_{z \to a} h(z) \cdot \dfrac{z-a}{k(z) - k(a)}$

$$= \frac{h(a)}{k'(a)}.$$

REMARK: In calculating residue of $\dfrac{h(z)}{k(z)}$ it is good practice to put all factors of $k(z)$ which do not contribute to a zero at 'a' into numerator.

DETERMINING MULTIPLICITY OF A POLE.

§: Characterization of a pole of order m.

Let f have a pole of order m at 'a'.

$$f(z) = \frac{1}{(z-a)^m} \cdot g(z), \quad g(a) \neq 0$$

Then $\lim_{z \to a} (z-a)^n f(z) = 0 \;\; \forall \; n > m$

and $\lim_{z \to a} (z-a)^m f(z) \neq 0$. (Pf: easy)

§ Relation between poles and zeroes

Suppose f is holomorphic in an open disc $B_r(a)$. We have seen that its zeros are isolated; infact $f(z) = (z-a)^m \underbrace{\sum_{n=m+1}^{\infty} a_n \cdot (z-a)^n}_{g(z) \,\ni\, g(a) \neq 0}$

Theorem: With notations as above
f has a zero of order m at 'a' if and only if $1/f$ has a pole of order m at 'a'.

f has a zero of order m $\Big\}$ $\dfrac{1}{f(z)} = \dfrac{1}{(z-a)^m} \cdot \dfrac{1}{g(z)} \quad \forall \; 0 < |z-a| < r$

f has a pole of order m $\Big\}$ $\dfrac{1}{f(z)} = \dfrac{1}{(z-a)^m} \cdot h(z) \quad \forall \; 0 < |z-a| < r$

$h(a) \neq 0$

$$\Longrightarrow \quad f(z) = \boxed{\frac{1}{h(z)}} (z-a)^m \quad f.$$

§ Counting order of a pole

Let f have a pole of order m at 'a'.

(a) Let g be holomorphic at 'a'. Then

(i) if g has a zero of order 'n' at 'a',

fg has a pole of order $m-n$ (if $m>n$)

fg has a zero of order $n-m$ (if $n\geq m$)

fg has a removable singularity (if $n=m$

(ii) $1/g$ has a pole of order 'n'. at 'a'.

Then f/g has a pole of order $n+m$ at a

$$\operatorname{res}\left(\frac{1}{(2-z)(z^2+4)}; 2\right)$$

$$= \lim_{z\to 2} (z-2)\cdot\frac{1}{(2-z)(z^2+4)} = -\frac{1}{8}$$

$$\operatorname{res}\left(\frac{1}{(2-z)(z^2+4)}; 2i\right) = \lim_{z\to 2i}\frac{(z-2i)}{(2-z)(z+2i)(z-2i)}$$

$$= \frac{1}{(2-2i)(4i)} = \frac{1+i}{16}$$

$$\operatorname{res}\left(\frac{e^{iz}}{z^4}; 0\right) = \frac{1}{3!}\frac{d^3(e^{iz})}{dz^3} = \frac{(i)^3 e^{iz}}{3!} = \frac{-ie^{iz}}{6}$$

$z - 1/4 = w$

Ex $\operatorname{res}\left(\frac{e^{i\pi z}}{z(\underbrace{4z}_{w}-1)^3}; 1/4\right) = \operatorname{res}\left(\frac{e^{i\pi(w+1/4)}}{(w+1/4)(4w)^3}; 0\right)$

$\sum c_n (z-1/4)^n = \sum c_n w^n$

$$= \frac{e^{i\pi/4}\cdot e^{i\pi w}}{(w+1/4)16w^3}$$

Rmk $f(z)$

$g(z+a)$

$$4\,\frac{e^{i\pi/4}\left(1+i\pi w+\frac{(i\pi w)^2}{2!}+\cdots\right)}{(4w+1)\cdot 16w^3}$$

$$e^{i\pi/4}\cdot\left(1-4w+16w^2-\cdots\right)$$