Practice problems 1: The Real Number System

1. Let $x_0 \in \mathbb{R}$ and $x_0 \geq 0$. If $x_0 < \epsilon$ for every positive real number ϵ, show that $x_0 = 0$.

2. Prove Bernoulli's inequality: for $x > -1$, $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

3. Let E be a non-empty bounded above subset of $\mathbb{R}$. If α and β are supremums of E, show that $\alpha = \beta$.

4. Suppose that α and β are any two real numbers satisfying $\alpha < \beta$. Show that there exists $n \in \mathbb{N}$ such that $\alpha < \alpha + \frac{1}{n} < \beta$. Similarly, show that for any two real numbers s and t satisfying $s < t$, there exists $n \in \mathbb{N}$ such that $s < t - \frac{1}{n} < t$.

5. Let A be a non-empty subset of $\mathbb{R}$ and $\alpha \in \mathbb{R}$ be an upper bound of A. Suppose for every $n \in \mathbb{N}$, there exists $a_n \in A$ such that $a_n \geq \alpha - \frac{1}{n}$. Show that α is the supremum of A.

6. Find the supremum and infimum of the set $\left\{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\right\}$.

7. Let E be a non-empty bounded above subset of $\mathbb{R}$. If $\alpha \in \mathbb{R}$ is an upper bound of E and $\alpha \in E$, show that α is the l.u.b. of E.

8. Let $x \in \mathbb{R}$. Show that there exists an integer m such that $m \leq x < m+1$ and an integer l such that $x < l \leq x+1$.

9. Let A be a non empty subset of $\mathbb{R}$ and $x \in \mathbb{R}$. Define the distance $d(x, A)$ between x and A by $d(x, A) = inf\{|x-a| : a \in A\}$. If $\alpha \in \mathbb{R}$ is the l.u.b. of A, show that $d(\alpha, A) = 0$.

10. **(*)**
    (a) Let $x \in \mathbb{Q}$ and $x > 0$. If $x^2 < 2$, show that there exists $n \in \mathbb{N}$ such that $(x + \frac{1}{n})^2 < 2$. Similarly, if $x^2 > 2$, show that there exists $n \in \mathbb{N}$ such that $(x - \frac{1}{n})^2 > 2$.
    (b) Show that the set $A = \{r \in \mathbb{Q} : r > 0,\ r^2 < 2\}$ is bounded above in $\mathbb{Q}$ but it does not have the l.u.b. in $\mathbb{Q}$.
    (c) From (b), conclude that $\mathbb{Q}$ does not posses the l.u.b. property.
    (d) Let A be the set defined in (b) and $\alpha \in \mathbb{R}$ such that $\alpha = \sup A$. Show that $\alpha^2 = 2$.

11. **(*)** For a subset A of $\mathbb{R}$, define $-A = \{-x : x \in A\}$. Suppose that S is a nonempty bounded above subset of $\mathbb{R}$.
    (a) Show that $-S$ is bounded below.
    (b) Show that $\inf(-S) = -\sup(S)$.
    (c) From (b) conclude that the l.u.b. property of $\mathbb{R}$ implies the g.l.b. property of $\mathbb{R}$ and vice versa.

12. **(*)** Let k be a positive integer and $x = \sqrt{k}$. Suppose x is rational and $x = \frac{m}{n}$ such that $m \in \mathbb{Z}$ and n is the least positive integer such that nx is an integer. Define $n' = n(x - [x])$ where $[x]$ is the integer part of x.
    (a) Show that $0 \leq n' < n$ and $n'x$ is an integer.
    (b) Show that $n' = 0$.
    (c) From (a) and (b) conclude that $\sqrt{k}$ is either a positive integer or irrational.

Hints/Solutions

1. Suppose $x_0 \neq 0$. Then for $\epsilon_0 = \frac{x_0}{2}$, $x_0 > \epsilon_0 > 0$ which is a contradiction.

2. Use Mathematical induction.

3. Since α is a l.u.b. of E and β is an u.b. of E, $\alpha \leq \beta$. Similarly $\beta \leq \alpha$.

4. Since $\beta - \alpha > 0$, by Archimedian property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\beta - \alpha}$.

5. If α is not the l.u.b then there exists an u.b. β of A such that $\beta < \alpha$. Find $n \in \mathbb{N}$ such that $\beta < \alpha - \frac{1}{n}$. Since $\exists\ a_n \in A$ such that $\alpha - \frac{1}{n} < a_n$, β is not an u.b. which is a contradiction.

6. sup = 1 and inf = -1.

7. If α is not the l.u.b. of E, then there exists an u.b. β of E such that $\beta < \alpha$. But $\alpha \in E$ which contradicts the fact that β is an u.b. of E.

8. Using the Archimedian property, find $m, n \in \mathbb{N}$ such that $-m < x < n$. Let $[x]$ be the largest integer between $-m$ and n such that $[x] \leq x$. So, $[x] \leq x < [x] + 1$. This implies that $x < [x] + 1 \leq x + 1$. Take $l = [x] + 1$. ($[x]$ is called the integer part of x).

9. If $d(\alpha, A) > 0$, then find $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < d(\alpha, A)$. So $\alpha - a > \epsilon$ for all $a \in A$. That is $a < \alpha - \epsilon$ for all $a \in A$. Hence $\alpha - \epsilon$ is an u.b. of A which is contradiction.

10. (a) Suppose $x^2 < 2$. Observe that $(x + \frac{1}{n})^2 < x^2 + \frac{1}{n} + \frac{2x}{n}$ for any $n \in \mathbb{N}$. Using the Archimedian property, find n such that $x^2 + \frac{1}{n} + \frac{2x}{n} < 2$. This n will do.

    (b) Note that 2 is an u.b. of A. If $m \in \mathbb{Q}$ such that $m = \sup A$, then there are three possibilities: i. $m^2 < 2$ ii. $m^2 = 2$ iii. $m^2 > 2$. Using (a) show that this is not possible.

    (c) The set A defined in (b) is bounded above in $\mathbb{Q}$ but does not have the l.u.b. in $\mathbb{Q}$.

    (d) Using (a), justify that the following cases cannot occur: (i) $\alpha^2 < 2$ and (ii) $\alpha^2 > 2$.

11. (a) Trivial.

    (b) Let $\alpha = \sup S$. We claim that $-\alpha = \inf(-S)$. Since $\alpha = \sup S$, $a \leq \alpha$ for all $a \in S$. This implies that $-a \geq -\alpha$ for all $a \in S$. Hence $-\alpha$ is a l.b. of $-S$. If $-\alpha$ is not the g.l.b.of $-S$ then there exists a lower bound β of A such that $-\alpha < \beta$. Verify that $-\beta$ is an u.b. of S and $-\beta < \alpha$ which is a contradiction.

    (c) Assume that $\mathbb{R}$ has the l.u.b. property and S is a non empty bounded below set. Then from (b) or the proof of (b), we conclude that $\inf S$ exists and is equal to $-\sup(-S)$.

12. Trivial.