LECTURE - 12

Application of Taylor's theorem.

→ Zeros of analytic functions

→ Identity theorem

→ Maximum Modulus principle.

§ Zeros of analytic functions:

Let f be an analytic function in a domain $D \supset B_R(z_0)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$.

Suppose $f(z_0) = 0 \Rightarrow a_0 = 0$

($f \not\equiv 0$ on $B_R(z_0)$) Let $a_m \neq 0 \ni a_i = 0 \ \forall \ i < m$

Then $f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n$

$$= (z-z_0)^m \underbrace{\sum_{n=0}^{\infty} a_{n+m} (z-z_0)^n}_{g(z)}.$$

$$f(z) = (z-z_0)^m g(z), \quad g(z_0) = a_m \neq 0$$

Now, $g(z)$ is analytic, hence continuous

$$\Rightarrow g(z_0) \neq 0 \Rightarrow g(z) \neq 0 \;\forall\, z \in B_\varepsilon(z_0)$$

$$\therefore f(z) \neq 0 \;\forall\, z \neq z_0,\; z \in B_\varepsilon(z_0)$$

ie Zeros of f are isolated.

§ Identity theorem

Suppose f is analytic on D (domain). If $\{z_n\} \subset D \ni z_n \to z_0 \in D \ni f(z_n) = 0 \;\forall\, n$
(assume that infinitely many z_n's are distinct, in other words that $\{z_n\}$ is not eventually constant sequence)
Then $f \equiv 0$ on D.

Pf: $z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

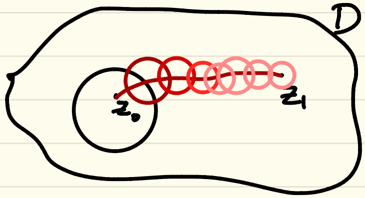
$$\therefore f(z_0) = 0.$$

Suppose $f \not\equiv 0$ on $B_R(z_0)$
By previous result, $\exists\, \varepsilon > 0 \ni f(z) \neq 0$
$\forall\, z \neq z_0,\; z \in B_\varepsilon(z_0)$

but $z_n \to z_0 \Rightarrow \exists\, N > 0 \ni z_n \in B_\varepsilon(z_0)$
$\forall\, n > N$

$\Rightarrow\!\Leftarrow$

Thus, $f \equiv 0$ on $B_R(z_0)$



Idea of proof.

Take $|z|=\varepsilon$, $\exists \{\zeta_n\} \to z \ni f(\zeta_n)=0 \ \forall n$ by similar argument as above $f \equiv 0$ on $B_{\varepsilon_2}(z)$. Proceeding thus we get $f(z_1)=0$ ▨

<u>Cor</u>: (Uniqueness theorem) Let f, g be analytic on $B_R(z_0) \ni f(z_n)=g(z_n)$ for $\{\cancel{z}_n\} \to z$.

Then $f=g$ on $B_R(z_0)$. (Pf: apply above to $f-g$)

§ <u>Maximum-Modulus principle</u>.

Let f be a non-constant analytic $\underline{fn}$. on a domain G. Then $|f|$ does not attain a local maximum "in" G.

<u>Pf:</u> Suppose $\exists z_0 \in G \ni |f(z)| \le |f(z_0)| \ \forall z \in B_R(z_0)$

Now $f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_{S_r(z_0)} \dfrac{f(\omega)\,d\omega}{\omega - z_0}$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}}\, rie^{it}\, dt$$

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt}$$ "mean value property".

$B_r(z_0) \subset B_R(z_0)$

$$\Rightarrow \quad |f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|\, dt$$

$$\le |f(z_0)|$$

$$\therefore \quad \frac{1}{2\pi} \int_0^{2\pi} \Big| |f(z_0)| - |f(z_0 + re^{it})| \Big|\, dt = 0$$

$$\Rightarrow \quad |f(z_0)| = |f(z_0 + re^{it})|$$

$\Rightarrow$ $|f|$ is constant on $B_R(z_0)$

$\Rightarrow$ f is constant on $B_R(z_0)$

$\Rightarrow$ f = constant on G (by Uniqueness thm).

<u>Cor</u>: If f is analytic inside and on a simple closed curve C, then $|f|$ attains its maximum on the boundary.