


LECTURE - 17

Linear Fractional
Transformations



①

Möbius transformation:

Let $a, b, c, d \in \mathbb{C} \Rightarrow ad - bc \neq 0$.

Then $f(z) = \frac{az+b}{cz+d}$ is called a Möbius transformation.

Properties: (1) f is diff'ble for $z \neq -d/c$

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

(2) Composition of Möbius transformation is again a Möbius transformation

$$g(z) = \frac{Az+B}{Cz+D}$$

$$g \circ f(z) = \frac{(Aa+Bc)z + Ab+Bd}{(Ca+Dc)z + Cb+Dd}.$$

$$\begin{aligned} (Aa+Bc)(Cb+Dd) - (Ab+Bd)(Ca+Dc) \\ = (AD-BC)(ad-bc) \neq 0. \end{aligned}$$

(2)

③ Möbius transformation takes circles and straight lines to circles & straight lines

$$\text{Equation of a line : } \frac{|z-p|}{|z-q|} = 1$$

(gives the \perp^r bisector joining a to b).

$$\text{Equation of a circle : } \left| \frac{z-p}{z-q} \right| = k. \quad (k \neq 1)$$

$$(x-a_1)^2 + (y-a_2)^2 = k^2 [(x-b_1)^2 + (y-b_2)^2]$$

$$x^2(1-k^2) + y^2(1-k^2) - 2a_1x + k^2(2b_1x)$$

$$- 2a_2y + k^2(2b_2y) = k^2(b_1^2 + b_2^2 - a_1^2 - a_2^2)$$

$$\frac{x^2 + y^2 - 2(a_1 + k^2b_1)x - 2(a_2 + k^2b_2)y}{1-k^2} = k^2$$

$$\left(x - \frac{a_1 + k^2b_1}{1-k^2} \right)^2 + \left(y - \frac{a_2 + k^2b_2}{1-k^2} \right)^2 = k^2 + \frac{(a_1 + k^2b_1)^2}{(1-k^2)^2} + \frac{(a_2 + k^2b_2)^2}{(1-k^2)^2}$$

(3)

$$\left| \frac{z-p}{z-q} \right| = k \quad \& \quad f(z) = w$$

$$\Rightarrow \left| \frac{w-p}{w-q} \right| = k$$

- (4) Möbius transformations are invertible. The inverse is also a Möbius transformation. (on $\mathbb{C} \setminus \{-d/c\}$)

$$\frac{az+b}{cz+d} = w \Rightarrow z = \left(\frac{dw-b}{-cz+d} \right) \frac{1}{ad-bc}.$$

- (5) Every Möbius transformation is a composition of translation, inversion, rotation and magnification.

$$T_a(z) = z + a \rightarrow \text{translation}$$

$$P_\theta(z) = e^{i\theta} z \quad - \text{rotation}$$

$$m_\alpha(z) = \alpha z \quad (\alpha > 0) \rightarrow \text{magnification}$$

$$j(z) = \frac{1}{z} \quad (z \neq 0) \rightarrow \text{inversion.}$$

(Note: all the above are Möbius transformations).

$$\frac{az+b}{cz+d} = t_2 \circ r \circ m \circ j \circ t_1(z)$$

$$c \neq 0 \quad t_1(z) = z + d/c, \quad m(z) = \left| \frac{ad-bc}{c^2} \right| \cdot z$$

④

$$r(z) = \frac{ad-bc}{|ad-bc|} \frac{|c|^2}{c^2} \cdot z$$

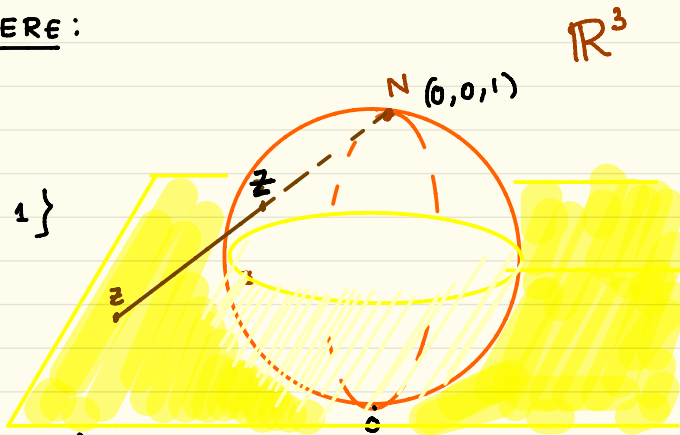
$$t_2(z) = z + \frac{a}{c}$$

$$c \neq 0: \quad r \circ t_1(z) = \frac{az+b}{d}$$

RIEMANN SPHERE:

Eqn of sphere:

$$\{(x, y, z) / x^2 + y^2 + z^2 = 1\}$$



Eqn of line through $(0,0,1)$

$$\text{and } (x, y, 0) : tN + (1-t)z \quad 0 \leq t$$

$$= \{(1-t)x, (1-t)y, t\}$$

Intersection of the line with sphere.

$$(1-t)^2 x^2 + (1-t)^2 y^2 + t^2 = 1$$

$$|z|^2(1-t^2) + t^2 = 1 \Rightarrow t = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Thus, $(x, y, z) = Z$

$$= \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

Every line ^{on \mathbb{C}} corresponds to a circle on the sphere under stereographic projection. ^{though N .}

North pole = point at ∞ .

Line: P, Q, ∞ .

$0, 1, \infty$ real

$0, i, \infty$ im.

Circle: P, Q, R .

$i, -i, 1$ - unit circle.

Convention: $\frac{az+b}{cz+d} = \infty \iff \frac{cz+d}{az+b} = 0, \frac{a\infty+b}{c\infty+d} = \frac{a+b \cdot 0}{c+d \cdot 0}$

THEOREM: There is a unique Möbius transformation taking a triplet z_1, z_2, z_3 to $0, 1, \infty$

Hence, there is a unique Möbius transf taking 1 triplet to another.

Pf: $\left(\frac{z - z_1}{z - z_3} \right) \left(\frac{z_2 - z_3}{z_2 - z_1} \right)$

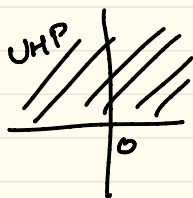
$$\left[\begin{array}{l} f(z) = \infty \iff \frac{1}{f(z)} = 0 \\ f(\infty) = w \iff f\left(\frac{1}{z}\right) = w \text{ for } z=0 \end{array} \right.$$

CONVENTION: if one of z_i 's is ∞ the the ratio of terms containing it is 1.

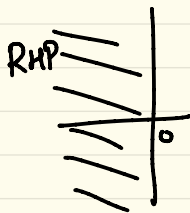
⑥

The above theorem enables us to also map regions under Möbius transformation to other regions

Eg 1: A Möbius transformation taking



to



real axis
 $0, 1, \infty$

imaginary axis
 $0, i, -i$

$$\left(\frac{z-0}{z+i} \right) \left(\frac{i+i}{i} \right) \quad \longleftarrow$$

$$\frac{2z}{z+i}$$

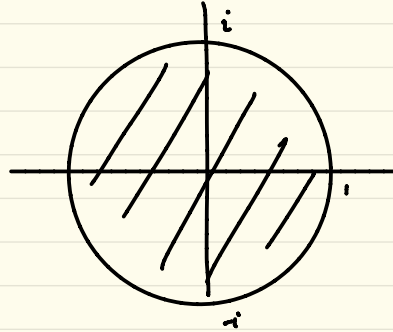
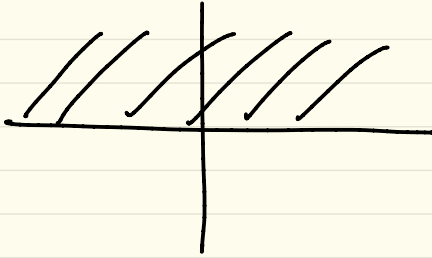
$$\frac{-2}{-1+i}$$

$$\longleftarrow -1$$

$$= \frac{-2(-1-i)}{2}$$

$$= 1+i \in \text{UHP}$$

eg 2:



$$\underbrace{\left(\frac{z-i}{z+i} \right) \left(\frac{1+i}{1-i} \right)}_{i \left(\frac{z-i}{z+i} \right)}$$

$$1 \longleftarrow \frac{i \left(\frac{z-i}{z+i} \right)}{0}$$

Exercises:

① Consider the inversion map $z \mapsto 1/z$ ($z \neq 0$)


Determine the image of the line

$$|z| = |z-2| \text{ under the inversion.}$$

Soln: let $w = \frac{1}{z}$, i.e. $z = \frac{1}{w}$

$$\therefore \left| \frac{1}{w} \right| = \left| \frac{1}{w} - 2 \right|$$

$$\Rightarrow |2w - 1| = 1 \quad (\forall |w| \neq 0)$$

 denotes a circle with radius 1 and center at $1/2$.

② Show the $\left| \frac{z-7}{z-4} \right| = 2$ represents the circle

$$|z-3| = 2.$$

(Check).

③ Find Möbius transformation which take

(i) $-1, i, 1$ to $-1, -i, i$ (Determine its poles)

(ii) $-1-i, 0, 1+i$ to $0, 1, \infty$

(4) (a) Obtain a Möbius transformation

taking $0, 1, \infty$ to $1, 1+i, i$.

(b) Under this determine the image of
(i) the arc through -1 and $-i$

(ii) real axis

(iii) imaginary axis

(5) Find the inverse of the following Möbius transformations and find the fixed points of each (i.e. find the pts $z \Rightarrow f(z) = z$)

(a) $\frac{z-1}{z+1}$

(b) $\frac{3z-4}{z-1}$

(c) iz

(d) $\frac{2iz}{z+i}$

Solⁿ (a) $w = \frac{z-1}{z+1}$

$$w(z+1) = z-1 \Rightarrow z(w-1) = -w-1$$

$$z = \frac{1+w}{1-w}, \text{ inverse: } w \mapsto \frac{1+w}{1-w}$$

Fixed pts: $\frac{z-1}{z+1} = z$

$$\Rightarrow z-1 = z^2+z \Rightarrow z^2 = 1, z = \pm 1.$$

