MSO202A COMPLEX VARIABLES
Solutions–3

Problems for Discussion:

1. Determine *all* $z \in \mathbb{C}$ for which the following series is convergent.
   a)$\sum \frac{z^n}{n^2}$ b)$\sum \frac{z^n}{n!}$ c)$\sum \frac{z^n}{2^n}$ d)$\sum \frac{1}{2^n}\frac{1}{z^n}$.
   Solution:

   (a) Here $\frac{a_n}{a_{n+1}} \to 1 \Rightarrow R = 1$. The series converges for $|z| < 1$ and diverges for $|z| > 1$. For $|z| = 1$, the series converges as it is then dominated by $\sum \frac{1}{n^2}$.

   (b) As $\frac{a_n}{a_{n+1}} \to \infty \Rightarrow R = \infty$ and so the series converges for all z.

   (c) As $\frac{a_n}{a_{n+1}} \to 2 \Rightarrow R = 2$. The series converges for $|z| < 2$ and diverges for $|z| > 2$. Also it diverges for $|z| = 2$ as $n-$ th term does not goto zero.

   (d) Let $w = \frac{1}{z}$, where $z \neq 0$ and apply previous solution to conclude that the series will converges for $|z| > 1/2$, and diverges for all other values.

2. Express the following complex numbers in the standard form $x + iy$ and find their principal value. (a) $(-1 + i\sqrt{3})^i$ (b) $\tan^{-1}(2i)$ (c) $\tan^{-1}(-\dfrac{i\pi}{2})$

   Solution: (a) As $z = -1+i\sqrt{3} = 2e^{2i\pi/3}$, so $(-1+i\sqrt{3})^i = e^{iLnz} = e^{i(\ln 2+i(\frac{2\pi}{3}+2k\pi)} = e^{i\ln 2}e^{-\frac{2\pi}{3}-2k\pi}$, with principal value $e^{-\frac{2\pi}{3}}(\cos(\ln 2) + i\sin(\ln 2))$. Here k is an integer.

   (b) $z = \tan^{-1}(2i) \Leftrightarrow \tan z = 2i \Leftrightarrow \dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -2 \Leftrightarrow e^{-2iz} = -3 = 3e^{i\pi} = e^{\ln 3+i\pi}$. Hence $-2iz = \ln 3 + i\pi + i2k\pi \Rightarrow z = i\dfrac{\ln 3}{2} - \dfrac{\pi}{2} - k\pi$. Here k is an integer. Principal value is $z = i\frac{\ln 3}{2} - \frac{\pi}{2}$.

   (c) $\tan z = -\dfrac{i\pi}{2} \Rightarrow \dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \dfrac{\pi}{2} \Leftrightarrow e^{2iz} = \dfrac{1 + \pi/2}{1 - \pi/2} = a$ say. Then $e^{2iz} = e^{\ln a} \Rightarrow 2iz = \ln a + i2k\pi$ or $z = \dfrac{1}{2i}\ln a + k\pi = \dfrac{-i}{2}\ln a + k\pi$, with principal value $\frac{-i}{2}\ln a$. Here k is an integer.

3. Find all $z \in \mathbb{C}$ such that $|e^z| \leq 1$.

   Solution: For $z = x + iy$, $|e^z| = e^x \leq 1 \Rightarrow x \leq 0$.

4. Find $\cosh(\ln 4)$.

   Solution: $\cosh(\ln 4) = \frac{e^{\ln 4}+e^{-\ln 4}}{2} = \frac{17}{8}$.

5. Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$. Conclude that cos function is not bounded in $\mathbb{C}$.

   Solution: The first part follows from : $\cos z = \cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y$. Since $\sinh^2(y) \geq \frac{e^{2y}}{4}$, $\cos z$ is unbounded on $\mathbb{C}$.

6. Using the method of parametric representation, evaluate

$$\oint_C f(z)\, dz$$

for (a) $f(z) = \overline{z}$, (b) $f(z) = z + \frac{1}{z}$, (c) $f(z) = \text{Re } z$ and C is the unit circle centered at origin oriented counterclockwise.

Solution: Let $z = e^{i\theta}$, $-\pi < \theta \leq \pi$. Then

(a)

$$\oint \overline{z}\, dz = \int_{-\pi}^{\pi} e^{-i\theta}\, ie^{i\theta}\, d\theta = 2\pi i.$$

(b)

$$\oint (z+\frac{1}{z})\, dz = \int_{-\pi}^{\pi} (e^{i\theta}+e^{-i\theta})\, ie^{i\theta}\, d\theta = i\int_{-\pi}^{\pi} (e^{2i\theta}+1)\, d\theta. = i(\frac{e^{2i\theta}}{2i}+\theta)|_{-\pi}^{\pi} = 2\pi i.$$

(c)

$$\oint \text{Re } z\, dz = \int_{-\pi}^{\pi} \cos\theta\, ie^{i\theta}\, d\theta = i\int_{-\pi}^{\pi} (\cos^2\theta + i\cos\theta\sin\theta)\ \ d\theta = i\pi.$$

7. Verify Cauchy's integral theorem for $f(z) = z^2$ over the boundary of the square with vertices $1+i$, $-1+i$, $-1-i$ and $1-i$, counterclockwise.

   Solution : Let $C = \cup_{j=1}^{4} C_j$, where C_j, $j = 1, 2, 3, 4$, are the four sides of the square represented as $C_1 : z = x - i$, $dz = dx$, x goes from -1 to 1. $C_2 : z = 1 + iy$, y goes from -1 to 1. $C_3 : z = x + i$, x goes from 1 to -1 $C_4 : z = -1 + iy$, y goes from 1 to -1. Therefore,

$$\oint_C f(z)\, dz = \int_{-1}^{1} (x-i)^2\, dx + \int_{-1}^{1} (1+iy)^2\, idy + \int_{1}^{-1} (x+i)^2\, dx + \int_{1}^{-1} (-1+iy)^2\, idy.$$

$$= \int_{-1}^{1} [(x^2-1-2ix)dx + (1-y^2+2iy)idy - (x^2-1-2ix)dx - i(1-y^2+2iy)dy] = 0.$$

Problem for Tutorial:

1. Let R be the radius of convergence of $\sum_n a_n z^n$. Find the radius of convergence of (a) $\sum {a_n}^k z^n$, (b) $\sum a_n z^{kn}$.

   Solution: (a) R^k (b) $R^{\frac{1}{k}}$.

2. Show that $\sin\overline{z}$ and $\cos\overline{z}$ are not analytic function on any domain.

   Solution: recall f is analytic iff $\frac{\partial}{\partial \overline{z}} f = 0$, f being thought of as a function of z and $\overline{z}$. Here $\frac{\partial}{\partial \overline{z}} \sin\overline{z} = \cos\overline{z}$, which does not vanish identically in any domain, so $\sin\overline{z}$ is nowhere analytic. Similarly, for $\cos\overline{z}$. Alternatively, use CR eqns.

3. Express i^i in the standard form $x + iy$ and find its principal value.

   Solution: $i = e^{i(\pi/2+2k\pi)} \Rightarrow i^i = e^{i[\ln 1 + i(\pi/2+2k\pi)]} = e^{-\pi/2-2k\pi}$. Here k is an integer.

   Its principal value is $e^{-\pi/2}$.

4. Find the roots of the equation $\sin z = 2$.

   Solution: $\sin z = 2 \Leftrightarrow \dfrac{e^{iz} - e^{-iz}}{2i} = 2 \Leftrightarrow e^{iz} - e^{-iz} = 4i$. Set $w = e^{iz}$, to get $w^2 - 4iw - 1 = 0$. So $w = i(2 \pm \sqrt{3})$ now proceed as in previous parts to get $z = \pi/2 + 2k\pi - i\ln(2 \pm \sqrt{3})$.

5. Evaluate the following integrals by parametrizing the contour

(a) $\int_{\mathcal{C}} x dz$ where $\mathcal{C}$ is the line segment joining 1 to i.

(b) $\int_{\mathcal{C}} (z-1) dz$ where $\mathcal{C}$ is the semicircle (in the lower half plane) joining 0 to 2.

(c) $\int_{\mathcal{C}} \cos(\frac{z}{2}) dz$ where $\mathcal{C}$ is the line segment joining 0 to $\pi + 2i$.

Solution: (a) Let $z = x + iy = x + i(1-x)$, $dz = (1-i)dx$ with x goes from 0 to 1. Then $\int_{\mathcal{C}} x dz = \int_0^1 x(1-i)\, dx = \frac{1-i}{2}$.
(b) Use the parametrisation $z = 1 + e^{i\theta}$, θ goes from $-\pi$ to 0, and $dz = ie^{i\theta}\, d\theta$. $\int_{\mathcal{C}} (z-1)\, dz = \int_{-\pi}^{0} e^{i\theta}(ie^{i\theta})\, d\theta = 0$.
(c) Using $z = x + iy = x(1 + 2i/\pi)$, $dz = (1 + \frac{2i}{\pi})\, dx$ x goes from 0 to π, we get the value to be $e + e^{-1}$.