ESC 201T : Introduction to Electronics

Lecture32: Digital Circuits-2

B. Mazhari
Dept. of EE, IIT Kanpur

Numbers

Every number system is associated with a base or radix

A positional notation is commonly used to express numbers

$$(a_5a_4a_3a_2a_1a_0)_r = a_5r^5 + a_4r^4 + a_3r^3 + a_2r^2 + a_1r^1 + a_0r^0$$

The decimal system has a base of 10 and uses symbols (0,1,2,3,4,5,6,7,8,9) to represent numbers

$$(2009)_{10} = 2\times10^3 + 0\times10^2 + 0\times10^1 + 9\times10^0$$

$$(123.24)_{10} = 1\times10^2 + 2\times10^1 + 3\times10^0 + 2\times10^{-1} + 4\times10^{-2}$$

An octal number system has a base 8 and uses symbols (0,1,2,3,4,5,6,7)

$$(2007)_8 = 2\times8^3 + 0\times8^2 + 0\times8^1 + 7\times8^0$$

What decimal number does it represent?

$$(2007)_8 = 2\times512 + 0\times64 + 0\times8^1 + 7\times8^0 = 1033$$

A hexadecimal system has a base of 16

Number	Symbol
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
10	A
11	B
12	C
13	D
14	E
15	F

$$(2BC9)_{10} = 2\times 16^3 + B\times 16^2 + C\times 16^1 + 9\times 16^0$$

How do we convert it into decimal number?

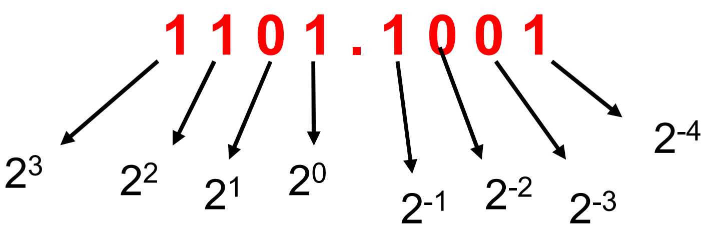
$$(2BC9)_{10} = 2\times 4096 + 11\times 256 + 12\times 16^1 + 9\times 16^0 = 11209$$

A Binary system has a base 2 and uses only two symbols 0, 1 to represent all the numbers

$$(1101)_2 = 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0$$

Which decimal number does this correspond to ?

$$(1101)_2 = 1\times 8 + 1\times 4 + 0\times 2^1 + 1\times 2^0 = 13$$



2^0	1
2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64
2^7	128
2^8	256
2^9	512
2^{10}	1024(K)
2^{20}	1048576(M)

2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
0.5	0.25	0.125	0.0625	0.03125	0.015625

Developing Fluency with Binary Numbers

1 1 0 0 1 = ? 25

1100001 = ? 64+32+1=97

0.101 = ? 0.5+0.125=0.625

11.001 = ? 3+0.125=3.125

Converting decimal to binary number

Convert 45 to binary number

$$(45)_{10} = b_n b_{n-1} \ldots\ldots. b_0$$

$$45 = b_n 2^n + b_{n-1} 2^{n-1} \ldots\ldots. b_1 2^1 + b_0$$

Divide both sides by 2

$$\frac{45}{2} = 22.5 = b_n 2^{n-1} + b_{n-1} 2^{n-2} \ldots\ldots. b_1 2^0 + b_0 \times 0.5$$

$$22 + 0.5 = b_n 2^{n-1} + b_{n-1} 2^{n-2} \ldots\ldots. + b_1 2^0 + b_0 \times 0.5$$

$$\Rightarrow \; b_0 = 1$$

$$22 + 0.5 = b_n 2^{n-1} + b_{n-1} 2^{n-2} \ldots\ldots + b_1 2^0 + b_0 \times 0.5 \qquad \Rightarrow\ b_0 = 1$$

$$22 = b_n 2^{n-1} + b_{n-1} 2^{n-2} \ldots\ldots b_2 2^1 + b_1 2^0$$

Divide both sides by 2

$$\frac{22}{2} = 11 = b_n 2^{n-2} + b_{n-1} 2^{n-3} \ldots\ldots b_2 2^0 + b_1 \times 0.5 \qquad \Rightarrow\ b_1 = 0$$

$$11 = b_n 2^{n-2} + b_{n-1} 2^{n-3} \ldots\ldots + b_3 2^1 + b_2 2^0$$

$$5.5 = b_n 2^{n-3} + b_{n-1} 2^{n-4} \ldots\ldots + b_3 2^0 + 0.5 b_2 \qquad \Rightarrow\ b_2 = 1$$

$$5 = b_n 2^{n-3} + b_{n-1} 2^{n-4} \ldots\ldots b_4 2^1 + b_3 2^0$$

$$5 = b_n 2^{n-3} + b_{n-1} 2^{n-4} \ldots\ldots b_4 2^1 + b_3 2^0$$

$$2.5 = b_n 2^{n-4} + b_{n-1} 2^{n-5} \ldots\ldots b_4 2^0 + 0.5 b_3$$ $$\Rightarrow \; b_3 = 1$$

$$2 = b_n 2^{n-4} + b_{n-1} 2^{n-5} \ldots\ldots b_5 2^1 + b_4 2^0$$

$$1 = b_n 2^{n-5} + b_{n-1} 2^{n-6} \ldots\ldots b_5 2^0 + 0.5 b_4$$ $$\Rightarrow \; b_4 = 0$$

$$\Rightarrow \; b_5 = 1$$

$$(45)_{10} = b_5 b_4 b_3 b_2 b_1 b_0 = 101101$$

Converting decimal to binary number

Method of successive division by 2

45	remainder
22	1
11	0
5	1
2	1
1	0
0	1

45 = 1 0 1 1 0 1

Convert $(153)_{10}$ to octal number system

$$(153)_{10} = (b_n b_{n-1} ....b_0)_8$$

$$(153)_{10} = b_n 8^n + b_{n-1} 8^{n-1} ....b_1 8^1 + b_0$$

Divide both sides by 8

$$\frac{153}{8} = 19.125 = b_n 8^{n-1} + b_{n-1} 8^{n-2} ....b_1 8^0 + \frac{b_0}{8} \quad \Rightarrow \frac{b_0}{8} = 0.125 \quad \Rightarrow b_0 = 1$$

153	remainder
19	1
2	3
0	2

153 = $(231)_8$

Converting decimal to binary number

Convert $(0.35)_{10}$ to binary number

$$(0.35)_{10} = 0.b_{-1}b_{-2}b_{-3}.......b_{-n}$$

$$0.35 = 0 + b_{-1}2^{-1} + b_{-2}2^{-2} + ...b_{-n}2^{-n}$$

How do we find the b_{-1} b_{-2} …coefficients?

Multiply both sides by 2

$$0.7 = b_{-1} + b_{-2}2^{-1} + ...b_{-n}2^{-n+1}$$

$$\Rightarrow b_{-1} = 0$$

$$0.7 = b_{-2}2^{-1} + b_{-3}2^{-2} + ...b_{-n}2^{-n+1}$$

$$0.7 = b_{-2}2^{-1} + b_{-3}2^{-2} + ...b_{-n}2^{-n+1}$$

Multiply both sides by 2

$$1.4 = b_{-2} + b_{-3}2^{-1} + ...b_{-n}2^{-n+2}$$

Note that ½+1/4+1/8+……≤1

$$\Rightarrow b_{-2} = 1$$

$$0.4 = b_{-3}2^{-1} + b_{-4}2^{-2}.......b_{-n}2^{-n+2}$$

$$0.8 = b_{-3} + b_{-4}2^{-1}.......b_{-n}2^{-n+3}$$

$$\Rightarrow b_{-3} = 0$$

Converting decimal to binary number

0.125 = ?

0 .	125	
		x2
0 .	25	
		x2
0 .	5	
		x2
1.	0	

$0.125 = (.001)_2$

0.8125 = ?

0 .	8125	
		x2
1 .	625	
		x2
1 .	25	
		x2
0.	5	
		x2
1.	0	

$0.8125 = (.1101)_2$

Binary numbers

Most significant bit or MSB

1011000111

Least significant bit or LSB

This is a 10 bit number

Binary digit = bit

decimal	2bit	3bit	4bit	5bit
0	00	000	0000	00000
1	01	001	0001	00001
2	10	010	0010	00010
3	11	011	0011	00011
4		100	0100	00100
5		101	0101	00101
6		110	0110	00110
7		111	0111	00111
8			1000	01000
9			1001	01001
10			1010	01010
11			1011	01011
12			1100	01100
13			1101	01101
14			1110	01110
15			1111	01111

N-bit binary number can represent numbers from 0 to 2^N -1

Converting Binary to Hex and Hex to Binary

$$(b_7 b_6 b_5 b_4 b_3 b_2 b_1 b_0)_b = (h_1, h_0)_{Hex}$$

$$b_7 2^7 + b_6 2^6 + b_5 2^5 + b_4 2^4 + b_3 2^3 + b_2 2^2 b_1 2^1 + b_0 = h_1 16^1 + h_0$$

$$\underbrace{(b_7 2^3 + b_6 2^2 + b_5 2^1 + b_4)}_{h_1} 2^4 + \underbrace{(b_3 2^3 + b_2 2^2 b_1 2^1 + b_0)}_{h_0} = h_1 16^1 + h_0$$

$$(10110011)_b = (1011)(0011) = (B3)_{Hex}$$

$$(110011)_b = (11)(0011) = (33)_{Hex}$$

$$(EC)_{Hex} = (1110)(1100) = (11101100)_b$$

Number	Symbol
0(0000)	0
1(0001)	1
2(0010)	2
3(0011)	3
4(0100)	4
5(0101)	5
6(0110)	6
7(0111)	7
8(1000)	8
9(1001)	9
10(1010)	A
11(1011)	B
12(1100)	C
13(1101)	D
14(1110)	E
15(1111)	F

Binary Addition/Subtraction

$$\begin{array}{r} 0 \\ \underline{0} \\ \underline{0} \end{array} \qquad \begin{array}{r} 1 \\ \underline{0} \\ \underline{1} \end{array} \qquad \begin{array}{r} 0 \\ \underline{1} \\ \underline{1} \end{array} \qquad \begin{array}{r} 1 \\ \underline{1} \\ \underline{1\ 0} \end{array} \qquad \begin{array}{r} 1 \\ 1 \\ \underline{1} \\ \underline{1\ 1} \end{array}$$

$$\begin{array}{r} 1\ 0\ 1 \\ \underline{1\ 1\ 0} \\ \underline{1\ 0\ 1\ 1} \end{array} \qquad \begin{array}{r} 1\ 1\ 0\ 1 \\ \underline{+\ \ 1\ 1\ 1\ 0} \\ 1\ \underline{1\ 0\ 1\ 1} \end{array}$$

Complement of a number

Decimal system: → 9's complement

Decimal system: → 10's complement

9's complement of n-digit number x is $10^n - 1 - x$

10's complement of n-digit number x is $10^n - x$

9's complement of 85 ? $10^2 - 1 - 85$ $99 - 85 = 14$

$$\text{9's complement of } 123 = 999 - 123 = 876$$

$$\text{10's complement of } 123 = \text{9's complement of } 123 + 1 = 877$$

Complement of a binary number

Binary system: → 1's complement
→ 2's complement

1's complement of n-bit number x is $2^n - 1 - x$

2's complement of n-bit number x is $2^n - x$

1's complement of 1011 ? $2^4 - 1 - 1011$ $1111 - 1011 = 0100$

1's complement is simply obtained by flipping a bit (changing 1 to 0 and 0 to 1)

$$1\text{'s complement of } 1001101 = ?$$

$$0110010$$

$$2\text{'s complement of } 1010 = 1\text{'s complement of } 1010 + 1 = 0110$$

$$2\text{'s complement of } 110010 =$$

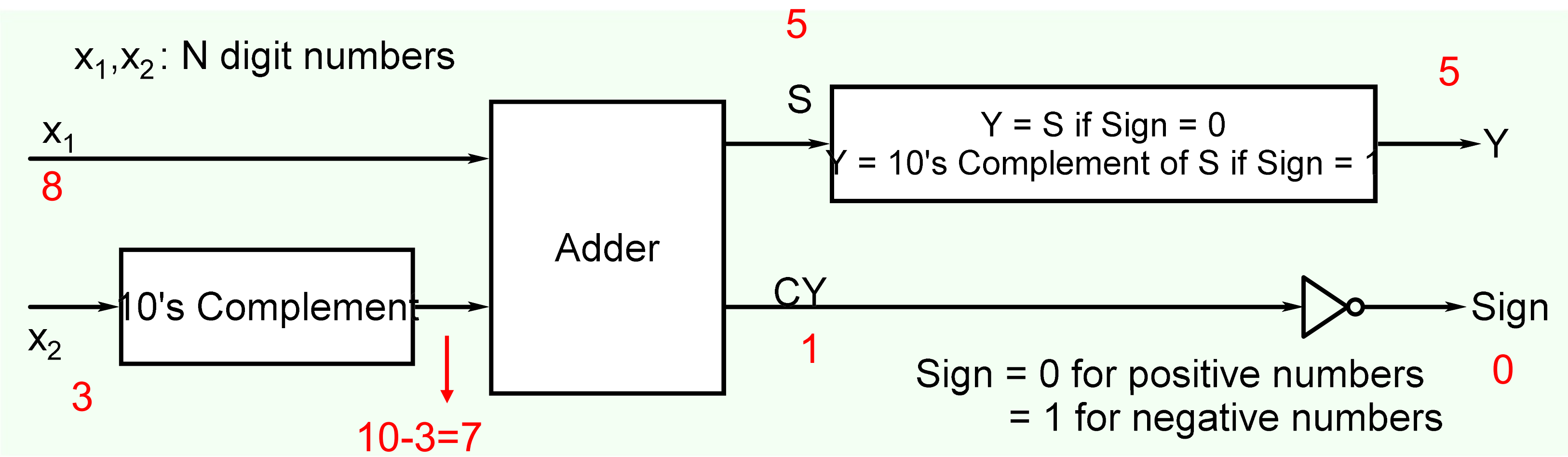
Leave all least significant 0’s as they are, leave first 1 unchanged and then flip all subsequent bits

$$001110$$

$$1011 \;\rightarrow 0101$$

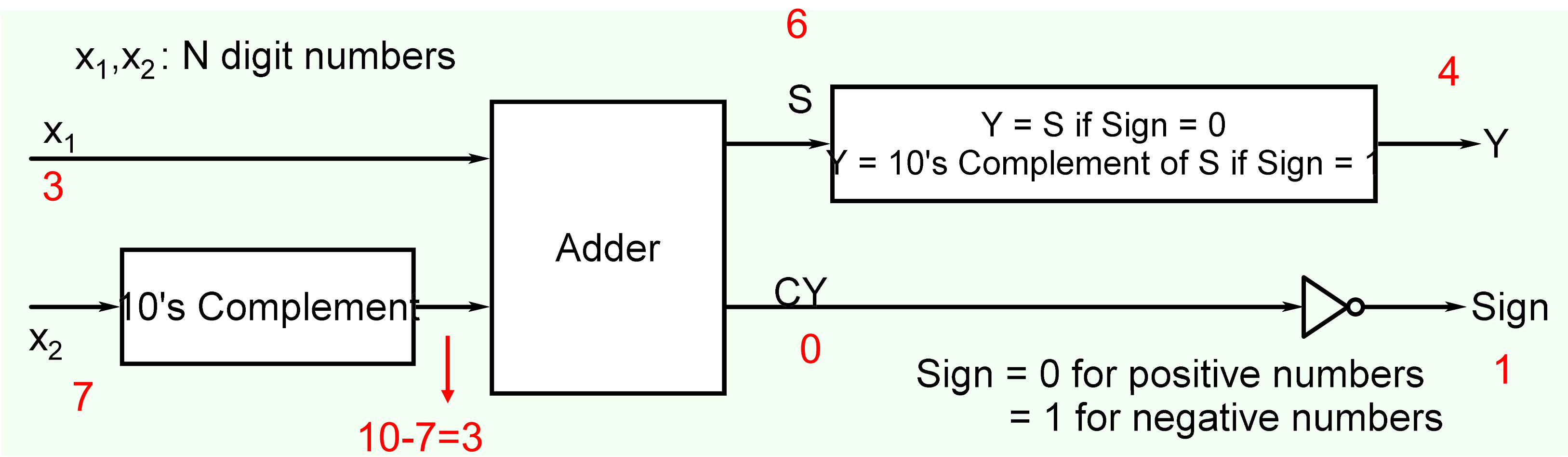
$$101101100 \;\rightarrow 010010100$$

Subtraction using 10's complement



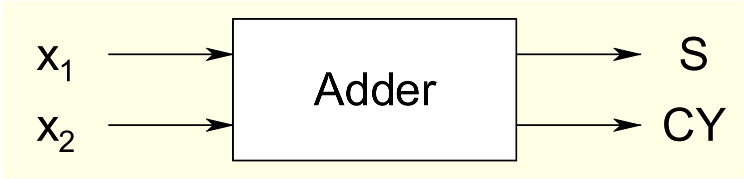
This way of subtraction would make sense only if subtracting a number x_2 from 10^N is much simpler than directly subtracting it directly from x_1

Subtraction using 10's complement

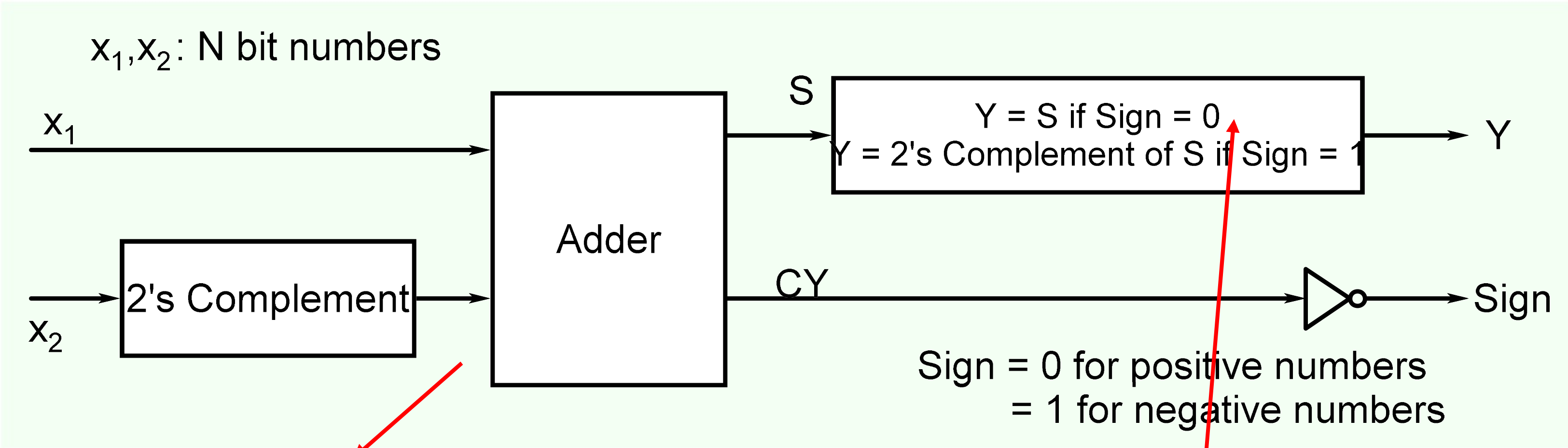


This way of subtraction would make sense only if subtracting a number x_2 from 10^N is much simpler than directly subtracting it directly from x_1

Advantages of using 2's complement



Can we carry out Y = $X_1 - X_2$ using such an adder?

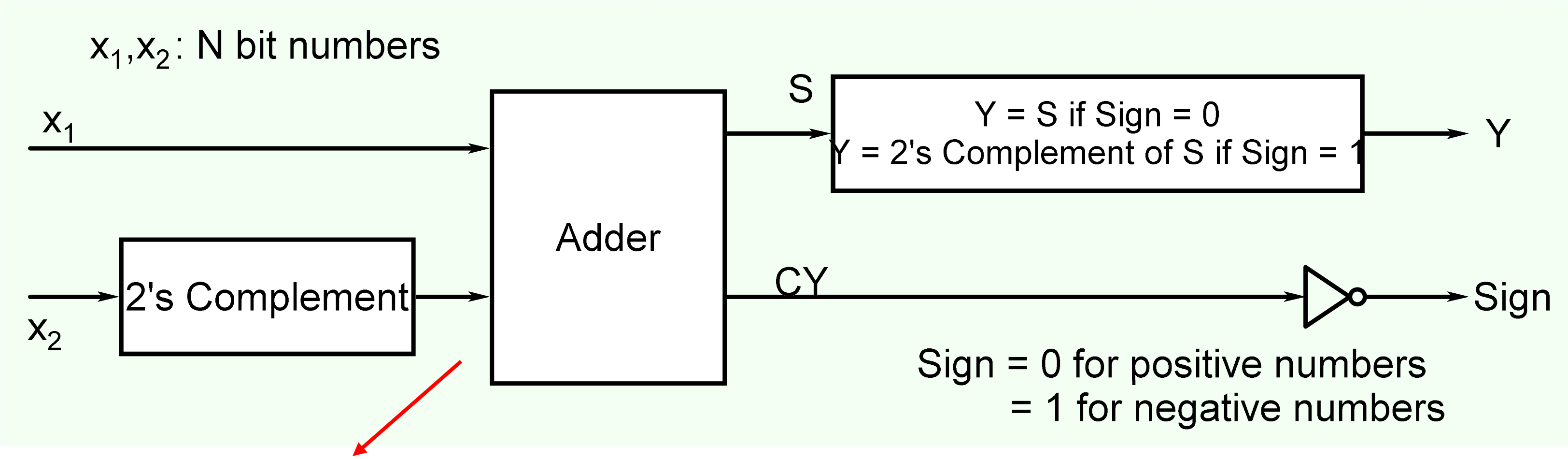


$$2^N - x_2$$

$$(CY, S) = x_1 + 2^N - x_2$$

Note that carry will be there only if $x_1 - x_2$ is positive as 2^N is N+1 bits (1 followed by N zeros)

Advantages of using 2's complement



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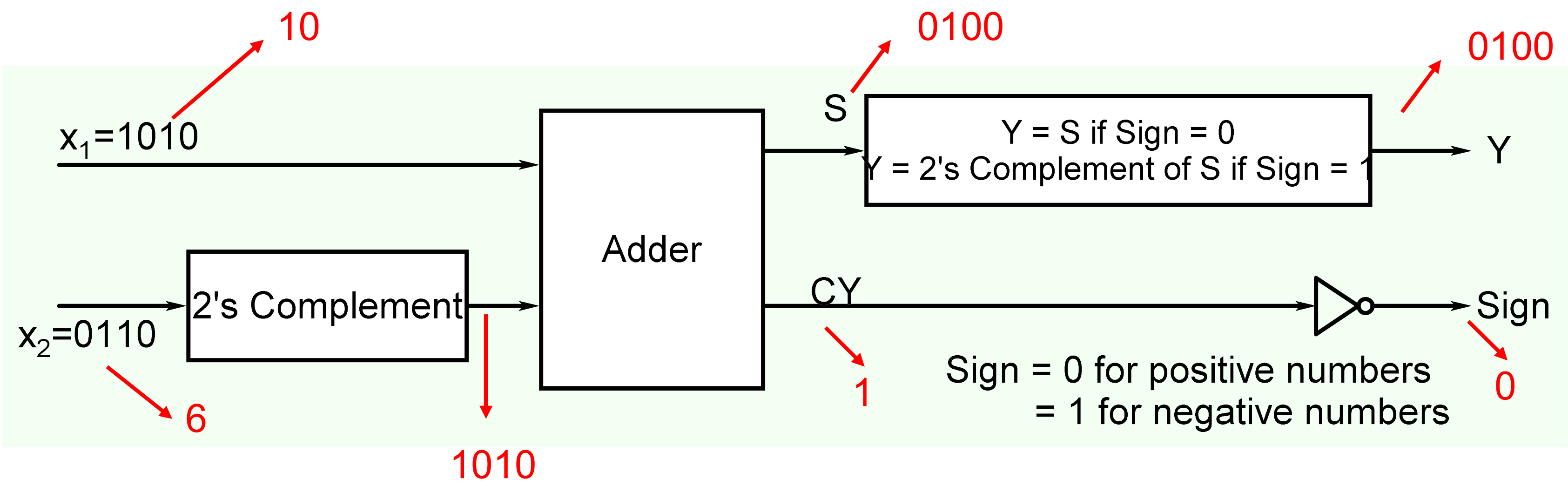
Note that carry will be there only if $x_1 - x_2$ is positive as 2^N is N+1 bits (1 followed by N zeros)

A zero carry implies a negative number whose magnitude $(x_2 - x_1)$ can be found as follows:

$$S = x_1 + 2^N - x_2$$

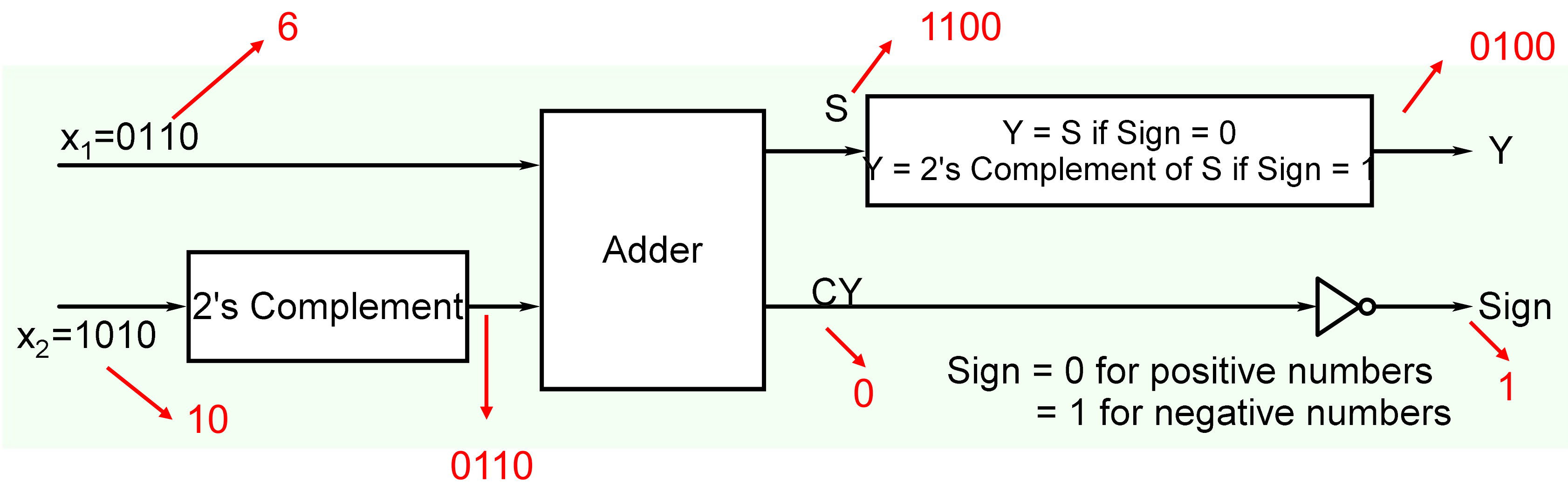
$$\text{2'scomplement of } S = 2^N - (x_1 + 2^N - x_2) = x_2 - x_1$$

Example



$$\begin{array}{r} 1\,0\,1\,0 \\ +\;1\,0\,1\,0 \\ \hline 1\,0\,1\,0\,0 \\ \hline \end{array}$$

Example



$$
\begin{array}{r}
0\,1\,1\,0 \\
+\,0\,1\,1\,0 \\
\hline
1\,1\,0\,0 \\
\hline
\end{array}
$$

It makes sense to use adder as a subtractor as well provided additional circuit required for carrying out 2's complement is simple

Representing positive and negative binary numbers

One extra bit is required to carry sign information. Sign bit = 0 represents positive number and Sign bit = 1 represents negative number

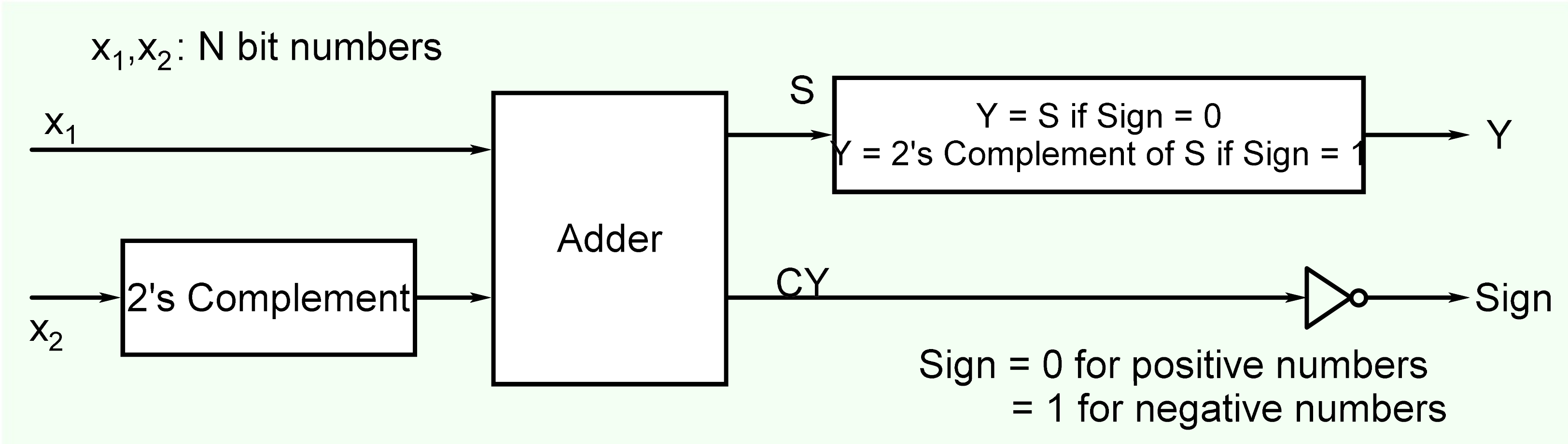
decimal	Signed Magnitude
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
-0	1000
-1	1001
-2	1010
-3	1011
-4	1100
-5	1101
-6	1110
-7	1111

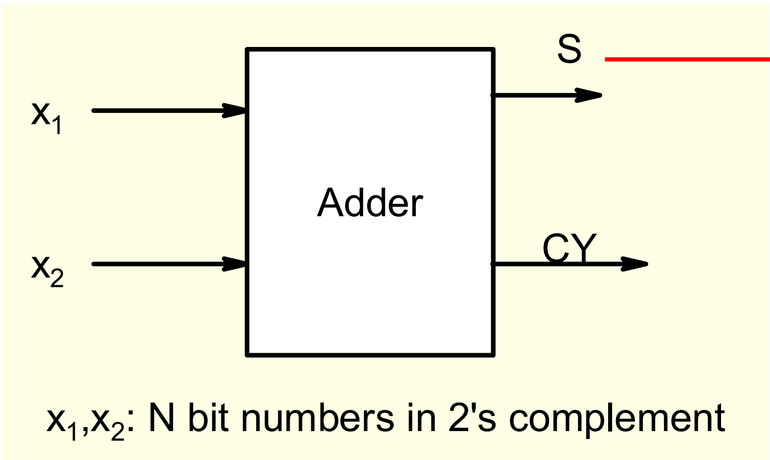
decimal	Signed 1's complement
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
-0	1111
-1	1110
-2	1101
-3	1100
-4	1011
-5	1010
-6	1001
-7	1000

decimal	Signed 2's complement
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
-1	1111
-2	1110
-3	1101
-4	1100
-5	1011
-6	1010
-7	1001

If we represent numbers in 2's complement form carrying out subtraction is same as addition

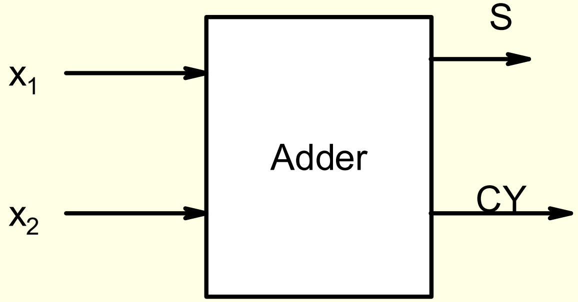
decimal	Signed 2's complement
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
-1	1111
-2	1110
-3	1101
-4	1100
-5	1011
-6	1010
-7	1001





Answer is in 2's complement form

Example



x_1, x_2: N bit numbers in 2's complement

$$\begin{array}{r} +5 \\ +2 \\ \hline +7 \end{array} \qquad \begin{array}{r} 0101 \\ +\,0010 \\ \hline 0111 \end{array}$$

$$\begin{array}{r} +5 \\ -2 \\ \hline +3 \end{array} \qquad \begin{array}{r} 0101 \\ +\,1110 \\ \hline 0011 \end{array}$$

$$\begin{array}{r} -5 \\ +2 \\ \hline -3 \end{array} \qquad \begin{array}{r} 1011 \\ +\,0010 \\ \hline 1101 \end{array}$$

2’s complement is 0011 = 3

$$\begin{array}{r} -5 \\ -2 \\ \hline -7 \end{array} \qquad \begin{array}{r} 1011 \\ +\,1110 \\ \hline 1001 \end{array}$$

2’s complement is 0111 = 7

Boolean Algebra

Algebra on Binary numbers

A variable x can take two values {0,1}

0 → False, No, Low voltage

1 → True, Yes, High voltage

Basic operations:

$$\text{AND:} \quad y = x_1 . x_2$$

Y is 1 if and only if both x_1 and x_2 are 1, otherwise zero

Truth Table

x_1	x_2	y
0	0	0
0	1	0
1	0	0
1	1	1

Basic operations:

$$\text{OR:} \quad y = x_1 + x_2$$

Y is 1 if either x_1 and x_2 is 1. Or y= 0 if and only if both variables are zero

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	1

$$\text{NOT:} \quad y = \overline{x}$$

x	y
0	1
1	0

Boolean Algebra

Basic Postulates

P1: $x + 0 = x$	P1: $x \cdot 1 = x$
P2: $x + y = y + x$	P2: $x \cdot y = y \cdot x$
P3: $x.(y+z) = x.y+x.z$	P3: $x+y.z = (x+y).(x+z)$
P4: $x + \overline{x} = 1$	P4: $x \cdot \overline{x} = 0$

Basic Theorems

T1: $x + x = x$	T1: $x \cdot x = x$
T2: $x + 1 = 1$	T2: $x \cdot 0 = 0$
T3: $\overline{(\overline{x})} = x$	
T4: $x + (y+z) = (x+y)+z$	T4: $x \cdot (y.z) = (x.y).z$
T5 $\overline{(x+y)} = \overline{x} \cdot \overline{y}$ (DeMorgan's theorem)	T5 $\overline{(x.y)} = \overline{x} + \overline{y}$ (DeMorgan's theorem)
T6: $x + x.y = x$	T6: $x.(x+y) = x$

Proving theorems

P1: $x + 0 = x$

P2: $x + y = y + x$

P3: $x.(y+z) = x.y+x.z$

P4: $x + \overline{x} = 1$

P1: $x . 1 = x$

P2: $x . y = y . x$

P3: $x+y.z = (x+y).(x+z)$

P4: $x . \overline{x} = 0$

Prove T1: $x + x = x$

$x + x = (x+x). 1$ (P1)

$= (x+x). (x+\overline{x})$ (P4)

$= x + x.\overline{x}$ (P3)

$= x + 0$ (P4)

$= x$ (P1)

Prove T1: $x . x = x$

$x . x = x.x+ 0$ (P1)

$= x.x + x.\overline{x}$ (P4)

$= x . (x+\overline{x})$ (P3)

$= x . 1$ (P4)

$= x$ (P1)

Proving theorems

$P1: \quad x + 0 = x$

$P2: \quad x + y = y + x$

$P3: \quad x.(y+z) = x.y+x.z$

$P4: \quad x + \overline{x} = 1$

$P1: \quad x \,.\, 1 = x$

$P2: \quad x \,.\, y = y \,.\, x$

$P3: \quad x+y.z = (x+y).(x+z)$

$P4: \quad x \,.\, \overline{x} = 0$

Prove : $x + 1 = 1$

$x + 1 = x+(x+\overline{x})$

$= (x+x)+\overline{x}$

$= x + \overline{x}$

$= 1$

$x + x.y = x$

$= x \,.\, 1 + x.\, y$

$= x.\,(1+ y)$

$= x \,.\, 1$

$=x$

$x + \overline{x}.y = x+y$

$= (x + \overline{x}).\,(x+ y)$

$= 1.\,(x+ y)$

$= x + y$

DeMorgan's theorem

$$\overline{(x_1 + x_2 + x_3 + \ldots)} = \overline{x_1} \,.\, \overline{x_2} \,.\, \overline{x_3} \,.$$

$$\overline{(x_1 .\, x_2 .\, x_3 \ldots)} = (\overline{x_1} + \overline{x_2} + \overline{x_3} + \ldots)$$

Simplification of Boolean expressions

$$\overline{(x_1 + x_2 + x_3 + \dots)} = \overline{x}_1 \,.\, \overline{x}_2 \,.\, \overline{x}_3 \,.$$

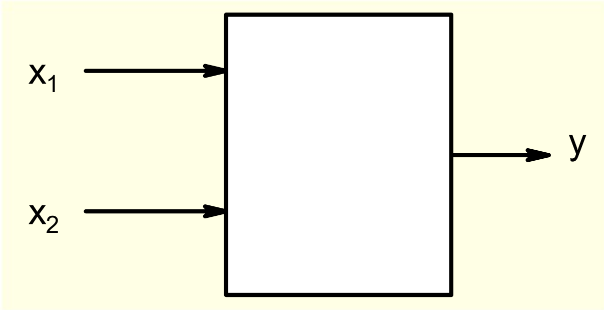
$$\overline{(\overline{x}_1 . x_2 + \overline{x}_2 . x_3)} = ?$$

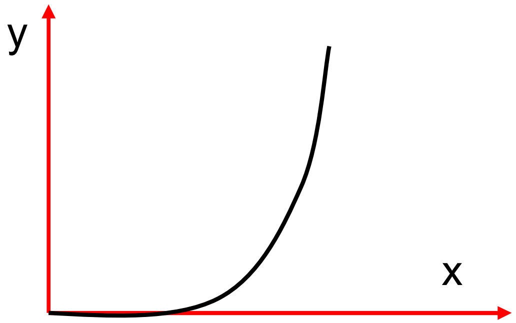
$$\overline{(x_1 .\, x_2 .\, x_3 \dots)} = (\overline{x}_1 + \overline{x}_2 + \overline{x}_3 + \dots)$$

$$= (x_1 + \overline{x}_2) \,.\, (x_2 + \overline{x}_3)$$

$$= x_1 .\, x_2 + x_1 \,.\, \overline{x}_3 + \overline{x}_2 .\, \overline{x}_3$$

Function of Boolean variables





$$y = x^2$$

x_1	x_2	y
0	0	0
0	1	1
1	0	0
1	1	0

Y = 1 when x_1 is 0 and x_2 is 1

$$y = \overline{x_1} \cdot x_2$$

Boolean expression

Obtaining Boolean expressions from truth Table

x_1	x_2	y
0	0	1
0	1	0
1	0	0
1	1	0

$$y = \overline{x_1} \,.\, \overline{x_2}$$

x_1	x_2	y
0	0	0
0	1	0
1	0	1
1	1	0

$$y = x_1 \,.\, \overline{x_2}$$

x_1	x_2	y
0	0	1
0	1	0
1	0	0
1	1	1

$$\overline{x_1} \,.\, \overline{x_2}$$

$$x_1 \,.\, x_2$$

$$y = \overline{x_1} \,.\, \overline{x_2} + x_1 . x_2$$

Obtaining Boolean expressions from truth Table

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

$$y = \overline{x_1} \, . \, x_2 + x_1 . \overline{x_2}$$

Instead of writing expressions as sum of terms that make y equal to 1, we can also write expressions using terms that make y equal to 0

x_1	x_2	y
0	0	1
0	1	1
1	0	1
1	1	0

$$y = \overline{x_1} \, . \, \overline{x_2} + \overline{x_1} \, . \, x_2 + x_1 . \overline{x_2}$$

$$y = \overline{x_1} + \overline{x_2}$$

x_1	x_2	y
0	0	1
0	1	0
1	0	1
1	1	1

$$y = x_1 + \overline{x_2}$$

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	1

$$y = x_1 + x_2$$

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

$$x_1 + x_2$$

$$\overline{x_1} + \overline{x_2}$$

$$y = (x_1 + x_2).(\overline{x_1} + \overline{x_2})$$

Obtaining Boolean expressions from truth Table

x_1	x_2	x_3	y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$y = \overline{x_1}.\overline{x_2}.x_3 + \overline{x_1}.x_2.x_3 + x_1.\overline{x_2}.x_3 + x_1.x_2.x_3$$

Sum of Products (SOP) form

$$y = (x_1 + x_2 + x_3).(x_1 + \overline{x_2} + x_3).(\overline{x_1} + x_2 + x_3).(\overline{x_1} + \overline{x_2} + x_3)$$

Product of Sum (POS) form