MSO202A COMPLEX ANALYSIS
Solutions–1

Exercise Problems:

1. For any $z, w \in \mathbb{C}$, show that (a) $\overline{z+w} = \overline{z} + \overline{w}$,(b) $\overline{zw} = \overline{z}\,\overline{w}$, (c) $\overline{\overline{z}} = z$, (d) $|\overline{z}| = |z|$ and (e) $|zw| = |z||w|$.

2. Show that $(a)|z+w|^2 = |z|^2 + |w|^2 + 2\text{Re}(z\overline{w})$

   $(b)|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$.

   $(c)|z+w| = |z| + |w|$ if and only if either $zw = 0$ or $z = cw$ for some positive real number c.

3. Let α be any of the n th roots of unity except 1. Show that $1+\alpha+\alpha^2+\ldots+\alpha^{n-1} = 0$.

4. Express in polar form: (a) $1+i$ (b) $-1-i$ (c) $\sqrt{3}+i$ (d) $1+\cos\theta+i\sin\theta$. Determine the value of $\text{Arg}(z^2)$ in each of the cases.

5. Let z be a nonzero complex number and n a positive integer. If $z = r(\cos\theta + i\sin\theta)$, show that $z^{-n} = r^{-n}(\cos n\theta - \sin n\theta)$.

6. Find the roots of each of the following in the form $x+iy$. Indicate the principal root (a) $\sqrt{2i}$, (b) $(-1)^{1/3}$ and (c) $(-16)^{1/4}$.

7. Determine the values of the following:

   (a) $(1+i)^{20} - (1-i)^{20}$.

   (b) $\cos\dfrac{\pi}{4} + i\cos\dfrac{3}{4}\pi + \ldots + i^n\cos\dfrac{2n+1}{4}\pi + \ldots + i^{40}\cos\dfrac{81}{4}\pi$.

8. Find the roots of $z^4+4=0$. Use these roots to factor z^4+4 as a product of two quadratics with real coefficients.

9. Determine whether the following sets describe domains (open and connected sets) in $\mathbb{C}$: (a) Re $z>1$ (b) $0 \le \text{Arg}z \le \dfrac{\pi}{4}$ (c) Im $(z)=1$, (d) $|z-2+i|<1$ (e) $|2z+3|>4$.

Problem for Tutorial:

1. Give a geometric description of the following sets:

   (a) $\{z \in \mathbb{C} : |z+i| \ge |z-i|\}$

   (b) $\{z \in \mathbb{C} : |z-i| + |z+i| = 2\}$.

2. Discuss the convergence of the following sequences: (a) (z^n), (b) $(\frac{z^n}{n!})$, (c) $(i^n \sin\frac{n\pi}{4})$ and (d) $(\frac{1}{n} + i^n)$.

3. Determine if the following series converge or diverge: (a) $\sum_{n=0}^{\infty} \left(\frac{1+i}{4}\right)^n$ (b) $\sum_{n=0}^{\infty} \left(\frac{1}{n+in^2}\right)$

4. *Limit at infinity: Let $f : \mathbb{C} \to \mathbb{C}$ be a function. The limit of f at infinity is said to be l if, given any $\epsilon > 0$ there exists a $R > 0$ such that $|f(z) - l| < \epsilon$ for all z such that $|z| > R$.*
(a) Show that $\lim_{z\to\infty} \frac{1}{z^2} = 0$.
Infinite limit: Let $f : D \to \mathbb{C}$ be a function defined around z_0 (except possibly at z_0). The limit of f at z_0 is said to be ∞ if, given any $R > 0$ there exists a $\delta > 0$ such that $|f(z)| > R$ for all z such that $0 < |z| < \delta$.
(b) Show that $\lim_{z\to a} \frac{1}{z-a} = \infty$

5. Verify if the following functions can be given a value at $z = 0$, so that they become continuous: (a)$f(z) = \frac{|z|^2}{z}$, (b) $f(z) = \frac{z+1}{|z|-1}$, (c) $f(z) = \frac{\bar{z}}{z}$, (d) $\frac{\text{Im } (z^2)}{|z|}$, (e) $\frac{\text{Im } z}{1-|z|}$.