MTH 101 Topic 5

Section 1

Q1 Show that $1 - \frac{x^2}{2} \leq \cos x \quad \forall\, x \in \mathbb{R}$

Q2 Show that the function f defined by

$$f = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not analytic at $x = 0$.

Q3 Consider the Harmonic Series H_n given by $H_n = \sum_{n \geq 1} \frac{1}{n}$.

Show that the series diverges

Q4 What is the radius of convergence of following series ?

a> $\sum_{n \geq 1} n!\, x^n$ b> $\sum_{n \geq 1} \frac{x^n}{n!}$

Q5 Consider the sequence 0.2, 0.22, 0.222, 0.2222 ....
Write this sequence as ~~as~~ a sequence of partial

sums of a series and find its limit.

Section 2

1. Determine the values of x for which the series $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$ converges absolutely.

2. Show the convergence of Maclaurin series of $\ln(1+x)$ in $0 \le x \le 1$.

3. Show that every sequence is a sequence of partial sums of a series.

MTH101 Questions

Topic - 5

1. Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that

$$x - \frac{1}{2}x^2 + \cdots - \frac{1}{2k}x^{2k} < \log(1+x) < x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}$$

2. Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim_{n \to \infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1 + a_2 + \cdots a_n}{n}$.

Show that $\sum_{n \geq 1} (-1)^n b_n$ converges.

3. In each of the following cases, determine the values of x for which the power series converges.

(a). $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n^n}$.

(b). $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$

(c). $\displaystyle\sum_{n=0}^{\infty} (-1)^n n 2^n x^n$.

(d). $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{n 3^n}$

(e). $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(10)^n}{n!} (x-10)^n$.

4. Show that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $x > 0$.

5. Let $a_n \geq 0$. Then show that both the series $\sum_{n \geq 1} a_n$ and $\sum_{n \geq 1} \frac{a_n}{a_n + 1}$ converge or diverge together.

6. Let $x_0 \in (a, b)$ and $n \geq 2$. Suppose $f', f'', \ldots, f^{(n)}$ are continuous on (a, b) and $f'(x_0) = \ldots = f^{(n-1)}(x_0) = 0$. Then, if n is even and $f^n(x_0) > 0$, then f has a local minimum at x_0. Similarily, if n is even and $f^n(x_0) < 0$, then f has a local maximum at x_0.

Q. 7. Let $a_n, b_n \in \mathbb{R}$ for all n and $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge, show that

$$\sum_{n=1}^{\infty} (a_n - b_n)^p \text{ converges for all } p \geq 2.$$