Problem Set 5

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Let $S = \{\mathbf{e}_1 + \mathbf{e}_4, -\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3\} \subset \mathbb{R}^4$. Find $S^{\perp}$.

2. Show that there are infinitely many orthonormal bases of $\mathbb{R}^2$.

3. **(T)** What is the projection of $\mathbf{v} = \mathbf{e}_1 + 2\mathbf{e}_2 - 3\mathbf{e}_3$ on $H := \{(x_1, x_2, x_3, x_4) : x_1 + 2x_2 + 4x_4 = 0\}$?

4. Let $\mathbb{V}$ be a subspace of $\mathbb{R}^n$. Then show that $\dim \mathbb{V} = n - 1$ if and only if $\mathbb{V} = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = 0\}$ for some $\mathbf{a} \neq \mathbf{0}$.

5. **(T)** Does there exist a real matrix A, for which, the Row space and column space are same but the null-space and left null-space are different?

6. **(T)** Consider two real systems, say $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{y} = \mathbf{d}$. If the two systems have the same **nonempty** solution set, then, is it necessary that row space(A) = row space(C)?

7. Show that the system of equations $A\mathbf{x} = \mathbf{b}$ given below

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 5 \\ 2x_1 + 2x_2 + 3x_3 &= 5 \\ 3x_1 + 4x_2 + 5x_3 &= 9 \end{aligned}$$

   has no solution by finding $\mathbf{y} \in \mathcal{N}(A^T)$ such that $\mathbf{y}^T\mathbf{b} \neq 0$.

8. **(T)** Suppose A is an n by n real invertible matrix. Describe the subspace of the row space of A which is orthogonal to the first column of A^{-1}.

   Solution: Let $A[:, j]$ (respectively, $A[i, :]$) denote the j-th column (respectively, the i-th row) of A. Then, $AA^{-1} = I_n$ implies $\langle A[i, :], A^{-1}[:, 1]\rangle = 0$ for $2 \leq i \leq n$. So, the row subspace of A which is orthogonal to the first column of A^{-1} equals $\mathrm{LS}(A[2, :], A[3, :], \ldots, A[n, :])$.

9. **(T)** Let $A_{n\times n}$ be any matrix. Then, the following statements are equivalent.

   (i) A is unitary.

   (ii) For any orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of $\mathbb{C}^n$, the set $\{A\mathbf{u}_1, \ldots, A\mathbf{u}_n\}$ is also an orthonormal basis.

10. Let $\mathbb{V}$ be an inner product space and S be a nonempty subset of $\mathbb{V}$. Show that

    (i) $S \subset (S^{\perp})^{\perp}$.

    (ii) If $\mathbb{V}$ is finite dimensional and S is a subspace then $(S^{\perp})^{\perp} = S$.

    (iii) If $S \subset T \subset \mathbb{V}$, then $S^{\perp} \supset T^{\perp}$.

    (iv) If S is a subspace then $S \cap S^{\perp} = \{0\}$.

11. Let $A_1, \cdots, A_k$ be k real symmetric matrices of order n such that $\sum A_i^2 = 0$. Show that each $A_i = 0$.

12. Let $\mathbb{V}$ be a normed linear space and $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. Is it true that $\Big| \|\mathbf{x}\| - \|\mathbf{y}\| \Big| \leq \|\mathbf{x} - \mathbf{y}\|$?

13. **(T) Polar Identity**: The following identity holds in an inner product space.
    - Complex IPS : $4\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 + i\|\mathbf{x}+i\mathbf{y}\|^2 - i\|\mathbf{x}-i\mathbf{y}\|^2$.
    - Real IPS : $4\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2$

14. **Just for knowledge, will NOT be asked** Let $\|\cdot\|$ be a norm on $\mathbb{V}$. Then $\|\cdot\|$ is induced by some inner product if and only if $\|\cdot\|$ satisfies the parallelogram law:
$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

15. Show that an orthonormal set in an inner product space is linearly independent.

16. Let A be unitary equivalent to B (that is $A = U^*BU$ for some unitary matrix U). Then $\sum\limits_{ij} |a_{ij}|^2 = \sum\limits_{ij} |b_{ij}|^2$.

17. For the following questions, find a projection matrix P that projects $\mathbf{b}$ onto the column space of A, that is, $P\mathbf{b} \in \operatorname{col}(A)$ and $\mathbf{b} - P\mathbf{b}$ is orthogonal to $\operatorname{col}(A)$.
$$(i)\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad (ii)\quad A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

18. We are looking for the parabola $y = c + dt + et^2$ that gives the least squares fit to these four measurements:
$y = 1$ at $t = -2$, $y = 1$ at $t = -1$, $y = 1$ at $t = 1$ and $y = 0$ at $t = 2$.
    (a) Write down the four equations ($A\mathbf{x} = \mathbf{b}$) for the parabola $c + dt + et^2$ to go through the given four points. Prove that $A\mathbf{x} = \mathbf{b}$ has no solution.
    (b) For finding a least square fit of $A\mathbf{x} = \mathbf{b}$, *i.e.*, of $A \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \mathbf{b}$, what equations would you solve?
    (c) Compute A^TA. Compute its determinant. Compute its inverse.
    (d) Now, determine the parabola $y = c + dt + et^2$ that gives the least squares fit.
    (e) The first two columns of A are already orthogonal. From column 3, subtract its projection onto the plane of the first two columns to get the third orthogonal vector $\mathbf{v}$. Normalize $\mathbf{v}$ to find the third orthonormal vector $\mathbf{w}_3$ from Gram-Schmidt.
    (f) Now compute $\mathbf{x} = A \begin{bmatrix} c \\ d \\ e \end{bmatrix}$ to verify that $\mathbf{x}$ is indeed the projection vector onto the column space of the matrix A.