Problem Set 1

Problems marked **(T)** are for discussions in Tutorial sessions.

1. If A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix, then show that $A(BD) = (AB)D$ (Associativity holds).

2. If A is an $m \times n$ matrix, B and C are $n \times p$ matrices and D is a $p \times s$ matrix, then show that

   (a) $A(B + C) = AB + AC$ (Distributive law holds).

   (b) $(B + C)D = BD + CD$ (Distributive law holds).

3. **(T)** Let A and B be 2×2 real matrices such that $A \begin{bmatrix} x \\ y \end{bmatrix} = B \begin{bmatrix} x \\ y \end{bmatrix}$ for all $(x, y) \in \mathbb{R}^2$. Prove that $A = B$.

4. Let A and B be $m \times n$ real matrices such that $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then, $A = B$

5. $(A + B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$ whenever $A + B$ and AB are defined.

6. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then $A = S + T$, where $S^* = S$ (Hermitian matrix) and $T^* = -T$ (skew-Hermitian matrix).

7. Give examples of 3×3 non zero matrices A and B such that $A^2 = 0$ and $B^3 = B$.

8. Show by an example that if $AB \neq BA$ then $(A + B)^2 = A^2 + 2AB + B^2$ need not hold.

9. Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be invertible matrices. Then $(AB)^{-1} = B^{-1}A^{-1}$.

10. Let $A \in \mathbb{M}_n(\mathbb{C})$ be a nilpotent matrix. Then show that $I + A$ is invertible.

11. **(T)** Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Define Tr $(A) = \sum_{i=1}^{n} a_{ii}$. Then show that Tr $(AB) =$ Tr (BA). Hence or otherwise, show that if A is invertible then Tr $(ABA^{-1}) =$ Tr (B). Furthermore, show that there do not exist matrices A and B such that $AB - BA = cI$, for any $c \neq 0$.

12. Let $A \in \mathbb{M}_n(\mathbb{C})$. If $AA^* = \mathbf{0}$ then show that $A = \mathbf{0}$.

13. **(T)** The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$, $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

14. **(T)** Let $J = \mathbf{1}\mathbf{1}^*$. Then each entry of J equals 1. Determine condition(s) on a and b such that $bJ + (a - b)I_n$ is invertible. Find α and β in terms of a and b such that the inverse has the form $\alpha J + \beta I$.

15. **(T)** Let $\mathbf{x} \in \mathbb{M}_{3,1}(\mathbb{R})$. Then find $\mathbf{y}, \mathbf{z} \in \mathbb{M}_{3,1}(\mathbb{R})$ such that $\mathbf{x}^T\mathbf{y} = 0$ and $\mathbf{x}^T\mathbf{z} = 0$.

16. **(T)** Let A be an upper triangular matrix. If $AA^* = AA^*$ then A is a diagonal matrix.