

Lecture 15-16 : Riemann Integration

Integration is concerned with the problem of finding the area of a region under a curve.

Let us start with a simple problem : *Find the area A of the region enclosed by a circle of radius r .* For an arbitrary n , consider the n equal inscribed and superscribed triangles as shown in Figure 1.

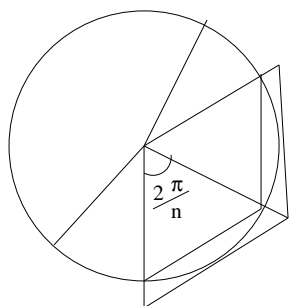


Figure 1

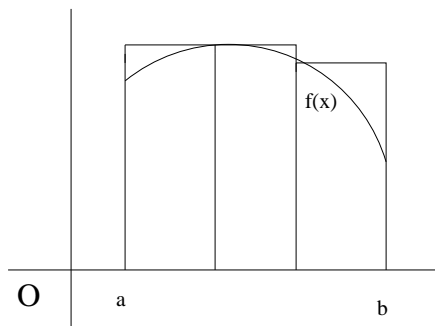
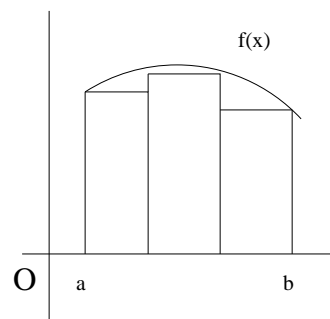


Figure 2



Since A is between the total areas of the inscribed and superscribed triangles, we have

$$nr^2 \sin(\pi/n) \cos(\pi/n) \leq A \leq nr^2 \tan(\pi/n).$$

By sandwich theorem, $A = \pi r^2$. **We will use this idea to define and evaluate the area of the region under a graph of a function.**

Suppose f is a non-negative function defined on the interval $[a, b]$. We first subdivide the interval into a finite number of subintervals. Then we squeeze the area of the region under the graph of f between the areas of the inscribed and superscribed rectangles constructed over the subintervals as shown in Figure 2. **If the total areas of the inscribed and superscribed rectangles converge to the same limit as we make the partition of $[a, b]$ finer and finer then the area of the region under the graph of f can be defined as this limit and f is said to be integrable.**

Let us define whatever has been explained above formally.

The Riemann Integral

Let $[a, b]$ be a given interval. A *partition* P of $[a, b]$ is a finite set of points $x_0, x_1, x_2, \dots, x_n$ such that $a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b$. We write $P = \{x_0, x_1, x_2, \dots, x_n\}$.

If P is a partition of $[a, b]$ we write $\Delta x_i = x_i - x_{i-1}$ for $1 \leq i \leq n$. Let f be a bounded real valued function on $[a, b]$. For a partition P of $[a, b]$, we define

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \text{ and } m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i \text{ and } L(P, f) = \sum_{i=1}^n m_i \Delta x_i.$$

The numbers $U(P, f)$ and $L(P, f)$ are called *upper and lower Riemann sums* for the partition P (see Figure 2).

Since f is bounded, there exist real numbers m and M such that $m \leq f(x) \leq M$, for all $x \in [a, b]$. Thus for every partition P ,

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a).$$

We define

$$\overline{\int_a^b} f dx = \inf U(P, f) \quad (1)$$

and

$$\underline{\int_a^b} f dx = \sup L(P, f). \quad (2)$$

(1) and (2) are called *upper and lower Riemann integrals* of f over $[a, b]$ respectively.

If the upper and lower integrals are equal, we say that f is *Riemann integrable* or *integrable*. In this case the common value of (1) and (2) is called the Riemann integral of f and is denoted by $\int_a^b f dx$ or $\int_a^b f(x) dx$.



Examples : 1. Consider the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f\left(\frac{1}{2}\right) = 1 \text{ and } f(x) = 0 \text{ for all } x \in [0, 1] \setminus \left\{\frac{1}{2}\right\}.$$

Then f is integrable. We show this using the definition as follows. For any partition P of $[0, 1]$, $L(P, f)$ is always 0 and hence the lower integral is 0. Let us evaluate the upper integral. Let $P = \{x_1, x_2, \dots, x_n\}$ be any partition of $[0, 1]$ and $\frac{1}{2} \in [x_i, x_{i+1}]$ for some i . Then $U(P, f) \leq 2 \max \Delta x_j$. Since we can always choose a partition P such that $\max \Delta x_j$ is as small as possible, the upper integral, which is the infimum of $U(P, f)$'s, is 0. Hence, f is integrable and $\int_0^1 f(x) dx = 0$.

2. *Not every bounded function is integrable.* For example the function

$$f(x) = 1 \text{ if } x \text{ is rational and } 0 \text{ otherwise}$$

is not integrable over any interval $[a, b]$ (Check this).

In general, determining whether a bounded function on $[a, b]$ is integrable, using the definition, is difficult. For the purpose of checking the integrability, we give a criterion for integrability, called Riemann criterion, which is analogous to the Cauchy criterion for the convergence of a sequence.

Let us define some concepts and results before presenting the criterion. Throughout, we will assume that f is a bounded real function on $[a, b]$.

Definition: A partition P_2 of $[a, b]$ is said to be finer than a partition P_1 if $P_2 \supset P_1$. In this case we say that P_2 is a *refinement* of P_1 . Given two partition P_1 and P_2 , the partition $P_1 \cup P_2 = P$ is called their common refinement.

The following theorem illustrates that refining partition increases lower terms and decreases upper terms.

Theorem 1 : Let P_2 be a refinement of P_1 then $L(P_1, f) \leq L(P_2, f)$ and $U(P_2, f) \leq U(P_1, f)$.

Proof (*): First we assume that P_2 contains just one more point than P_1 . Let this extra point be x^* . Suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are consecutive points of P_1 . Let

$$\begin{aligned} w_1 &= \inf\{f(x) : x_{i-1} \leq x \leq x^*\} \text{ and} \\ w_2 &= \inf\{f(x) : x^* \leq x \leq x_i\} \end{aligned}$$

Then $w_1 \geq m_i$ and $w_2 \geq m_i$ where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$. Then

$$L(P_2, f) - L(P_1, f) = w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i(x_i - x_{i-1}) \geq 0.$$

If P_2 contains k more points then we repeat this process k -times. The other inequality is analogously proved. (Prove it). \square

The geometric interpretation suggests that the lower integral is less than or equal to the upper integral. So the next result is also anticipated.

Corollary 2 : $\int_a^b f dx \geq \int_a^b f dx$.

Proof (*) : Let P_1, P_2 be two partitions and let P be their common refinement. Then

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f).$$

Thus for any two partitions P_1 and P_2 , we have $L(P_1, f) \leq U(P_2, f)$.

Fix P_2 and take sup over all P_1 . Then $\int_a^b f dx \leq U(P_2, f)$. Now take inf over all P_2 to get the desired result. \square

In the following result we present the Reimann criterion (a necessary and sufficient condition for the existence of the integral of a bounded function).

Theorem 3 : (Riemann's criterion for integrability): f is integrable on $[a, b] \Leftrightarrow$ for every $\epsilon > 0$ there exists a partition P such that

$$U(P, f) - L(P, f) < \epsilon. \quad (3)$$

Proof (*) : For any P , we have

$$L(P, f) \leq \int_a^b f dx \leq \int_a^b f dx \leq U(P, f).$$

Therefore (3) implies that

$$\int_a^b f dx - \int_a^b f dx < \epsilon, \quad \forall \epsilon > 0.$$

Hence $\int_a^b f dx = \int_a^b f dx$ i.e. f is integrable. Conversely, suppose f is integrable and $\epsilon > 0$. Then there exist partitions P_1 and P_2 such that

$$U(P_2, f) - \int_a^b f dx < \epsilon/2 \quad \text{and} \quad \int_a^b f dx - L(P_1, f) < \epsilon/2$$

Let P be the common refinement of P_1 and P_2 . Then $U(P, f) - L(P, f) < \epsilon$. \square

The proof of the following corollary is immediate from the previous theorem.

Corollary 3 : Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Suppose (P_n) is a sequence of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \rightarrow 0$, then f is integrable.

Problem : Let $f : [0, 1] \rightarrow \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$ Show that f is integrable and find $\int_0^1 f(x) dx$.

Solution : We will use the Riemann criterion to show that f is integrable on $[0, 1]$. Let $\epsilon > 0$ be given. We will choose a partition P such that $U(P, f) - L(P, f) < \epsilon$. Since $1/n \rightarrow 0$, there exists N such that $1/n \in [0, \epsilon]$ for all $n > N$. So only finite number of $\frac{1}{n}$'s lie in the interval $[\epsilon, 1]$. Cover these finite number of $\frac{1}{n}$'s by the intervals $[x_1, x_2], [x_3, x_4], \dots, [x_{m-1}, x_m]$ such that $x_i \in [\epsilon, 1]$ for all

$i = 1, 2, \dots, m$ and the sum of the length of these m intervals is less than ε . Consider the partition $P = \{0, \varepsilon, x_1, x_2, \dots, x_m\}$. It is clear that $U(P, f) - L(P, f) < 2\varepsilon$. Hence by the Riemann criterion the function is integrable. Since the lower integral is 0 and the function is integrable, $\int_0^1 f(x)dx = 0$.

We will apply the Riemann criterion for integrability to prove the following two existence theorems.

Theorem 4: *If f is continuous on $[a, b]$ then f is integrable.*

Proof : Let $\epsilon > 0$. Since f is uniformly continuous, choose $\delta > 0$ such that $|s - t| < \delta \Rightarrow |f(s) - f(t)| < \epsilon$ for $s, t \in [a, b]$.

Let P be a partition of $[a, b]$ such that $\Delta x_i < \delta \forall i = 1, 2, \dots, n$. Then

$$M_i - m_i \leq \epsilon \quad \forall i = 1, 2, \dots, n.$$

Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i) \Delta x_i \leq \epsilon(b - a).$$

This implies that f is integrable. □

Theorem 5: *If f is a monotone function then f is integrable.*

Proof : Suppose f is monotonically increasing (the proof is similar in the other case.) Choose a partition P such that $\Delta x_i = \frac{b-a}{n}$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore

$$\begin{aligned} U(P, f) - L(P, f) &= \frac{b-a}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \\ &= \frac{b-a}{n} [f(b) - f(a)] \\ &< \epsilon \quad \text{for large } n. \end{aligned}$$

Hence f is integrable. □

In the following problem we will see that limit and integral cannot be interchanged.

Problem : Let $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y} & \text{if } 0 \leq y < 1 \\ 0 & \text{if } y = 1 \end{cases}$. Then prove that $\lim_{n \rightarrow \infty} \int_0^1 g_n(y) dy = \frac{1}{2}$ whereas

$$\int_0^1 \lim_{n \rightarrow \infty} g_n(y) dy = 0.$$

Solution : From the ratio test for sequences we can show that $\lim_{n \rightarrow \infty} \frac{ny^{n-1}}{1+y} = 0$, for each $0 < y < 1$. Therefore $\int_0^1 \lim_{n \rightarrow \infty} g_n(y) dy = 0$.

For the other part, use integration by parts to see that $\int_0^1 \frac{ny^{n-1}}{1+y} dy = \frac{1}{2} + \int_0^1 \frac{y^n}{(1+y)^2} dy$. Note that $\int_0^1 \frac{y^n}{(1+y)^2} dy \leq \int_0^1 y^n = \frac{1}{n+1} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\lim_{n \rightarrow \infty} \int_0^1 g_n(y) dy = \frac{1}{2}$.