MSO202A COMPLEX VARIABLES
Solution -4

Problems for Discussion:

1. Use the ML-inequality to prove the following inequalities:

   (a) $\left|\int_\gamma \frac{1}{1+z^2}\,dz\right| \le \frac{\pi}{3}$, γ is the arc of $|z|=2$ from 2 to $2i$.

   (b) $\left|\int_\gamma (1+z^2)\,dz\right| \le \pi R(R^2+1)$, γ is the semicircular arc of $|z|=R$.

   Solution:

   (a) $\left|\dfrac{1}{1+z^2}\right| \le \dfrac{1}{|z|^2-1} = \dfrac{1}{3}, L = \pi.$

   (b) $|(1+z^2)| \le |z|^2+1 = R^2+1, L=\pi R.$

2. Evaluate, by parametrizing the path if you must, or otherwise:

   (a) $\int_C \tan z\,dz$, where C is the circle $|z|=1$ oriented counter -clockwise.

   (b) $\int_C \text{Re } z^2\,dz$, C is the circle $|z|=1$ oriented counter -clockwise.

   (c) $\int_C e^{4z}\,dz$, C is the shortest path from $8-3i$ to $8-(3+\pi)i$.

   Solution:

   (a) 0, as $\tan z$ is analytic in a disc containing the unit circle $|z|=1$.

   (b) $\int_C \text{Re } z^2\,dz = \int_0^{2\pi} \cos 2\theta\,d\theta = 0.$

   (c) As e^{4z} has primitive $F(z) = \frac{e^{4z}}{4}$, $\int_C e^{4z}\,dz = F(8-(3+\pi)i.) - F(8-3i).$

3. For what closed contours C will it follow from Cauchy's integral theorem that $(a)\ \displaystyle\int_C \frac{1}{z}\,dz = 0,\quad (b)\ \int_C \frac{e^{1/z}}{z^2+9}\,dz = 0?$

   Solution: (a) Any closed contours C which does not enclose 0. (b) Any closed contours C which does not enclose $0, \pm 3i$.

4. Integrate $\frac{z^2}{z^4-1}$ counter -clockwise around the circle $(a)|z+1|=1$ (b) $|z+i|=1$.

   Solution: (a) $z^4-1=(z^2+1)(z^2-1)$, $\frac{z^2}{z^4-1} = \frac{z^2-1+1}{z^4-1} = \frac{1}{z^2+1}+\frac{1}{z^4-1}$, $\int_C \frac{z^2}{z^4-1}\,dz = 0+2\pi i f(-1) = -\frac{\pi i}{2}$, where $f(z) = \frac{1}{(z-1)(z^2+1)}$. (b) Similar.

5. Integrate the following functions counter -clockwise on the unit circle $|z|=1$: $(a)\dfrac{z^3}{2z-i}\ (b)\ \dfrac{\cosh 3z}{2z}\ (c)\ \dfrac{z^3\sin z}{3z-1}.$

   Solution: (a)$2\pi i\frac{z^3}{2}|_{z=i/2}$. (b) $\frac{\cosh 3z}{2}|_{z=0} = 1$. (c) $\frac{z^3\sin z}{3}|_{z=1/3}$.

6. Let Γ denote the positively oriented boundary of the square whose sides lie on the lines $x=\pm 2$ and $y=\pm 2$. Using Cauchy integral formula, evaluate the following integrals:

   $$(a)\int_\Gamma \frac{e^{-z}}{z-2\pi i}\,dz (b)\int_\Gamma \frac{\cos z}{z(z^2+8)}\,dz (c)\int_\Gamma \frac{z}{2z+1}\,dz (d)\int_\Gamma \frac{\cosh z}{z^4}\,dz.$$

   Solution: Using the Cauchy integral formula :

(a) 0.

(b) $i\pi/4$.

(c) $\displaystyle\int_\gamma \frac{z/2}{z+1/2}\,dz = 2i\pi(-1/4) = -i\pi/2.$

(d) $2\pi i \sinh 0/(3!) = 0$.

7. Let C be the positively oriented circle $|z| = 3$. If $f(w) = \int_C \frac{2z^2-z-2}{z-w}\,dz$, $|w| \neq 3$, then show that $f(2) = 8i\pi$. What is $f(w)$, if $|w| > 3$?

Solution: $f(2) = \displaystyle\int_C \frac{2z^2-z-2}{z-2}\,dz = 2\pi i(2z^2-z-2)|_{z=2} = 8\pi i$. When $|w| > 3$, the integrand is analytic in a open set containing C (since w lies outside C) and is hence 0.

Problems for Tutorial:

1. Evaluate, by parametrizing the path if you must, or otherwise:

(a) $\displaystyle\int_{-i}^{i} \frac{1}{z}\,dz$

(b) $\int_C \sin^2 z\,dz$, C is from $-\pi i$ to πi along $|z| = \pi$ oriented counter -clockwise.

Solution:

(a) The integral is well defined as the function $\frac{1}{z}$ is analytic in a simply connected domain $\mathbb{C} \setminus$ the negative real axis containing $\pm i$. Therefore, $\displaystyle\int_{-i}^{i} \frac{1}{z}\,dz = Ln(i) - Ln(-i) = i\pi.$

(b) As $\sin^2 z = \frac{1-\cos 2z}{2}$ has primitive $F(z) = \frac{z}{2} - \frac{\sin 2z}{4}$, $\int_C \sin^2 z\,dz = F(\pi i) - F(-\pi i) = \pi i$.

2. For what closed contours C will it follow from Cauchy's integral theorem that (a) $\displaystyle\int_C Ln(z)\,dz = 0$ (b) $\displaystyle\int_C \frac{\cos z}{z^6 - z^2}\,dz = 0$?

Solution: (a) Any closed contours C which is contained in the simply connected domain $\mathbb{C} \setminus$ the negative real axis.

(b) Any closed contours C which does not enclose $0, \pm 1, \pm i$.

3. Apply Cauchy's theorem to $f(z) = e^{z^2}$ on the rectangle with vertices at $\pm a$ and $\pm a + ib$, $a, b \in \mathbb{R}^+$ to show that $\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx\,dx = \sqrt{\pi}e^{-b^2}$.

Solution: Let the oriented contour be $C_1 \cup C_2 \cup C_3 \cup C_4$ where $C_1 : z = x, dz = dx$ x varies from $-a$ to a: $C_2 : z = a + iy, dz = idy$ y goes from 0 to b, $C_3 : z = x + ib, dz = dx$ with x goes from a to $-a$ and for $C_4 : z = -a + iy, dz = idy$ with y going from b to 0.

By Cauchy's theorem, $\sum_{i=1}^{4} \int_{C_i} f = 0$.

$\left|\int_{C_2} e^{z^2}\,dz\right| = \left|\int_0^b e^{-(a^2-y^2)} e^{-2ayi} i\,dy\right| \leq e^{-a^2} \int_0^b e^{y^2}\,dy \to 0$ as $a \to \infty$. Similarly, $\left|\int_{C_4} e^{z^2}\,dz\right| \to 0$ as $a \to \infty$. Thus $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} = \int_{-\infty}^{\infty} e^{-(x^2+b^2)} e^{-2ixb}\,dx$.

The result follows by comparing the real parts.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\{z \in \mathbb{C} : z \neq 0\}$ and bounded on the set $\{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. Prove that $\int_{|z|=R} f(z)dz = 0$ for every $R > 0$.

   Solution: Let $r < \frac{1}{2}$. Using Cauchys theorem for multiply connected domains, $\displaystyle\int_{|z|=R} f(z)\,dz = \int_{|z|=r} f(z)\,dz$, which using ML inequality satisfy $\displaystyle\left|\int_{|z|=R} f(z)\,dz\right| = \left|\int_{|z|=r} f(z)\,dz\right| \leq \sup_{|z|=r} |f(z)| 2\pi r \to 0$, as $r \to 0$.

5. Using Cauchy integral formula integrate counterclockwise:

$$\oint_C \frac{\text{Ln }(z+1)}{z^2+1}\,dz, \quad C: \ |z-2i| = 2.$$

   Solution :

$$\oint_C \frac{\text{Ln }(z+1)}{z^2+1}\,dz = \frac{i}{2}\oint_C \text{Ln }(z{+}1)\left[\frac{1}{z-i} - \frac{1}{z+i}\right]\,dz = \frac{i}{2}\oint_C \frac{\text{Ln }(z+1)}{z-i}\,dz = \pi\text{Ln}(1{+}i).$$

   as $z = -i$ lies out side $|z - 2i| = 2$ and hence the intergral of that term is zero by Cauchy's intergal theorem.