Lowpass FIR Digital Filters

- The time-domain input-output relation of the simplest causal first-order lowpass FIR digital filter is given by

$$\boxed{y[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}x[n-1]}$$

- Note: Above is simply a 2-point moving average filter

Lowpass FIR Digital Filters

- The transfer function of this filter is given by

$$\boxed{H_{LP}(z) = \tfrac{1}{2}(1+z^{-1})}$$

- Its frequency response is given by

$$H_{LP}(e^{j\omega}) = \tfrac{1}{2}(1+e^{-j\omega})$$
$$= \tfrac{1}{2}e^{-j\omega/2}(e^{j\omega/2}+e^{-j\omega/2})$$
$$= e^{-j\omega/2}\cos(\omega/2)$$

Lowpass FIR Digital Filters

- The magnitude function is thus

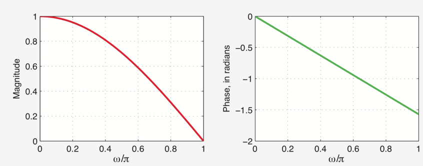
$$|H_{LP}(e^{j\omega})| = \cos(\omega/2)$$

- Note: The magnitude function is a monotonically decreasing function of ω with a maximum value of 1 at ω = 0 and a minimum value of 0 at ω = π
- The phase function is

$$\theta(\omega) = \arg\{H_{LP}(e^{j\omega})\} = -\omega/2$$

Lowpass FIR Digital Filters

- Thus, the FIR filter has a lowpass magnitude response and a linear phase
- Plots of the magnitude and phase functions



Lowpass FIR Digital Filters

- Usually the cutoff frequency ω_c of the lowpass filter is given by

$$|H_{LP}(e^{j\omega_c})|^2 = \frac{1}{2}|H_{LP}(e^{j0})|^2$$

- The gain $\mathcal{G}_{LP}(\omega_c)$ in dB at $\omega = \omega_c$ is given by

$$\mathcal{G}_{LP}(\omega_c) = 10\log_{10}|H_{LP}(e^{j\omega})|^2$$

Lowpass FIR Digital Filters

- Hence,

$$\mathcal{G}_{LP}(\omega_c) = 10\log_{10}|H_{LP}(e^{j0})|^2 + 10\log_{10}\left(\frac{1}{2}\right)$$
$$= -3.0133 \cong -3 \ dB$$

as the dc gain $20\log_{10}|H_{LP}(e^{j0})| = 0$

- Thus, the gain at $\omega = \omega_c$ is approximately 3 dB below the dc gain
- ω_c is called the 3-dB cutoff frequency of the lowpass filter

Lowpass FIR Digital Filters

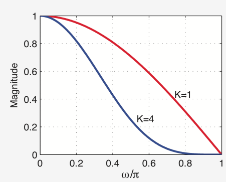
- Exact value of ω_c can be determined by solving
$$\left|H_{LP}(e^{j\omega_c})\right|^2 = \cos^2(\omega_c/2) = 1/2$$
which yields $\boxed{\omega_c = \pi/2}$
- Passband: $0 \le \omega \le \pi/2$
- Stopband: $\pi/2 < \omega < \pi$

Lowpass FIR Digital Filters

- A straightforward approach to achieve a sharper magnitude response is to cascade several first-order lowpass FIR filters
- A cascade of K first-order lowpass FIR filters has a transfer function given by
$$\boxed{G_K(z) = \frac{1}{2^K}(1+z^{-1})^K}$$
with a 3-dB cutoff frequency given by
$$\boxed{\omega_c = 2\cos^{-1}(2^{-1/2K})}$$

Lowpass FIR Digital Filters

- A plot of the magnitude responses for $K = 4$ and $K = 1$ is shown below illustrating a sharper drop off of the cascaded filter



Lowpass FIR Digital Filters

- For $K = 4$, the 3-dB cutoff frequency is
$$\omega_c = 0.2612\pi$$
- Thus the width of the passband of the cascaded structure is much smaller while the width of the stopband is much larger than those of a single section

Lowpass FIR Digital Filters

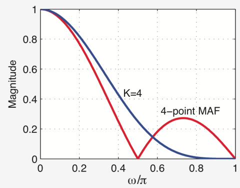
- The total number of first-order FIR lowpass sections needed to design a lowpass filter with a given 3-dB cutoff frequency ω_c is
$$\boxed{K = \frac{1}{2\log_2(\cos(\omega_c/2))}}$$

Lowpass FIR Digital Filters

- Higher order lowpass FIR filter with improved magnitude response than that of a cascade of first order lowpass FIR digital filters is provided by an M-point moving-average filter with a transfer function
$$\boxed{H_{LP}(z) = \frac{1}{M}\sum_{\ell=0}^{M-1} z^{-\ell}}$$

Lowpass FIR Digital Filters

- Figure below shows the magnitude responses of a 4-point moving average filter and a cascade of 4 first-order lowpass FIR filters



Highpass FIR Digital Filters

- The simplest causal highpass FIR digital filter is described in the time-domain by a first-order difference equation given by

$$\boxed{y[n] = \tfrac{1}{2}x[n] - \tfrac{1}{2}x[n-1]}$$

- Its transfer function is given by

$$\boxed{H_{HP}(z) = \tfrac{1}{2}(1 - z^{-1})}$$

Highpass FIR Digital Filters

- Its frequency response is given by

$$H_{HP}(e^{j\omega}) = \tfrac{1}{2}(1 - e^{-j\omega})$$

$$= \tfrac{1}{2}e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})$$

$$= je^{-j\omega/2}\sin(\omega/2)$$

- The magnitude function is thus

$$\left|H_{HP}(e^{j\omega})\right| = \sin(\omega/2)$$

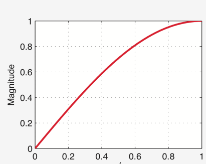
Highpass FIR Digital Filters

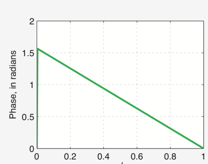
- Note: The magnitude function is a monotonically increasing function of ω with a minimum value of 0 at ω = 0 and a maximum value of 1 at ω = π
- The phase function is

$$\theta(\omega) = \arg\{H_{HP}(e^{j\omega})\} = \frac{\pi - \omega}{2}$$

Highpass FIR Digital Filters

- Thus, the FIR filter has a highpass magnitude response and a linear phase
- Plots of the magnitude and phase functions





Highpass FIR Digital Filters

- It can be shown that the 3-dB cutoff frequency of the above highpass filter is also

$$\boxed{\omega_c = \pi/2}$$

- Passband: $\pi/2 \le \omega \le \pi$
- Stopband: $0 \le \omega < \pi/2$

Highpass FIR Digital Filters

- A highpass FIR filter with a sharper magnitude response can be designed by cascading several sections of the first-order highpass FIR digital filter
- Here the 3-dB cutoff frequency ω_c of a cascade of K sections is given by

$$\boxed{\omega_c = 2\sin^{-1}(2^{-1/2K})}$$

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Highpass FIR Digital Filters

- The value of the number of first-order FIR highpass sections K needed to realize an FIR highpass filter in a cascade form for a specified 3-dB cutoff frequency ω_c is given by

$$\boxed{K = -\frac{1}{2\log_2(\sin(\omega_c/2))}}$$

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Highpass FIR Digital Filters

- Alternately, a higher-order highpass FIR filter can be designed using

$$\boxed{H_{LP}(z) = \frac{1}{M}\sum_{\ell=0}^{M-1}(-1)^{\ell} z^{-\ell}}$$

- An application of the highpass FIR filters described above is in moving-target-indicator (MTI) radars

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Highpass FIR Digital Filters

- In this type of radars, signals, called clutters, generated from objects that are fixed in the radar path beam, such as ground echoes and weather returns, can interfere in the detection of moving objects
- The clutters can be removed by passing the received radar signal through a highpass filter

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Highpass FIR Digital Filters

- One such filter is the first-order highpass FIR filter without the scale factor of 1/2,

$$H(z) = 1 - z^{-1}$$

commonly known as the two-pulse canceller

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Highpass FIR Digital Filters

- For a more effective removal of the clutter a cascade of two first-order highpass FIR filters without the scakle factor of 1/4,

$$H(z) = 1 - 2z^{-1} + z^{-2}$$

called three-pulse canceller, is used
- The above filter is also known as the 1D discrete Laplacian operator, sometimes used for signal enhancement

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Lowpass IIR Digital Filters

- The causal first-order lowpass IIR digital filter is described in the time-domain by the input-output equation

$$y[n] = \alpha y[n-1] + \left(\tfrac{1-\alpha}{2}\right)x[n] + \left(\tfrac{1-\alpha}{2}\right)x[n-1]$$

- Its transfer function is

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right), \quad 0 < |\alpha| < 1$$

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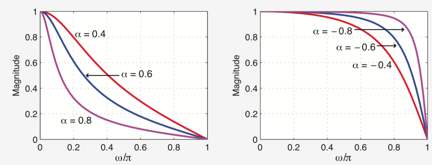
Lowpass IIR Digital Filters

- Plots of the magnitude responses of the above filter are shown below for several positive and negative values of α



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Lowpass IIR Digital Filters

- For a given 3-dB cutoff frequency ω_c, the value of α for a stable filter is given by

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

- Passband: $0 \le \omega \le \omega_c$
- Stopband: $\omega_c < \omega \le \pi$

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Lowpass IIR Digital Filters

Example – Design a first-order lowpass IIR digital filter with a 3-dB cutoff frequency at $\omega_c = 0.4\pi$

- Thus, $\alpha = \dfrac{1-\sin\omega_c}{\cos\omega_c} = \dfrac{1-\sin(0.4\pi)}{\cos(0.4\pi)} = 0.1584$
- Hence, the desired transfer function is

$$H_{LP}(z) = 0.4208\left(\frac{1+z^{-1}}{1-0.1584z^{-1}}\right)$$

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Lowpass IIR Digital Filters

- Lowpass IIR digital filters with sharper magnitude responses can be designed by cascading several first-order lowpass IIR digital filters
- For a cascade of K such filters, the transfer function of the cascade structure is given by

$$G_{LP}(z) = \left(\frac{1-\alpha}{2}\cdot\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)^K, \quad 0 < |\alpha| < 1$$

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Lowpass IIR Digital Filters

- For a given 3-dB cutoff frequency ω_c of the cascade, the filter parameter α of the first-order section in the cascade is given by

$$\alpha = \frac{1+(1-C)\cos\omega_c - (\sqrt{2C-C^2})\sin\omega_c}{1-C+\cos\omega_c}$$

where

$$C = 2^{(K-1)/K}$$

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Lowpass IR Digital Filters

Example - Consider the design of a lowpass IIR digital filter using a cascade of five first-order lowpass IIR digital filters for a 3-dB cutoff frequency of $\omega_c = 0.6\pi$

- Substituting $K = 5$ and $\omega_c = 0.6\pi$ in the previous two equations we get $\alpha = -0.5623$
- Hence,

$$G_{LP}(z) = 0.5792\left(\frac{1+z^{-1}}{1+0.5623z^{-1}}\right)^5$$

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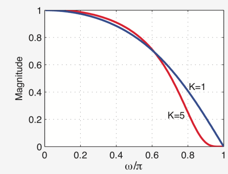
Lowpass IIR Digital Filters

- Magnitude responses of a single first-order section with a 3-dB cutoff frequency at 0.6π and a cascade of 5 first-order sections are shown below



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Highpass IIR Digital Filters

- The causal first-order highpass IIR digital filter is characterized in the time-domain by the input-output relation

$$y[n] = \alpha y[n-1] + \left(\tfrac{1+\alpha}{2}\right)x[n] - \left(\tfrac{1+\alpha}{2}\right)x[n-1]$$

- Its transfer function is

$$H_{LP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right),\quad 0 < |\alpha| < 1$$

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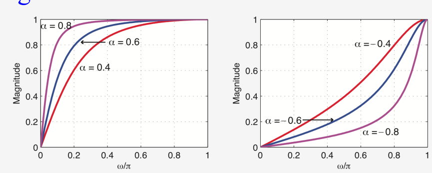
Highpass IIR Digital Filters

- Plots of the magnitude response of the above highpass filter for several positive and negative values of α are shown below



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Highpass IIR Digital Filters

- For a given 3-dB cutoff frequency ω_c, the value of α for a stable filter is given by

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

- Passband: $\omega_c \le \omega \le \pi$
- Stopband: $0 \le \omega < \omega_c$

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Highpass IIR Digital Filters

- Highpass IIR digital filters with sharper magnitude responses can be designed by cascading several first-order highpass IIR digital filters
- For a cascade of K such filters, the transfer function of the cascade structure is given by

$$G_{HP}(z) = \left(\frac{1+\alpha}{2}\cdot\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)^K,\quad 0 < |\alpha| < 1$$

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Highpass IIR Digital Filters

- For a given 3-dB cutoff frequency ω_c of the cascade, the filter parameter α of the first-order section in the cascade is given by

$$\boxed{\alpha = \frac{(\sqrt{2C - C^2})\sin\omega_c - 1 + (1 - C)\cos\omega_c}{1 - C - \cos\omega_c}}$$

where

$$\boxed{C = 2^{(K-1)/K}}$$

Bandpass IIR Digital Filters

- The simplest causal bandpass IIR digital filter is characterized in the time-domain by a second-order difference equation given by

$$\boxed{\begin{aligned} y[n] = \beta(1+\alpha)y[n-1] - \alpha y[n-2] \\ + \left(\tfrac{1-\alpha}{2}\right)x[n] - \left(\tfrac{1-\alpha}{2}\right)x[n-2] \end{aligned}}$$

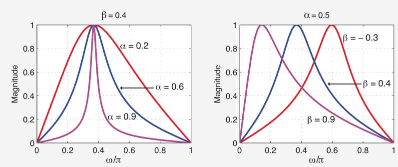
Bandpass IIR Digital Filters

- Its transfer function is given by

$$\boxed{H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}\right)}$$

  where $|\alpha| < 1$ and $|\beta| < 1$ for stability
- Plots of the magnitude response of the above bandpass filter for several values of the parameters α and β are shown in the next slide

Bandpass IIR Digital Filters



- The frequency ω_o at which the magnitude function assumes its maximum value of unity, known as the center frequency of the bandpass filter, is given by $\boxed{\omega_o = \cos^{-1}(\beta)}$

Bandpass IIR Digital Filters

- The frequencies ω_{c1} and ω_{c2}, at which the square magnitude function takes the value of 1/2 are called the 3-dB cutoff frequencies
- The 3-dB bandwidth of the bandpass filter is given by

$$\boxed{B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)}$$

Bandpass IIR Digital Filters

Example - Consider the design of a causal second-order bandpass IIR digital filter with a center frequency at $\omega_o = 0.6\pi$ and a 3-dB bandwidth of $B_w = 0.2\pi$

- Hence, $\beta = \cos(0.6\pi) = -0.3090$ and

$$\frac{2\alpha}{1+\alpha^2} = \cos(0.2\pi) = 0.809$$

Bandpass IIR Digital Filters

- The solution of the quadratic equation yields two values of α: 1.9626 and 0.5095
- The stable bandpass filter is for $\alpha = 0.5095$ which leads to the transfer function:

$$H_{BP}(z) = \frac{0.2452(1 - z^{-2})}{1 + 0.4665 z^{-1} + 0.5095 z^{-2}}$$

- As the magnitudes of both α and β are less than 1, the above bandpass filter is stable

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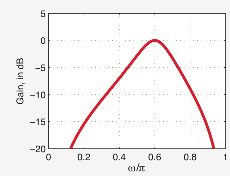
Bandpass IIR Digital Filters

- A plot of the gain response of the designed filter is shown below



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Bandpass IIR Digital Filters

- The second-order bandpass filter can also be designed using the function `iirpeak` of MATLAB
- IIR bandpass filters with sharper magnitude responses, that is, with narrower bandwidths, can be realized by cascading several identical second-order bandpass filters

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Bandpass IIR Digital Filters

- Figure in the next slide shows the magnitude responses of a single section of a second-order bandpass digital filter (plot marked 1), a cascade of three sections of the same second-order bandpass filter (plot marked 3), and a cascade of six sections of the same second-order bandpass filter (plot marked 6)

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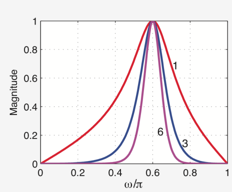
Bandpass IIR Digital Filters



- As can be seen from these plots, the 3-dB bandwidth decreases with an increase in the number of sections

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Bandstop IIR Digital Filters

- The simplest causal bandstop IIR digital filter is characterized in the time-domain by a second-order difference equation given by

$$y[n] = \beta(1+\alpha)y[n-1] - \alpha y[n-2] + \left(\tfrac{1+\alpha}{2}\right)x[n] - 2\beta\left(\tfrac{1+\alpha}{2}\right)x[n-1] + \left(\tfrac{1+\alpha}{2}\right)x[n-2]$$

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Bandstop IIR Digital Filters

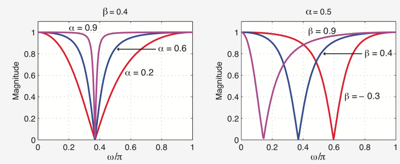
- Its transfer function is given by

$$\boxed{H_{BS}(z) = \frac{1+\alpha}{2}\left(\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)}$$

where $|\alpha| < 1$ and $|\beta| < 1$ for stability

- Plots of the magnitude response of the above bandpass filter for several values of the parameters α and β are shown in the next slide

Bandstop IIR Digital Filters



- The frequency ω_o at which the magnitude function has a value 0, known as the notch frequency of the bandpass filter, is given by

$$\boxed{\omega_o = \cos^{-1}(\beta)}$$

Bandstop IIR Digital Filters

- If ω_{c1} and ω_{c2} denote the frequencies at which the gain of the bandstop filter is 3-dB below the 0 dB gain at dc and at ω = π, the 3-dB notch bandwidth B_w is given by

$$\boxed{B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)}$$

- The second-order bandstop filter is also known as the notch filter

Bandstop IIR Digital Filters

- The second-order bandstop IIR digital filter can be designed using the function `iirnotch` of MATLAB

Example – We design a bandstop filter with a notch frequency at 0.6π and a 3-dB notch bandwidth of 0.2π

- The code fragments for the design of this filter are given in the next slide

Bandstop IIR Digital Filters

```
wo = 0.6;
BW = 0.2;
[num,den] = iirnotch(wo,BW);
```

which yield

```
num =
0.7548   0.4665   0.7548
den =
1.0000   0.4665   0.5095
```

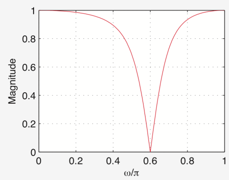

Bandstop IIR Digital Filters

- The transfer function of the designed bandstop filter is

$$H_{BS}(z) = \frac{0.7548 + 0.4665 z^{-1} + 0.7548 z^{-2}}{1 + 0.4665 z^{-1} + 0.5095 z^{-2}}$$

- A plot of its magnitude response is shown in the next slide

Bandstop IIR Digital Filters



- It can be shown that for this example $\alpha = 0.5095$ and $\beta = -0.309$

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