MTH 102A ASSIGNMENT-10 (ODE) **Mar 24 & 26, 2020.**

1. (a) Show that the k-th order linear differential operator $L := \sum_{i=0}^{k} a_i(x) \frac{d^i}{dx^i}$ is a linear transformation from $C^k(\bar{I})$ to $C(\bar{I})$.
   (b) Show that the solution to the homogeneous ODE $Ly = 0$ is a subspace of $C(\bar{I})$.
   (c) Recall that we have stated a theorem (without proof) that the dimesion of $N(L)$ is k. Use this information to give an expression for the general solution of $Ly = 0$.
   (d) Classify the set of all solutions of the inhomogeneous ODE $Ly = f$, for any given non-zero $f \in C(\bar{I})$, in terms of the $N(L)$.

2. Check that both $y = x^2 \sin x$ and $y = 0$ are solutions of the initial value problem

$$\begin{cases} x^2y'' - 4xy' + (x^2+6)y &= 0 \\ y(0) = y'(0) &= 0. \end{cases}$$

   Does the IVP violate the uniqueness result?

3. Solve the following second order ODE by reducing them to a first order ODE:
   (a) $xy'' + y' = (y')^2$.
   (b) $yy'' + (y')^2 + 1 = 0$.
   (c) $y'' - 2y' \coth x = 0$.

4. Find the curve $y = y(x)$ such that $y'' = y'$ and the line $y = x$ is tangent at the origin.

5. Check the linear dependence and independence of the following functions on the given intervals:
   (a) $\sin 4x$ and $\cos 4x$ in $\mathbb{R}$.
   (b) $\ln x$ and $\ln x^3$ in $(0, \infty)$.
   (c) $\cos 2x$ and $\sin^2 x$ in $(0, \infty)$.
   (d) x^3 and $x^2|x|$ in $[-1, 1]$.

6. Show that the following functions are linearly independent both using the definition of linear independence and using the Wronskian property:
   (a) e^x and e^{3x} in any $[a, b] \subset \mathbb{R}$.
   (b) x^2 and x^{-2} in any $[a, b] \subset \mathbb{R}$ not containing zero.

7. Let $L := \frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)$ be the second order linear differential operator where P and Q are continuous functions on $\mathbb{R}$.
   (a) Show that a solution to $Ly = 0$ with x-axis as tangent at any point in I must be identically zero in I.
   (b) Show that if y_1 and y_2 are two solutions of $Ly = 0$ with a common zero at any point in I then y_1 and y_2 are linearly dependent on I.
   (c) Show that $y_1 = x$ and $y_2 = \sin x$ are not solutions of $Ly = 0$.

(d) Let y_1 and y_2 be two linearly independent solutions of $Ly = 0$. Show that

i. between two consecutive zero of y_1, there exists a unique zero of y_2;

ii. Two different linear combinations $ay_1 + by_2$ and $cy_1 + dy_2$ are linearly independent solutions of $Ly = 0$ iff $ad - bc \neq 0$.

8. (a) Let $y_1, y_2 \in C^2(\bar{I})$. Show that if the Wronskian $W(y_1, y_2) \not\equiv 0$ in I then there exists a unique pair of continuous functions (P, Q) on I such that y_1 and y_2 span the solution space of $y'' + P(x)y' + Q(x) = 0$.

(b) Construct the second order linear ODE from the following pairs of solutions:

i. e^{-x} and xe^{-x}.

ii. $e^{-x} \sin 2x$ and $e^{-x} \cos 2x$.

iii. x and $x^2 + 1$.