Operations on Digital Signals

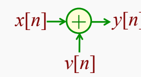
Elementary Operations

- A variety of operations is carried out on digital signals in discrete time
- In most cases, the operation implemented by a digital system is composed of some elementary operations

Operations on Digital Signals

Addition

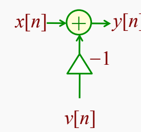
$$\boxed{y[n] = x[n] + v[n]}$$



Adder

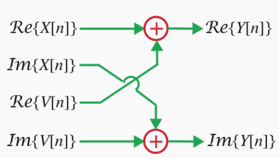
Subtraction

$$\boxed{y[n] = x[n] - v[n]}$$



Operations on Digital Signals

- Figure below shows the implementation of $Y[n] = X[n] + V[n]$ where $X[n]$ and $V[n]$ are complex-valued signals



Operations on Digital Signals

- The model for a sequence $x[n]$ corrupted by an additive random digital signal $\eta[n]$:

$$y[n] = x[n] + \eta[n]$$

where $y[n]$ is the noise-corrupted sequence

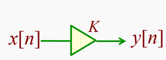
- Ensemble average operation is another application of the addition operation

Operations on Digital Signals

Amplitude Scaling

$$\boxed{y[n] = K \cdot x[n]}$$

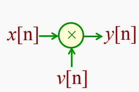
constant



Multiplier

Modulation

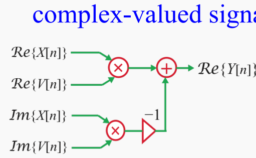
$$\boxed{y[n] = x[n] \cdot v[n] = x[n]v[n]}$$

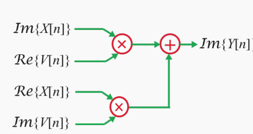


Modulator

Operations on Digital Signals

- Figure below shows the implementation of $Y[n] = X[n] \cdot V[n]$ where $X[n]$ and $V[n]$ are complex-valued signals





Operations on Digital Signals

- An application of the modulation operation is in the representation of a causal sequence:

$$y[n] = \underbrace{\sin(\omega_o n)\mu[n]}_{\tilde{x}[n]}$$

- Other applications include the implementation of the amplitude modulation scheme and the frequency-division multiplexing scheme

Operations on Digital Signals

Division

$$\boxed{y[n] = \frac{x[n]}{v[n]}}$$

- One application is to determine whether two sequences, which visually may look different, have the same shape
- The sequences have the same shape if their sample-wise ratio is approximately the same number for all values n in the range the sequences are defined

Operations on Digital Signals

Time-Shifting

$$\boxed{y[n] = x[n - N_o]}$$

- Shifts the sequence $x[n]$ in time by a fixed integer value N_o to generate another sequence $y[n]$
- If $N_o > 0$, then $y[n]$ is a delayed version of $x[n]$
- If $N_o < 0$, then $y[n]$ is an advanced version of $x[n]$

Operations on Digital Signals

- In practice, it is more common to make use of a device implementing the time-shifting operation by one sample period, that is, $N_o = \pm 1$
- For $N_o = +1$, the device is known as the unit delay
- For $N_o = -1$, the device is known as the unit advance operator

Operations on Digital Signals

- The schematic representations of the unit delay and the unit advance operations are usually depicted using the z-transform relation as indicated below

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n] \qquad\qquad x[n] \longrightarrow \boxed{z} \longrightarrow y[n]$$

Unit delay

$$y[n] = x[n-1]$$

Unit advance operator

$$y[n] = x[n+1]$$

Operations on Digital Signals

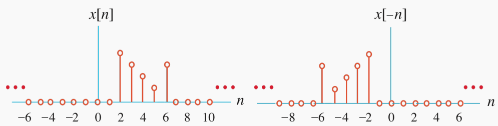
Time-Reversal

$$\boxed{y[n] = x[-n]}$$

- The time reversal operation, also called time folding operation, on a digital signal $x[n]$, develops another digital signal $y[n]$ that is a reflected version around the time index $n = 0$

Operations on Digital Signals

- Figure below illustrates the time-reversal operation on a sequence



Sampling Rate Alteration

Down-Sampling

- Operation employed to reduce the sampling rate of a sequence by an integer factor M

$$\boxed{y[n] = x[nM]}$$

- Schematic representation

$$\underset{F_T}{x[n]} \longrightarrow \boxed{\downarrow M} \longrightarrow \underset{F_T'}{y[n]}$$

Down-sampler

Sampling Rate Alteration

- Operation keeps every M-th sample of $x[n]$ and removes $M-1$ samples between the samples that are being kept
- Relation between the two sampling rates is

$$F_T' = \frac{F_T}{M}$$

- In most cases, a digital system is placed before the down-sampler for implementing the decimation operation to prevent aliasing

Sampling Rate Alteration

Up-Sampling

- Employed to increase the sampling rate of a sequence by an integer factor L

$$\boxed{y[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}}$$

- Schematic representation

$$\underset{F_T}{x[n]} \longrightarrow \boxed{\uparrow L} \longrightarrow \underset{F_T'}{y[n]}$$

Up-sampler

Sampling Rate Alteration

- Operation inserts $L-1$ equidistant zero-valued samples between two consecutive samples of $x[n]$
- Relation between the two sampling rates is

$$F_T' = L \cdot F_T$$

- In most cases, a digital system is placed after the up-sampler to replace the inserted zero-valued samples with more appropriate values

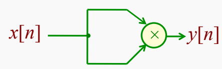
Nonlinear Operations

- The modulation and division operations are nonlinear operations
- We describe next three additional nonlinear operations that find applications in digital speech processing

Nonlinear Operations

- **Squarer**

$$y[n] = x^2[n]$$



- Used in the computation of the short-time energy of a speech signal

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Nonlinear Operations

- **Absolute Value Generator**

$$y[n] = |x[n]|$$

- Used in the computation of short-time average magnitude function of a speech signal

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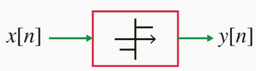
Nonlinear Operations

- **Signal Function Generator**

$$y[n] = \begin{cases} 1, & x[n] \geq 0 \\ -1, & x[n] < 0 \end{cases}$$



- Used in the computation of short-time zero-crossing rate of a speech signal

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Combination of Operations

- The elementary operations are used to generate sequences with more desirable waveforms

Generation of Periodic Sequences

- Let

$$\tilde{x}_1[n] = \sin(0.05\pi n)$$
$$\tilde{x}_2[n] = \sin(0.15\pi n)$$
$$\tilde{x}_3[n] = \sin(0.25\pi n)$$

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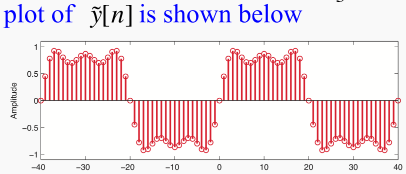
Combination of Operations

- Consider

$$\tilde{y}[n] = \sin(0.05\pi n) + \tfrac{1}{3}\sin(0.15\pi n) + \tfrac{1}{5}\sin(0.25\pi n)$$

- A plot of $\tilde{y}[n]$ is shown below



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Combination of Operations

Representation of an Arbitrary Sequence

- An important application of the basic sequence is in the modeling of an arbitrary sequence
- For example, an arbitrary sequence can be represented as a weighted sum of the unit sample sequence $\delta[n]$ and its time-shifted versions

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Combination of Operations

- Consider
$$\{x[n]\} = \{-2.1,\ 0,\ 0,\ 3.3,\ -0.9,\ 0,\ 0.56,\ 1.06\},$$
$$-2 \le n \le 5$$
- We can express the sequence $x[n]$ as
$$x[n] = -2.1\delta[n+2] + 3.3\delta[n-1] - 0.9\delta[n-2]$$
$$+0.5\delta[n-4] + 1.06\delta[n-5]$$

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Combination of Operations

- Such a representation allows us to express the output sequence of certain types of digital systems for an arbitrary input sequence as a function of its response to an unit sample sequence $\delta[n]$
- Certain class of digital signals and systems can be represented as weighted combinations of complex exponential sequences of the form $e^{j\omega n}$

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The Sampling Process

Relation Between the Analog Signal and Its Sampled Version

- In many applications we convert an analog signal $x_a(t)$ by sampling it uniformly at time intervals $t = nT$, generating the digital signal $x[n]$ according to
$$x[n] = x_a(t)\big|_{t=nT} = x_a(nT),$$
$$n = \ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots$$

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The Sampling Process

- The relation between the analog time variable t and the discrete time instants nT is given by
$$\boxed{nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}}$$
where $F_T = 1/T$ is the sampling frequency and $\Omega_T = 2\pi F_T$ is the sampling angular frequency

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The Sampling Process

- Consider the sampling of an analog sinusoidal signal
$$\tilde{x}_a(t) = A\sin(2\pi f_o t + \phi) = A\sin(\Omega_o t + \phi)$$
- The digital signal $x[n]$ generated by sampling $\tilde{x}_a(t)$ is given by
$$x[n] = A\sin(\Omega_o nT + \phi)$$
$$= A\sin\left(\frac{2\pi\Omega_o}{\Omega_T}n + \phi\right) = A\sin(\omega_o n + \phi)$$

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The Sampling Process

- In the last equation, ω_o is the normalized angular frequency of $x[n]$ and is related to the angular frequency Ω_o of $\tilde{x}_a(t)$ through
$$\boxed{\omega_o = \frac{2\pi\Omega_o}{\Omega_T} = \Omega_o T}$$
- If T is in seconds, then the unit of Ω_o is radians/second, the unit of f_o is Hz, and the unit of ω_o is radians/sample

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The Sampling Process

Non-uniqueness in the Discrete-Time Representation

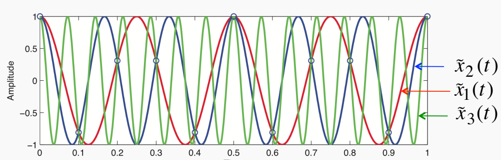
- An infinite number of analog signals after sampling may have an identical discrete-time representation
- The non-uniqueness problem is illustrated next by considering the sampling of three sinusoidal analog signals with different frequencies

The Sampling Process

- Consider $\tilde{x}_1(t) = \cos(8\pi t/\text{sec})$
  $\tilde{x}_2(t) = \cos(12\pi t/\text{sec})$
  $\tilde{x}_3(t) = \cos(28\pi t/\text{sec})$
- After sampling at a rate of T = 0.1 sec, we get
  $\tilde{x}_1[n] = \cos(0.8\pi n)$
  $\tilde{x}_2[n] = \cos(1.2\pi n)$
  $\tilde{x}_3[n] = \cos(2.8\pi n)$

The Sampling Process

- Figure below shows the three analog signals along with their sampled versions (shown with circles)



The Sampling Process

- Note: All three digital signals have the same sample value for a given time index n
- To verify this feature we apply trigonometric identities resulting in

$$\tilde{x}_2[n] = \cos(1.2\pi n) = \cos\big((2\pi - 0.8\pi)n\big) = \cos(0.8\pi n) = \tilde{x}_1[n]$$

$$\tilde{x}_3[n] = \cos(2.8\pi n) = \cos\big((2\pi + 0.8\pi)n\big) = \cos(0.8\pi n) = \tilde{x}_1[n]$$

The Sampling Process

- It can be shown that all sinusoidal analog signals $\tilde{x}_i(t) = \cos\big((10k \pm 4)\pi t/\text{sec}\big)$ with k any positive integer have identical discrete-time version given by $\tilde{x}_i[n] = \cos(0.8\pi n)$ after sampling at the rate $T = 0.1$ sec
- Thus, an analog sinusoidal signal with a higher frequency can appear as a digital sinusoidal signal of lower frequency if sampled at an inappropriate rate

The Sampling Process

- This type of phenomenon is commonly known as aliasing
- To associate a unique sinusoidal digital signal $\tilde{x}[n]$ to a given analog sinusoidal signal $\tilde{x}_a(t)$ we need to impose an additional condition on the allowable sampling rates

The Sampling Process

- If this condition is satisfied then we can recover the original analog signal from its digital version

Example – Consider

$$\tilde{x}_a(t) = 5\cos(30\pi t/\text{sec}) + 7\cos(80\pi t/\text{sec}) + 3\cos(230\pi t/\text{sec}) - 4\cos(320\pi t/\text{sec})$$

- $\tilde{x}_a(t)$ is sampled at a rate of 100 Hz generating $\tilde{x}[n]$

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The Sampling Process

- Here $T = 1/100 = 0.01$ sec
- Hence,

$$x[n] = 5\cos(30\times 0.01\pi n) + 7\cos(80\times 0.01\pi n) + 3\cos(230\times 0.01\pi n) - 4\cos(320\times 0.01\pi n)$$
$$= 5\cos(0.3\pi n) + 7\cos(0.8\pi n) + 3\cos(2.3\pi n) - 4\cos(3.2\pi n)$$

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The Sampling Process

- Using the trigonometric identity we have

$$\cos(2.3\pi n) = \cos((2\pi + 0.3\pi)n)$$
$$= \underbrace{\cos(2\pi n)}_{=1}\cos(0.3\pi n) - \underbrace{\sin(2\pi n)}_{=0}\sin(0.3\pi n)$$
$$= \cos(0.3\pi n)$$

- Similarly, it can be shown that

$$\cos(3.2\pi n) = \cos((4\pi - 0.8\pi)n) = \cos(0.8\pi n)$$

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The Sampling Process

- As a result, the two sequences $\cos(2.3\pi n)$ and $\cos(3.2\pi n)$ have aliased into sequences of lower frequencies as $\cos(0.3\pi n)$ and $\cos(0.8\pi n)$
- Hence, $x[n] = 5\cos(0.3\pi n) + 7\cos(0.8\pi n) + 3\cos(0.3\pi n) - 4\cos(0.8\pi n)$

$$= 8\cos(0.3\pi n) + 3\cos(0.8\pi n)$$

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Sampling Theorem

- Recall that the normalized angular frequency ω_o is restricted to the range $0 \le \omega_o < \pi$
- Therefore, it follows from the relation

$$\boxed{\omega_o = \frac{2\pi\Omega_o}{\Omega_T} = \Omega_o T}$$

that we need to ensure

$$\boxed{\frac{2\pi\Omega_o}{\Omega_T} < \pi}$$

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Sampling Theorem

- or, equivalently

$$\boxed{\Omega_T > 2\Omega_o}$$

to ensure no aliasing

- If $\Omega_T < 2\Omega_o$, then ω_o will be greater than π and the normalized digital angular frequency will get folded over to a value less than π causing aliasing

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Sampling Theorem

- Thus, to prevent aliasing, the sampling angular frequency Ω_T should be greater than twice the angular frequency of the sinusoidal analog signal that is being sampled

Sampling Theorem

- As in practice, an analog signal $x_a(t)$ is composed of a weighted sum of many sinusoidal signals
- Hence, to prevent aliasing the analog signal should be sampled at a rate greater than twice the highest frequency component present in the signal to ensure no aliasing

Sampling Theorem

- If the highest angular frequency present in the analog signal $x_a(t)$ is Ω_m, then the analog signal should be sampled at a sampling rate $\Omega_T = 2\pi / T$ where

$$\boxed{\Omega_T > 2\Omega_m}$$

- This condition is more commonly known as the sampling theorem, a formal proof of which is beyond the scope of this course

Digital Processing of Analog Signals

- For the digital processing of an analog signal, the analog signal first has to be transformed into a digital form using an analog-to-digital (A/D) converter
- In some applications, the processed digital signal needs to be transformed back into an analog signal using a digital-to-analog (D/A) converter

Digital Processing of Analog Signals

- Several additional circuits are needed in addition to the above two devices
- An analog lowpass filter, called the anti-aliasing filter, is placed before the sampler to ensure no aliasing by choosing the cutoff frequency of the filter to satisfy the condition of the sampling theorem

Digital Processing of Analog Signals

- As the conversion from analog to digital form by the A/D converter takes some time, to minimize the error in the digital representation, a sample-and-hold (S/H) circuit is placed before the A/D converter
- Finally, the output of the D/A converter is a continuous-time analog signal with a staircase type waveform

Digital Processing of Analog Signals

- The D/A converter output signal is smoothed by passing it through an analog lowpass filter, called the reconstruction filter
- Figure below shows the schematic representation of the digital processing of analog signals

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