Let $a,b,c,d \in \mathbb{C} \ni ad-bc \neq 0$.

Then $f(z) = \dfrac{az+b}{cz+d}$ is called a

Möbius transformation.

Let $a,b,c,d \in \mathbb{C} \ni ad-bc \neq 0$.

Then $f(z) = \dfrac{az+b}{cz+d}$ is called a

Möbius transformation.

Properties: (1) f is diff'ble for $z \neq -d/c$

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

LINEAR FRACTIONAL TRANSFORMATION

Let $a, b, c, d \in \mathbb{C} \ni ad - bc \neq 0$.

Then $f(z) = \dfrac{az+b}{cz+d}$ is called a Möbius transformation.

Properties: (1) f is diff'ble for $z \neq -d/c$

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

(2) Möbius transformation takes "circles and straight lines to circles & straight lines"

Equation of a line : $\dfrac{|z-p|}{|z-q|} = 1$

(gives the $\perp^r$ bisector joining p to q).

Equation of a circle : $\left|\dfrac{z-p}{z-q}\right| = k.$ $(k \neq 1)$

③ Möbius transformation takes circles and straight lines to circles & straight lines

Equation of a line : $\dfrac{|z-p|}{|z-q|} = 1$

(gives the $\perp^r$ bisector joining a to b).

Equation of a circle : $\left|\dfrac{z-p}{z-q}\right| = k.$ $(k \neq 1)$

$$(x-a_1)^2 + (y-a_2)^2 = k^2\left[(x-b_1)^2 + (y-b_2)^2\right]$$

$$x^2(1-k^2) + y^2(1-k^2) - 2a_1x + k^2(2b_1x)$$

$$-2a_2y + k^2(2b_2y) = k^2(b_1^2+b_2^2) - (a_1^2+a_2^2)$$

$$x^2 + y^2 - 2\frac{(a_1+k^2b_1)}{1-k^2}x - 2\frac{(a_2+k^2b_2)}{1-k^2}y = K$$

$$\left(x - \frac{a_1+k^2b_1}{1-k^2}\right)^2 + \left(y - \frac{a_2+k^2b_2}{1-k^2}\right)^2 = K + \frac{(a_1+k^2b_1)^2}{(1-k^2)^2} + \frac{(a_2+k^2b_2)^2}{(1-k^2)^2}$$

③ Möbius transformation takes circles and straight lines to circles & straight lines

Equation of a line : $\dfrac{|z-p|}{|z-q|} = 1$

(gives the $\perp^r$ bisector joining a to b).

Equation of a circle : $\left|\dfrac{z-p}{z-q}\right| = k.$ $(k \neq 1)$

$$\left|\frac{z-p}{z-q}\right| = k \quad \& \quad f(z) = \omega$$

$$\Rightarrow \left|\frac{\omega - P}{\omega - Q}\right| = K \qquad \left(\because z = \frac{-d\omega + b}{c\omega - a}\right)$$

$$P = b + pa / d + pc$$

$$Q = b + qa / d + qc$$

$$K = k\,|d + qc| / |d + pc|$$

$$f(z) = \frac{az+b}{cz+d} = \underline{\underline{w}} \qquad (z \neq -d/c)$$

$$az+b = w(cz+d)$$

$$z(a-cw) = dw-b$$

$$z = \frac{dw-b}{-cw+a}$$

$$\left|\frac{z-p}{z-q}\right| = k$$

$$\Leftrightarrow \left|\frac{\frac{dw-b}{-cw+a} - p}{\frac{dw-b}{-cw+a} - q}\right| = k$$

$$\Leftrightarrow \left|\frac{dw-b-p(-cw+a)}{dw-b-q(-cw+a)}\right| = k$$

$$\Leftrightarrow \left.\left|\frac{(cp+d)w-ap-b}{(qc+d)w-aq-b}\right| = k\right\}$$

$$\frac{|cp+d|\,|w-f(p)|}{|cq+d|\,|w-f(q)|} = k.$$

$$\left|\frac{w-f(p)}{w-f(q)}\right| = k\left|\frac{cq+d}{cp+d}\right|$$

③ Composition of möbius transformation is again a Möbius transformation

$$g(z) = \frac{Az+B}{Cz+D}$$

$$g\circ f(z) = \frac{(Aa+Bc)z + Ab+Bd}{(Ca+Dc)z + Cb+Dd}.$$

$$(Aa+Bc)(Cb+Dd) - (Ab+Bd)(Ca+Dc)$$
$$= (AD-BC)(ad-bc) \neq 0.$$

④ Möbius transformation are invertible (on $\mathbb{C}\setminus\{-d/c\}$). The inverse is also a Möbius transformation.

$$\frac{az+b}{cz+d} = w \implies z \mapsto \left(\frac{dz-b}{-cz+a}\right).$$

Infact, $\frac{az+b}{cz+d} : \mathbb{C}\setminus\left\{\frac{-d}{c}\right\} \longrightarrow \mathbb{C}\setminus\left\{a/c\right\}$

is 1-1 and onto

(5) Every Möbius transformation is a composition of translation, inversion, rotation and magnification

$\tau_a(z) = z + a \rightarrow$ translation

$\rho_\theta(z) = e^{i\theta} z$ - rotation

$m_\alpha(z) = \alpha z \quad (\alpha > 0) \rightarrow$ magnification

$j(z) = \frac{1}{z} \quad (z \neq 0) \rightarrow$ inversion.

(Note: all the above are Möbius transformations)

(5) Every Möbius transformation is a composition of translation, inversion, rotation and magnification

$\tau_a(z) = z + a \rightarrow$ translation

$\rho_\theta(z) = e^{i\theta} z$ - rotation

$m_\alpha(z) = \alpha z \quad (\alpha > 0) \rightarrow$ magnification

$j(z) = \frac{1}{z} \quad (z \neq 0) \rightarrow$ inversion.

(Note: all the above are Möbius transformations)

$$\frac{az+b}{cz+d} = t_2 \circ r \circ m \circ j \circ t_1(z)$$

$c \neq 0 \quad t_1(z) = z + d/c,$

$$r \circ m(z) = -\left(\frac{ad-bc}{c^2}\right) z$$

$$t_2(z) = z + \frac{a}{c}.$$

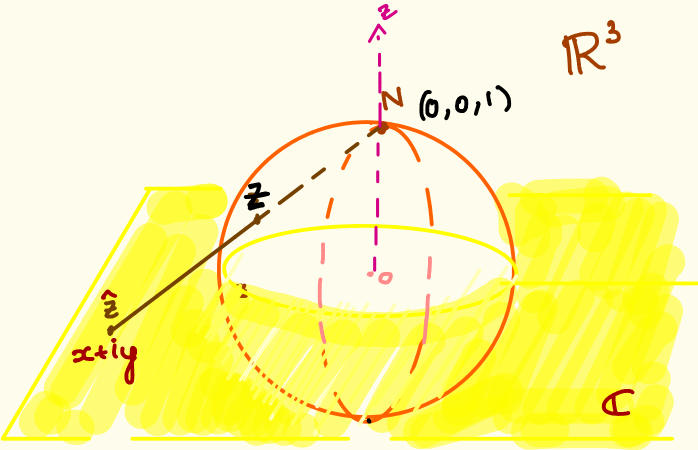
$$-\frac{\left(\frac{ad-bc}{c^2}\right) 1}{\left(z + \frac{d}{c}\right)}$$

$$= \frac{-ad+bc}{c^{\cancel{2}}} \cdot \frac{\cancel{c}}{cz+d} + a/c$$

$$= \frac{-ad+bc}{c(cz+d)} + \frac{a(cz+d)}{}$$

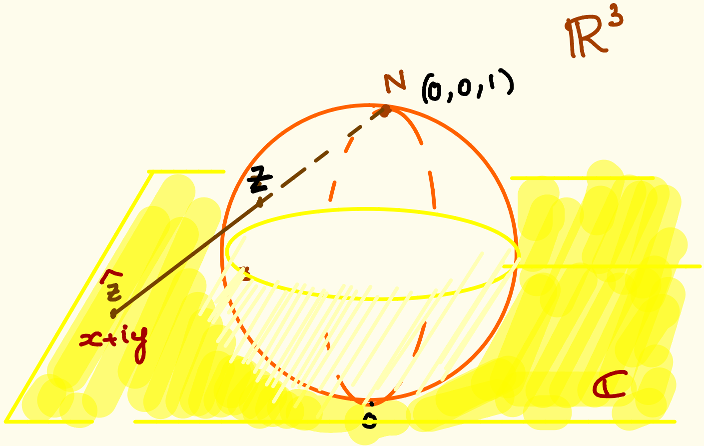
$$= \frac{acz+bc}{c(cz+d)} = \frac{az+b}{cz+d}$$

RIEMANN SPHERE



Eqn of sphere:

$$\{(x,y,z) / x^2+y^2+z^2=1\}$$



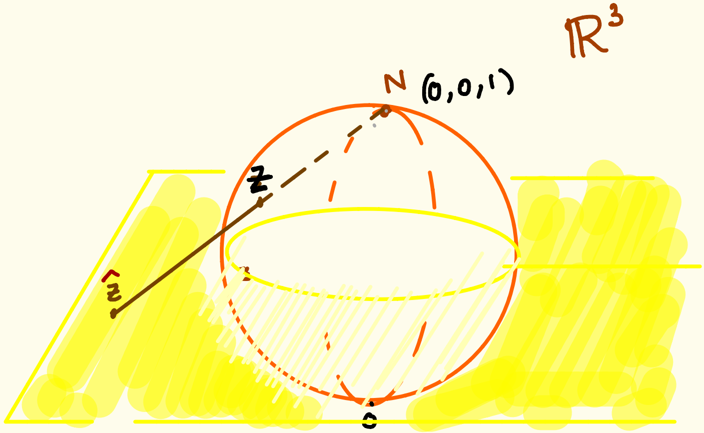
Eqn of sphere:

$\{(x,y,z) / x^2+y^2+z^2=1\}$

Eqn of line through $\hat{z}$ $(0,0,1)$

and $(x,y,0)$: $t\hat{z}+(1-t)N$ $\quad 0 \le t$

$= \{(tx, ty, (1-t) \quad)\}$



$t^2x^2 + t^2y^2 + (1-t)^2 = 1$

$t^2x^2 + t^2y^2 + \cancel{1} - 2t + t^2 = \cancel{1}$

$t^2(x^2+y^2+1) - 2t = 0$

$t\,[t(x^2+y^2+1) - 2] = 0$

Eqn of sphere:

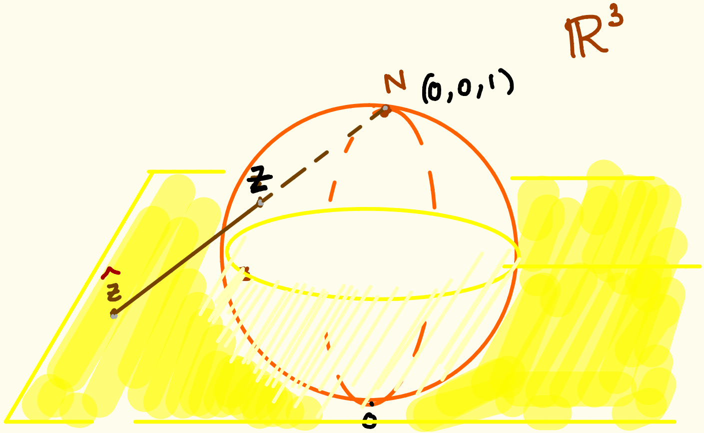
$\{(x,y,z) / x^2+y^2+z^2 = 1\}$

Eqn of line through $(0,0,1)$ and $(x,y,0)$: $t\hat{z} + (1-t)N$ $\quad 0 \le t$

$= \{(tx, ty, (1-t))\}$

$\in S^1$

$\Rightarrow \boxed{t = \dfrac{2}{|\hat{z}|^2+1}}$



Eqn of sphere:

$$\{(x,y,z) / x^2+y^2+z^2 = 1\}$$

Eqn of line through $(0,0,1)$ and $(x,y,0)$: $t\hat{z} + (1-t)N \qquad 0 \leq t$

$$= \{(tx, ty, 1-t) / t \geq 0\}$$

$$(t\hat{z} + (1-t)N) \cap S^1 \qquad \Rightarrow t = \frac{2}{|\hat{z}|^2+1} \quad (\text{or } t=0)$$

Thus, $(x,y,z) = Z$

is $\boxed{\left(\frac{2x}{|\hat{z}|^2+1}, \frac{2y}{|\hat{z}|^2+1}, \frac{|\hat{z}|^2-1}{|\hat{z}|^2+1}\right)}$ $\cdot Z$

North pole = "point at ∞".

Line : P, Q, ∞.

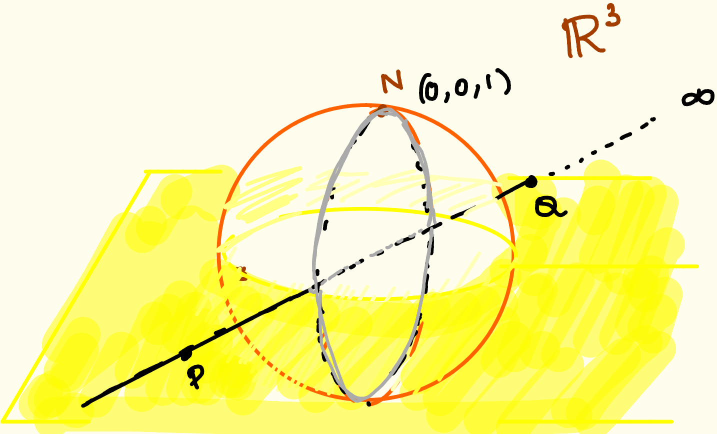
Circle : P, Q, R.

Eg: $0, 1, \infty$ real

$0, i, \infty$ im.

$i, -i, 1$ - unit circle

$$\boxed{\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}}$$



North pole = point at ∞.

Line : P, Q, ∞.

Circle : P, Q, R.

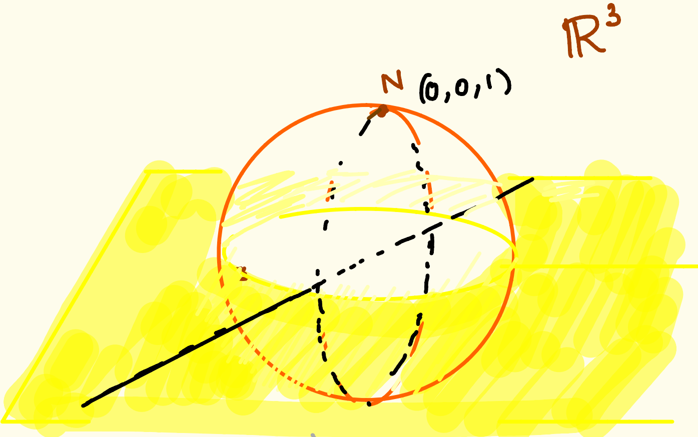
$0, 1, \infty$ real

$0, i, \infty$ im.

$i\ -i, 1$ - unit circle

Convention: $\boxed{\frac{az+b}{cz+d} = \infty \text{ if } \frac{cz+d}{az+b} = 0}$, $\boxed{\frac{a\infty+b}{c\infty+d} := \frac{a+b\cdot 0}{c+d\cdot 0}}$

extending möbius transformation to $\hat{\mathbb{C}}$.



$z \longmapsto \frac{az+b}{cz+d}$

$\mathbb{C}\setminus\{-d/c\} \xrightarrow{\text{bij}} \mathbb{C}\setminus\{a/c\}$

$z \longmapsto \frac{az+b}{cz+d}$

$\hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$

$-d/c \longmapsto \infty$

$\infty \longmapsto a/c$

North pole = point at ∞.

Line : P, Q, ∞. $\qquad$ $0, 1, \infty$ real

$0, i, \infty$ im.

Circle : P, Q, R. $\qquad$ $i, -i, 1$ - unit circle

Convention: $\frac{az+b}{cz+d} = \infty$ if $\frac{cz+d}{az+b} = 0$, $\frac{a\infty+b}{c\infty+d} = \frac{a+b\cdot 0}{c+d\cdot 0}$

THEOREM: There is a unique Möbius transformation taking a triplet z_1, z_2, z_3 to $0, 1, \infty$

Hence, there is a unique Möbius transf taking 1 triplet to another. (in $\hat{\mathbb{C}}$)

Pf: $f(z) = \left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right)$. $\quad$ takes : $z_1 \mapsto 0$, $z_2 \mapsto 1$, $z_3 \mapsto \infty$

Uniqueness: $\quad \phi : z_1, z_2, z_3 \longmapsto 0, 1, \infty$

$\psi : z_1, z_2, z_3 \longmapsto 0, 1, \infty$

$\Rightarrow \underbrace{\psi \circ \phi^{-1}} : 0, 1, \infty \longmapsto 0, 1, \infty$

$\Rightarrow \left(\frac{az+b}{cz+d}\right)\Big|_{z=0} = 0$

$z = 1 \quad = 1$

$z = \infty \quad = \infty$

$\left.\right\}$ $a = 1, b = 0$, $c = 0, d = 1$.

$\therefore \psi \circ \phi^{-1} = \text{identity}$

ie $\psi = \phi$ ■

North pole = point at ∞.

Line : P, Q, ∞.

$0, 1, \infty$ real

$0, i, \infty$ im.

Circle : P, Q, R.

$i, -i, 1$ - unit circle

Convention: $\frac{az+b}{cz+d} = \infty$ if $\frac{cz+d}{az+b} = 0$, $\frac{a\infty+b}{c\infty+d} = \frac{a+b\cdot 0}{c+d\cdot 0}$

THEOREM: There is a unique Möbius transformation taking a triplet z_1, z_2, z_3 to $0, 1, \infty$

Hence, there is a unique Möbius transf taking 1 triplet to another.

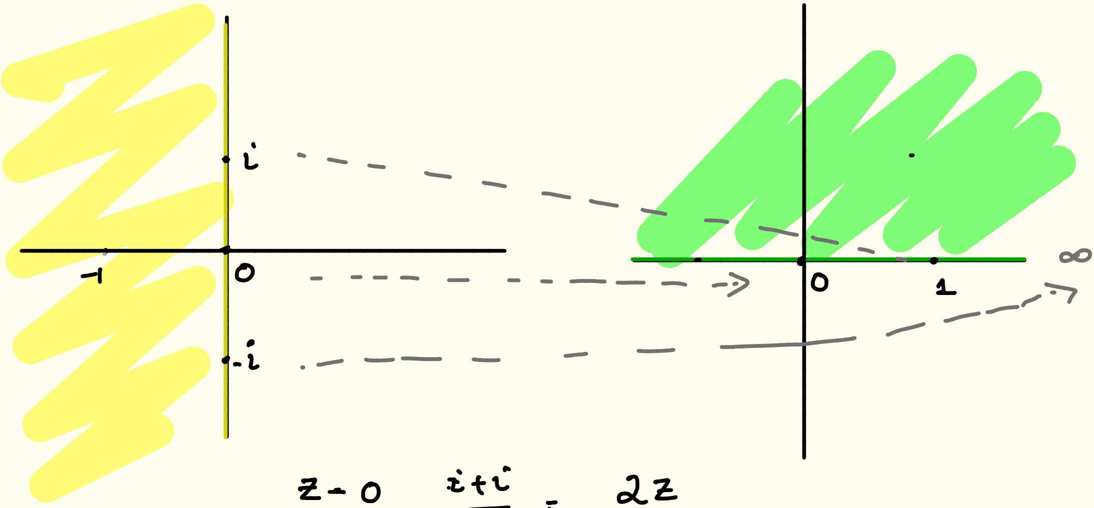
Pf: $$\left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right).$$

Uniqueness: $\phi : z_1, z_2, z_3 \longmapsto 0, 1, \infty$

$\psi : z_1, z_2, z_3 \longmapsto 0, 1, \infty$

$\Rightarrow \psi \circ \phi^{-1} : 0, 1, \infty \longmapsto 0, 1, \infty$

The above theorem enables us to also map regions under Möbius transformation to other regions



$$\frac{z-0}{z+i}\,\frac{i+i}{i} = \frac{2z}{z+i}$$

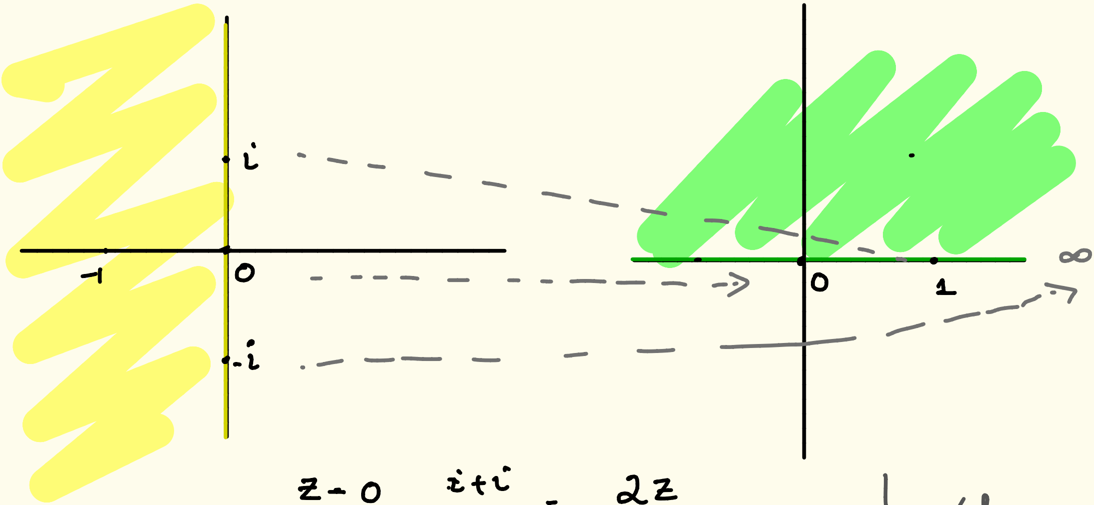
1-1
onto

$\Downarrow$

$\mathbb{C} \setminus \{\text{imaginary axis}\} \xrightarrow{\text{bij}} \mathbb{C} \setminus \{\text{real axis}\}$

left component $\longmapsto$ top/down (not both (by connectedness)

bij $\therefore$



$$\frac{z-0}{z+i}\;\frac{i+i}{i} = \frac{2z}{z+i}$$

$$-1 \longmapsto \frac{-2}{-1+i} = \frac{-2(-1-i)}{2}$$

$$= \underline{\underline{1+i}}$$

ϕ has a pole at "$-i$"

Inverse: $\left[\begin{array}{l} \omega = \dfrac{2z}{z+i} \\ z \longleftarrow \omega \end{array}\right.$ $\Rightarrow$ $\omega(z+i) = 2z$

$\Rightarrow z(\omega-2) = -i\omega$

$z = \dfrac{-i\omega}{\omega-2}$

$a = 2 \quad b = 0$

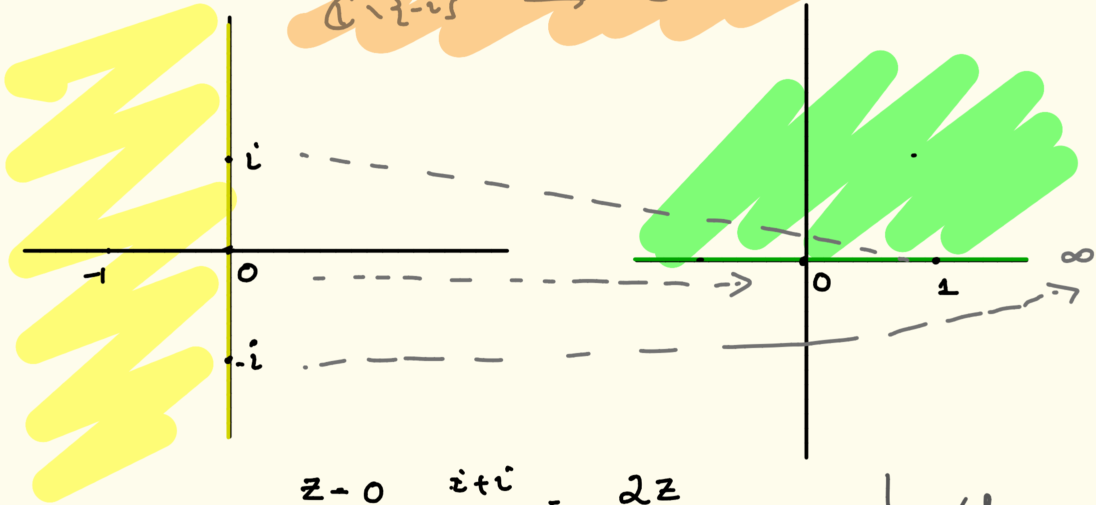
$c = 1 \quad d = i$

$0 \longmapsto 0$

$1 \longmapsto i$

has a pole at $\omega = 2$.

$\mathbb{C} \setminus \{-i\} \longrightarrow \mathbb{C} - \{2\}$



$$\frac{z-0}{z+i}\;\frac{i+i}{i} = \frac{2z}{z+i}$$

$$-1 \longmapsto \frac{-2}{-1+i} = \frac{-2(-1-i)}{2} = \underline{\underline{1+i}}$$

ϕ has a pole at "$-i$"

Inverse: $\left[\; w = \dfrac{2z}{z+i} \qquad z \longleftarrow\!\mapsto w \right.$ $\Rightarrow$ $w(z+i) = 2z$

$\Rightarrow z(w-2) = -iw$

$$z = \frac{-iw}{w-2}$$

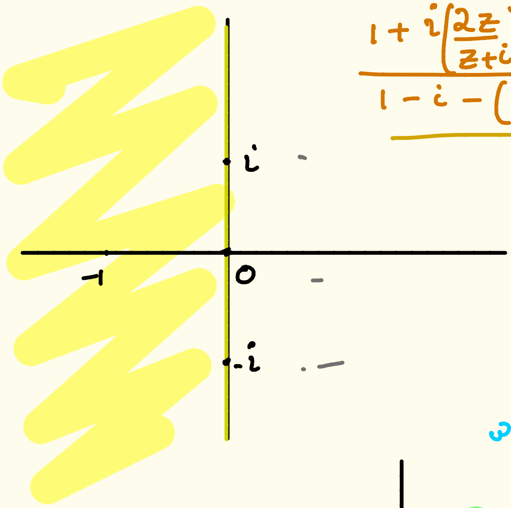
$a = 2 \quad b = 0$

$c = 1 \quad d = i$

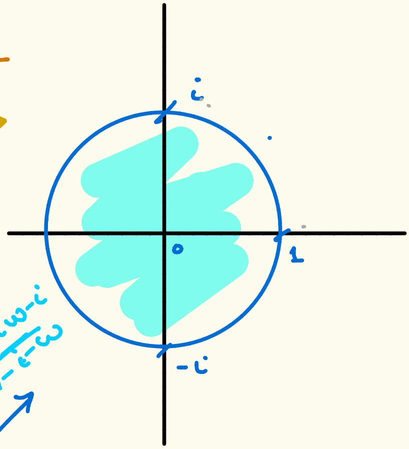
$0 \longmapsto 0$

$1 \longmapsto i$

has a pole at $w = 2$.

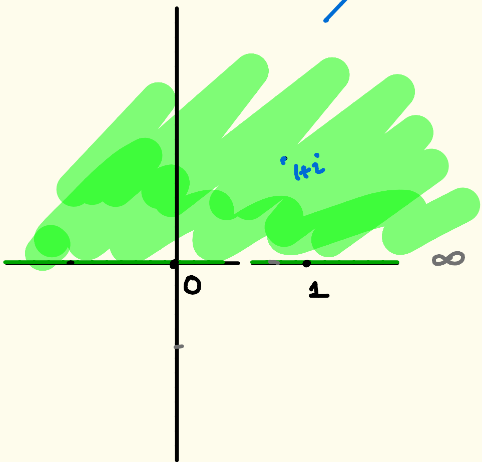


$$\frac{1+i\left(\frac{2z}{z+i}\right)-i}{1-i-\left(\frac{2z}{z+i}\right)}$$



$$w \mapsto \frac{1+iw-i}{1-i-w}$$

$$\frac{2z}{z+i}$$



$$\frac{z-1}{z+i}\left(\frac{2i}{i-1}\right),$$

$$= \frac{z-1}{z+i}(1-i)$$

$$0 \mapsto \frac{-1}{i}(1-i)$$

$$w = \frac{z-1}{z+i}(1-i)$$

$$w(z+i) = (z-1)(1-i)$$

$$z(w-(1-i)) = -iw-1+i$$

$$z = \frac{1+iw-i}{1-i-w}$$

Prop$\underline{\underline{n}}$: The LFT that takes

$z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$ is

given by

$$\boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$

$\underline{Pf}$: $T: z_1 \mapsto 0$, $z_2 \mapsto 1$, $z_3 \mapsto \infty$ is

given by $z \mapsto \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

$S: w_1 \mapsto 0$, $w_2 \mapsto 1$, $w_3 \mapsto \infty$ is

given by

$$w \mapsto \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

Then $S^{-1} \circ T: z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$

Let $(S^{-1} \circ T)(z) = w$

ie w is given by : $\boxed{T(z) = S(w)}$ ∎

THE END