Chapter 7

Sturm's Separation, and Comparison theorems

Most of the times we can not have explict solutions to ODE when the ODE has variable coefficients. However we can understand some of the qualitative properties of solutions of such equations, for example nature of zeroes of a solution. This is the goal of this section.

We consider a second order ODE of the form

$$y'' + q(x)y = 0, \tag{7.1}$$

where q is a continuous function. It is true that any second order ODE is essentially of this form, via a change of variable, we shall not discuss further on this point. Further details can be found in Myint-U's book [20]. The results that we are going to prove also hold for operators in self-adjoint form.

7.1 Sturm's separation theorem

Theorem 7.1 (Separation) *Suppose that ϕ_1 and ϕ_2 be a fundamental pair of solutions (and, hence are linearly independent) of*

$$y'' + q(x)y = 0. \tag{7.2}$$

Then

(i) *The zeroes of nontrivial solutions of (7.2) are isolated.*

(ii) *Let x_1 and x_2 be two consecutive zeros of ϕ_1. Then ϕ_2 has exactly one zero in (x_1, x_2).*

Proof :
Proof of (i) is an exercise; assuming that the zero set of a solution has a limit point, one can show that the solution is a trivial solution by virtue of satisfying an IVP with initial values zero.

Proof of (ii) is by contradiction. Assuming that ϕ_2 does not have a zero on (x_1, x_2), we conclude that ϕ_2 does not have a zero on $[x_1, x_2]$. This is due to the fact that ϕ_1 and ϕ_2 is a fundamental pair (equivalently, linearly independent).
Therefore we can define a function ψ on $[x_1, x_2]$ by

$$\psi(x) = \frac{\phi_1(x)}{\phi_2(x)}. \tag{7.3}$$

Then the function ψ has the properties:

(i) ψ is continuous on $[x_1, x_2]$.

(ii) ψ' exists on (x_1, x_2).

(iii) $\psi(x_1) = \psi(x_2) = 0$.

Therefore by Rolle's theorem, there exists a c such that $x_1 < c < x_2$ such that $\psi'(c) = 0$. This means wronskian is zero at c and this is not possible as ϕ_1 and ϕ_2 is a fundamental pair (equivalently, linearly independent). Therefore, ϕ_2 has at least one zero on (x_1, x_2).

We claim that ϕ_2 does not have more than one zero in (x_1, x_2). If the claim were not true, then there would exist at least two zeroes of ϕ_2 in (x_1, x_2). With out loss of generality, let x_3 and x_4 be two consecutive zeroes of ϕ_2 in (x_1, x_2), such that $x_1 < x_3 < x_4 < x_2$. Then by previous arguments, ϕ_1 will have a zero in (x_3, x_4) and thus contradicting the fact that x_1 and x_2 are two consecutive zeros of ϕ_1. ■

Corollary 7.2 *If ϕ_1 and ϕ_2 are as in the above theorem, then zeros of ϕ_1 separate and are separated by those of ϕ_2.* ■

However the above theorem does not address the existentce of zeros! The following examples illustrate this point.

Example 7.3

$$y'' - y = 0. \tag{7.4}$$

(i) $\phi_1(x) = e^x$ and $\phi_2(x) = e^{-x}$ are linearly independent solutions. None of them have zeros! (ii)$\psi_1(x) = \sinh x$ and $\psi_2(x) = \cosh x$ are linearly independent solutions. ψ_1 has one zero, ψ_2 has no zero on $\mathbb{R}$.

Theorem 7.4 *If $q(x) \le 0$ on an interval $\mathbb{I}$, then no non-trivial solution of $y'' + qy = 0$ can have two zeros on $\mathbb{I}$.*

PROOF :

Suppose if possible, that a non-trivial soluton y has at least two zeroes on $\mathbb{I}$. WLOG let $x_1 < x_2$ be two consecutive zeroes of y, and that $y > 0$ on (x_1, x_2).

This implies that

$$y'(x_1) > 0, \quad \text{and} \quad y'(x_2) < 0. \tag{7.5}$$

The inequalities in (7.5) follow from continuity of y'. We indicate the proof of the first inequality, for instance. We prove that $y'(x_1) \le 0$ is not possible, for, if $y'(x_1) = 0$ then since $y(x_1) = 0$ already, it follows that $y \equiv 0$, a contradiction. On the other hand, if $y'(x_1) < 0$ then y will be negative for $x > x_1$ and x near-by x_1, again contradicting the assumption that $y > 0$ on (x_1, x_2).

Thus

$$y''(x) = -q(x)y(x) \ge 0 \quad \text{on} \quad (x_1, x_2). \tag{7.6}$$

The last inequality implies that y' is an increasing function on (x_1, x_2). Hence

$$0 > y'(x_2) \ge y'(x_1) > 0. \tag{7.7}$$

This is a contradiction, and thus the non-trivial solution y can not have two zeroes. ■

Remark 7.5 *Trivial solution is the only solution of $y'' + qy = 0$ with $q(x) \le 0$ on an interval $\mathbb{I}$, which has two zeroes.*

7.2 Sturm's comparison theorem

Theorem 7.6 (Comparison) *Let ϕ_1 and ϕ_2 be non-trivial solutions of equations*

$$y'' + q_1(x)y = 0 \tag{7.8}$$

$$\textit{and} \quad y'' + q_2(x)y = 0 \tag{7.9}$$

respectively, on an interval $\mathbb{I}$; where q_1 and q_2 are continuous functions such that $q_1(x) \le q_2(x)$ on $\mathbb{I}$.

Then between any two consecutive zeroes x_1 and x_2 of ϕ_1, there exists at least one zero of ϕ_2 unless $q_1(x) \equiv q_2(x)$ on (x_1, x_2).

PROOF :
Let x_1 and x_2 with $x_1 < x_2$ be consecutive zeroes of ϕ_1. Assume WLOG $\phi_1 > 0$ on (x_1, x_2) (if not, consider $-\phi_1$ which has these properties). Consequently, by arguments in the proof of Theorem 7.4,

$$\phi_1'(x_1) > 0, \quad \text{and} \quad \phi_1'(x_2) < 0. \tag{7.10}$$

Suppose that ϕ_2 does not have a zero on (x_1, x_2). WLOG let $\phi_2 > 0$ on (x_1, x_2). Multiplying the equation satisfied by ϕ_1, with ϕ_2, and vice versa, and then subtract the two equations we get

$$\phi_1''\phi_2 - \phi_1\phi_2'' + (q_1 - q_2)\phi_1\phi_2 = 0$$

Rewriting the last equation as

$$\left(\phi_1'\phi_2 - \phi_1\phi_2'\right)' = -(q_1 - q_2)\phi_1\phi_2 \tag{7.11}$$

Integrating on both sides of the last equation from x_1 to x_2, we get

$$\left[\phi_1'\phi_2 - \phi_1\phi_2'\right]_{x_1}^{x_2} = -\int_{x_1}^{x_2} (q_1(x) - q_2(x))\phi_1(x)\phi_2(x)\,dx. \tag{7.12}$$

Note that LHS of (7.12) is non-positive. The RHS is strictly positive unless $q_1(x) \equiv q_2(x)$ on (x_1, x_2). Therefore, if $q_1(x) \not\equiv q_2(x)$ on (x_1, x_2), we arrive at a contradiction. This finishes the proof of theorem. ■

Exercise 7.7 *Can we prove that LHS of* (7.12) *is negative? If yes, we can weaken the hypothesis of our theorem, thereby strengthening the result.*

Definition 7.8 *With q_1 and q_2 as in the statement of Theorem 7.6, Equation* (7.9) *is said to be a Sturm majorant of equation* (7.8) *on $\mathbb{I}$. Equation* (7.8) *is said to be a Sturm minorant of equation* (7.9) *on $\mathbb{I}$.*

Remark 7.9 *A non-trivial solution of the equation $(py')' + qy = 0$ has infinitely many zeros on an interval $\mathbb{I}$ if and only if every (non-trivial) solution has infinitely many zeros on $\mathbb{I}$. This follows from Sturm's comparison theorem.*

Definition 7.10 *The equation $(py')' + qy = 0$ is said to be oscillatory on an interval $\mathbb{I}$ if there exists a non-trivial solution of the equation with infinitely many zeros on $\mathbb{I}$.*

Remark 7.11 *An equation $(py')' + qy = 0$ is oscillatory on an interval $\mathbb{I}$ if and only if all solutions of the equation have infinitely many zeros on $\mathbb{I}$.*

Exercise 7.12 (i) *The equation $y'' + y = 0$ is oscillatory on $\mathbb{R}$.*

(ii) *The equation $y'' - y = 0$ is non-oscillatory.*

(iii) *The equation $y'' = 0$ is non-oscillatory.*

(iv) *The equation $y'' + xy = 0$ is oscillatory on $(1, \infty)$. Prove also that any of its non-trivial solutions have a zero between $n\pi$ and $(n+1)\pi$ for any integer n.*

An important application of Sturm's comparison theorem is in understanding zero set of non-trivial solutions of Bessel's equation. Recall that Bessel's equation is given by

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad (\nu \geq 0). \tag{7.13}$$

For $x > 0$, making a change of variable $y = \frac{v}{\sqrt{x}}$, the equation (7.13) transforms into

$$v'' + \left(1 + \frac{1 - 4\nu^2}{4x^2}\right)v = 0 \quad (x > 0). \tag{7.14}$$

(to obtain the above equation, start differentiating the equation $\sqrt{x}y = v$).

Case 1: $0 < \nu < 1/2$ In this case, compare (7.14) with

$$y'' + y = 0, \tag{7.15}$$

which has a solution $\sin x$ with zeros at $x = n\pi$, $n \in \mathbb{N}$. Therefore a solution v of (7.14) has at least one zero on each of the open intervals $((n-1)\pi, n\pi)$, $n \in \mathbb{N}$.

Case 2: $\nu > 1/2$ In this case, compare once again (7.14) with

$$y'' + y = 0, \tag{7.16}$$

and conclude that between any two consecutive zeros, α and β of v, there exists at least one zero of $\sin x$. Thus, we have $\alpha < n\pi < \beta$ for some $n \in \mathbb{N}$.

Remark 7.13 (i) *Sturm's comparison theorem guarantees the existence of at least one zero.*

(ii) *The assumption $q_2(x) \geq q_1(x)$ can not be dropped. Consider the equations on $x \geq 0$, $y''+y = 0$ ($q_1(x) = 1$), and $v'' - v = 0$ ($q_2(x) = -1$); and let y and v be their non-trivial solutions respectively. Between any two zeros of y, v does not admit a zero.*

(iii) *Consider the equations on $x \in \mathbb{R}$, $y'' + y = 0$ ($q_1(x) = 1$), and $v'' + 4v = 0$ ($q_2(x) = 4$); and let $y = \sin x$ and $v = \sin 2x$ are their non-trivial solutions respectively. But there is no zero of y between two consecutive zeros of v.*