MTH-102A | **November 01, 2020**

Total Marks: 58 — COMPENSATION EXAM — **Duration: 3 Hours**

Part-3

1. (a) (3 marks) Write the general solution of $xy' + y + xe^{-xy} = 0$.

Solution: Set $z = xy$. Then $z' = xy' + y$ and $z' + xe^{-z} = 0$ or $e^z dz + xdx = 0$ or $d(e^z + x^2/2) = 0$. Hence, $e^{xy} + x^2/2 = \text{constant}$.

(b) (7 marks) Find the two linearly independent power series solution of the ODE $x^2y'' - 2xy' + (2 + x)y = 0$.

Solution: Note that $x = 0$ is a regular singular point. We seek solution of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{r+j}$$

such that $a_0 \neq 0$. Equating like powers, we get

$$\begin{aligned} [r(r-1) - 2r + 2]a_0 &= 0 \\ [(r+j)(r+j-1) - 2(r+j) + 2]\, a_j &= -a_{j-1} \text{ for } j = 1, 2, 3, \dots . \end{aligned}$$

$r(r - 1) - 2r + 2 = 0$, $r = 2, 1$ and

$$y_1 = x^2 \left[1 - \frac{x}{1^2 + 1} + \frac{x^2}{(1^2 + 1)(2^2 + 2)} - \dots\right].$$

For $r = 1$, $(j^2 - j)a_j = -a_{j-1}$. For $j = 1$ we get $0a_1 = -a_0 = -1$ is never satisfied. Thus, the second linearly independent solution is

$$y_2 = |x| \sum_{k=0}^{\infty} c_k x^k + dy_1 \ln |x|.$$

Part-4

2. (5 marks) Find the general solution of the ODE $y' = y^2 + 1 - x^2$.

Solution: A good guess, by inspection, is that $y_1(x) = x$ is a solution. If y is a general solution then set $w := y - x$. Using this in the ODE, we see that w satisfies the Bernoulli ODE

$$w' = w^2 + 2wx.$$

Set $u = w^{-1}$, then $u' + 2xu = -1$. Its integrating factor is $e^{\int 2x\,dx} = e^{x^2}$ and

$$u(x) = -e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$$

and

$$y(x) = x + \frac{e^{x^2}}{c - \int e^{x^2}\,dx}.$$

3. ~~(5 marks)~~ Solve for (y_1, y_2) in the ODEs

4 Marks

$$\begin{cases} y_1''(x) + 2y_1(x) & = y_2(x) \\ y_1(x) & = y_2''(x) + y_2(x). \end{cases}$$

Solution: The second equation gives $y_1(x) = y_2'' + y_2(x)$ which on substitution in first equation gives $y_2^{(4)} + 3y_2^{(2)} + y_2 = 0$. The characteristic equation is $m^4 + 3m^2 + 1 = 0$ with roots $m^2 = \frac{-3\pm\sqrt{5}}{2}$. Call the two roots as $-a^2$ and $-b^2$. Then the four roots are $m = \pm\imath a$ and $\pm\imath b$. Thus,

$$y_2(x) = c_1\cos ax + c_2\sin ax + c_3\cos bx + c_4\sin bx$$

and

$$y_1(x) = c_1\cos ax(a^2 - 1) - c_2\sin ax(a^2 + 1) + c_3\cos bx(b^2 - 1) - c_4\sin bx(b^2 + 1).$$

Part-5

4. (9 marks) Find the solution in $(0, \infty)$ of the IVP

$$\begin{cases} y'' + 2y' + 2y &= \begin{cases} 1 & \pi \leq x \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \\ y(0) &= 0 \\ y'(0) &= 1. \end{cases}$$

Solution: The Laplace transform of the RHS data is

$$\mathcal{L}[H(x-\pi)] - \mathcal{L}[H(x-2\pi)] = \frac{e^{-p\pi} - e^{-2p\pi}}{p}.$$

1 mark

Applying Laplace transform to the given ODE and using the initial values we get

$$\mathcal{L}(y)(p) = \frac{1}{(p+1)^2+1}\left[1 + \frac{e^{-p\pi} - e^{-2p\pi}}{p}\right].$$

2 marks

Using partial fractions we obtain

$$\frac{1}{[(p+1)^2+1]p} = \frac{-p/2-1}{(p+1)^2+1} + \frac{1/2}{p} = \frac{-(p+1)/2 - 1/2}{(p+1)^2+1} + \frac{1/2}{p}.$$

2 marks

Taking inverse Laplace transform, we get

$$\begin{aligned} y(x) \quad &= \quad e^{-x}\sin x + \frac{1}{2}\left[1 - e^{-(x-\pi)}(\sin(x-\pi) + \cos(x-\pi))\right] H(x-\pi) \\ &\quad -\frac{1}{2}\left[1 - e^{-(x-2\pi)}(\sin(x-2\pi) + \cos(x-2\pi))\right] H(x-2\pi). \end{aligned}$$

3 marks

Use the fact that $\sin(x - k\pi) = (-1)^k \sin x$ for integer k, and the values of Heaviside function, we get

$$y(x) = \begin{cases} e^{-x}\sin x & 0 \leq x \leq \pi \\ \frac{1}{2} + (1 + \frac{e^{\pi}}{2})e^{-x}\sin x + \frac{e^{\pi}}{2}e^{-x}\cos x & \pi \leq x \leq 2\pi \\ (1 + \frac{e^{\pi}}{2} + \frac{e^{2\pi}}{2})e^{-x}\sin x + (\frac{e^{\pi}}{2} + \frac{e^{2\pi}}{2})e^{-x}\cos x & x \geq 2\pi. \end{cases}$$

1 mark

Part-6

10 marks

5. ~~(9 marks)~~ Find the two linearly independent power series solution of the ODE $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$.

Solution: Note that $x = 0$ is a regular singular point. We seek solution of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^{r+j}$$

such that $a_0 \neq 0$. Equating like powers, we get

$$\begin{aligned} (r^2 - \frac{1}{9})a_0 &= 0 \\ \left[(r+1)^2 - \frac{1}{9}\right]a_1 &= 0 \\ \left[(r+j)^2 - \frac{1}{9}\right]a_j &= -a_{j-2} \text{ for } j = 2, 3, \ldots. \end{aligned}$$

$r^2 - \frac{1}{9} = 0$, $r = \pm 1/3$ and choosing $a_0 = 1$

$$y_1 = x^{1/3}\left[1 - \frac{x^2}{4(1+1/3)} + \frac{x^4}{4^2(1+1/3)(4+2/3)} - \ldots\right]$$

and

$$y_2 = x^{-1/3}\left[1 - \frac{x^2}{4(1-1/3)} + \frac{x^4}{4^2(1-1/3)(4-2/3)} - \ldots\right].$$

If $\alpha_1 y_1 + \alpha_2 y_2 = 0$, for all $x \neq 0$, then $\alpha_1 x^{1/3} y_1 + \alpha_2 x^{1/3} y_2 = 0$, for all $x \neq 0$. Thus, as $x \to 0$, we have $\alpha_2 = 0$ and, hence $\alpha_1 = 0$. The solutions as linealry independent.

Part-7

6. (5 marks) Find the general solution of the ODE $x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2}\cos x$ if $\frac{\sin x}{\sqrt{x}}$ and $\frac{\cos x}{\sqrt{x}}$ are two independent solutions of the corresponding homogeneous problem for $x > 0$.

Solution: The particular solution is $y_p(x) = v_1(x)\frac{\sin x}{\sqrt{x}} + v_2(x)\frac{\cos x}{\sqrt{x}}$. The Wronskian $W(\frac{\sin x}{\sqrt{x}}, \frac{\cos x}{\sqrt{x}}) = \frac{-1}{x}$. By the method of variation of parameters, we get $v_1' = \cos^2 x$ and $v_2' = -\sin x \cos x$. Thus, $v_1 = \frac{x}{2} + \frac{\sin 2x}{4}$ and $v_2(x) = \frac{\cos 2x}{4}$. Then $y_p = \frac{\sqrt{x}\sin x}{2} + \frac{\cos x}{4\sqrt{x}}$. Then

$$y(x) = \frac{\sqrt{x}\sin x}{2} + c_1\frac{\sin x}{\sqrt{x}} + c_2\frac{\cos x}{\sqrt{x}}.$$

7. Derive the inverse Laplace transform of:

(a) (3 marks) $\frac{e^{-p}}{p^2+3p+2}$.

Solution: $\frac{1}{p^2+3p+2} = \frac{1}{p+1} - \frac{1}{p+2}$. Then

$$\mathcal{L}^{-1}\left(\frac{e^{-p}}{p^2+3p+2}\right) = H(x-1)[e^{1-x} - e^{2-2x}].$$ 2 marks

(b) (2 marks) $\frac{e^{-p}-e^{-3p}}{p}$.

Solution:

$$\mathcal{L}^{-1}\left(\frac{e^{-p}}{p} - \frac{e^{-3p}}{p}\right) = H(x-1) - H(x-3).$$ 2 marks

Part-8

8. Find the curves with the following property:

   (a) (2 marks) The tangent to the curve is such that the segment of the tangent line lying in the first quadrant is bisected by its point of tangency.

   > **Solution:** The slope of the tangent line satisfies $y' = -\frac{y}{x}$. Thus, $xy = c$ is the general solution, the family of hyperbolae.

   (b) (3 marks) The segment of the normal to the curve lying between the curve and x-axis is of the constant length one.

   > **Solution:** The slope of the normal is the negative reciprocal of the slope of the tangent. If the tangent has zero slope, we get the constant lines $y = \pm 1$ as the required curve. For non-zero slope of the tangent, we get $\frac{-1}{y'} = \frac{y}{\sqrt{1-y^2}}$, a separable ODE. The general solution is, $\sqrt{1-y^2} = x + c$ or $(x+c)^2 + y^2 = 1$, family of unit circle centred at $(-c, 0)$ in the x-axis.

   (c) (2 marks) The segment of the normal to the curve lying between the curve and x-axis is bisected by the y-axis.

   > **Solution:** The slope of the normal is the negative reciprocal of the slope of the tangent. For non-zero slope of the tangent, we get $\frac{-1}{y'} = \frac{y}{2x}$, a separable ODE. The general solution is $x^2 + \frac{y^2}{2} = c$, family of ellipses.

   (d) (3 marks) The area under the continuous curve between a and x, for a fixed $a \in \mathbb{R}$, is proportional to $y(x) - y(a)$.

   > **Solution:** $\int_a^x y(t)\, dt = \alpha[y(x) - y(a)]$. Then $y(x) = \alpha y'(x)$ and $y(x) = ce^{x/k}$, family of exponential curves.