LECTURE - 15 & 16

Calculus of residues
Evaluation of integrals.

1 Evaluating Integrals $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

$$\frac{z+z^{-1}}{2} \qquad \frac{z-z^{-1}}{2i}$$

$$z = e^{i\theta}.$$

Eg: $$\int_0^{2\pi} \frac{1}{1+8\cos^2\theta}\, d\theta$$

$$\gamma(\theta) = e^{i\theta}, \quad \gamma'(\theta) = ie^{i\theta}, \quad \text{If } z = e^{i\theta}, \quad \gamma'(\theta) = iz$$

$$\cos\theta = \frac{z + z^{-1}}{2}$$

$$\int_{\gamma(0,1)} \frac{1}{1+8\left[\frac{z+z^{-1}}{2}\right]^2} \cdot \frac{dz}{iz} = \int_\gamma \frac{1}{1+2(z^2+2+z^{-2})} \frac{dz}{iz}$$

$$= \int_\gamma \frac{z}{z^2+2z^4+4z^2+2} \frac{dz}{i}$$

$$\int_{\gamma(0)} f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) dz$$

$$= \int_0^{2\pi} \underbrace{f(\cos\theta, \sin\theta)\, ie^{i\theta}}_{F(\cos\theta, \sin\theta)} d\theta$$

$$= \int_\gamma \frac{z}{(2z^4+5z^2+2)\, i}\, dz$$

$$2z^4+4z^2+z^2+2 = 2z^2(z^2+2)+1(z^2+2)$$

$$\therefore f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) = \frac{F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)}{iz}$$

$$= \int_\gamma \frac{z}{i(2z^2+1)(z^2+2)}\, dz, \quad z = \frac{\pm i}{\sqrt{2}}, \quad z = \pm i\sqrt{2}$$

Eg:
$$\int_0^{2\pi} \frac{1}{1+8\cos^2\theta}\, d\theta$$

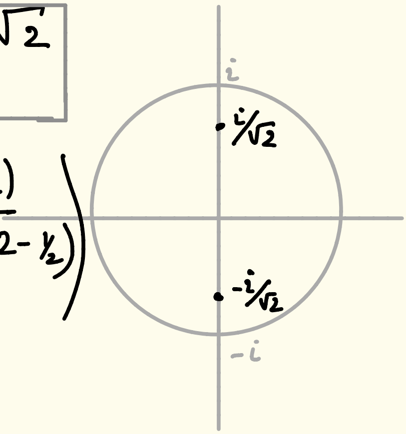
$$= \int_{C_1(0)} \frac{1}{1+8\left(\frac{z+z^{-1}}{2}\right)^2} \cdot \frac{dz}{iz}$$

$$= \int_{C_1(0)} \frac{z^{\cancel{2}}}{2z^4+2+4z^2+z^2} \cdot \frac{dz}{i\cancel{z}}$$

$$= \int_{C_1(0)} \frac{-iz}{\underbrace{2z^4+5z^2+2}_{(2z^2+1)(z^2+2)}}\, dz$$

Tip:
$$\frac{h}{k} = \frac{\left(-iz/z^2+2\right)}{2z^2+1}$$
$$= \frac{h(a)}{k'(a)}$$

$$\boxed{\text{poles}: \; z = \frac{\pm i}{\sqrt{2}}, \; \pm i\sqrt{2}}$$



$$= \frac{2\pi i}{2}\left(\frac{-i\left(i/\sqrt{2}\right)}{\left(2\frac{i}{\sqrt{2}}\right)\left(2-1/2\right)} + \frac{-i\left(-i/\sqrt{2}\right)}{2\frac{(-i)}{\sqrt{2}}\left(2-1/2\right)}\right)$$

$$= 2\pi/3 .$$

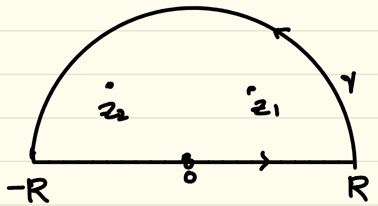
2 EVALUATING REAL INTEGRAL $\int_0^\infty f(x)\,dx$.

$$\int_0^\infty \frac{1}{x^4+1}\,dx$$

even $\quad = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

$$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{x^4+1}\,dx.$$

Find a suitable contour



$$z^4+1=0$$
$$z^4=-1$$
$$z_1 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$
$$z_2 = \frac{-1+i}{\sqrt{2}}$$

Find the complex-valued fn.

$$\int_0^\pi \frac{1}{z^4+1}\,dz + \int_{-R}^{R} \frac{1}{1+x^4}\,dx$$

$$= 2\pi i\left[\mathrm{Res}(f; z_1) + \mathrm{Res}(f: z_2)\right]$$

$$= \frac{\pi}{\sqrt{2}}.$$

estimate

$$\left|\int_0^\pi \frac{1}{R^4e^{i4t}+1}\cdot Rie^{it}\cdot dt\right| \le \frac{2\pi\cdot R}{R(R^3-1)} = \frac{2\pi}{R^3-1} \to 0 \text{ as } R\to\infty.$$

$$|R^4e^{i4t}+1| \geq |R^4e^{i4t}| - |1| \geq R^4-R$$

$$\operatorname{Res}\left(\frac{1}{1+z^4}, \frac{1+i}{\sqrt{2}}\right)$$

$$\frac{h(z)}{k(z)} \quad , \; h(a) \neq 0$$

$$k(a) = 0 \; , \; k'(a) \neq 0$$

$$\operatorname{Res}\left(\frac{h}{k}; a\right) = \frac{h(a)}{k'(a)} = \left.\frac{1}{4z^3}\right|_{z=a}$$

<u>OBSERVE!</u>- $z^4 = -1$

$$\Rightarrow z = -1/z^3$$

$$= \left. -\frac{1}{4} z \right|_{z=a}$$

$$\leq \frac{R\pi}{R^4 - 1} \to 0 \text{ as } R \to \infty$$

$$\left| \int_0^{\pi} \frac{1}{1 + R^4 e^{i4\theta}} R i e^{i\theta} d\theta \right| + \int_{-R}^{R} \frac{1}{1+x^4} dx = 2\pi i \left(-\frac{1}{4} \left(\frac{-1+i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}} \right) \right)$$

$$= \frac{2\pi i}{4}\left(-\frac{2i}{\sqrt{2}}\right) = \frac{\pi\sqrt{2}}{2}$$

$$\therefore \lim_{R \to \infty} \frac{1}{2} \int_{-R}^{R} \frac{1}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}.$$

§ Integrals of rational functions

$$\int_0^\infty \frac{1}{(x^2+1)^2(x^2+4)}\,dx$$

(this works since $f(x)$ is even)

$$= \lim_{R\to\infty} 2\int_{-R}^{R} \cdots$$

For even functions, this approach is effective!

Consider $\lim_{R\to\infty} \int_{\gamma_R(0)} \cdots$

if we took semi circle then we would have atleast 5 poles to take care of!

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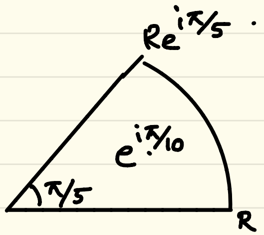
$$\int_0^\infty \frac{1}{1+x^{10}}\,dx \qquad \text{roots} : e^{i(2k+1)\pi/10}$$

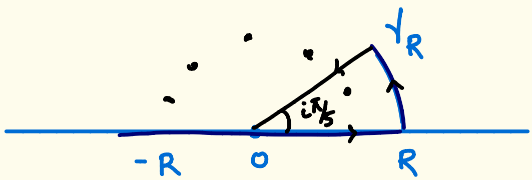


If we took the semicircular contour, then we would have to consider more poles. Choosing sector of angle $\pi/5$, we ensure that there is only one pole inside.

Eg: $\displaystyle\int_0^\infty \frac{1}{1+x^{10}}\,dx$

$\displaystyle\frac{1}{1+z^{10}} = f(z)$

poles $e^{\frac{i\pi}{10}}, e^{\frac{i3\pi}{10}}, \dots e^{\frac{i9\pi}{10}}$



$$\int_0^R f(x)\,dx + \int_0^{\pi/5} \frac{1}{1+R^{10}e^{i10\theta}}\left(iRe^{i\theta}\right)d\theta + \int_R^0 \frac{1}{1+t^{10}e^{2\pi i}}\left(e^{i\pi/5}\right)dt$$

$$= \int_0^R \frac{1-e^{i\pi/5}}{1+t^{10}}\,dt + \left|\int_0^{\pi/5} \frac{iRe^{i\theta}}{1+R^{10}e^{i10\theta}}\,d\theta\right| = \frac{-2\pi i e^{i\pi/10}}{10}$$

$\leq \frac{R}{R^{10}-1} \cdot \pi/5 \cdot$

$$\lim_{R\to\infty} \int_0^R \frac{1}{1+t^{10}}\,dt = \frac{-2\pi i e^{i\pi/10}}{10(1-e^{i\pi/5})}$$

§ EVALUATION of $\int_0^\infty \varphi(x) \begin{Bmatrix} \sin mx \\ \cos mx \end{Bmatrix} dx$

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fn

$$\int_{-\infty}^{\infty} \varphi(x) \{ \cdots \} dx$$

$|\cos Re^{i\theta}|^2 = \cosh^2(R\sin\theta) - \sin^2(R\cos\theta)$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+x+1} dx$$

$\frac{\cos z}{z^2+z+1}$ wouldn't have been easy to estimate

$$\int_{\gamma_R(0)} \frac{e^{iz}}{z^2+z+1} dz$$

(when $z \in \mathbb{R}$, real part of integral is the required integral as $R \to \infty$)

not even function

the integral evaluates the Principal value i.e. $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$.
integral

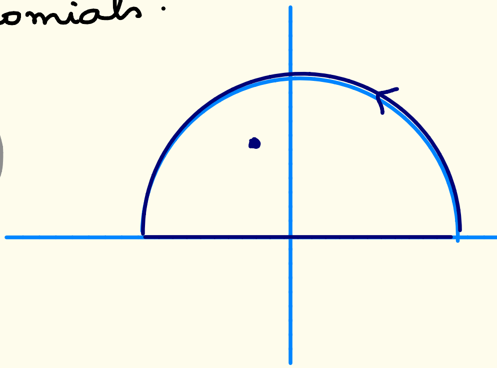
the improper integral $\lim_{R,S\to\infty} \int_{-S}^{R} f(x)\,dx$

is equal to $PV \int_{-\infty}^{\infty} f(x)\,dx$, if it exists.

$\boxed{\text{It does, if } f(x) = O(x^{-p}),\ p > 1}$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot \begin{Bmatrix} \sin mx \\ \cos mx \end{Bmatrix} dx$$

p, q are polynomials.



Eg: $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+x+1}\, dx$$

$\gamma_R = Re^{i\theta}$; $\dfrac{e^{iz}}{z^2+z+1}$

$$\int_{\gamma_R} \frac{e^{iz}}{z^2+z+1}\, dz = 2\pi i \left(\frac{e^{iz}}{z^2+z+1} ; e^{2\pi i/3} \right)$$

$$= 2\pi i \left. \frac{e^{iz}}{(2z+1)} \right|_{z = e^{2\pi i/3}}$$

$$= 2\pi i \frac{e^{i(\cos 2\pi/3 + i \sin 2\pi/3)}}{2e^{2\pi i/3}+1}$$

$$\int_{-R}^{R} \frac{\cos x + i \sin x}{x^2+x+1}\, dx + \int_{0}^{2\pi} \frac{e^{iRe^{i\theta}} \cdot Re^{i\theta} i\, d\theta}{R^2 e^{i2\theta} + Re^{i\theta} + 1}$$

Lt real part
$R \to \infty$

real part

$$= \frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}/2} \cos \tfrac{1}{2}$$

Indentation lemma (for simple poles).

Let f be analytic in $B_r(a) \setminus \{a\}$ with a simple pole of residue b at a.

Let $\gamma_\varepsilon(\theta) = a + \varepsilon e^{i\theta}$, $\quad 0 \le \theta_1 \le \theta \le \theta_2 \le 2\pi$.

Then $$\boxed{\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\,dz = ib(\theta_2 - \theta_1).}$$

Pf:

$$b = \lim_{z \to a} (z-a) f(z)$$

$\therefore$ Given $\eta > 0$ $\exists\, \delta > 0 \ni \left| (z-a)f(z) - b \right| < \eta \quad \forall\, |z-a| < \delta$

$$\left| \int_{\gamma_\varepsilon} f(z)\,dz - ib(\theta_2 - \theta_1) \right|$$

$$= \left| \int_{\theta_1}^{\theta_2} \Big(f(\underbrace{\gamma_\varepsilon(\theta)}_{z}) (\underbrace{\varepsilon i e^{i\theta}}_{i(z-a)}) - ib \Big)\, d\theta \right|$$

$$\le \eta \int_{\theta_1}^{\theta_2} d\theta = \eta(\theta_2 - \theta_1).$$

$$\therefore \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\,dz = ib(\theta_2 - \theta_1) \qquad \blacksquare$$

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Sol

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

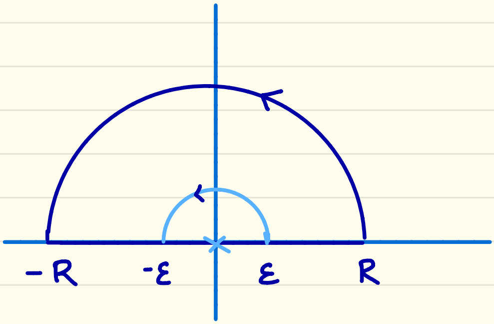
$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx \qquad \longrightarrow \frac{1-\cos 2x}{2}$$

Γ_R : semicircle.

$$\frac{1}{2}\int_{\gamma_R(0)} \frac{1-e^{i2z}}{z^2}\,dz$$

real part

pole.



$$\int_{-R}^{-\varepsilon} - \int_{\Gamma_\varepsilon} + \int_{\varepsilon}^{R} + \int_{\Gamma_R} = 0$$

$$2\int_\varepsilon^R \frac{1-e^{2ix}}{x^2}\,dx = 2\int_\varepsilon^R \frac{1-\cos 2x - i\sin 2x}{x^2}\,dx$$

the real part

$$= \int_\varepsilon^R \frac{4\sin^2 x}{x^2}\,dx$$

$\frac{1-e^{i2z}}{z^2}$ has a simple pole at '0'.

Indentation Lemma $\Rightarrow$ $$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} f(z)\,dz = i(\pi - 0)\,\mathrm{res}(f;0)$$
$$= 2\pi.$$

Estimation: $$\left| \int_{\Gamma_R} f(z)\,dz \right| \le \int_0^\pi \left| \frac{1 + e^{-2R\sin\theta}}{R^2} \right| R\,d\theta \le \frac{2}{R}.\pi.$$
$\to 0$ as $R \to \infty$

$$\therefore \lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \int_\varepsilon^R \frac{\sin^2 x}{x^2}\,dx = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$\text{i.e.} \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \pi/2$$

Up V/s Down.

(4) (ct)

$$\int_{-\infty}^{\infty} \frac{e^{-2ix}}{1+x^4}\,dx$$

$$\int_{-\infty}^{\infty} \frac{e^{-2iz}}{1+z^4}\,dz \qquad z = Re^{i\theta}$$

$$\left|e^{-2iz}\right| = \left|e^{2R\sin\theta}\right|$$

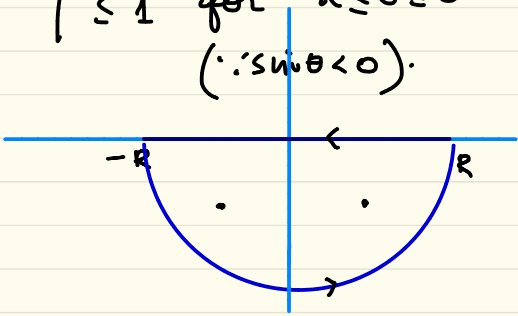
is not bounded.

So we use the lower semicircle.

$$\left|e^{-2iz}\right| = \left|e^{2R\sin\theta}\right| \leq 1 \text{ for } -\pi \leq \theta \leq 0$$

$(\because \sin\theta < 0)$.

$$\int_{\gamma_R(0)} \frac{e^{-i2z}}{1+z^4}\,dz$$



$$-\int_{-R}^{R} \frac{e^{-2ix}}{1+x^4}\,dx - \int_{0}^{\pi} \frac{e^{-iR(\cos\theta - i\sin\theta)}\, Rie^{-i\theta}(-d\theta)}{1+R^4e^{-i4\theta}}$$

$$\left| -\int_0^{\pi} \frac{e^{-iR(\cos\theta - i\sin\theta)}}{1+R^4 e^{-i4\theta}} iRe^{-i\theta}(-d\theta) \right|$$

bdd.

$$\leq \frac{e^{-R\sin\theta} R}{R^4 - 1}$$

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-2ix}}{1+x^4} dx = -\left[2\pi i \operatorname{Res}\left(\frac{e^{i2z}}{1+z^4} ; e^{-i\pi/4} \right) + 2\pi i \operatorname{Res}\left(\frac{e^{i2z}}{1+z^4} ; e^{-3\pi i/4} \right) \right]$$

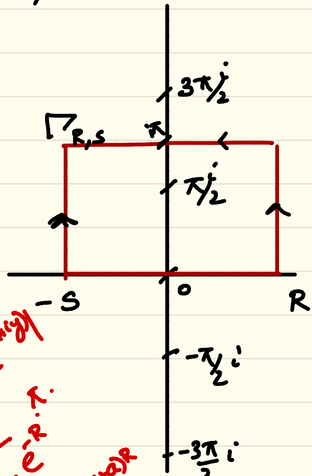
④ Ch

A contour excluding infinitely many poles

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\,dx \qquad (-1<a<1)$$

zeros of $\cosh z = \cos(iz)$

$= (2n+1)\frac{\pi}{2} i$



$$\int_{\Gamma_{R,S}} \frac{e^{az}}{\cosh z}\,dz$$

$$= \int_{-S}^{R} \frac{e^{ax}}{\cosh x}\,dx + \int_0^{\pi} \frac{e^{a(R+iy)}}{\cosh(R+iy)}\,i\,dy$$

$$\le \left|\frac{e^{aR}}{e^{R+iy}+e^{-(R+iy)}}\right|\cdot\pi \le \frac{e^{aR}}{e^{R}-e^{-R}}\cdot\pi \le \frac{1}{e^{(1-a)R}-e^{-(1+a)R}}$$

$$+ \int_{R}^{-S} \frac{e^{a(\pi i+x)}}{\underbrace{\cosh(x+i\pi)}_{-\cosh x}}\,dx + \int_{\pi}^{0} \frac{e^{a(-S+iy)}}{\cosh(-S+iy)}\,dy$$

$$\le \left|\frac{2e^{-aS}}{e^{-S+iy}+e^{S-iy}}\right|\cdot\pi \le \frac{2e^{-aS}}{e^{-S}+e^{S}}\cdot\pi$$

$$= 2\pi i\,\mathrm{Res}\left(\frac{e^{az}}{\cosh z};\ \frac{i\pi}{2}\right)$$

↑ simple pole

$$= 2\pi i\,\frac{e^{a(i\pi/2)}}{\underbrace{\sinh(i\pi/2)}_{i\sin(-\pi/2)}}$$

realpart

$$= 2\pi e^{ia\pi/2}$$

$\therefore$ as $R \to \infty$ & $S \to \infty$

$$\left(1 + e^{a\pi i}\right) \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\, dx = 2\pi e^{\frac{a\pi i}{2}}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\, dx = \frac{2\pi e^{\frac{a\pi i}{2}}}{e^{\frac{a\pi i}{2}}\left(e^{-\frac{a\pi i}{2}} + a^{\frac{a\pi i}{2}}\right)}$$

$$= \frac{\pi}{\cos\left(\frac{a\pi}{2}\right)}.$$