Digital Signals in the Frequency Domain

- Objective: Decomposition of a periodic digital signal as a weighted combination of sinusoidal digital signals of different angular frequencies
- Process involves the determination of the angular frequencies of each individual sinusoidal component and its peak amplitude

Digital Signals in the Frequency Domain

- Such a decomposition is possible if the frequencies of the constituent sinusoidal digital signals are harmonically related to an angular frequency of smaller value
- The resulting decomposition is obtained by developing the discrete-time Fourier series representation of the periodic digital signal

Digital Signals in the Frequency Domain

- A frequency domain representation of an aperiodic digital signal is obtained by the discrete-time Fourier transform, a generalization of the discrete-time Fourier series decomposition
- Here, the aperiodic digital signal is represented as a weighted sum of an infinite number of sinusoidal digital signals

Sum of Periodic Sequences

- Note: Unlike analog signals, a weighted sum of periodic digital signals is always a periodic signal

Fundamental Period

- Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ denote two periodic sequences with integer valued fundamental periods N_1 and N_2, respectively

Sum of Periodic Sequences

- Since $\tilde{x}_1[n]$ is a periodic sequence with a fundamental period N_1, we have
$$\tilde{x}_1[n] = \tilde{x}_1[n + k_1 N_1]$$
where k_1 is a positive integer
- Likewise, $\tilde{x}_2[n]$ being a periodic sequence with a fundamental period N_2, we have
$$\tilde{x}_2[n] = \tilde{x}_2[n + k_2 N_2]$$
where k_2 is a positive integer

Sum of Periodic Sequences

- The sequence $\tilde{x}[n] = \tilde{x}_1[n] + \tilde{x}_2[n]$ will be a periodic sequence with a period N_o if
$$\tilde{x}[n + N_o] = \tilde{x}_1[n + N_o] + \tilde{x}_2[n + N_o]$$
$$= \tilde{x}_1[n + k_1 N_1] + \tilde{x}_2[n + k_2 N_2]$$
- The above condition will hold if
$$N_o = k_1 N_1 = k_2 N_2$$

Sum of Periodic Sequences

- Thus, the fundamental period N_o of the sequence $\tilde{x}[n]$ is given by

$$\boxed{N_o = \mathrm{LCM}(N_1, N_2)}$$

- Now consider M periodic sequences $\tilde{x}_i[n]$, $1 \le i \le M$ with integer valued fundamental periods N_i, $1 \le i \le M$
- Let $\tilde{x}[n]$ denote a periodic sequence composed of a weighted sum of the M sequences $\tilde{x}_i[n]$

Sum of Periodic Sequences

- The fundamental period of $\tilde{x}[n]$ is given by

$$\boxed{N_o = \mathrm{LCM}(N_1, \ldots, N_M)}$$

Example – Consider $\tilde{x}[n] = \tilde{x}_1[n] + \tilde{x}_2[n] + \tilde{x}_3[n]$

where $\tilde{x}_1[n] = \cos(\pi n/6)$

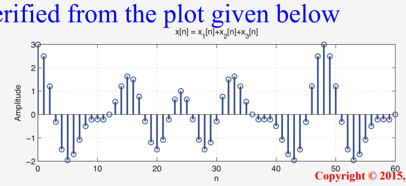
$\tilde{x}_2[n] = \cos(0.125\pi n)$

$\tilde{x}_3[n] = \cos(0.25\pi n)$

- Fundamental periods of the above signals are

Sum of Periodic Sequences

- $N_1 = \frac{1}{f_1} = 12$, $N_2 = \frac{1}{f_2} = 16$, $N_3 = \frac{1}{f_3} = 8$
- Hence, the fundamental period N_o of $\tilde{x}[n]$ is $N_o = \mathrm{LCM}(12,16,8) = 48$ which can be verified from the plot given below



Harmonically Related Sinusoidal Sequences

- Let $\tilde{x}_1[n] = \sin(\omega_1 n)$ and $\tilde{x}_2[n] = \sin(\omega_2 n)$
- These two sinusoidal sequences are harmonically related if their angular frequencies are integer multiples of an angular frequency ω_o with a smaller value

Harmonically Related Sinusoidal Sequences

- That is, $\omega_1 = k_1\omega_o$, $\omega_2 = k_2\omega_o$ where k_1 and k_2 are positive integers
- The smallest value of ω_o satisfying the above relations is called the fundamental frequency
- The angular frequencies ω_1 and ω_2 are called the harmonics

Harmonically Related Sinusoidal Sequences

- The angular frequency $2\omega_o$ is called the second harmonic
- The angular frequency $3\omega_o$ is called the third harmonic and so on
- The angular frequency ω_o is sometimes called the first harmonic

Harmonically Related Sinusoidal Sequences

- A linearly weighted sum of sinusoidal sequences with harmonically related angular frequencies is a periodic sequence
- Its fundamental period N_o is the fundamental period of the sinusoidal sequence with the smallest angular frequency ω_o which may or may not be present in the summed digital signal

Harmonically Related Sinusoidal Sequences

- The fundamental period N_o of the sum of the k-th harmonic and the ℓ-th harmonic is thus given by

$$\boxed{N_o = \mathrm{LCM}(N_o / k, N_o / \ell)}$$

Example – Consider the weighted sum

$$\tilde{x}[n] = \sin(0.05\pi n) + \tfrac{1}{2}\sin(0.10\pi n) + \tfrac{1}{5}\sin(0.25\pi n)$$

with cyclic frequencies 0.025, 0.05, and 0.0125, respectively

Harmonically Related Sinusoidal Sequences

- Note that the three cyclic frequencies are harmonically related
- The fundamental periods of the above three sinusoidal sequences are, respectively,

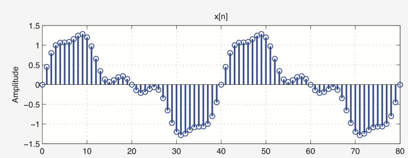
$$N_1 = \frac{1}{0.025} = 40\,,\ N_2 = \frac{1}{0.05} = 20\,,\ N_3 = \frac{1}{0.0125} = 8$$

- The fundamental period N_o of the sum is

$$N_o = \mathrm{LCM}(40, 20, 8) = 40$$

Harmonically Related Sinusoidal Sequences

- The fundamental period $N_o = 40$ can also be verified from the plot of the summed signal given below



Negative Frequency

- An equivalent representation of $\tilde{x}[n]$ given in Slide No. 14 is

$$\tilde{x}[n] = j\tfrac{1}{10}e^{-j0.25\pi n} + j\tfrac{1}{4}e^{-j0.1\pi n} + j\tfrac{1}{2}e^{-j0.05\pi n}$$
$$- j\tfrac{1}{2}e^{j0.05\pi n} - j\tfrac{1}{4}e^{j0.1\pi n} - j\tfrac{1}{10}e^{j0.25\pi n}$$

obtained using the inverse Euler's formula

Negative Frequency

- The the two expressions of the real periodic sequence given in Slide No. 14 and Slide No. 17 look very different, even though mathematically, both represent the same real periodic sequence
- The sequence given in Slide No. 14 is composed of real sinusoidal sequences with positive cyclic frequencies

Negative Frequency

- The sequence given in Slide No. 17, on the other hand, is composed of complex exponential sequences of both positive and negative cyclic frequencies
- The cyclic frequency of a real periodic sinusoidal sequence is a real positive physical quantity

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Negative Frequency

- The negative cyclic frequency shown in Slide No. 17 has been included for mathematical convenience
- Consider the real sinusoidal sequence $\cos(\omega_o n)$ with a positive normalized angular frequency ω_o and the real sinusoidal sequence $\cos(-\omega_o n)$ with a negative normalized angular frequency $-\omega_o$

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Negative Frequency

- The two sinusoidal sequences are identical as $\cos(-\omega_o n) = \cos(\omega_o n)$
- By looking at a sinusoidal sequence, it is not possible to infer the sign of its frequency

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Discrete-Time Fourier Series

- A mathematical representation of a periodic sequence as a weighted sum of sinusoidal sequences is possible if the cyclic frequencies of the constituent sinusoidal sequences are harmonically related
- The decomposition is given by the discrete-time Fourier series expansion

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Discrete-Time Fourier Series

- There is a major difference between the expressions for the discrete-time Fourier series representation of a real periodic sequence and the continuous-time Fourier series representation of a real periodic analog signal

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Discrete-Time Fourier Series

- In the case of an analog signal, the range of frequency Ω is from $0 \le \Omega < +\infty$
- As a result, the Fourier series expansion, in general, consists of an infinite number of sinusoidal analog signals with normalized angular frequencies $k\Omega_o$, $0 \le k < +\infty$

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Discrete-Time Fourier Series

- The difference between the frequencies of two successive harmonics of a continuous-time Fourier series expansion is $\Omega = 2\pi / T_o$, with T_o denoting the fundamental period
- In the case of a digital signal, the range of frequency ω is from $0 \le \omega < 2\pi$

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Discrete-Time Fourier Series

- Consequently, a real periodic sequence with a fundamental frequency $\omega_o = 2\pi / N_o$ with N_o denoting the fundamental period can contain at most N_o sinusoidal sequences of frequencies $k\omega_o$, $0 \le k \le N_o - 1$
- The difference between the frequencies of two successive harmonics here is $\omega = 2\pi / N_o$

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Exponential Form of Discrete-Time Fourier Series

- The representation of a real periodic sequence $\tilde{x}[n]$ with a fundamental period N_o is given by

$$\boxed{\tilde{x}[n] = \sum_{k=0}^{N_o-1} c_k e^{j2\pi kn/N_o}}$$

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Exponential Form

where the Fourier coefficients $\{c_k\}$ are obtained using

$$\boxed{c_k = \frac{1}{N_o} \sum_{k=0}^{N_o-1} \tilde{x}[n] e^{-j2\pi kn/N_o}, \; 0 \le k \le N_o - 1}$$

- The constants c_k are complex numbers, except c_0 which is a real number

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Exponential Form

- For a real periodic sequence,

$$c_{N_o-k} = c_k^*, \; 0 \le k \le N_o - 1$$

- For real-valued Fourier coefficients, the above relation reduces to

$$c_{N_o-k} = c_k, \; 0 \le k \le N_o - 1$$

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Exponential Form

- The set of Fourier coefficients $\{c_k\}$ is also a periodic sequence with a fundamental period N_o:

$$e^{j2\pi(n+\ell N_o)/N_o} = e^{j2\pi\ell} e^{j2\pi n/N_o} = e^{j2\pi n/N_o}$$

as $e^{j2\pi\ell} = 1$ for any integer ℓ

Example – Develop the Fourier series representation of $\tilde{x}[n] = \cos(\pi n/5)$

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Exponential Form

- Since $\cos(\pi n/5) = \cos(2\pi n/10)$, the fundamental period of $\tilde{x}[n]$ is $N_o = 10$
- Hence, the Fourier coefficients of $\tilde{x}[n]$ are obtained using

$$c_k = \frac{1}{10}\sum_{k=0}^{9}\tilde{x}[n]e^{-j2\pi kn/10}\ ,\ 0 \le k \le 9$$

Exponential Form

- Using the inverse Euler's formula we can express $\tilde{x}[n] = \cos(\pi n/5)$ as

$$\tilde{x}[n] = \tfrac{1}{2}(e^{j\pi n/5} + e^{-j\pi n/5})$$
$$= \tfrac{1}{2}(e^{j2\pi n/10}) + \tfrac{1}{2}(e^{-j2\pi n/10})$$

- Now

$$e^{-j2\pi n/10} = e^{-j(18-20)\pi n/10} = e^{j18\pi n/10}$$

Exponential Form

- Therefore

$$\tilde{x}[n] = \tfrac{1}{2}(e^{j2\pi n/10}) + \tfrac{1}{2}(e^{j18\pi n/10})$$

- Comparing the above expression with that in Slide No. 31, we conclude

$$c_0 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$$
$$c_1 = c_9 = 0.5$$

- It can be seen from the above values of the Fourier coefficients that they satisfy the relation $c_{10-k} = c_k$ for $1 \le k \le 9$

Trigonometric Form of Discrete-Time Fourier Series

- An alternate form is given by weighted sum of sinusoidal sequences

$$\tilde{x}[n] = a_0 + \sum_{k=1}^{M_o}\left(a_k\cos\left(\frac{2kn}{N_o}\right) + b_k\sin\left(\frac{2kn}{N_o}\right)\right)$$

where

$$a_0 = c_0\ ,\quad a_k = 2|c_k|\cos(\theta_k)\ ,\ b_k = 2|c_k|\sin(\theta_k)$$

and $M_o = \begin{cases} N_o/2, & \text{for } N_o \text{ even} \\ (N_o-1)/2, & \text{for } N_o \text{ odd} \end{cases}$

Power Density Spectrum

- The average power of a periodic sequence can be computed using

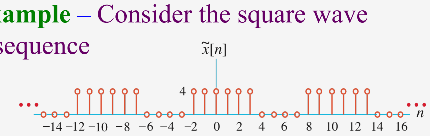
$$\mathcal{P}_{\tilde{x}} = \frac{1}{N_o}\sum_{n=0}^{N_o-1}|\tilde{x}[n]|^2 = \sum_{k=0}^{N_o-1}|c_k|^2$$

- The above equality is sometimes referred to as the Parseval's relation for periodic sequences

Power Density Spectrum

- A plot of $|c_k|^2$ as a function of the frequency index k in the range $0 \le k \le N_o - 1$ is known as the power density spectrum of the periodic sequence

Example – Consider the square wave sequence



Power Density Spectrum

- The fundamental period of the periodic square wave sequence in the previous slide is $N_o = 10$
- It can be shown that the Fourier coefficients in the exponential form of its discrete-time Fourier series expansion are given by

$$c_0 = 12/5$$

$$c_k = \frac{2}{5} e^{-j\pi k/2} \left(\frac{\sin(3\pi k/5)}{\sin(\pi k/10)} \right), \quad 1 \le k \le 9$$

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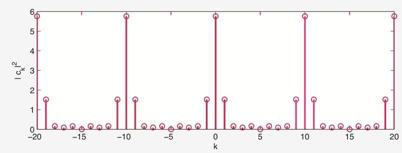
Power Density Spectrum

- Figure below shows the power density spectrum of the periodic square wave sequence



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