MSO202A COMPLEX ANALYSIS
Solutions–1

Exercise Problems:

1. For any $z, w \in \mathbb{C}$, show that (a) $\overline{z+w} = \overline{z} + \overline{w}$,(b) $\overline{zw} = \overline{z}\,\overline{w}$, (c) $\overline{\overline{z}} = z$, (d) $|\overline{z}| = |z|$ and (e) $|zw| = |z||w|$.

   Proof: Easy.

2. Show that $(a)|z+w|^2 = |z|^2 + |w|^2 + 2\text{Re}(z\overline{w})$

   Proof: $|z+w|^2 = (z+w)\overline{(z+w)} = |z|^2 + |w|^2 + (z\overline{w} + \overline{z}w) = |z|^2 + |w|^2 + 2\text{Re}(z\overline{w})$.

   $(b)|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$.

   Proof: Follows by applying (a) to $|z+w|^2$ and $|z+(-w)|^2$ and adding.

   $(c)|z+w| = |z| + |w|$ if and only if either $zw = 0$ or $z = cw$ for some positive real number c.

   Proof: If $|z+w| = |z|+|w|$ and $zw \neq 0$, then from 2(a) we obtain that $\text{Re}(z\overline{w}) = |zw|$. It follows from here that $\text{Im}(z\overline{w}) = 0$. Hence, $z\overline{w}$ is a positive real, say c. Thus $z = c\frac{w}{|w|^2}$. Conversely, if $zw = 0$, then either $z = 0$ or $w = 0$, in which case the equality holds. If $z = cw$, then $|z+w| = (1+c)|w| = |z| + |w|$.
   Note, the above means that if neither z nor w is 0 and equality holds in the triangle inequality then 0, z, w and $z+w$ are collinear.

3. Let α be any of the n th roots of unity except 1. Show that $1+\alpha+\alpha^2+\ldots+\alpha^{n-1} = 0$.

   Proof: For any $z \neq 1$, we know that $1+z+z^2+\ldots+z^k = \frac{z^{k+1}-1}{z-1}$. The result follows by applying the above relation to α different from 1.

4. Express in polar form: (a) $1+i$ (b) $-1-i$ (c) $\sqrt{3}+i$ (d) $1+\cos\theta+i\sin\theta$. Determine the value of $\text{Arg}(z^2)$ in each of the cases.

   Proof:

   (a) $1+i = \sqrt{2}e^{i(\pi/4+2n\pi)}$; $\text{Arg}(z) = \pi/4$; $\text{Arg}(z^2) = \pi/2$
   (b) $-1-i = \sqrt{2}e^{i(-3\pi/4+2n\pi)}$; $\text{Arg}(z) = -3\pi/4$; $\text{Arg}(z^2) = \pi/2$
   (c) $\sqrt{3}+i = 2e^{i(\pi/3+2n\pi)}$; $\text{Arg}(z) = \pi/3$; $\text{Arg}(z^2) = 2\pi/3$
   (d) $1+\cos\theta+i\sin\theta = 2\cos^2(\theta/2)+i(2\sin(\theta/2)\cos(\theta/2)) = 2\cos(\theta/2)e^{i\theta/2}$; $\text{Arg}(z^2) = \theta + 2n\pi$ such that $-\pi < \theta + 2n\pi \leq \pi$

5. Let z be a nonzero complex number and n a positive integer. If $z = r(\cos\theta + i\sin\theta)$, show that $z^{-n} = r^{-n}(\cos n\theta - \sin n\theta)$.

Proof: $z = r(\cos\theta + i\sin\theta)$. For $n > 0, z^n = r^n(\cos n\theta + i\sin n\theta)$, so $z^{-n} = \dfrac{1}{z^n} = \dfrac{1}{r^n(\cos n\theta + i\sin\theta)} = r^{-n}(\cos n\theta - \sin n\theta)$.

6. Find the roots of each of the following in the form $x + iy$. Indicate the principal root (a) $\sqrt{2i}$, (b) $(-1)^{1/3}$ and (c) $(-16)^{1/4}$.

Proof:

(a) $2i = 2e^{i(\frac{\pi}{2}+2k\pi)} \Rightarrow \sqrt{2i} = \sqrt{2}e^{i(\frac{\pi}{4}+k\pi)} = 1 + i$, when $k = 0$ and is $-1 - i$ when $k = 1$. $k = 0$ corresponds to the principal root.

(b) $-1 = e^{i(\pi+2k\pi)} \Rightarrow (-1)^{\frac{1}{3}} = e^{i(\frac{\pi}{3}+2k\frac{\pi}{3})}$. When $k = 0$ this is $\frac{1+i\sqrt{3}}{2}$, which corresponds to the principal root and when $k = 1$ this is -1, when $k = 2$ this is $\frac{1-i\sqrt{3}}{2}$.

(c) $(-16) = 16e^{i(\pi+2k\pi)} \Rightarrow (-16)^{\frac{1}{4}} = 2e^{i(\pi/4+k\pi/2)}$. For $k = 0$ this is $\sqrt{2}(1 + i)$, when $k = 1$ this is $\sqrt{2}(-1 + i)$, when $k = 2$ this is $\sqrt{2}(-1 - i)$, when $k = 3$ this is $\sqrt{2}(1 - i)$. When $k = 0$ the corresponding root is the principal root.

7. Determine the values of the following:

(a) $(1 + i)^{20} - (1 - i)^{20}$.

Proof: $1 + i = \sqrt{2}e^{i\pi/4}$, so $(1 + i)^{20} = \sqrt{2}^{20} e^{i5\pi} = \sqrt{2}^{20}$. Similarly, $(1 - i)^{20} = \sqrt{2}^{20}$. Thus $(1 + i)^{20} - (1 - i)^{20} = 0$.

(b) $\cos\dfrac{\pi}{4} + i\cos\dfrac{3}{4}\pi + \ldots + i^n \cos\dfrac{2n+1}{4}\pi + \ldots + i^{40}\cos\dfrac{81}{4}\pi$.

Proof: Let $a_n = i^n \cos\frac{2n+1}{4}\pi$ Then $a_{n+2} = -i^n \cos\left(\dfrac{2n+1}{4}\pi + \pi\right) = a_n$. Thus, $a_0 = a_2 = \ldots = a_{40}$ and $a_1 = a_3 = \ldots = a_{39}$. So, $a_0 + \ldots + a_{40} = 21a_0 + 20a_1 = \dfrac{\sqrt{2}}{2}(21 - 20i)$.

8. Find the roots of $z^4 + 4 = 0$. Use these roots to factor $z^4 + 4$ as a product of two quadratics with real coefficients.

Proof: $z = \sqrt{2}e^{i(\frac{\pi}{4}+\frac{k\pi}{2})}, k = 0, 1, 2, 3$. So the roots are $z_0 = 1 + i, z_1 = -1 + i, z_2 = -1-i, z_3 = 1-i$. Thus $z^4+4 = (z-z_0)(z-z_1)(z-z_2)(z-z_3) = (z^2-2z+2)(z^2+2z+2)$.

9. Determine whether the following sets describe domains (open and connected sets) in $\mathbb{C}$: (a) Re $z > 1$ (b) $0 \leq \text{Arg}z \leq \dfrac{\pi}{4}$ (c) Im $(z) = 1$, (d) $|z-2+i| < 1$ (e) $|2z+3| > 4$.

   Proof:

   (a) Re $z > 1$. This implies $x > 1$, the half plane, which is open and connected.

   (b) (b) $0 \leq \text{Arg}z \leq \frac{\pi}{4}$. This is connected but not open and hence not a domain.

   (c) Im $(z) = 1$. This is the line $y = 1$ which is not open and hence not a domain.

   (d) $|z-2+i| < 1$. Interior of the circle with center $(2, -1)$ and has radius 1. Hence, it is a domain.

   (e) $|2z+3| > 4$. The exterior of the circle of radius 2 and center $(-3/2, 0)$. This is a domain.

Problem for Tutorial:

1. Give a geometric description of the following sets:

   (a) $\{z \in \mathbb{C} : |z+i| \geq |z-i|\}$

   Proof: $\{z \in \mathbb{C} : |z+i| = |z-i|\}$ describes the set of points equidistant from $-i$ and i which are just the points on the x-axis. The set $\{z \in \mathbb{C} : |z+i| \geq |z-i|\} = \{x+iy \in \mathbb{C} : |x+i(y+1)|^2 \geq |x+i(y-1)|^2\} = \{x+iy \in \mathbb{C} : y \geq 0\}$, is the upper half plane.

   (b) $\{z \in \mathbb{C} : |z-i| + |z+i| = 2\}$.

   Proof: Note that the distance between i and $-i$ is 2. Since any three points in $\mathbb{C}$ should satisfy the triangle inequality. By Ex. 2(c) above, the points z such that $|z-i|+|z+i| = 2 = |(z+i)-(z-i)|$ is either i, $-i$ or 0, $-(z-i)$, $z+i$ and $2i$ are collinear. Hence, $z+i = c(2i)$ for some $c \in \mathbb{R}$. Now, it is easy to see that the only points on the imaginary axis satisfying the relation $|z-i|+|z+i| = 2$ are points lying in between i and $-i$.

2. Discuss the convergence of the following sequences: (a) (z^n), (b) $(\frac{z^n}{n!})$, (c) $(i^n \sin \frac{n\pi}{4})$ and (d) $(\frac{1}{n} + i^n)$.

   Proof: (a) Recall that, if $\{a_n\}$ converges to l then $\{|a_n|\}$ converges to $|l|$. So, since $|z|^n$ does not converge for $|z| > 1$, so does (z^n) whenever $|z| > 1$. If $|z| < 1$ then $|z|^n \to 0$ as $n \to \infty$,i.e., given $\epsilon > 0$ there exists a $N > 0$ such that $||z|^n| < \epsilon$ for all $n > N$. Hence, we also get $|z^n| \to 0$ as $n \to \infty$, i.e., $\lim_{n\to\infty} z^n = 0$. If $z = 1$ then $z^n \to 1$. Let $|z| = 1$ and $z \neq 1$. Suppose $\lim_{n\to\infty} z^n = l \Rightarrow |l| = 1$. Now $z^{n+1} - z^n \to l - l = 0$ while $z^{n+1} - z^n = z^n(1-z) \to l(1-z) \Rightarrow l(1-z) = 0$. Thus $l = 0$, which is a contradiction. (b) $|\frac{z^n}{n!}|$ converges to 0, using Ratio test for real sequences applied to $\frac{|z^n|}{n!}$. Hence, we deduce that $\frac{z^n}{n!}$ also converges to 0. (c) and (d) do not converge (look at values taken at $n = 4k, 4k+1, 4k+2, 4k+3$ to see that they oscillate).

3. Determine if the following series converge or diverge: (a) $\sum_{n=0}^{\infty} \left(\frac{1+i}{4}\right)^n$ (b) $\sum_{n=1}^{\infty} \left(\frac{1}{n+in^2}\right)$

Proof: (a) $\left|\left(\frac{1+i}{4}\right)^n\right| = \left|\left(\frac{1}{2\sqrt{2}}\right)^n\right|$, so by Comparison Test (a) converges.
(b)
$$\left|\left(\frac{1}{n+in^2}\right)\right| = \frac{1}{\sqrt{n^2+n^4}} = \frac{1}{n\sqrt{1+n^2}} < \frac{1}{n\sqrt{n^2}} = \frac{1}{n^2}$$
so by Comparison Test, since the latter converges so does the given series.

4. *Limit at infinity: Let $f : \mathbb{C} \to \mathbb{C}$ be a function. The limit of f at infinity is said to be l if, given any $\epsilon > 0$ there exists a $R > 0$ such that $|f(z) - l| < \epsilon$ for all z such that $|z| > R$.*
(a) Show that $\lim_{z\to\infty} \frac{1}{z^2} = 0$.
Infinite limit: Let $f : D \to \mathbb{C}$ be a function defined around z_0 (except possibly at z_0). The limit of f at z_0 is said to be ∞ if, given any $R > 0$ there exists a $\delta > 0$ such that $|f(z)| > R$ for all z such that $0 < |z| < \delta$.
(b) Show that $\lim_{z\to a} \frac{1}{z-a} = \infty$

Proof:(a) Given $\epsilon > 0$, choose $R > 1/\sqrt{\epsilon}$. Then for $|z| > R$ we have $1/|z|^2 < \epsilon$, so $\lim_{z\to\infty} 1/|z|^2 = 0$.
(b) Given $R > 0$, let $\delta < 1/R$. Then for $0 < |z-a| < \delta$ we have $1/|z-a| > 1/\delta > R$, so $\lim_{z\to a} 1/(z-a) = \infty$

5. Verify if the following functions can be given a value at $z = 0$, so that they become continuous: (a)$f(z) = \frac{|z|^2}{z}$, *(b)* $f(z) = \frac{z+1}{|z|-1}$, (c) $f(z) = \frac{\bar{z}}{z}$, (d) $\frac{\text{Im }(z^2)}{|z|}$, (e) $\frac{\text{Im } z}{1-|z|}$.

Proof:

(a) $\lim_{z\to 0} f(z) = 0$, since $\left|\frac{|z|^2}{z}\right| = \frac{|z|^2}{|z|} = |z|$.

(b) $|z| - 1 \to -1$ as $z \to 0$, so $\frac{1}{|z|-1} \to -1$ as $z \to 0 \Rightarrow (z+1)\frac{1}{|z|-1} \to -1$ as $z \to 0$.

(c) the limit does not exist, since along the x-axis and y-axis the limit is 1 and -1 respectively.

(d)
$$f(z) = \frac{2xy}{\sqrt{x^2+y^2}} + i0 \to 0 + i0 = \frac{r^2 \sin 2\theta}{r} + i0 \quad r \to 0,$$
hence assigning $f(0) = 0$ makes f continuous at $z = 0$.

(e) we have

$$f(z) = \frac{y}{1 - \sqrt{x^2 + y^2}} + i0 = \frac{r \sin \theta}{1 - r} + i0.$$

Given $\epsilon > 0$, choose $r < \min\{1/2, \epsilon/2\}$. Then we have $1/1 - r < 2$ and $\left|\frac{r \sin \theta}{1-r}\right| < 2r < \epsilon$. Hence assigning $f(0) = 0$ makes f continuous at $z = 0$.