Problem Set 5

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Let $S = \{\mathbf{e}_1 + \mathbf{e}_4, -\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3\} \subset \mathbb{R}^4$. Find $S^\perp$.

   Solution: $(\mathbf{e}_1+\mathbf{e}_4)^\perp$ is the set of all vectors that are orthogonal to $\mathbf{e}_1+\mathbf{e}_4$. That is, the set of all $\mathbf{x}^T = (x_1, \ldots, x_4)$ such that $x_1+x_4 = 0$. So $S^\perp$ is the solution space of $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 0 \\ -1 & 3 & -1 & 0 & 0 \end{array}\right]$. Apply GJE and get it.

   Otherwise apply GS with $\{\mathbf{e}_1 + \mathbf{e}_4, -\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. Linear span of the last two vectors of the orthonormal basis is $S^\perp$.

2. Show that there are infinitely many orthonormal bases of $\mathbb{R}^2$.

   Solution: Columns of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, for $0 \le \theta < 2\pi$, form bases of $\mathbb{R}^2$. Idea is that take $\{e_1, e_2\}$ and then counter-clockwise rotate the set by an angle θ.

3. **(T)** What is the projection of $\mathbf{v} = \mathbf{e}_1+2\mathbf{e}_2-3\mathbf{e}_3$ on $H := \{(x_1, x_2, x_3, x_4) : x_1+2x_2+4x_4 = 0\}$?

   Solution: Basis for H: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$.

   Orthonormalize: $\left\{ \mathbf{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{w}_3 = \frac{1}{\sqrt{105}} \begin{bmatrix} 4 \\ 8 \\ 0 \\ -5 \end{bmatrix} \right\}$.

   The projection is $\langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle \mathbf{v}, \mathbf{w}_2 \rangle \mathbf{w}_2 + \langle \mathbf{v}, \mathbf{w}_3 \rangle \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} + 0\mathbf{w}_2 + \frac{20}{105} \begin{bmatrix} 4 \\ 8 \\ 0 \\ -5 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 16 \\ 32 \\ -63 \\ -20 \end{bmatrix}$.

   Alternately: Let $\mathbf{x}$ be the projection. Then $\mathbf{v} - \mathbf{x}$ is parallel to $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$, the normal vector of H.

   As $\widehat{\mathbf{u}}$ is the unit vector in the direction of the vector $\mathbf{u}$, we get

   $\mathbf{v} - \mathbf{x} = \langle \mathbf{v}, \widehat{\mathbf{v} - \mathbf{x}} \rangle \widehat{\mathbf{v} - \mathbf{x}} = \frac{5}{21} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$. So $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix} - \frac{5}{21} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 16 \\ 32 \\ -63 \\ -20 \end{bmatrix}$.

4. Let $\mathbb{V}$ be a subspace of $\mathbb{R}^n$. Then show that $\dim \mathbb{V} = n - 1$ if and only if $\mathbb{V} = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = 0\}$ for some $\mathbf{a} \neq \mathbf{0}$.

   Solution: Let $\mathbb{V} = \{\mathbf{x} : \mathbf{a}^T\mathbf{x} = 0\} = \mathcal{N}(A)$, where $A = \mathbf{a}^T$. Since $\mathbf{a} \neq \mathbf{0}$, we see that $\text{rank}(A) = 1$ and hence $\dim \mathbb{V} = n - 1$ (use $\dim(\mathcal{N}(A)) + \dim(\text{col space}(A)) = n$).

Conversely, suppose that $\dim \mathbb{V} = n-1$. Get an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}$ of $\mathbb{V}$ and extend it to an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of $\mathbb{R}^n$. Then $\mathbb{V} = \{\mathbf{x} : \mathbf{u}_n^T \mathbf{x} = 0\}$.

5. **(T)** Does there exist a real matrix A, for which, the Row space and column space are same but the null-space and left null-space are different?

   Solution: Not possible. Use the fundamental theorem of linear algebra which states that

   $$\mathcal{N}(A) = \left(\text{col space}(A^T)\right)^{\perp} \text{ and } \mathcal{N}(A^T) = \left(\text{col space}(A)\right)^{\perp}.$$

   That is, same row and column spaces require us to have a square matrix. This further implies that the dimension of null-spaces have to be same. Now, null-space and left-null-space are orthogonal to row and column spaces, respectively (which are same in this case). Hence, the Null-spaces are also same.

6. **(T)** Consider two real systems, say $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{y} = \mathbf{d}$. If the two systems have the same **nonempty** solution set, then, is it necessary that row space(A) = row space(C)?

   Solution: Yes. Observe that they have to be systems with the same number of variables. So, the two matrices A and C have the same number of columns. If they have the unique solution then $\mathcal{N}(A) = \{\mathbf{0}\} = \mathcal{N}(C)$.

   If it has infinite number of solutions then let S_h be the solution set of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$ and $C\mathbf{y} = \mathbf{0}$. Thus, $\mathcal{N}(A) = \mathcal{N}(C)$.

   So, by fundamental theorem of linear algebra, col space(A^T) = col space(C^T). That is, row space(A) = row space(C).

7. Show that the system of equations $A\mathbf{x} = \mathbf{b}$ given below

   $$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 5 \\ 2x_1 + 2x_2 + 3x_3 &= 5 \\ 3x_1 + 4x_2 + 5x_3 &= 9 \end{aligned}$$

   has no solution by finding $\mathbf{y} \in \mathcal{N}(A^T)$ such that $\mathbf{y}^T\mathbf{b} \neq 0$.

   Solution: Note that if the system has a solution $\mathbf{x}_0$ then, we get $A\mathbf{x}_0 = \mathbf{b}$. Thus, for any $\mathbf{y} \in \mathcal{N}(A^T)$, we have

   $$\mathbf{y}^T\mathbf{b} = \mathbf{y}^T(A\mathbf{x}_0) = (\mathbf{y}^T A)\mathbf{x}_0 = (A^T\mathbf{y})^T\mathbf{x}_0 = \mathbf{0}^T\mathbf{b} = 0. \tag{1}$$

   But, it is easy to check that $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is in $\mathcal{N}(A^T)$ and $\begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix} = -1$. A contradiction to Equation (1). Thus, the given system has no solution.

8. **(T)** Suppose A is an n by n real invertible matrix. Describe the subspace of the row space of A which is orthogonal to the first column of A^{-1}.

   Solution: Let $A[:, j]$ (respectively, $A[i, :]$) denote the j-th column (respectively, the i-th row) of A. Then, $AA^{-1} = I_n$ implies $\langle A[i, :], A^{-1}[:, 1]\rangle = 0$ for $2 \leq i \leq n$. So, the row subspace of A which is orthogonal to the first column of A^{-1} equals $\mathrm{LS}(A[2, :], A[3, :], \ldots, A[n, :])$.

9. **(T)** Let $A_{n\times n}$ be any matrix. Then, the following statements are equivalent.

   (i) A is unitary.

   (ii) For any orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of $\mathbb{C}^n$, the set $\{A\mathbf{u}_1, \ldots, A\mathbf{u}_n\}$ is also an orthonormal basis.

   Solution: $(i) \Rightarrow (ii)$: Suppose A is unitary. Then $\langle A\mathbf{u}_i, A\mathbf{u}_j\rangle = \langle \mathbf{u}_i, A^*A\mathbf{u}_j\rangle = \langle \mathbf{u}_i, \mathbf{u}_j\rangle$. It follows that $\{A\mathbf{u}_1, \ldots, A\mathbf{u}_n\}$ is orthonormal, hence a basis of $\mathbb{C}^n$.

   $(ii) \Rightarrow (i)$: Suppose (ii) is satisfied by A. Consider the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. By hypothesis $\{A\mathbf{e}_1, \ldots, A\mathbf{e}_n\}$ is an orthonormal basis. That is the columns of A form an orthonormal basis, that is, $A^*A = I$.

10. Let $\mathbb{V}$ be an inner product space and S be a nonempty subset of $\mathbb{V}$. Show that

    (i) $S \subset (S^\perp)^\perp$.

    (ii) If $\mathbb{V}$ is finite dimensional and S is a subspace then $(S^\perp)^\perp = S$.

    (iii) If $S \subset T \subset \mathbb{V}$, then $S^\perp \supset T^\perp$.

    (iv) If S is a subspace then $S \cap S^\perp = \{0\}$.

    Solution: (i) $\mathbf{x} \in S \Rightarrow \langle \mathbf{w}, \mathbf{x}\rangle = 0$,for all $\mathbf{w} \in S^\perp \Rightarrow \mathbf{x} \perp S^\perp \Rightarrow \mathbf{x} \in (S^\perp)^\perp$.

    (ii) If $S = \{\mathbf{0}\}, \mathbb{V}$ we have nothing to show. So let $S \neq \{\mathbf{0}\}, \mathbb{V}$. Take a basis of S, apply GS to get an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ of S. Extend that to an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of $\mathbb{V}$. It is easy to show that $\mathbf{w}_i \in S^\perp$.

    Now let $\mathbf{x} \in (S^\perp)^\perp \subset \mathbb{V}$. Thus $\mathbf{x} = \sum \alpha_i \mathbf{u}_i + \sum \beta_j \mathbf{w}_j$, for some $\alpha_i, \beta_j \in \mathbb{C}$. As $\mathbf{x} \in (S^\perp)^\perp$, we have $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, for all $\mathbf{y} \in S^\perp$. In particular $\langle \mathbf{x}, \mathbf{w}_j\rangle = 0$,for all j. Thus $\beta_j = 0$, for all j. Thus $\mathbf{x} = \sum \alpha_i \mathbf{u}_i \in S$.

    (iii) Obvious.

    (iv) Let $\mathbf{x} \in S \cap S^\perp$. Then $\mathbf{x} \perp S$. In particular $\langle \mathbf{x}, \mathbf{x}\rangle = 0$. Thus $\mathbf{x} = \mathbf{0}$.

11. Let $A_1, \cdots, A_k$ be k real symmetric matrices of order n such that $\sum A_i^2 = 0$. Show that each $A_i = 0$.

    Solution: For each $\mathbf{x} \in \mathbb{R}^n$ we have

    $$0 = \mathbf{x}^T \left(\sum A_i^2\right) \mathbf{x} = \sum \mathbf{x}^T A_i^2 \mathbf{x} = \sum \mathbf{x}^T A_i^T A_i \mathbf{x} = \sum \|A_i x\|^2.$$

    Hence, $A_i\mathbf{x} = 0$ for each i and for each $\mathbf{x}$. In particular, $A_i\mathbf{e}_1 = \mathbf{0}, A_i\mathbf{e}_2 = \mathbf{0}, \ldots, A_i\mathbf{e}_n = \mathbf{0} \Rightarrow A_i = \mathbf{0}$.

12. Let $\mathbb{V}$ be a normed linear space and $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. Is it true that $\Big|\|\mathbf{x}\| - \|\mathbf{y}\|\Big| \leq \|\mathbf{x} - \mathbf{y}\|$?

13. **(T) Polar Identity**: The following identity holds in an inner product space.

    - Complex IPS : $4\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 + i\|\mathbf{x} + i\mathbf{y}\|^2 - i\|\mathbf{x} - i\mathbf{y}\|^2$.

- Real IPS : $4\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2$

Solution: We see that $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle$,
$\|\mathbf{x}-\mathbf{y}\|^2 = \langle \mathbf{x}, \mathbf{x}\rangle - \langle \mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle$
$i\|\mathbf{x}+i\mathbf{y}\|^2 = i\langle \mathbf{x}, \mathbf{x}\rangle + i\langle \mathbf{x}, i\mathbf{y}\rangle + i\langle i\mathbf{y}, \mathbf{x}\rangle + \langle i\mathbf{y}, i\mathbf{y}\rangle$ and
$i\|\mathbf{x}-i\mathbf{y}\|^2 = i\langle \mathbf{x}, \mathbf{x}\rangle - i\langle \mathbf{x}, i\mathbf{y}\rangle - i\langle i\mathbf{y}, \mathbf{x}\rangle + \langle i\mathbf{y}, i\mathbf{y}\rangle$. Hence

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 + i\|\mathbf{x}+i\mathbf{y}\|^2 - i\|\mathbf{x}-i\mathbf{y}\|^2 \\
&= 2\langle \mathbf{x}, \mathbf{y}\rangle + 2\langle \mathbf{y}, \mathbf{x}\rangle + 2i\langle \mathbf{x}, i\mathbf{y}\rangle + 2i\langle i\mathbf{y}, \mathbf{x}\rangle \\
&= 2\langle \mathbf{x}, \mathbf{y}\rangle + 2\langle \mathbf{y}, \mathbf{x}\rangle - 2i^2\langle \mathbf{x}, \mathbf{y}\rangle + 2i^2\langle \mathbf{y}, \mathbf{x}\rangle = 4\langle \mathbf{x}, \mathbf{y}\rangle.
\end{aligned}$$

14. **Just for knowledge, will NOT be asked** Let $\|\cdot\|$ be a norm on $\mathbb{V}$. Then $\|\cdot\|$ is induced by some inner product if and only if $\|\cdot\|$ satisfies the parallelogram law:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

Solution: See the appendix in my notes.

15. Show that an orthonormal set in an inner product space is linearly independent.

Solution: Let S be an orthonormal set and suppose that $\sum\limits_{i=1}^{n} \alpha_i \mathbf{x}_i = \mathbf{0}$, for some $\mathbf{x}_i \in S$. Then $\alpha_i = \langle \mathbf{x}_i, \sum\limits_{j=1}^{n} \alpha_j x_j\rangle = \langle \mathbf{x}_i, \mathbf{0}\rangle = 0$, for each i. Thus, S is linearly independent.

16. Let A be unitarily equivalent to B (that is $A = U^*BU$ for some unitary matrix U). Then $\sum\limits_{ij} |a_{ij}|^2 = \sum\limits_{ij} |b_{ij}|^2$.

Solution: We have

$$\sum_{ij} |a_{ij}|^2 = \operatorname{tr}(A^*A) = \operatorname{tr}(U^*B^*UU^*BU) = \operatorname{tr}(U^*B^*BU) = \operatorname{tr}(B^*BU^*U) = \operatorname{tr}(B^*B) = \sum_{ij} |b_{ij}|^2.$$

17. For the following questions, find a projection matrix P that projects $\mathbf{b}$ onto the column space of A, that is, $P\mathbf{b} \in \operatorname{col}(A)$ and $\mathbf{b} - P\mathbf{b}$ is orthogonal to $\operatorname{col}(A)$.

$$(i)\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad (ii)\quad A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: Note that an orthonormal basis of $\operatorname{col}(A)$ is given by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \mathbb{R}^4$. Hence, the projection matrix equals

$$P = \mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_2\mathbf{e}_2^T + \mathbf{e}_3\mathbf{e}_3^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the second question, we know $\mathbf{w}_1 = \frac{1}{2}(1,1,1,1)^T, \mathbf{w}_2 = \frac{1}{\sqrt{10}}(-2,-1,1,2)^T$ and $\mathbf{w}_3 = \frac{1}{2}(1,-1,-1,1)^T$ form an orthonormal basis of col(A). Thus, the projection matrix equals

$$P = \mathbf{w}_1\mathbf{w}_1^T + \mathbf{w}_2\mathbf{w}_2^T + \mathbf{w}_3\mathbf{w}_3^T = \frac{1}{10}\begin{bmatrix} 9 & 2 & -2 & 1 \\ 2 & 6 & 4 & -2 \\ -2 & 4 & 6 & 2 \\ 1 & -2 & 2 & 9 \end{bmatrix}.$$

Alternate:

$$\begin{aligned} P &= A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Similarly, it can be verified that

$$P = A(A^TA)^{-1}A^T = \frac{1}{10}\begin{bmatrix} 9 & 2 & -2 & 1 \\ 2 & 6 & 4 & -2 \\ -2 & 4 & 6 & 2 \\ 1 & -2 & 2 & 9 \end{bmatrix}.$$

18. We are looking for the parabola $y = c + dt + et^2$ that gives the least squares fit to these four measurements:
$y = 1$ at $t = -2$, $y = 1$ at $t = -1$, $y = 1$ at $t = 1$ and $y = 0$ at $t = 2$.

  (a) Write down the four equations ($A\mathbf{x} = \mathbf{b}$) for the parabola $c + dt + et^2$ to go through the given four points. Prove that $A\mathbf{x} = \mathbf{b}$ has no solution.

  Solution: Verify: $A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $RREF([A\ \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$.

  (b) For finding a least square fit of $A\mathbf{x} = \mathbf{b}$, *i.e.*, of $A\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \mathbf{b}$, what equations would you solve?

  Solution: Let $\mathbf{y} = A\mathbf{x} - \mathbf{b}$ be the error vector. Then, the sum of squared errors equals

$$f(x_1, x_2, x_3) = \mathbf{y}^T\mathbf{y} = (A\mathbf{x} - \mathbf{b})^T (A\mathbf{x} - \mathbf{b}) = \mathbf{x}^TA^TA\mathbf{x} - 2\mathbf{x}^TA^T\mathbf{b} + \mathbf{b}^T\mathbf{b}.$$

Thus, differentiating w.r.t x_1, x_2 and x_3, we get

$$\left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right]^T = 2A^TA\mathbf{x} - 2A^T\mathbf{b}.$$

Now, equating it to zero gives $A^TA\mathbf{x} = A^T\mathbf{b}$. Thus, we want to solve $A^TA\begin{bmatrix} c \\ d \\ e \end{bmatrix} = A^T\mathbf{b}$.

(c) Compute A^TA. Compute its determinant. Compute its inverse.

Solution: $A^TA = \begin{bmatrix} 4 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$, $\det(A^TA) = 4 \times 10 \times 34 - 10 \times 10 \times 10 = 360$ and $(A^TA)^{-1} = \frac{1}{det(A^TA)}C^T$, where C (cofactor matrix, symmetric in this case) is given by:

$$C = \begin{bmatrix} 340 & 0 & -100 \\ 0 & 36 & 0 \\ -100 & 0 & 40 \end{bmatrix}.$$

(d) Now, determine the parabola $y = c + dt + et^2$ that gives the least squares fit.

Solution: Using the previous two parts, we see that

$$\begin{bmatrix} c \\ d \\ e \end{bmatrix} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{30}\begin{bmatrix} 35 \\ -6 \\ -5 \end{bmatrix}.$$

(e) The first two columns of A are already orthogonal. From column 3, subtract its projection onto the plane of the first two columns to get the third orthogonal vector $\mathbf{v}$. Normalize $\mathbf{v}$ to find the third orthonormal vector $\mathbf{w}_3$ from Gram-Schmidt.

Solution: Since third and second columns are already orthogonal, suffices to subtract from the third column its projection onto the first column:

$$\mathbf{v}^T = (4, 1, 1, 4) - \frac{5}{2}(1, 1, 1, 1) = (3/2, -3/2, -3/2, 3/2).$$

To find $\mathbf{w}_3$, just divide $\mathbf{v}$ by its length, 3. So,

$$\mathbf{w}_3 = (1/2, -1/2, -1/2, 1/2).$$

(f) Now compute $\mathbf{x} = A\begin{bmatrix} c \\ d \\ e \end{bmatrix}$ to verify that $\mathbf{x}$ is indeed the projection vector onto the column space of the matrix A.

Solution: Verify that for the value of P computed in the previous problem which corresponds to $\mathbf{w}_3$ in the previous part, we have

$$\mathbf{x}^T = \frac{1}{10}\begin{bmatrix} 9 & 12 & 8 & 1 \end{bmatrix} = (P\mathbf{b})^T.$$