MSO202A COMPLEX ANALYSIS
Assignment 5

Exercise Problems:

1. Evaluate the integral $\frac{1}{2\pi i}\int_C \frac{ze^{zt}}{(z+1)^3}\,dz$ where C is a counter-clockwise oriented simple closed contour enclosing $z=-1$.

   Proof: Using the Cauchy integral formula for derivatives of an analytic function, the above integral is $\frac{f^{(2)}(-1)}{2!} = \frac{1}{2}(2t-t^2)e^{-t}$.

2. Write down the Taylor series centred at the given point for the following functions and find its disc of convergence:
   $(i)\ f(z) = \frac{1}{z^2}$ at $z_0 \neq 0$ $\quad (ii)\ f(z) = \frac{6z+8}{(2z+3)(4z+5)}$ at $z_0 = 1$
   $(iii)\ \ f(z) = \frac{e^z}{z+1}$ at $z_0 = 1$.

   Proof: (i). If a function g is given by a power series expansion, then g' is given by term by term differentiation. Since we know, $\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n, |z| < 1$, $\Rightarrow \frac{1}{(1+z)^2} = \sum_{n=1}^{\infty}(-1)^{n+1}nz^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+1)z^n$, which is valid in the disc $\{z : |z| < 1\}$. We get

$$\begin{aligned}
\frac{1}{z^2} &= \frac{1}{(z-z_0+z_0)^2} \\
&= \frac{1}{z_0^2}\frac{1}{\left[1+\frac{z-z_0}{z_0}\right]^2} \\
&= \sum_{n=0}^{\infty}(-1)^n(n+1)\frac{1}{z_0^{n+2}}(z-z_0)^n.
\end{aligned}$$

   The disc of convergence is $\{z : |z-z_0| < 1\}$.

   (ii) Let $t = z-1$. $f(z) = \frac{1}{2z+3} + \frac{1}{4z+5} = \frac{1}{2t+5} + \frac{1}{4t+9}$. This is equal to $\sum_{0}^{\infty}\frac{(-2)^n(z-1)^n}{5^{n+1}} + \sum_{0}^{\infty}\frac{(-4)^n(z-1)^n}{9^{n+1}}$. The disc of convergence is $\{z : |z-1| < 9/4\}$

   (iii) $f(z) = \frac{e^z}{z+1} = \frac{e}{2}\left[\sum_{0}^{\infty}\frac{(z-1)^n}{n!}\right]\left[\sum_{0}^{\infty}\frac{(-1)^n(z-1)^n}{2^n}\right]$. The coefficient of $(z-1)^n$ is $\frac{e}{2}\sum_{j=0}^{n}\frac{(-1)^{n-j}}{j!2^{n-j}}$. The disc of convergence is $\{z : |z-1| < 2\}$.

3. Let $f, g\colon \mathbb{C} \to \mathbb{C}$ be analytic functions such that $f(a_n) = g(a_n), n = 1, 2, \ldots$ for a bounded sequence of distinct complex numbers. Show that $f \equiv g$ on $\mathbb{C}$.

   Proof: Let $h(z) = f(z) - g(z)$. Then, $h(a_n) = 0, \forall n$. Since a_n is bounded sequence of distinct numbers, it has a convergent subsequence a_{n_k}, of distinct numbers which converges to a (why?). Then, $h(a) = 0$ and thus a is not an isolated zero of h. By the identity(uniqueness) theorem, $h \equiv 0$.

4. Derive the Taylor series representation of $\frac{1}{1-z}$ around i.

   $$\frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \text{ where } |z-i| < \sqrt{2}.$$

   Proof: As $1 - z = \left[1 - \frac{z-i}{1-i}\right](1-i)$, and $|\frac{z-i}{1-i}| < 1$, the solution follows from the geometric series.

5. Let f be analytic in a simply connected domain D and γ be a simple closed curve in D oriented counterclockwise. Suppose z_0 is the only zero of f in the region enclosed by γ. Show that $\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i m$ where m is the order of zero of f at z_0. (*If $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$ then f is said to have a zero of order m.*)

   Proof: Let $f(z) = (z - z_0)^m g(z)$ inside a disc $B_R(z_0)$ in the region enclosed by γ, where $g(z)$ is analytic and $g(z_0) \neq 0$. Since z_0 is the only zero of f in the region enclosed by γ, we get $g(z) \neq 0$ there. Now

   $$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m} g(z) = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}.$$

   Since $\frac{f'(z)}{f(z)}$ is analytic in D except possibly at z_0 we get for $r < R$,

   $$\int_\gamma \frac{f'(z)}{f(z)} dz = \int_{C_r(z_0)} \frac{f'(z)}{f(z)} dz = \int_{C_r(z_0)} \left[\frac{m}{z - z_0} + \frac{g'(z)}{g(z)}\right] dz.$$

   As $g'(z)/g(z)$ is analytic inside $C_r(z_0)$ we get $\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i m$

6. (Mean Value Theorem) Let D be a simply connected domain and $f : D \to \mathbb{C}$ be an analytic function. Then prove that $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$ for every $r > 0$ such that $B(z_0, r)$ is contained in D.

   Proof: Apply Cauchy Integral Formula.

7. Find the maximum of the function $|f|$ on $\overline{\mathbb{D}}$ if (a) $f(z) = z^2 - z$ (b) $f(z) = \sin z$.

Proof: By the Maximum Modulus principle, it is sufficient to look only at $|z| = 1$. (a) $|f(z)| = |z||z-1| \leq |z|(|z|+1) \leq 2$ and $f(-1) = 2$ and so maximum is 2.

(b) $|\sin z| \leq \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} = \frac{1}{2}(e - \frac{1}{e})$.
When $z = i, |\sin z| = \frac{1}{2}(e - \frac{1}{e})$.

Problem for Tutorial:

8. Let f be entire and $|f(z)| \leq a + b|z|^n$ for some positive real numbers a and b and $n \in \mathbb{N}$. Show that f is a polynomial of degree at most n.

Proof: As f is entire, $f(z) = \sum_{k=0}^{\infty} a_k z^k, z \in \mathbb{C}$. Applying Cauchy's estimate, we get $|a_k| \leq \dfrac{1}{2\pi} \dfrac{\sup_{z \in C_R(0)} |f(z)|}{R^{k+1}} \times 2\pi R \leq \dfrac{a + bR^n}{R^k} \leq M \dfrac{1}{R^{k-n}} \to 0$ as $R \to \infty$, as $k \geq n+1$. Hence f is a polynomial of degree at most n.

9. Let $f \colon \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Let $z_0 \in \mathbb{C}$ and $r > 0$ be arbitrary. Show that the image of f intersects the disc $B_r(z_0) = \{z : |z - z_0| < r\}$. (Hence image of a non-constant analytic function intersects every disc in $\mathbb{C}$.)

Proof: Assume that the image of f does not intersect the disc $B_r(z_0)$. Then $|f(z) - z_0| \geq r, \ \forall z \in \mathbb{C}$. Define $g(z) = \frac{1}{f(z) - z_0}, z \in \mathbb{C}$. Then g is a well-defined analytic function on $\mathbb{C}$ and is bounded by $1/r$. By Liouville's theorem g is a constant function i.e., f is a constant function, a contradiction.

10. Let f and g be nonzero analytic functions defined on the disc $\mathbb{D}$ with $|f(z)| \leq |g(z)| \forall z$. Assume that z_0 is a zero for $g(z)$ of order n. Show that z_0 is a zero for $f(z)$ of order at least n.

Proof: Assume that a is a zero for g of order n. Then $g(z) = (z-a)^n h(z)$, where h is analytic on $B_{r_1}(a) \subset \mathbb{D}$ and $h(a) \neq 0$. Since $|f(a)| \leq |g(a)| = 0$, a is a zero of f. Let m be its order. Then $f(z) = (z-a)^m \phi(z)$, on $B_{r_2}(a) \subset B_{r_1}(a)$ for some analytic function ϕ with $\phi(a) \neq 0$. As $|f(z)| \leq |g(z)| \Rightarrow |(z-a)|^m |\phi(z)| \leq |z-a|^n |h(z)|$. If $m < n$, it implies that for $z \neq a, |\phi(z)| \leq |z-a|^{n-m} |h(z)|$. By taking limit as $z \to a$, it follows that $|\phi(a)| \leq 0$ or $\phi(a) = 0$, a contradiction. Therefore, $m \geq n$.