Classification of Second order Linear PDE

General form: (2-dim)

$$\begin{cases} A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} \\ \quad + D(x,y)\,u_x + E(x,y)\,u_y + F(x,y)\,u = k \end{cases}$$

Classification

1. Elliptic : If
$$B^2 - 4AC < 0$$

2. Hyperbolic : $B^2 - 4AC > 0$

3. Parabolic : $B^2 - 4AC = 0$

Ex. 1 $\Delta u = 0$ (Laplace) 2o

$$u_{xx} + u_{yy} = 0$$

$A = 1$, $B = 0$, $C = 1$

$$B^2 - 4AC = -4 \ (<0).$$

Ex. 2 Heat Eqn.

$$u_t = u_{xx} \Leftrightarrow u_{xx} - u_t = 0$$

$A = 1$ $B = 0$ $C = 0$

$$B^2 - 4AC = 0$$

(Parabolic)

Ex. 3 Wave Eq.

$$\left.\begin{array}{l} u_{tt} - u_{xx} = 0 \\ A = -1 \quad B = 0 \quad C = 1 \end{array}\right\} B^2 - 4AC = 4$$

(Hyperbolic)

: Reduction to Cannonical form :

$\boxed{3}$

Idea :

$$u_{tt} - u_{xx} = 0$$

$$\begin{cases} \xi(x,y) = x+t & \Rightarrow \xi_x = 1 \\ \eta(x,y) = x-t \end{cases}$$

$$v(\xi,\eta) = u(x,t) \quad \text{(change of variables)}$$

$$\Rightarrow u_x = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= v_\xi \xi_x + v_\eta \eta_x = v_\xi + v_\eta .$$

$$u_{xx} = \frac{\partial}{\partial x}(v_\xi) + \frac{\partial}{\partial x}(v_\eta)$$

$$= (v_{\xi\xi}\xi_x + v_{\xi\eta}\eta_x) + v_{\eta\xi}\xi_x + v_{\eta\eta}\eta_x$$

$$= v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}$$

$$u_{tt} = v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta}$$ 4

$$\underbrace{u_{tt} - u_{xx}}_{=0} = 4v_{\xi\eta}$$

$$\Rightarrow \boxed{v_{\xi\eta} = 0} \quad (v_\xi)_\eta = 0$$

$$v_\xi = f(\xi)$$

$$\boxed{v(\xi,\eta) = f(\xi) + g(\eta)}$$

$$\Rightarrow \left\{ \begin{array}{l} 2xu_{xx} + 5u_{xy} + 7u_{yy} \\ + 10x\,u_x + 15y\,u_y = 0 \end{array} \right.$$

Easier

Reduction to cannonical of 2nd order linear PDE

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$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$

A, B, C, D, E, F, G are "smooth" fn. of x, and y.

CHANGE THE VARIABLE :

$\xi = \xi(x, y)$, $\eta = \eta(x, y)$ (nice fn)

$u(x, y) = v(\xi, \eta)$

u_{xx}, u_x, u_{xy}, u_y, u_{yy}

v_ξ, $v_{\xi\xi}$ ...

6.

$$u_x(x,y) = \frac{\partial v}{\partial \xi}\xi_x + \frac{\partial v}{\partial \eta}\eta_x \qquad u(x,y) = v(\xi,\eta)$$

$$u_{xx} = v_{\xi\xi}\xi_x^2 + 2v_{\xi\eta}\xi_x\eta_y + v_{\eta\eta}\eta_x^2 + v_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_y = v_\xi \xi_y + v_\eta \eta_y.$$

$$u_{yy} = v_{\xi\xi}\xi_y^2 + 2v_{\xi\eta}\xi_y\eta_y + v_{\eta\eta}\eta_y^2 + v_\xi \xi_{yy} + v_\eta \eta_{yy}.$$

$$u_{xy} = v_{\xi\xi}\xi_x\xi_y + 2v_{\xi\eta}(\xi_x\eta_y + \eta_x\xi_y) + v_{\eta\eta}\eta_x\eta_y + v_\xi \xi_{xy} + v_\eta \eta_{xy}$$

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

7o

$$\Downarrow$$

$$\bar{A} v_{\xi\xi} + \bar{B} v_{\xi\eta} + \bar{C} v_{\eta\eta} + \bar{D} v_{\xi} + \bar{E} v_{\eta} + \bar{F} u = \bar{G}.$$

where

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$\bar{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$

$$\bar{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$

$$\bar{F} = F, \quad \bar{G} = G$$

Reduction to cannonical form

Examples :- $u_{tt} - u_{xx} = 0$

(Hyperbolic)

$A = -1, B = 0 \quad C = 1$

$$\Rightarrow \frac{dt}{dx} = \frac{B \pm \sqrt{B^2 - 4CA}}{2A}$$

$$= \frac{\pm 2}{-2} = \pm 1$$

$t = \pm x + c$

$$\left.\begin{aligned} t &= x + c_1 \\ t &= -x + c_2 \end{aligned}\right\} \quad \begin{aligned} \eta(x,y) &= t - x \\ \xi(x,y) &= x + t \end{aligned}$$

$\bar{A}, \bar{B}, \bar{C}$ $\boxed{v_{\xi\eta} = 0}$

Parabolic

2.

$$\underbrace{x^2}_{A} u_{xx} - \underbrace{2xy}_{B} u_{xy} + y^2 u_{yy} = e^{x} \quad G$$

$A = x^2$, $\quad B(x,y) = -2xy$

$C = y^2$.

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2$$
$$= 0 \quad \text{(Parabolic)}$$

$\longrightarrow$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = -B/2A$$

$$= \quad \frac{dy}{dx} = \frac{-\cancel{2}xy}{\cancel{2}x^2} = -y/x$$

$\Rightarrow \boxed{xy = C} \qquad \xi(x,y) = xy.$

$$\boxed{\eta(x,y) = y}$$

$\underline{\underline{3}}$

$$x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = e^x$$

$\bar{A}, \bar{B}, \bar{C}$

$\bar{A} = 0$ $\quad \bar{G} = G = e^x$

$\bar{B} = 0$ $\quad \bar{D} = -2xy$

$\bar{C} = y^2$

$$y^2 v_{\eta\eta} - 2xy\, v_\xi = e^x$$

$$\eta^2 v_{\eta\eta} - 2\xi v_\xi = e^{\xi/\eta}$$

$$\boxed{v_{\eta\eta} = \frac{2\xi}{\eta^2} v_\xi + \frac{1}{\eta^2} e^{\xi/\eta}}$$

Wave Equations

Kreyszig (544)

1-dim Wave Equations

$$\boxed{u_{tt} = c^2 u_{xx}}$$

$c \in \mathbb{R} \setminus \{0\}$

$\boxed{t > 0}$

time

space variable

$$\boxed{u_{tt} = c^2 \Delta u}$$

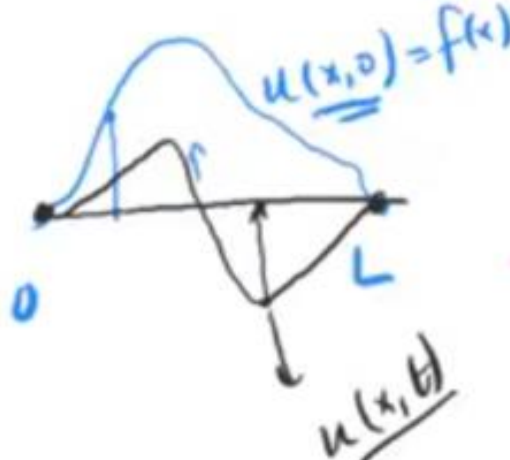
higher dim wave eqn.

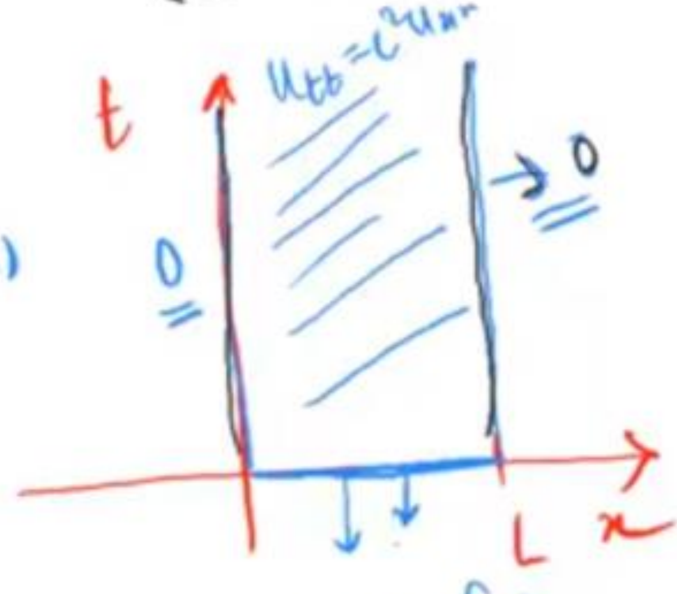
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 u_{xx} \\ u(0,t)=0, \quad u(L,t)=0, \quad t>0 \\ u(x,0)=f(x), \quad u_t(x,0)=g(x) \end{cases}$$

$\underline{L>0}$ $\quad$ 2:

(Boundary cond^n)

(Initial condition)





$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

$u(0,t) = 0$

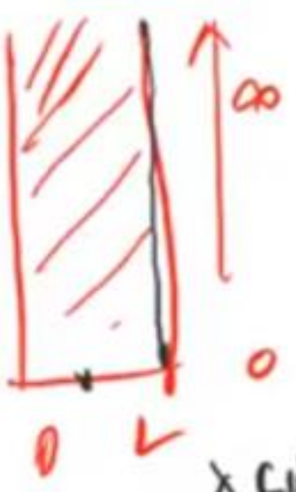
$u(L,t) = 0$

Idea to solve the 1-dim wave Eqn is "Seperation of variables". 3.

$$\boxed{u(x,t) = F(x)\,G(t)}$$

Let 'u' solves the wave eqn.

$(0,L)\times[0,\infty)$



$$u_{tt} = F(x)\,\ddot{G}(t)$$

$$u_{xx} = F''(x)\,G(t)$$

$$F(x)\,\ddot{G}(t) = c^2 F''(x)\,G(t)$$

$$\Longleftrightarrow \left[\frac{F''(x)}{F(x)} = \frac{\ddot{G}(t)}{c^2 G(t)}\right] = -\lambda \qquad \lambda \in \mathbb{R}$$

$$\frac{\ddot{G}(t)}{G(t)} \qquad \frac{F''(x)}{F(x)}$$

$$\boxed{\frac{F''(x)}{F(x)} = \frac{\ddot{G}(t)}{G(t)c^2} = -\lambda_n} \longrightarrow 1$$

$\Rightarrow$ 1st Eqn $\left\{\begin{array}{l} F''(x) + \lambda F(x) = 0 \\ F(0) = 0 \quad F(L) = 0 \end{array}\right\}$ Eq in X variable

Boundary Cond $u(0,t) = 0 \Rightarrow F(0)\,G(t) = 0$

$G(t) \neq 0$

$\Rightarrow F(0) = 0$

Similarly $F(L) = 0$

$$F_n(x) = \sin\left(\frac{\pi n}{L}x\right), \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

$n \in \mathbb{N}$.

$\boxed{\begin{array}{l}\text{Equation in } t \text{ variable} \\ \dfrac{\ddot{G}_n(t)}{c^2 G_n(t)} = -\lambda_n\end{array}}$

S.

$$\ddot{G}_n(t) = -\frac{n^2\pi^2}{L^2}c^2 G_n(t)$$

$u(x,0) = f$

General Soln

$$\boxed{G_n(t) = B_n \cos\left(\frac{n\pi}{L}c\,t\right) + B_n^* \sin\left(\frac{n\pi}{L}ct\right)}$$

General soln in t variable

Defn

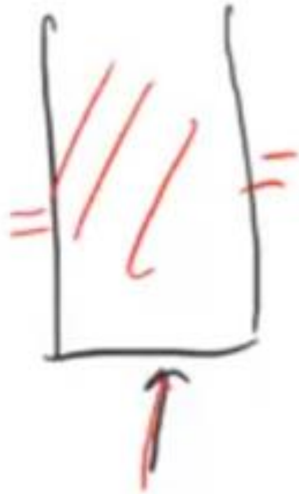
$$u_n(x,t) = F_n(x)G_n(t) = \left[B_n \cos\left(\frac{n\pi}{L}ct\right) + B_n^* \sin\left(\frac{n\pi c}{L}\right)t\right]\sin\frac{n\pi}{L}x$$

$(u_n)_{tt} = c^2 (u_n)_{xx}$

$\Rightarrow$ Method of Super position

$\underline{\underline{6}}$

$u_n(x,t) \quad \forall n \in \mathbb{N}$



Idea (Fourier Series)

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$\boxed{u_{tt} - c^2 u_{xx} = 0}$ (Linearity)

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$$\boxed{u(x,0) = f(x)}$$

$$\sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{L} c t\right) = f(x)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

If is differentiable

cos fourier coefficient have exp

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\boxed{u_t(x,0) = g(x)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty}\left(-B_n \frac{n\pi}{L} c \sin\left(\frac{n\pi}{L} ct\right) + B_n^{*} \frac{n\pi}{L} c \cos\left(\frac{n\pi}{L} ct\right)\right) \sin\frac{n\pi x}{L}\right]_{t=0}$$

$$\| \quad g(x)$$

$$= \sum_{n=1}^{\infty} \boxed{B_n^{*} \frac{n\pi}{L} c} \sin\frac{n\pi x}{L} = g(x)$$

$$\boxed{B_n^{*} \frac{n\pi}{L} c = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

1 sin Fourier coefficient

wave Eqn

WAVE EQUATION :

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & \end{cases}$$

D-ALEMBERT'S PRINCIPAL

$$\boxed{u(x,t) = \varphi(x+ct) + \psi(x-ct)}$$

$\boxed{c=1}$

$$u_{tt} - u_{xx} = 0$$

$$\begin{cases} \xi = x+t \\ \eta = x-t \end{cases}$$

$$v_{\xi\eta} = 0$$

$$v(\xi,\eta) = \varphi(\xi) + \psi(\eta)$$

$$u(x,t) = \varphi(x+t) + \psi(x-t)$$

$u(x,t) = \varphi(\underline{x+ct}) + \psi(x-ct)$. $\underline{\underline{2.}}$

at $t=0$

$\underline{u(x,0)} = \underline{\varphi(x) + \psi(x)} = f(x) \xrightarrow[\forall x \in \mathbb{R}]{} (1)$

$u_t(x,t) = c\varphi'(x+ct) - c\psi'(x-ct)$

at $t=0$

$u_t(x,0) = g(x) = c\varphi'(x) - c\psi'(x) \xrightarrow[\forall x \in \mathbb{R}]{} (2)$

IC.

Integrate 2. from '0' to x, to get

$$\int_0^x g(s)\,ds = c[\varphi(x) - \varphi(0)] - c[\psi(x) - \psi(0)]$$

$$\frac{1}{c}\int_0^x g(s)\,ds = \underline{[\varphi(x) - \psi(x)]} + \boxed{[\psi(0) - \varphi(0)]} \xrightarrow[\forall x \in \mathbb{R}]{} (3)$$

$$\varphi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{1}{2}[\varphi(0) - \psi(0)] \quad \longrightarrow (4)$$

3.

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(s)\,ds + \frac{1}{2}[\varphi(0) - \psi(0)] \quad \longrightarrow (5)$$

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$\boxed{u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

D-Alembert's formula

$$u(x_0,t_0) = \frac{1}{2}\left(f(x_0+ct_0) + f(x_0-ct_0)\right) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds$$ 4.