Digital Integrators

- The frequency response of an ideal digital integrator is given by

$$H_{INT}(e^{j\omega}) = \frac{1}{j\omega}$$

- As it is not possible to design an ideal digital integrator, digital systems with a frequency response approximating that given above are designed

IIR Digital Integrators

- Several simple IIR digital integrators that are based on numerical integration methods have been proposed

Forward rectangular integrator

- Based on the forward rectangular method of numerical integration

IIR Digital Integrators

- Described in the time-domain by the input-output equation

$$y[n] = y[n-1] + T \cdot x[n-1]$$

where T is the sampling period

- Its transfer function is given by

$$H_{FR}(z) = T\left(\frac{z^{-1}}{1-z^{-1}}\right)$$

IIR Digital Integrators

Backward Rectangular Integrator

- Based on the backward rectangular method of numerical integration
- Described in the time-domain by the input-output equation

$$y[n] = y[n-1] + T \cdot x[n]$$

where T is the sampling period

IIR Digital Integrators

- Its transfer function is given by

$$H_{BR}(z) = T\left(\frac{1}{1-z^{-1}}\right)$$

- Both the forward rectangular integrator and backward rectangular integrator have the same magnitude function

$$\left|H_{FR}(e^{j\omega})\right| = \left|H_{BR}(e^{j\omega})\right| = \frac{T}{2\cos(\omega/2)}$$

IIR Digital Integrators

Trapezoidal Integrator

- Based on the trapezoidal method of numerical integration
- Described in the time-domain by the input-output equation

$$y[n] = y[n-1] + \tfrac{T}{2}(x[n] + x[n-1])$$

where T is the sampling period

IIR Digital Integrators

- Its transfer function is given by

$$\boxed{H_{TR}(z) = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)}$$

- The magnitude responses of the ideal, the rectangular and the trapezoidal integrators for $T = 1$ are shown in the next slide

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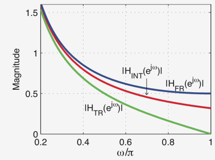
IIR Digital Integrators



- Note: The plot of $\left|H_{FR}(e^{j\omega})\right|$ is above the plot of $\left|H_{INT}(e^{j\omega})\right|$ and the plot of $\left|H_{TR}(e^{j\omega})\right|$ is below the plot of $\left|H_{INT}(e^{j\omega})\right|$

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Digital Differentiators

- An ideal digital differentiator is characterized by the frequency response

$$\boxed{H_{DIF}(e^{j\omega}) = j\omega, \; 0 \le \omega \le \pi}$$

- Its magnitude function is given by

$$\boxed{\left|H_{DIF}(e^{j\omega})\right| = \omega}$$

which is a linear function of ω in the frequency range from dc to π as shown in the next slide

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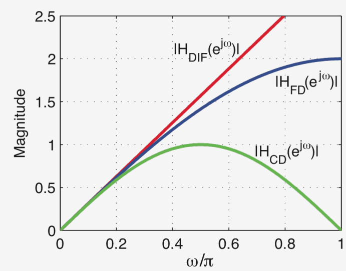
Digital Differentiators



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Digital Differentiators

- In practice, a digital differentiator is designed to have a linear magnitude response from dc to a frequency much smaller than π, as it is employed to implement the differentiation operation in the low frequency range

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Digital Differentiators

First-Difference Differentiator

- Characterized in the time-domain by the input-output equation

$$\boxed{y[n] = x[n] - x[n-1]}$$

- Its transfer function is given by

$$\boxed{H_{FD}(z) = 1 - z^{-1}}$$

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Digital Differentiators

Central-Difference Differentiator

- Characterized in the time-domain by the input-output equation

$$y[n] = \tfrac{1}{2}x[n] - \tfrac{1}{2}x[n-2]$$

- Its transfer function is given by

$$H_{CD}(z) = \tfrac{1}{2}(1 - z^{-2})$$

Digital Differentiators

- Plots of the magnitude responses of the ideal differentiator (red line), the first-difference differentiator (blue line), and the central-difference differentiator (green line) are shown in Slide No. 10

Digital Differentiators

- Note: The first-difference differentiator has a very high gain at high frequencies and as a result, it amplifies the high-frequency noise often present in many digital signals
- Problem is avoided in the central-difference differentiator which has lower gains at high frequencies and has a linear magnitude response in the frequency range $0 \le \omega \le 0.16\pi$

DC Blockers

- It removes the dc bias that is present in a digital signal before the necessary signal processing algorithms are applied to it
- The magnitude response of an ideal dc blocker has a zero value at $\omega = 0$ and passes all nonzero frequency components without distortion

DC Blockers

- In a sense, the ideal dc blocker is a highpass filter with a magnitude response given by

$$\left|H(e^{j\omega})\right| = \begin{cases} 0, & \omega = 0 \\ 1, & \omega \neq 0 \end{cases}$$

- A preferred dc blocker has an input-output relation $y[n] = \alpha y[n-1] + x[n] - x[n-1]$ where α is a non-zero real number with a value less than 1

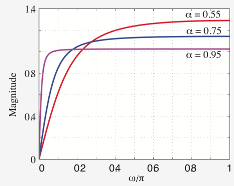
DC Blockers

- Its transfer function is $H_{DC}(z) = \dfrac{1 - z^{-1}}{1 - \alpha z^{-1}}$
- The above system can be considered as a cascade of the first-difference differentiator $H_{FD}(z)$ with a leaky integrator with a transfer function given by

$$H_{leaky}(z) = \frac{1}{1 - \alpha z^{-1}}$$

DC Blockers

- The magnitude responses of the IIR dc blocker for three different values of α are shown below



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DC Blockers

- The IIR dc blocker is used in musical sound synthesis and also to suppress clutters in MTI radars

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Comb Filter

- Has a frequency response that is a periodic function of ω with a period $2\pi/K$, where K is a positive integer
- Let $H(e^{j\omega})$ denote the frequency of the prototype causal LTI digital filter with a single passband and/or single stopband

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Comb Filters

- By replacing each unit delay in the realization of $H(z)$ with K unit delays, we can realize a comb filter with a transfer function

$$\boxed{G_{comb}(z) = H(z^K)}$$

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FIR Comb Filters

FIR Comb Filters

- Consider an FIR digital filter with a transfer function

$$\boxed{H(z) = 1 + \alpha z^{-1}}$$

- The above prototype filter is a lowpass filter for positive values of α and is a highpass filter for negative values of α

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FIR Comb Filters

- The comb filter generated from this prototype filter has a transfer function given by

$$\boxed{G_{comb}(z) = H(z^K) = 1 + \alpha z^{-K}}$$

- It is described in the time-domain by the difference equation

$$\boxed{y[n] = x[n] + \alpha x[n-K]}$$

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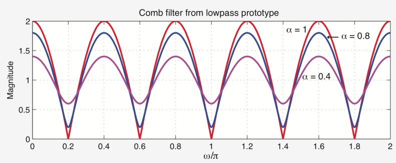
FIR Comb Filters

- Plots of the magnitude responses for $K = 5$ and $\alpha > 0$ are shown below



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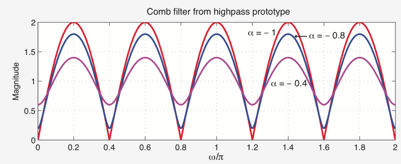
FIR Comb Filters

- Plots of the magnitude responses for $K = 5$ and $\alpha < 0$ are shown below



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FIR Comb Filters

- The FIR comb filter is also known as the feedforward comb filter
- It models the direct sound represented by $x[n]$ and a single echo appearing after K sample periods which is the time taken by the sound wave to travel to the listener from the source after it is reflected from the wall

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FIR Comb Filters

- The parameter α for $|\alpha| < 1$ represents the loss of signal due to travel
- A modified form of the FIR comb filter has been used to simulate the flanging effect in sound recording

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FIR Comb Filters

- A fixed number of multiple echoes with exponentially decaying amplitudes and spaced K samples apart can be realized using a modified type of comb filter with a transfer function given by

$$\boxed{G_{comb}(z) = \sum_{\ell=0}^{K-1} \alpha^{\ell} z^{-\ell} = \frac{1-\alpha^K z^{-K}}{1-\alpha z^{-1}}}$$

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FIR Comb Filters

- Various other types of FIR comb filters can be designed with an appropriately chosen prototype FIR filter
- One possible prototype FIR filter is the M-point moving average filter with a transfer function given by

$$\boxed{H(z) = \frac{1}{M}\sum_{\ell=0}^{M-1} z^{-\ell} = \frac{1-z^{-M}}{M(1-z^{-1})}}$$

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FIR Comb Filters

- The transfer function of the comb filter generated from the above prototype FIR filter is given by

$$\boxed{G_{comb}(z) = \frac{1 - z^{-MK}}{M(1 - z^{-K})}}$$

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FIR Comb Filters

- By choosing M and K appropriately we can design a comb filter with a magnitude response having peaks and notches at desired frequency locations
- Has been used to separate weak and strong solar spectral components in ionospheric electron concentration measurements

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IIR Comb Filters

IIR Comb Filters

- The simplest IIR comb filter has a transfer function

$$\boxed{G_{comb}(z) = \frac{1}{1 - \alpha z^{-1}}, |\alpha| < 1}$$

with an input-output relation given by

$$\boxed{y[n] = \alpha y[n-K] + x[n]}$$

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IIR Comb Filters

- This IIR comb filter is known as the feedback comb filter and it generates an infinite number of echoes spaced K samples apart with exponentially decaying amplitudes
- Has been proposed to artificially introduce reverberation in a sound recorded inside an inert studio to make it appear as a more naturally sounding music

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Allpass Digital Filters

- The allpass digital filter is a causal stable IIR digital system with a square-magnitude response equal to 1 for all values of the frequency ω:

$$\boxed{\left|\mathcal{A}(e^{j\omega})\right|^2 = 1 \text{ for all } \omega}$$

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Allpass Digital Filters

- The transfer function of an M-th order real coefficient allpass digital filter is given by

$$\boxed{\mathrm{A}_M(z) = \pm \frac{q_M + q_{M-1}z^{-1} + \cdots + q_1 z^{-(M-1)} + z^{-M}}{1 + q_1 z^{-1} + \cdots + q_{M-1}z^{-(M-1)} + q_M z^{-M}}}$$

where $\{q_i\}$ are real numbers

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Allpass Digital Filters

- The transfer function in factored form is given by

$$\boxed{\mathcal{A}_M(z) = \pm\prod_{i=1}^{M}\left(\frac{-\lambda_i^* + z^{-1}}{1-\lambda_i z^{-1}}\right)}$$

where $|\lambda_i| < 1$, $1 \le i \le M$, for stability

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Allpass Digital Filters

- If $Y(e^{j\omega})$ and $X(e^{j\omega})$ denote the DTFTs of the output and input sequences, $y[n]$ and $x[n]$, respectively, of the all pass filter, then it follows from the definition

$$|Y(e^{j\omega})|^2 = |X(e^{j\omega})|^2$$

- Using the Parseval's relation we arrive at

$$\boxed{\sum_{n=-\infty}^{\infty}|y[n]|^2 = \sum_{n=-\infty}^{\infty}|x[n]|^2}$$

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Allpass Digital Filters

- In other words, a stable allpass digital filter is a lossless digital system
- A first-order causal allpass digital filter with real coefficients has a transfer function given by $\boxed{\mathcal{A}_1(z) = \dfrac{q_1 + z^{-1}}{1+q_1 z^{-1}}}$

where q_1 is a real number, and for stability

$$|q_1| < 1$$

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Allpass Digital Filters

- A first-order allpass digital filter is described in the time-domain by an input-output relation

$$\boxed{y[n] = -q_1 y[n-1] + q_1 x[n] + x[n-1]}$$

- The transfer function of a real coefficient causal allpass digital second-order allpass digital filter is

$$\boxed{\mathcal{A}_2(z) = \frac{q_2 + q_1 z^{-1} + z^{-2}}{1 + q_1 z^{-1} + q_2 z^{-2}}}$$

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Allpass Digital Filters

where q_1 and q_2 are real numbers, and for stability

$$\boxed{|q_2| < 1 \quad \text{and} \quad |q_1| < 1 + q_2}$$

- A second-order allpass digital filter is described in the time-domain by an input-output relation

$$\boxed{\begin{aligned} y[n] &= -q_1 y[n-1] - q_2 y[n-2] \\ &\quad + q_2 x[n] + q_1 x[n-1] + x[n-2] \end{aligned}}$$

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Allpass Digital Filters

- A higher-order allpass filter can be realized as a cascade of first and/or second order allpass sections
- We describe a few applications of the allpass digital filter later

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Nonlinear Digital Filters

- We describe next two simple nonlinear digital filters that are used in certain applications

Teager Operator

$$\boxed{y[n] = x^2[n] - x[n-1]x[n+1]}$$

- The Teager operator behaves like the product of a Laplacian highpass FIR filter with a local mean lowpass filter

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Nonlinear Digital Filters

Median Filter

- Based on determining the median of a set of odd numbers
- The median value of a set of $2L+1$ numbers is given by the number X in the set so that L numbers have values smaller than X and L numbers have values greater than X

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Nonlinear Digital Filters

- Example – Let

$$\{x[n]\} = \{5,\ -4,\ 20,\ 2,\ 8\}$$

- We reorder the numbers in this set from the smallest to the largest resulting resulting in a rank-ordered set

$$\{-4,\ 2,\ 5,\ 8,\ 20\}$$

whose median is 5

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Nonlinear Digital Filters

- Hence

$$med\{x[n]\} = med\{-4,\ 2,\ 5,\ 8,\ 20\} = 5$$

- The median filter is applied to an input sequence of finite length
- For an 1D sequence $\{x[n]\}, 0 \le n \le N-1$, a median filter of length $2L+1$ with $L << N$, replaces the n-th sample with

$$y[n] = med\{x[n-L],\ \cdots,\ x[n-1],\ x[n],\ x[n+1],\ \cdots,\ x[n+L]\}$$

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Nonlinear Digital Filters

- To generate an output sequence $\{y[n]\}$ of length N, the input sequence is extended in length by $2L$ samples usually by appending with L zeros to the left of the sample $x[0]$ and L zeros to the right of $x[N-1]$

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Nonlinear Digital Filters

- The median filter works very well in masking impulse noises of the noise-corrupted samples that have the maximum and minimum values in the dynamic ranges of the uncorrupted input sequence

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Nonlinear Digital Filters

- In the case of black-and-white images, the corrupted samples show up as either solid black or pure white pixels, and are more popularly known as salt-and-pepper noise
- The MATLAB functions `medfilt1` and `medfilt2` can be used for the masking of impulse noise in one-dimensional and two-dimensional sequences, respectively

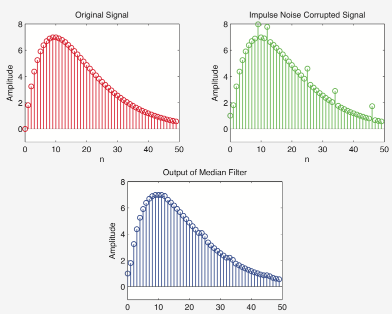
Nonlinear Digital Filters

- Example – Let the original uncorrupted sequence is given by

$$s[n] = 2[n(0.9)^n]$$

- The plots of the original uncorrupted sequence, its impulse noise corrupted version, and the output of an one-dimensional median filter of length 3 are shown in the next slide

Nonlinear Digital Filters



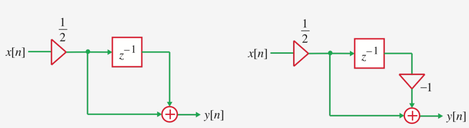
FIR Digital Filter Structures

- Block-diagram representations of the first-order FIR lowpass and highpass digital filters

$$\boxed{\begin{aligned} H_{LP}(z) &= \tfrac{1}{2}(1 + z^{-1}) \\ H_{HP}(z) &= \tfrac{1}{2}(1 - z^{-1}) \end{aligned}}$$

are shown in the next slide

FIR Digital Filter Structures



Lowpass Highpass

- Note: These FIR filters can be implemented without a hardware multiplier

FIR Digital Filter Structures

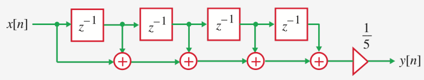
- The block-diagram representation of an M-point moving average lowpass FIR digital filter follows directly from its transfer function

$$\boxed{H_{MA}(z) = \frac{1}{M}\sum_{i=0}^{M-1} z^{-i}}$$

FIR Digital Filter Structures

- For example, figure below shows the representation of a 5-point moving average lowpass filter with a transfer function

$$H_{MA}(z) = \tfrac{1}{5}(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4})$$



Allpass Digital Filter Structures

- The transfer function of an M-th order real coefficient allpass digital filter is characterized by M unique coefficients
- A block-diagram representation using only M multipliers can be obtained

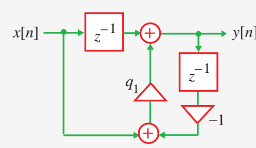
Allpass Digital Filter Structures

- To this end we need to rewrite the input-output relation by sharing common coefficients
- For example, the time-domain input-output representation of the first-order allpass digital filter can be rewritten as

$$y[n] = q_1(x[n] - y[n-1]) + x[n-1]$$

Allpass Digital Filter Structures

- Its block-diagram representation with one multiplier and two unit delays is shown below



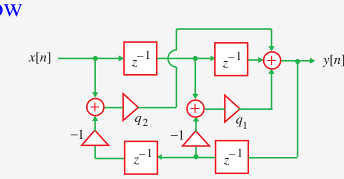
Allpass Digital Filter Structures

- Likewise, the time-domain input-output representation of the second-order allpass digital filter can be rewritten as

$$\begin{aligned} y[n] &= q_1(x[n-1] - y[n-1]) \\ &\quad + q_2(x[n] - y[n-2]) + x[n-2] \end{aligned}$$

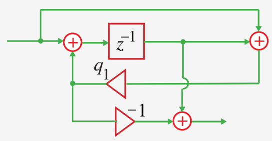
Allpass Digital Filter Structures

- Its block-diagram representation with two multipliers and four unit delays is shown below



Allpass Digital Filter Structures

- Block-diagram representation using one multiplier and one unit delay of a first-order allpass digital filter is shown below

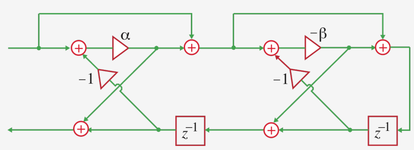


Allpass Digital Filter Structures

- Block-diagram representation using two multipliers and two unit delays of a second-order allpass digital filter shown in the next slide is based on the transfer function

$$\mathcal{A}_2(z) = \frac{\alpha - \beta(1+\alpha)z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

Allpass Digital Filter Structures



Computationally Efficient IIR Digital Filter Structures

First-Order Lowpass and Highpass IIR Digital Filter Pair

- Consider the first-order lowpass IIR digital filter:

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

Computationally Efficient IIR Digital Filter Structures

- The transfer function $H_{LP}(z)$ can be rewritten in the form

$$H_{LP}(z) = \frac{1}{2}\left[\frac{1-\alpha+z^{-1}-\alpha z^{-1}}{1-\alpha z^{-1}}\right]$$

$$= \frac{1}{2}\left[1 + \underbrace{\frac{-\alpha+z^{-1}}{1-\alpha z^{-1}}}_{\mathcal{A}_1(z)}\right] = \frac{1}{2}[1+\mathcal{A}_1(z)]$$

Computationally Efficient IIR Digital Filter Structures

- Similarly, we can show that the first-order highpass transfer function

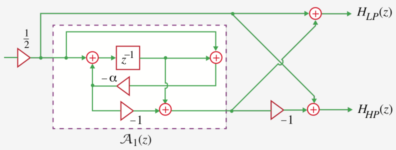
$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

can be rewritten as

$$H_{HP}(z) = \tfrac{1}{2}[1-\mathcal{A}_1(z)]$$

Computationally Efficient IIR Digital Filter Structures

- Figure below shows a block-diagram representation of the first-order lowpass and highpass IIR transfer functions requiring only one multiplier with coefficient α



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Computationally Efficient IIR Digital Filter Structures

Second-Order Bandpass and Bandstop IIR Digital Filter Pair

- Consider the second-order bandpass and bandstop IIR digital filters:

$$H_{BP}(z) = \frac{1-\alpha}{2}\left[\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right]$$

$$H_{BS}(z) = \frac{1+\alpha}{2}\left[\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right]$$

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Computationally Efficient IIR Digital Filter Structures

- It can be shown that $H_{BP}(z)$ and $H_{BS}(z)$ can be rewritten as

$$H_{BP}(z) = \tfrac{1}{2}[1-\mathcal{A}_2(z)]$$

$$H_{BS}(z) = \tfrac{1}{2}[1+\mathcal{A}_2(z)]$$

where

$$\mathcal{A}_2(z) = \frac{\alpha-\beta(1+\alpha)z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

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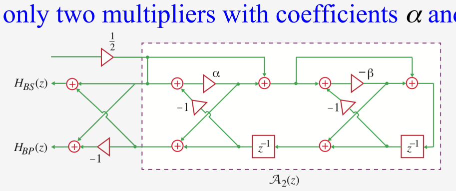
Computationally Efficient IIR Digital Filter Structures

- Figure below shows the block-diagram representation of $H_{BP}(z)$ and $H_{BS}(z)$ with only two multipliers with coefficients α and β



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Computationally Efficient IIR Digital Filter Structures

- Note: The parameter β controls the center frequency the bandpass filter and the notch frequency of the bandstop filter
- Note: The parameter α controls the 3-dB bandwidth of the bandpass filter and the 3-dB notch bandwidth of the bandstop filter
- Hence, it is a parametrically tunable filter structure

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