1. Find the Laplace transform of $f_a(x) := f(ax)$, for given $a > 0$, in terms of the Laplace transform of f.

2. Find the Laplace transform of the following:
   (a) $\lfloor x \rfloor$, i.e. assigning the greatest integer less than or equal to x.
   (b) $x^m \cosh bx$, for any $m \in \mathbb{N} \cup \{0\}$ and $b \in \mathbb{R}$.
   (c) $e^x \sin ax$, for any real a.
   (d)
   $$f(x) = \begin{cases} \sin 3x, & 0 < x < \pi, \\ 0, & x > \pi. \end{cases}$$

3. Let f be of exponential order. Show that $\mathcal{L}(f)(p) \to 0$ as $p \to \infty$.

4. In the lectures we have computed the derivative of Laplace transform. Use it to compute the integral of Laplace transform, i.e. show that $\int_p^\infty \mathcal{L}(f)(s)\,ds = \mathcal{L}\left(\frac{f(x)}{x}\right)(p)$.

   Use the above properties to find the Laplace transform of:
   (a) $\frac{e^x \sin ax}{x}$.
   (b) $\frac{\sin x \cosh x}{x}$.

5. Find the Laplace transform of the following using the second shifting theorem:
   (a)
   $$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ 0, & \pi < x < 2\pi, \\ \cos x, & x > 2\pi. \end{cases}$$
   (b)
   $$f(x) = \begin{cases} 0, & 0 < x < 1, \\ \cos(\pi x), & 1 < x < 2, \\ 0, & x > 2. \end{cases}$$

6. Find the inverse Laplace transform of the following:
   (a) $\tan^{-1}(a/p)$.
   (b) $\ln\left(\frac{p^2+1}{(p+1)^2}\right)$.
   (c) $\frac{p+2}{(p^2+4p-5)^2}$.
   (d) $\frac{pe^{-\pi p}}{p^2+4}$.
   (e) $\frac{(1-e^{-2p})(1-3e^{-2p})}{p^2}$.

7. Use Laplace transform to solve the following IVP:

(a) $y'' + 4y = \cos 2x$ with $y(0) = 0$ and $y'(0) = 1$.

(b)

$$y'' + 3y' + 2y = \begin{cases} 4x & \text{when } 0 < x < 1 \\ 8 & \text{when } x > 1 \end{cases}$$

with $y(0) = y'(0) = 0$.

(c)

$$y'' + 9y = \begin{cases} 8\sin x & \text{when } 0 < x < \pi \\ 0 & \text{when } x > \pi \end{cases}$$

with $y(0) = 0$ and $y'(0) = 4$.

(d)

$$\begin{aligned} y_1' + 2y_1 + 6\int_0^x y_2(\tau)\,d\tau &= 2H(x) \\ y_1' + y_2' &= -y_2 \end{aligned}$$

with $y_1(0) = -5$ and $y_2(0) = 6$.

8. Solve for the unknown function y in the following integral equations:
   (a) $y(x) + \int_0^x y(\tau)\,d\tau = H(x-a) + H(x-b)$.
   (b) $e^{-x} = y(x) + 2\int_0^x \cos(x-\tau)y(\tau)\,d\tau$.
   (c) $3\sin 2x = y(x) + \int_0^x (x-\tau)y(\tau)\,d\tau$.