Topic Nine: Motion in Fields

Review of Gravitational Fields and Introduction to Projectiles

Projectile motion is motion under the influence of a uniform field.
In this course we study projectile motion in a uniform gravitational field.
A uniform field is one which is the same size everywhere and is always in the same direction.

We assume a uniform gravitational field at and around the surface of the Earth.

Gravitational field lines are the lines followed by a test mass placed in the field.
At the surface of the Earth, the field lines are considered to be parallel and point towards the Earth, at 90° to the flat, horizontal surface. This is consistent with a uniform field.

Gravitational Field strength is defined to be the force per unit mass acting on a test mass when placed in the field.

(A test mass is one small enough not to affect the shape of the field it is in)

Hence: $g = \frac{F}{m}$

g, often known simply as "gravity" is better termed "gravitational field strength".

g at the surface of the Earth is known to be $9.81\,Nkg^{-1}$
(It can be shown that g also measures acceleration due to gravity and thus can also have units of acceleration – hence, we also know g as $9.81\,ms^{-2}$)

Do not confuse the constant, $g = 9.81\,Nkg^{-1} = 9.81\,ms^{-2}$, the gravitational field strength or acceleration due to gravity ***at the surface of the Earth***, with the general quantity g, which measures the general gravitational field strength for any position in the proximity of any mass.

Hence, if the Earth is treated as a planet of finite size in an infinitely larger universe, the field strength is in fact variable and decreases as one moves away from the planet. Field lines are no longer considered parallel, but radial and non-uniform, pointing towards the centre of the Earth.

We have learned (Topic 6) how to find this value (field strength) at any position.

On the surface of the Earth, masses experience $g = 9.81\,Nkg^{-1}$, and so this is the field strength.

The force due to gravity, using the above equation, is $F = mg$.

This force is also given the name weight (and often loosely called "gravity"). Hence the equation "weight = mass x gravity" – weight is the force, "gravity", here, is gravitational field strength.

Projectile Motion.

A projectile is a mass moving through a uniform (gravitational) field.
This type of motion only commences once the mass has departed from the contact force causing its projection.
The path taken by a projectile is called a trajectory.

It can be shown that all trajectories are parabolic in shape (except when the projectile is thrown vertically upwards or downwards).

The essence of projectile motion is that, regardless of the direction and speed of a projectile, the force acting on it is constant (since the mass is constant and the field strength is constant) and $F = mg$.

So any object "flying through the air" only has one force acting on it: its weight, acting vertically downwards.

This, of course, assumes zero air resistance, which is not completely accurate but serves as a means for quite accurate velocity and displacement calculations provided that the mass is fairly dense and therefore air resistance forces are small compared with its weight.

To solve projectile problems we use the following method –we split the motion into horizontal and vertical motion and we take horizontal motion to be at uniform speed and vertical motion to be at uniform acceleration. Thus:

For horizontal motion the only equation we can use is: $speed = \frac{distance}{time}$

For vertical motion we use the equations of motion and (assuming we are on the surface of the Earth) that acceleration, g is equal to $9.81\,ms^{-2}$

The time for the horizontal motion is always the same as the time for the vertical motion for any trajectory.

Example T9.1

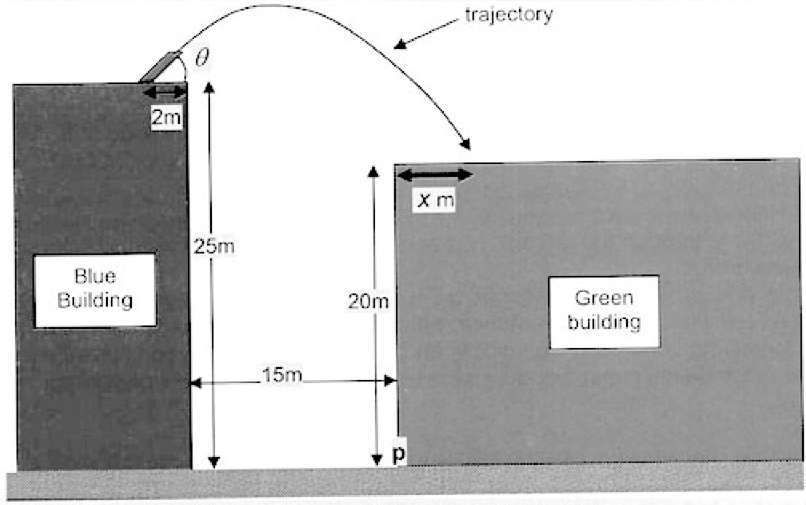
A ball is projected horizontally from a point 5.0 metres above the ground at a speed of $8.0 ms^{-1}$ What is the range of the ball and what is its speed as it makes initial impact with the ground. State any assumptions.

Example T9.2

What is the optimum angle from the horizontal in order to maximize the range of a projectile, assuming negligible air resistance?

Example T9.3

An experiment is carried out, projecting a ball from one building to another. The angle of projection and speed of projection can be varied. The following diagram illustrates the situation:



The ball is projected 2m from the edge of the blue building, at an angle θ from the horizontal. Assume that the green building is very wide and that the height from which the ball is projected is at the top of the blue building (the projector is not raised above the roof of the blue building)

(a) If the ball is projected at 40° to the horizontal, at a speed of $12ms^{-1}$:
 i) find the maximum height above the ground reached by the ball
 ii) find the value of x, giving the position where it hits the green building
 iii) find the speed at which it hits the building

(b) At the same angle (40°), find the speed that the ball should be projected in order for it to land just at the base of the green building, labelled **P** on the diagram

Gravitational and Electric Field, Potential and Energy

Gravitational fields and Electric fields are almost identical, in concept and in mathematical treatment.

To illustrate this, and to help you to be able to draw comparisons, I shall deal with them simultaneously.

Gravitational Potential Energy

The GPE is a quantity associated with a mass at a certain position in a gravitational field.

Hence, a mass in a gravitational field will have a potential energy due to its presence in the field.

Definition: The GPE of a mass is the energy required (work done) to move it from infinity (ie outside the field) to its position in the field.

Since gravitational fields are always attractive, work will not be required to move masses into the field – in fact, energy is liberated (work is done to move them out of the field).

Hence, the gravitational potential energy of a mass in a field is always negative. The maximum GPE a mass can have is zero – and this is when it has escaped the field.

The average human being (mass 65kg) has a gravitational potential of approximately –4000 *million joules*

Electric Potential Energy

Electric Potential energy is the energy of a charge due to its position (in an electric field).

Definition: The electric potential energy (of a charge) is the work done (or energy required) to bring the charge in question from infinity (ie from outside the field) to the point in question.

For example a proton placed next to a positive charge would require work to be done to get it there, so it has positive potential energy. However, an electron placed at the same position would have actually accelerated there (if allowed to), and so would have released energy (rather than having to be worked on) so the electron would have a negative potential energy.

Like masses, charges at infinite distance have zero potential energy.

Comparison: charges in an attractive field have negative potential energies, in a similar way to masses in gravitational fields (always attractive).

Gravitational Potential

Gravitational Potential is a quantity associated with a position in a gravitational field.

It allows us to calculate the GPE of a mass at that position.

Definition: Gravitational Potential (of a position) is the work done per unit mass in bringing a test mass from infinity to that position in the field.

So gravitational potential is simply GPE per kilogram.

Note, though, that the quantity measures the field position rather than any measurement of a mass.

To convert potential into potential energy:

$$E_p = mV_g \text{ (not given in data booklet)}$$

i.e. Gravitational Potential Energy = mass x gravitational potential.

Note that the IB data booklet simply uses the symbol V (not V_g) for gravitational potential – which is a little confusing!

The surface of the Earth has a gravitational potential of around $-63\ MJkg^{-1}$

We can calculate gravitational potential (and hence GPE) as follows:

$$V = -\frac{Gm}{r}$$

G is the universal constant of gravitation
m is the mass
V is the (grav.) potential, distance r from the centre of the mass

To find the ***potential energy*** of another mass at this position, just multiply the above expression by the other mass.

(Note that $\Delta E_p = mg\Delta h$ only applies when g is constant)

Note also: $\Delta V = \frac{\Delta E_p}{m}$

Finally note that work done = change in potential energy (ie there will be a corresponding energy change elsewhere if we have a change in GPE)

Example T9.4

Universal constant of gravitation (G) $= 6.67 \times 10^{-11}\ Nm^2kg^{-2}$
Mass of Earth $= 6.0 \times 10^{24}\ kg$
Radius of Earth $= 6380\ km$

(i) Calculate the gravitational potential:

(a) at the surface of the earth
(b) 50 km above the surface of the earth
(c) 5000 km above the surface of the earth

(ii) Using (i) calculate the gravitational potential energy of a 50kg mass

(a) at the surface,
(b) 50 km above the surface
(c) 5000 km above the surface

(iii) Using (ii) (and by calculating the GPE change) Calculate the work done in moving a 50kg mass from the surface of the earth to:

(a) 50 km above the surface of the earth (b) 5000 km above the surface of the earth

(iv) Use the equation $\Delta E_p = mg\Delta h$ to re-solve the questions in (iii). Compare answers using these two methods and comment.

Notes:

- the work done in moving a mass between two points in a gravitational field is independent of the path taken – so it is the same as long as you start at the same place and finish at the same place.
- gravitational field strength is a vector quantity
- gravitational potential (and potential energy) is a scalar quantity

Electric Potential

Electric Potential is a quantity associated with a position in an electric field.

It allows us to calculate the EPE (electric potential energy) of a charge at that position.

Definition: Electric Potential (of a position) is the work done, per unit charge, in bringing a positive test charge from infinity to that position in the field.

So electric potential is simply EPE per coulomb.

Note, though, that the quantity measures the field ***position*** rather than any measurement of a charge.

Note also that, purely by convention, to define electric potential we imagine that we are moving a ***positive*** test charge from infinity to the point.

To convert potential into potential energy:

$$E_p = qV_E \text{ (not given in data book)}$$

i.e. Electric Potential Energy = charge x electric potential.

Note that the IB data booklet simply uses the symbol V (not V_E) for electric potential

We can calculate electric potential (and hence EPE) as follows:

$$V = \frac{kq}{r}$$

k is the coulomb constant

q is the charge

V is the (elect.) potential, distance r from the centre of the charge

To find the ***potential energy*** of another charge at this position, just multiply the above expression by the charge.

Note: $\Delta V = \dfrac{\Delta E_p}{q}$ where ΔV is the potential difference (voltage)

Finally note that work done = change in potential energy (ie there will be a corresponding energy change elsewhere if we have a change in EPE)

Example T9.5

Coulomb constant (k)	$= 8.99 \times 10^{9}\ Nm^{2}C^{-2}$
Charge of an electron	$= -1.6 \times 10^{-19} C$
Charge of a proton	$= 1.6 \times 10^{-19} C$

(i) Calculate the electric potential:

(a) 1.0mm from an electron
(b) 1.0mm from a proton
(c) at the surface of an alpha particle (charge 2+), of radius $1.4\ fm$

(ii) Using (i), where possible, calculate the electric potential energy of:

(a) a proton 1.0mm from an electron,
(b) a proton 1.0mm from a proton
(c) an electron $10^{-12}\ m$ from the centre of an alpha particle

(iii) Calculate the work done by an electron when it moves from a position 1 mm from an isolated electron to another position 3mm from an isolated proton

Notes:

- the work done in moving a charge between two points in an electric field is independent of the path taken – so it is the same as long as you start at the same place and finish at the same place.
- electric field strength is a vector quantity
- electric potential (and potential energy) is a scalar quantity

Hopefully, you can see that the mathematical treatment for potential and potential energy is almost identical for charges and for masses – except that for charges, you can have positive of negative values, whereas with masses the values can only be negative.

<u>Further examples:</u>

Example T9.6

(a) Find the potential energy difference of a +2.00nC charge moving between two plates with voltage across them (potential difference) of 400V
(b) assuming that the charge has a mass of 4.15 x 10^{-15} kg, and that it is placed at the positive plate, find the speed when it reaches the negative plate

Example T9.7

The potential energy difference between an electron at point A and electron at point B is 1.08 x 10^{-17} J. Find the potential difference between the points.

Potential Gradient and Equipotentials.

Potential Gradient

This quantity measures how quickly potential changes, with distance.

Definition: Potential gradient is the rate of change of potential with distance between two points in a (gravitational / electric) field.

Gravitational potential gradient is given the symbol: $\frac{\Delta V}{\Delta r}$ in the IB data booklet.

Equipotentials

Points at the same distance above the Earth's surface are all at the same (gravitational) potential – such surfaces are called equipotentials, or equipotential surfaces.

So the potential gradient along points placed "horizontally" is zero.

However, vertically, potential changes.

Since the energy of a 1 kg mass increases by 9.81 joules per metre (using $E = mg\Delta h$), we can say that the potential above the Earth's surface increases by $9.81\ Jkg^{-1}$ per metre, and, therefore, the potential gradient above the Earth's surface is $9.81\ Jkg^{-1}m^{-1}$

Example T9.8

By finding the gravitational potential at the Earth's surface and at 1m above the Earth's surface, find the (gravitational) potential gradient around the surface of the Earth.

Example T9.9

Find the electric potential gradient between a $+100V$ charged plate and a $-100V$ charged plate that are parallel and placed 1.5mm apart

Relationship between field strength and potential gradient

We can see in the above example (T9.9) that the potential gradient is equal in magnitude (size) to field strength. It can be shown that this is always true. Hence:

$g = -\frac{\Delta V}{\Delta r}$ in words, gravitational field strength is equal (in magnitude) to potential gradient (the negative sign indicates that the direction of the field is opposite to the direction of increasing potential)

And, for electric fields:

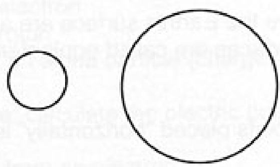
$E = -\frac{\Delta V}{\Delta r}$ in words, electric field strength is equal (in magnitude) to potential gradient (the negative sign indicates that the direction of the field is opposite to the direction of increasing potential)

Another relationship between fields and potential is that equipotentials are always at right angles (perpendicular) to field lines.

We can use this fact to plot equipotentials by ensuring that they are always perpendicular to field lines.

Example T9.10

For the following two-planet system, show the field lines (using lines with arrows) and the equipotential surfaces:



Calculation of the potential due to two or more masses (or charges)

Example T9.11

Given that the moon has a mass of $7.36 \times 10^{22}kg$ $(Earth: 6.0 \times 10^{24}kg)$ and the distance between centres of moon and Earth is $384000km$, find the gravitational potential midway between the centres of the moon and earth.

Strictly, when we calculate the potential at the surface of the earth, we should take into account the potential due to the presence of the moon – but the moon's presence would have an insignificant effect due to it being so far away.

Example T9.12

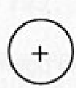
Calculate the electric potential midway between:

(i) two electrons placed 1mm apart
(ii) an electron and a proton placed 1mm apart

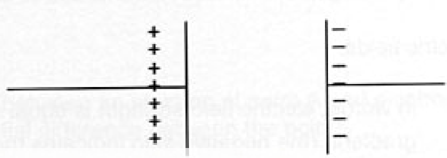
Example T9.13

Draw the field lines (in blue) and equipotential surfaces (in red) for the following charged surfaces

(i) A positively charged sphere



(ii) Two parallel plates



Escape Speed

In order for a mass to escape from a planet, it must be given enough energy for it to be able to increase its gravitational potential energy to zero.

There is thus an energy change: the energy it possesses is converted into gravitational energy: it moves 'upwards" away from the planet.

This energy source could be, for example, chemical energy. For instance, a rocket burns fuel to produce thrust to power it upwards. The associated energy conversion is from chemical energy to gravitational potential energy (note that if it is not actually getting faster, there is no continual transfer to kinetic energy).

An alternative way to give an object enough energy to escape is to give it kinetic energy. The speed associated with this kinetic energy – the energy required for it to escape the planet – is called the escape velocity, or escape speed.

Derivation of escape speed formula:

Energy required to escape = change in GPE

Final necessary potential = zero

Initial potential = $V = -\frac{GM}{r}$ where M is the mass of the planet and r,the distance from the planet = planet radius.

Change in potential = $\Delta V = final - initial = 0 - -\frac{GM}{r} = +\frac{GM}{r}$

Change in potential energy, $\Delta E_P = m\Delta V$ where m is the mass of the object escaping

$$\therefore \Delta E_P = m\frac{GM}{r}$$

If we provide the object with this amount of energy as kinetic energy, its kinetic energy will therefore equate to the above energy. Hence:

kinetic energy = $m\frac{GM}{r}$

$$\therefore \frac{1}{2}mv^2 = m\frac{GM}{r} \Rightarrow v^2 = 2\frac{GM}{r} \Rightarrow v = \sqrt{\frac{2GM}{r}}$$

$v_{esc} = \sqrt{\frac{2GM}{r}}$ where v_{esc} is the velocity required for a mass to escape from a planet of mass M and radius r

(formula not given in data booklet!)

Note that escape speed is independent of the size of the object escaping from the planet.

Example T9.14

Find the theoretical escape speed of a mass:

(i) on the moon
(ii) on Earth

Radius of Earth = 6380 km Radius of Moon = 1740 km Mass of Earth = $6.0 \times 10^{24}\ kg$ Mass of Moon = $7.4 \times 10^{22}\ kg$ G = $6.67 \times 10^{-11}\ Nm^2kg^{-2}$

Orbital Motion

An orbit is the path followed by an object around another object, or around a point. For example the moon is said to orbit the Earth.
We only consider circular orbits.

Objects moving in circles at constant speed are continually accelerating, because they are continually changing direction and therefore velocity. It can be shown that this acceleration is always directed towards the centre of the circle. This acceleration is called centripetal acceleration. There is therefore a resultant force required to keep a mass moving in a circle. This force is called centripetal force. This force can come from a variety of sources. For example, it can be the tension in a piece of string (with a mass on the end, twirling in a circle).

In the case of planets or satellites (such as moons – natural satellites) centripetal force is provided by the gravitational attraction between the planet and star (for planets orbiting stars), or between planets and satellites (for satellites orbiting planets).

Kepler's Third Law:

This law describes the relationship between the time taken for a planet to orbit the sun and its average (mean) distance from the sun. As the distance (orbit radius) increases it is not surprising that the orbit time also increases but the exact relationship is not immediately obvious.

The law states that the square of the time of revolution of a planet (ie its period time, T) is proportional to the cube of the mean distance (r) from it.

$$\text{ie: } r^3 \propto T^2 \text{ or } r^3/T^2 = \text{a constant}$$

Example T9.15

If planet A has an average orbit radius twice that of B, by what factor is its orbit time period greater than that of planet B?

Example T9.16

Given that Earth has an orbit time period about the sun of 1 year (3.16×10^7 seconds) and that Uranus has an orbit time period of 84.0 years, and that the mean distance from the Earth to the sun is 1.49×10^{11} metres, calculate an expected mean orbit radius for Uranus, and state this as a ratio to that for Earth.

Kepler's Third Law – derivation

An orbiting satellite can be assumed to be moving in an approximate circle. It therefore requires a force, called centripetal force, to maintain its path. The force is provided by the gravitational attraction between the satellite and the planet that it is orbiting. So centripetal force required = gravitational force provided. (If gravitational force is larger than centripetal force required, satellite would spiral into the planet, if is it smaller, satellite would spiral outwards into space).

Algebraically:

gravitational force = centripetal force

$$\Rightarrow \frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{GM}{r}$$

(where m is mass of satellite; M is mass of planet, r is distance between centres)

but, since speed is constant, $v = \frac{\text{distance}}{\text{time}} = \frac{2\pi r}{T}$

(if we take distance as one complete orbit, radius r, T = time period)

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r} \Rightarrow \frac{4\pi^2 r^2}{T^2} = \frac{GM}{r} \Rightarrow \frac{r^3}{T^2} = \frac{GM}{4\pi^2} \approx 1.01\times10^{13} \text{ (a constant)}$$

Therefore $\frac{r^3}{T^2}$ = constant

An interesting consequence of Kepler's third Law is the fact that geostationary satellites have to all be the same distance above the Earth. Since $r^3 \propto T^2$, it follows that at a certain distance from the Earth, a satellite will have a certain time period (orbit time). A geostationary satellite has to follow a point on the Earth, so it has to have a time period the same as the time taken for a single rotation of the Earth about its axis – ie. 1 day. Therefore we can easily calculate such a satellite's distance above the Earth.

Example T9.17

Calculate the orbit height of an Earth geostationary satellite.

<u>**Satellite Energy**</u>

A satellite will have gravitational potential energy due to its position in a gravitational field, and kinetic energy due to its movement. Once it is in place, orbiting the planet, its total energy will remain constant – there is no resistance, since it will be out of the planet's atmosphere. It is therefore easy to deduce that if the orbit is not perfectly circular, then as it moves closer to the planet, its potential energy will decrease and therefore its kinetic energy will increase. As a satellite moves closer to a planet its speed therefore increases.

Satellite Energies – derivation

$E_k = \frac{1}{2}mv^2$, where m is mass of satellite and v, speed

but $\frac{mv^2}{r} = \frac{GMm}{r^2}$ (equating centripetal and gravitational forces, as earlier)

where M is mass of planet being orbited

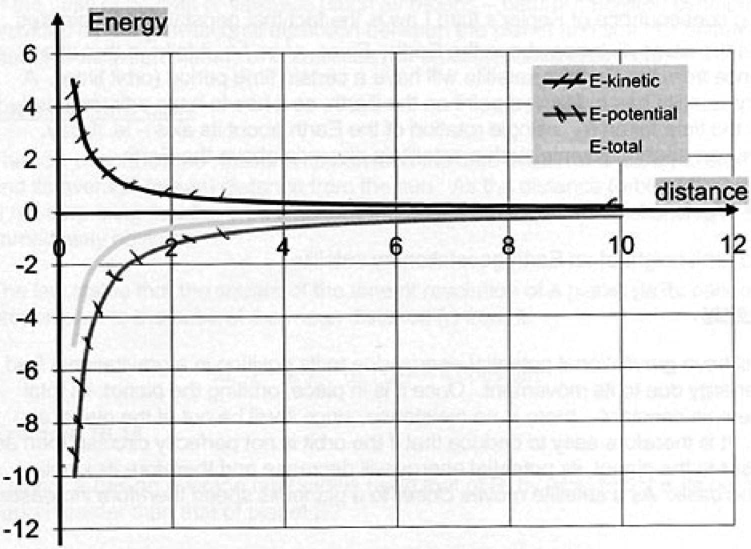
$$\Rightarrow mv^2 = \frac{rGMm}{r^2} = \frac{GMm}{r} \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r} = \frac{GMm}{2r}$$

So, $E_k = \frac{GMm}{2r}$ (kinetic energy)

We already have that $E_p = -\frac{GMm}{r}$ so, $E_T = \frac{GMm}{2r} + -\frac{GMm}{r} = -\frac{GMm}{2r}$

$$Summary: E_k = \frac{GMm}{2r},\ E_p = -\frac{GMm}{r},\ E_{Total} = -\frac{GMm}{2r}$$

The following graph shows how kinetic energy, (gravitational) potential energy and total energy for a satellite vary as its distance (number of planet radii) from a planet increases.



You need to be able to produce these sketch graphs.

Notes

- Potential energy is always negative – its maximum value is zero when it escapes the planet's field.

- Total energy is always negative – this is because kinetic energy is never great enough to overcome (negative) potential energy – if it were, the satellite would escape the planet, and fly off into space with zero potential (out of the field) and positive kinetic energy (moving). The graph refers only to satellites in orbit, so total energy must be negative.

Example T9.18

Calculate the kinetic energy, potential energy and total energy of a 650kg geostationary satellite. (Hint – use the distance from centre of Earth as calculated in Example T9.17)

Weightlessness

There are two kinds of weightlessness: true weightlessness, where an object has no weight, and apparent weightlessness.

True weightlessness can only be experienced by a mass when it is in a zero-gravitational field. This can happen in deep space or when the net field is zero – for example, at a certain point between the moon and Earth, where the attractive field from each cancel each other out.
More common and practically achievable, and with the same sensation, is apparent weightlessness. Apparent weightlessness is experienced by masses for which the only force acting on them is their weight. To help explain this concept, consider a man standing on the ground. He only feels his weight because of the contact force of the ground (normal reaction force) pushing up on him. The full weight of a mass is only experienced when resultant force is zero – ie when there is an opposing force equal to weight. A person jumping from an airplane experiences weightlessness to some extent – but air resistance prevents true weightlessness, except at the moment the person begins to fall, when air resistance is effectively zero.

We consider two examples of apparent weightlessness. The first situation is when a mass is falling and in a state of free fall as described above, with the only force being weight. The second situation is when a satellite is in orbit. Again, the only force acting is its weight. The motion of the satellite makes this concept difficult to understand. It moves in a circle, but it is always accelerating towards the centre of the circle, and its weight provides the force to do this. Its weight (ie. the gravitational force of attraction between the planet and the satellite) is the centripetal force required to allow it to maintain a circle.