Mathematical Representation of the Cumulative Link Mixed Model (CLMM)

1 Model Setup

Assume we have:

- N subjects (indexed by $i = 1, \ldots, N$),
- Each subject has n_i observations (indexed by $j = 1, \ldots, n_i$),
- An ordinal response Y_{ij} with categories $\{1, 2, \ldots, K\}$,
- Predictors/covariates in the vector $\mathbf{x}_{ij}$.

We let b_i be a random intercept for subject i.

2 Cumulative Link Model

A cumulative link model assumes:

$$P(Y_{ij} \leq k \mid \mathbf{x}_{ij}, b_i) = F\Big(\alpha_k - (\mathbf{x}_{ij}^\top \boldsymbol{\beta} + b_i)\Big), \quad k = 1, 2, \ldots, K-1, \tag{1}$$

where:

- F is a cumulative distribution function (CDF) corresponding to the chosen link function.
  - **Logit link**: $F(z) = \frac{1}{1+e^{-z}}$,
  - **Probit link**: $F(z) = \Phi(z)$,
  - **Cloglog link**: $F(z) = 1 - e^{-e^z}$.
- α_k are the thresholds or cut-points, with

$$\alpha_1 < \alpha_2 < \cdots < \alpha_{K-1}.$$

- $\mathbf{x}_{ij}^\top \boldsymbol{\beta}$ is the fixed-effect linear predictor for observation (i, j).
- b_i is the random intercept for subject i, capturing how subject i's baseline outcome may shift up or down on the latent scale.

3 Random Effects Distribution

The random intercept b_i is commonly assumed to be normally distributed:

$$b_i \sim \mathcal{N}(0, \sigma_b^2), \tag{2}$$

and is independent across subjects. This distribution introduces intra-subject correlation among the repeated ordinal outcomes $(Y_{i1}, Y_{i2}, \ldots, Y_{in_i})$.

4 Latent Variable Interpretation

Each ordinal response Y_{ij} can be viewed as arising from a latent continuous variable Z_{ij} such that:

$$Y_{ij} = k \quad \Longleftrightarrow \quad \alpha_{k-1} < Z_{ij} \leq \alpha_k, \tag{3}$$

with

$$Z_{ij} = \mathbf{x}_{ij}^\top \boldsymbol{\beta} + b_i + \varepsilon_{ij}, \tag{4}$$

where ε_{ij} follows a distribution whose CDF is F. The logistic distribution for ε_{ij} yields the proportional-odds logit link.

5 Example: Logit (Proportional Odds) Link

Specifically, for the **logistic link**, we write the model as:

$$\operatorname{logit}\bigl(P(Y_{ij} \leq k)\bigr) = \log\left(\frac{P(Y_{ij} \leq k)}{1 - P(Y_{ij} \leq k)}\right) = \alpha_k - (\mathbf{x}_{ij}^\top \boldsymbol{\beta} + b_i), \tag{5}$$

for $k = 1, \ldots, K-1$. Here:

- α_k are increasing threshold parameters.
- $\mathbf{x}_{ij}^\top \boldsymbol{\beta} + b_i$ is the linear predictor with both fixed and random effects.

6 Interpretation

- **Thresholds** $\alpha_1 < \cdots < \alpha_{K-1}$ partition the latent scale into K regions (one for each ordinal category).
- **Fixed effects** $\boldsymbol{\beta}$ shift the latent scale based on covariates $\mathbf{x}_{ij}$.
- **Random intercept** b_i lets each subject i have a unique baseline shift, accounting for correlation within subjects.
- By inverting F (the link function), one obtains predicted probabilities for each category of Y_{ij}.