Appendix 1 Equivalence of Induction Axioms

Recall the three induction axioms: Let $a \in Z$ and Z_a denote $\{n \in Z | n \geq a\}$ – the universe of this discussion

Weak Induction if $S \subseteq Z_a$ such that

1') $a \in S$

and 2') $\forall_n (n \in S \rightarrow n+1 \in S)$

then $S = Z_a$

Strong Induction if $S \subseteq Z_a$ such that

1) $a \in S$

and 2)$\forall_n (a, a+1,..,n \in S \rightarrow n+1 \in S)$

then $S = Z_a$

Well-Ordering if $T \subseteq Z_a$ and $T \neq \emptyset$ then $\exists\, t_0 \in T$ such that $\forall_n (n \in T \rightarrow t_0 \leq n)$

We shall establish the string of implications:

Strong Induction $\Rightarrow$ Weak Induction $\Rightarrow$ Well-Ordering $\Rightarrow$ Strong Induction

Strong Induction$\Rightarrow$Weak Induction: Here we prove that under the assumption of the strong induction axiom, the weak axiom follows. Hence we begin by assuming the following statement to be true:

$(\cdot)$ if $S \subseteq Z_a$ satisfies

(1) $a \in S$

and (2)$\forall n (a, a+1,..,n \in S \rightarrow n+1 \in S)$

then $S = Z_a$

We want to prove the nest statement to be true:

$(\cdot\cdot)$ if $S \in Z_a$ satisfies

(1') $a \in S$

and (2')$\forall n (n \in S \rightarrow n+1 \in S)$

then $S=Z_a$

To do this we assume (1') and (2') to be true and endeavor to prove $S=Z_a$. Thus

(1') $a \in S$ is true

and (2') $\forall_n \left(n \in S \rightarrow n+1 \in S\right)$ is true

Since (1) is true we need only prove (2), i.e.

$$\forall_n \left(a,\ a+1, \ldots\ n \in S \rightarrow n+1 \in S\right)$$

for then the truth of (*) allows us to conclude that $Z_a = S$. To prove (2) is true we assume $a, a+1, .., n \in S$ to be true. But then $n \in S$ is true and it follows by the truth of (2') that $n+1 \in S$. Hence (2) is true.

Finally then we conclude by the truth of (*) that $Z_a = S$.

<u>Weak</u> <u>Induction</u> $\Rightarrow$ <u>Well</u> – <u>Ordering</u>: Here we assume the truth of (**) and prove the following (well-ordering statement) to be true by contradiction:

(***) if $T \subseteq Z_a$ and $T \neq \emptyset$ then $\exists t_0 \in T$
such that $\forall_n \left(n \in T \rightarrow t_0 \leq n\right)$

Suppose (***) is false, i.e. $\exists$ T such that $T \neq \emptyset$ and $\forall t_0 \in T\ \exists n$ such that $n \in T$ and $n < t_0$.
Let $S = \{n \mid a, a+1, \ldots, n \notin T\}$. Observe that $a \in S$. Also suppose $n \in S$. Then, by the definition of S, $a, a+1, \ldots, n \notin T$. If $n+1 \in T$ then it folows that $t \in T \rightarrow n+1 \leq t$ i.e. n+1 is the SMALLEST element of T-a contradiction. Hence $n+1 \notin T$ and so $a, a+1, \ldots, n, n+1 \notin T$. Thus $n+1 \in S$ and we may conclude by (**) that $S=Z_a$. Finally we assert that the <u>contradiction</u> $T=\emptyset$ follows. Indeed, if $n \in Z_a$ then $n \in S$ and so $a, \ldots, n \notin T$, i.e. $T = \emptyset$.

<u>Well</u>-<u>Ordering</u> $\Rightarrow$ <u>Strong</u> <u>Induction</u>: Here we assume that (***) hold and we prove (*). Thus let $S \subseteq Z_a$ such that

(1) $a \in S$

and (2) $\forall n \left(a, \ldots, n \in S \rightarrow n+1 \in S\right)$

Our job is to conclude that $S=Z_a$ by using (***). Indeed, consider $T=Z_a - S$. If $T \neq \emptyset$ then $\exists\ t_0 \in T$ such that

(***) $\forall_n \left(n \in T \rightarrow t_0 \leq n\right)$

Case (1) $t_0 = a$ - contradiction to (1).

Case (2) $t_0 > a$. Consider $n=t_0 - 1 \geq a$.

Now we have $a, \ldots, t_0 - 1 \notin T$ because of (***) so that $a, \cdots, t_0 - 1 \in S$.
But then $t_0 \in S$ by (2), i. e $t_0 \notin T$. This contradiction forces $T = \emptyset$ and $S = Z_a$ follows.