



<http://algs4.cs.princeton.edu>

## 1.4 ANALYSIS OF ALGORITHMS

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- ▶ *introduction*
- ▶ *observations*
- ▶ *mathematical models*
- ▶ *order-of-growth classifications*
- ▶ *theory of algorithms*
- ▶ *memory*

# Cast of characters

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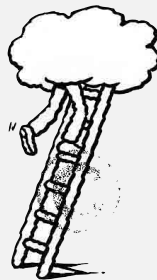
**Programmer** needs to develop a working solution.



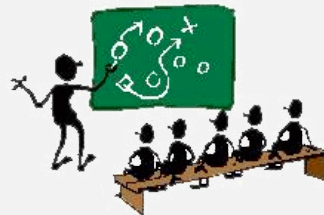
**Student** might play any or all of these roles someday.



**Client** wants to solve problem efficiently.



**Theoretician** wants to understand.

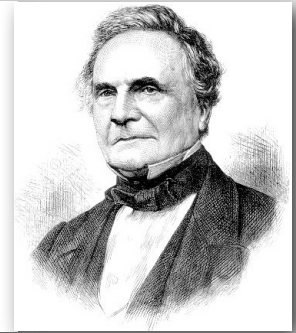


Basic **blocking and tackling** is sometimes necessary.  
[this lecture]

# Running time

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*“ As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time? ” — Charles Babbage (1864)*



how many times do you have to turn the crank?

Analytic Engine

# Reasons to analyze algorithms

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Predict performance.

Compare algorithms.

Provide guarantees.

Understand theoretical basis.

this course

theory of algorithms

**Primary practical reason:** avoid performance bugs.



**client gets poor performance because programmer  
did not understand performance characteristics**



# Some algorithmic successes

## Discrete Fourier transform.

- Break down waveform of  $N$  samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ....
- Brute force:  $N^2$  steps.
- FFT algorithm:  $N \log N$  steps, **enables new technology.**



Friedrich Gauss  
1805



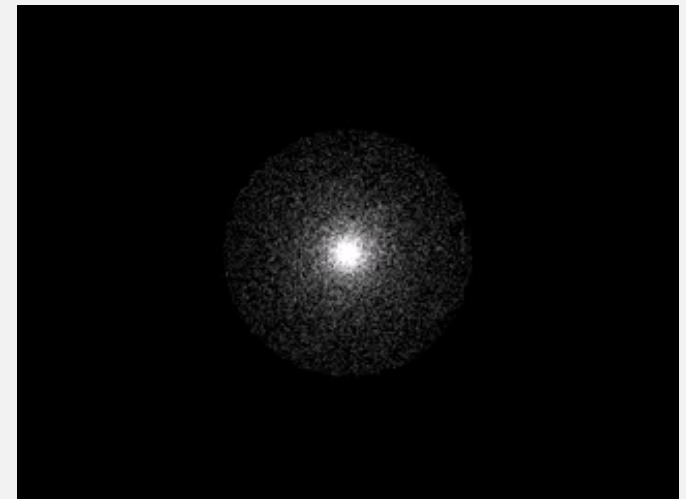
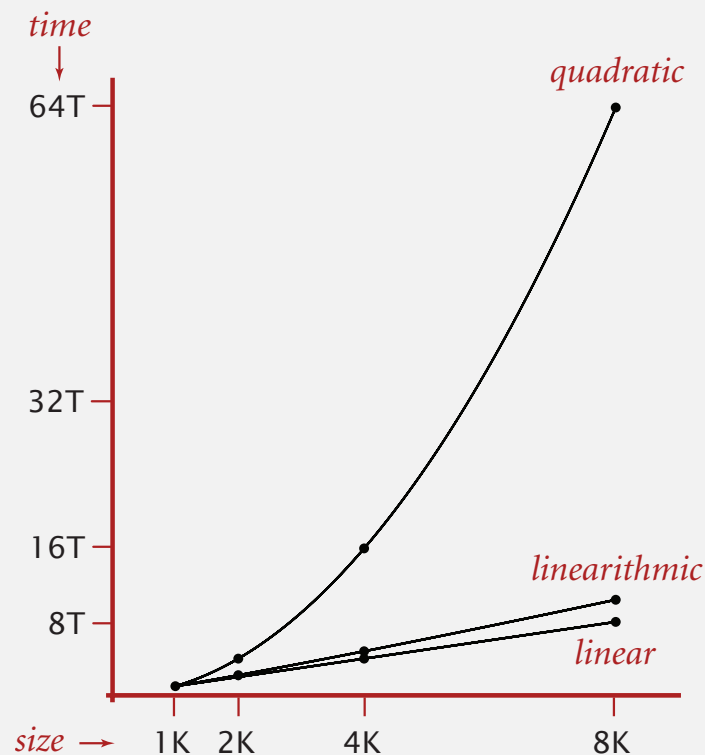
# Some algorithmic successes

## N-body simulation.

- Simulate gravitational interactions among  $N$  bodies.
- Brute force:  $N^2$  steps.
- Barnes-Hut algorithm:  $N \log N$  steps, **enables new research**.



Andrew Appel  
PU '81



# The challenge

---

Q. Will my program be able to solve a large practical input?

Why is my program so slow ?

Why does it run out of memory ?



**Insight.** [Knuth 1970s] Use **scientific method** to understand performance.

# Scientific method applied to analysis of algorithms

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A framework for predicting performance and comparing algorithms.

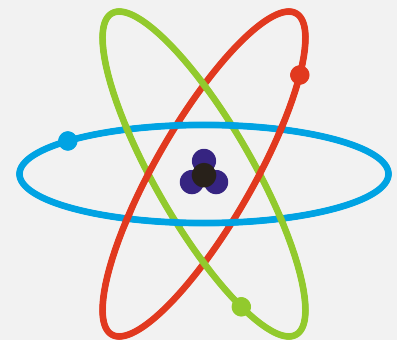
## Scientific method.

- **Observe** some feature of the natural world.
- **Hypothesize** a model that is consistent with the observations.
- **Predict** events using the hypothesis.
- **Verify** the predictions by making further observations.
- **Validate** by repeating until the hypothesis and observations agree.

## Principles.

- Experiments must be **reproducible**.
- Hypotheses must be **falsifiable**.

Feature of the natural world. Computer itself.







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## Example: 3-SUM

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**3-SUM.** Given  $N$  distinct integers, how many triples sum to exactly zero?

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

% java ThreeSum 8ints.txt
4
```

|   | a[i] | a[j] | a[k] | sum |
|---|------|------|------|-----|
| 1 | 30   | -40  | 10   | 0   |
| 2 | 30   | -20  | -10  | 0   |
| 3 | -40  | 40   | 0    | 0   |
| 4 | -10  | 0    | 10   | 0   |

**Context.** Deeply related to problems in computational geometry.

## 3-SUM: brute-force algorithm

---

```
public class ThreeSum
{
    public static int count(int[] a)
    {
        int N = a.length;
        int count = 0;
        for (int i = 0; i < N; i++)
            for (int j = i+1; j < N; j++)
                for (int k = j+1; k < N; k++)
                    if (a[i] + a[j] + a[k] == 0)
                        count++;
        return count;
    }

    public static void main(String[] args)
    {
        int[] a = In.readInts(args[0]);
        StdOut.println(count(a));
    }
}
```

← check each triple  
← for simplicity, ignore integer overflow

## Measuring the running time

## Q. How to time a program?

### A. Manual.



```
% java ThreeSum 1Kints.txt
```



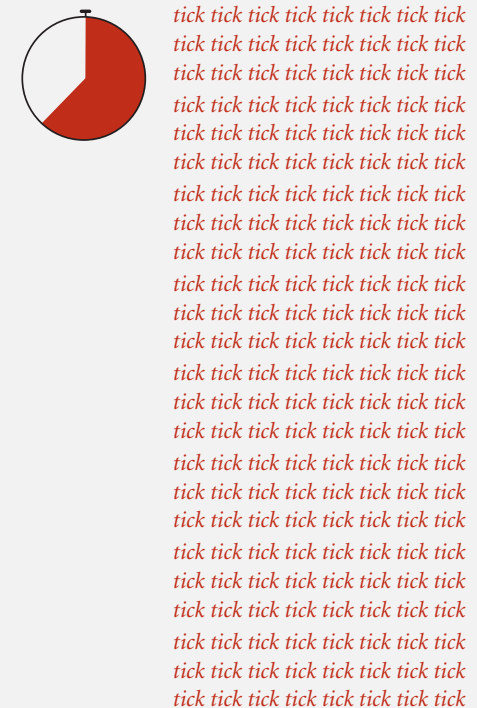
70

```
% java ThreeSum 2Kints.txt
```



528

```
% java ThreeSum 4Kints.txt
```



4039

# Measuring the running time

---

Q. How to time a program?

A. Automatic.

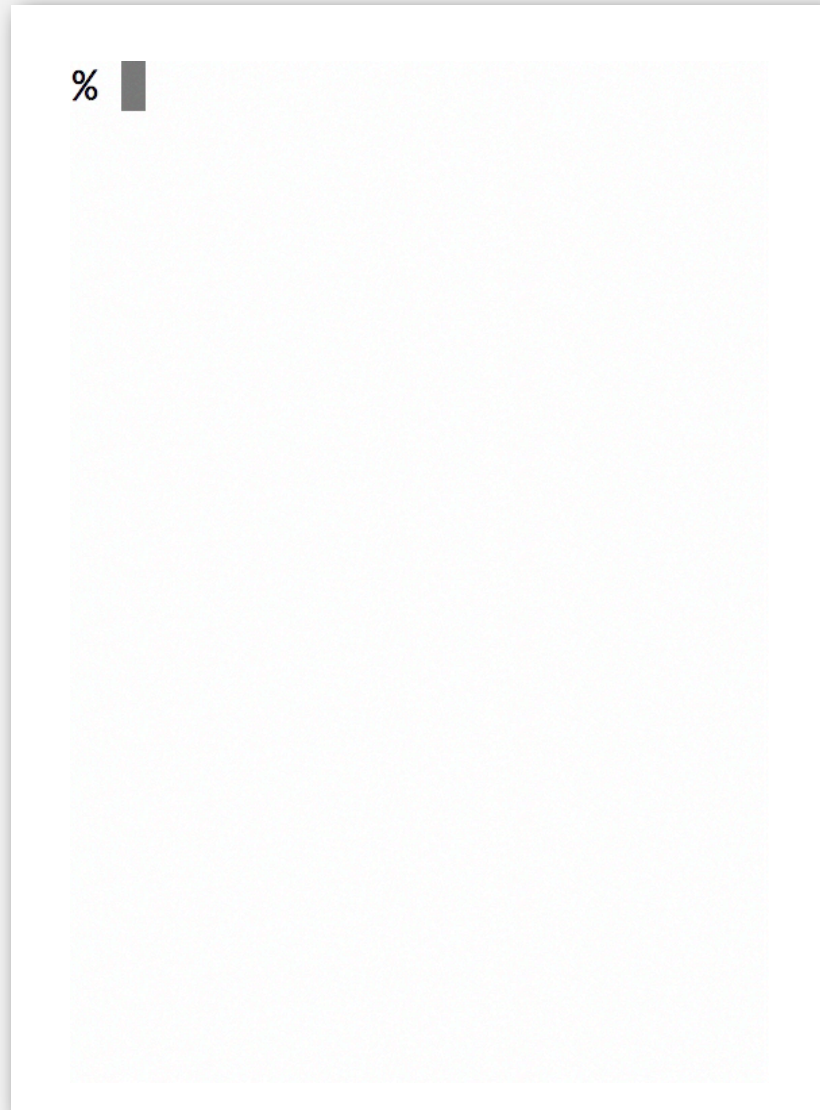
|                                     |   |
|-------------------------------------|---|
| <code>public class Stopwatch</code> | <i>(part of stdlib.jar)</i>             |
| <code>Stopwatch()</code>            | <i>create a new stopwatch</i>           |
| <code>double elapsedTime()</code>   | <i>time since creation (in seconds)</i> |

```
public static void main(String[] args)
{
    int[] a = In.readInts(args[0]);
    Stopwatch stopwatch = new Stopwatch();
    StdOut.println(ThreeSum.count(a));
    double time = stopwatch.elapsedTime();
}
```

# Empirical analysis

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Run the program for various input sizes and measure running time.



# Empirical analysis

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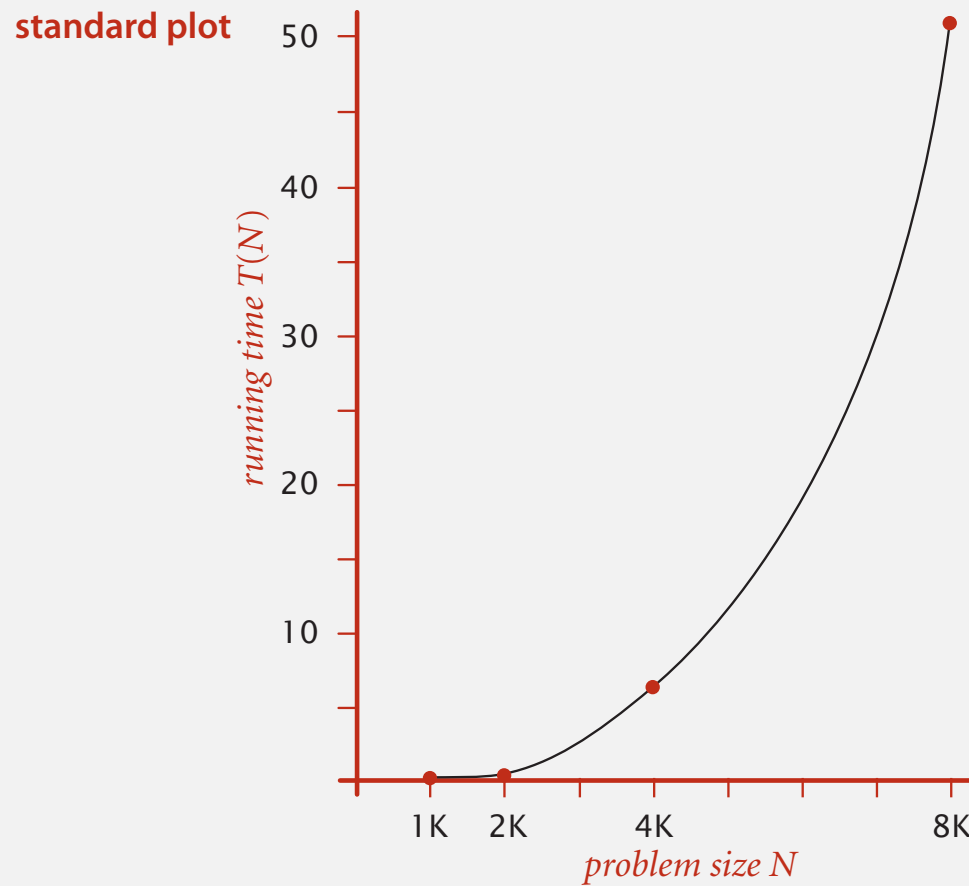
Run the program for various input sizes and measure running time.

| N      | time (seconds) † |
|--------|------------------|
| 250    | 0.0              |
| 500    | 0.0              |
| 1,000  | 0.1              |
| 2,000  | 0.8              |
| 4,000  | 6.4              |
| 8,000  | 51.1             |
| 16,000 | ?                |

# Data analysis

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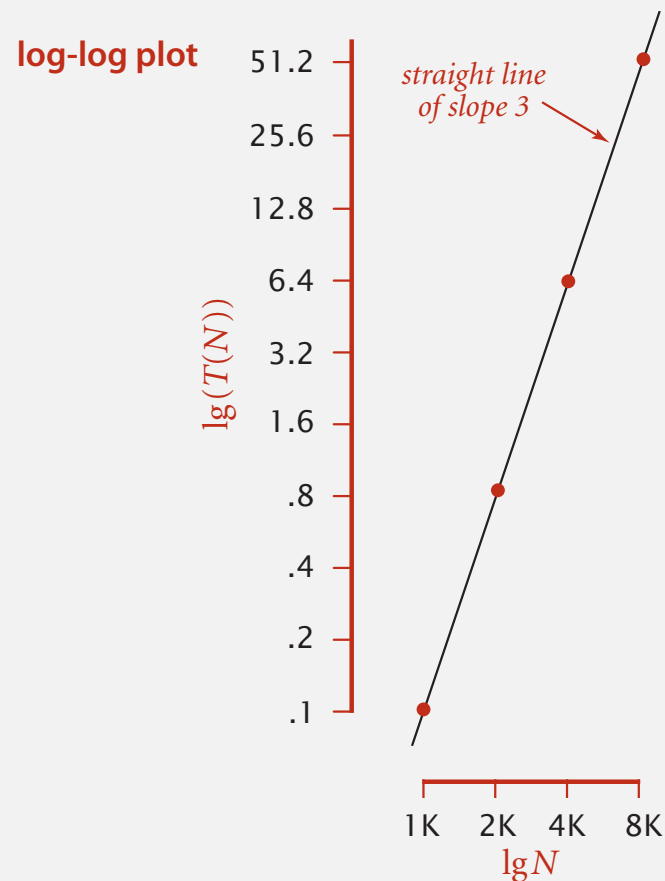
Standard plot. Plot running time  $T(N)$  vs. input size  $N$ .





# Data analysis

**Log-log plot.** Plot running time  $T(N)$  vs. input size  $N$  using **log-log scale**.



$$\lg(T(N)) = b \lg N + c$$

$$b = 2.999$$

$$c = -33.2103$$

$$T(N) = a N^b, \text{ where } a = 2^c$$

**Regression.** Fit straight line through data points:  $a N^b$ .

**Hypothesis.** The running time is about  $1.006 \times 10^{-10} \times N^{2.999}$  seconds.

power law  
slope

# Prediction and validation

---

**Hypothesis.** The running time is about  $1.006 \times 10^{-10} \times N^{2.999}$  seconds.

↑  
"order of growth" of running  
time is about  $N^3$  [stay tuned]

## Predictions.

- 51.0 seconds for  $N = 8,000$ .
- 408.1 seconds for  $N = 16,000$ .

## Observations.

| N      | time (seconds) † |
|--------|------------------|
| 8,000  | 51.1             |
| 8,000  | 51.0             |
| 8,000  | 51.1             |
| 16,000 | 410.8            |

**validates hypothesis!**

# Doubling hypothesis

---

**Doubling hypothesis.** Quick way to estimate  $b$  in a power-law relationship.

Run program, **doubling** the size of the input.

| N     | time (seconds) † | ratio | lg ratio |
|-------|------------------|-------|----------|
| 250   | 0.0              |       | –        |
| 500   | 0.0              | 4.8   | 2.3      |
| 1,000 | 0.1              | 6.9   | 2.8      |
| 2,000 | 0.8              | 7.7   | 2.9      |
| 4,000 | 6.4              | 8.0   | 3.0      |
| 8,000 | 51.1             | 8.0   | 3.0      |

↑  
seems to converge to a constant  $b \approx 3$

**Hypothesis.** Running time is about  $a N^b$  with  $b = \lg \text{ratio}$ .

**Caveat.** Cannot identify logarithmic factors with doubling hypothesis.

# Doubling hypothesis

---

**Doubling hypothesis.** Quick way to estimate  $b$  in a power-law relationship.

**Q.** How to estimate  $a$  (assuming we know  $b$ ) ?

**A.** Run the program (for a sufficient large value of  $N$ ) and solve for  $a$ .

| N     | time (seconds) † |
|-------|------------------|
| 8,000 | 51.1             |
| 8,000 | 51.0             |
| 8,000 | 51.1             |

$$51.1 = a \times 8000^3$$

$$\Rightarrow a = 0.998 \times 10^{-10}$$

**Hypothesis.** Running time is about  $0.998 \times 10^{-10} \times N^3$  seconds.



almost identical hypothesis  
to one obtained via linear regression

# Experimental algorithmics

---

## System independent effects.


- Algorithm.
  - Input data.
- 
- determines exponent  $b$   
in power law

## System dependent effects.

- Hardware: CPU, memory, cache, ...
  - Software: compiler, interpreter, garbage collector, ...
  - System: operating system, network, other apps, ...
- 
- determines constant  $a$   
in power law

**Bad news.** Difficult to get precise measurements.

**Good news.** Much easier and cheaper than other sciences.



e.g., can run huge number of experiments



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# Mathematical models for running time

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**Total running time:** sum of cost  $\times$  frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.



Donald Knuth  
1974 Turing Award

**In principle**, accurate mathematical models are available.



# Cost of basic operations

---

| operation               | example                       | nanoseconds † |
|-------------------------|-------------------------------|---------------|
| integer add             | <code>a + b</code>            | 2.1           |
| integer multiply        | <code>a * b</code>            | 2.4           |
| integer divide          | <code>a / b</code>            | 5.4           |
| floating-point add      | <code>a + b</code>            | 4.6           |
| floating-point multiply | <code>a * b</code>            | 4.2           |
| floating-point divide   | <code>a / b</code>            | 13.5          |
| sine                    | <code>Math.sin(theta)</code>  | 91.3          |
| arctangent              | <code>Math.atan2(y, x)</code> | 129.0         |
| ...                     | ...                           | ...           |

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

# Cost of basic operations

---

| operation            | example                          | nanoseconds <sup>†</sup> |
|----------------------|----------------------------------|--------------------------|
| variable declaration | <code>int a</code>               | $c_1$                    |
| assignment statement | <code>a = b</code>               | $c_2$                    |
| integer compare      | <code>a &lt; b</code>            | $c_3$                    |
| array element access | <code>a[i]</code>                | $c_4$                    |
| array length         | <code>a.length</code>            | $c_5$                    |
| 1D array allocation  | <code>new int[N]</code>          | $c_6 N$                  |
| 2D array allocation  | <code>new int[N][N]</code>       | $c_7 N^2$                |
| string length        | <code>s.length()</code>          | $c_8$                    |
| substring extraction | <code>s.substring(N/2, N)</code> | $c_9$                    |
| string concatenation | <code>s + t</code>               | $c_{10} N$               |

**Novice mistake.** Abusive string concatenation.

## Example: 1-SUM

---

Q. How many instructions as a function of input size  $N$ ?

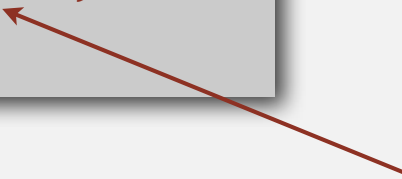
```
int count = 0;
for (int i = 0; i < N; i++)
    if (a[i] == 0)
        count++;
```

| operation            | frequency   |
|----------------------|-------------|
| variable declaration | 2           |
| assignment statement | 2           |
| less than compare    | $N + 1$     |
| equal to compare     | $N$         |
| array access         | $N$         |
| increment            | $N$ to $2N$ |

## Example: 2-SUM

Q. How many instructions as a function of input size  $N$ ?

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        if (a[i] + a[j] == 0)
            count++;
```


$$0 + 1 + 2 + \dots + (N - 1) = \frac{1}{2} N (N - 1) \\ = \binom{N}{2}$$

| operation            | frequency                                     |
|----------------------|---|
| variable declaration | $N + 2$                                       |
| assignment statement | $N + 2$                                       |
| less than compare    | $\frac{1}{2} (N + 1) (N + 2)$                 |
| equal to compare     | $\frac{1}{2} N (N - 1)$                       |
| array access         | $N (N - 1)$                                   |
| increment            | $\frac{1}{2} N (N - 1) \text{ to } N (N - 1)$ |

} tedious to count exactly

# Simplifying the calculations

---

*“ It is convenient to have a **measure of the amount of work involved in a computing process**, even though it be a very **crude** one. We may count up the number of times that various elementary operations are applied in the whole process and then given them various weights. We might, for instance, count the number of additions, subtractions, multiplications, divisions, recording of numbers, and extractions of figures from tables. In the case of computing with matrices most of the work consists of multiplications and writing down numbers, and **we shall therefore only attempt to count the number of multiplications and recordings.** ” — Alan Turing*

## ROUNDING-OFF ERRORS IN MATRIX PROCESSES

By A. M. TURING

(National Physical Laboratory, Teddington, Middlesex)

[Received 4 November 1947]

### SUMMARY

A number of methods of solving sets of linear equations and inverting matrices are discussed. The theory of the rounding-off errors involved is investigated for some of the methods. In all cases examined, including the well-known ‘Gauss elimination process’, it is found that the errors are normally quite moderate: no exponential build-up need occur.



# Simplification 1: cost model

**Cost model.** Use some basic operation as a proxy for running time.

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        if (a[i] + a[j] == 0)
            count++;
```

$$0 + 1 + 2 + \dots + (N - 1) = \frac{1}{2} N (N - 1) \\ = \binom{N}{2}$$

| operation            | frequency                              |
|----------------------|--|
| variable declaration | $N + 2$                                |
| assignment statement | $N + 2$                                |
| less than compare    | $\frac{1}{2} (N + 1) (N + 2)$          |
| equal to compare     | $\frac{1}{2} N (N - 1)$                |
| array access         | $N (N - 1)$                            |
| increment            | $\frac{1}{2} N (N - 1)$ to $N (N - 1)$ |

cost model = array accesses

(we assume compiler/JVM do not optimize array accesses away!)

## Simplification 2: tilde notation

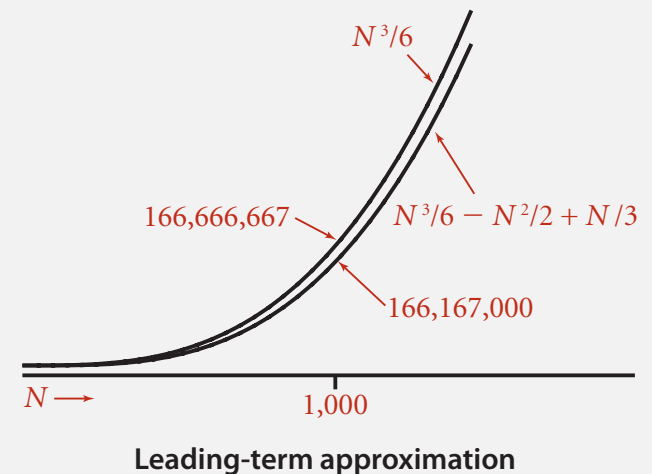
- Estimate running time (or memory) as a function of input size  $N$ .
- Ignore lower order terms.
  - when  $N$  is large, terms are negligible
  - when  $N$  is small, we don't care

Ex 1.  $\frac{1}{6} N^3 + 20 N + 16 \sim \frac{1}{6} N^3$

Ex 2.  $\frac{1}{6} N^3 + 100 N^{4/3} + 56 \sim \frac{1}{6} N^3$

Ex 3.  $\frac{1}{6} N^3 - \underbrace{\frac{1}{2} N^2 + \frac{1}{3} N}_{\text{discard lower-order terms}} \sim \frac{1}{6} N^3$

(e.g.,  $N = 1000$ : 500 thousand vs. 166 million)



Technical definition.  $f(N) \sim g(N)$  means  $\lim_{N \rightarrow \infty} \frac{f(N)}{g(N)} = 1$

## Simplification 2: tilde notation

---

- Estimate running time (or memory) as a function of input size  $N$ .
- Ignore lower order terms.
  - when  $N$  is large, terms are negligible
  - when  $N$  is small, we don't care

| operation            | frequency                              | tilde notation                       |
|----------------------|--|--------------------------------------|
| variable declaration | $N + 2$                                | $\sim N$                             |
| assignment statement | $N + 2$                                | $\sim N$                             |
| less than compare    | $\frac{1}{2} (N + 1) (N + 2)$          | $\sim \frac{1}{2} N^2$               |
| equal to compare     | $\frac{1}{2} N (N - 1)$                | $\sim \frac{1}{2} N^2$               |
| array access         | $N (N - 1)$                            | $\sim N^2$                           |
| increment            | $\frac{1}{2} N (N - 1)$ to $N (N - 1)$ | $\sim \frac{1}{2} N^2$ to $\sim N^2$ |



## Example: 2-SUM

Q. Approximately how many array accesses as a function of input size  $N$ ?

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        if (a[i] + a[j] == 0)
            count++;
```

"inner loop"

A.  $\sim N^2$  array accesses.

$$\begin{aligned} 0 + 1 + 2 + \dots + (N-1) &= \frac{1}{2} N(N-1) \\ &= \binom{N}{2} \end{aligned}$$

Bottom line. Use cost model and tilde notation to simplify counts.

## Example: 3-SUM

Q. Approximately how many array accesses as a function of input size  $N$ ?

```
int count = 0;
for (int i = 0; i < N; i++)
    for (int j = i+1; j < N; j++)
        for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
                count++;
```

"inner loop"

A.  $\sim \frac{1}{2} N^3$  array accesses.

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!}$$
$$\sim \frac{1}{6} N^3$$

Bottom line. Use cost model and tilde notation to simplify counts.

# Estimating a discrete sum

---

Q. How to estimate a discrete sum?

A1. Take discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 1.  $1 + 2 + \dots + N$ .

$$\sum_{i=1}^N i \sim \int_{x=1}^N x \, dx \sim \frac{1}{2} N^2$$

Ex 2.  $1^k + 2^k + \dots + N^k$ .

$$\sum_{i=1}^N i^k \sim \int_{x=1}^N x^k \, dx \sim \frac{1}{k+1} N^{k+1}$$

Ex 3.  $1 + 1/2 + 1/3 + \dots + 1/N$ .

$$\sum_{i=1}^N \frac{1}{i} \sim \int_{x=1}^N \frac{1}{x} \, dx = \ln N$$

Ex 4. 3-sum triple loop.

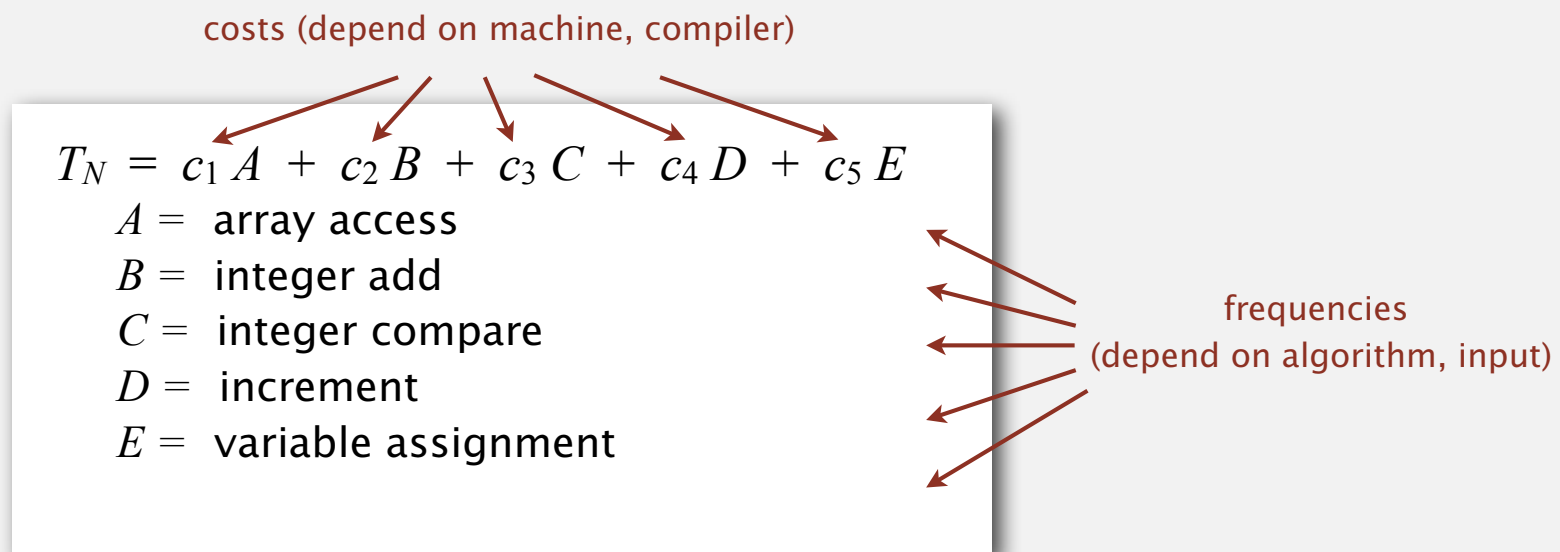
$$\sum_{i=1}^N \sum_{j=i}^N \sum_{k=j}^N 1 \sim \int_{x=1}^N \int_{y=x}^N \int_{z=y}^N dz \, dy \, dx \sim \frac{1}{6} N^3$$

# Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.



Bottom line. We use **approximate** models in this course:  $T(N) \sim c N^3$ .



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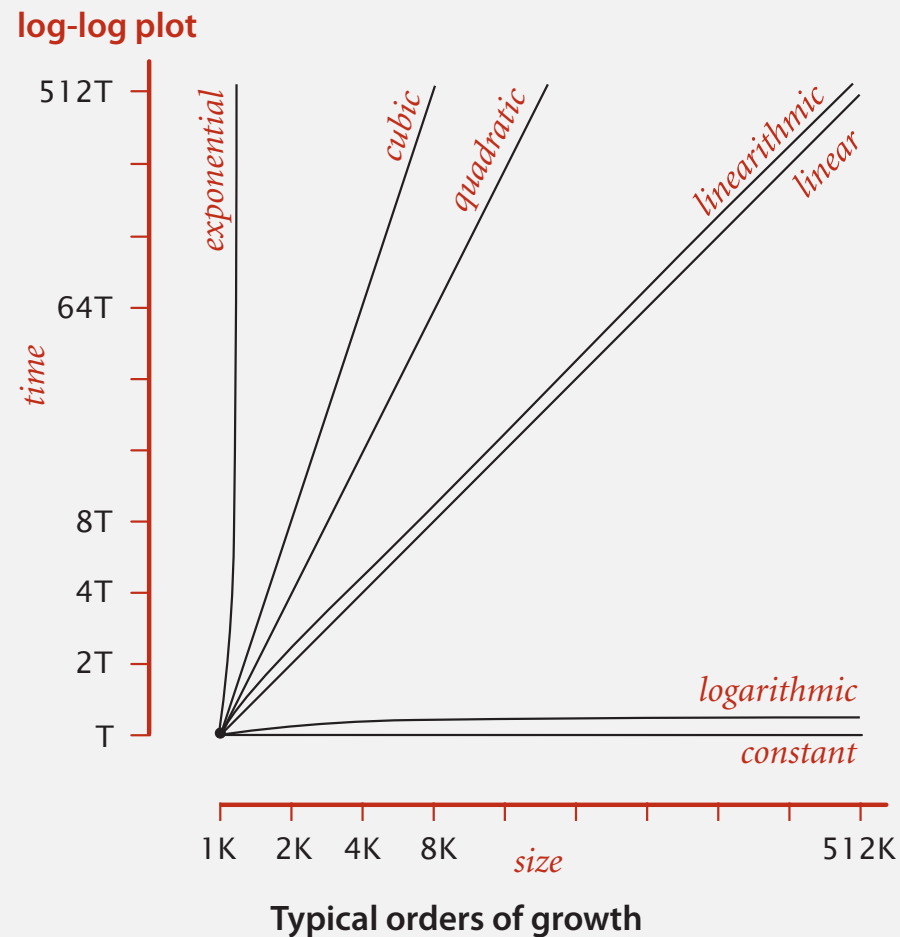
# Common order-of-growth classifications

Good news. the small set of functions

$1$ ,  $\log N$ ,  $N$ ,  $N \log N$ ,  $N^2$ ,  $N^3$ , and  $2^N$

suffices to describe order-of-growth of typical algorithms.

order of growth discards  
leading coefficient



# Common order-of-growth classifications

| order of growth | name         | typical code framework  | description        | example           | $T(2N) / T(N)$ |
|-----------------|--------------|---|--------------------|-------------------|----------------|
| 1               | constant     | <code>a = b + c;</code>   | statement          | add two numbers   | 1              |
| $\log N$        | logarithmic  | <code>while (N &gt; 1)<br/>{ N = N / 2; ... }</code>  | divide in half     | binary search     | $\sim 1$       |
| $N$             | linear       | <code>for (int i = 0; i &lt; N; i++)<br/>{ ... }</code>   | loop               | find the maximum  | 2              |
| $N \log N$      | linearithmic | [see mergesort lecture]   | divide and conquer | mergesort         | $\sim 2$       |
| $N^2$           | quadratic    | <code>for (int i = 0; i &lt; N; i++)<br/>  for (int j = 0; j &lt; N; j++)<br/>  { ... }</code>  | double loop        | check all pairs   | 4              |
| $N^3$           | cubic        | <code>for (int i = 0; i &lt; N; i++)<br/>  for (int j = 0; j &lt; N; j++)<br/>    for (int k = 0; k &lt; N; k++)<br/>    { ... }</code> | triple loop        | check all triples | 8              |
| $2^N$           | exponential  | [see combinatorial search lecture]  | exhaustive search  | check all subsets | $T(N)$         |



# Practical implications of order-of-growth

---

| growth rate | problem size solvable in minutes |                  |                      |                      |
|-------------|----------------------------------|------------------|----------------------|----------------------|
|             | 1970s                            | 1980s            | 1990s                | 2000s                |
| 1           | any                              | any              | any                  | any                  |
| $\log N$    | any                              | any              | any                  | any                  |
| $N$         | millions                         | tens of millions | hundreds of millions | billions             |
| $N \log N$  | hundreds of thousands            | millions         | millions             | hundreds of millions |
| $N^2$       | hundreds                         | thousand         | thousands            | tens of thousands    |
| $N^3$       | hundred                          | hundreds         | thousand             | thousands            |
| $2^N$       | 20                               | 20s              | 20s                  | 30                   |

**Bottom line.** Need linear or linearithmic alg to keep pace with Moore's law.

# Binary search demo

---

**Goal.** Given a sorted array and a key, find index of the key in the array?

**Binary search.** Compare key against middle entry.

- Too small, go left.
- Too big, go right.
- Equal, found.



**successful search for 33**

|         |    |    |    |    |    |    |    |    |    |    |    |    |    |         |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|----|---------|
| 6       | 13 | 14 | 25 | 33 | 43 | 51 | 53 | 64 | 72 | 84 | 93 | 95 | 96 | 97      |
| 0       | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14      |
| ↑<br>lo |    |    |    |    |    |    |    |    |    |    |    |    |    | ↑<br>hi |

# Binary search: Java implementation

---

## Trivial to implement?

- First binary search published in 1946; first bug-free one in 1962.
- Bug in Java's `Arrays.binarySearch()` discovered in 2006.

```
public static int binarySearch(int[] a, int key)
{
    int lo = 0, hi = a.length-1;
    while (lo <= hi)
    {
        int mid = lo + (hi - lo) / 2;
        if      (key < a[mid]) hi = mid - 1;
        else if (key > a[mid]) lo = mid + 1;
        else return mid;
    }
    return -1;
}
```

← one "3-way compare"

**Invariant.** If key appears in the array `a[]`, then  $a[lo] \leq key \leq a[hi]$ .

# Binary search: mathematical analysis

---

**Proposition.** Binary search uses at most  $1 + \lg N$  key compares to search in a sorted array of size  $N$ .

**Def.**  $T(N) \equiv$  # key compares to binary search a sorted subarray of size  $\leq N$ .

**Binary search recurrence.**  $T(N) \leq T(N/2) + 1$  for  $N > 1$ , with  $T(1) = 1$ .

$\uparrow$  left or right half       $\uparrow$  possible to implement with one 2-way compare (instead of 3-way)

**Pf sketch.**

$$\begin{aligned} T(N) &\leq T(N/2) + 1 && \text{given} \\ &\leq T(N/4) + 1 + 1 && \text{apply recurrence to first term} \\ &\leq T(N/8) + 1 + 1 + 1 && \text{apply recurrence to first term} \\ &\dots \\ &\leq T(N/N) + 1 + 1 + \dots + 1 && \text{stop applying, } T(1) = 1 \\ &= 1 + \lg N \end{aligned}$$

# An $N^2 \log N$ algorithm for 3-SUM

## Sorting-based algorithm.

- Step 1: **Sort** the  $N$  (distinct) numbers.
- Step 2: For each pair of numbers  $a[i]$  and  $a[j]$ , **binary search** for  $-(a[i] + a[j])$ .

**Analysis.** Order of growth is  $N^2 \log N$ .

- Step 1:  $N^2$  with insertion sort.
- Step 2:  $N^2 \log N$  with binary search.

input

30 -40 -20 -10 40 0 10 5

sort

-40 -20 -10 0 5 10 30 40

binary search

|            |                |
|------------|----------------|
| (-40, -20) | 60             |
| (-40, -10) | 50             |
| (-40, 0)   | 40             |
| (-40, 5)   | 35             |
| (-40, 10)  | 30             |
| $\vdots$   | $\vdots$       |
| (-40, 40)  | <del>0</del>   |
| $\vdots$   | $\vdots$       |
| (-20, -10) | 30             |
| $\vdots$   | $\vdots$       |
| (-10, 0)   | 10             |
| $\vdots$   | $\vdots$       |
| (10, 30)   | <del>-40</del> |
| (10, 40)   | -50            |
| (30, 40)   | -70            |

only count if  
 $a[i] < a[j] < a[k]$   
to avoid  
double counting

## Comparing programs

---

**Hypothesis.** The sorting-based  $N^2 \log N$  algorithm for 3-SUM is significantly faster in practice than the brute-force  $N^3$  algorithm.

| N     | time (seconds) |
|-------|----------------|
| 1,000 | 0.1            |
| 2,000 | 0.8            |
| 4,000 | 6.4            |
| 8,000 | 51.1           |

ThreeSum.java

| N      | time (seconds) |
|--------|----------------|
| 1,000  | 0.14           |
| 2,000  | 0.18           |
| 4,000  | 0.34           |
| 8,000  | 0.96           |
| 16,000 | 3.67           |
| 32,000 | 14.88          |
| 64,000 | 59.16          |

ThreeSumDeluxe.java

**Guiding principle.** Typically, better order of growth  $\Rightarrow$  faster in practice.



## 1.4 ANALYSIS OF ALGORITHMS

---

- ▶ *introduction*
- ▶ *observations*
- ▶ *mathematical models*
- ▶ *order-of-growth classifications*
- ▶ *theory of algorithms*
- ▶ *memory*



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# Types of analyses

---

**Best case.** Lower bound on cost.

- Determined by “easiest” input.
- Provides a goal for all inputs.

**Worst case.** Upper bound on cost.

- Determined by “most difficult” input.
- Provides a guarantee for all inputs.

**Average case.** Expected cost for random input.

- Need a model for “random” input.
- Provides a way to predict performance.

**Ex 1.** Array accesses for brute-force 3-SUM.

Best:  $\sim \frac{1}{2} N^3$

Average:  $\sim \frac{1}{2} N^3$

Worst:  $\sim \frac{1}{2} N^3$

**Ex 2.** Compares for binary search.

Best:  $\sim 1$

Average:  $\sim \lg N$

Worst:  $\sim \lg N$

# Types of analyses

---

**Best case.** Lower bound on cost.

**Worst case.** Upper bound on cost.

**Average case.** “Expected” cost.

## Actual data might not match input model?

- Need to understand input to effectively process it.
- Approach 1: design for the worst case.
- Approach 2: randomize, depend on probabilistic guarantee.

# Theory of algorithms

---

## Goals.

- Establish “difficulty” of a problem.
- Develop “optimal” algorithms.

## Approach.

- Suppress details in analysis: analyze “to within a constant factor”.
- Eliminate variability in input model by focusing on the worst case.

## Optimal algorithm.

- Performance guarantee (to within a constant factor) for any input.
- No algorithm can provide a better performance guarantee.

# Commonly-used notations in the theory of algorithms

---

| notation  | provides                      | example       | shorthand for   | used to                 |
|-----------|-------------------------------|---------------|---|-------------------------|
| Big Theta | asymptotic<br>order of growth | $\Theta(N^2)$ | $\frac{1}{2} N^2$<br>$10 N^2$<br>$5 N^2 + 22 N \log N + 3N$<br>$\vdots$ | classify<br>algorithms  |
| Big Oh    | $\Theta(N^2)$ and smaller     | $O(N^2)$      | $10 N^2$<br>$100 N$<br>$22 N \log N + 3 N$<br>$\vdots$                  | develop<br>upper bounds |
| Big Omega | $\Theta(N^2)$ and larger      | $\Omega(N^2)$ | $\frac{1}{2} N^2$<br>$N^5$<br>$N^3 + 22 N \log N + 3 N$<br>$\vdots$     | develop<br>lower bounds |

# Theory of algorithms: example 1

---

## Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 1-SUM = “*Is there a 0 in the array?*”

## Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 1-SUM: Look at every array entry.
- Running time of the optimal algorithm for 1-SUM is  $O(N)$ .

## Lower bound. Proof that no algorithm can do better.

- Ex. Have to examine all  $N$  entries (any unexamined one might be 0).
- Running time of the optimal algorithm for 1-SUM is  $\Omega(N)$ .

## Optimal algorithm.

- Lower bound equals upper bound (to within a constant factor).
- Ex. Brute-force algorithm for 1-SUM is optimal: its running time is  $\Theta(N)$ .

# Theory of algorithms: example 2

---

## Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

## Upper bound. A specific algorithm.

- Ex. Brute-force algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is  $O(N^3)$ .

# Theory of algorithms: example 2

---

## Goals.

- Establish “difficulty” of a problem and develop “optimal” algorithms.
- Ex. 3-SUM.

## Upper bound. A specific algorithm.

- Ex. **Improved** algorithm for 3-SUM.
- Running time of the optimal algorithm for 3-SUM is  $O(N^2 \log N)$ .

## Lower bound. Proof that no algorithm can do better.

- Ex. Have to examine all  $N$  entries to solve 3-SUM.
- Running time of the optimal algorithm for solving 3-SUM is  $\Omega(N)$ .

## Open problems.

- Optimal algorithm for 3-SUM?
- Subquadratic algorithm for 3-SUM?
- Quadratic lower bound for 3-SUM?

# Algorithm design approach

---

## Start.

- Develop an algorithm.
- Prove a lower bound.

## Gap?

- Lower the upper bound (discover a new algorithm).
- Raise the lower bound (more difficult).

## Golden Age of Algorithm Design.

- 1970s–.
- Steadily decreasing upper bounds for many important problems.
- Many known optimal algorithms.

## Caveats.

- Overly pessimistic to focus on worst case?
- Need better than “to within a constant factor” to predict performance.



# Commonly-used notations

| notation  | provides                  | example       | shorthand for   | used to                   |
|-----------|---------------------------|---------------|---|---------------------------|
| Tilde     | leading term              | $\sim 10 N^2$ | $10 N^2$<br>$10 N^2 + 22 N \log N$<br>$10 N^2 + 2 N + 37$   | provide approximate model |
| Big Theta | asymptotic growth rate    | $\Theta(N^2)$ | $\frac{1}{2} N^2$<br>$10 N^2$<br>$5 N^2 + 22 N \log N + 3N$ | classify algorithms       |
| Big Oh    | $\Theta(N^2)$ and smaller | $O(N^2)$      | $10 N^2$<br>$100 N$<br>$22 N \log N + 3 N$                  | develop upper bounds      |
| Big Omega | $\Theta(N^2)$ and larger  | $\Omega(N^2)$ | $\frac{1}{2} N^2$<br>$N^5$<br>$N^3 + 22 N \log N + 3 N$     | develop lower bounds      |

**Common mistake.** Interpreting big-Oh as an approximate model.

**This course.** Focus on approximate models: use Tilde-notation



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# Basics

---

Bit. 0 or 1.

Byte. 8 bits.

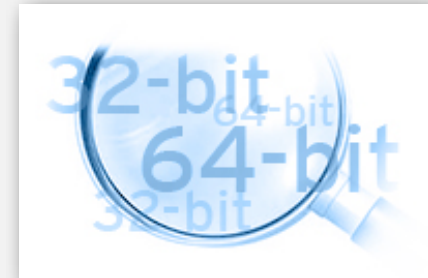
Megabyte (MB). 1 million or  $2^{20}$  bytes.

Gigabyte (GB). 1 billion or  $2^{30}$  bytes.

NIST



most computer scientists



64-bit machine. We assume a 64-bit machine with 8 byte pointers.

- Can address more memory.
- Pointers use more space.



some JVMs "compress" ordinary object pointers to 4 bytes to avoid this cost

# Typical memory usage for primitive types and arrays

---

| type    | bytes |
|---------|-------|
| boolean | 1     |
| byte    | 1     |
| char    | 2     |
| int     | 4     |
| float   | 4     |
| long    | 8     |
| double  | 8     |

**for primitive types**

| type     | bytes     |
|----------|-----------|
| char[]   | $2N + 24$ |
| int[]    | $4N + 24$ |
| double[] | $8N + 24$ |

**for one-dimensional arrays**

| type       | bytes        |
|------------|--------------|
| char[][]   | $\sim 2 M N$ |
| int[][]    | $\sim 4 M N$ |
| double[][] | $\sim 8 M N$ |

**for two-dimensional arrays**

# Typical memory usage for objects in Java

---

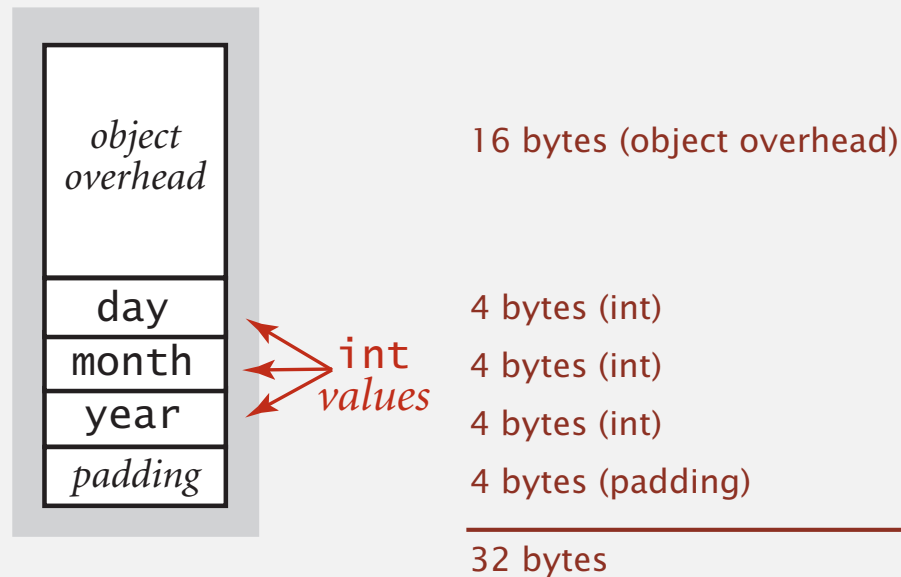
Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Each object uses a multiple of 8 bytes.

Ex 1. A Date object uses 32 bytes of memory.

```
public class Date
{
    private int day;
    private int month;
    private int year;
    ...
}
```



# Typical memory usage for objects in Java

---

**Object overhead.** 16 bytes.

**Reference.** 8 bytes.

**Padding.** Each object uses a multiple of 8 bytes.

**Ex 2.** A virgin String of length  $N$  uses  $\sim 2N$  bytes of memory.

```
public class String
{
    private char[] value;
    private int offset;
    private int count;
    private int hash;
    ...
}
```



16 bytes (object overhead)

8 bytes (reference to array)  
2N + 24 bytes (char[] array)

4 bytes (int)

4 bytes (int)

4 bytes (int)

4 bytes (padding)

---

2N + 64 bytes

# Typical memory usage summary

---

## Total memory usage for a data type value:

- Primitive type: 4 bytes for int, 8 bytes for double, ...
- Object reference: 8 bytes.
- Array: 24 bytes + memory for each array entry.
- Object: 16 bytes + memory for each instance variable + 8 bytes if inner class (for pointer to enclosing class).
- Padding: round up to multiple of 8 bytes.

**Shallow memory usage:** Don't count referenced objects.

**Deep memory usage:** If array entry or instance variable is a reference, add memory (recursively) for referenced object.



## Example

Q. How much memory does `WeightedQuickUnionUF` use as a function of  $N$ ?  
Use tilde notation to simplify your answer.

```
public class WeightedQuickUnionUF
{
```

```
    private int[] id;
    private int[] sz;
    private int count;
```

```
    public WeightedQuickUnionUF(int N)
    {
```

```
        id = new int[N];
        sz = new int[N];
        for (int i = 0; i < N; i++) id[i] = i;
        for (int i = 0; i < N; i++) sz[i] = 1;
```

```
    }
```

```
    ...
```

```
}
```

← 16 bytes  
(object overhead)

← 8 + (4N + 24) each  
reference + int[] array

← 4 bytes (int)

← 4 bytes (padding)

A.  $8N + 88 \sim 8N$  bytes.



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# Turning the crank: summary

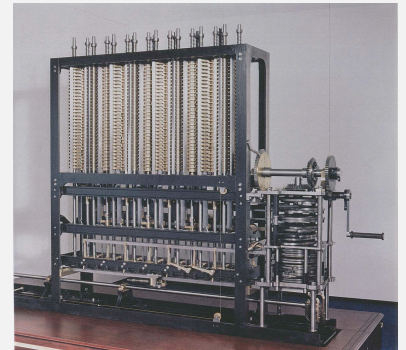
---

## Empirical analysis.

- Execute program to perform experiments.
- Assume power law and formulate a hypothesis for running time.
- Model enables us to **make predictions**.

## Mathematical analysis.

- Analyze algorithm to count frequency of operations.
- Use tilde notation to simplify analysis.
- Model enables us to **explain behavior**.



## Scientific method.

- Mathematical model is independent of a particular system; applies to machines not yet built.
- Empirical analysis is necessary to validate mathematical models and to make predictions.



<http://algs4.cs.princeton.edu>

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