

Momentum Distribution of a Fermi Gas in the Random Phase Approximation

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Abstract

We consider a system of interacting fermions on the three-dimensional torus in the mean-field scaling limit. Our objective is computing the occupation number of the Fourier modes in a trial state obtained through the random phase approximation for the ground state. Our result shows that – in the trial state – the Fermi momentum does not depend on the interaction potential (it is universal), while the jump at the Fermi surface is as predicted by Daniel and Vosko.

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1 Introduction and Main Result

We consider a quantum system of N spinless fermionic particles on the torus $\mathbb{T}^3 := [0, 2\pi]^3$. This system is described by the Hamilton operator

$$H_N := \sum_{j=1}^N -\hbar^2 \Delta_{x_j} + \lambda \sum_{i < j}^N V(x_i - x_j) \quad (1.1)$$

acting on wave functions in the antisymmetric tensor product $L_a^2(\mathbb{T}^{3N}) = \bigwedge_{i=1}^N L^2(\mathbb{T}^3)$. As the general case with $N \simeq 10^{23}$ is too difficult to analyze, we will consider the asymptotics for particle number $N \rightarrow \infty$ in the *mean-field scaling limit* introduced by [13], i. e., we set

$$\hbar := N^{-\frac{1}{3}}, \quad \text{and} \quad \lambda := N^{-1}. \quad (1.2)$$

The ground state energy of the system is defined as the infimum of the spectrum

$$E_N := \inf \sigma(H_N) = \inf_{\substack{\psi \in L_a^2(\mathbb{T}^{3N}) \\ \|\psi\|=1}} \langle \psi, H_N \psi \rangle.$$

Any eigenvector of H_N with eigenvalue E_N is called a ground state. In the present paper we analyze the momentum distribution (i. e., the Fourier transform of the one-particle reduced density matrix) of the random phase approximation of the ground state.

In the non-interacting case $V = 0$, it is well-known that one ground state is given by a Slater determinant comprising all plane waves with N different momenta $k_j \in \mathbb{Z}^3$ of minimal kinetic energy $|k_j|^2$, i. e.,

$$\psi(x_1, x_2, \dots, x_N) = \frac{1}{\sqrt{N!}} \det \left(\frac{1}{(2\pi)^{3/2}} e^{ik_j \cdot x_i} \right)_{j,i=1}^N. \quad (1.3)$$

This is (up to a phase) unique if we impose that the number of particles exactly fills a ball in momentum space; i. e., if

$$N = |B_F| \quad \text{for} \quad B_F := \{k \in \mathbb{Z}^3 : |k| \leq k_F\} \quad \text{with some} \quad k_F > 0. \quad (1.4)$$

This means that the Fermi momentum k_F scales like¹

$$k_F = \kappa N^{\frac{1}{3}} \quad \text{with} \quad \kappa = \left(\frac{3}{4\pi} \right)^{\frac{1}{3}} + \mathcal{O}(N^{-\frac{1}{3}}) . \quad (1.5)$$

The set of momenta B_F is called the Fermi ball. We also define its complement

$$B_F^c := \mathbb{Z}^3 \setminus B_F .$$

As a first step towards including the effects of a non-vanishing interaction potential V one may consider the Hartree–Fock approximation. In this approximation, the expectation value $\langle \psi, H_N \psi \rangle$ is minimized over the choice of orbitals $\{\varphi_j : j = 1, \dots, N\} \subset L^2(\mathbb{T}^3)$ in the Slater determinant

$$\psi(x_1, x_2, \dots, x_N) = \frac{1}{\sqrt{N!}} \det (\varphi_j(x_i))_{j,i=1}^N .$$

(This is to be compared to the general quantum many-body problem, in which also linear combinations of Slater determinants are permitted.) In general, the Hartree–Fock minimizer will not have plane waves as orbitals. However, with our particular assumptions on the potential, the scaling limit, and the particle number, one can show [2, Appendix A] that (1.3) is the (unique up to a phase) Hartree–Fock minimizer. Therefore, in the Hartree–Fock approximation the momentum distribution is simply

$$\langle \psi, a_q^* a_q \psi \rangle = \begin{cases} 0 & \text{for } q \in B_F^c \\ 1 & \text{for } q \in B_F . \end{cases} \quad (1.6)$$

It is highly non-trivial to understand if this jump in the momentum distribution survives the presence of an interaction potential, and if it does, how its location and height are affected by the interaction. In physics, it has become known as *Luttinger’s theorem* that “[the] interaction may deform the FS [Fermi surface], but it cannot change its volume. In the isotropic case, where symmetry requires the FS to remain a sphere, its radius must then remain k_F (the Fermi momentum of the unperturbed system)” [12]. In other words, the Fermi momentum is conjectured to be universal, i. e., independent of the interaction potential V . This is in contrast to the height of the jump, called *quasiparticle weight* Z , which generally depends on V . As discussed, the Hartree–Fock approximation predicts that k_F is independent of V , but also that the quasiparticle weight is $Z = 1$ independent of V . So to observe any effect of the interaction, we have to employ a more precise approximation for the ground state. In [1, 2, 4] it has been shown that the ground state energy can be approximated to higher precision using the random phase approximation, in its formulation as bosonization of particle–hole pair excitations. Our goal in this paper is to exhibit the prediction of the random phase

¹In [2], κ is defined as $(\frac{3}{4}\pi)^{\frac{1}{3}}$, so in that notation $\kappa = \hbar k_F(1 + \mathcal{O}(\hbar))$.

approximation for the momentum distribution. We will take a trial state constructed by bosonization and compute the deviation of its momentum distribution from (1.6),

$$n_q := \begin{cases} \langle \psi, a_q^* a_q \psi \rangle & \text{for } q \in B_F^c \\ 1 - \langle \psi, a_q^* a_q \psi \rangle & \text{for } q \in B_F . \end{cases} \quad (1.7)$$

We only consider interaction potentials V that have compactly supported non-negative Fourier transform \hat{V} ,². Let $R > 0$ such that $\text{supp}(\hat{V}) \subset B_R(0)$. To state our main theorem, we need to define the set of momenta

$$\mathcal{C}^q := \begin{cases} B_R(0) \cap H^{\text{nor}} \cap (B_F - q \cup B_F + q) & \text{for } q \in B_F^c \\ B_R(0) \cap H^{\text{nor}} \cap (B_F^c - q \cup B_F^c + q) & \text{for } q \in B_F , \end{cases} \quad (1.8)$$

with the northern half-space defined as

$$H^{\text{nor}} := \{k \in \mathbb{R}^3 : k_3 > 0 \text{ or } (k_3 = 0 \text{ and } k_2 > 0) \text{ or } (k_3 = k_2 = 0 \text{ and } k_1 > 0)\} . \quad (1.9)$$

In the following, let us adopt the convention that C is a positive constant not depending on N, V or q but whose value may change from line to line.

Theorem 1.1 (Main Result). *Assume that V has compactly supported non-negative Fourier transform, and let $R > 0$ such that $\text{supp}(\hat{V}) \subset B_R(0)$. Then, there exists a sequence of trial states $(\psi_N) \subset L_a^2(\mathbb{T}^{3N})$ with particle numbers corresponding to completely filled Fermi balls as in (1.4) such that*

- *the sequence of trial states is energetically close to the ground state in the sense that there exists some $\alpha > 0$ such that*

$$\langle \psi_N, H_N \psi_N \rangle - \inf \sigma(H_N) \leq CN^{-\frac{1}{3}-\alpha} ; \quad (1.10)$$

- *and for any $\epsilon, \varepsilon > 0$ and all q in the set*

$$\mathcal{Q}_\epsilon := \left\{ q \in \mathbb{Z}^3 \mid \nexists k \in B_R(0) : \frac{|k \cdot q|}{|k||q|} \in (0, \epsilon) \right\} \quad (1.11)$$

its momentum distribution n_q can be estimated as

$$0 \leq n_q \leq N^{-\frac{2}{3}} \sum_{k \in \mathcal{C}^q \cap \mathbb{Z}^3} \frac{\hat{V}_k}{2\kappa|k|} \frac{1}{\pi} \int_0^\infty \frac{(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu + \mathcal{E} , \quad (1.12)$$

where

$$\lambda_{q,k} := \frac{|k \cdot q|}{|k||q|} , \quad Q_k^{(0)}(\mu) := \frac{3\hat{V}_k}{2\kappa\hbar k_F} \left(1 - \mu \arctan \left(\frac{1}{\mu} \right) \right) , \quad (1.13)$$

²The interaction $V(x-y)$ being a two-particle multiplication operator, we use the convention $V(x) = \sum_{k \in \mathbb{Z}^3} \hat{V}_k e^{ik \cdot x}$ for its Fourier transform. This is in contrast to wave functions, whose Fourier transform is defined to be L^2 -unitary, $\psi(x_1, \dots, x_N) = (2\pi)^{-\frac{3N}{2}} \sum_{k_1, \dots, k_N \in \mathbb{Z}^3} \hat{\psi}(k_1, \dots, k_N) e^{i(k_1 \cdot x_1 + \dots + k_N \cdot x_N)}$.

with error term \mathcal{E} bounded by

$$|\mathcal{E}| \leq C_\epsilon \|\hat{V}\|_1 e^{C_\epsilon \|\hat{V}\|_1} N^{-\frac{2}{3} - \frac{2}{27} + \epsilon}, \quad (1.14)$$

for some C_ϵ depending on ϵ , and some C_ϵ depending on ϵ .

The theorem is proven in Section 10, based on the strategy explained in Section 3.

As a measure for the height of the jump at the Fermi surface we define the *quasiparticle weight* as

$$Z := 1 - \sup_{q \in B_F} n_q - \sup_{q \in B_F^c} n_q.$$

As a corollary of our main theorem, we obtain an estimate for Z .

Theorem 1.2 (Jump at the Fermi Surface). *Assume that V has compactly supported non-negative Fourier transform. Then there exists a sequence of trial states $(\psi_N) \subset L_a^2(\mathbb{T}^{3N})$ satisfying the assertions of Theorem 1.1, whose occupation density for large N exhibits a jump at the Fermi surface, in the sense that*

$$Z \geq 1 - C \|\hat{V}\|_1 e^{C \|\hat{V}\|_1} N^{-\frac{2}{3} + \frac{2}{27}}. \quad (1.15)$$

The proof is given in Section 10. Note that (1.15) does not depend on ϵ or ϵ .

Remarks. 1. *Optimality:* In the trial state ψ_N which we construct in Section 2, for most $q \in \mathbb{Z}^3$, the upper bound in (1.12) is an equality. The precise meaning of “most” is given by Proposition 2.1.

2. *The condition $q \in \mathcal{Q}_\epsilon$:* This avoids situations in which $\lambda_{q,k} < \epsilon$. As we will see in the proof of Proposition 10.4, we have $\mathcal{E} \sim \lambda_{q,k}^{-2}$ for $\lambda_{q,k} \rightarrow 0$, which for $q \in \mathcal{Q}_\epsilon$ is bounded by $\epsilon^{-2} = \mathcal{O}(1)$. Without the condition $q \in \mathcal{Q}_\epsilon$, the best bound would be $\lambda_{q,k}^{-2} \leq N^{\frac{2}{3}}$.

For every $q \in \mathbb{Z}^3$ (actually even $q \in \mathbb{R}^3$), we have

$$\min\{\lambda_{q,k} : k \in B_R(0), k \cdot q \neq 0\} > 0, \quad (1.16)$$

so $q \in \mathcal{Q}_\epsilon$ can be achieved by choosing ϵ below the l.h.s. of (1.16). Further, the set of $q \in \mathbb{R}^3$ excluded by the condition $\{k \in B_R(0) : \lambda_{q,k} \in (0, \epsilon)\}$ is a union of rays starting at 0, and converges to the empty set as $\epsilon \rightarrow 0$. Thus, out of all points $q \in \mathbb{Z}^3$ with distance $< R$ to the Fermi surface (i.e., those for which our result (1.12) becomes nontrivial), we only exclude a proportion which is $\mathcal{O}(1)$ and can be made arbitrarily small by choosing ϵ small.

The restriction $q \in \mathcal{Q}_\epsilon$ is not required in Theorem 1.2.

3. *Agreement with the physics literature:* To our knowledge, the momentum distribution in the random phase approximation has first been computed by Daniel and Vosko in 1960 [9] by a Hellmann–Feynman argument. In Appendix B we show that their result agrees with our leading-order term in the upper bound of (1.12).

4. *Bosonization approximation:* Heuristically we can obtain the formula of the upper bound by introducing operators for the creation of particle–hole pairs as in Section 4. These behave approximately as bosonic creation operators. If we assume them to satisfy *exactly* bosonic commutator relations then we find the momentum distribution $n_q^{(b)}$ as in (3.2); from there one proceeds with elementary estimates to get (1.12). This bosonization computation is detailed in Section 5.
5. *One-particle reduced density matrix and correlation function:* If the ground state is translation invariant, then its one-particle density matrix $\gamma(x, y) = \langle \psi, a_y^* a_x \psi \rangle$ ($x, y \in \mathbb{T}^3$) can also be written in translation invariant form as $\gamma(x - y)$. Its Fourier transform is then given by the momentum distribution $\langle \psi, a_q^* a_q \psi \rangle$. The one-particle density matrix is also the same as the two-point correlation function.
6. *Validity for the ground state:* Our result concerns a trial state, not the actual ground state. It would be very interesting to understand if similar statements holds for the ground state. This is a very subtle question. It is expected that due to the Kohn–Luttinger instability the ground state always has a superconducting part which smoothens out the jump at the Fermi surface. We conjecture that the Kohn–Luttinger effect affects the momentum distribution only on a much smaller (even exponentially small) scale. A rigorous proof seems very challenging because one cannot use a-priori bounds obtainable from the ground state energy; in fact, a single pair excitation $a_p^* a_h^*$ may change the momentum distribution from $\langle \psi_N, a_p^* a_p \psi_N \rangle = 0$ to $\langle \psi_N, a_p^* a_p \psi_N \rangle = 1$ at a kinetic energy cost as small as order $\hbar^2 = N^{-2/3}$; this has to be compared to the resolution of the ground state energy (1.10) that is only $\hbar N^{-\alpha} = N^{-\frac{1}{3}-\alpha}$.

1.1 Comparison with the Thermodynamic Limit

The physics literature [9, 10, 11] considers the system in the thermodynamic limit, where sums over momenta become integrals. Our estimates are not uniform in the system’s volume, but we can formally extrapolate (1.12) to the thermodynamic limit. To do so, we rescale the torus \mathbb{T}^3 to the torus $L\mathbb{T}^3 = [0, 2\pi L]^3$. The corresponding momentum space is $L^{-1}\mathbb{Z}^3$ and we can replace sums over \mathbb{Z}^3 by sums over $L^{-1}\mathbb{Z}^3$. The number of momenta in the Fermi ball $B_F := \{k \in L^{-1}\mathbb{Z}^3 : |k| \leq k_F\}$ is now

$$N = |B_F| \approx \frac{4\pi}{3} k_F^3 L^3. \quad (1.17)$$

We consider $L \rightarrow \infty$ followed by the high density limit $k_F \rightarrow \infty$. The density is

$$\rho := \frac{N}{(2\pi L)^3} = \frac{k_F^3}{6\pi^2} (1 + \mathcal{O}(k_F^{-1})). \quad (1.18)$$

Setting $\hbar := k_F^{-1}$, we can define $Q_k^{(0)}(\mu)$ via (1.13), and the right-hand side (r.h.s.) of (1.12) becomes

$$n_q(k_F, L) \approx \sum_{k \in \mathcal{C}^q \cap L^{-1}\mathbb{Z}^3} \frac{1}{\pi} \frac{\hat{V}_k}{2\hbar\kappa N|k|} \int_0^\infty \frac{(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu. \quad (1.19)$$

In Appendix B.1 we compute that, with $R_q := ||q| - k_F|$,

$$\lim_{L \rightarrow \infty} n_q(k_F, L) \approx \int_{R_q}^R d|k| |k| \int_0^\infty \frac{3\hat{V}_k}{4\pi\hbar k_F^3 \kappa} \left(\frac{1}{1 + \mu^2} - \frac{R_q |k|^{-1}}{R_q^2 |k|^{-2} + \mu^2} \right) \frac{d\mu}{1 + Q_k^{(0)}(\mu)}. \quad (1.20)$$

This agrees with [9], as we argue in Appendix B.2.

2 Construction of the Trial State

The definition of the trial state ψ_N uses second quantization. That means, we extend the N -particle space $L_a^2(\mathbb{T}^{3N})$ by introducing the fermionic Fock space

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} L_a^2(\mathbb{T}^{3n}).$$

To each momentum mode $q \in \mathbb{Z}^3$, we assign the plane wave

$$f_q \in L^2(\mathbb{T}^3), \quad f_q(x) := (2\pi)^{-\frac{3}{2}} e^{iq \cdot x}$$

and the respective creation and annihilation operators on Fock space

$$a_q^* := a^*(f_q), \quad a_q := a(f_q)$$

which satisfy the canonical anticommutation relations (CAR)

$$\{a_q, a_{q'}^*\} = \delta_{q,q'}, \quad \{a_q, a_{q'}\} = \{a_q^*, a_{q'}^*\} = 0 \quad \text{for all } q, q' \in \mathbb{Z}^3. \quad (2.1)$$

The number operator on Fock space is defined as

$$\mathcal{N} := \sum_{q \in \mathbb{Z}^3} a_q^* a_q \quad (2.2)$$

and the vacuum vector $\Omega \in \mathcal{F}$ is $\Omega := (1, 0, 0, \dots)$, which satisfies $a_q \Omega = 0$ for all $q \in \mathbb{Z}^3$.

The trial state As a trial state $\psi_N \in L_a^2(\mathbb{T}^{3N}) \subset \mathcal{F}$ for Theorem 1.1, we use the state constructed by means of the random phase approximation (in its formulation as *bosonization* of particle-hole excitations) in [1, (4.20)], i. e.,

$$\psi_N := RT\Omega, \quad (2.3)$$

with a particle-hole transformation $R : \mathcal{F} \rightarrow \mathcal{F}$ and an almost-bosonic Bogoliubov transformation $T : \mathcal{F} \rightarrow \mathcal{F}$ that are both defined below. According to [1, 2, 4], the state (2.3) is energetically close to the ground state.

The particle–hole transformation The particle–hole transformation is the unitary operator $R : \mathcal{F} \rightarrow \mathcal{F}$ defined by its action on creation operators

$$R^* a_q^* R := \begin{cases} a_q^* & \text{if } q \in B_F^c \\ a_q & \text{if } q \in B_F \end{cases} \quad (2.4)$$

and its action on the vacuum

$$R^* \Omega := \prod_{k_j \in B_F} a_{k_j}^* \Omega.$$

The latter product is (up to an irrelevant phase $e^{i\pi}$) uniquely defined and one easily verifies that it is a Slater determinant of plane waves as in (1.3). The particle–hole transform satisfies $R^* = R$.

Particle–hole pair operators on patches The key observation [1, 2, 4] motivating the choice of T is that after the transformation R , the Hamiltonian H becomes almost quadratic in some almost–bosonic operators c^* and c . For their definition we use a patch decomposition of a shell around the Fermi surface. The general requirements for that decomposition are described in the following; an example for the construction of such a patch decomposition was given in [1]. We divide half of the Fermi surface $\partial B_F := \{k \in \mathbb{R}^3 : |k| = k_F\}$ into a number $M/2 \in \mathbb{N}$ of patches \tilde{P}_α , each of surface area $\sigma(\tilde{P}_\alpha) = 4\pi k_F^2/M$. The number of patches M is a parameter that will eventually be chosen as a function of the particle number N , subject to the constraint

$$N^{2\delta} \ll M \ll N^{\frac{2}{3}-2\delta}, \quad \text{where } 0 < \delta < \frac{1}{6}. \quad (2.5)$$

We assume that the patches do not degenerate into very long and narrow shapes as $N \rightarrow \infty$, or more precisely we assume that always

$$\text{diam}(\tilde{P}_\alpha) \leq CN^{\frac{1}{3}} M^{-\frac{1}{2}}. \quad (2.6)$$

Inside each \tilde{P}_α , we now choose a slightly smaller patch P_α such that the distance between two adjacent patches is at least $2R$, that is, twice the interaction radius. By radially extending P_α , we obtain the final patches B_α with thickness $2R$ (see Figure 1):

$$B_\alpha := \left\{ r q \in \mathbb{R}^3 : q \in P_\alpha, r \in \left[1 - \frac{R}{k_F}, 1 + \frac{R}{k_F} \right] \right\}. \quad (2.7)$$

The patches are separated by corridors wider than $2R$. To cover also the southern hemisphere, we define the patch $B_{\alpha+\frac{M}{2}}$ by applying the reflection $k \mapsto -k$ to B_α .

The center of patch B_α , a vector in \mathbb{R}^3 , will be denoted $\omega_\alpha \in P_\alpha$, with associated direction vector $\hat{\omega}_\alpha := \omega_\alpha/|\omega_\alpha|$. For any $k \in \mathbb{Z}^3 \cap B_R(0)$, we define the index sets

$$\begin{aligned} \mathcal{I}_k^+ &:= \{ \alpha \in \{1, \dots, M\} : k \cdot \hat{\omega}_\alpha \geq N^{-\delta} \}, \\ \mathcal{I}_k^- &:= \{ \alpha \in \{1, \dots, M\} : k \cdot \hat{\omega}_\alpha \leq -N^{-\delta} \}, \\ \mathcal{I}_k &:= \mathcal{I}_k^+ \cup \mathcal{I}_k^-. \end{aligned} \quad (2.8)$$

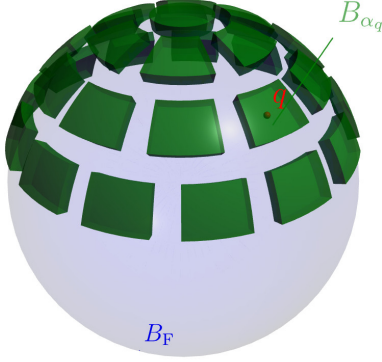


Figure 1: Patches on the Fermi ball in momentum space, with patch B_{α_q} including q .

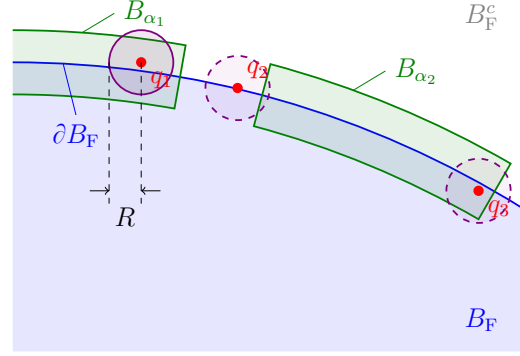


Figure 2: Close-up of a patch: in (2.16), q_1 is an included momentum, whereas q_2 and q_3 are excluded.

Visually speaking, \mathcal{I}_k excludes a belt of patches near the equator of the Fermi ball (if the direction of k is taken as north).

Generally, we assume all momenta k, p, h to be in \mathbb{Z}^3 and do not write this condition under summations. Moreover, we adopt the following convention: whenever a momentum is denoted by a lowercase p (“particle”), then we abbreviate the condition $p \in B_F^c \cap B_\alpha$ by $p : \alpha$ and say “ p is compatible with B_α ”. Likewise, for a momentum denoted by h (“hole”), the condition $h \in B_F \cap B_\alpha$ is abbreviated as $h : \alpha$. For $k \in \mathbb{Z}^3 \cap B_R(0)$ and $\alpha \in \mathcal{I}_k^+$, we now define the *particle–hole pair creation operator*

$$b_\alpha^*(k) := \frac{1}{n_{\alpha,k}} \sum_{p,h:\alpha} \delta_{p,h+k} a_p^* a_h^* = \frac{1}{n_{\alpha,k}} \sum_{\substack{p:p \in B_F^c \cap B_\alpha \\ p-k \in B_F \cap B_\alpha}} a_p^* a_{p-k}^* \quad (2.9)$$

with normalization constant $n_{\alpha,k}$ defined by

$$n_{\alpha,k}^2 := \sum_{p,h:\alpha} \delta_{p,h+k} = \sum_{\substack{p:p \in B_F^c \cap B_\alpha \\ p-k \in B_F \cap B_\alpha}} 1. \quad (2.10)$$

Observe that if $\alpha \notin \mathcal{I}_k^+$, then $b_\alpha^*(k)$ will usually be an empty sum, in which case we understand it as the zero operator. (Strictly speaking near the equator the sums may still contain a small number of summands. The index set \mathcal{I}_k^+ is defined such that they contain a number of summands that grows sufficiently fast as $N \rightarrow \infty$; this is quantified in Lemma 6.2.) Therefore we introduce the half-ball

$$\Gamma^{\text{nor}} := H^{\text{nor}} \cap \mathbb{Z}^3 \cap B_R(0) \quad (2.11)$$

with H^{nor} defined in (1.9), and then, for $k \in \Gamma^{\text{nor}}$, define

$$c_\alpha^*(k) := \begin{cases} b_\alpha^*(k) & \text{for } \alpha \in \mathcal{I}_k^+ \\ b_\alpha^*(-k) & \text{for } \alpha \in \mathcal{I}_k^- \end{cases}. \quad (2.12)$$

In Lemma 4.3 we are going to show that these pair operators satisfy approximately the canonical commutation relations (CCR) of bosons

$$[c_\alpha(k), c_\beta(\ell)] = 0, \quad [c_\alpha(k), c_\beta^*(\ell)] \approx \delta_{\alpha,\beta} \delta_{k,\ell}.$$

The almost-bosonic Bogoliubov transformation The almost-bosonic Bogoliubov transformation is the unitary operator on fermionic Fock space $T : \mathcal{F} \rightarrow \mathcal{F}$ chosen such that it would diagonalize an effective quadratic Hamiltonian

$$h_{\text{eff}} = \sum_{k \in \Gamma^{\text{nor}}} h_{\text{eff}}(k)$$

where

$$h_{\text{eff}}(k) = \sum_{\alpha, \beta \in \mathcal{I}_k} \left((D(k) + W(k))_{\alpha, \beta} c_\alpha^*(k) c_\beta(k) + \frac{1}{2} \widetilde{W}(k)_{\alpha, \beta} (c_\alpha^*(k) c_\beta^*(k) + c_\beta(k) c_\alpha(k)) \right),$$

if the c^* and c operators would exactly satisfy the CCR of bosons (see [2, (1.47)]). Here, $D(k)$, $W(k)$, and $\widetilde{W}(k)$ are symmetric matrices in $\mathbb{R}^{|\mathcal{I}_k| \times |\mathcal{I}_k|}$ given in block form

$$D(k) = \begin{pmatrix} d(k) & 0 \\ 0 & d(k) \end{pmatrix}, \quad W(k) = \begin{pmatrix} b(k) & 0 \\ 0 & b(k) \end{pmatrix}, \quad \widetilde{W}(k) = \begin{pmatrix} 0 & b(k) \\ b(k) & 0 \end{pmatrix},$$

with the smaller matrices $d(k)$ and $b(k)$ in $\mathbb{R}^{|\mathcal{I}_k^+| \times |\mathcal{I}_k^+|}$ given by

$$d(k) := \sum_{\alpha \in \mathcal{I}_k^+} |\hat{k} \cdot \hat{\omega}_\alpha| |\alpha\rangle\langle\alpha|, \quad b(k) := \sum_{\alpha, \beta \in \mathcal{I}_k^+} \frac{\hat{V}_k}{2\hbar\kappa N|k|} n_{\alpha,k} n_{\beta,k} |\alpha\rangle\langle\beta|, \quad (2.13)$$

where $|\alpha\rangle$ is the α -th canonical basis vector of $\mathbb{R}^{|\mathcal{I}_k^+|}$ (and $\hat{k} := k/|k|$). We are going to define T by the explicit formula which was given in [1, 2, 4] as

$$T := e^{-S} \quad S := -\frac{1}{2} \sum_{k \in \Gamma^{\text{nor}}} \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} (c_\alpha^*(k) c_\beta^*(k) - \text{h.c.}). \quad (2.14)$$

The operator S is anti-self-adjoint (i.e., $S^* = -S$) and the matrix $K(k) \in \mathbb{R}^{|\mathcal{I}_k| \times |\mathcal{I}_k|}$ is defined via

$$K(k) := \log |S_1(k)^T|, \quad (2.15)$$

$$S_1(k) := (D(k) + W(k) - \widetilde{W}(k))^{\frac{1}{2}} E(k)^{-\frac{1}{2}},$$

$$E(k) := \left((D(k) + W(k) - \widetilde{W}(k))^{\frac{1}{2}} (D(k) + W(k) + \widetilde{W}(k)) (D(k) + W(k) - \widetilde{W}(k))^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

This concludes the construction of the trial state.

2.1 Optimality of the Main Result

The upper bound (1.12) in our main result is sharp if $q \in \mathcal{Q}_\epsilon$ (compare (1.11)) is located in the interior of a patch. The precise statement is as follows.

Proposition 2.1 (Optimality). *Under the assumptions of Theorem 1.1, whenever $q \in \mathcal{Q}_\epsilon$ is in the interior of a patch B_{α_q} in the sense that*

$$\begin{aligned} \text{for } q \in B_F^c \quad & \text{we have } B_R(q) \cap B_F \subset B_{\alpha_q} , \\ \text{for } q \in B_F \quad & \text{we have } B_R(q) \cap B_F^c \subset B_{\alpha_q} , \end{aligned} \quad (2.16)$$

(this is represented in Fig. 2) the upper bound (1.12) becomes an equality:

$$n_q = N^{-\frac{2}{3}} \sum_{k \in \mathcal{C}^q \cap \mathbb{Z}^3} \frac{\hat{V}_k}{2\kappa|k|} \frac{1}{\pi} \int_0^\infty \frac{(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu + \mathcal{E} , \quad (2.17)$$

where the error term \mathcal{E} is bounded as in (1.14).

The proof is given in the end of Section 10.

Since $Q_k^{(0)} \geq 0$, the integral in the leading order term is bounded by

$$\begin{aligned} \left| \int_0^\infty \frac{(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu \right| & \leq \int_0^{\lambda_{q,k}} \frac{\mu^2 - \lambda_{q,k}^2}{(\mu^2 + \lambda_{q,k}^2)^2} d\mu + \int_{\lambda_{q,k}}^\infty \frac{\lambda_{q,k}^2 - \mu^2}{(\mu^2 + \lambda_{q,k}^2)^2} d\mu \\ & = \left[\frac{\mu}{\mu^2 + \lambda_{q,k}^2} \right]_{\mu=0}^{\lambda_{q,k}} + \left[-\frac{\mu}{\mu^2 + \lambda_{q,k}^2} \right]_{\mu=\lambda_{q,k}}^\infty = \lambda_{q,k}^{-1} \leq \epsilon^{-1} . \end{aligned} \quad (2.18)$$

So for $q \in \mathcal{Q}_\epsilon$, the leading order term is $\mathcal{O}(N^{-\frac{2}{3}})$, which we believe to be optimal. Further, the upper bound in (2.17) is trivially optimal if q is at a distance larger than the range of the potential from the Fermi surface, i. e., if $||q| - k_F| > R$, because then both sides of the equality vanish.

3 Strategy of Proof of the Main Theorem

The statement (1.10) in Theorem 1.1, that ψ_N replicates the ground state energy, was proven in [1, 2, 4]. So we have to establish the formula (1.12) for the momentum distribution. This is done in three steps.

Step 1: We employ a bosonization technique as in [2] in order to show that n_q is approximately given by a bosonized approximation $n_q^{(b)}$. In fact, n_q vanishes whenever q is not inside some patch B_{α_q} , so we will set $n_q^{(b)} = 0$ in that case. For all other q , we will see that n_q amounts to a sum over contributions depending on the momentum exchange k .

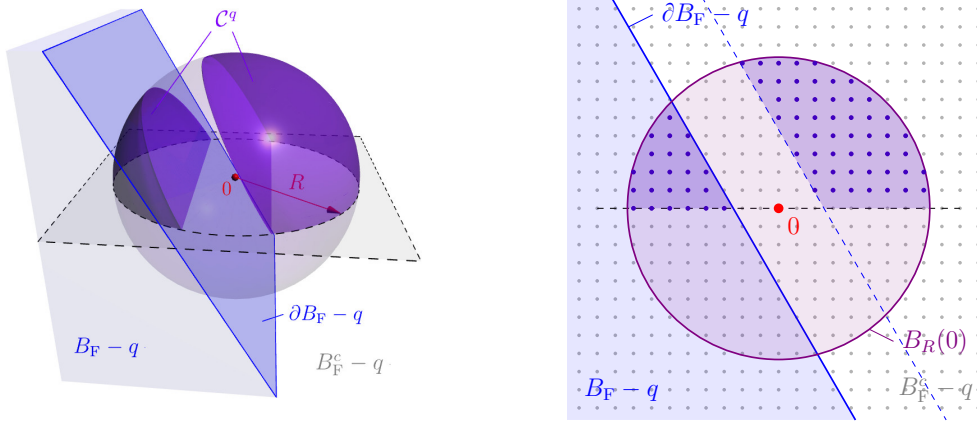


Figure 3: **Left:** The set \mathcal{C}^q consists of two parts which are cut off by the displaced Fermi surfaces $\partial B_F - q$ and $\partial B_F + q$. Since the figure depicts the local situation around a point of the very large Fermi surface, the piece of the Fermi surface (in blue) appears flat. **Right:** The relevant sum over k runs over the lattice points in \mathcal{C}^q .

The first commutator for the evaluation of this contribution is computed in Lemma 4.2 and vanishes whenever k is not inside the set

$$\tilde{\mathcal{C}}^q := \begin{cases} B_R(0) \cap H^{\text{nor}} \cap (((B_F \cap B_{\alpha_q}) - q) \cup (-(B_F \cap B_{\alpha_q}) + q)) \\ \quad \cap \{k : |k \cdot \hat{\omega}_{\alpha_q}| \geq N^{-\delta}\} & \text{for } q \in B_F^c \\ B_R(0) \cap H^{\text{nor}} \cap (((B_F^c \cap B_{\alpha_q}) - q) \cup (-(B_F^c \cap B_{\alpha_q}) + q)) \\ \quad \cap \{k : |k \cdot \hat{\omega}_{\alpha_q}| \geq N^{-\delta}\} & \text{for } q \in B_F. \end{cases} \quad (3.1)$$

The reason for using $\tilde{\mathcal{C}}^q$ is explained after Lemma 4.2. Note that $\tilde{\mathcal{C}}^q$ agrees with \mathcal{C}^q , defined in (1.8) and depicted in Fig. 3, up to the exclusion of $|k \cdot \hat{\omega}_{\alpha_q}| < N^{-\delta}$ and the compatibility restriction with B_{α_q} .

We then find the approximate momentum distribution in the exactly bosonic approximation to be

$$n_q^{(b)} := \frac{1}{2} \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{n_{\alpha_q, k}^2} (\cosh(2K(k)) - 1)_{\alpha_q, \alpha_q}. \quad (3.2)$$

The final closeness statement $n_q \approx n_q^{(b)}$ is given in the next theorem. Its proof can be found in Section 8.

Theorem 3.1 (Bosonized Momentum Distribution). *Assume that \hat{V} is compactly supported and non-negative. Then for every $\varepsilon > 0$ there exists some $\tilde{C}_\varepsilon > 0$ such that*

$$|n_q - n_q^{(b)}| \leq C \|\hat{V}\|_1 e^{\tilde{C}_\varepsilon \|\hat{V}\|_1} N^{-\frac{5}{6} + \frac{5}{4}\delta + \varepsilon}. \quad (3.3)$$

Step 2: We evaluate the hyperbolic cosine of the matrix $K(k)$ in (3.2) by functional calculus, which brings us close to the final form (1.12). The computation is given in Section 9, the result is the next proposition.

Proposition 3.2. *If $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$, then*

$$n_q^{(b)} = \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \frac{\hat{V}_k}{2\hbar\kappa N|k|} \int_0^\infty \frac{(\mu^2 - \lambda_{\alpha_q}^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + Q_k(\mu)} d\mu \quad (3.4)$$

with

$$\lambda_\alpha := |\hat{k} \cdot \hat{\omega}_\alpha|, \quad Q_k(\mu) := \frac{\hat{V}_k}{\hbar\kappa N|k|} \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha,k}^2 (\mu^2 + \lambda_\alpha^2)^{-1} \lambda_\alpha. \quad (3.5)$$

Step 3: We approximate the sum within $Q_k(\mu)$ by a surface integral over the half-sphere, which renders $Q_k(\mu) \approx Q_k^{(0)}(\mu)$. We arrive at a formula resembling (1.12), but with $\tilde{\mathcal{C}}^q$ instead of \mathcal{C}^q . Eq. (1.12) then follows by bounding the contributions from $k \in (\tilde{\mathcal{C}}^q \setminus \mathcal{C}^q) \cap \mathbb{Z}^3$. The computations are done in Section 10.

The innovative part is Step 1, its idea being the following: We evaluate n_q in the trial state $\psi_N = R e^{-S} \Omega$ as a Lie–Schwinger series

$$n_q = \langle \Omega, e^S a_q^* a_q e^{-S} \Omega \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Omega, \text{ad}_S^n(a_q^* a_q) \Omega \rangle, \quad (3.6)$$

where $\text{ad}_A^n(B) := [A, \dots [A, [A, B]] \dots]$ denotes the n -fold commutator (for $n = 0$, we have $\text{ad}_A^0(B) = B$). When computing the first commutator, we find that

$$\text{ad}_S^1(a_q^* a_q) = [S, a_q^* a_q] \sim c^* c^* - cc,$$

where c^* stands for some generalized pair operator $c^*(g) = \sum_{p,h} g(p,h) a_p^* a_h^*$ containing a weight function $g : B_F^c \times B_F \rightarrow \mathbb{C}$.

In the bosonization approximation we assume that the operators c^*, c exactly satisfy canonical commutation relations. In this approximation, we denote the result of the multi-commutator evaluation as $\text{ad}_{q,(b)}^n$; precise definitions will be given in (5.2)–(5.4). When computing $\text{ad}_{q,(b)}^n$, we find that

- commuting $S \sim (c^* c^* - cc)$ with $B \sim (c^* c^* - cc)$ results in $[S, B] \sim (c^* c + \text{const})$,
- commuting $S \sim (c^* c^* - cc)$ with $B \sim (c^* c + \text{const})$ results in $[S, B] \sim (c^* c^* - cc)$.

So for $n \geq 1$, the form of $\text{ad}_{q,(b)}^n$ alternates:

$$\text{ad}_{q,(b)}^n \sim \begin{cases} c^* c^* - cc & \text{for } n \text{ odd} \\ c^* c + \text{const} & \text{for } n \text{ even} . \end{cases} \quad (3.7)$$

Computing the vacuum expectation value $\frac{1}{n!} \langle \Omega, \text{ad}_{q,(b)}^n \Omega \rangle$ will only give a nonzero contribution if $n \geq 2$ and n is even. These contributions sum up to a cosh-series and render $n_q^{(b)}$ in (3.2).

The proof of Theorem 3.1 then amounts to showing that the error in this bosonization approximation is small. This will be done by a bootstrap bound: Applying Duhamel's formula, we may express the error $|n_q - n_q^{(b)}|$ in terms of expectation values in the parametrized states

$$\xi_t := e^{-tS}\Omega, \quad t \in [-1, 1]. \quad (3.8)$$

Employing some lemmas from Section 6 we may bound these expectation values using $\langle \xi_t, a_q^* a_q \xi_t \rangle$ (see Section 7). The bootstrapping is then based on the “bootstrap Lemma” 8.1. Initially we know that $0 \leq \langle \xi_t, a_q^* a_q \xi_t \rangle \leq 1$, which we write with $r := 0$ as

$$\langle \xi_t, a_q^* a_q \xi_t \rangle = \mathcal{O}(N^{-r}). \quad (3.9)$$

Using this bound within Duhamel's formula, Lemma 8.1 provides us with

$$|n_q - n_q^{(b)}| = |\langle \xi_1, a_q^* a_q \xi_1 \rangle - n_q^{(b)}| = \mathcal{O}(N^{-r'}) \quad (3.10)$$

for some $r' > r$. The same bound holds if the trial state $Re^{-S}\Omega$ is replaced with $Re^{-tS}\Omega$. Together with the observation (8.8) that $n_q^{(b)} = \mathcal{O}(N^{-\frac{2}{3}+\delta})$ (and the same if the trial state is replaced by its t -dependent version), this implies again (3.9), but with an improved exponent $\min\{r', \frac{2}{3} - \delta\}$ in place of r . Iteration yields increasing exponents r' and r until $r = \frac{2}{3} - \delta$ is reached after a finite number of steps. In that case r' is the claimed error exponent from Theorem 3.1 and we have concluded Step 1.

4 Generalized Pair Operators

Recall the definition (2.12) of the bosonized pair creation operator $c_\alpha^*(k)$. It will be convenient to use more general pair operators, similar to the weighted pair operators from [4, Lemma 5.3]. For a function $g : B_F^c \times B_F \rightarrow \mathbb{C}$, we define

$$c^*(g) := \sum_{\substack{p \in B_F^c \\ h \in B_F}} g(p, h) a_p^* a_h^*, \quad c(g) := \sum_{\substack{p \in B_F^c \\ h \in B_F}} \overline{g(p, h)} a_h a_p. \quad (4.1)$$

We may then identify

$$c_\alpha^*(k) = c^*(d_{\alpha,k}), \quad \text{where} \quad d_{\alpha,k}(p, h) := \begin{cases} \delta_{p, h+k} \frac{1}{n_{\alpha,k}} \chi(p, h : \alpha) & \text{if } \alpha \in \mathcal{I}_k^+ \\ \delta_{p, h-k} \frac{1}{n_{\alpha,k}} \chi(p, h : \alpha) & \text{if } \alpha \in \mathcal{I}_k^- \end{cases} \quad (4.2)$$

and

$$\chi(p, h : \alpha) := \chi_{B_F^c \cap B_\alpha}(p) \chi_{B_F \cap B_\alpha}(h).$$

In the following, we will adopt the shorthand notation:

$$\pm k := \begin{cases} +k & \text{if } \alpha \in \mathcal{I}_k^+ \\ -k & \text{if } \alpha \in \mathcal{I}_k^- \end{cases}, \quad \mp k := \begin{cases} -k & \text{if } \alpha \in \mathcal{I}_k^+ \\ +k & \text{if } \alpha \in \mathcal{I}_k^- \end{cases}.$$

So

$$d_{\alpha,k}(p, h) = \delta_{p, h \pm k} \frac{1}{n_{\alpha,k}} \chi(p, h : \alpha) . \quad (4.3)$$

The generalized pair operators satisfy the following commutation relations.

Lemma 4.1 (Generalized approximate CCR). *Consider $g, \tilde{g} : B_F^c \times B_F \rightarrow \mathbb{C}$. Then*

$$[c(g), c(\tilde{g})] = [c^*(g), c^*(\tilde{g})] = 0 \quad (4.4)$$

and

$$[c(g), c^*(\tilde{g})] = \langle g, \tilde{g} \rangle + \mathcal{E}(g, \tilde{g}) , \quad (4.5)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\ell^2(B_F^c \times B_F)$, and where

$$\mathcal{E}(g, \tilde{g}) := - \sum_{\substack{p \in B_F^c \\ h_1, h_2 \in B_F}} \overline{g(p, h_1)} \tilde{g}(p, h_2) a_{h_2}^* a_{h_1} - \sum_{\substack{p_1, p_2 \in B_F^c \\ h \in B_F}} \overline{g(p_1, h)} \tilde{g}(p_2, h) a_{p_2}^* a_{p_1} . \quad (4.6)$$

Proof. Direct computation using the CAR (2.1). \square

The generalized pair operators are convenient when evaluating the first commutator $\text{ad}_S^1(a_q^* a_q) = [S, a_q^* a_q]$, which involves $[c_\alpha^*(k), a_q^* a_q]$.

Lemma 4.2 (Occupation number of a single mode in $c_\alpha^*(k)$). *Let $k \in \Gamma^{\text{nor}}$, $\alpha \in \mathcal{I}_k$ and $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$. Then, we have*

$$[c_\alpha^*(k), a_q^* a_q] = \begin{cases} 0 & \text{if } \alpha \neq \alpha_q \\ -c^*(g_{q,k}) & \text{if } \alpha = \alpha_q \end{cases}$$

with

$$g_{q,k}(p, h) := \delta_{p, h \pm k} \frac{1}{n_{\alpha_q, k}} \chi(p, h : \alpha_q) (\delta_{h, q} + \delta_{p, q}) . \quad (4.7)$$

In particular, $[c_\alpha^*(k), a_q^* a_q] = 0$ whenever $\alpha_q \notin \mathcal{I}_k$.

Proof. Direct computation using the CAR (2.1). \square

Note that Lemma 4.2 motivates the definition of $\tilde{\mathcal{C}}^q$ in (3.1), which is chosen such that $[c_\alpha^*(k), a_q^* a_q]$ does not vanish:

- For $k \notin \Gamma^{\text{nor}} = H^{\text{nor}} \cap \mathbb{Z}^3 \cap B_R(0)$, this commutator would not even be defined.
- The condition $|k \cdot \hat{\omega}_{\alpha_q}| \geq N^{-\delta}$ ensures $\alpha_q \in \mathcal{I}_k$ as otherwise $[c_\alpha^*(k), a_q^* a_q] = 0$.
- And finally, $k \in ((B_F^{(c)} \cap B_{\alpha_q}) - q) \cup (-(B_F^{(c)} \cap B_{\alpha_q}) + q)$ guarantees that the factor $\chi(p, h : \alpha_q)$ in $g_{q,k}(p, h)$ does not vanish. Here, $B_F^{(c)}$ is either B_F^c or B_F depending on whether $q \in B_F$ or $q \in B_F^c$.

The simplest case of Lemma 4.1 are the approximate canonical commutation relations introduced in [1, Lemma 4.1].

Lemma 4.3 (Approximate CCR for c^*, c). *Let $k, \ell \in \Gamma^{\text{nor}}$ and $\alpha \in \mathcal{I}_k, \beta \in \mathcal{I}_\ell$. Then, we have*

$$[c_\alpha(k), c_\beta^*(\ell)] = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \delta_{k,\ell} + \mathcal{E}_\alpha(k, \ell) & \text{if } \alpha = \beta \end{cases} \quad (4.8)$$

with³

$$\mathcal{E}_\alpha(k, \ell) := - \sum_{p, h_1, h_2: \alpha} \frac{\delta_{h_1, p \mp k} \delta_{h_2, p \mp \ell}}{n_{\alpha, k} n_{\alpha, \ell}} a_{h_2}^* a_{h_1} - \sum_{p_1, p_2, h: \alpha} \frac{\delta_{h, p_1 \mp k} \delta_{h, p_2 \mp \ell}}{n_{\alpha, k} n_{\alpha, \ell}} a_{p_2}^* a_{p_1}, \quad (4.9)$$

where $\mathcal{E}_\alpha(k, \ell)^* = \mathcal{E}_\alpha(\ell, k)$.

Proof. Follows from (4.2) with $g = d_{\alpha, k}$ and $\tilde{g} = d_{\beta, \ell}$. \square

To compute the iterated commutators $\text{ad}_S^{n+1}(a_q^* a_q) = [S, \text{ad}_S^n(a_q^* a_q)]$ the following commutator is useful.

Lemma 4.4. *Let $k, \ell \in \Gamma^{\text{nor}}$, $\alpha \in \mathcal{I}_k$, $\alpha_q \in \mathcal{I}_\ell$, $q \in B_{\alpha_q}$ and $g_{q, \ell}$ as in (4.7). Then,*

$$[c_\alpha(k), c^*(g_{q, \ell})] = \begin{cases} 0 & \text{if } \alpha \neq \alpha_q \\ \delta_{k, \ell} \rho_{q, k} + \mathcal{E}_q^{(g)}(k, \ell) & \text{if } \alpha = \alpha_q \end{cases}. \quad (4.10)$$

with

$$\rho_{q, k} := \begin{cases} \frac{1}{n_{\alpha_q, k}^2} \chi(q \mp k \in B_F \cap B_{\alpha_q}) & \text{if } q \in B_F^c \\ \frac{1}{n_{\alpha_q, k}^2} \chi(q \pm k \in B_F^c \cap B_{\alpha_q}) & \text{if } q \in B_F \end{cases} \quad (4.11)$$

and

$$\mathcal{E}_q^{(g)}(k, \ell) := \begin{cases} -\frac{1}{n_{\alpha_q, k} n_{\alpha_q, \ell}} \left(\chi\left(\begin{smallmatrix} q \mp k \in B_F \cap B_{\alpha_q} \\ q \mp \ell \in B_F \cap B_{\alpha_q} \end{smallmatrix}\right) a_{q \mp \ell}^* a_{q \mp k} - \chi\left(\begin{smallmatrix} q \mp \ell \in B_F \cap B_{\alpha_q} \\ q \mp \ell \pm k \in B_F^c \cap B_{\alpha_q} \end{smallmatrix}\right) a_q^* a_{q \mp \ell \pm k} \right) & \text{if } q \in B_F^c, \\ -\frac{1}{n_{\alpha_q, k} n_{\alpha_q, \ell}} \left(\chi\left(\begin{smallmatrix} q \mp \ell \in B_F^c \cap B_{\alpha_q} \\ q \pm \ell \mp k \in B_F \cap B_{\alpha_q} \end{smallmatrix}\right) a_q^* a_{q \pm \ell \mp k} - \chi\left(\begin{smallmatrix} q \pm \ell \in B_F^c \cap B_{\alpha_q} \\ q \pm k \in B_F^c \cap B_{\alpha_q} \end{smallmatrix}\right) a_{q \pm \ell}^* a_{q \pm k} \right) & \text{if } q \in B_F. \end{cases} \quad (4.12)$$

Note also that we have $\mathcal{E}_q^{(g)}(k, k)^* = \mathcal{E}_q^{(g)}(k, k)$.

³Note that $\mathcal{E}_\alpha(k, \ell)$ might be ill-defined for $\alpha \neq \beta$, as for $\alpha \notin \mathcal{I}_\ell$, the denominator $n_{\alpha, \ell}$ might become 0.

5 Momentum Distribution from Bosonization

Recall that the exact momentum distribution, according to (3.6), is given by

$$n_q = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Omega, \text{ad}_S^n(a_q^* a_q) \Omega \rangle . \quad (5.1)$$

The evaluation of multi-commutators $\text{ad}_S^n(a_q^* a_q)$ results in a rather involved expression. However, the dominant contribution is obtained pretending that bosonization was exact, i.e., if we drop the term $\mathcal{E}_\alpha(k, \ell)$ in the approximate CCR (4.8). The exactly bosonic computation may be found in Lemma A.1.

For $n = 0$, we choose $\text{ad}_{q,(b)}^0 = a_q^* a_q = \text{ad}_S^0(a_q^* a_q)$, so the bosonization approximation is exact.

For $n \geq 1$, the bosonized multi-commutator $\text{ad}_{q,(b)}^n$ is expressed in contributions of six different terms (**A**, **B**, **C**, **D**, **E** and **F**). If $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$, we define

$$\text{ad}_{q,(b)}^n := \begin{cases} 2^{n-1} \mathbf{A}_n + \mathbf{B}_n + \mathbf{B}_n^* + \sum_{m=1}^{n-1} \binom{n}{m} \mathbf{C}_{n-m,m} & \text{if } n \text{ is even} \\ \mathbf{E}_n + \mathbf{E}_n^* + \sum_{m=1}^s \binom{n}{m} \mathbf{D}_{n-m,m} + \sum_{m=1}^s \binom{n}{m} \mathbf{F}_{m,n-m} & \text{if } n \text{ is odd,} \\ & n = 2s + 1. \end{cases} \quad (5.2)$$

If q is not inside any patch we set $\text{ad}_{q,(b)}^n := 0$. With $\rho_{q,k}$ from (4.11) the terms are

$$\begin{aligned} \mathbf{A}_n &:= \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} (K(k)^n)_{\alpha_q, \alpha_q} \rho_{q,k} \in \mathbb{C} , \\ \mathbf{B}_n &:= \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} c^*(g_{q,k}) c_{\alpha_1}(k) , \\ \mathbf{C}_{m,m'} &:= \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \sum_{\alpha_1, \alpha_2 \in \mathcal{I}_k} (K(k)^m)_{\alpha_q, \alpha_1} (K(k)^{m'})_{\alpha_q, \alpha_2} \rho_{q,k} c_{\alpha_1}^*(k) c_{\alpha_2}(k) , \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \mathbf{D}_{m,m'} &:= \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \sum_{\alpha_1, \alpha_2 \in \mathcal{I}_k} (K(k)^m)_{\alpha_q, \alpha_1} (K(k)^{m'})_{\alpha_q, \alpha_2} \rho_{q,k} c_{\alpha_1}(k) c_{\alpha_2}(k) , \\ \mathbf{E}_n &:= \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} c^*(g_{q,k}) c_{\alpha_1}^*(k) , \\ \mathbf{F}_{m,m'} &:= \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \sum_{\alpha_1, \alpha_2 \in \mathcal{I}_k} (K(k)^m)_{\alpha_q, \alpha_1} (K(k)^{m'})_{\alpha_q, \alpha_2} \rho_{q,k} c_{\alpha_1}^*(k) c_{\alpha_2}^*(k) . \end{aligned} \quad (5.4)$$

Since $\rho_{q,k}$ is real, and since $K(k)$ is a real and symmetric matrix, we have

$$\mathbf{A}_n^* = \mathbf{A}_n , \quad \mathbf{C}_{m,m'}^* = \mathbf{C}_{m',m} , \quad (5.5)$$

$$\mathbf{D}_{m,m'} = \mathbf{D}_{m',m} , \quad \mathbf{F}_{m,m'} = \mathbf{F}_{m',m} , \quad \mathbf{D}_{m,m'}^* = \mathbf{F}_{m',m} . \quad (5.6)$$

Replacing $\text{ad}_S^n(a_q^* a_q)$ by $\text{ad}_{q,(b)}^n$ in (5.1) yields the bosonization approximation $n_q^{(b)}$: If $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$, then⁴

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Omega, \text{ad}_{q,(b)}^n \Omega \rangle &= \sum_{m=1}^{\infty} \frac{2^{2m-1}}{(2m)!} \langle \Omega, \mathbf{A}_{2m} \Omega \rangle = \sum_{m=1}^{\infty} \frac{2^{2m-1}}{(2m)!} \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} (K(k)^{2m})_{\alpha_q, \alpha_q} \rho_{q,k} \\ &= \frac{1}{2} \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{n_{\alpha_q, k}^2} (\cosh(2K(k)) - 1)_{\alpha_q, \alpha_q} = n_q^{(b)} . \end{aligned} \quad (5.7)$$

Otherwise, $\text{ad}_{q,(b)}^n = n_q^{(b)} = 0$. Both is in agreement with (3.2).

6 Controlling the Bosonization Error

In this section we compile the basic estimates required to control the bosonization.

Lemma 6.1 (Bound on Powers of K). *Suppose that \hat{V} is non-negative. Then there is $C > 0$ such that for all $k \in \Gamma^{\text{nor}}$, $\alpha, \beta \in \mathcal{I}_k$ and $n \in \mathbb{N}$, we have*

$$|(K(k)^n)_{\alpha, \beta}| \leq (C\hat{V}_k)^n M^{-1} . \quad (6.1)$$

Proof. This follows easily using [4, Lemma 7.1], which states that

$$|K(k)_{\alpha, \beta}| \leq C\hat{V}_k M^{-1} . \quad \square$$

The small quantity controlling the bosonization approximation is $n_{\alpha, k}^{-1}$.

Lemma 6.2 (Bounds on $n_{\alpha, k}$). *For all $k \in \Gamma^{\text{nor}}$ and $\alpha \in \mathcal{I}_k$ we have*

$$n_{\alpha, k} \geq C\mathbf{n} , \quad \text{for} \quad \mathbf{n} := N^{\frac{1}{3} - \frac{\delta}{2}} M^{-\frac{1}{2}} . \quad (6.2)$$

Proof. This is just [1, Eq. (3.18)]. (In [1] it is assumed that $M = N^{\frac{1}{3} + \varepsilon}$ with $\varepsilon > 0$. We only assume $M \gg N^{2\delta}$ but the proof of this lemma is nevertheless true.) \square

By (2.5) we conclude that $\mathbf{n} \rightarrow \infty$ as $N \rightarrow \infty$ at least as fast as $\mathbf{n} \geq CN^{\frac{\delta}{2}}$.

Next, we compile estimates on the bosonization errors $\mathcal{E}_\alpha(k, \ell)$ (4.9) and $\mathcal{E}_q^{(g)}(k, \ell)$ (4.12), which are partly based on [1, Lemma 4.1].

⁴Note that $k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3$ automatically enforces $\alpha_q \in \mathcal{I}_k$, so the denominator $n_{\alpha_q, k}^2$ does not vanish.

Lemma 6.3 (Bounds on \mathcal{E}_α and $\mathcal{E}^{(g)}$). *Let $k, \ell \in \Gamma^{\text{nor}}$, $\alpha, \alpha_q \in \mathcal{I}_k \cap \mathcal{I}_\ell$, and $q \in B_{\alpha_q}$. Then for all $\psi \in \mathcal{F}$ we have, for all choices of $\sharp \in \{\cdot, *\}^5$,*

$$\|\mathcal{E}_\alpha(k, \ell)^\sharp \psi\| \leq \frac{2}{n_{\alpha, k} n_{\alpha, \ell}} \|\mathcal{N} \psi\|, \quad \|\mathcal{E}_q^{(g)}(k, \ell)^\sharp \psi\| \leq \frac{2}{n_{\alpha_q, k} n_{\alpha_q, \ell}} \|\psi\|, \quad (6.3)$$

and

$$\sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \|\mathcal{E}_\beta(k, \ell)^\sharp \psi\|^2 \leq \frac{C}{n^2} \|\mathcal{N}^{\frac{1}{2}} \psi\|^2. \quad (6.4)$$

Note that $\mathcal{E}_q^{(g)}(k, \ell)^\sharp$ satisfies a sharper bound than $\mathcal{E}_\alpha(k, \ell)^\sharp$, not requiring the number operator \mathcal{N} on the r. h. s. because (4.12) does not contain any sum.

Proof. The first bound in (6.3) was already given in [1, Lemma 4.1] for $\mathcal{E}_\alpha(k, \ell)$. The statement then follows for $\mathcal{E}_\alpha(k, \ell)^* = \mathcal{E}_\alpha(\ell, k)$.

For the bound on $\mathcal{E}_q^{(g)}(k, \ell)^\sharp$ in (6.3), recall the definition (4.12) of $\mathcal{E}_q^{(g)}(k, \ell)$ and use the operator norm bound $\|a_{q'}^\sharp\|_{\text{op}} \leq 1$.

It remains to establish (6.4). Eq. (4.9) renders

$$\begin{aligned} & \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \|\mathcal{E}_\beta(k, \ell) \psi\|^2 \\ & \leq \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \frac{2}{n_{\beta, k}^2 n_{\beta, \ell}^2} \left(\left\| \sum_{p: \beta} \chi_{\left(\begin{smallmatrix} p \mp k \in B_F \cap B_\beta \\ p \mp \ell \in B_F \cap B_\beta \end{smallmatrix} \right)} a_{p \mp \ell}^* a_{p \mp k} \psi \right\|^2 + \left\| \sum_{h: \beta} \chi_{\left(\begin{smallmatrix} h \pm k \in B_F^c \cap B_\beta \\ h \pm \ell \in B_F^c \cap B_\beta \end{smallmatrix} \right)} a_{h \pm \ell}^* a_{h \pm k} \psi \right\|^2 \right). \end{aligned} \quad (6.5)$$

Let us introduce the set $\mathcal{S}_\beta := \{p: \beta \mid p \mp k \in B_F \cap B_\beta \text{ and } p \mp \ell \in B_F \cap B_\beta\}$. The first term in (6.5) then becomes

$$\begin{aligned} & \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \frac{2}{n_{\beta, k}^2 n_{\beta, \ell}^2} \left\| \sum_{p \in \mathcal{S}_\beta} a_{p \mp \ell}^* a_{p \mp k} \psi \right\|^2 \leq \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \frac{2}{n_{\beta, k}^2 n_{\beta, \ell}^2} \left(\sum_{p \in \mathcal{S}_\beta} \|a_{p \mp \ell}^* a_{p \mp k} \psi\| \right)^2 \\ & \leq \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \frac{2}{n_{\beta, k}^2 n_{\beta, \ell}^2} \left(\sum_{p \in \mathcal{S}_\beta} 1 \right) \left(\sum_{p \in \mathcal{S}_\beta} \|a_{p \mp \ell}^* a_{p \mp k} \psi\|^2 \right) \leq \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \frac{2}{n_{\beta, k} n_{\beta, \ell}} \sum_{p \in \mathcal{S}_\beta} \|a_{p \mp \ell}^* a_{p \mp k} \psi\|^2 \\ & \leq \frac{C}{n^2} \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \sum_{p \in \mathcal{S}_\beta} \|a_{p \mp k} \psi\|^2 \leq \frac{C}{n^2} \|\mathcal{N}^{\frac{1}{2}} \psi\|^2, \end{aligned}$$

where we used the Cauchy–Schwartz inequality, $\sum_{p \in \mathcal{S}_\beta} 1 \leq \min\{n_{\beta, k}^2, n_{\beta, \ell}^2\} \leq n_{\beta, k} n_{\beta, \ell}$ and $\|a_h^\sharp\|_{\text{op}} \leq 1$. The second term on the r. h. s. of (6.5) is bounded analogously and the proof for $\mathcal{E}_\beta(k, \ell)^* = \mathcal{E}_\beta(\ell, k)$ works the same way. \square

⁵The meaning of \sharp as “adjoint” or “non adjoint” may vary between every appearance of the symbol, even within the same formula; we mean that the statement holds for all possible combinations.

In the trial state $\xi_t \in \mathcal{F}$ from (3.8), the estimate (6.3) for $\mathcal{E}_q^{(g)}(k, \ell)$ is far from optimal: $\|a_{q'}^\sharp\|_{\text{op}} \leq 1$ means that we bound the q -mode as if it was fully occupied. We will later establish an “initial bootstrap bound” $\langle \xi_t, a_q^* a_q \xi_t \rangle = \mathcal{O}(N^{-r})$ with $r = 0$. Once this bound is established, we can iteratively apply the next lemma to improve the exponent up to $r = \frac{2}{3} - \delta$.

Lemma 6.4 (Bootstrap Bounds on $\mathcal{E}^{(g)}$). *Let $k, \ell \in \Gamma^{\text{nor}}$, $\alpha_q \in \mathcal{I}_k \cap \mathcal{I}_\ell$, and $q \in B_{\alpha_q}$. Assume that there is $r \geq 0$ such that for all $t \in [-1, 1]$ and all $q' \in \mathbb{Z}^3$ it is known that (with C independent of t, q')*

$$\langle \xi_t, a_{q'}^* a_{q'} \xi_t \rangle \leq CN^{-r}. \quad (6.6)$$

Then (with $\sharp \in \{\cdot, *\}$) for all $t \in [-1, 1]$ we have

$$\|\mathcal{E}_q^{(g)}(k, \ell)^\sharp \xi_t\| \leq \frac{C}{n_{\alpha_q, k} n_{\alpha_q, \ell}} N^{-\frac{r}{4}}. \quad (6.7)$$

Proof. From (4.12) we have

$$\|\mathcal{E}_q^{(g)}(k, \ell) \xi_t\| \leq \frac{1}{n_{\alpha_q, k} n_{\alpha_q, \ell}} (\|a_{q \mp \ell}^* a_{q \mp k} \xi_t\| + \|a_q^* a_{q \mp \ell \pm k} \xi_t\|). \quad (6.8)$$

Regarding the first norm on the r. h. s., for $k \notin \{\ell, -\ell\}$, the CAR and $\|a_q^\sharp\|_{\text{op}} \leq 1$ imply

$$\begin{aligned} \|a_{q \mp \ell}^* a_{q \mp k} \xi_t\|^2 &= \langle \xi_t, a_{q \mp k}^* a_{q \mp \ell} a_{q \mp \ell}^* a_{q \mp k} \xi_t \rangle \\ &\leq \|a_{q \mp k}^* a_{q \mp k} \xi_t\| \|a_{q \mp \ell} a_{q \mp \ell}^* \xi_t\| \leq \|a_{q \mp k}^* a_{q \mp k} \xi_t\|. \end{aligned} \quad (6.9)$$

We use $a_{q'}^* a_{q'} a_{q'}^* a_{q'} = a_{q'}^* a_{q'}$ and then apply the bootstrap assumption (6.6):

$$\|a_{q \mp k}^* a_{q \mp k} \xi_t\| \leq \langle \xi_t, a_{q \mp k}^* a_{q \mp k} a_{q \mp k}^* a_{q \mp k} \xi_t \rangle^{\frac{1}{2}} = \langle \xi_t, a_{q \mp k}^* a_{q \mp k} \xi_t \rangle^{\frac{1}{2}} \leq C^{\frac{1}{2}} N^{-\frac{r}{2}}. \quad (6.10)$$

So the first norm on the r. h. s. of (6.8) is $\leq C^{\frac{1}{4}} N^{-\frac{r}{4}}$. The bound for the second norm is obtained analogously.

In case $k \in \{\ell, -\ell\}$ we may skip (6.9), the rest of the proof works as before. \square

We will also need to bound the c^*, c -operators. For this purpose, we adopt the gapped number operator from [2, Eq. (5.6)]. (In the present paper we could generally use the standard number operator $\mathcal{N} = \sum_{q \in \mathbb{Z}^3} a_q^* a_q$ instead of the gapped number operator, but in view of future generalizations we go through the little extra work of getting bounds in terms of the gapped number operator.) With κ defined in (1.5) let

$$e(q) := |\hbar^2 |q|^2 - \kappa^2|$$

and define the δ -gap as the following shell around the Fermi sphere:

$$\mathcal{G} := \left\{ q \in \mathbb{Z}^3 : e(q) < \frac{1}{4} N^{-\frac{1}{3} - \delta} \right\}. \quad (6.11)$$

Then the *gapped number operator* is defined as

$$\mathcal{N}_\delta := \sum_{q \in \mathbb{Z}^3 \setminus \mathcal{G}} a_q^* a_q. \quad (6.12)$$

Lemma 6.5 (Bounds for c^*, c). *Let $k \in \Gamma^{\text{nor}}$ and consider a family of bounded functions $(g^{(\alpha)})_{\alpha \in \mathcal{I}_k}$ with $g^{(\alpha)} : B_F^c \times B_F \rightarrow \mathbb{R}$ such that*

$$\text{supp}(g^{(\alpha)}) \subseteq \{(p, h : \alpha) : p = h \pm k\} .$$

Then for all $f \in \ell^2(\mathcal{I}_k)$ we have

$$\begin{aligned} \left\| \sum_{\alpha \in \mathcal{I}_k} f_\alpha c^*(g^{(\alpha)}) \psi \right\| &\leq \|f\|_2 \max_{\alpha \in \mathcal{I}_k} (n_{\alpha, k} \|g^{(\alpha)}\|_\infty) \left\| (\mathcal{N}_\delta + 1)^{\frac{1}{2}} \psi \right\| , \\ \left\| \sum_{\alpha \in \mathcal{I}_k} f_\alpha c(g^{(\alpha)}) \psi \right\| &\leq \|f\|_2 \max_{\alpha \in \mathcal{I}_k} (n_{\alpha, k} \|g^{(\alpha)}\|_\infty) \|\mathcal{N}_\delta^{\frac{1}{2}} \psi\| . \end{aligned} \quad (6.13)$$

In particular

$$\left\| \sum_{\alpha \in \mathcal{I}_k} f_\alpha c_\alpha^*(k) \psi \right\| \leq \|f\|_2 \|(\mathcal{N}_\delta + 1)^{\frac{1}{2}} \psi\| , \quad \left\| \sum_{\alpha \in \mathcal{I}_k} f_\alpha c_\alpha(k) \psi \right\| \leq \|f\|_2 \|\mathcal{N}_\delta^{\frac{1}{2}} \psi\| . \quad (6.14)$$

Proof. For (6.13) see [2, Lemma 5.4] (note that in [2], a factor of $n_{\alpha, k}^{-1}$ is included in the definition of $c^\sharp(g)$ instead of the weight function g). The bounds (6.14) follow setting $g^{(\alpha)} = d_{\alpha, k}$, defined in (4.3). \square

Analogous bounds could be derived for the operators $c^\sharp(g_{q, k})$, which just differ from $c_{\alpha_q}^\sharp(k)$ by an additional factor of $(\delta_{q, p} + \delta_{q, h})$, see (4.7). However, these Kronecker deltas significantly reduce the number of summands, from $\sim n_{\alpha_q, k}$ to ~ 1 . Accordingly the following lemma provides sharper bounds on $c^\sharp(g_{q, k})$. In particular, applied to ξ_t one may achieve an even better bound depending on $\langle \xi_t, a_{q'}^* a_{q'} \xi_t \rangle$, which will also be used in the bootstrap argument.

Lemma 6.6 (Bootstrap Bounds on c^*, c). *Let $k \in \Gamma^{\text{nor}}$, $\alpha_q \in \mathcal{I}_k$, and $q \in B_{\alpha_q}$. Let $g_{q, k}$ be defined as in (4.7). Then for all $\psi \in \mathcal{F}$ and any choice $\sharp \in \{\cdot, *\}$ we have*

$$\|c^\sharp(g_{q, k}) \psi\| \leq \frac{1}{n_{\alpha_q, k}} . \quad (6.15)$$

Further, suppose there is $r \geq 0$ and $C > 0$ such that for all $t \in [-1, 1]$ and all $q' \in \mathbb{Z}^3$ we have

$$\langle \xi_t, a_{q'}^* a_{q'} \xi_t \rangle \leq C N^{-r} . \quad (6.16)$$

Then there exists $C > 0$ such that for all $t \in [-1, 1]$ we have

$$\|c(g_{q, k}) \xi_t\| \leq \frac{C}{n_{\alpha_q, k}} N^{-\frac{r}{2}} . \quad (6.17)$$

We caution the reader that a bound like (6.17) does not hold for $c^*(g_{q, k})$.

Proof. Statement (6.15) follows from definition (4.7) and $\|a_q^\sharp\|_{\text{op}} \leq 1$: For $q \in B_F^c$

$$\|c^*(g_{q,k})\xi_t\| \leq \frac{1}{n_{\alpha_q,k}} \|a_q^* a_{q\mp k}^* \xi_t\| \leq \frac{1}{n_{\alpha_q,k}} \|\xi_t\| = \frac{1}{n_{\alpha_q,k}}. \quad (6.18)$$

The same argument applies to $q \in B_F$ and $\|c(g_{q,k})\xi_t\|$.

Concerning the stronger bound (6.17), in case $q \in B_F^c$, we start again from

$$\|c(g_{q,k})\xi_t\| \leq \frac{1}{n_{\alpha_q,k}} \|a_{q\mp k} a_q \xi_t\| = \frac{1}{n_{\alpha,k}} \sqrt{\langle \xi_t, a_q^* a_{q\mp k}^* a_{q\mp k} a_q \xi_t \rangle}. \quad (6.19)$$

Using $a_{q\mp k}^* a_{q\mp k} \leq 1$ we get

$$\|c(g_{q,k})\xi_t\| \leq \frac{1}{n_{\alpha,k}} \sqrt{\langle \xi_t, a_q^* a_q \xi_t \rangle} \leq \frac{1}{n_{\alpha,k}} \sqrt{CN^{-r}}.$$

The case $q \in B_F$ is treated analogously. \square

Finally, we also need to bound combinations of c^* , c - and \mathcal{E}_α -operators.

Lemma 6.7 (Bounds on Combinations of c^* , c and \mathcal{E}_α). *Let $k, \ell \in \Gamma^{\text{nor}}$, $\alpha \in \mathcal{I}_\ell$, $\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell$ and suppose that \hat{V} is non-negative. Then there is $C > 0$ such that for all $n, m \in \mathbb{N}$ and all choices of $\sharp \in \{\cdot, *\}$ we have*

$$\begin{aligned} & \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha_q, \beta} (K(\ell)^m)_{\alpha, \beta} c_\alpha^\sharp(\ell) \mathcal{E}_\beta(k, \ell)^\sharp \psi \right\| \leq (C\hat{V}_k)^n (C\hat{V}_\ell)^m \mathbf{n}^{-1} M^{-1} \|(\mathcal{N} + 1)\psi\|, \\ & \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha_q, \beta} (K(\ell)^m)_{\alpha, \beta} \mathcal{E}_\beta(k, \ell)^\sharp c_\alpha^\sharp(\ell) \psi \right\| \leq (C\hat{V}_k)^n (C\hat{V}_\ell)^m \mathbf{n}^{-1} M^{-1} \|(\mathcal{N} + 1)\psi\|. \end{aligned} \quad (6.20)$$

Proof. To establish the first bound, we use $\mathcal{N}_\delta \leq \mathcal{N}$ and the fact that $\mathcal{E}_\beta(k, \ell)^\sharp$ commutes with \mathcal{N} :

$$\begin{aligned} & \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha_q, \beta} (K(\ell)^m)_{\alpha, \beta} c_\alpha^\sharp(\ell) \mathcal{E}_\beta(k, \ell)^\sharp \psi \right\| \\ & \leq \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} |(K(k)^n)_{\alpha_q, \beta}| \left\| \sum_{\alpha \in \mathcal{I}_\ell} (K(\ell)^m)_{\alpha, \beta} c_\alpha^\sharp(\ell) \mathcal{E}_\beta(k, \ell)^\sharp \psi \right\| \\ & \stackrel{(6.14)}{\leq} \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} |(K(k)^n)_{\alpha_q, \beta}| \left(\sum_{\alpha \in \mathcal{I}_\ell} |(K(\ell)^m)_{\alpha, \beta}|^2 \right)^{\frac{1}{2}} \|(\mathcal{N}_\delta + 1)^{\frac{1}{2}} \mathcal{E}_\beta(k, \ell)^\sharp \psi\| \\ & \stackrel{(6.1)}{\leq} (C\hat{V}_k)^n (C\hat{V}_\ell)^m M^{-\frac{2}{3}} \sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \|\mathcal{E}_\beta(k, \ell)^\sharp (\mathcal{N} + 1)^{\frac{1}{2}} \psi\| \\ & \leq (C\hat{V}_k)^n (C\hat{V}_\ell)^m M^{-\frac{2}{3}} \left(\sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} 1 \right)^{\frac{1}{2}} \left(\sum_{\beta \in \mathcal{I}_k \cap \mathcal{I}_\ell} \|\mathcal{E}_\beta(k, \ell)^\sharp (\mathcal{N} + 1)^{\frac{1}{2}} \psi\|^2 \right)^{\frac{1}{2}} \\ & \stackrel{(6.4)}{\leq} (C\hat{V}_k)^n (C\hat{V}_\ell)^m \mathbf{n}^{-1} M^{-1} \|(\mathcal{N} + 1)\psi\|. \end{aligned} \quad (6.21)$$

For the second line of (6.20), we start with

$$\begin{aligned}
& \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha q, \beta} (K(\ell)^m)_{\alpha, \beta} \mathcal{E}_\beta(k, \ell)^\# c_\alpha^\#(\ell) \psi \right\| \\
& \leq \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha q, \beta} (K(\ell)^m)_{\alpha, \beta} c_\alpha^\#(\ell) \mathcal{E}_\beta(k, \ell)^\# \psi \right\| \\
& \quad + \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha q, \beta} (K(\ell)^m)_{\alpha, \beta} [\mathcal{E}_\beta(k, \ell)^\#, c_\alpha^\#(\ell)] \psi \right\|.
\end{aligned} \tag{6.22}$$

The first term is bounded by (6.21). Within the second term, the commutator can be explicitly evaluated:

$$\begin{aligned}
& [\mathcal{E}_\beta(k, k'), c_\alpha(k'')] \\
& = -\frac{1}{n_{\alpha, k} n_{\alpha, k'} n_{\alpha, k''}} \left(\sum_{h: \alpha} f_{k, k', k''}^{(\alpha)}(h) a_{h \pm k''} a_{h \pm k' \mp k} - \sum_{p: \alpha} g_{k, k', k''}^{(\alpha)}(p) a_{p \mp k''} a_{p \mp k' \pm k} \right) \delta_{\alpha, \beta},
\end{aligned}$$

with

$$f_{k, k', k''}^{(\alpha)}(h) := \chi \left(\begin{array}{l} h \pm k'' \in B_\alpha \cap B_F^c \\ h \pm k' \in B_\alpha \cap B_F^c \\ h \pm k' \mp k \in B_\alpha \cap B_F \end{array} \right), \quad g_{k, k', k''}^{(\alpha)}(p) := \chi \left(\begin{array}{l} p \mp k'' \in B_\alpha \cap B_F \\ p \mp k' \in B_\alpha \cap B_F \\ p \mp k' \pm k \in B_\alpha \cap B_F^c \end{array} \right). \tag{6.23}$$

The three commutators for different choices of $\# \in \{\cdot, *\}$ can easily be deduced via $\mathcal{E}_\beta(k, \ell)^* = \mathcal{E}_\beta(\ell, k)$ and $[\mathcal{E}_\beta(k, \ell)^*, c_\alpha(\ell)^*] = -([\mathcal{E}_\beta(k, \ell), c_\alpha(\ell)])^*$. If in both instances $\# = *$ is chosen, we can bound

$$\begin{aligned}
& \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha q, \beta} (K(\ell)^m)_{\alpha, \beta} [\mathcal{E}_\beta(k, \ell)^*, c_\alpha^*(\ell)] \psi \right\| \\
& \leq \left\| \sum_{\alpha \in \mathcal{I}_k \cap \mathcal{I}_\ell} \frac{(K(k)^n)_{\alpha q, \alpha} (K(\ell)^m)_{\alpha, \alpha}}{n_{\alpha, k} n_{\alpha, \ell}^2} \left(\sum_{h: \alpha} f_{k, \ell, \ell}^{(\alpha)}(h) a_{h \pm \ell \mp k}^* a_{h \pm \ell}^* - \sum_{p: \alpha} g_{k, \ell, \ell}^{(\alpha)}(p) a_{p \mp \ell \pm k}^* a_{p \mp \ell}^* \right) \psi \right\|.
\end{aligned}$$

To control the $a^* a^*$ -term (or aa -term later on), we use the following elementary bounds for $f \in \ell^2(\mathbb{Z}^3 \times \mathbb{Z}^3)$, which are established as in [5, Lemma 3.1]

$$\begin{aligned}
& \left\| \sum_{k_1, k_2 \in \mathbb{Z}^3} f(k_1, k_2) a_{k_1}^* a_{k_2}^* \psi \right\| \leq 2 \|f\|_2 (\mathcal{N} + 1)^{\frac{1}{2}} \|\psi\|, \\
& \left\| \sum_{k_1, k_2 \in \mathbb{Z}^3} f(k_1, k_2) a_{k_1} a_{k_2} \psi \right\| \leq \|f\|_2 \mathcal{N}^{\frac{1}{2}} \|\psi\|.
\end{aligned} \tag{6.24}$$

Introducing the functions

$$\begin{aligned}
f_{k, k', k''}(h) &:= \sum_{\alpha \in \mathcal{I}_k \cap \mathcal{I}_\ell} \frac{\chi(h: \alpha)}{n_{\alpha, k} n_{\alpha, k'} n_{\alpha, k''}} (K(k)^n)_{\alpha q, \alpha} (K(k'')^m)_{\alpha, \alpha} f_{k, k', k''}^{(\alpha)}(h), \\
g_{k, k', k''}(p) &:= \sum_{\alpha \in \mathcal{I}_k \cap \mathcal{I}_\ell} \frac{\chi(p: \alpha)}{n_{\alpha, k} n_{\alpha, k'} n_{\alpha, k''}} (K(k)^n)_{\alpha q, \alpha} (K(k'')^m)_{\alpha, \alpha} g_{k, k', k''}^{(\alpha)}(p),
\end{aligned} \tag{6.25}$$

we may then estimate

$$\begin{aligned}
& \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha q, \beta} (K(\ell)^m)_{\alpha, \beta} [\mathcal{E}_\beta(k, \ell)^*, c_\alpha^*(\ell)] \psi \right\| \\
& \leq \left\| \sum_{h \in B_F} f_{k, \ell, \ell}(h) a_{h \pm \ell \mp k}^* a_{h \pm \ell}^* \psi \right\| + \left\| \sum_{p \in B_F^c} g_{k, \ell, \ell}(p) a_{p \mp \ell \pm k}^* a_{p \mp \ell}^* \psi \right\| \\
& \leq 2(\|f_{k, \ell, \ell}\|_2 + \|g_{k, \ell, \ell}\|_2) \|(\mathcal{N} + 1)^{\frac{1}{2}} \psi\|,
\end{aligned} \tag{6.26}$$

where in the last line we used (6.24) with $f(h \pm \ell \mp k, h \pm \ell) = f_{k, \ell, \ell}(h)$, so $\|f\|_2 = \|f_{k, \ell, \ell}\|_2$, and $f(p \mp \ell \pm k, p \mp \ell) = g_{k, \ell, \ell}(h)$, so $\|f\|_2 = \|g_{k, \ell, \ell}\|_2$. To estimate $\|f_{k, \ell, \ell}\|_2$, note that

$$|f_{k, \ell, \ell}^{(\alpha)}(h)| \leq 1 \quad \Rightarrow \quad |f_{k, \ell, \ell}(h)| \stackrel{(6.1)}{\leq} (C\hat{V}_k)^n (C\hat{V}_\ell)^m \mathbf{n}^{-3} M^{-2}. \tag{6.27}$$

Further, the support of $f_{k, \ell, \ell}$ only contains holes with a distance of $\leq R$ to the Fermi surface ∂B_F , whose surface area is $\sim N^{\frac{2}{3}}$. Thus,

$$\begin{aligned}
\|f_{k, \ell, \ell}\|_2 & \leq (C\hat{V}_k)^n (C\hat{V}_\ell)^m \mathbf{n}^{-3} M^{-2} |\text{supp}(f_{k, \ell, \ell})|^{\frac{1}{2}} \\
& \leq (C\hat{V}_k)^n (C\hat{V}_\ell)^m \mathbf{n}^{-3} M^{-2} N^{\frac{1}{3}} = (C\hat{V}_k)^n (C\hat{V}_\ell)^m \mathbf{n}^{-1} M^{-1} N^{-\frac{1}{3} + \delta}.
\end{aligned} \tag{6.28}$$

The same bound applies to $\|g_{k, \ell, \ell}\|_2$. Therefore, the second term on the r. h. s. of (6.22) for $\sharp = *$ is bounded by

$$\begin{aligned}
& \left\| \sum_{\substack{\alpha \in \mathcal{I}_\ell \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_\ell}} (K(k)^n)_{\alpha q, \beta} (K(\ell)^m)_{\alpha, \beta} [\mathcal{E}_\beta(k, \ell)^*, c_\alpha^*(\ell)] \psi \right\| \\
& \leq (C\hat{V}_k)^n (C\hat{V}_\ell)^m \mathbf{n}^{-1} M^{-1} N^{-\frac{1}{3} + \delta} \|(\mathcal{N} + 1)^{\frac{1}{2}} \psi\|.
\end{aligned} \tag{6.29}$$

Since $\delta < \frac{1}{6} < \frac{1}{3}$ and $(\mathcal{N} + 1)^{\frac{1}{2}} \leq (\mathcal{N} + 1)$, this is smaller than the required bound in the second line of (6.20). So we established the second line of (6.20) for $\sharp = *$ in both places. The bound for the three other choices of $\sharp \in \{\cdot, *\}$ is obtained analogously. \square

We need to show that the products of \mathcal{N}_δ and \mathcal{N} appearing in the previous lemma are essentially invariant under conjugation with the approximate Bogoliubov transformation e^{-S} introduced in (2.14). This is the content of the following lemma, which was partly already given in [2, Lemma 7.2].

Lemma 6.8 (Stability of Number Operators). *Assume that \hat{V} is compactly supported and non-negative. Then for all $m \in \mathbb{N}_0$ there exists $C_m > 0$ such that for all $t \in [-1, 1]$*

$$\begin{aligned}
e^{tS} (\mathcal{N} + 1)^m e^{-tS} & \leq e^{C_m |t| \|\hat{V}\|_1} (\mathcal{N} + 1)^m, \\
e^{tS} (\mathcal{N}_\delta + 1) (\mathcal{N} + 1)^m e^{-tS} & \leq e^{C_m |t| \|\hat{V}\|_1} (\mathcal{N}_\delta + 1) (\mathcal{N} + 1)^m, \\
e^{tS} (\mathcal{N}_\delta + 1)^m e^{-tS} & \leq e^{C_m |t| \|\hat{V}\|_1} (\mathcal{N}_\delta + 1)^m.
\end{aligned} \tag{6.30}$$

The proof is based on Grönwall's lemma and will be given further down since it employs Lemmas 6.9 and 6.10. Recall from (6.12) that \mathcal{N}_δ counts fermions outside the gap $\mathcal{G} = \{q \in \mathbb{Z}^3 : e(q) < \frac{1}{4}N^{-\frac{1}{3}-\delta}\}$. Let us write

$$\mathcal{G}^c := \mathbb{Z}^3 \setminus \mathcal{G}.$$

For $k \in \Gamma^{\text{nor}}$ and $\alpha \in \mathcal{I}_k$, we define the sequence $(g_j^{(\alpha,k)})_{j \in \mathbb{N}_0}$ of non-negative auxiliary weight functions

$$g_j^{(\alpha,k)}(p, h) := \delta_{p, h \pm k} \frac{\chi(p, h : \alpha)}{n_{\alpha,k}} (\chi_{\mathcal{G}^c}(p) + \chi_{\mathcal{G}^c}(h))^j. \quad (6.31)$$

Comparing with (4.2) and (4.3), note that $g_0^{(\alpha,k)} = d_{\alpha,k}$, which is just the weight function of $c_\alpha(k) = c(d_{\alpha,k})$.

Lemma 6.9. *Let $k \in \Gamma^{\text{nor}}$ and $\alpha \in \mathcal{I}_k$. Then, for any $j \in \mathbb{N}_0$ and $a \in \mathbb{R}$, we have*

$$[\mathcal{N}_\delta + a, c^*(g_j^{(\alpha,k)})] = c^*(g_{j+1}^{(\alpha,k)}). \quad (6.32)$$

Further, for all $m \in \mathbb{N}$,

$$\begin{aligned} & [(\mathcal{N}_\delta + a)^m, c^*(g_j^{(\alpha,k)})] \\ &= \sum_{\ell=1}^m (-1)^{\ell+1} \binom{m}{\ell} (\mathcal{N}_\delta + a)^{m-\ell} c^*(g_{j+\ell}^{(\alpha,k)}) = \sum_{\ell=1}^m \binom{m}{\ell} c^*(g_{j+\ell}^{(\alpha,k)}) (\mathcal{N}_\delta + a)^{m-\ell}. \end{aligned} \quad (6.33)$$

Proof. The first commutator can be computed directly to be

$$\begin{aligned} & [\mathcal{N}_\delta + a, c^*(g_j^{(\alpha,k)})] = [\mathcal{N}_\delta, c^*(g_j^{(\alpha,k)})] \\ &= \sum_{q \in \mathcal{G}^c} \sum_{p, h : \alpha} \frac{\delta_{p, h \pm k}}{n_{\alpha,k}} (\chi_{\mathcal{G}^c}(p) + \chi_{\mathcal{G}^c}(h))^j ([a_q^* a_q, a_p^*] a_h^* + a_p^* [a_q^* a_q, a_h^*]) \\ &= \sum_{q \in \mathbb{Z}^3} \sum_{p, h : \alpha} \frac{\delta_{p, h \pm k}}{n_{\alpha,k}} (\chi_{\mathcal{G}^c}(p) + \chi_{\mathcal{G}^c}(h))^j \chi_{\mathcal{G}^c}(q) (\delta_{q,p} a_p^* a_h^* + \delta_{q,h} a_p^* a_h^*) \\ &= \sum_{p, h : \alpha} \frac{\delta_{p, h \pm k}}{n_{\alpha,k}} (\chi_{\mathcal{G}^c}(p) + \chi_{\mathcal{G}^c}(h))^{j+1} a_p^* a_h^*. \end{aligned} \quad (6.34)$$

The second commutator can be proved by induction. For brevity we only write the case $a = 0$, the generalization to $a \neq 0$ is trivial. The case $m = 1$ is just (6.32). For the inductive step from $m - 1$ to m we have, with the help of the recursion relation for

binomial coefficients,

$$\begin{aligned}
[\mathcal{N}_\delta^m, c^*(g_j^{(\alpha,k)})] &= [\mathcal{N}_\delta^{m-1}, c^*(g_j^{(\alpha,k)})]\mathcal{N}_\delta + \mathcal{N}_\delta^{m-1}[\mathcal{N}_\delta, c^*(g_j^{(\alpha,k)})] \\
&= \sum_{\ell=1}^{m-1} (-1)^{\ell+1} \binom{m-1}{\ell} \mathcal{N}_\delta^{m-1-\ell} c^*(g_{j+\ell}^{(\alpha,k)}) \mathcal{N}_\delta + \mathcal{N}_\delta^{m-1} c^*(g_{j+1}^{(\alpha,k)}) \\
&= \sum_{\ell=1}^{m-1} (-1)^{\ell+1} \binom{m-1}{\ell} \mathcal{N}_\delta^{m-1-\ell} (\mathcal{N}_\delta c^*(g_{j+\ell}^{(\alpha,k)}) - c^*(g_{j+\ell+1}^{(\alpha,k)})) + \mathcal{N}_\delta^{m-1} c^*(g_{\ell+1}^{(\alpha,k)}) \\
&= \sum_{\ell=1}^{m-1} (-1)^{\ell+1} \binom{m-1}{\ell} \mathcal{N}_\delta^{m-\ell} c^*(g_{j+\ell}^{(\alpha,k)}) + \sum_{\ell=1}^m (-1)^{\ell+1} \binom{m-1}{\ell-1} \mathcal{N}_\delta^{m-\ell} c^*(g_{j+\ell}^{(\alpha,k)}) \quad (6.35) \\
&= \sum_{\ell=1}^{m-1} (-1)^{\ell+1} \binom{m}{\ell} \mathcal{N}_\delta^{m-\ell} c^*(g_{j+\ell}^{(\alpha,k)}) + (-1)^{m+1} \binom{m-1}{m-1} c^*(g_{j+m}^{(\alpha,k)}) \\
&= \sum_{\ell=1}^m (-1)^{\ell+1} \binom{m}{\ell} \mathcal{N}_\delta^{m-\ell} c^*(g_{j+\ell}^{(\alpha,k)}) .
\end{aligned}$$

The second expression in (6.33) is obtained analogously. \square

The next lemma generalizes estimates from [2, Lemmas 5.3 and 5.4].

Lemma 6.10. *Let $k \in \Gamma^{\text{nor}}$ and $\alpha \in \mathcal{I}_k$ and suppose that \hat{V} is non-negative. Then there exists $C > 0$ such that for all $s \in \mathbb{R}$ and all $a \geq 0$ we have*

$$\begin{aligned}
&\left\| (\mathcal{N}_\delta + a)^s \sum_{\beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} c_\beta(k) \psi \right\| \\
&\leq \begin{cases} C \hat{V}_k M^{-\frac{1}{2}} \|(\mathcal{N}_\delta + a)^{s+\frac{1}{2}} \psi\| & \text{if } s \geq 0 \\ C \hat{V}_k M^{-\frac{1}{2}} \|\mathcal{N}_\delta^{\frac{1}{2}} (\mathcal{N}_\delta + a - 2)^s \psi\| & \text{if } s < 0 \end{cases} \quad (6.36)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| (\mathcal{N}_\delta + a)^s \sum_{\beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} c_\beta^*(k) \psi \right\| \\
&\leq \begin{cases} C \hat{V}_k M^{-\frac{1}{2}} \|(\mathcal{N}_\delta + a + 2)^{s+\frac{1}{2}} \psi\| & \text{if } s \geq 0 \\ C \hat{V}_k M^{-\frac{1}{2}} \|(\mathcal{N}_\delta + 1)^{\frac{1}{2}} (\mathcal{N}_\delta + a)^s \psi\| & \text{if } s < 0 . \end{cases} \quad (6.37)
\end{aligned}$$

Moreover for $j \in \mathbb{N}_0$ we have

$$\sum_{\alpha \in \mathcal{I}_k} \|(\mathcal{N}_\delta + a)^s c(g_j^{(\alpha,k)}) \psi\| \leq \begin{cases} 3 \cdot 2^j M^{\frac{1}{2}} \|(\mathcal{N}_\delta + a)^{s+\frac{1}{2}} \psi\| & \text{if } s \geq 0 \\ 3 \cdot 2^j M^{\frac{1}{2}} \|\mathcal{N}_\delta^{\frac{1}{2}} (\mathcal{N}_\delta + a - 2)^s \psi\| & \text{if } s < 0 . \end{cases} \quad (6.38)$$

Proof. To establish (6.36), we use $f_\beta \in \ell^2(\mathcal{I}_k)$ to denote $f_\beta(\alpha) := K(k)_{(\alpha), \beta}$. We use the spectral decomposition $\mathcal{N}_\delta = \sum_{N_\delta \in \mathbb{N}_0} N_\delta P_{N_\delta}$, where P_{N_δ} are the orthogonal projections

on the corresponding spectral subspaces. By orthogonality of the spectral subspaces, we can pull the sum over N_δ out of the norm squares. So for both choices of $\sharp \in \{\cdot, *\}$

$$\begin{aligned} \left\| (\mathcal{N}_\delta + a)^s \sum_{\beta \in \mathcal{I}_k} f_\beta c_\beta^\sharp(k) \psi \right\|^2 &= \sum_{N_\delta \in \mathbb{N}_0} \left\| (N_\delta + a)^s P_{N_\delta} \sum_{\beta \in \mathcal{I}_k} f_\beta c_\beta^\sharp(k) \psi \right\|^2 \\ &= \sum_{N_\delta \in \mathbb{N}_0} \left\| \sum_{\Delta=0}^2 (N_\delta + a)^s P_{N_\delta} \sum_{\beta \in \mathcal{I}_k} f_\beta c_\beta^\sharp(k) P_{N_\delta \pm \Delta} \psi \right\|^2. \end{aligned}$$

In the second step we inserted $\sum_{N'_\delta \in \mathbb{N}_0} P_{N'_\delta} = \text{Id}$ between $c_\beta^\sharp(k)$ and ψ ; since $c_\beta^\sharp(k)$ annihilates or creates between 0 and 2 particles in \mathcal{G}^c there are only three non-vanishing contributions, which we wrote using the sum over $\Delta := \pm(N'_\delta - N_\delta)$. We used $P_{N_\delta \pm \Delta}$ to denote $P_{N_\delta + \Delta}$ for the case $c_\beta^\sharp(k) = c_\beta(k)$, and to denote $P_{N_\delta - \Delta}$ for the case $c_\beta^\sharp(k) = c_\beta^*(k)$.

To pull also the sum over Δ out of the norm squares we use that by the Cauchy–Schwartz inequality

$$\left(\sum_{i=1}^n c_i \right)^2 \leq n \sum_{i=1}^n c_i^2. \quad (6.39)$$

Thus

$$\begin{aligned} \left\| (\mathcal{N}_\delta + a)^s \sum_{\beta \in \mathcal{I}_k} f_\beta c_\beta^\sharp(k) \psi \right\|^2 &\leq 3 \sum_{N_\delta \in \mathbb{N}_0} \sum_{\Delta=0}^2 (N_\delta + a)^{2s} \left\| P_{N_\delta} \sum_{\beta \in \mathcal{I}_k} f_\beta c_\beta^\sharp(k) P_{N_\delta \pm \Delta} \psi \right\|^2 \\ &\leq 3 \sum_{N_\delta \in \mathbb{N}_0} \sum_{\Delta=0}^2 (N_\delta + a)^{2s} \left\| \sum_{\beta \in \mathcal{I}_k} f_\beta c_\beta^\sharp(k) P_{N_\delta \pm \Delta} \psi \right\|^2. \end{aligned} \quad (6.40)$$

The application of (6.14) to $c_\beta^\sharp(k) = c_\beta(k)$ now renders

$$\begin{aligned} &\left\| (\mathcal{N}_\delta + a)^s \sum_{\beta \in \mathcal{I}_k} f_\beta c_\beta^\sharp(k) \psi \right\|^2 \\ &\leq 3 \sum_{N_\delta \in \mathbb{N}_0} \sum_{\Delta=0}^2 (N_\delta + a)^{2s} \|f\|_2^2 \|\mathcal{N}_\delta^{\frac{1}{2}} P_{N_\delta + \Delta} \psi\|^2 \\ &\leq \begin{cases} 9 \|f\|_2^2 \|(\mathcal{N}_\delta + a)^{s+\frac{1}{2}} \psi\|^2 & \text{if } s \geq 0 \\ 9 \|f\|_2^2 \|\mathcal{N}_\delta^{\frac{1}{2}} (\mathcal{N}_\delta + a - 2)^s \psi\|^2 & \text{if } s < 0. \end{cases} \end{aligned} \quad (6.41)$$

By Lemma 6.1 we have

$$\|f\|_2^2 = \|K(k)_{\alpha,\cdot}\|_2^2 = \sum_{\beta \in \mathcal{I}_k} |K(k)_{\alpha,\beta}|^2 \leq M(C\hat{V}_k M^{-1})^2 = C^2 \hat{V}_k^2 M^{-1}. \quad (6.42)$$

This concludes the proof of the first estimate (6.36).

The estimate (6.37) is obtained analogously by applying (6.14) to $c_\beta^\sharp(k) = c_\beta^*(k)$.

$$\begin{aligned}
& \left\| (\mathcal{N}_\delta + a)^s \sum_{\beta \in \mathcal{I}_k} f_\beta c_\beta^*(k) \psi \right\|^2 \\
& \leq 3 \sum_{N_\delta \in \mathbb{N}_0} \sum_{\Delta=0}^2 (N_\delta + a)^{2s} \|f\|_2^2 (\mathcal{N}_\delta + 1)^{\frac{1}{2}} P_{N_\delta - \Delta} \psi \|^2 \\
& \leq \begin{cases} 9 \|f\|_2^2 \|(\mathcal{N}_\delta + a + 2)^{s+\frac{1}{2}} \psi\|^2 & \text{if } s \geq 0 \\ 9 \|f\|_2^2 (\mathcal{N}_\delta + 1)^{\frac{1}{2}} (\mathcal{N}_\delta + a)^s \|\psi\|^2 & \text{if } s < 0. \end{cases}
\end{aligned} \tag{6.43}$$

To prove (6.38) we proceed similarly as in [2, Proof of Lemma 5.3]:

$$\begin{aligned}
\|(\mathcal{N}_\delta + a)^s c(g_j^{(\alpha,k)}) \psi\| &= \frac{1}{n_{\alpha,k}} \left\| \sum_{p,h:\alpha} \delta_{p,h\pm k} (\chi_{\mathcal{G}^c}(p) + \chi_{\mathcal{G}^c}(h))^j (\mathcal{N}_\delta + a)^s a_h a_p \psi \right\| \\
&\leq \frac{2^j}{n_{\alpha,k}} \sum_{p,h:\alpha} \delta_{p,h\pm k} \|(\mathcal{N}_\delta + a)^s a_h a_p \psi\| \\
&\leq \frac{2^j}{n_{\alpha,k}} \left(\sum_{p,h:\alpha} \delta_{p,h\pm k} \right)^{\frac{1}{2}} \left(\sum_{p,h:\alpha} \delta_{p,h\pm k} \|(\mathcal{N}_\delta + a)^s a_h a_p \psi\|^2 \right)^{\frac{1}{2}} \\
&= 2^j \left(\sum_{p,h:\alpha} \delta_{p,h\pm k} \|(\mathcal{N}_\delta + a)^s a_h a_p \psi\|^2 \right)^{\frac{1}{2}},
\end{aligned} \tag{6.44}$$

where we used the definition (2.10) of $n_{\alpha,k}$. Employing again the spectral decomposition of \mathcal{N}_δ , proceeding exactly as for the first part of the lemma, this implies

$$\begin{aligned}
\|(\mathcal{N}_\delta + a)^s c(g_j^{(\alpha,k)}) \psi\|^2 &\leq 2^{2j} \sum_{p,h:\alpha} \delta_{p,h\pm k} \left\| \sum_{N_\delta, N'_\delta \in \mathbb{N}_0} (N_\delta + a)^s P_{N_\delta} a_h a_p P_{N'_\delta} \psi \right\|^2 \\
&= 2^{2j} \sum_{p,h:\alpha} \delta_{p,h\pm k} \left\| \sum_{N_\delta \in \mathbb{N}_0} \sum_{\Delta=0}^2 (N_\delta + a)^s P_{N_\delta} a_h a_p P_{N_\delta + \Delta} \psi \right\|^2 \\
&\leq 3 \cdot 2^{2j} \sum_{p,h:\alpha} \delta_{p,h\pm k} \sum_{N_\delta \in \mathbb{N}_0} (N_\delta + a)^{2s} \sum_{\Delta=0}^2 \|P_{N_\delta} a_h a_p P_{N_\delta + \Delta} \psi\|^2 \\
&\leq 3 \cdot 2^{2j} \sum_{N_\delta \in \mathbb{N}_0} \sum_{\Delta=0}^2 (N_\delta + a)^{2s} \sum_{p,h:\alpha} \delta_{p,h\pm k} \|a_h a_p P_{N_\delta + \Delta} \psi\|^2.
\end{aligned} \tag{6.45}$$

The condition $\alpha \in \mathcal{I}_k$, by a geometric consideration given in [2, Proof of Lemma 5.3], implies that at least one of the two momenta p and h are outside the gap \mathcal{G} , and therefore

$$\sum_{p,h:\alpha} \delta_{p,h\pm k} \|a_h a_p P_{N_\delta + \Delta} \psi\|^2 \leq \sum_{q \in B_\alpha \cap \mathcal{G}^c} \|a_q P_{N_\delta + \Delta} \psi\|^2. \tag{6.46}$$

Thus

$$\begin{aligned}
& \sum_{\alpha \in \mathcal{I}_k} \|(\mathcal{N}_\delta + a)^s c(g_j^{(\alpha, k)}) \psi\|^2 \\
& \leq 3 \cdot 2^{2j} \sum_{N_\delta} \sum_{\Delta=0}^2 (N_\delta + a)^{2s} \sum_{\alpha \in \mathcal{I}_k} \sum_{q \in B_\alpha \cap \mathcal{G}^c} \|a_q P_{N_\delta + \Delta} \psi\|^2 \\
& \leq 3 \cdot 2^{2j} \sum_{N_\delta} \sum_{\Delta=0}^2 (N_\delta + a)^{2s} \sum_{q \in \mathcal{G}^c} \langle P_{N_\delta + \Delta} \psi, a_q^* a_q P_{N_\delta + \Delta} \psi \rangle \\
& = 3 \cdot 2^{2j} \sum_{N_\delta} \sum_{\Delta=0}^2 (N_\delta + a)^{2s} \|\mathcal{N}_\delta^{\frac{1}{2}} P_{N_\delta + \Delta} \psi\|^2 \\
& \leq 3 \cdot 2^{2j} \sum_{\Delta=0}^2 \sum_{N_\delta} \|\mathcal{N}_\delta^{\frac{1}{2}} (\mathcal{N}_\delta + a - \Delta)^s P_{N_\delta + \Delta} \psi\|^2 \\
& \leq 9 \cdot 2^{2j} \max \left\{ \|\mathcal{N}_\delta^{\frac{1}{2}} (\mathcal{N}_\delta + a)^s \psi\|^2, \|\mathcal{N}_\delta^{\frac{1}{2}} (\mathcal{N}_\delta + a - 2)^s \psi\|^2 \right\}.
\end{aligned} \tag{6.47}$$

By the Cauchy–Schwartz inequality we obtain

$$\begin{aligned}
\sum_{\alpha \in \mathcal{I}_k} \|(\mathcal{N}_\delta + a)^s c(g_j^{(\alpha, k)}) \psi\| & \leq \left(\sum_{\alpha \in \mathcal{I}_k} 1 \right)^{\frac{1}{2}} \left(\sum_{\alpha \in \mathcal{I}_k} \|(\mathcal{N}_\delta + a)^s c(g_j^{(\alpha, k)}) \psi\|^2 \right)^{\frac{1}{2}} \\
& \leq \begin{cases} 3 \cdot 2^j M^{\frac{1}{2}} \|(\mathcal{N}_\delta + a)^{s+\frac{1}{2}} \psi\| & \text{if } s \geq 0 \\ 3 \cdot 2^j M^{\frac{1}{2}} \|\mathcal{N}_\delta^{\frac{1}{2}} (\mathcal{N}_\delta + a - 2)^s \psi\| & \text{if } s < 0. \end{cases}
\end{aligned}$$

This is the intended inequality. \square

Proof of Lemma 6.8. The first two estimates are given, without specifying explicitly the $\|\hat{V}\|_1$ –dependence (which however can be read from the proof), in [2, Lemma 7.2]. The proof for the third estimate is based on the same idea; we give a brief argument. Consider the real–valued function

$$f_m(t) := \langle \psi, e^{tS} (\mathcal{N}_\delta + 1)^m e^{-tS} \psi \rangle. \tag{6.48}$$

Our goal is to establish $|\partial_t f_m(t)| \leq c \|\hat{V}\|_1 f_m(t)$, so that Grönwall’s lemma implies the intended $f_m(t) \leq e^{c\|t\|\|\hat{V}\|_1} f_m(0)$. For brevity, we write $\phi_t := e^{-tS} \psi$. The derivative is

$$\begin{aligned}
\partial_t f_m(t) & = \langle \phi_t, [S, (\mathcal{N}_\delta + 1)^m] \phi_t \rangle \\
& = \operatorname{Re} \sum_{k \in \Gamma^{\text{nor}}} \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} \langle \phi_t, [(\mathcal{N}_\delta + 1)^m, c_\alpha^*(k) c_\beta^*(k)] \phi_t \rangle.
\end{aligned} \tag{6.49}$$

Using (6.33) and the symmetry $K(k)_{\alpha,\beta} = K(k)_{\beta,\alpha}$, we get

$$\begin{aligned}
& |\partial_t f_m(t)| \\
& \leq \sum_{k \in \Gamma^{\text{nor}}} \left| \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha,\beta} \langle \phi_t, \left([(\mathcal{N}_\delta + 1)^m, c_\alpha^*(k)] c_\beta^*(k) + c_\alpha^*(k) [(\mathcal{N}_\delta + 1)^m, c_\beta^*(k)] \right) \phi_t \rangle \right| \\
& \leq \sum_{k \in \Gamma^{\text{nor}}} \sum_{j=1}^m \binom{m}{j} \left| \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha,\beta} \langle \phi_t, c^*(g_j^{(\alpha,k)})(\mathcal{N}_\delta + 1)^{m-j} c_\beta^*(k) \phi_t \rangle \right. \\
& \quad \left. + \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha,\beta} \langle \phi_t, c_\beta^*(k) c^*(g_j^{(\alpha,k)})(\mathcal{N}_\delta + 1)^{m-j} \phi_t \rangle \right| \\
& \leq \sum_{k \in \Gamma^{\text{nor}}} \sum_{j=1}^m \binom{m}{j} \left(\left| \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha,\beta} \langle \phi_t, c^*(g_j^{(\alpha,k)})(\mathcal{N}_\delta + 1)^{m-j} c_\beta^*(k) \phi_t \rangle \right| \right. \\
& \quad \left. + \left| \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha,\beta} \langle \phi_t, c_\beta^*(k) c^*(g_j^{(\alpha,k)})(\mathcal{N}_\delta + 1)^{m-j} \phi_t \rangle \right| \right). \tag{6.50}
\end{aligned}$$

The first absolute value on the r. h. s. of (6.50) is bounded using Lemma 6.10: We split $(\mathcal{N}_\delta + 1)^{m-j} = (\mathcal{N}_\delta + 1)^r (\mathcal{N}_\delta + 1)^s$ where $r, s \in [0, \frac{m}{2} - \frac{1}{2}]$. An appropriate choice of r and s is always possible, since $0 \leq m - j \leq m - 1$. We conclude that

$$\begin{aligned}
& \left| \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha,\beta} \langle \phi_t, c^*(g_j^{(\alpha,k)})(\mathcal{N}_\delta + 1)^{m-j} c_\beta^*(k) \phi_t \rangle \right| \\
& \leq \sum_{\alpha \in \mathcal{I}_k} \|(\mathcal{N}_\delta + 1)^r c(g_j^{(\alpha,k)}) \phi_t\| \max_{\alpha' \in \mathcal{I}_k} \left\| (\mathcal{N}_\delta + 1)^s \sum_{\beta \in \mathcal{I}_k} K(k)_{\alpha',\beta} c_\beta^*(k) \phi_t \right\| \\
& \stackrel{(6.36)}{\leq} \sum_{\alpha \in \mathcal{I}_k} \|(\mathcal{N}_\delta + 1)^r c(g_j^{(\alpha,k)}) \phi_t\| C \hat{V}_k M^{-\frac{1}{2}} \|(\mathcal{N}_\delta + 3)^{s+\frac{1}{2}} \phi_t\| \\
& \stackrel{(6.38)}{\leq} C \hat{V}_k 2^j \|(\mathcal{N}_\delta + 1)^{r+\frac{1}{2}} \phi_t\| \|(\mathcal{N}_\delta + 3)^{s+\frac{1}{2}} \phi_t\| \\
& \leq C \hat{V}_k 2^j \langle \phi_t, (\mathcal{N}_\delta + 3)^m \phi_t \rangle. \tag{6.51}
\end{aligned}$$

Thus there exists $C_m > 0$ such that the first absolute value of the r. h. s. of (6.50) is bounded by $C_m f_m(t)$.

The second absolute value on the r. h. s. of (6.50) is bounded similarly, with the difference that $c^*(g_j^{(\alpha,k)})$ must be moved to the left in order to apply (6.38). This is easily done using $[c_\beta^*(k), c^*(g_j^{(\alpha,k)})] = 0$. In the following, for $j < m$, we pick any $s \in [\frac{1}{2}, \frac{m-j}{2}]$ and

estimate

$$\begin{aligned}
& \left| \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} \langle \phi_t, c_\beta^*(k) c^*(g_j^{(\alpha, k)}) (\mathcal{N}_\delta + 1)^{m-j} \phi_t \rangle \right| \\
& \leq \sum_{\alpha \in \mathcal{I}_k} \max_{\alpha' \in \mathcal{I}_k} \left| \langle c(g_j^{(\alpha, k)}) \phi_t, \sum_{\beta \in \mathcal{I}_k} K(k)_{\alpha', \beta} c_\beta^*(k) (\mathcal{N}_\delta + 1)^{m-j} \phi_t \rangle \right| \\
& \leq \sum_{\alpha \in \mathcal{I}_k} \|(\mathcal{N}_\delta + 1)^s c(g_j^{(\alpha, k)}) \phi_t\| \\
& \quad \times \max_{\alpha' \in \mathcal{I}_k} \|(\mathcal{N}_\delta + 1)^{-s} \sum_{\beta \in \mathcal{I}_k} K(k)_{\alpha', \beta} c_\beta^*(k) (\mathcal{N}_\delta + 1)^{m-j} \phi_t\| \tag{6.52} \\
& \stackrel{(6.36)}{\leq} \sum_{\alpha \in \mathcal{I}_k} \|(\mathcal{N}_\delta + 3)^s c(g_j^{(\alpha, k)}) \phi_t\| C \hat{V}_k M^{-\frac{1}{2}} \|(\mathcal{N}_\delta + 1)^{m-j-s} \phi_t\| \\
& \stackrel{(6.38)}{\leq} C \hat{V}_k 2^j \|(\mathcal{N}_\delta + 1)^{s+\frac{1}{2}} \phi_t\| \|(\mathcal{N}_\delta + 3)^{m-j-s} \phi_t\| \\
& \leq C \hat{V}_k 2^j \langle \phi_t, (\mathcal{N}_\delta + 3)^m \phi_t \rangle.
\end{aligned}$$

In case $j = m$, we immediately get with (6.36) and (6.38):

$$\begin{aligned}
& \left| \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} \langle \phi_t, c_\beta^*(k) c^*(g_j^{(\alpha, k)}) \phi_t \rangle \right| \leq \sum_{\alpha \in \mathcal{I}_k} \|c(g_j^{(\alpha, k)}) \phi_t\| \max_{\alpha' \in \mathcal{I}_k} \left\| \sum_{\beta \in \mathcal{I}_k} K(k)_{\alpha', \beta} c_\beta^*(k) \phi_t \right\| \\
& \leq C \hat{V}_k 2^j \|\mathcal{N}_\delta^{\frac{1}{2}} \phi_t\| \|(\mathcal{N}_\delta + 2)^{\frac{1}{2}} \phi_t\| \leq C \hat{V}_k 2^j \langle \phi_t, (\mathcal{N}_\delta + 2)^m \phi_t \rangle. \quad \square
\end{aligned}$$

We also need bounds on combinations of \mathcal{N}_δ^- and $\mathcal{E}^{(g)}$ -operators. The following lemma is the analogue of Lemma 6.4.

Lemma 6.11 (Bootstrap Bound on $\mathcal{N}_\delta^{1/2} \mathcal{E}^{(g)}$). *Let $k, \ell \in \Gamma^{\text{nor}}$, $\alpha_q \in \mathcal{I}_k \cap \mathcal{I}_\ell$, and $q \in B_{\alpha_q}$. Suppose that \hat{V} is non-negative and has compact support. Suppose there is $0 \leq r \leq r_{\max} < \infty$ and $C > 0$ such that for all $t \in [-1, 1]$ and $q' \in \mathbb{Z}^3$ we have*

$$\langle \xi_t, a_{q'}^* a_{q'} \xi_t \rangle \leq C N^{-r}. \tag{6.53}$$

*Then, for all $\varepsilon > 0$, there exists $c_\varepsilon > 0$, independent of r (but possibly depending on r_{\max}), such that for all $t \in [-1, 1]$ we have, for both choices of $\sharp \in \{\cdot, *\}$, the bound*

$$\|\mathcal{N}_\delta^{\frac{1}{2}} \mathcal{E}_q^{(g)}(k, \ell)^\sharp \xi_t\| \leq C \frac{e^{c_\varepsilon |t|} \|\hat{V}\|_1}{n_{\alpha_q, k} n_{\alpha_q, \ell}} N^{-\frac{r}{4} + \varepsilon}. \tag{6.54}$$

Remark. The assumption $r \leq r_{\max}$ is required so that we can take c_ε independent of r . The proof of Lemma 6.11 employs an iterative procedure on r . The upper bound $r \leq r_{\max}$ makes sure that both the number of iteration steps and c_ε stay bounded. Later on, we will see that $r_{\max} = \frac{2}{3}$ is a suitable choice.

Proof. By the Cauchy–Schwartz inequality, for all $a \geq 0$ we have

$$\begin{aligned} \|\mathcal{N}_\delta^a \mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\| &= \langle \mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t, \mathcal{N}_\delta^{2a} \mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t \rangle^{\frac{1}{2}} \\ &\leq \|\mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\|^{\frac{1}{2}} \|\mathcal{N}_\delta^{2a} \mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\|^{\frac{1}{2}}. \end{aligned} \quad (6.55)$$

Applying this n times to $\|\mathcal{N}_\delta^{\frac{1}{2}} \mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\|$ yields

$$\begin{aligned} \|\mathcal{N}_\delta^{\frac{1}{2}} \mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\| &\leq \|\mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\|^{1-2^{-n}} \|\mathcal{N}_\delta^{2^{n-1}} \mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\|^{2^{-n}} \\ &\leq \|\mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\|^{1-2^{-n}} \left(\|\mathcal{E}_q^{(g)}(k, \ell)^\# \mathcal{N}_\delta^{2^{n-1}} \xi_t\| + \|[\mathcal{N}_\delta^{2^{n-1}}, \mathcal{E}_q^{(g)}(k, \ell)^\#] \xi_t\| \right)^{2^{-n}}. \end{aligned} \quad (6.56)$$

The first norm on the r. h. s. of (6.56) is bounded by Lemma 6.4:

$$\|\mathcal{E}_q^{(g)}(k, \ell)^\# \xi_t\|^{1-2^{-n}} \leq \left(\frac{CN^{-\frac{r}{4}}}{n_{\alpha_q, k} n_{\alpha_q, \ell}} \right)^{1-2^{-n}}. \quad (6.57)$$

For the second norm, we use (6.3) and then the stability of number operators under conjugation from Lemma 6.8 (and $\xi_0 = \Omega$, $\|\Omega\| = 1$), yielding

$$\|\mathcal{E}_q^{(g)}(k, \ell)^\# \mathcal{N}_\delta^{2^{n-1}} \xi_t\| \leq \frac{2}{n_{\alpha_q, k} n_{\alpha_q, \ell}} \|\mathcal{N}_\delta^{2^{n-1}} \xi_t\| \leq \frac{2e^{\frac{1}{2}C_{2n}|t|\|\hat{V}\|_1}}{n_{\alpha_q, k} n_{\alpha_q, \ell}}. \quad (6.58)$$

Finally, for the third norm, we explicitly evaluate the commutator in the case $q \in B_F^c$ and $\mathcal{E}_q^{(g)}(k, \ell)^\# = \mathcal{E}_q^{(g)}(k, \ell)$, using for the latter its explicit formula (4.12). (The cases $q \in B_F$ and $\mathcal{E}_q^{(g)}(k, \ell)^*$ are treated exactly in the same way.) We find

$$\|[\mathcal{N}_\delta^{2^{n-1}}, \mathcal{E}_q^{(g)}(k, \ell)] \xi_t\| \leq \frac{1}{n_{\alpha_q, k} n_{\alpha_q, \ell}} \left(\|[\mathcal{N}_\delta^{2^{n-1}}, a_{q \mp k}^* a_{q \mp k}] \xi_t\| + \|[\mathcal{N}_\delta^{2^{n-1}}, a_q^* a_{q \mp \ell \pm k}] \xi_t\| \right).$$

Depending on whether the momenta $q \mp \ell$ and $q \mp k$ lie inside the gap \mathcal{G}_δ , the first commutator norm amounts to either 0 or

$$\|[\mathcal{N}_\delta^{2^{n-1}}, a_{q \mp \ell}^* a_{q \mp k}] \xi_t\| = \|a_{q \mp \ell}^* a_{q \mp k} ((\mathcal{N}_\delta \pm 1)^{2^{n-1}} - \mathcal{N}_\delta^{2^{n-1}}) \xi_t\| \leq 2\|(\mathcal{N}_\delta + 1)^{2^{n-1}} \xi_t\|. \quad (6.59)$$

The same holds for the second commutator norm. Applying Lemma 6.8, we find

$$\|[\mathcal{N}_\delta^{2^{n-1}}, \mathcal{E}_q^{(g)}(k, \ell)] \xi_t\| \leq \frac{4e^{\frac{1}{2}C_{2n}|t|\|\hat{V}\|_1}}{n_{\alpha_q, k} n_{\alpha_q, \ell}}. \quad (6.60)$$

Plugging the estimates (6.57), (6.58), and (6.60) into (6.56) renders

$$\begin{aligned} \|\mathcal{N}_\delta^{\frac{1}{2}} \mathcal{E}_q^{(g)}(k, \ell) \xi_t\| &\leq \frac{1}{n_{\alpha_q, k} n_{\alpha_q, \ell}} (CN^{-\frac{r}{4}})^{1-2^{-n}} \left(6e^{\frac{1}{2}C_{2n}|t|\|\hat{V}\|_1} \right)^{2^{-n}} \\ &\leq \frac{1}{n_{\alpha_q, k} n_{\alpha_q, \ell}} C^{1-2^{-n}} e^{2^{-n}(\log(6) + \frac{1}{2}C_{2n}|t|\|\hat{V}\|_1)} N^{-\frac{r}{4} + 2^{-n-2}r}. \end{aligned}$$

Choosing $n = n(\varepsilon)$ large enough, such that $2^{-n-2}r_{\max} < \varepsilon \Rightarrow 2^{-n-2}r < \varepsilon$, we obtain the desired bound. \square

7 Controlling the Bosonized Terms

We now use the bounds from the previous section to derive estimates on the expectation values of $\text{ad}_{q,(b)}^n$ (defined in (5.2)) and on the commutation error

$$\mathcal{E}_{n,q} := [S, \text{ad}_{q,(b)}^{n-1}] - \text{ad}_{q,(b)}^n, \quad n \in \mathbb{N}. \quad (7.1)$$

Lemma 7.1 (Bound on the Commutation Error). *Suppose there is $r \in [0, \frac{2}{3}]$ such that for all $t \in [-1, 1]$ and $q' \in \mathbb{Z}^3$ we have (with C independent of t, q')*

$$\langle \xi_t, a_{q'}^* a_{q'} \xi_t \rangle \leq CN^{-r}. \quad (7.2)$$

Further, suppose that \hat{V} is non-negative and compactly supported. Then, there exist constants $\mathfrak{c}_2, \mathfrak{C}_2 > 0$, and for every $\varepsilon > 0$ a constant $\tilde{c}_\varepsilon > 0$, such that

$$|\langle \xi_t, \mathcal{E}_{n,q} \xi_t \rangle| \leq \mathfrak{c}_2 \mathfrak{C}_2^n e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} \|\hat{V}\|_1^{n+1} N^{-\frac{2}{3} + \delta - \frac{r}{4} + \varepsilon} \quad (7.3)$$

for all $n, N \in \mathbb{N}$ and $t \in [-1, 1]$.

Remark. Later in (8.5), we will take an infinite sum over n of error bounds like (7.3). To ensure convergence, it is important to track the dependence of constants on n . For this reason, we introduced the separate constants $\mathfrak{c}_2, \mathfrak{C}_2$. Further, in the following C is not only independent of N, V and q but also of n and m .

Proof. If q is not inside any patch, then the statement is trivial since $\text{ad}_{q,(b)}^{n-1} = 0 = \text{ad}_{q,(b)}^n$, thus also $\mathcal{E}_{n,q} = 0$. So we may assume that $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$. Recall that $\text{ad}_{q,(b)}^n$ is given by \mathbf{A}, \mathbf{B} and \mathbf{C} (for even n) or \mathbf{D}, \mathbf{E} and \mathbf{F} (for odd n) as defined in (5.3) and (5.4). We define⁶

$$\begin{aligned} \mathcal{E}_n^{(A)} &:= [S, \mathbf{A}_n] = 0 \\ \mathcal{E}_n^{(B)} &:= [S, \mathbf{B}_n] - \mathbf{D}_{n,1} - \mathbf{E}_{n+1} \\ \mathcal{E}_{n-m,m}^{(C)} &:= [S, \mathbf{C}_{n-m,m}] - \mathbf{D}_{n-m+1,m} - \mathbf{F}_{n-m,m+1} \\ \mathcal{E}_{n-m,m}^{(D)} &:= [S, \mathbf{D}_{n-m,m}] - \mathbf{A}_{n+1} - \mathbf{C}_{n-m+1,m} - \mathbf{C}_{n-m,m+1} \\ \mathcal{E}_n^{(E)} &:= [S, \mathbf{E}_n] - \mathbf{A}_{n+1} - \mathbf{C}_{n,1} - \mathbf{B}_{n+1} \\ \mathcal{E}_{n-m,m}^{(F)} &:= [S, \mathbf{F}_{n-m,m}] - \mathbf{A}_{n+1} - \mathbf{C}_{n-m+1,m} - \mathbf{C}_{n-m,m+1}. \end{aligned} \quad (7.4)$$

Then we may express the commutation error for $n \geq 1$ as

$$\mathcal{E}_{n+1,q} = \begin{cases} \mathcal{E}_n^{(B)} + (\mathcal{E}_n^{(B)})^* + \sum_{m=1}^{n-1} \binom{n}{m} \mathcal{E}_{n-m,m}^{(C)} & \text{for } n : \text{even} \\ \mathcal{E}_n^{(E)} + (\mathcal{E}_n^{(E)})^* + \sum_{m=1}^s \binom{n}{m} \mathcal{E}_{n-m,m}^{(D)} + \sum_{m=1}^s \binom{n}{m} \mathcal{E}_{m,n-m}^{(F)} & \text{for } n : \text{odd}, \end{cases} \quad (7.5)$$

⁶In the proof of Lemma A.1, we will see that the subtracted terms (such as $\mathbf{D}_{n,1} + \mathbf{E}_{n+1}$ for $\mathcal{E}^{(B)}$) are the bosonization approximation of the commutators. So $\mathcal{E}_n^{(A)}, \dots, \mathcal{E}_{n-m,m}^{(F)}$ are the deviations from the bosonization approximation.

where $n = 2s + 1$ for odd n . For $n = 0$, we have $\mathcal{E}_1 = 0$, since $\text{ad}_{q,(b)}^1 = [S, a_q^* a_q] = [S, \text{ad}_{q,(b)}^0]$ (see also Appendix A).

Bounding $\mathcal{E}^{(B)}$:

Recall that $K(k)$ is symmetric and that $k \in \tilde{\mathcal{C}}^q$ implies $\alpha_q \in \mathcal{I}_k$. For this proof, let us adopt the convention that \sum_k denotes a sum over $k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3$ and $\sum_{k'}$ a sum over all $k' \in \Gamma^{\text{nor}}$ such that $\alpha_q \in \mathcal{I}_{k'}$. A straightforward computation starting from the definitions (5.3) and (5.4) and using Lemmas 4.3 and 4.4 renders

$$\begin{aligned} & |\langle \xi_t, \mathcal{E}_n^{(B)} \xi_t \rangle| \\ &= \left| \frac{1}{2} \sum_{k,k'} \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \alpha_1 \in \mathcal{I}_k}} K(k')_{\alpha, \alpha_q} (K(k)^n)_{\alpha_q, \alpha_1} \langle \xi_t, (c_\alpha(k') \mathcal{E}_q^{(g)}(k', k) c_{\alpha_1}(k) + \mathcal{E}_q^{(g)}(k', k) c_\alpha(k') c_{\alpha_1}(k)) \xi_t \rangle \right. \\ & \quad \left. + \frac{1}{2} \sum_{k,k'} \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^n)_{\alpha_q, \beta} \langle \xi_t, (c^*(g_{q,k}) c_\alpha^*(k') \mathcal{E}_\beta(k, k') + c^*(g_{q,k}) \mathcal{E}_\beta(k, k') c_\alpha^*(k')) \xi_t \rangle \right|. \end{aligned}$$

By the Cauchy–Schwartz inequality

$$\begin{aligned} & |\langle \xi_t, \mathcal{E}_n^{(B)} \xi_t \rangle| \\ & \leq \frac{1}{2} \sum_{k,k'} \left\| \sum_{\alpha \in \mathcal{I}_{k'}} K(k')_{\alpha_q, \alpha} c_\alpha^*(k') \xi_t \right\| \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} \mathcal{E}_q^{(g)}(k', k) c_{\alpha_1}(k) \xi_t \right\| \\ & \quad + \frac{1}{2} \sum_{k,k'} \left\| \sum_{\alpha \in \mathcal{I}_{k'}} K(k')_{\alpha_q, \alpha} c_\alpha^*(k') \mathcal{E}_q^{(g)}(k', k)^* \xi_t \right\| \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} c_{\alpha_1}(k) \xi_t \right\| \\ & \quad + \frac{1}{2} \sum_{k,k'} \|c(g_{q,k}) \xi_t\| \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^n)_{\alpha_q, \beta} c_\alpha^*(k') \mathcal{E}_\beta(k, k') \xi_t \right\| \\ & \quad + \frac{1}{2} \sum_{k,k'} \|c(g_{q,k}) \xi_t\| \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^n)_{\alpha_q, \beta} \mathcal{E}_\beta(k, k') c_\alpha^*(k') \xi_t \right\| \\ & =: (\text{I}^{(B)}) + (\text{II}^{(B)}) + (\text{III}^{(B)}) + (\text{IV}^{(B)}) . \end{aligned} \tag{7.6}$$

We start with bounding $(\text{II}^{(B)})$, which is slightly easier than bounding $(\text{I}^{(B)})$. We apply Lemma 6.5:

$$(\text{II}^{(B)}) \leq \frac{1}{2} \sum_{k,k'} \|K(k')_{\alpha_q, \cdot}\|_2 \|(\mathcal{N}_\delta + 1)^{\frac{1}{2}} \mathcal{E}_q^{(g)}(k', k)^* \xi_t\| \|(K(k)^n)_{\alpha_q, \cdot}\|_2 \|\mathcal{N}_\delta^{\frac{1}{2}} \xi_t\| . \tag{7.7}$$

Lemma 6.1 yields

$$\|(K(k)^n)_{\alpha_q, \cdot}\|_2 = \left(\sum_{\alpha} |(K(k)^n)_{\alpha_q, \alpha}|^2 \right)^{\frac{1}{2}} \leq \left(M(C\hat{V}_k)^{2n} M^{-2} \right)^{\frac{1}{2}} \leq (C\hat{V}_k)^n M^{-\frac{1}{2}} . \tag{7.8}$$

The second factor containing \mathcal{N}_δ is controlled using Lemma 6.8

$$\|\mathcal{N}_\delta^{\frac{1}{2}} \xi_t\|^2 \leq \langle \Omega, e^{tS} (\mathcal{N}_\delta + 1) e^{-tS} \Omega \rangle \leq \langle \Omega, e^{C|t|\|\hat{V}\|_1} (\mathcal{N}_\delta + 1) \Omega \rangle = e^{C|t|\|\hat{V}\|_1} . \quad (7.9)$$

Concerning the first factor containing \mathcal{N}_δ in (7.7), we apply Lemmas 6.11 and 6.4 together with $n_{\alpha_q, k} \geq C\mathbf{n}$:

$$\|(\mathcal{N}_\delta + 1)^{\frac{1}{2}} \mathcal{E}_q^{(g)}(k', k)^* \xi_t\| = \|\mathcal{N}_\delta^{\frac{1}{2}} \mathcal{E}_q^{(g)}(k', k)^* \xi_t\| + \|\mathcal{E}_q^{(g)}(k', k)^* \xi_t\| \leq C \frac{e^{c_\varepsilon |t|\|\hat{V}\|_1}}{\mathbf{n}^2} N^{-\frac{r}{4} + \varepsilon} , \quad (7.10)$$

for an arbitrarily small $\varepsilon > 0$ and some $c_\varepsilon > 0$ depending on ε . So finally, choosing a fixed ε small enough,

$$(\Pi^{(B)}) \leq e^{\tilde{c}_\varepsilon |t|\|\hat{V}\|_1} \sum_{k, k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-2} M^{-1} N^{-\frac{r}{4} + \varepsilon} \quad (7.11)$$

with

$$\tilde{c}_\varepsilon := C + c_\varepsilon .$$

(Note that \tilde{c}_ε and c_ε are independent of r, t, q' and n .)

In $(\mathbf{I}^{(B)})$, using Lemmas 6.1 and 6.5, the first norm can be bounded by

$$\left\| \sum_{\alpha \in \mathcal{I}_{k'}} K(k')_{\alpha_q, \alpha} c_\alpha^*(k') \xi_t \right\| \leq \|K(k')_{\alpha_q, \cdot}\|_2 \|(\mathcal{N}_\delta + 1)^{\frac{1}{2}} \xi_t\| \leq C \hat{V}_{k'} M^{-\frac{1}{2}} e^{C|t|\|\hat{V}\|_1} . \quad (7.12)$$

For the second norm in $(\mathbf{I}^{(B)})$, we have

$$\begin{aligned} & \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} \mathcal{E}_q^{(g)}(k', k) c_{\alpha_1}(k) \xi_t \right\| \\ & \leq \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} c_{\alpha_1}(k) \mathcal{E}_q^{(g)}(k', k) \xi_t \right\| + \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} [\mathcal{E}_q^{(g)}(k', k), c_{\alpha_1}(k)] \xi_t \right\| . \end{aligned} \quad (7.13)$$

The first norm on the r. h. s. is bounded just as the second norm in $(\Pi^{(B)})$ by

$$\left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} c_{\alpha_1}(k) \mathcal{E}_q^{(g)}(k', k) \xi_t \right\| \leq e^{c_\varepsilon |t|\|\hat{V}\|_1} (C\hat{V}_k)^n \mathbf{n}^{-2} M^{-\frac{1}{2}} N^{-\frac{r}{4} + \varepsilon} . \quad (7.14)$$

For the second norm on the r. h. s., in order to control the commutator term, we use the explicit form of $\mathcal{E}_q^{(g)}(k', k)$ as in (4.12). We restrict to the case $q \in B_F^c$ ($q \in B_F$ can be

treated analogously) and use that $[\mathcal{E}_q^{(g)}(k', k), c_\alpha^\sharp(k)] = 0$ whenever $\alpha \neq \alpha_q$:

$$\begin{aligned}
& \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} [\mathcal{E}_q^{(g)}(k', k), c_{\alpha_1}(k)] \xi_t \right\| \\
& \leq (K(k)^n)_{\alpha_q, \alpha_q} \|\mathcal{E}_q^{(g)}(k', k), c_{\alpha_q}(k)\| \xi_t \leq (C\hat{V}_k)^n M^{-1} \|\mathcal{E}_q^{(g)}(k', k), c_{\alpha_q}(k)\| \xi_t \\
& \leq \frac{(C\hat{V}_k)^n M^{-1}}{n_{\alpha_q, k}^2 n_{\alpha_q, k'}} \sum_{p: \alpha_q} (\| [a_{q\mp k}^* a_{q\mp k'}, a_{p\mp k} a_p] \xi_t \| + \| [a_q^* a_{q\mp k \pm k'}, a_{p\mp k} a_p] \xi_t \|) \\
& = \frac{(C\hat{V}_k)^n M^{-1}}{n_{\alpha_q, k}^2 n_{\alpha_q, k'}} (\| a_{q\mp k'} a_q \xi_t \| + \| a_{q\mp k \pm k'} a_{q \pm k} \xi_t \|) \\
& \leq (C\hat{V}_k)^n \mathbf{n}^{-3} M^{-1}.
\end{aligned} \tag{7.15}$$

By definition (6.2) of \mathbf{n} , (7.15) scales like $\mathbf{n}^{-3} M^{-1} = \mathbf{n}^{-2} M^{-\frac{1}{2}} N^{-\frac{1}{3} + \frac{\delta}{2}}$, whereas (7.14) scales like $\mathbf{n}^{-2} M^{-\frac{1}{2}} N^{-\frac{r}{4} + \varepsilon}$. Since $r \leq \frac{2}{3}$ and $\delta < \frac{1}{6}$, the latter is smaller than the first and thus, remembering the factor (7.12), we have

$$(\text{I}^{(B)}) \leq e^{(C+c_\varepsilon)|t|\|\hat{V}\|_1} \sum_{k, k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-2} M^{-1} N^{-\frac{r}{4} + \varepsilon}. \tag{7.16}$$

Concerning $(\text{III}^{(B)})$, the second norm is bounded by Lemma 6.7:

$$\left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^n)_{\alpha_q, \beta} c_\alpha^*(k') \mathcal{E}_\beta(k, k') \xi_t \right\| \leq (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-1} M^{-1} \|(\mathcal{N} + 1) \xi_t\|.$$

Estimating the first norm in $(\text{III}^{(B)})$ with (6.17), we then obtain

$$(\text{III}^{(B)}) \leq \sum_{k, k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-2} M^{-1} N^{-\frac{r}{2}} \|(\mathcal{N} + 1) \xi_t\|. \tag{7.17}$$

By Lemma 6.8 we have

$$\|(\mathcal{N} + 1) \xi_t\| = \langle \Omega, e^{tS} (\mathcal{N} + 1)^2 e^{-tS} \Omega \rangle \leq e^{C|t|\|\hat{V}\|_1} \langle \Omega, (\mathcal{N} + 1)^2 \Omega \rangle \leq e^{C|t|\|\hat{V}\|_1}. \tag{7.18}$$

Consequently

$$(\text{III}^{(B)}) \leq e^{C|t|\|\hat{V}\|_1} \sum_{k, k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-2} M^{-1} N^{-\frac{r}{2}}. \tag{7.19}$$

The term $(\text{IV}^{(B)})$ is bounded analogously by

$$(\text{IV}^{(B)}) \leq e^{C|t|\|\hat{V}\|_1} \sum_{k, k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-2} M^{-1} N^{-\frac{r}{2}}. \tag{7.20}$$

We add up all four contributions to $\mathcal{E}_n^{(B)}$ and choose a fixed ε :

$$|\langle \xi_t, \mathcal{E}_n^{(B)} \xi_t \rangle| \leq e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} \sum_{k, k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-2} M^{-1} N^{-\frac{r}{4} + \varepsilon}. \tag{7.21}$$

The same bound applies to $|\langle \xi_t, (\mathcal{E}_n^{(B)})^* \xi_t \rangle| = |\langle \xi_t, \mathcal{E}_n^{(B)} \xi_t \rangle|$.

Bounding $\mathcal{E}^{(C)}$:

As before, from the definitions (5.3) and (5.4), using Lemmas 4.3 and 4.4, we obtain

$$\begin{aligned}
& |\langle \xi_t, \mathcal{E}_{n-m,m}^{(C)} \xi_t \rangle| \\
&= \left| \frac{1}{2} \sum_{k,k'} \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \alpha_2 \in \mathcal{I}_k \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha,\beta} (K(k)^{n-m})_{\alpha_q,\beta} (K(k)^m)_{\alpha_q,\alpha_2} \rho_{q,k} \times \right. \\
&\quad \times \langle \xi_t, (c_\alpha(k') \mathcal{E}_\beta(k', k) c_{\alpha_2}(k) + \mathcal{E}_\beta(k', k) c_\alpha(k') c_{\alpha_2}(k)) \xi_t \rangle \\
&\quad + \frac{1}{2} \sum_{k,k'} \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \alpha_1 \in \mathcal{I}_k \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha,\beta} (K(k)^{n-m})_{\alpha_q,\alpha_1} (K(k)^m)_{\alpha_q,\beta} \rho_{q,k} \times \\
&\quad \times \langle \xi_t, (c_{\alpha_1}^*(k) c_\alpha^*(k') \mathcal{E}_\beta(k, k') + c_{\alpha_1}^*(k) \mathcal{E}_\beta(k, k') c_\alpha^*(k')) \xi_t \rangle \left. \right|. \tag{7.22}
\end{aligned}$$

By the Cauchy–Schwarz inequality

$$\begin{aligned}
& |\langle \xi_t, \mathcal{E}_{n-m,m}^{(C)} \xi_t \rangle| \\
&\leq \frac{1}{2} \sum_{k,k'} \rho_{q,k} \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha,\beta} (K(k)^{n-m})_{\alpha_q,\beta} \mathcal{E}_\beta(k', k)^* c_\alpha^*(k') \xi_t \right\| \left\| \sum_{\alpha_2 \in \mathcal{I}_k} (K(k)^m)_{\alpha_q,\alpha_2} c_{\alpha_2}(k) \xi_t \right\| \\
&\quad + \frac{1}{2} \sum_{k,k'} \rho_{q,k} \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha,\beta} (K(k)^{n-m})_{\alpha_q,\beta} c_\alpha^*(k') \mathcal{E}_\beta(k', k)^* \xi_t \right\| \left\| \sum_{\alpha_2 \in \mathcal{I}_k} (K(k)^m)_{\alpha_q,\alpha_2} c_{\alpha_2}(k) \xi_t \right\| \\
&\quad + \frac{1}{2} \sum_{k,k'} \rho_{q,k} \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q,\alpha_1} c_{\alpha_1}(k) \xi_t \right\| \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha,\beta} (K(k)^m)_{\alpha_q,\beta} c_\alpha^*(k') \mathcal{E}_\beta(k, k') \xi_t \right\| \\
&\quad + \frac{1}{2} \sum_{k,k'} \rho_{q,k} \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q,\alpha_1} c_{\alpha_1}(k) \xi_t \right\| \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha,\beta} (K(k)^m)_{\alpha_q,\beta} \mathcal{E}_\beta(k, k') c_\alpha^*(k') \xi_t \right\| \\
&=: (\text{I}^{(C)}) + (\text{II}^{(C)}) + (\text{III}^{(C)}) + (\text{IV}^{(C)}) .
\end{aligned}$$

In contribution $(\text{I}^{(C)})$, the first norm is bounded by Lemma 6.7 and the second norm as the second norm of $(\text{II}^{(B)})$. We end up with

$$(\text{I}^{(C)}) \leq e^{C\|t\|\|\hat{V}\|_1} \sum_{k,k'} \rho_{q,k} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-1} M^{-\frac{3}{2}} . \tag{7.23}$$

The contribution $(\text{II}^{(C)})$ is bounded analogously and the last two contributions are identical to the first two, under replacement $m \mapsto n - m$. Thus,

$$(\text{II}^{(C)}), (\text{III}^{(C)}), (\text{IV}^{(C)}) \leq e^{C\|t\|\|\hat{V}\|_1} \sum_{k,k'} \rho_{q,k} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-1} M^{-\frac{3}{2}}. \quad (7.24)$$

From the definition (4.11) of $\rho_{q,k}$ and from (6.2), and recalling that $k \in \tilde{\mathcal{C}}^q$ implies $\alpha_q \in \mathcal{I}_k$, it becomes clear that

$$\rho_{q,k} \leq \frac{1}{n_{\alpha_q,k}^2} \leq C \mathbf{n}^{-2}. \quad (7.25)$$

Adding up all contributions renders

$$|\langle \xi_t, \mathcal{E}_{n-m,m}^{(C)} \xi_t \rangle| \leq e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} \sum_{k,k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-3} M^{-\frac{3}{2}}. \quad (7.26)$$

(This bound is independent of the bootstrap assumption.)

Bounding $\mathcal{E}^{(D)}$:

Again, from the definitions (5.3) and (5.4), using Lemmas 4.3 and 4.4, we obtain

$$\begin{aligned} & |\langle \xi_t, \mathcal{E}_{n-m,m}^{(D)} \xi_t \rangle| \\ &= \left| \frac{1}{2} \sum_{k,k'} \rho_{q,k} \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \alpha_1 \in \mathcal{I}_k \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha,\beta} (K(k)^{n-m})_{\alpha_q,\alpha_1} (K(k)^m)_{\alpha_q,\beta} \right. \\ &\quad \times \langle \xi_t, (c_{\alpha_1}(k) \mathcal{E}_\beta(k, k') c_\alpha^*(k') + c_{\alpha_1}(k) c_\alpha^*(k') \mathcal{E}_\beta(k, k')) \xi_t \rangle \\ &\quad + \frac{1}{2} \sum_{k,k'} \rho_{q,k} \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \alpha_2 \in \mathcal{I}_k \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha,\beta} (K(k)^{n-m})_{\alpha_q,\beta} (K(k)^m)_{\alpha_q,\alpha_2} \\ &\quad \times \langle \xi_t, (c_\alpha^*(k') \mathcal{E}_\beta(k, k') c_{\alpha_2}(k) + \mathcal{E}_\beta(k, k') c_\alpha^*(k') c_{\alpha_2}(k)) \xi_t \rangle \\ &\quad \left. + \sum_k \rho_{q,k} \sum_{\alpha,\beta \in \mathcal{I}_k} K(k)_{\alpha,\beta} (K(k)^{n-m})_{\alpha_q,\beta} (K(k)^m)_{\alpha_q,\alpha} \langle \xi_t, \mathcal{E}_\beta(k, k) \xi_t \rangle \right|. \end{aligned} \quad (7.27)$$

By the Cauchy–Schwarz inequality

$$\begin{aligned}
& |\langle \xi_t, \mathcal{E}_{n-m,m}^{(D)} \xi_t \rangle| \\
& \leq \frac{1}{2} \sum_{k,k'} \rho_{q,k} \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q, \alpha_1} c_{\alpha_1}^*(k) \xi_t \right\| \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^m)_{\alpha_q, \beta} \mathcal{E}_\beta(k, k') c_\alpha^*(k') \xi_t \right\| \\
& + \frac{1}{2} \sum_{k,k'} \rho_{q,k} \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q, \alpha_1} c_{\alpha_1}^*(k) \xi_t \right\| \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^m)_{\alpha_q, \beta} c_\alpha^*(k') \mathcal{E}_\beta(k, k') \xi_t \right\| \\
& + \frac{1}{2} \sum_{k,k'} \rho_{q,k} \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^{n-m})_{\alpha_q, \beta} \mathcal{E}_\beta(k, k')^* c_\alpha(k') \xi_t \right\| \left\| \sum_{\alpha_2 \in \mathcal{I}_k} (K(k)^m)_{\alpha_q, \alpha_2} c_{\alpha_2}(k) \xi_t \right\| \\
& + \frac{1}{2} \sum_{k,k'} \rho_{q,k} \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^{n-m})_{\alpha_q, \beta} c_\alpha(k') \mathcal{E}_\beta(k, k')^* \xi_t \right\| \left\| \sum_{\alpha_2 \in \mathcal{I}_k} (K(k)^m)_{\alpha_q, \alpha_2} c_{\alpha_2}(k) \xi_t \right\| \\
& + \sum_k \rho_{q,k} \left\| \sum_{\beta \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q, \beta} (K(k)^{m+1})_{\alpha_q, \beta} \mathcal{E}_\beta(k, k) \xi_t \right\| \|\xi_t\| \\
& =: (\text{I}^{(D)}) + (\text{II}^{(D)}) + (\text{III}^{(D)}) + (\text{IV}^{(D)}) + (\text{V}^{(D)}) .
\end{aligned}$$

The first four contributions are exactly bounded as the four contributions of $\mathcal{E}_{n-m,m}^{(C)}$. For $(\text{V}^{(D)})$, we use $\rho_{q,k} \leq C \mathbf{n}^{-2}$, $\|\xi_t\| = 1$ and apply the Cauchy–Schwarz inequality:

$$\begin{aligned}
(\text{V}^{(D)}) & \leq C \sum_k \mathbf{n}^{-2} \left\| \sum_{\beta \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q, \beta} (K(k)^{m+1})_{\alpha_q, \beta} \mathcal{E}_\beta(k, k) \xi_t \right\| \\
& \leq C \sum_k \mathbf{n}^{-2} \left(\sum_{\beta \in \mathcal{I}_k} |(K(k)^{n-m})_{\alpha_q, \beta} (K(k)^{m+1})_{\alpha_q, \beta}|^2 \right)^{\frac{1}{2}} \left(\sum_{\beta \in \mathcal{I}_k} \|\mathcal{E}_\beta(k, k) \xi_t\|^2 \right)^{\frac{1}{2}} \\
& \leq \sum_k \mathbf{n}^{-2} \left(\sum_{\beta \in \mathcal{I}_k} (C \hat{V}_k)^{2n+2} M^{-4} \right)^{\frac{1}{2}} \left(\mathbf{n}^{-2} \|\mathcal{N}^{\frac{1}{2}} \xi_t\|^2 \right)^{\frac{1}{2}} \tag{7.28} \\
& \leq e^{C\|t\|\|\hat{V}\|_1} \sum_k (C \hat{V}_k)^{n+1} \mathbf{n}^{-3} M^{-\frac{3}{2}} \\
& \leq e^{\tilde{C}_\varepsilon \|t\| \|\hat{V}\|_1} \sum_{k,k'} (C \hat{V}_k)^n (C \hat{V}_{k'}) \mathbf{n}^{-3} M^{-\frac{3}{2}}
\end{aligned}$$

where we used Lemmas 6.1 and 6.3 in the third line. Thus $\mathcal{E}_{n-m,m}^{(D)}$ has the same bound as $\mathcal{E}_{n-m,m}^{(C)}$, namely

$$|\langle \xi_t, \mathcal{E}_{n-m,m}^{(D)} \xi_t \rangle| \leq e^{\tilde{C}_\varepsilon \|t\| \|\hat{V}\|_1} \sum_{k,k'} (C \hat{V}_k)^n (C \hat{V}_{k'}) \mathbf{n}^{-3} M^{-\frac{3}{2}} . \tag{7.29}$$

Bounding $\mathcal{E}^{(E)}$:

Finally, again from (5.3) and (5.4), using Lemmas 4.3 and 4.4, we obtain

$$\begin{aligned}
& |\langle \xi_t, \mathcal{E}_n^{(E)} \xi_t \rangle| \\
&= \left| \frac{1}{2} \sum_{k, k'} \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \alpha_1 \in \mathcal{I}_k}} K(k')_{\alpha, \alpha_q} (K(k)^n)_{\alpha_q, \alpha_1} \times \right. \\
&\quad \times \langle \xi_t, (c_\alpha(k') \mathcal{E}_q^{(g)}(k', k) c_{\alpha_1}^*(k) + \mathcal{E}_q^{(g)}(k', k) c_\alpha(k') c_{\alpha_1}^*(k)) \xi_t \rangle \\
&\quad + \frac{1}{2} \sum_{k, k'} \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^n)_{\alpha_q, \beta} \times \\
&\quad \times \langle \xi_t, (c^*(g_{q,k}) c_\alpha(k') \mathcal{E}_\beta(k', k) + c^*(g_{q,k}) \mathcal{E}_\beta(k', k) c_\alpha(k')) \xi_t \rangle \\
&\quad \left. + \sum_k \rho_{q,k} \sum_{\beta \in \mathcal{I}_k} K(k)_{\alpha_q, \beta} (K(k)^n)_{\alpha_q, \beta} \langle \xi_t, \mathcal{E}_\beta(k, k) \xi_t \rangle \right|. \tag{7.30}
\end{aligned}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}
& |\langle \xi_t, \mathcal{E}_n^{(E)} \xi_t \rangle| \\
&\leq \frac{1}{2} \sum_{k, k'} \left\| \sum_{\alpha \in \mathcal{I}_{k'}} K(k')_{\alpha, \alpha_q} c_\alpha^*(k') \xi_t \right\| \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} \mathcal{E}_q^{(g)}(k', k) c_{\alpha_1}^*(k) \xi_t \right\| \\
&\quad + \frac{1}{2} \sum_{k, k'} \left\| \sum_{\alpha \in \mathcal{I}_{k'}} K(k')_{\alpha, \alpha_q} c_\alpha^*(k') \mathcal{E}_q^{(g)}(k', k) \xi_t \right\| \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} c_{\alpha_1}^*(k) \xi_t \right\| \\
&\quad + \frac{1}{2} \sum_{k, k'} \|c(g_{q,k}) \xi_t\| \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^n)_{\alpha_q, \beta} c_\alpha(k') \mathcal{E}_\beta(k', k) \xi_t \right\| \\
&\quad + \frac{1}{2} \sum_{k, k'} \|c(g_{q,k}) \xi_t\| \left\| \sum_{\substack{\alpha \in \mathcal{I}_{k'} \\ \beta \in \mathcal{I}_k \cap \mathcal{I}_{k'}}} K(k')_{\alpha, \beta} (K(k)^n)_{\alpha_q, \beta} \mathcal{E}_\beta(k', k) c_\alpha(k') \xi_t \right\| \\
&\quad + \sum_k \rho_{q,k} \left\| \sum_{\beta \in \mathcal{I}_k} K(k)_{\alpha_q, \beta} (K(k)^n)_{\alpha_q, \beta} \mathcal{E}_\beta(k, k) \xi_t \right\| \|\xi_t\| \\
&=: (\text{I}^{(E)}) + (\text{II}^{(E)}) + (\text{III}^{(E)}) + (\text{IV}^{(E)}) + (\text{V}^{(E)}) . \tag{7.31}
\end{aligned}$$

The first four contributions are bounded like the four contributions of $\mathcal{E}_n^{(B)}$. The fifth contribution is bounded by the same steps as $(\text{V}^{(D)})$:

$$\begin{aligned}
(\text{V}^{(E)}) &\leq C \sum_k \mathbf{n}^{-2} \left(\sum_{\beta \in \mathcal{I}_k} |K(k)_{\alpha_q, \beta} (K(k)^n)_{\alpha_q, \beta}|^2 \right)^{\frac{1}{2}} \left(\sum_{\beta \in \mathcal{I}_k} \|\mathcal{E}_\beta(k, k) \xi_t\|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_k (C \hat{V}_k)^{n+1} \mathbf{n}^{-2} M^{-\frac{3}{2}} \left(\mathbf{n}^{-2} \|\mathcal{N}^{\frac{1}{2}} \xi_t\|^2 \right)^{\frac{1}{2}} \leq e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} \sum_{k, k'} (C \hat{V}_k)^n (C \hat{V}_{k'}) \mathbf{n}^{-3} M^{-\frac{3}{2}} .
\end{aligned}$$

Recalling $\mathbf{n} = N^{\frac{1}{3}-\frac{\delta}{2}}M^{-\frac{1}{2}}$, the final bound for $|\langle \xi_t, \mathcal{E}_n^{(E)} \xi_t \rangle|$ is thus

$$|\langle \xi_t, \mathcal{E}_n^{(E)} \xi_t \rangle| \leq e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} \sum_{k,k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-2} M^{-1} N^{-\frac{r}{4}+\varepsilon}. \quad (7.32)$$

and the same bound holds for $|\langle \xi_t, (\mathcal{E}_n^{(E)})^* \xi_t \rangle| = |\langle \xi_t, \mathcal{E}_n^{(E)} \xi_t \rangle|$.

Bounding $\mathcal{E}^{(F)}$:

Since $S^* = -S$, $\mathbf{F}_{m,m'}^* = \mathbf{D}_{m',m}$, and $\mathbf{A}_n^* = \mathbf{A}_n$, $\mathbf{C}_{m,m'}^* = \mathbf{C}_{m',m}$, we obtain

$$\begin{aligned} (\mathcal{E}_{n-m,m}^{(F)})^* &\stackrel{(7.4)}{=} -[S^*, \mathbf{F}_{n-m,m}^*] - \mathbf{A}_{n+1}^* - \mathbf{C}_{n-m+1,m}^* - \mathbf{C}_{n-m,m+1}^* \\ &= [S, \mathbf{D}_{m,n-m}] - \mathbf{A}_{n+1} - \mathbf{C}_{m,n-m+1} - \mathbf{C}_{m+1,n-m} \stackrel{(7.4)}{=} \mathcal{E}_{m,n-m}^{(D)}. \end{aligned} \quad (7.33)$$

Hence, $|\langle \xi_t, \mathcal{E}_{n-m,m}^{(F)} \xi_t \rangle| = |\langle \xi_t, (\mathcal{E}_{n-m,m}^{(F)})^* \xi_t \rangle| = |\langle \xi_t, \mathcal{E}_{m,n-m}^{(D)} \xi_t \rangle|$, so with (7.29),

$$|\langle \xi_t, \mathcal{E}_{n-m,m}^{(F)} \xi_t \rangle| \leq e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} \sum_{k,k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-3} M^{-\frac{3}{2}}. \quad (7.34)$$

Summing up the bounds:

Consider again (7.5). If n is even, then there are $1 + 1 + \sum_{m=1}^{n-1} \binom{n}{m} = \sum_{m=0}^n \binom{n}{m} = 2^n$ terms to bound. Since $r \leq \frac{2}{3}$ and $\delta < \frac{1}{6}$, we have $-\frac{r}{4} \geq -\frac{1}{3} + \frac{\delta}{2}$, so

$$\begin{aligned} |\langle \xi_t, \mathcal{E}_{n+1,q} \xi_t \rangle| &\leq 2^n e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} \sum_{k,k'} (C\hat{V}_k)^n (C\hat{V}_{k'}) \mathbf{n}^{-2} M^{-1} N^{-\frac{r}{4}+\varepsilon} \\ &\leq e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} (2C)^{n+1} \|\hat{V}\|_1^{n+1} N^{-\frac{2}{3}+\delta-\frac{r}{4}+\varepsilon} \\ &\leq \mathfrak{C}_2^{n+1} e^{\tilde{c}_\varepsilon |t| \|\hat{V}\|_1} \|\hat{V}\|_1^{n+1} N^{-\frac{2}{3}+\delta-\frac{r}{4}+\varepsilon}. \end{aligned} \quad (7.35)$$

This is (7.3), which we wanted to prove. If $n = 2s + 1$ is odd, then the number of occurring terms is $1 + 1 + 2 \sum_{m=1}^s \binom{n}{m} = \sum_{m=0}^n \binom{n}{m} = 2^n$, and (7.35) still holds. \square

The second bound required to prove Lemma 8.1 is the following.

Lemma 7.2 (Bound on the Bosonized Commutator). *Suppose \hat{V} is non-negative and compactly supported. Suppose there is $r \geq 0$ and $C > 0$ such that for all $t \in [-1, 1]$ and all $q' \in \mathbb{Z}^3$ we have*

$$\langle \xi_t, a_{q'}^* a_{q'} \xi_t \rangle \leq CN^{-r}. \quad (7.36)$$

Then there exist constants $\mathfrak{c}_3, \mathfrak{C}_3 > 0$ such that

$$|\langle \xi_t, \text{ad}_{q,(b)}^n \xi_t \rangle| \leq \mathfrak{c}_3 \mathfrak{C}_3^n e^{C|t| \|\hat{V}\|_1} \|\hat{V}\|_1^n, \quad (7.37)$$

for all $n, N \in \mathbb{N}$, $q \in \mathbb{Z}^3$ and $t \in [-1, 1]$.

Remark. It becomes clear from the following proof that we could also obtain

$$|\langle \xi_t, \text{ad}_{q,(b)}^n \xi_t \rangle| \leq \mathfrak{c}_3 \mathfrak{C}_3^n e^{C|t|\|\hat{V}\|_1} \|\hat{V}\|_1^n \max\{N^{-\frac{1}{3}-\frac{r}{2}+\frac{\delta}{2}}, N^{-\frac{2}{3}+\delta}\}. \quad (7.38)$$

However, applying Lemma 7.2 in (8.5), we only need a bound which is $\mathcal{O}(1)$ in N .

Proof. Recall (5.2). If q is not inside any patch, then $\text{ad}_{q,(b)}^n = 0$, so the statement is trivially satisfied. We may therefore assume that $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$.

We adopt the convention $\sum_k = \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3}$ for this proof. By Lemma 6.1 and by (7.25) we obtain

$$|\langle \xi_t, \mathbf{A}_n \xi_t \rangle| = \sum_k (K(k)^n)_{\alpha_q, \alpha_q} \rho_{q,k} \langle \xi_t, \xi_t \rangle \leq \sum_k (C\hat{V}_k)^n M^{-1} \mathbf{n}^{-2} = \sum_k (C\hat{V}_k)^n N^{-\frac{2}{3}+\delta}. \quad (7.39)$$

For \mathbf{B}_n we use Lemmas 6.5 and 6.6:

$$\begin{aligned} |\langle \xi_t, \mathbf{B}_n \xi_t \rangle| &\leq \sum_k \|c(g_{q,k}) \xi_t\| \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha_1} c_{\alpha_1}(k) \xi_t \right\| \\ &\leq \sum_k \frac{C}{n_{\alpha_q, k}} N^{-\frac{r}{2}} \|(K(k)^n)_{\alpha_q, \cdot}\|_2 \|\mathcal{N}_\delta^{\frac{1}{2}} \xi_t\|. \end{aligned} \quad (7.40)$$

Now, by Lemma 6.2, we have $n_{\alpha_q, k} \geq C\mathbf{n}$ (recall that $k \in \tilde{\mathcal{C}}^q$ implies $\alpha_k \in \mathcal{I}_k$), and (7.9) allows us to bound $\|\mathcal{N}_\delta^{\frac{1}{2}} \xi_t\| \leq e^{C|t|\|\hat{V}\|_1}$, so together with (7.8) we obtain

$$|\langle \xi_t, \mathbf{B}_n \xi_t \rangle| \leq e^{C|t|\|\hat{V}\|_1} \sum_k (C\hat{V}_k)^n \mathbf{n}^{-1} N^{-\frac{r}{2}} M^{-\frac{1}{2}} = e^{C|t|\|\hat{V}\|_1} \sum_k (C\hat{V}_k)^n N^{-\frac{1}{3}-\frac{r}{2}+\frac{\delta}{2}}. \quad (7.41)$$

The same holds for $|\langle \xi_t, (\mathbf{B}_n)^* \xi_t \rangle| = |\langle \xi_t, \mathbf{B}_n \xi_t \rangle|$.

The bound on $\mathbf{C}_{n-m, m}$ is obtained by the same steps, together with (7.25):

$$\begin{aligned} |\langle \xi_t, \mathbf{C}_{n-m, m} \xi_t \rangle| &\leq \sum_k \rho_{q,k} \left\| \sum_{\alpha_1 \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q, \alpha_1} c_{\alpha_1}(k) \xi_t \right\| \left\| \sum_{\alpha_2 \in \mathcal{I}_k} (K(k)^m)_{\alpha_q, \alpha_2} c_{\alpha_2}(k) \xi_t \right\| \\ &\leq \sum_k \rho_{q,k} \|(K(k)^{n-m})_{\alpha_q, \cdot}\|_2 \|\mathcal{N}_\delta^{\frac{1}{2}} \xi_t\| \|(K(k)^m)_{\alpha_q, \cdot}\|_2 \|\mathcal{N}_\delta^{\frac{1}{2}} \xi_t\| \\ &\leq e^{C|t|\|\hat{V}\|_1} \sum_k (C\hat{V}_k)^n M^{-1} \mathbf{n}^{-2} = e^{C|t|\|\hat{V}\|_1} \sum_k (C\hat{V}_k)^n N^{-\frac{2}{3}+\delta}. \end{aligned} \quad (7.42)$$

For $\mathbf{D}_{n-m, m}$, since $\mathbf{D}_{n-m, m}$ is obtained from $\mathbf{C}_{n-m, m}$ by replacing $c_{\alpha_1}(k)$ by $c_{\alpha_1}^*(k)$, so we only need to replace $\|\mathcal{N}_\delta^{\frac{1}{2}} \xi_t\|^2 \leq e^{C|t|\|\hat{V}\|_1}$ by $\|(\mathcal{N}_\delta + 1)^{\frac{1}{2}} \xi_t\|^2 \leq e^{C|t|\|\hat{V}\|_1}$, yielding

$$|\langle \xi_t, \mathbf{D}_{n-m, m} \xi_t \rangle| \leq e^{C|t|\|\hat{V}\|_1} \sum_k (C\hat{V}_k)^n N^{-\frac{2}{3}+\delta}. \quad (7.43)$$

Since $\mathbf{F}_{n-m, m}^* = \mathbf{D}_{m, n-m}$ the same bound also applies to $|\langle \xi_t, \mathbf{F}_{m, n-m} \xi_t \rangle|$.

Finally, \mathbf{E}_n is obtained from \mathbf{B}_n by replacing $c_{\alpha_1}(k)$ by $c_{\alpha_1}^*(k)$. So again

$$|\langle \xi_t, \mathbf{E}_n \xi_t \rangle| \leq e^{C\|t\|\hat{V}\|_1} \sum_k (C\hat{V}_k)^n N^{-\frac{1}{3}-\frac{r}{2}+\frac{\delta}{2}}. \quad (7.44)$$

The total number of terms involved for even n is now $2^{n-1} + 1 + 1 + \sum_{m=1}^{n-1} \binom{n}{m} = 2^{n-1} + 2^n < 2^{n+1}$, while for odd $n = 2s + 1$, it is $1 + 1 + \sum_{m=1}^s \binom{n}{m} + \sum_{m=1}^s \binom{n}{m} = 2^n$. So in any case, we have

$$|\langle \xi_t, \text{ad}_{q,(b)}^n \xi_t \rangle| \leq e^{C\|t\|\hat{V}\|_1} (2C)^n \|\hat{V}\|_1^n \max\{N^{-\frac{1}{3}-\frac{r}{2}+\frac{\delta}{2}}, N^{-\frac{2}{3}+\delta}\}. \quad \square$$

As the exponents on N are negative, this implies (7.37).

8 Proof of Theorem 3.1

In this section we prove Theorem 3.1, which states that the bosonized excitation density $n_q^{(b)}$ is a good approximation for the true excitation density n_q . The proof employs a bootstrap iteration.

Lemma 8.1 (Bootstrap Step). *Suppose there is $r \in [0, \frac{2}{3})$ and $C > 0$ such that for all $t \in [-1, 1]$ and $q' \in \mathbb{Z}^3$ we have*

$$\langle \xi_t, a_{q'}^* a_{q'} \xi_t \rangle \leq CN^{-r}. \quad (8.1)$$

Further, suppose that \hat{V} is non-negative and compactly supported. Then, for every $\varepsilon > 0$, there exists a $\tilde{C}_\varepsilon > 0$ such that

$$\left| \langle \xi_t, a_q^* a_q \xi_t \rangle - \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle \Omega, \text{ad}_{q,(b)}^n \Omega \rangle \right| \leq C \|\hat{V}\|_1 e^{\tilde{C}_\varepsilon \|t\| \|\hat{V}\|_1} N^{-\frac{2}{3}+\delta-\frac{r}{4}+\varepsilon} \quad (8.2)$$

for all $t \in [-1, 1]$ and all $q \in \mathbb{Z}^3$.

Proof. If q is not inside any patch the bound is trivial since then $\text{ad}_{q,(b)}^n = 0$ and $\langle \xi_t, a_q^* a_q \xi_t \rangle = 0$. So we may assume that $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$. Further, without loss of generality $t \geq 0$.

We use the abbreviations $\langle A \rangle_\psi := \langle \psi, A\psi \rangle$ and $\text{ad}_q^n := \text{ad}_S^n(a_q^* a_q)$. Further, we denote the scaled n -dimensional unit simplex by $t\Delta^{(n)} := \{(t_1, \dots, t_n) : 0 \leq t_n \leq t_{n-1} \leq \dots \leq t_2 \leq t_1 \leq t\}$.

Recall that $\mathcal{E}_{n,q} = [S, \text{ad}_{q,(b)}^{n-1}] - \text{ad}_{q,(b)}^n$. We expand

$$\begin{aligned} e^{tS} a_q^* a_q e^{-tS} &= \sum_{n=0}^{n_*} \frac{t^n}{n!} \text{ad}_{q,(b)}^n + \sum_{n=1}^{n_*} \int_{t\Delta^{(n)}} dt_1 \dots dt_n e^{t_n S} \mathcal{E}_{n,q} e^{-t_n S} \\ &\quad + \int_{t\Delta^{(n_*+1)}} dt_1 \dots dt_{n_*+1} e^{t_{n_*+1} S} [S, \text{ad}_{q,(b)}^{n_*}] e^{-t_{n_*+1} S}. \end{aligned} \quad (8.3)$$

This expansion can be checked inductively: The case $n_* = 0$ is just the Duhamel formula $e^{tS} B e^{-tS} = B + \int_0^t dt_1 e^{t_1 S} [S, B] e^{-t_1 S}$ with $B = \text{ad}_{q, (b)}^0 = a_q^* a_q$. For the induction step from n_* to $n_* + 1$ we write $[S, \text{ad}_{q, (b)}^{n_*}] = \text{ad}_{q, (b)}^{n_*+1} + \mathcal{E}_{n_*+1, q}$. Then, we apply the Duhamel formula to $\text{ad}_{q, (b)}^{n_*+1}$ and use $\int_{t\Delta^{(n_*+1)}} dt_1 \dots dt_n \text{ad}_{q, (b)}^{n_*+1} = \frac{t^{n_*+1}}{(n_*+1)!} \text{ad}_{q, (b)}^{n_*+1}$.

Now, using $\xi_t = e^{-tS} \Omega$, this expansion renders

$$\begin{aligned} & \langle \xi_t, a_q^* a_q \xi_t \rangle - \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle \Omega, \text{ad}_{q, (b)}^n \Omega \rangle \\ &= - \sum_{n=n_*+1}^{\infty} \frac{t^n}{n!} \langle \text{ad}_{q, (b)}^n \rangle_{\Omega} + \sum_{n=1}^{n_*} \int_{t\Delta^{(n)}} dt_1 \dots dt_n \langle \mathcal{E}_{n, q} \rangle_{\xi_{t_n}} \\ &+ \int_{t\Delta^{(n_*+1)}} dt_1 \dots dt_{n_*+1} \left(\langle \text{ad}_{q, (b)}^{n_*+1} \rangle_{\xi_{t_{n_*+1}}} + \langle \mathcal{E}_{n_*+1, q} \rangle_{\xi_{t_{n_*+1}}} \right). \end{aligned} \quad (8.4)$$

Applying Lemmas 7.1 and 7.2, as well as $\int_{t\Delta^{(n)}} dt_1 \dots dt_n 1 = \frac{t^n}{n!}$, we obtain

$$\begin{aligned} & \left| \langle \xi_t, a_q^* a_q \xi_t \rangle - \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle \Omega, \text{ad}_{q, (b)}^n \Omega \rangle \right| \\ & \leq \mathfrak{c}_3 \sum_{n=n_*+1}^{\infty} \frac{(t\mathfrak{C}_3 \|\hat{V}\|_1)^n}{n!} + \mathfrak{c}_2 \|\hat{V}\|_1 e^{\tilde{c}_\varepsilon t \|\hat{V}\|_1} \sum_{n=0}^{n_*} \frac{t^n (\mathfrak{C}_2 \|\hat{V}\|_1)^n}{n!} N^{-\frac{2}{3} + \delta - \frac{r}{4} + \varepsilon} \\ &+ \frac{\mathfrak{c}_3 e^{Ct \|\hat{V}\|_1} (t\mathfrak{C}_3 \|\hat{V}\|_1)^{n_*+1} + \mathfrak{c}_2 \|\hat{V}\|_1 e^{\tilde{c}_\varepsilon t \|\hat{V}\|_1} (t\mathfrak{C}_2 \|\hat{V}\|_1)^{n_*+1} N^{-\frac{2}{3} + \delta - \frac{r}{4} + \varepsilon}}{(n_* + 1)!} \\ & \leq \mathfrak{c}_2 \|\hat{V}\|_1 e^{(\mathfrak{C}_2 + \tilde{c}_\varepsilon) t \|\hat{V}\|_1} N^{-\frac{2}{3} + \delta - \frac{r}{4} + \varepsilon} \\ &+ \mathfrak{c}_3 \left(\sum_{n=n_*+1}^{\infty} \frac{(t\mathfrak{C}_3 \|\hat{V}\|_1)^n}{n!} + e^{Ct \|\hat{V}\|_1} \frac{(t\mathfrak{C}_3 \|\hat{V}\|_1)^{n_*+1}}{(n_* + 1)!} \right). \end{aligned} \quad (8.5)$$

The last line vanishes as $n_* \rightarrow \infty$. Defining $\tilde{C}_\varepsilon := \mathfrak{C}_2 + \tilde{c}_\varepsilon$ completes the proof. \square

Proof of Theorem 3.1. First, note that the statement is trivial if q is not inside any patch, since then $n_q = n_q^{(b)} = 0$. So we can assume that $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$. Now, recall that

$$n_q = \langle \Omega, e^S a_q^* a_q e^{-S} \Omega \rangle = \langle \xi_1, a_q^* a_q \xi_1 \rangle. \quad (8.6)$$

Using Lemma 8.1 (the bootstrap step) with $t = 1$ provides us with

$$|n_q - n_q^{(b)}| = \left| \langle \xi_1, a_q^* a_q \xi_1 \rangle - \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Omega, \text{ad}_{q, (b)}^n \Omega \rangle \right| \leq C \|\hat{V}\|_1 e^{\tilde{C}_\varepsilon \|\hat{V}\|_1} N^{-\frac{2}{3} + \delta - \frac{r}{4} + \varepsilon} \quad (8.7)$$

whenever the bootstrap assumption $\langle \xi_t, a_q^* a_q \xi_t \rangle \leq C N^{-r}$ holds.

By $\|a_{q'}^\sharp\|_{\text{op}} = 1$, it is obvious that $\langle \xi_t, a_{q'}^* a_{q'} \xi_t \rangle \leq 1$, so the bootstrap assumption is initially fulfilled for $r = 0$.

Using (5.7) and applying the estimates (6.1) and (7.25) we obtain

$$\begin{aligned} n_q^{(b)} &= \sum_{m=1}^{\infty} \frac{2^{2m-1}}{(2m)!} \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} (K(k)^{2m})_{\alpha_q, \alpha_q} \rho_{q,k} \\ &\leq \sum_{m=1}^{\infty} \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{2^{2m-1}}{(2m)!} (C\hat{V}_k)^{2m} M^{-1} \mathbf{n}^{-2} = \mathcal{O}(N^{-\frac{2}{3}+\delta}). \end{aligned} \quad (8.8)$$

Thus

$$n_q = |n_q - n_q^{(b)}| + n_q^{(b)} \leq CN^{-\frac{2}{3}+\delta-\frac{r}{4}+\varepsilon} + CN^{-\frac{2}{3}+\delta}. \quad (8.9)$$

So the best exponent that repeated application of the bootstrap step can yield is $r = \frac{2}{3} - \delta$. And indeed, this degree is reached within the bootstrap after finitely many steps. To see this, observe that the map describing the improvement of the bootstrap constant in one step, $f : r \mapsto \frac{2}{3} - \delta + \frac{r}{4} - \varepsilon$, is a contraction which has a fixed point at

$$r^* = \frac{2}{3} - \delta + \frac{r^*}{4} - \varepsilon \quad \Leftrightarrow \quad r^* = \frac{2}{3} + \frac{4}{3} \left(\frac{1}{6} - \delta - \varepsilon \right). \quad (8.10)$$

Since $\delta < \frac{1}{6}$, we can always write $\delta = \frac{1}{6} - \kappa$ with $\kappa > 0$ and set $\varepsilon := \frac{\kappa}{2}$. Then, the fixed point is at

$$r^* = \frac{2}{3} + \frac{2}{3}\kappa > \frac{2}{3} > \frac{2}{3} - \delta. \quad (8.11)$$

Since the n -fold iteration $f^n(r)$ converges to r^* for $n \rightarrow \infty$ and $r^* > \frac{2}{3} - \delta$, there exists $n_0 \in \mathbb{N}$ such that $f^n(r) > \frac{2}{3} - \delta$ for all $n \geq n_0$. I.e., after at most n_0 steps, the term $n_q^{(b)} \sim N^{-\frac{2}{3}+\delta}$ becomes dominant in (8.9). So after at most n_0 bootstrap steps we have

$$n_q \leq CN^{-\frac{2}{3}+\delta}.$$

Now, we apply (8.7) once more with $r = \frac{2}{3} - \delta$ and obtain

$$|n_q - n_q^{(b)}| \leq C \|\hat{V}\|_1 e^{\tilde{C}_\varepsilon \|\hat{V}\|_1} N^{-\frac{5}{6} + \frac{5}{4}\delta + \varepsilon}. \quad \square$$

Remark. We remark that the number of bootstrap steps is uniformly bounded in the choice of $\delta \in (0, \frac{1}{6})$, as the distance of the desired r to the fixed point is always $r^* - \frac{2}{3} - \delta > \frac{1}{9}$ and the Lipschitz constant of the contraction f is $1/4$. Thus, despite δ being fixed throughout all proofs, the constant C in (8.2) is even uniform in all choices of δ .

9 Proof of Proposition 3.2

Proof of Proposition 3.2. Starting from (3.2), we have to compute the diagonal matrix elements of $\cosh(2K(k)) - 1$. We use the identities [4, (7.4)], where in all the proof we

suppress the k -dependence.

$$\cosh(K) = \frac{|S_1^T| + |S_1^T|^{-1}}{2}, \quad \sinh(K) = \frac{|S_1^T| - |S_1^T|^{-1}}{2}, \quad (9.1)$$

This yields

$$\cosh(2K) - 1 = 2 \sinh(K)^2 = \frac{1}{2}(|S_1^T|^2 - 2 + |S_1^T|^{-2}). \quad (9.2)$$

The $|\mathcal{I}_k| \times |\mathcal{I}_k|$ -matrices $|S_1^T|^2$ and $|S_1^T|^{-2}$ can be diagonalized following the steps in [4, (7.7)] and thereafter: We introduce

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & -\mathbb{I} \end{pmatrix}, \quad (9.3)$$

with \mathbb{I} being the $|\mathcal{I}_k^+| \times |\mathcal{I}_k^+|$ identity matrix, and obtain

$$\begin{aligned} U^T |S_1^T|^2 U &= \begin{pmatrix} d^{\frac{1}{2}}(d^{\frac{1}{2}}(d+2b)d^{\frac{1}{2}})^{-\frac{1}{2}}d^{\frac{1}{2}} & 0 \\ 0 & (d+2b)^{\frac{1}{2}}((d+2b)^{\frac{1}{2}}d(d+2b)^{\frac{1}{2}})^{-\frac{1}{2}}(d+2b)^{\frac{1}{2}} \end{pmatrix} \\ U^T |S_1^T|^{-2} U &= \begin{pmatrix} d^{-\frac{1}{2}}(d^{\frac{1}{2}}(d+2b)d^{\frac{1}{2}})^{\frac{1}{2}}d^{-\frac{1}{2}} & 0 \\ 0 & (d+2b)^{-\frac{1}{2}}((d+2b)^{\frac{1}{2}}d(d+2b)^{\frac{1}{2}})^{\frac{1}{2}}(d+2b)^{-\frac{1}{2}} \end{pmatrix} \end{aligned}$$

in terms of the matrices d and b defined in (2.13). The required diagonal matrix element of (9.2) is thus

$$\begin{aligned} (\cosh(2K) - 1)_{\alpha_q, \alpha_q} &= \frac{1}{4} \left(-4 + \langle \alpha_q | d^{\frac{1}{2}}(d^{\frac{1}{2}}(d+2b)d^{\frac{1}{2}})^{-\frac{1}{2}}d^{\frac{1}{2}} | \alpha_q \rangle \right. \\ &\quad + \langle \alpha_q | (d+2b)^{\frac{1}{2}}((d+2b)^{\frac{1}{2}}d(d+2b)^{\frac{1}{2}})^{-\frac{1}{2}}(d+2b)^{\frac{1}{2}} | \alpha_q \rangle \\ &\quad + \langle \alpha_q | d^{-\frac{1}{2}}(d^{\frac{1}{2}}(d+2b)d^{\frac{1}{2}})^{\frac{1}{2}}d^{-\frac{1}{2}} | \alpha_q \rangle \\ &\quad \left. + \langle \alpha_q | (d+2b)^{-\frac{1}{2}}((d+2b)^{\frac{1}{2}}d(d+2b)^{\frac{1}{2}})^{\frac{1}{2}}(d+2b)^{-\frac{1}{2}} | \alpha_q \rangle \right) \\ &=: \frac{1}{4} (-4 + (A) + (B) + (C) + (D)), \end{aligned} \quad (9.4)$$

with $|\alpha_q\rangle \in \mathbb{C}^{|\mathcal{I}_k^+|}$ denoting the canonical basis vector corresponding to the patch B_{α_q} if $\alpha_q \in \mathcal{I}_k^+$ or to the patch opposite to B_{α_q} if $\alpha_q \in \mathcal{I}_k^-$. Using the abbreviations (3.5) and

$$g_k := \frac{\hat{V}_k}{2\hbar\kappa N|k|}$$

we can write

$$\begin{aligned} d &= \sum_{\alpha \in \mathcal{I}_k^+} \lambda_\alpha |\alpha\rangle\langle\alpha|, \quad \text{therefore} \quad d^s |\alpha\rangle = \lambda_\alpha^s |\alpha\rangle \quad \forall s \in \mathbb{R}; \\ b &= g_k |n\rangle\langle n| \quad \text{with} \quad |n\rangle = \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha,k} |\alpha\rangle. \end{aligned} \quad (9.5)$$

To evaluate (A), (B), (C), and (D), we make use of the integral identities

$$A^{\frac{1}{2}} = \frac{2}{\pi} \int_0^\infty \left(1 - \frac{\mu^2}{A + \mu^2}\right) d\mu, \quad A^{-\frac{1}{2}} = \frac{2}{\pi} \int_0^\infty \frac{1}{A + \mu^2} d\mu, \quad (9.6)$$

which are valid for any symmetric matrix A , as well as the Sherman–Morrison formula

$$(A + |v\rangle\langle w|)^{-1} = A^{-1} - \frac{A^{-1}|v\rangle\langle w|A^{-1}}{1 + \langle w|A^{-1}|v\rangle}. \quad (9.7)$$

Evaluation of (A): Applying (9.5), we may simplify

$$(A) = \lambda_{\alpha_q} \langle \alpha_q | (d^{\frac{1}{2}}(d + 2b)d^{\frac{1}{2}})^{-\frac{1}{2}} | \alpha_q \rangle = \lambda_{\alpha_q} \langle \alpha_q | (d^2 + 2\tilde{b})^{-\frac{1}{2}} | \alpha_q \rangle \quad (9.8)$$

with the rank-one operator

$$\tilde{b} := d^{\frac{1}{2}}bd^{\frac{1}{2}} = g_k d^{\frac{1}{2}}|n\rangle\langle n|d^{\frac{1}{2}} = g_k|\tilde{n}\rangle\langle\tilde{n}| \quad \text{where} \quad |\tilde{n}\rangle := d^{\frac{1}{2}}|n\rangle = \sum_{\alpha \in \mathcal{I}_k^+} \lambda_{\alpha}^{\frac{1}{2}} n_{\alpha,k} |\alpha\rangle. \quad (9.9)$$

Applying the integral identities (9.6) and the Sherman–Morrison formula, we obtain

$$\begin{aligned} (d^2 + 2\tilde{b})^{-\frac{1}{2}} &= \frac{2}{\pi} \int_0^\infty \frac{d\mu}{\mu^2 + d^2 + 2\tilde{b}} \\ &= \frac{2}{\pi} \int_0^\infty \left(\frac{1}{\mu^2 + d^2} - \frac{2g_k(\mu^2 + d^2)^{-1}|\tilde{n}\rangle\langle\tilde{n}|(\mu^2 + d^2)^{-1}}{1 + 2g_k\langle\tilde{n}|(\mu^2 + d^2)^{-1}|\tilde{n}\rangle} \right) d\mu \\ &= d^{-1} - \frac{2}{\pi} \int_0^\infty \frac{2g_k(\mu^2 + d^2)^{-1}|\tilde{n}\rangle\langle\tilde{n}|(\mu^2 + d^2)^{-1}}{1 + 2g_k\langle\tilde{n}|(\mu^2 + d^2)^{-1}|\tilde{n}\rangle} d\mu. \end{aligned} \quad (9.10)$$

Sandwiching this expression with $|\alpha_q\rangle$ and multiplying by λ_{α_q} renders

$$\begin{aligned} (A) &= \lambda_{\alpha_q} \langle \alpha_q | d^{-1} | \alpha_q \rangle - \frac{2}{\pi} \int_0^\infty \frac{2g_k \lambda_{\alpha_q} |\langle \alpha_q | (\mu^2 + d^2)^{-1} |\tilde{n}\rangle|^2}{1 + 2g_k \langle \tilde{n} | (\mu^2 + d^2)^{-1} |\tilde{n}\rangle} d\mu \\ &= 1 - \frac{2}{\pi} \int_0^\infty \frac{2g_k n_{\alpha_q,k}^2 \lambda_{\alpha_q}^2 (\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + 2g_k \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha,k}^2 (\mu^2 + \lambda_{\alpha}^2)^{-1} \lambda_{\alpha}} d\mu. \end{aligned} \quad (9.11)$$

Evaluation of (B): By means of the integral identities (9.6) we get

$$\begin{aligned} ((d + 2b)^{\frac{1}{2}}d(d + 2b)^{\frac{1}{2}})^{-\frac{1}{2}} &= \frac{2}{\pi} \int_0^\infty \frac{d\mu}{(d + 2b)^{\frac{1}{2}}d(d + 2b)^{\frac{1}{2}} + \mu^2} \\ &= \frac{2}{\pi} \int_0^\infty \left((d + 2b)^{\frac{1}{2}} (d + \mu^2(d + 2b)^{-1}) (d + 2b)^{\frac{1}{2}} \right)^{-1} d\mu. \end{aligned} \quad (9.12)$$

Thus

$$\begin{aligned} (B) &= \langle \alpha_q | (d + 2b)^{\frac{1}{2}} ((d + 2b)^{\frac{1}{2}}d(d + 2b)^{\frac{1}{2}})^{-\frac{1}{2}} (d + 2b)^{\frac{1}{2}} | \alpha_q \rangle \\ &= \frac{2}{\pi} \int_0^\infty \langle \alpha_q | (d + \mu^2(d + 2b)^{-1})^{-1} | \alpha_q \rangle d\mu. \end{aligned} \quad (9.13)$$

To compute the integrand, we use the Sherman–Morrison formula twice. First

$$(d + 2b)^{-1} = (d + 2g_k|n\rangle\langle n|)^{-1} = d^{-1} - \frac{2g_k d^{-1}|n\rangle\langle n|d^{-1}}{1 + 2g_k\langle n|d^{-1}|n\rangle}$$

and in the second step then

$$\begin{aligned} (d + \mu^2(d + 2b)^{-1})^{-1} &= \left(d + \mu^2 d^{-1} - \frac{2g_k \mu^2}{1 + 2g_k\langle n|d^{-1}|n\rangle} d^{-1}|n\rangle\langle n|d^{-1} \right)^{-1} \\ &= (d + \mu^2 d^{-1})^{-1} + \frac{2g_k \mu^2 (d^2 + \mu^2)^{-1} |n\rangle\langle n| (d^2 + \mu^2)^{-1}}{1 + 2g_k\langle n|d^{-1}|n\rangle - 2g_k \mu^2 \langle n|d^{-1}(d^2 + \mu^2)^{-1}|n\rangle}. \end{aligned}$$

Using this in the integral and using the integral identities (9.6) for the first summand, we obtain

$$\begin{aligned} \text{(B)} &= \frac{2}{\pi} \int_0^\infty \frac{d\mu}{\lambda_{\alpha_q}^2 + \mu^2} \lambda_{\alpha_q} + \frac{2}{\pi} \int_0^\infty \frac{2g_k \mu^2 |\langle \alpha_q | (d^2 + \mu^2)^{-1} | n \rangle|^2}{1 + 2g_k \langle n | d^{-1} | n \rangle - 2g_k \mu^2 \langle n | d^{-1} (d^2 + \mu^2)^{-1} | n \rangle} d\mu \\ &= 1 + \frac{2}{\pi} \int_0^\infty \frac{2g_k \mu^2 n_{\alpha_q, k}^2 (\lambda_{\alpha_q}^2 + \mu^2)^{-2}}{1 + 2g_k \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha, k}^2 \lambda_\alpha^{-1} - 2g_k \mu^2 \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha, k}^2 (\lambda_\alpha^2 + \mu^2)^{-1} \lambda_\alpha^{-1}} d\mu \\ &= 1 + \frac{2}{\pi} \int_0^\infty \frac{2g_k n_{\alpha_q, k}^2 \mu^2 (\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + 2g_k \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha, k}^2 (\mu^2 + \lambda_\alpha^2)^{-1} \lambda_\alpha} d\mu. \end{aligned} \quad (9.14)$$

This agrees with (A) up to a replacement of $-2g_k n_{\alpha_q, k}^2 \lambda_{\alpha_q}^2$ by $2g_k n_{\alpha_q, k}^2 \mu^2$ in the numerator.

Evaluation of (C): As in (A) we use (9.5) to simplify

$$(C) = \lambda_{\alpha_q}^{-1} \langle \alpha_q | (d^2 + 2\tilde{b})^{\frac{1}{2}} | \alpha_q \rangle. \quad (9.15)$$

Applying the integral identities (9.6) and the Sherman–Morrison formula, we obtain

$$\begin{aligned} (d^2 + 2\tilde{b})^{\frac{1}{2}} &= \frac{2}{\pi} \int_0^\infty \left(1 - \frac{\mu^2}{\mu^2 + d^2 + 2g_k |\tilde{n}\rangle\langle \tilde{n}|} \right) d\mu \\ &= \frac{2}{\pi} \int_0^\infty \left(1 - \frac{\mu^2}{\mu^2 + d^2} + \mu^2 \frac{2g_k (\mu^2 + d^2)^{-1} |\tilde{n}\rangle\langle \tilde{n}| (\mu^2 + d^2)^{-1}}{1 + 2g_k \langle \tilde{n} | (\mu^2 + d^2)^{-1} | \tilde{n} \rangle} \right) d\mu \\ &= d + \frac{2}{\pi} \int_0^\infty \mu^2 \frac{2g_k (\mu^2 + d^2)^{-1} |\tilde{n}\rangle\langle \tilde{n}| (\mu^2 + d^2)^{-1}}{1 + 2g_k \langle \tilde{n} | (\mu^2 + d^2)^{-1} | \tilde{n} \rangle} d\mu. \end{aligned} \quad (9.16)$$

Plugging this into (9.15) yields

$$\begin{aligned} (C) &= \lambda_{\alpha_q}^{-1} \langle \alpha_q | d | \alpha_q \rangle + \frac{2}{\pi} \int_0^\infty \mu^2 \lambda_{\alpha_q}^{-1} \frac{2g_k |\langle \alpha_q | (\mu^2 + d^2)^{-1} | \tilde{n} \rangle|^2}{1 + 2g_k \langle \tilde{n} | (\mu^2 + d^2)^{-1} | \tilde{n} \rangle} d\mu \\ &= 1 + \frac{2}{\pi} \int_0^\infty \frac{2g_k n_{\alpha_q, k}^2 \mu^2 (\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + 2g_k \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha, k}^2 (\mu^2 + \lambda_\alpha^2)^{-1} \lambda_\alpha} d\mu = \text{(B)}. \end{aligned} \quad (9.17)$$

Evaluation of (D): As for (B), we will make the factors of $(d+2b)^{\frac{1}{2}}$ cancel. Let this time $A := (d+2b)^{\frac{1}{2}}$ and $B := d$. Then

$$\begin{aligned} ((d+2b)^{\frac{1}{2}}d(d+2b)^{\frac{1}{2}})^{\frac{1}{2}} &= (ABA)^{\frac{1}{2}} = \frac{2}{\pi} \int_0^\infty \left(1 - \frac{\mu^2}{ABA + \mu^2}\right) d\mu \\ &= \frac{2}{\pi} \int_0^\infty ABA(ABA + \mu^2)^{-1} d\mu. \end{aligned} \quad (9.18)$$

Now, using $YX^{-1}Y = (Y^{-1}XY^{-1})^{-1}$ for any matrices X and Y , we get

$$\begin{aligned} ABA(ABA + \mu^2)^{-1} &= ABA((ABA)^2 + \mu^2(ABA))^{-1}ABA \\ &= AB(BA^2B + \mu^2B)^{-1}BA = A(A^2 + \mu^2B^{-1})^{-1}A. \end{aligned} \quad (9.19)$$

Thus,

$$\begin{aligned} (d+2b)^{-\frac{1}{2}}((d+2b)^{\frac{1}{2}}d(d+2b)^{\frac{1}{2}})^{\frac{1}{2}}(d+2b)^{-\frac{1}{2}} &= \frac{2}{\pi} \int_0^\infty (A^2 + \mu^2B^{-1})^{-1} d\mu \\ &= d \frac{2}{\pi} \int_0^\infty \frac{d\mu}{\mu^2 + d^2 + 2db} = d \frac{2}{\pi} \int_0^\infty \left(\frac{1}{\mu^2 + d^2} - \frac{2g_k(\mu^2 + d^2)^{-1}d|n\rangle\langle n|(\mu^2 + d^2)^{-1}}{1 + 2g_k\langle n|(\mu^2 + d^2)^{-1}d|n\rangle} \right) d\mu \\ &= 1 - \frac{2}{\pi} \int_0^\infty \frac{2g_k(\mu^2 + d^2)^{-1}d^2|n\rangle\langle n|(\mu^2 + d^2)^{-1}}{1 + 2g_k\langle n|(\mu^2 + d^2)^{-1}d|n\rangle} d\mu. \end{aligned} \quad (9.20)$$

In the expectation value of $|\alpha_q\rangle$ we obtain

$$(D) = 1 - \frac{2}{\pi} \int_0^\infty \frac{2g_k n_{\alpha_q, k}^2 \lambda_{\alpha_q}^2 (\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + 2g_k \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha, k}^2 (\mu^2 + \lambda_\alpha^2)^{-1} \lambda_\alpha} d\mu = (A). \quad (9.21)$$

Plugging (9.11), (9.14), (9.17) and (9.21) into (9.4) results in

$$(\cosh(2K) - 1)_{\alpha_q, \alpha_q} = \frac{1}{\pi} \int_0^\infty \frac{2g_k n_{\alpha_q, k}^2 (\mu^2 - \lambda_{\alpha_q}^2) (\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + 2g_k \sum_{\alpha \in \mathcal{I}_k^+} n_{\alpha, k}^2 (\mu^2 + \lambda_\alpha^2)^{-1} \lambda_\alpha} d\mu. \quad (9.22)$$

Proposition 3.2 now follows by inserting this expression into (3.2). \square

10 Conclusion of the Proof of Theorems 1.1 and 1.2

With Theorem 3.1 and Proposition 3.2 at hand, it remains to show that the replacements $Q_k(\mu) \mapsto Q_k^{(0)}(\mu)$, $\lambda_{\alpha_q} \mapsto \lambda_{q, k}$, and $\tilde{\mathcal{C}}^q \rightarrow \mathcal{C}^q$ have a small impact. To evaluate $Q_k(\mu)$ (see (3.5)), we approximate $\sum_{\alpha \in \mathcal{I}_k^+}$ by an integral over the half-sphere, as around [1, (5.17)]:

$$\frac{2}{M} \sum_{\alpha \in \mathcal{I}_k^+} f(\lambda_\alpha) \approx \int_0^{\frac{\pi}{2}} f(\cos \theta) \sin \theta d\theta. \quad (10.1)$$

We estimate the approximation error with the following lemma.

Lemma 10.1. *Let $k \in \Gamma^{\text{nor}}$. Then, for any Lipschitz-continuous function $f : [0, 1] \rightarrow \mathbb{R}$ with Lipschitz constant \mathfrak{L} , we have*

$$\frac{2}{M} \sum_{\alpha=1}^{\frac{M}{2}} f(\lambda_\alpha) = \int_0^{\frac{\pi}{2}} f(\cos \theta) \sin \theta \, d\theta + \mathfrak{L} \mathcal{O}(M^{-\frac{1}{2}}), \quad (10.2)$$

where $\lambda_\alpha = |\hat{k} \cdot \hat{\omega}_\alpha|$ and we have

$$\frac{2}{M} \sum_{\alpha \in \mathcal{I}_k^+} f(\lambda_\alpha) = \int_0^{\frac{\pi}{2}} f(\cos \theta) \sin \theta \, d\theta + (\mathfrak{L} + \|f\|_\infty) \mathcal{O}(M^{-\frac{1}{2}}) + \|f\|_\infty \mathcal{O}(N^{-\delta}). \quad (10.3)$$

Proof. Recall that \mathcal{I}_k^+ excludes patches with $k \cdot \hat{\omega}_\alpha < N^{-\delta}$. We can write the sum in (10.2) as an integral over a sum of indicator functions: Let $\hat{\omega}_{\theta, \varphi}$ be the unit vector with radial coordinates (θ, φ) and define, with a slight abuse of notation,

$$g(\theta, \varphi) := \sum_{\alpha=1}^{\frac{M}{2}} f(\lambda_\alpha) \chi_{\tilde{P}_\alpha}(\theta, \varphi), \quad \chi_{\tilde{P}_\alpha}(\theta, \varphi) := \begin{cases} 1 & \text{if } k_F \hat{\omega}_{\theta, \varphi} \in \tilde{P}_\alpha \\ 0 & \text{else,} \end{cases} \quad (10.4)$$

where \tilde{P}_α are the disjoint cells on ∂B_F with surface volume $\sigma(\tilde{P}_\alpha) = \frac{4\pi k_F^2}{M}$, defined above (2.5). The l. h. s. of (10.2) then becomes

$$\frac{2}{M} \sum_{\alpha=1}^{\frac{M}{2}} f(\lambda_\alpha) = \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} g(\theta, \varphi) \sin \theta \, d\varphi \, d\theta. \quad (10.5)$$

So (10.2) follows if we can show that

$$|f(\cos \theta) - g(\theta, \varphi)| = \mathfrak{L} \mathcal{O}(M^{-\frac{1}{2}}). \quad (10.6)$$

For almost every (θ, φ) in the integration domain there is now some cell \tilde{P}_α with $k_F \hat{\omega}_{\theta, \varphi} \in \tilde{P}_\alpha$. Denote by $(\theta_\alpha, \varphi_\alpha)$ the center of this cell (so $\hat{\omega}_{\theta_\alpha, \varphi_\alpha} = \hat{\omega}_\alpha$). Obviously, if $(\theta, \varphi) = (\theta_\alpha, \varphi_\alpha)$, then $g(\theta, \varphi) = f(\lambda_\alpha) = f(\cos \theta)$, so the difference in (10.6) is 0. For a general (θ, φ) , by regularity of the cells (2.6),

$$\begin{aligned} |\theta - \theta_\alpha| = \mathcal{O}(M^{-\frac{1}{2}}) &\Rightarrow |\cos \theta - \cos \theta_\alpha| = \mathcal{O}(M^{-\frac{1}{2}}) \\ \Rightarrow |f(\cos \theta) - f(\cos \theta_\alpha)| &= \mathfrak{L} \mathcal{O}(M^{-\frac{1}{2}}). \end{aligned} \quad (10.7)$$

Since $f(\cos \theta_\alpha) = f(\lambda_\alpha) = g(\theta, \varphi)$, this implies (10.6) and hence the first claim, (10.2).

For the second claim (10.3), we need to put an upper bound on the number of excluded patches. Excluding a patch requires $|\hat{k} \cdot \hat{\omega}_\alpha| < |k|^{-1} N^{-\delta} \leq N^{-\delta}$. So the “angle of thickness” $\Delta\theta$ (see Figure 4) of the excluded (inner) belt on the unit sphere is $\sim N^{-\delta}$.

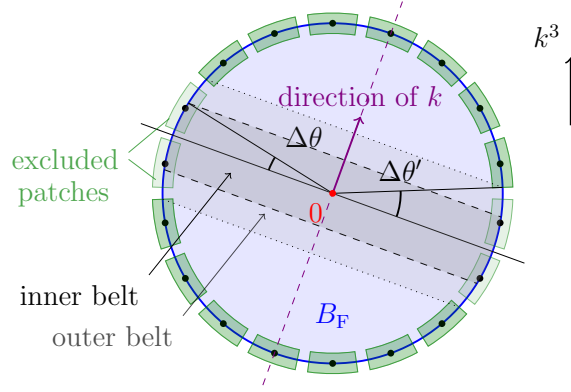


Figure 4: Depiction of the two angles of thickness $\Delta\theta$ and $\Delta\theta'$ of the excluded belts.

By regularity (2.6), the cells \tilde{P}_α corresponding to excluded patches B_α are now contained in an (outer) belt on ∂B_F whose angle of thickness is

$$\Delta\theta' = \mathcal{O}(N^{-\delta}) + \mathcal{O}(M^{-\frac{1}{2}}). \quad (10.8)$$

The outer belt itself takes up a surface area of $\leq 2\pi\Delta\theta'k_F^2$. On the other hand, each cell takes up an area of exactly $\frac{4\pi k_F^2}{M}$, so the number of excluded patches is $\leq \frac{M}{2}\Delta\theta'$. Thus,

$$\left| \frac{2}{M} \sum_{\alpha=1}^{\frac{M}{2}} f(\lambda_\alpha) - \frac{2}{M} \sum_{\alpha \in \mathcal{I}_k^+} f(\lambda_\alpha) \right| \leq \|f\|_\infty \frac{2}{M} \frac{M}{2} \Delta\theta' = \|f\|_\infty \Delta\theta'. \quad (10.9)$$

Combining this result with (10.2) and (10.8) yields (10.3). \square

To apply (10.3) to the evaluation of $Q_k(\mu)$, we need to extract a factor $\frac{2}{M}$ from $n_{\alpha,k}^2$.

Lemma 10.2. *For $k \in \Gamma^{\text{nor}}$ and $\alpha \in \mathcal{I}_k$, the number of particle-hole pairs with momentum difference k in the patch is given by*

$$n_{\alpha,k}^2 = 2\pi k_F^2 \frac{2}{M} |k| \lambda_\alpha \left(1 + \mathcal{O}\left(M^{\frac{1}{2}} N^{-\frac{1}{3}+\delta}\right) \right). \quad (10.10)$$

Proof. In [1, Sect. 3.2], it was shown that the surface area covered by a patch P_α is

$$\sigma(P_\alpha) = \frac{4\pi k_F^2}{M} + \mathcal{O}(M^{-\frac{1}{2}} N^{\frac{1}{3}}) = \frac{4\pi k_F^2}{M} \left(1 + \mathcal{O}\left(M^{\frac{1}{2}} N^{-\frac{1}{3}}\right) \right) \quad (10.11)$$

This can also easily be seen as the widths of the corridors are ~ 1 and the sides of the patches have lengths $\sim k_F M^{-\frac{1}{2}} \sim N^{\frac{1}{3}} M^{-\frac{1}{2}}$. Further, by⁷ [1, Eq. (6.2)]:

$$n_{\alpha,k}^2 = \lambda_\alpha \sigma(P_\alpha) |k| \left(1 + \mathcal{O}\left(M^{\frac{1}{2}} N^{-\frac{1}{3}+\delta}\right) \right). \quad (10.12)$$

This yields the desired result. \square

⁷In [1], λ_α is called $u_\alpha(k)^2$ and $\sigma(P_\alpha)$ is called $k_F^2 \sigma(p_\alpha)$. Note that in [1, (6.2)], it was assumed that $\delta \leq \frac{1}{6} - \frac{\varepsilon}{2}$ with $\varepsilon \in \mathbb{R}$ being the parameter in the choice $M \sim N^{\frac{1}{3}+\varepsilon}$. Since $M \leq CN^{\frac{2}{3}-2\delta}$, this constraint is always satisfied here.

Now we can approximate $Q_k(\mu)$ in (3.5) by an integral.

Lemma 10.3. *For all $k \in \Gamma^{\text{nor}}$, we have*

$$Q_k(\mu) = \frac{3\hat{V}_k}{2\kappa\hbar k_F} \left(1 - \mu \arctan\left(\frac{1}{\mu}\right) + (1 + \mu^{-1})\mathcal{O}(M^{-\frac{1}{2}}) + \mathcal{O}(N^{-\delta}) \right) \times \left(1 + \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}\right) \right). \quad (10.13)$$

Proof. We start by applying Lemma 10.2 to $Q_k(\mu)$ as given in (3.5)

$$Q_k(\mu) = 4\pi g_k k_F^2 |k| \frac{2}{M} \sum_{\alpha \in \mathcal{I}_k^+} \lambda_\alpha^2 (\mu^2 + \lambda_\alpha^2)^{-1} \left(1 + \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}\right) \right). \quad (10.14)$$

Next, we apply Lemma 10.1 with $f(x) = \frac{x^2}{x^2 + \mu^2}$. Here, $\|f\|_\infty \leq 1$ and a short computation reveals that the optimal Lipschitz constant for $x \in [0, 1]$ is

$$\mathfrak{L} = \begin{cases} f'\left(\frac{\mu}{\sqrt{3}}\right) = \frac{3\sqrt{3}}{8\mu} & \text{for } \mu \leq \sqrt{3} \\ f'(1) = \frac{2\mu^2}{(1+\mu^2)^2} & \text{for } \mu > \sqrt{3} \end{cases} \Rightarrow \mathfrak{L} \leq \max\{1, \mu^{-1}\} \leq 1 + \mu^{-1}. \quad (10.15)$$

So applying Lemma 10.1 renders

$$\begin{aligned} Q_k(\mu) &= 4\pi g_k k_F^2 |k| \left(\int_0^{\frac{\pi}{2}} \frac{\cos^2 \theta}{\cos^2 \theta + \mu^2} \sin \theta \, d\theta + \mathcal{O}(M^{-\frac{1}{2}})(1 + \mu^{-1}) + \mathcal{O}(N^{-\delta}) \right) \\ &\quad \times \left(1 + \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}\right) \right) \\ &= 4\pi g_k k_F^2 |k| \left(1 - \mu \arctan\left(\frac{1}{\mu}\right) + (1 + \mu^{-1})\mathcal{O}(M^{-\frac{1}{2}}) + \mathcal{O}(N^{-\delta}) \right) \\ &\quad \times \left(1 + \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}\right) \right). \end{aligned}$$

Finally, we plug in (1.13) $g_k = \frac{\hat{V}_k}{2\hbar\kappa N|k|}$ and (1.18) $N = 8\pi^3 \rho = \frac{4\pi k_F^3}{3}(1 + \mathcal{O}(N^{-\frac{1}{3}}))$ to obtain

$$4\pi g_k k_F^2 |k| = \frac{2\pi k_F^2 \hat{V}_k}{\hbar\kappa N} = \frac{3\hat{V}_k}{2\kappa\hbar k_F} (1 + \mathcal{O}(N^{-\frac{1}{3}})). \quad (10.16)$$

Together with $N^{-\frac{1}{3}} \leq CM^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}$, this renders the claimed result. \square

We are now in the position to establish the following central proposition, which is a refinement of the claim (1.12) of our main result, with a replacement of \mathcal{C}^q by $\tilde{\mathcal{C}}^q$.

Proposition 10.4. *Under the same assumptions as in Theorem 1.1, we have*

$$n_q = \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty \frac{g_k(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} \, d\mu + \mathcal{E}, \quad (10.17)$$

with an error bounded by

$$\pm \mathcal{E} \leq C_\epsilon \|\hat{V}\|_1 e^{C_\epsilon \|\hat{V}\|_1} N^{-\frac{2}{3} - \frac{2}{27} + \epsilon}. \quad (10.18)$$

Proof. The error \mathcal{E} can be decomposed into three sub-errors:

$$\begin{aligned} \pm \mathcal{E} &\leq |n_q - n_q^{(b)}| + \left| n_q^{(b)} - \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty \frac{g_k(\mu^2 - \lambda_{\alpha_q}^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu \right| \\ &\quad + \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty \left(\frac{g_k(\mu^2 - \lambda_{\alpha_q}^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} - \frac{g_k(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} \right) d\mu \right| \\ &=: \mathcal{E}_{(I)} + \mathcal{E}_{(II)} + \mathcal{E}_{(III)}. \end{aligned} \quad (10.19)$$

Theorem 3.1 already bounds

$$\mathcal{E}_{(I)} \leq C \|\hat{V}\|_1 e^{\tilde{C}_\varepsilon \|\hat{V}\|_1} N^{-\frac{5}{6} + \frac{5}{4}\delta + \varepsilon}. \quad (10.20)$$

The two other errors are essentially integrals over differences of functions in μ : We define

$$f_k^\bullet(\mu) := \frac{g_k(\mu^2 - \lambda_{\alpha_q}^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + Q_k^\bullet(\mu)}, \quad \tilde{f}_k^\bullet(\mu) := \frac{g_k(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^\bullet(\mu)}, \quad (10.21)$$

where $\bullet \in \{\cdot, (0)\}$. Then, Proposition 3.2 implies

$$\begin{aligned} \mathcal{E}_{(II)} &= \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty g_k(\mu^2 - \lambda_{\alpha_q}^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2} \left((1 + Q_k(\mu))^{-1} - (1 + Q_k^{(0)}(\mu))^{-1} \right) d\mu \right| \\ &= \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty (f_k(\mu) - f_k^{(0)}(\mu)) d\mu \right| \end{aligned} \quad (10.22)$$

and the error $\mathcal{E}_{(III)}$ becomes

$$\mathcal{E}_{(III)} = \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty (f_k^{(0)}(\mu) - \tilde{f}_k^{(0)}(\mu)) d\mu \right|. \quad (10.23)$$

We start by estimating $\mathcal{E}_{(II)}$. To obtain a bound on $f_k - f_k^{(0)}$, we employ Lemma 10.3, taking into account that $\hbar k_F \sim 1$:

$$\begin{aligned} Q_k(\mu) &= \left(Q_k^{(0)}(\mu) + \hat{V}_k(1 + \mu^{-1})\mathcal{O}(M^{-\frac{1}{2}}) + \hat{V}_k\mathcal{O}(N^{-\delta}) \right) \left(1 + \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}\right) \right) \\ &= \left(Q_k^{(0)}(\mu) + A_1(\mu) + A_2 \right) (1 + A), \end{aligned}$$

where we abbreviated

$$A_1(\mu) := \hat{V}_k(1 + \mu^{-1})\mathcal{O}(M^{-\frac{1}{2}}), \quad A_2 := \hat{V}_k\mathcal{O}(N^{-\delta}), \quad A := \mathcal{O}\left(M^{\frac{1}{2}}N^{-\frac{1}{3}+\delta}\right).$$

We are going to investigate separately how these three errors affect $\mathcal{E}_{(II)}$. Let

$$Q_k^{(1)}(\mu) := \left(Q_k^{(0)}(\mu) + A_1(\mu) \right), \quad Q_k^{(2)}(\mu) := \left(Q_k^{(0)}(\mu) + A_1(\mu) + A_2 \right), \quad (10.24)$$

and define the corresponding functions $f_k^{(1)}, f_k^{(2)}$, by setting $\bullet \in \{(1), (2)\}$ in (10.21). Before splitting $\mathcal{E}_{(\text{II})}$, observe that $(1 + Q_k^{(0)}(\mu)) \geq 1$, since $Q_k(\mu) \geq 0$ and that $Q_k^{(0)}$ is uniformly bounded in μ . This means, no matter how large N is, there exists always a regime $0 \leq \mu \leq cM^{-\frac{1}{2}}$ for some $c > 0$, where the error $A_1(\mu)$ dominates $1 + Q_k^{(0)}(\mu)$. So we must treat this regime separately, otherwise relative errors may blow up. Now, the separate error in the regime of small μ , together with the errors caused by A, A_2 , and $A_1(\mu)$ lead to four error terms in total, with an arbitrary $c > 0$:

$$\begin{aligned} \mathcal{E}_{(\text{II})} &\leq \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^{cM^{-\frac{1}{2}}} (f_k(\mu) - f_k^{(0)}(\mu)) \, d\mu \right| + \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_{cM^{-\frac{1}{2}}}^\infty (f_k(\mu) - f_k^{(2)}(\mu)) \, d\mu \right| \\ &+ \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_{cM^{-\frac{1}{2}}}^\infty (f_k^{(2)}(\mu) - f_k^{(1)}(\mu)) \, d\mu \right| + \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^{cM^{-\frac{1}{2}}} (f_k^{(1)}(\mu) - f_k^{(0)}(\mu)) \, d\mu \right| \\ &=: \mathcal{E}_{(\text{IIa})} + \mathcal{E}_{(\text{IIb})} + \mathcal{E}_{(\text{IIc})} + \mathcal{E}_{(\text{IId})} . \end{aligned} \quad (10.25)$$

For $\mathcal{E}_{(\text{IIa})}$, as the error dominates $(1 + Q_k^{(0)}(\mu))$, we use that $Q_k(\mu), Q_k^{(0)}(\mu) \geq 0$ implies the very coarse bound

$$(1 + Q_k(\mu))^{-1}, (1 + Q_k^{(0)}(\mu))^{-1} \in (0, 1] \quad \Rightarrow \quad \left| (1 + Q_k(\mu))^{-1} - (1 + Q_k^{(0)}(\mu))^{-1} \right| \leq 1 .$$

As $M \gg N^{2\delta}$, we have $M^{-\frac{1}{2}} \ll N^{-\delta} \leq R\lambda_{\alpha_q}$, so for N large enough, we get the estimate

$$\begin{aligned} \mathcal{E}_{(\text{IIa})} &\leq \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} g_k \int_0^{cM^{-\frac{1}{2}}} |\mu^2 - \lambda_{\alpha_q}^2| (\mu^2 + \lambda_{\alpha_q}^2)^{-2} \, d\mu \\ &\leq C \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} g_k \int_0^{\lambda_{\alpha_q}} \frac{\lambda_{\alpha_q}^2 - \mu^2}{(\mu^2 + \lambda_{\alpha_q}^2)^2} \, d\mu \\ &= C \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} g_k \left[\frac{\mu}{\mu^2 + \lambda_{\alpha_q}^2} \right]_{\mu=0}^{\lambda_{\alpha_q}} = C \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{2\pi} g_k \lambda_{\alpha_q}^{-1} . \end{aligned} \quad (10.26)$$

Here, $g_k = \frac{\hat{V}_k}{2\hbar\kappa N|k|} \sim \hat{V}_k N^{-\frac{2}{3}}$. Further, the opening angle of a patch is $\sim M^{-\frac{1}{2}} \ll \lambda_{\alpha_q}$ (see (10.7)), so

$$\lambda_{\alpha_q} = |\hat{k} \cdot \hat{\omega}_{\alpha_q}| = |\hat{k} \cdot \hat{q}| (1 + \mathcal{O}(M^{-\frac{1}{2}})) = \lambda_{q,k} (1 + \mathcal{O}(M^{-\frac{1}{2}})) \quad (10.27)$$

and in particular, $\lambda_{\alpha_q}^{-1} \geq \lambda_{q,k}^{-1} (1 + \mathcal{O}(M^{-\frac{1}{2}}))$. Now, the condition $q \in \mathcal{Q}_\epsilon$ from Theorem 1.1 implies $\lambda_{q,k} \geq \epsilon \Leftrightarrow \lambda_{q,k}^{-1} \leq \epsilon^{-1}$ for the fixed $\epsilon > 0$. As the sum over k comprises $\leq |B_R(0) \cap \mathbb{Z}^3| = \mathcal{O}(1)$ points, we conclude

$$\mathcal{E}_{(\text{IIa})} = \|\hat{V}\|_1 \epsilon^{-1} \mathcal{O}(N^{-\frac{2}{3}} M^{-\frac{1}{2}}) . \quad (10.28)$$

Concerning $\mathcal{E}_{(\text{Ib})}$, we have $(1 + Q_k(\mu)) \geq 1$, so the relative error caused by A on $(1 + Q_k(\mu))$ is $\leq A$, itself. By Taylor's formula, this also entails a relative error of the same order on $(1 + Q_k(\mu))^{-1}$ and thus in the function f_k :

$$\begin{aligned} \left| (1 + Q_k(\mu))^{-1} - (1 + Q_k^{(2)}(\mu))^{-1} \right| &\leq A (1 + Q_k(\mu))^{-1} \\ \Rightarrow \left| f_k(\mu) - f_k^{(2)}(\mu) \right| &\leq A |f_k(\mu)| . \end{aligned} \quad (10.29)$$

So we can effectively pull the relative error out of the integral:

$$\mathcal{E}_{(\text{Ib})} \leq A \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_{cM^{-\frac{1}{2}}}^{\infty} |f_k(\mu)| \, d\mu . \quad (10.30)$$

Now, let us investigate the sign of the function $f_k(\mu)$ in further detail. Since in (10.21), all factors are positive with the exception of $(\mu^2 - \lambda_{\alpha_q}^2) = (\mu - \lambda_{\alpha_q})(\mu + \lambda_{\alpha_q})$, we have $f_k(\mu) < 0 \Leftrightarrow \mu < \lambda_{\alpha_q}$. Thus, after making the integral start at 0 instead of $cM^{-\frac{1}{2}}$, we obtain

$$\mathcal{E}_{(\text{Ib})} \leq A \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \left(\int_0^{\lambda_{\alpha_q}} |f_k(\mu)| \, d\mu + \int_{\lambda_{\alpha_q}}^{\infty} f_k(\mu) \, d\mu \right) . \quad (10.31)$$

The first integral can be estimated as in (10.26)

$$\int_0^{\lambda_{\alpha_q}} |f_k(\mu)| \, d\mu = g_k \int_0^{\lambda_{\alpha_q}} \frac{(\lambda_{\alpha_q}^2 - \mu^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + Q_k(\mu)} \, d\mu \leq \frac{g_k}{2} \lambda_{\alpha_q}^{-1} . \quad (10.32)$$

As $g_k = \hat{V}_k \mathcal{O}(N^{-\frac{2}{3}})$, the first integral in (10.31) is also $\hat{V}_k \epsilon^{-1} \mathcal{O}(N^{-\frac{2}{3}})$. For the second integral, we have the same bound,

$$\int_{\lambda_{\alpha_q}}^{\infty} f_k(\mu) \, d\mu \leq g_k \int_{\lambda_{\alpha_q}}^{\infty} \frac{\mu^2 - \lambda_{\alpha_q}^2}{(\mu^2 + \lambda_{\alpha_q}^2)^2} \, d\mu = g_k \left[-\frac{\mu}{\mu^2 + \lambda_{\alpha_q}^2} \right]_{\mu=\lambda_{\alpha_q}}^{\infty} = \frac{g_k}{2} \lambda_{\alpha_q}^{-1} . \quad (10.33)$$

So in total, we get

$$\mathcal{E}_{(\text{Ib})} = A \|\hat{V}\|_1 \epsilon^{-1} \mathcal{O}(N^{-\frac{2}{3}}) = \|\hat{V}\|_1 \epsilon^{-1} \mathcal{O}\left(M^{\frac{1}{2}} N^{-1+\delta}\right) . \quad (10.34)$$

For $\mathcal{E}_{(\text{Ic})}$, the estimation works similarly: We have

$$(1 + Q_k^{(2)}(\mu)) - (1 + Q_k^{(1)}(\mu)) = A_2 \quad (10.35)$$

and

$$\mathcal{E}_{(\text{Ic})} = A_2 \epsilon^{-1} \mathcal{O}(N^{-\frac{2}{3}}) = \|\hat{V}\|_1 \epsilon^{-1} \mathcal{O}(N^{-\frac{2}{3}-\delta}) . \quad (10.36)$$

For $\mathcal{E}_{(\text{Id})}$, the relative error caused by $A_1(\mu)$ on $(1 + Q_k^{(0)}(\mu))$ depends on⁸ μ : As long as $\mu \geq c$, it is $\mathcal{O}(M^{-\frac{1}{2}})$ and as μ decreases towards $cM^{-\frac{1}{2}}$, it gradually increases to $\mathcal{O}(1)$. Thus, for an $n \in \mathbb{N}$, we define the intervals I_1, \dots, I_n as

$$I_\ell := [cM^{-\frac{\ell}{2n}}, cM^{-\frac{\ell-1}{2n}}]. \quad (10.37)$$

We can then write

$$\mathcal{E}_{(\text{Id})} \leq \sum_{\ell=1}^n \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_{I_\ell} (f_k^{(1)}(\mu) - f_k^{(0)}(\mu)) \, d\mu \right| + \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_c^\infty (f_k^{(1)}(\mu) - f_k^{(0)}(\mu)) \, d\mu \right|.$$

For $\mu \in I_\ell$, the relative error caused by $A_1(\mu)$ on $\hat{V}_k^{-1}(1 + Q_k^{(0)}(\mu))$ and thus on $f_k^{(0)}(\mu)$ is of order

$$A_1(\mu) = \hat{V}_k(1 + \mu^{-1})\mathcal{O}(M^{-\frac{1}{2}}) = \hat{V}_k\mathcal{O}(M^{-\frac{1}{2} + \frac{\ell}{2n}}). \quad (10.38)$$

On the other hand, we have

$$|f_k(\mu)| \leq g_k \lambda_{\alpha_q}^{-2} = \hat{V}_k \epsilon^{-2} \mathcal{O}(N^{-\frac{2}{3}}) \quad (10.39)$$

uniformly in $\mu \in I_\ell$. As the interval has length $|I_\ell| \sim M^{-\frac{\ell-1}{2n}}$, we conclude

$$\sum_{\ell=1}^n \left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_{I_\ell} (f_k^{(1)}(\mu) - f_k^{(0)}(\mu)) \, d\mu \right| = n \|\hat{V}\|_1 \epsilon^{-2} \mathcal{O}(N^{-\frac{2}{3}} M^{-\frac{1}{2} + \frac{1}{2n}}). \quad (10.40)$$

In the regime $\mu \geq c$, the term $A_1(\mu)$ causes a relative error of $\mathcal{O}(M^{-\frac{1}{2}})$ on $\hat{V}_k^{-1}(1 + Q_k^{(0)}(\mu))$, so following the same steps as in (10.29)–(10.34), we may bound the error by

$$\left| \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_c^\infty (f_k^{(1)}(\mu) - f_k^{(0)}(\mu)) \, d\mu \right| = \|\hat{V}\|_1 \epsilon^{-1} \mathcal{O}(N^{-\frac{2}{3}} M^{-\frac{1}{2}}). \quad (10.41)$$

Adding all errors, and choosing n large enough, for any given $\epsilon' > 0$ we obtain

$$\mathcal{E}_{(\text{Id})} = \|\hat{V}\|_1 \epsilon^{-2} \mathcal{O}(N^{-\frac{2}{3}} M^{-\frac{1}{2} + \epsilon'}). \quad (10.42)$$

It remains to estimate $\mathcal{E}_{(\text{III})}$ in (10.23), i.e., the error caused by the replacement of λ_{α_q} by $\lambda_{q,k}$. Again, we split the integral over μ according to the relative error induced by this replacement. The relative error is expected to blow up if either $f_k^{(0)}$ or $\tilde{f}_k^{(0)}$ becomes

⁸By “relative error” at a fixed μ , we mean the maximum of the quotients $\frac{|(1+Q_k^{(0)}(\mu))-(1+Q_k^{(1)}(\mu))|}{(1+Q_k^{(0)}(\mu))}$ and $\frac{|(1+Q_k^{(0)}(\mu))-(1+Q_k^{(1)}(\mu))|}{(1+Q_k^{(1)}(\mu))}$. Both quotients are bounded from above by $CM^{-\frac{1}{2}}$ uniformly in $\mu \geq c$. This upper bound increases as the lower bound for μ is decreased towards $cM^{-\frac{1}{2}}$.

0, which is at $\mu = \lambda_{\alpha_q}$ and $\mu = \lambda_{q,k}$, respectively. By (10.27), we have $|\lambda_{\alpha_q} - \lambda_{q,k}| = \mathcal{O}(M^{-\frac{1}{2}})$ and $\lambda_{\alpha_q}, \lambda_{q,k} \sim 1$. We therefore choose the “critical interval” to be

$$I := [\lambda_{q,k} - \tilde{c}M^{-\beta}, \lambda_{q,k} + \tilde{c}M^{-\beta}] , \quad (10.43)$$

with $\beta \in [0, \frac{1}{2}]$ to be optimized later and $\tilde{c} > 0$ small enough. Then, (10.23) can be further estimated as

$$\mathcal{E}_{\text{III}} \leq \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_I |f_k^{(0)}(\mu) - \tilde{f}_k^{(0)}(\mu)| d\mu + \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_{\mathbb{R}_+ \setminus I} |f_k^{(0)}(\mu) - \tilde{f}_k^{(0)}(\mu)| d\mu . \quad (10.44)$$

Outside the critical interval, we use the relative error induced by the replacement of λ_{α_q} by $\lambda_{q,k}$ to derive an estimate: On the one hand, as $\lambda_{\alpha_q}, \lambda_{q,k} \leq 1$, we have

$$|(\mu^2 \pm \lambda_{\alpha_q}^2) - (\mu^2 \pm \lambda_{q,k}^2)| = |\lambda_{\alpha_q} - \lambda_{q,k}|(\lambda_{\alpha_q} + \lambda_{q,k}) = \mathcal{O}(M^{-\frac{1}{2}}) . \quad (10.45)$$

On the other hand

$$(\mu^2 + \lambda_{\alpha_q}^2) \geq |\mu^2 - \lambda_{\alpha_q}^2| \geq |\mu - \lambda_{\alpha_q}| \lambda_{\alpha_q} \geq c\epsilon M^{-\beta} . \quad (10.46)$$

So each replacement induces a relative error of $\epsilon^{-1}\mathcal{O}(M^{-\frac{1}{2}+\beta})$, which, by Taylor’s formula, results in a relative error of $\epsilon^{-1}\mathcal{O}(M^{-\frac{1}{2}+\beta})$ on $f_k^{(0)}(\mu)$. Following the same arguments as in (10.29) through (10.34), the resulting total error is

$$\sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_{\mathbb{R}_+ \setminus I} |f_k^{(0)}(\mu) - \tilde{f}_k^{(0)}(\mu)| d\mu = \|\hat{V}\|_1 \epsilon^{-2} \mathcal{O}(N^{-\frac{2}{3}} M^{-\frac{1}{2}+\beta}) . \quad (10.47)$$

Within the critical interval (10.45) remains valid, resulting in an absolute error of

$$|f^{(0)}(\mu) - \tilde{f}^{(0)}(\mu)| = g_k \mathcal{O}(M^{-\frac{1}{2}}) = \hat{V}_k \mathcal{O}(N^{-\frac{2}{3}} M^{-\frac{1}{2}}) . \quad (10.48)$$

As the interval width is $|I| \sim M^{-\beta}$, we obtain

$$\sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} \int_I |f_k^{(0)}(\mu) - \tilde{f}_k^{(0)}(\mu)| d\mu = \|\hat{V}\|_1 \mathcal{O}(N^{-\frac{2}{3}} M^{-\frac{1}{2}-\beta}) . \quad (10.49)$$

Comparing (10.47) and (10.49), we see that the optimal bound is attained for $\beta = 0$, given by

$$\mathcal{E}_{\text{III}} = \|\hat{V}\|_1 \epsilon^{-2} \mathcal{O}(N^{-\frac{2}{3}} M^{-\frac{1}{2}}) . \quad (10.50)$$

Finally, we put together all error bounds (10.20), (10.28), (10.34), (10.36), (10.42), and (10.50). Recalling $M \ll N^{2\delta}$ and $\epsilon \sim 1$, it is easy to see that \mathcal{E}_{IIa} , \mathcal{E}_{IIb} and \mathcal{E}_{III} are subleading to \mathcal{E}_{IIc} . Setting $M = N^{2\tau}$, $\delta < \tau < \frac{1}{3} - \delta$ and optimizing the maximum of \mathcal{E}_{I} , \mathcal{E}_{IIb} and \mathcal{E}_{IIc} with respect to (δ, τ) , we find that the optimal error bound is found at $\delta = \frac{2}{27}$ and $\tau = \frac{5}{27}$ and reads

$$\begin{aligned} \pm \mathcal{E} &\leq \mathcal{E}_{\text{I}} + \mathcal{E}_{\text{IIa}} + \mathcal{E}_{\text{IIb}} + \mathcal{E}_{\text{IIc}} + \mathcal{E}_{\text{IIb}} + \mathcal{E}_{\text{III}} \\ &\leq C\epsilon^{-2} \|\hat{V}\|_1 e^{C_\epsilon \|\hat{V}\|_1} N^{-\frac{2}{3} - \frac{2}{27} + \epsilon} , \end{aligned} \quad (10.51)$$

where we used $\|\hat{V}\|_1 \leq \|\hat{V}\|_1 e^{C_\epsilon \|\hat{V}\|_1}$. This establishes (10.17) with $C_\epsilon = C\epsilon^{-2}$. \square

Proof of Theorem 1.1. Following [4, Thm. 1.1], the ground state energy satisfies

$$\inf \sigma(H_N) = E_N^{\text{HF}} + E_N^{\text{RPA}} + \mathcal{O}(N^{-\frac{1}{3}-\alpha}), \quad (10.52)$$

where E_N^{HF} and E_N^{RPA} are some explicit constants (the Hartree–Fock ground state energy and the Random Phase Approximation of the correlation energy) and some $\alpha > 0$. On the other hand, in [1, Sect. 5.5], the energy expectation within the trial state ψ_N was computed. There, the optimal error was attained for the same parameter choice as ours, i. e., $\delta = \frac{2}{27}$, $M = N^{\frac{10}{27}}$, and the result reads [1, Thm. 2.1]

$$\langle \psi_N, H_N \psi_N \rangle \geq E_N^{\text{HF}} + \tilde{E}_N^{\text{RPA}} + \mathcal{O}(N^{-\frac{1}{3}-\frac{1}{27}}), \quad (10.53)$$

where⁹ $|\tilde{E}_N^{\text{RPA}} - E_N^{\text{RPA}}| = \mathcal{O}(N^{-\frac{2}{3}})$. Putting together the bounds establishes (1.10).

It remains to establish (1.12). With Proposition 10.4 at hand, this follows if we can show that extending the sum from $k \in \tilde{\mathcal{C}}^q$ to $k \in \mathcal{C}^q$ never decreases the result by more than $C\epsilon^{-2}\|\hat{V}\|_1 e^{C_\epsilon\|\hat{V}\|_1} N^{-\frac{2}{3}-\frac{2}{27}+\epsilon}$. Now, as $q \in \mathcal{Q}_\epsilon$ still implies $\lambda_{q,k} \geq \epsilon \sim 1$ (compare (1.11) and (1.13)), the same arguments as above apply for any $k \in \mathcal{C}^q \setminus \tilde{\mathcal{C}}^q$, so recalling (5.7) we obtain

$$\begin{aligned} & \left| \frac{1}{\pi} \int_0^\infty \frac{g_k(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu - n_q \right| \\ &= \left| \frac{1}{\pi} \int_0^\infty \frac{g_k(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu - \frac{1}{2n_{\alpha_q,k}^2} (\cosh(2K(k)) - 1)_{\alpha_q, \alpha_q} \right| \\ &\stackrel{(10.19)}{\leq} \mathcal{E}_{\text{(II)}} + \mathcal{E}_{\text{(III)}} \leq C\epsilon^{-2}\|\hat{V}\|_1 e^{C_\epsilon\|\hat{V}\|_1} N^{-\frac{2}{3}-\frac{2}{27}+\epsilon}. \end{aligned} \quad (10.54)$$

Since $\cosh(2K) - 1$ is a positive matrix, we have $(\cosh(2K(k)) - 1)_{\alpha_q, \alpha_q} \geq 0$. Since the number of momenta $k \in (\mathcal{C}^q \setminus \tilde{\mathcal{C}}^q) \cap \mathbb{Z}^3$ is bounded by $|B_R(0) \cap \mathbb{Z}^3| \sim 1$, we conclude

$$\sum_{k \in (\mathcal{C}^q \setminus \tilde{\mathcal{C}}^q) \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty \frac{g_k(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu \geq -C\epsilon^{-2}\|\hat{V}\|_1 e^{C_\epsilon\|\hat{V}\|_1} N^{-\frac{2}{3}-\frac{2}{27}+\epsilon}. \quad (10.55)$$

Together with (10.17), this establishes (1.12). \square

Proof of Theorem 1.2. We just established that the sequence of trial states (ψ_N) defined in (2.3) satisfies the assertions of Theorem 1.1. It remains to establish the lower bound on the quasiparticle weight $Z = 1 - \sup_{q \in B_F} n_q - \sup_{q \in B_F^c} n_q \geq 1 - 2 \sup_{q \in \mathbb{Z}^3} n_q$. Recall that, by definition of ψ_N , we have $n_q = 0$ whenever q is not inside some patch B_{α_q} . So we

⁹This can easily be seen as $E_N^{\text{RPA}} = \mathcal{O}(N^{-\frac{1}{3}})$ and the replacement of κ by κ_0 (or vice versa) induces a relative error of $\mathcal{O}(N^{-\frac{1}{3}})$.

may focus on the case $q \in B_{\alpha_q}$ for some $1 \leq \alpha_q \leq M$. Theorem 3.1 and Proposition 3.2 can be combined to

$$n_q \leq \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} g_k \int_0^\infty \frac{(\mu^2 - \lambda_{\alpha_q}^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + Q_k(\mu)} d\mu + C \|\hat{V}\|_1 e^{\tilde{\mathcal{C}}_\varepsilon \|\hat{V}\|_1} N^{-\frac{5}{6} + \frac{5}{4}\delta + \varepsilon}. \quad (10.56)$$

The integral in the leading order term is estimated as in (10.32) and (10.33) by

$$\left| \int_0^\infty \frac{(\mu^2 - \lambda_{\alpha_q}^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + Q_k(\mu)} d\mu \right| \leq \int_0^{\lambda_{\alpha_q}} \frac{\lambda_{\alpha_q}^2 - \mu^2}{(\mu^2 + \lambda_{\alpha_q}^2)^2} d\mu + \int_{\lambda_{\alpha_q}}^\infty \frac{\mu^2 - \lambda_{\alpha_q}^2}{(\mu^2 + \lambda_{\alpha_q}^2)^2} d\mu \leq \lambda_{\alpha_q}^{-1}.$$

For $k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3$, the definition (3.1) of $\tilde{\mathcal{C}}^q$ entails

$$\lambda_{\alpha_q}^{-1} = |k| |k \cdot \hat{\omega}_\alpha|^{-1} \leq RN^{-\delta}. \quad (10.57)$$

As the sum over such k is finite and $g_k = \frac{\hat{V}_k}{2h\kappa N|k|} \leq C\hat{V}_k N^{-\frac{2}{3}}$, the leading order term in (10.56) is controlled by

$$\sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \frac{1}{\pi} g_k \int_0^\infty \frac{(\mu^2 - \lambda_{\alpha_q}^2)(\mu^2 + \lambda_{\alpha_q}^2)^{-2}}{1 + Q_k(\mu)} d\mu \leq C \|\hat{V}\|_1 N^{-\frac{2}{3} + \delta}. \quad (10.58)$$

Since $\delta < \frac{1}{6}$, the error term in (10.56) is subleading for a suitable finite choice of $\varepsilon > 0$. The optimal common upper bound is then given by

$$n_q \leq C \|\hat{V}\|_1 e^{C\|\hat{V}\|_1} N^{-\frac{2}{3} + \delta}, \quad (10.59)$$

where $\delta = \frac{2}{27}$. This bound is uniform in all $q \in \mathbb{Z}^3$. This renders the desired result. \square

Proof of Proposition 2.1. Comparing (10.17), which holds due to Proposition 10.4, and (2.17), which is what we want to show, we see that it suffices to bound the contributions from $k \in \mathcal{C}^q \setminus \tilde{\mathcal{C}}^q$ by

$$\left| \sum_{k \in (\mathcal{C}^q \setminus \tilde{\mathcal{C}}^q) \cap \mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty \frac{g_k(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu \right| \leq C_\varepsilon \|\hat{V}\|_1 e^{C_\varepsilon \|\hat{V}\|_1} N^{-\frac{2}{3} + \frac{2}{27} + \varepsilon}. \quad (10.60)$$

Further, comparing the definitions of \mathcal{C}^q (1.8) and of $\tilde{\mathcal{C}}^q$ (3.1), we see that $k \in \mathcal{C}^q \setminus \tilde{\mathcal{C}}^q$ can occur only if

- the momentum $q \pm k$, which is on the other side of ∂B_F with respect to q , is outside the patch B_{α_q}
- or if $|k \cdot \hat{\omega}_{\alpha_q}| < N^{-\delta}$

The first case is ruled out by condition (2.16). The second case can be ruled out via $q \in \mathcal{Q}_\varepsilon$ which implies $\lambda_{q,k} \geq \varepsilon$ (compare (1.11)), and $|k| \geq 1$, as

$$|k \cdot \hat{\omega}_{\alpha_q}| = |k| \lambda_{\alpha_q} \stackrel{(10.27)}{=} |k| \lambda_{q,k} (1 + \mathcal{O}(M^{-\frac{1}{2}})) \geq \varepsilon (1 + \mathcal{O}(M^{-\frac{1}{2}})), \quad (10.61)$$

provided N is large enough. In that case, the sum on the l.h.s. of (10.60) is empty. The set of N for which it is not empty is finite, so we conclude (10.60) and thus (2.17) by choosing C_ε large enough. \square

A Bosonization Approximation

In this section we show how $\text{ad}_{q,(b)}^n$, defined in (5.2), arises from a bosonization approximation. We replace the almost-bosonic operators $c^*(g)$, $c(g)$ defined in (4.1) by exactly bosonic operators $\tilde{c}^*(g)$, $\tilde{c}(g)$. Lemma A.1 then shows that the multi-commutator $\text{ad}_S^n(a_q^*a_q)$ becomes $\text{ad}_{q,(b)}^n$ with c^* , c replaced by \tilde{c}^* , \tilde{c} .

The exact bosonic operators \tilde{c}^* , \tilde{c} can be defined as elements of an abstract $*$ -algebra \mathcal{A} . More precisely, we define \mathcal{A} to be the $*$ -algebra generated by

$$\{\tilde{a}_q^*, \tilde{c}_{p,h}^* : q \in \mathbb{Z}^3, p \in B_F^c, h \in B_F\}, \quad (\text{A.1})$$

where we impose the (anti-)commutator relations

$$\begin{aligned} \{\tilde{a}_q, \tilde{a}_{q'}^*\} &= \delta_{q,q'}, \quad [\tilde{c}_{p,h}, \tilde{c}_{p',h'}^*] = \delta_{p,p'} \delta_{h,h'}, \\ \{\tilde{a}_q, \tilde{a}_{q'}\} &= \{\tilde{a}_q^*, \tilde{a}_{q'}^*\} = [\tilde{c}_{p,h}, \tilde{c}_{p',h'}] = [\tilde{c}_{p,h}^*, \tilde{c}_{p',h'}^*] = 0. \end{aligned} \quad (\text{A.2})$$

Further, we impose that $\tilde{c}_{p,h}^*$ behaves like a pair creation operator with respect to $a_q^*a_q$, that is

$$[\tilde{c}_{p,h}^*, \tilde{a}_q^* \tilde{a}_q] = -\tilde{c}_{p,h}^* (\delta_{h,q} + \delta_{p,q}). \quad (\text{A.3})$$

In analogy to (4.1), for $g : B_F^c \times B_F \rightarrow \mathbb{C}$ we define

$$\tilde{c}^*(g) := \sum_{\substack{p \in B_F^c \\ h \in B_F}} g(p, h) \tilde{c}_{p,h}^*, \quad \tilde{c}(g) := \sum_{\substack{p \in B_F^c \\ h \in B_F}} \overline{g(p, h)} \tilde{c}_{p,h}, \quad (\text{A.4})$$

and in analogy to (4.2), for $\alpha \in \mathcal{I}_k$ we define

$$\tilde{c}_\alpha^*(k) := \tilde{c}^*(d_{\alpha,k}) \quad \text{with} \quad d_{\alpha,k}(p, h) = \delta_{p, h \pm k} \frac{1}{n_{\alpha,k}} \chi(p, h : \alpha). \quad (\text{A.5})$$

The statement of Lemma 4.2 then holds true also for the modified operators, with $g_{q,k}$ as defined in (4.7):

$$[\tilde{c}_\alpha^*(k), \tilde{a}_q^* \tilde{a}_q] = -\delta_{\alpha, \alpha_q} \tilde{c}_\alpha^*(g_{q,k}). \quad (\text{A.6})$$

The approximate CCR from Lemmas 4.1, 4.3 and 4.4 become exact, i. e., with $\rho_{q,k}$ as defined in (4.11) we have

$$[\tilde{c}(g), \tilde{c}^*(\tilde{g})] = \langle g, \tilde{g} \rangle, \quad [\tilde{c}_\alpha(k), \tilde{c}_\beta^*(\ell)] = \delta_{\alpha, \beta} \delta_{k, \ell}, \quad [\tilde{c}_\alpha(k), \tilde{c}^*(g_{q, \ell})] = \delta_{\alpha, \alpha_q} \delta_{k, \ell} \rho_{q, k}. \quad (\text{A.7})$$

Accordingly we replace the exponent (2.14) in the transformation $T = e^{-S}$ by

$$\tilde{S} := -\frac{1}{2} \sum_{k \in \Gamma^{\text{nor}}} \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} (\tilde{c}_\alpha^*(k) \tilde{c}_\beta^*(k) - \text{h.c.}). \quad (\text{A.8})$$

This allows us to define an exact bosonic equivalent $\text{ad}_{\tilde{S}}^n(\tilde{a}_q^* \tilde{a}_q)$ of $\text{ad}_S^n(a_q^* a_q)$. In order to compare it with the bosonized multi-commutator $\text{ad}_{q,(b)}^n$ defined in (5.2), we introduce an exact bosonic equivalent $\widetilde{\text{ad}}_{q,(b)}^n$, given for $n = 0$ by $\widetilde{\text{ad}}_{q,(b)}^0 := \tilde{a}_q^* \tilde{a}_q$ and for $n \geq 1$ by

$$\widetilde{\text{ad}}_{q,(b)}^n := \begin{cases} 2^{n-1} \tilde{\mathbf{A}}_n + \tilde{\mathbf{B}}_n + \tilde{\mathbf{B}}_n^* + \sum_{m=1}^{n-1} \binom{n}{m} \tilde{\mathbf{C}}_{n-m,m} & \text{if } n \text{ is even} \\ \tilde{\mathbf{E}}_n + \tilde{\mathbf{E}}_n^* + \sum_{m=1}^s \binom{n}{m} \tilde{\mathbf{D}}_{n-m,m} + \sum_{m=1}^s \binom{n}{m} \tilde{\mathbf{F}}_{m,n-m} & \text{if } n \text{ is odd,} \\ & n = 2s + 1, \end{cases} \quad (\text{A.9})$$

where $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}}, \tilde{\mathbf{E}}$, and $\tilde{\mathbf{F}}$ are defined by replacing c^\sharp by \tilde{c}^\sharp in (5.3) and (5.4).

Lemma A.1. *Under the replacements of a^\sharp by \tilde{a}^\sharp and c^\sharp by \tilde{c}^\sharp , the multi-commutator $\text{ad}_S^n(a_q^* a_q)$ becomes*

$$\text{ad}_{\tilde{S}}^n(\tilde{a}_q^* \tilde{a}_q) = [\tilde{S}, \dots, [\tilde{S}, \tilde{a}_q^* \tilde{a}_q] \dots] = \widetilde{\text{ad}}_{q,(b)}^n. \quad (\text{A.10})$$

This motivates the definition of $\text{ad}_{q,(b)}^n$ we used in Section 5.

Proof. We use induction in n . The case $n = 0$ is trivial, since $\text{ad}_{\tilde{S}}^0(\tilde{a}_q^* \tilde{a}_q) = \tilde{a}_q^* \tilde{a}_q = \widetilde{\text{ad}}_{q,(b)}^0$.

Step $n - 1 \rightarrow n$ for $n = 1$: Here, in (A.9) we have $s = 0$, so $\widetilde{\text{ad}}_{q,(b)}^1 = \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_1^*$. On the other side, using (A.6) and $K(k) = K(k)^T$, we get

$$\begin{aligned} \text{ad}_{\tilde{S}}^1(\tilde{a}_q^* \tilde{a}_q) &= [\tilde{S}, \tilde{a}_q^* \tilde{a}_q] = -\frac{1}{2} \sum_{k \in \Gamma^{\text{nor}}} \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} [\tilde{c}_\alpha^*(k) \tilde{c}_\beta^*(k) - \text{h.c.}, \tilde{a}_q^* \tilde{a}_q] \\ &= -\frac{1}{2} \sum_{k \in \Gamma^{\text{nor}}} \sum_{\alpha, \beta \in \mathcal{I}_k} K(k)_{\alpha, \beta} (\tilde{c}_\alpha^*(k) [\tilde{c}_\beta^*(k), \tilde{a}_q^* \tilde{a}_q] + [\tilde{c}_\alpha^*(k), \tilde{a}_q^* \tilde{a}_q] \tilde{c}_\beta^*(k)) + \text{h.c.} \\ &= \sum_{k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3} \sum_{\alpha \in \mathcal{I}_k} K(k)_{\alpha, \alpha_q} \tilde{c}_\alpha^*(k) \tilde{c}^*(g_{q,k}) + \text{h.c.} = \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_1^*. \end{aligned}$$

Here, we were able to restrict to $\tilde{\mathcal{C}}^q \cap \mathbb{Z}^3$ since $g_{q,k} = 0$ otherwise. For the rest of this proof, we adopt the convention that \sum_k runs over $k \in \tilde{\mathcal{C}}^q \cap \mathbb{Z}^3$ while $\sum_{k'}$ over $k' \in \Gamma^{\text{nor}}$.

Step $n - 1 \rightarrow n$ for even n : On the l.h.s. of (A.10) we have

$$\begin{aligned} \text{ad}_{\tilde{S}}^n(\tilde{a}_q^* \tilde{a}_q) &= [\tilde{S}, \text{ad}_{\tilde{S}}^{n-1}(\tilde{a}_q^* \tilde{a}_q)] \\ &= [\tilde{S}, \tilde{\mathbf{E}}_{n-1}] + [\tilde{S}, \tilde{\mathbf{E}}_{n-1}^*] + \sum_{m=1}^s \binom{n-1}{m} [\tilde{S}, \tilde{\mathbf{D}}_{n-m-1,m}] + \sum_{m=1}^s \binom{n-1}{m} [\tilde{S}, \tilde{\mathbf{F}}_{m,n-m-1}], \end{aligned}$$

where $s = n/2 - 1$. The CCR (A.7) render

$$\begin{aligned}
[\tilde{S}, \tilde{\mathbf{E}}_{n-1}] &= \frac{1}{2} \sum_{k,k'} \sum_{\substack{\alpha, \beta \in \mathcal{I}_{k'} \\ \alpha_1 \in \mathcal{I}_k}} K(k')_{\alpha, \beta} (K(k)^{n-1})_{\alpha_q, \alpha_1} [\tilde{c}_\alpha(k') \tilde{c}_\beta(k'), \tilde{c}^*(g_{q,k}) \tilde{c}_{\alpha_1}^*(k)] \\
&= \sum_k \sum_{\alpha, \alpha_1 \in \mathcal{I}_k} (K(k)^{n-1})_{\alpha_q, \alpha_1} K(k)_{\alpha, \alpha_q} \rho_{q,k} \tilde{c}_{\alpha_1}^*(k) \tilde{c}_\alpha(k) \\
&\quad + \sum_k \sum_{\alpha \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha} \tilde{c}^*(g_{q,k}) \tilde{c}_\alpha(k) \\
&\quad + \sum_k (K(k)^n)_{\alpha_q, \alpha_q} \rho_{q,k} = \tilde{\mathbf{C}}_{n-1,1} + \tilde{\mathbf{B}}_n + \tilde{\mathbf{A}}_n .
\end{aligned} \tag{A.11}$$

It is easy to see that $\tilde{\mathbf{C}}_{m,m'}^* = \tilde{\mathbf{C}}_{m',m}$, $\tilde{\mathbf{A}}_n^* = \tilde{\mathbf{A}}_n = \mathbf{A}_n$ and $\tilde{S}^* = -\tilde{S}$, which implies

$$[\tilde{S}, \tilde{\mathbf{E}}_{n-1}^*] = \tilde{\mathbf{C}}_{1,n-1} + \tilde{\mathbf{B}}_n^* + \tilde{\mathbf{A}}_n . \tag{A.12}$$

The next commutator to evaluate is

$$\begin{aligned}
[\tilde{S}, \tilde{\mathbf{D}}_{n-m-1,m}] &= -\frac{1}{2} \sum_{k,k'} \sum_{\substack{\alpha, \beta \in \mathcal{I}_{k'} \\ \alpha_1, \alpha_2 \in \mathcal{I}_k}} K(k')_{\alpha, \beta} (K(k)^{n-m-1})_{\alpha_q, \alpha_1} (K(k)^m)_{\alpha_q, \alpha_2} \rho_{q,k} [\tilde{c}_\alpha^*(k') \tilde{c}_\beta^*(k'), \tilde{c}_{\alpha_1}(k) \tilde{c}_{\alpha_2}(k)] \\
&= \sum_k \sum_{\alpha, \alpha_2 \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q, \alpha} (K(k)^m)_{\alpha_q, \alpha_2} \rho_{q,k} \tilde{c}_\alpha^*(k) \tilde{c}_{\alpha_2}(k) \\
&\quad + \sum_k \sum_{\beta, \alpha_1 \in \mathcal{I}_k} (K(k)^{m+1})_{\alpha_q, \beta} (K(k)^{n-m-1})_{\alpha_q, \alpha_1} \rho_{q,k} \tilde{c}_\beta^*(k) \tilde{c}_{\alpha_1}(k) \\
&\quad + \sum_k (K(k)^n)_{\alpha_q, \alpha_q} \rho_{q,k} = \tilde{\mathbf{C}}_{n-m,m} + \tilde{\mathbf{C}}_{m+1,n-m-1} + \tilde{\mathbf{A}}_n .
\end{aligned}$$

It is easy to see that $\tilde{\mathbf{D}}_{m,m'}^* = \tilde{\mathbf{F}}_{m',m}$, which implies

$$[\tilde{S}, \tilde{\mathbf{F}}_{m,n-m-1}] = \tilde{\mathbf{C}}_{m,n-m} + \tilde{\mathbf{C}}_{n-m-1,m+1} + \tilde{\mathbf{A}}_n . \tag{A.13}$$

We add together all $1 + \sum_{m=1}^s \binom{n-1}{m} + \sum_{m=1}^s \binom{n-1}{m} + 1 = 2^{n-1}$ commutators and get

$$\begin{aligned}
\text{ad}_{\tilde{S}}^n(\tilde{a}_q^* \tilde{a}_q) &= 2^{n-1} \tilde{\mathbf{A}}_n + \tilde{\mathbf{B}}_n + \tilde{\mathbf{B}}_n^* + \tilde{\mathbf{C}}_{1,n-1} + \tilde{\mathbf{C}}_{n-1,1} \\
&\quad + \sum_{m=1}^s \binom{n-1}{m} (\tilde{\mathbf{C}}_{n-m,m} + \tilde{\mathbf{C}}_{m+1,n-m-1} + \tilde{\mathbf{C}}_{m,n-m} + \tilde{\mathbf{C}}_{n-m-1,m+1}) \\
&= 2^{n-1} \tilde{\mathbf{A}}_n + \tilde{\mathbf{B}}_n + \tilde{\mathbf{B}}_n^* + \sum_{m=1}^{n-1} \left(\binom{n-1}{m} + \binom{n-1}{m-1} \right) \tilde{\mathbf{C}}_{n-m,m} \\
&= 2^{n-1} \tilde{\mathbf{A}}_n + \tilde{\mathbf{B}}_n + \tilde{\mathbf{B}}_n^* + \sum_{m=1}^{n-1} \binom{n}{m} \tilde{\mathbf{C}}_{n-m,m} .
\end{aligned} \tag{A.14}$$

This is exactly the desired term $\widetilde{\text{ad}}_{q,(b)}^n$.

Step $n-1 \rightarrow n$ for odd $n \geq 3$: The l. h. s. of (A.10) is

$$\text{ad}_{\tilde{S}}^n(\tilde{a}_q^* \tilde{a}_q) = 2^{n-2} [\tilde{S}, \tilde{\mathbf{A}}_{n-1}] + [\tilde{S}, \tilde{\mathbf{B}}_{n-1}] + [\tilde{S}, \tilde{\mathbf{B}}_{n-1}^*] + \sum_{m=1}^{n-2} \binom{n-1}{m} [\tilde{S}, \tilde{\mathbf{C}}_{n-m-1,m}].$$

Since $\tilde{\mathbf{A}}_{n-1} \in \mathbb{C}$ we have $[\tilde{S}, \tilde{\mathbf{A}}_{n-1}] = 0$. The next commutator amounts to

$$\begin{aligned} & [\tilde{S}, \tilde{\mathbf{B}}_{n-1}] \\ &= \frac{1}{2} \sum_{k,k'} \sum_{\substack{\alpha, \beta \in \mathcal{I}_{k'} \\ \alpha_1 \in \mathcal{I}_k}} K(k')_{\alpha, \beta} (K(k)^{n-1})_{\alpha_q, \alpha_1} [\tilde{c}_\alpha(k') \tilde{c}_\beta(k') - \tilde{c}_\alpha^*(k') \tilde{c}_\beta^*(k'), \tilde{c}^*(g_{q,k}) \tilde{c}_{\alpha_1}(k)] \\ &= \sum_k \sum_{\alpha, \alpha_1 \in \mathcal{I}_k} K(k)_{\alpha, \alpha_q} (K(k)^{n-1})_{\alpha_q, \alpha_1} \rho_{q,k} \tilde{c}_\alpha(k) \tilde{c}_{\alpha_1}(k) \\ &\quad + \sum_k \sum_{\alpha \in \mathcal{I}_k} (K(k)^n)_{\alpha_q, \alpha} \tilde{c}_\alpha^*(k) \tilde{c}^*(g_{q,k}) = \tilde{\mathbf{D}}_{1,n-1} + \tilde{\mathbf{E}}_n; \end{aligned}$$

and $[\tilde{S}, \tilde{\mathbf{B}}_{n-1}^*] = \tilde{\mathbf{F}}_{n-1,1} + \tilde{\mathbf{E}}_n^*$. The last commutator is

$$\begin{aligned} [\tilde{S}, \tilde{\mathbf{C}}_{n-m-1,m}] &= \frac{1}{2} \sum_{k,k'} \sum_{\substack{\alpha, \beta \in \mathcal{I}_{k'} \\ \alpha_1, \alpha_2 \in \mathcal{I}_k}} K(k')_{\alpha, \beta} (K(k)^{n-m-1})_{\alpha_q, \alpha_1} (K(k)^m)_{\alpha_q, \alpha_2} \rho_{q,k} \times \\ &\quad \times [\tilde{c}_\alpha(k') \tilde{c}_\beta(k') - \tilde{c}_\alpha^*(k') \tilde{c}_\beta^*(k'), \tilde{c}_{\alpha_1}^*(k) \tilde{c}_{\alpha_2}(k)] \\ &= \sum_k \sum_{\alpha, \alpha_2 \in \mathcal{I}_k} (K(k)^{n-m})_{\alpha_q, \alpha} (K(k)^m)_{\alpha_q, \alpha_2} \rho_{q,k} \tilde{c}_\alpha(k) \tilde{c}_{\alpha_2}(k) \\ &\quad + \sum_k \sum_{\beta, \alpha_1 \in \mathcal{I}_k} (K(k)^{m+1})_{\alpha_q, \beta} (K(k)^{n-m-1})_{\alpha_q, \alpha_1} \rho_{q,k} \tilde{c}_\beta^*(k) \tilde{c}_{\alpha_1}^*(k) \\ &= \tilde{\mathbf{D}}_{n-m,m} + \tilde{\mathbf{F}}_{m+1,n-m-1}. \end{aligned}$$

Putting all terms together and using $\tilde{\mathbf{D}}_{m,m'} = \tilde{\mathbf{D}}_{m',m}$ and $\tilde{\mathbf{F}}_{m,m'} = \tilde{\mathbf{F}}_{m',m}$ completes the induction step with $s = (n-1)/2$:

$$\begin{aligned} \text{ad}_{\tilde{S}}^n(\tilde{a}_q^* \tilde{a}_q) &= \tilde{\mathbf{E}}_n + \tilde{\mathbf{E}}_n^* + \tilde{\mathbf{D}}_{1,n-1} + \tilde{\mathbf{F}}_{n-1,1} + \sum_{m=1}^{n-2} \binom{n-1}{m} (\tilde{\mathbf{D}}_{n-m,m} + \tilde{\mathbf{F}}_{m+1,n-m-1}) \\ &= \tilde{\mathbf{E}}_n + \tilde{\mathbf{E}}_n^* + \sum_{m=1}^s \left(\binom{n-1}{m} + \binom{n-1}{m-1} \right) (\tilde{\mathbf{D}}_{n-m,m} + \tilde{\mathbf{F}}_{m,n-m}) \\ &= \tilde{\mathbf{E}}_n + \tilde{\mathbf{E}}_n^* + \sum_{m=1}^s \binom{n}{m} (\tilde{\mathbf{D}}_{n-m,m} + \tilde{\mathbf{F}}_{m,n-m}) = \widetilde{\text{ad}}_{q,(b)}^n. \quad \square \end{aligned}$$

B Comparison with the Literature

In this appendix, we provide heuristics to show that our result is compatible with the physics literature.

B.1 Infinite Volume Limit

In this appendix, we assume that V and thus also \hat{V} are radial. As long as the side length L of the torus is fixed, the proof of Theorem 1.1 carries through unchanged, so

$$n_q(k_F, L) \approx \sum_{k \in \tilde{\mathcal{C}}^q \cap L^{-1}\mathbb{Z}^3} \frac{1}{\pi} \int_0^\infty \frac{g_k(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu. \quad (\text{B.1})$$

Still $g_k = \frac{\hat{V}_k}{2\hbar\kappa N|k|}$, and (1.18) renders $\frac{N}{L^3} = 8\pi^3\rho = \frac{4\pi k_F^3}{3}(1 + \mathcal{O}(N^{-\frac{1}{3}}))$, so

$$n_q(k_F, L) \approx L^{-3} \sum_{k \in \tilde{\mathcal{C}}^q \cap L^{-1}\mathbb{Z}^3} \frac{3\hat{V}_k}{8\pi^2\hbar k_F^3|k|\kappa} (1 + \mathcal{O}(k_F^{-1})) \int_0^\infty \frac{(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu.$$

Note that $\lambda_{q,k} = |\hat{k} \cdot \hat{q}|$ and $Q_k^{(0)}(\mu)$ both depend on k , but not on L . So we are able to take the limit $L \rightarrow \infty$, in which the Riemann sum $L^{-3} \sum_k$ becomes an integral

$$\begin{aligned} n_q(k_F) &:= \lim_{L \rightarrow \infty} n_q(k_F, L) \\ &\approx \int_{\tilde{\mathcal{C}}^q} dk \frac{3\hat{V}_k}{8\pi^2\hbar k_F^3|k|\kappa} (1 + \mathcal{O}(k_F^{-1})) \int_0^\infty \frac{(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k^{(0)}(\mu)} d\mu. \end{aligned} \quad (\text{B.2})$$

For the approximate evaluation of this integral, we assume that the Fermi surface is locally flat and that q keeps sufficient distance to the boundary of its patch, so $\tilde{\mathcal{C}} = \mathcal{C}^q$. The integral is then evaluated in spherical coordinates, as shown in Fig. 5: We consider only the case $q \in B_F^c$, as $q \in B_F$ can be treated analogously. The integrand is symmetric under reflection $k \mapsto -k$, so we can reflect the part of \mathcal{C}_q outside the Fermi ball to the inside. The integral over the radial component $|k|$ starts where the sphere of radius $|k|$ touches the Fermi surface, which is at

$$|k| = R_q := ||q| - k_F|.$$

It runs up to the maximum interaction range, which is $|k| = R$. The integration over θ ranges from 0 to θ_{\max} with $\cos \theta_{\max} \approx \frac{R_q}{|k|} =: \lambda_{\min}$. Thus

$$\begin{aligned} n_q(k_F) &\approx \int_{R_q}^R d|k| |k|^2 \frac{3\hat{V}_k}{8\pi^2\hbar k_F^3|k|\kappa} \int_0^{\theta_{\max}} d\theta \sin \theta \, 2\pi \int_0^\infty \frac{\mu^2 - \cos^2 \theta}{(\mu^2 + \cos^2 \theta)^2} \frac{d\mu}{1 + Q_k^{(0)}(\mu)} \\ &= \int_{R_q}^R d|k| |k| \frac{3\hat{V}_k}{4\pi\hbar k_F^3\kappa} \int_{\lambda_{\min}}^1 d\lambda \int_0^\infty \frac{\mu^2 - \lambda^2}{(\mu^2 + \lambda^2)^2} \frac{d\mu}{1 + Q_k^{(0)}(\mu)}. \end{aligned} \quad (\text{B.3})$$

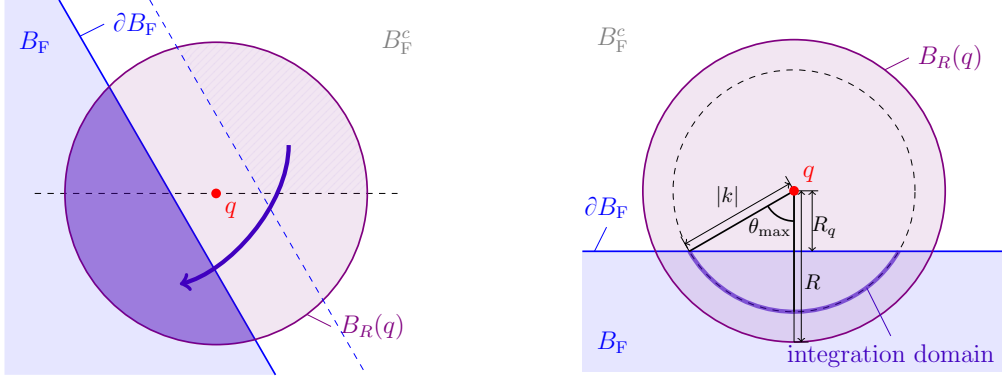


Figure 5: **Left:** Reflecting the part of \mathcal{C}^q outside the Fermi ball inside simplifies the integration (B.2). **Right:** The integration range for a fixed $|k|$ in spherical coordinates.

Since the potential is radial, $Q_k^{(0)}(\mu)$ depends only on $|k|$ and not on λ , so we may execute the integral over λ :

$$\int_{\lambda_{\min}}^1 \frac{\mu^2 - \lambda^2}{(\mu^2 + \lambda^2)^2} d\lambda = \left[\frac{\lambda}{\mu^2 + \lambda^2} \right]_{\lambda_{\min}}^1 = \frac{1}{1 + \mu^2} - \frac{R_q |k|^{-1}}{R_q^2 |k|^{-2} + \mu^2}. \quad (\text{B.4})$$

The final result for $n_q(k_F)$ is then the claimed (1.20).

B.2 Comparison with Computation of Daniel and Vosko

We now verify that (1.20) agrees with a computation by Daniel and Vosko. The momentum distribution was (for the Coulomb potential and in the thermodynamic limit) computed in [9, Eq. (8)]: For $q \in B_F^c$ we have¹⁰

$$n_q^{(\text{DV}, \text{out})} := \frac{\alpha}{|q|} \int_{|q| - k_F}^{|q| + k_F} d|k| |k| \int_0^\infty \left(\frac{|q| - \frac{|k|}{2}}{\left(|q| - \frac{|k|}{2}\right)^2 + k_F^2 \mu^2} - \frac{\frac{|q|^2 - k_F^2}{2|k|}}{\left(\frac{|q|^2 - k_F^2}{2|k|}\right)^2 + k_F^2 \mu^2} \right) \times \left(|k|^2 k_F^{-2} + \alpha Q_k^{(\text{DV})}(\mu) \right)^{-1} d\mu, \quad (\text{B.5})$$

with coupling constant $\alpha = \frac{e_{\text{Coul}}^2}{\pi^2 k_F}$ and

$$Q_k^{(\text{DV})}(\mu) := 2\pi \left[1 + \frac{k_F^2(1 + \mu^2) - \frac{|k|^2}{4}}{2|k|k_F} \log \left(\frac{\left(k_F + \frac{|k|}{2}\right)^2 + k_F^2 \mu^2}{\left(k_F - \frac{|k|}{2}\right)^2 + k_F^2 \mu^2} \right) - \mu \arctan \left(\frac{1 + \frac{|k|}{2k_F}}{\mu} \right) - \mu \arctan \left(\frac{1 - \frac{|k|}{2k_F}}{\mu} \right) \right]. \quad (\text{B.6})$$

¹⁰The variables q, k and u from [9] correspond to $\frac{|k|}{k_F}, \frac{|q|}{k_F}$ and μ in our notation.

Due to the long-range nature of the Coulomb potential, there is a separate formula [9, Eq. (9)] for momenta outside the Fermi ball¹¹:

$$n_q^{(\text{DV}, \text{in})} \quad (\text{B.7})$$

$$:= \frac{\alpha}{|q|} \int_{k_F - |q|}^{k_F + |q|} d|k| |k| \int_0^\infty \left(\frac{|q| + \frac{|k|}{2}}{\left(|q| + \frac{|k|}{2}\right)^2 + k_F^2 \mu^2} - \frac{\frac{k_F^2 - |q|^2}{2|k|}}{\left(\frac{k_F^2 - |q|^2}{2|k|}\right)^2 + k_F^2 \mu^2} \right) \frac{d\mu}{|k|^2 k_F^{-2} + \alpha Q_k^{(\text{DV})}(\mu)}$$

$$+ \frac{\alpha}{|q|} \int_{k_F + |q|}^\infty d|k| |k| \int_0^\infty \left(\frac{|q| + \frac{|k|}{2}}{\left(|q| + \frac{|k|}{2}\right)^2 + k_F^2 \mu^2} - \frac{\frac{|k|}{2} - |q|}{\left(\frac{|k|}{2} - |q|\right)^2 + k_F^2 \mu^2} \right) \frac{d\mu}{|k|^2 k_F^{-2} + \alpha Q_k^{(\text{DV})}(\mu)}.$$

We take a short-range approximation of (B.5) and (B.7) by cutting off the interaction at some R independent of N , so that in particular $|k| \leq R \ll k_F$. This allows for simplifying $Q_k^{(\text{DV})}(\mu)$; in fact, its contributions can be approximated as

$$\frac{k_F^2(1 + \mu^2) - \frac{|k|^2}{4}}{2|k|k_F} = \frac{k_F^2(1 + \mu^2) + \mathcal{O}(1)}{2|k|k_F} = \frac{k_F(1 + \mu^2)}{2|k|}(1 + \mathcal{O}(k_F^{-2})) \quad (\text{B.8})$$

and

$$\log \left(\frac{\left(k_F + \frac{|k|}{2}\right)^2 + k_F^2 \mu^2}{\left(k_F - \frac{|k|}{2}\right)^2 + k_F^2 \mu^2} \right) = \log \left(\frac{k_F^2(1 + \mu^2) + k_F|k| + \mathcal{O}(1)}{k_F^2(1 + \mu^2) - k_F|k| + \mathcal{O}(1)} \right)$$

$$= \log \left(1 + \frac{|k| + \mathcal{O}(k_F^{-1})}{k_F(1 + \mu^2)} \right) - \log \left(1 - \frac{|k| + \mathcal{O}(k_F^{-1})}{k_F(1 + \mu^2)} \right)$$

$$= 2 \frac{|k| + \mathcal{O}(k_F^{-1})}{k_F(1 + \mu^2)} + \mathcal{O}(k_F^{-2}) = \frac{2|k|}{k_F(1 + \mu^2)}(1 + \mathcal{O}(k_F^{-1}))$$

and

$$\mu \arctan \left(\frac{1 \pm \frac{|k|}{2k_F}}{\mu} \right) = \mu \arctan \left(\frac{1}{\mu} (1 + \mathcal{O}(k_F^{-1})) \right) = \mu \arctan \left(\frac{1}{\mu} \right) + \mathcal{O}(k_F^{-1}). \quad (\text{B.9})$$

Thus

$$Q_k^{(\text{DV})}(\mu) = Q_k^{(\text{SR})}(\mu) + \mathcal{O}(k_F^{-1}) \quad \text{with} \quad Q_k^{(\text{SR})}(\mu) := 4\pi \left(1 - \mu \arctan \left(\frac{1}{\mu} \right) \right). \quad (\text{B.10})$$

¹¹The momentum distribution $P(k)$ in [9, Eq. (9)] corresponds to $1 - n_q^{(\text{DV}, \text{in})}$ in our notation.

Then the momentum distribution outside the Fermi ball (B.5) becomes

$$\begin{aligned}
n_q^{(\text{DV,SR})} &:= \frac{\alpha}{|q|} \int_{R_q}^R d|k| |k| \int_0^\infty \left(\frac{|q| - \frac{|k|}{2}}{\left(|q| - \frac{|k|}{2}\right)^2 + k_F^2 \mu^2} - \frac{\frac{|q|^2 - k_F^2}{2|k|}}{\left(\frac{|q|^2 - k_F^2}{2|k|}\right)^2 + k_F^2 \mu^2} \right) \frac{d\mu}{|k|^2 k_F^{-2} + \alpha Q_k^{(\text{SR})}(\mu)} \\
&= \frac{\alpha}{|q|} \int_{R_q}^R d|k| |k| \int_0^\infty \left(\frac{k_F + \mathcal{O}(1)}{k_F^2 (1 + \mu^2) + \mathcal{O}(k_F)} - \frac{k_F R_q |k|^{-1} + \mathcal{O}(1)}{k_F^2 R_q^2 |k|^{-2} + k_F^2 \mu^2 + \mathcal{O}(k_F)} \right) \frac{d\mu}{|k|^2 k_F^{-2} + \alpha Q_k^{(\text{SR})}(\mu)} \\
&= \int_{R_q}^R d|k| |k| \frac{\alpha k_F}{|q| |k|^2} \int_0^\infty \left(\frac{1 + \mathcal{O}(k_F^{-1})}{1 + \mu^2} - \frac{R_q |k|^{-1} + \mathcal{O}(k_F^{-1})}{R_q^2 |k|^{-2} + \mu^2} \right) \frac{d\mu}{1 + \alpha |k|^{-2} k_F^2 Q_k^{(\text{SR})}(\mu)}.
\end{aligned}$$

A comparison with (1.20) and (1.13) suggests identifying $Q_k^{(0)}(\mu)$ with $\alpha |k|^{-2} k_F^2 Q_k^{(\text{SR})}(\mu)$, which corresponds to the following choice of the potential:

$$\frac{3\hat{V}_k}{2\kappa\hbar k_F} = \alpha |k|^{-2} k_F^2 4\pi \quad \Leftrightarrow \quad \hat{V}_k = \frac{8\pi\kappa\hbar k_F^3}{3} \alpha |k|^{-2} = \frac{8\kappa e_{\text{Coul}}^2 \hbar k_F^2}{3\pi |k|^2}. \quad (\text{B.11})$$

With this identification, the \hat{V}_k -dependent factor in (1.20) amounts to

$$\frac{3\hat{V}_k}{4\pi\hbar k_F^3 \kappa} = \frac{2e_{\text{Coul}}^2}{\pi^2 k_F |k|^2} = \frac{2\alpha}{|k|^2}. \quad (\text{B.12})$$

As $\frac{k_F}{|q|} = 1 + \mathcal{O}(k_F^{-1})$, we can equivalently write

$$\begin{aligned}
n_q^{(\text{DV,SR})} &= \int_{R_q}^R d|k| \frac{|k| 3\hat{V}_k}{4\pi\hbar k_F^3 \kappa} (1 + \mathcal{O}(k_F^{-1})) \int_0^\infty \left(\frac{1 + \mathcal{O}(k_F^{-1})}{1 + \mu^2} - \frac{R_q |k|^{-1} + \mathcal{O}(k_F^{-1})}{R_q^2 |k|^{-2} + \mu^2} \right) \frac{d\mu}{1 + Q_k^{(0)}(\mu)}.
\end{aligned}$$

So in the high-density limit $k_F \rightarrow \infty$, the short-range approximated momentum distribution outside the Fermi ball indeed converges to our approximate result (1.20).

Inside the Fermi ball, considering $n_q^{(\text{DV,in})}$ in (B.7), the second of the two integrals over $|k|$ vanishes in the short-range approximation $|k| \leq R$ as soon as k_F large enough. The first term is identical to $n_q^{(\text{DV,in})}$ up to a replacement:

- of $|q| - \frac{|k|}{2}$ by $|q| + \frac{|k|}{2}$ in two places. However, in the short-range approximation, these terms both become $k_F + \mathcal{O}(1)$.
- of $|q| - k_F$ by $k_F - |q|$ in the integral limits, with both terms amounting to R_q in the respective case.
- of $|q|^2 - k_F^2$ by $k_F^2 - |q|^2$ in two places, which both amount to $R_q(|q| + k_F)$ in the respective case.

Thus, the expansion of $n_q^{(\text{DV,in})}$ in the short-range approximation is identical to $n_q^{(\text{DV,SR})}$, agreeing with our approximate result (1.20) as $k_F \rightarrow \infty$.

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References

- [1] N. Benedikter, P. T. Nam, M. Porta, B. Schlein, R. Seiringer: Optimal Upper Bound for the Correlation Energy of a Fermi Gas in the Mean-Field Regime *Commun. Math. Phys.* **374**: 2097–2150 (2020)
- [2] N. Benedikter, P. T. Nam, M. Porta, B. Schlein, R. Seiringer: Correlation Energy of a Weakly Interacting Fermi Gas *Invent. Math.* **225**: 885–979 (2021)
- [3] N. Benedikter, P. T. Nam, M. Porta, B. Schlein, R. Seiringer: Bosonization of Fermionic Many-Body Dynamics *Ann. Henri Poincaré* **23**: 1725–1764 (2022)
- [4] N. Benedikter, M. Porta, B. Schlein, R. Seiringer: Correlation Energy of a Weakly Interacting Fermi Gas with Large Interaction Potential *Arch. Ration. Mech. Anal.* **247**: article number 65 (2023)
- [5] N. Benedikter, M. Porta, B. Schlein: Mean-Field Evolution of Fermionic Systems *Commun. Math. Phys.* **331**: 1087–1131 (2014)
- [6] K. O. Friedrichs: *Perturbation of Spectra in Hilbert Space. Lectures in Applied Mathematics* **3**, American Mathematical Society (1965)
- [7] M. Brooks, S. Lill: Friedrichs Diagrams—Bosonic and Fermionic *arXiv Preprint* <https://arxiv.org/abs/2303.13925>
- [8] J. Dereziński, C. Gérard: *Mathematics of Quantization and Quantum Fields* Cambridge University Press (2013)
- [9] E. Daniel, S. H. Vosko: Momentum Distribution of an Interacting Electron Gas *Phys. Rev.* **120**: 2041–2044 (1960)
- [10] J. Lam: Correlation Energy of the Electron Gas at Metallic Densities *Phys. Rev. B* **3(6)**: 1910–1918 (1971)
- [11] J. Lam: Momentum Distribution and Pair Correlation of the Electron Gas at Metallic Densities *Phys. Rev. B* **3(10)**: 3243–3248 (1971)
- [12] J. M. Luttinger: Fermi Surface and Some Simple Equilibrium Properties of a System of Interacting Fermions *Phys. Rev.* **119**: 1153–1163 (1960)

- [13] H. Narnhofer, G. L. Sewell: Vlasov hydrodynamics of a quantum mechanical model
Commun. Math. Phys. **79**: 9–24 (1981)
- [14] M. Gell-Mann, K. A. Brueckner: Correlation Energy of an Electron Gas at High
Density *Phys. Rev.* **106(2)**: 364–368 (1957)
- [15] K. Sawada: Correlation Energy of an Electron Gas at High Density *Phys. Rev.*
106(2): 372–383 (1957)