Occupation Density

Abstract

The

Keywords: keyword1, Keyword2, Keyword3, Keyword4

1 Introduction

We consider a quantum system of N spinless fermionic particles on $\mathbb{T}^3 := [0, 2\pi]^3$. The system is described by the Hamiltonian

$$H = -\hbar^2 \sum_{j=1}^{N} \Delta_{x_j} + \lambda \sum_{1 \leq i < j \leq N} V(x_i - x_j) \tag{1}$$

acting on the wave functions in the anti-symmetric tensor product $L^2_a(\mathbb{T}^{3N}) = \bigwedge_{i=1}^{N} L^2(\mathbb{T}^3)$. We want to find the occupation density in the asymptotic limit when $N \to \infty$ in the *mean-field scaling regime* i.e. we set

$$\hbar := N^{-\frac{1}{3}}, \quad \text{and} \quad \lambda := N^{-1}. \tag{2}$$

Then we have

$$\langle \Psi_{\text{trial}}, n_q \Psi_{\text{trial}} \rangle = \left\langle \Psi_{\text{trial}}, a_q^* a_q \Psi_{\text{trial}} \right\rangle . \tag{3}$$

Complete the introduction, creation annihilation operators and commutation relations, Bogoliubov transformation, density of lunes

2 Computations

Consider a trial state Ψ_{trial} such that $\langle \Psi_{\text{trial}}, H\Psi_{\text{trial}} \rangle = E_{\text{HF}} + E_{\text{RPA}} + o(\hbar)$, where E_{HF} is the Hartree-Fock energy and E_{RPA} is the correlation energy from *Random Phase Approximation*.
We need to calculate $\langle \Psi_{\text{trial}}, a_\ell^* a_\ell \Psi_{\text{trial}} \rangle$, $\ell \in \mathbb{Z}^3$. Here the trial state $\Psi_{\text{trial}} = Re^{\mathcal{K}}\Omega$, where

$$R\Omega = \frac{1}{\sqrt{N!}} \det \left(\frac{1}{(2\pi)^{3/2}} e^{ik_j \cdot x_i} \right)_{j,i=1}^{N}, \tag{4}$$

is the Slater determinant of all plane waves with N different momenta $k_j \in \mathbb{Z}^3$. We have the Fermi ball i.e. states filling up all the momenta up to the Fermi momentum as

$$B_{\text{F}} := \left\{ k \in \mathbb{Z}^3 : |k| \leq k_{\text{F}} \right\} \tag{5}$$

with $N := |B_{\text{F}}|$, for some $k_{\text{F}} > 0$ with the scaling

$$k_{\text{F}} \sim \left(\frac{3}{4\pi} \right)^{\frac{1}{3}} N^{\frac{1}{3}} + \mathcal{O}(1) \tag{6}$$

and we define its complement as

$$B_{\text{F}}^c = \mathbb{Z}^3 \setminus B_{\text{F}} \tag{7}$$

Similarly we define a set of momenta which are outside the Fermi ball but are constrained to be a certain distance away from the Fermi ball as

$$L_k \coloneqq \{p : p \in B_F^c \cap (B_F + k)\} \tag{8}$$

with the following symmetry $L_{-k} = -L_k \quad \forall k \in \mathbb{Z}^3$.

Definition 2.1 (Quasi-Bosonic Pair Creation and Annihilation Operators)**.** For $k \in \mathbb{Z}_*^3 \coloneqq \mathbb{Z}^3 \setminus \{0\}$ and $p \in L_k$, we define

$$b_p(k) = a_{p-k}a_p\,, \tag{9}$$
$$b_p^*(k) = a_p^* a_{p-k}^* \tag{10}$$

Lemma 2.2 (Quasi-Bosonic commutation relations)**.** *For $k, \ell \in \mathbb{Z}_*^3$ and, $p \in L_k$ and $q \in L_\ell$, we have*

$$[b_p(k), b_q(\ell)] = [b_p^*(k), b_q^*(\ell)] = 0\,, \tag{11}$$
$$[b_p(k), b_q^*(\ell)] = \delta_{p,q}\delta_{k,\ell} + \epsilon_{p,q}(k,\ell), \tag{12}$$

where

$$\epsilon_{p,q}(k,\ell) = -\left(\delta_{p,q}a_{q-\ell}^* a_{p-k} + \delta_{p-k,q-\ell}a_q^* a_p\right) \tag{13}$$

with $\epsilon_{p,q}(l,k) = \epsilon_{q,p}^*(k,l)$ and $\epsilon_{p,p}(k,k) \leq 0$

Proof. Using the CAR we find

$$\begin{aligned}
[b_p(k), b_q^*(\ell)] &= [a_{p-k}a_p, a_q^* a_{q-\ell}^*] \\
&= a_{p-k}[a_p, a_q^* a_{q-\ell}^*] + [a_{p-k}, a_q^* a_{q-\ell}^*]a_p \\
&= a_{p-k}\left\{a_p, a_q^*\right\} a_{q-\ell}^* - a_{p-k}a_q^*\{a_p, a_{q-\ell}^*\} \\
&\quad + \{a_{p-k}, a_q^*\}a_{q-\ell}^* a_p - a_q^*\{a_{p-k}, a_{q-\ell}^*\}a_p \\
&= \delta_{p,q}a_{p-k}a_{q-\ell}^* - \delta_{p-k,q-\ell}a_q^* a_p \\
&= \delta_{p,q}\delta_{k,\ell} - \left(\delta_{p,q}a_{q-\ell}^* a_{p-k} + \delta_{p-k,q-\ell}a_q^* a_p\right)
\end{aligned} \tag{14}$$

And we have the desired relation. As for the first commutation relation, we have it trivially by expanding the quasi-bosonic operators and using the properties of the commutator and CAR. ☐

Also, we have the following identity

$$[b_p^*(k), b_q(\ell)] = -[b_p(k), b_q^*(\ell)]^* \tag{15}$$

with the effect of the complex conjugate seen only on the error term as above.

Before we move on, we write some important commutation relations in order to facilitate further computations.

Lemma 2.3 (Commutation relation between $a_p^\sharp$,[1] and n_q)**.** *For $p, q \in \mathbb{Z}_*^3$, we have the number operator as $n_q = a_q^* a_q$ following the relations,*

$$\left[n_q, a_p^*\right] = \delta_{q,p}a_p^* \tag{16}$$
$$\left[n_q, a_p\right] = -\delta_{q,p}a_p \tag{17}$$

Proof.

$$\begin{aligned}
\left[n_q, a_p^*\right] = \left[a_q^* a_q, a_p^*\right] &= a_q^* a_q a_p^* - a_p^* a_q^* a_q \\
&= a_q^*\delta_{q,p} - a_q^* a_p^* a_q - a_p^* a_q^* a_q = \delta_{q,p}a_p^*
\end{aligned} \tag{18}$$

[1]Here $\sharp = \{\ ,*\}$

Here the second step follows from CAR for the fermionic creation and annihilation operators. For the second commutation relation, we observe that

$$[n_q, a_p] = -\left[n_q, a_p^*\right]^* . \tag{19}$$

Hence the commutation relation holds. ☐

Lemma 2.4 (Commutation relation between $b_p^\sharp$ and n_q)**.** *For $k \in \mathbb{Z}_*^3$ and $p, q \in L_k$,*

$$\left[n_q, b_p^*(k)\right] = (\delta_{q,p} + \delta_{q,p-k})\, b_p^*(k) \tag{20}$$

$$[n_q, b_p(k)] = -(\delta_{q,p} + \delta_{q,p-k})\, b_p(k). \tag{21}$$

Proof. We begin with the first commutation relation

$$\begin{aligned}[n_q, b_p^*(k)] = [n_q, a_p^* a_{p-k}^*] &= [n_q, a_p^*]a_{p-k}^* + a_p^*[n_q, a_{p-k}^*] \\ &= \big(\delta_{q,p} + \delta_{q,p-k}\big) b_p^*(k).\end{aligned} \tag{22}$$

It follows from the above Lemma 2.3. Similarly we observe

$$[n_q, b_p(k)] = -\left[n_q, b_p^*(k)\right]^* . \tag{23}$$

And we attain the said relation for the second commutator. ☐

Lemma 2.5. *For $p \in \mathbb{Z}_*^3$,*

$$(\mathcal{N}+1)a_p = a_p \mathcal{N} \tag{24}$$

$$\mathcal{N} a_p^* = a_p^*(\mathcal{N}+1) . \tag{25}$$

And for $k \in \mathbb{Z}_*^3$ and $p \in L_k$,

$$(\mathcal{N}+2)b_p(k) = b_p(k)\mathcal{N} \tag{26}$$

$$\mathcal{N} b_p^*(k) = b_p^*(k)(\mathcal{N}+2) . \tag{27}$$

Proof. We prove it using the CAR relations and the quasi-bosonic commutation relation. ☐

Consider a family of symmetric operators $K(\ell) : \ell^2(L_\ell) \to \ell^2(L_\ell), \ell \in \mathbb{Z}_*^3$. Then we define the associated Bogoliubov kernel $\mathcal{K} : \mathcal{H}_N \to \mathcal{H}_N$ by

$$\mathcal{K} = \frac{1}{2} \sum_{\ell \in \mathbb{Z}_*^3} \sum_{r,s \in L_\ell} K(\ell)_{r,s} \left(b_r(\ell) b_{-s}(-\ell) - b_{-s}^*(-\ell) b_r^*(\ell)\right) \tag{28}$$

Next, we define the Bogoliubov transformation $T_\lambda := e^{\lambda\mathcal{K}}$, where $\lambda \in \mathbb{R}$, with $T_1 = T$ which is a unitary due the fact that $\mathcal{K}$ is anti self-adjoint i.e. $\mathcal{K} = -\mathcal{K}^*$.

Lemma 2.6 (Symmetric property of K)**.** *For $\ell \in \mathbb{Z}_*^3$ and $r, s \in L_\ell$ we have,*

$$K(\ell)_{r,s} = K(-\ell)_{-r,-s} \tag{29}$$

Proof. to be filled ☐

Lemma 2.7 (Commutator between $\mathcal{K}$ and Pair Operators)**.** *For $k \in \mathbb{Z}_*^3$, and $p \in L_k$, we consider the above defined Bogoliubov kernel which implies the relations,*

$$[b_p^*(k), \mathcal{K}] = -\sum_{s \in L_k} K(k)_{p,s} b_{-s}(-k) + \mathcal{E}_p(k) \tag{30}$$

$$[b_p(k), \mathcal{K}] = -\sum_{s\in L_k} K(k)_{p,s} b^*_{-s}(-k) + \mathcal{E}_p(k)^*, \tag{31}$$

where

$$\mathcal{E}_p(k) = -\frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\left\{\epsilon_{r,p}(\ell,k), b_{-s}(-\ell)\right\} \tag{32}$$

Proof. We start with the first commutation relation.

$$\begin{aligned}
[b^*_p(k),\mathcal{K}] &= \left[b^*_p(k), \frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(b_r(\ell)b_{-s}(-\ell) - b^*_{-s}(-\ell)b^*_r(\ell)\big)\right]\\
&= \frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\left[b^*_p(k), b_r(\ell)b_{-s}(-\ell)\right]\\
&= \frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(\left[b^*_p(k), b_r(\ell)\right] b_{-s}(-\ell) + b_r(\ell)\left[b^*_p(k), b_{-s}(-\ell)\right]\big)\\
&= \frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\Big(\big(-\delta_{p,r}\delta_{k,\ell} - \epsilon_{r,p}(\ell,k)\big)b_{-s}(-\ell) + b_r(\ell)\big(-\delta_{p,-s}\delta_{k,-\ell} - \epsilon_{-s,p}(-\ell,k)\big)\Big)\\
&= -\frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(\delta_{p,r}\delta_{k,\ell}\big)b_{-s}(-\ell) - \frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(\epsilon_{r,p}(\ell,k)b_{-s}(-\ell)\big)\\
&\quad -\frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s} b_r(\ell)\big(\delta_{p,-s}\delta_{k,-\ell}\big) - \frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(b_r(\ell)\epsilon_{-s,p}(-\ell,k)\big)\\
&= -\frac{1}{2}\sum_{s\in L_k} K(k)_{p,s} b_{-s}(-k) - \frac{1}{2}\sum_{r\in L_{-k}} K(-k)_{r,-p} b_r(-k) \qquad (33)\\
&\quad -\frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(\epsilon_{r,p}(\ell,k)b_{-s}(-\ell)\big) - \frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(b_r(\ell)\epsilon_{-s,p}(-\ell,k)\big).
\end{aligned}$$

Consider the second summand, we know that $L_{-k} = -L_k$, then we identify r with $-s$ and we have

$$-\sum_{-s\in -L_k} K(-k)_{-s,-p} b_{-s}(-k) = -\sum_{s\in L_k} K(k)_{s,p} b_{-s}(-k). \tag{34}$$

Now, consider the fourth summand, first we exchange r and s and arrive at

$$-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(b_s(\ell)\epsilon_{-r,p}(-\ell,k)\big). \tag{35}$$

Second, we reflect all the summed over momenta (i.e. $\ell\to-\ell, r\to -r, s\to -s$) which provides us

$$(35) = -\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(b_{-s}(-\ell)\epsilon_{r,p}(\ell,k)\big). \tag{36}$$

Then substituting (34) and (36) in (33), we get

$$\begin{aligned}
(33) = &-\sum_{s\in L_k} K(k)_{p,s} b_{-s}(-k) &(37)\\
&-\frac{1}{2}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} K(\ell)_{r,s}\big(\epsilon_{r,p}(\ell,k)b_{-s}(-\ell) + b_{-s}(-\ell)\epsilon_{r,p}(\ell,k)\big) &(38)
\end{aligned}$$

Here, we observe (38) $= \mathcal{E}_p(k)$. $\square$

Next we define the quadratic operators.

Definition 2.8. Let A be a family of symmetric operators $A(\ell)$, for any $\ell \in \mathbb{Z}^3_*$, with $A(\ell) : \ell^2(L_\ell) \to \ell^2(L_\ell)$. We define the quadratic operators for A as

$$Q_1(A) \coloneqq \sum_{\ell\in\mathbb{Z}^3_*} \sum_{r,s\in L_\ell} A(\ell)_{r,s} \left(b^*_r(\ell)b_s(\ell) + b^*_s(\ell)b_r(\ell)\right) \tag{39}$$

$$Q_2(A) \coloneqq \sum_{\ell\in\mathbb{Z}^3_*} \sum_{r,s\in L_\ell} A(\ell)_{r,s} \left(b_r(\ell)b_{-s}(-\ell) + b^*_{-s}(-\ell)b^*_r(\ell)\right) \tag{40}$$

Remark 2.9. We assume that the symmetric operators are invariant under reflection of momenta, i.e., $A(\ell)_{s,r} = A(\ell)_{r,s} = A(-\ell)_{-r,-s}$.

Lemma 2.10 (Commutator between $\mathcal{K}$ and Q_1)**.** *We consider the above defined Bogoliubov kernel $\mathcal{K}$ and the quadratic operator $Q_1(A)$, with $A(\ell)_{s,r} = A(\ell)_{r,s} = A(-\ell)_{-r,-s}$., which implies the relation,*

$$[Q_1(A), \mathcal{K}] = -Q_2(\{A(\ell), K(\ell)\}) - E_{Q_1}(A) \tag{41}$$

where

$$E_{Q_1}(A) = -2\sum_{\ell\in\mathbb{Z}^3_*} \sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(\mathcal{E}_r(\ell)b_s(\ell) + b^*_s(\ell)\mathcal{E}^*_r(\ell)\Big). \tag{42}$$

Proof. We begin with $[Q_1(A), \mathcal{K}]$.

$$\begin{aligned}
[Q_1(A), \mathcal{K}] &= \sum_{\ell\in\mathbb{Z}^3_*} \sum_{r,s\in L_\ell} A(\ell)_{r,s} \Big[\Big(b^*_r(\ell)b_s(\ell) + b^*_s(\ell)b_r(\ell)\Big), \mathcal{K}\Big] \\
&= \sum_{\ell\in\mathbb{Z}^3_*} \sum_{r,s\in L_\ell} A(\ell)_{r,s} \Big(b^*_r(\ell)\,[b_s(\ell), \mathcal{K}] + [b^*_r(\ell), \mathcal{K}]\,b_s(\ell) \\
&\qquad + b^*_s(\ell)\,[b_r(\ell), \mathcal{K}] + [b^*_s(\ell), \mathcal{K}]\,b_r(\ell)\Big)
\end{aligned} \tag{43}$$

Now we use the commutation relation (30) and (31) to get

$$\begin{aligned}
(43) = \sum_{\ell\in\mathbb{Z}^3_*} \sum_{r,s\in L_\ell} A(\ell)_{r,s} \Bigg(& b^*_r(\ell)\Bigg(- \sum_{s'\in L_\ell} K(\ell)_{s,s'}b^*_{-s'}(-\ell) + \mathcal{E}^*_s(\ell)\Bigg) \\
&+ \Bigg(- \sum_{s'\in L_\ell} K(\ell)_{r,s'}b_{-s'}(-\ell) + \mathcal{E}_r(\ell)\Bigg) b_s(\ell) \\
&+ b^*_s(\ell)\Bigg(- \sum_{s'\in L_\ell} K(\ell)_{r,s'}b^*_{-s'}(-\ell) + \mathcal{E}^*_r(\ell)\Bigg) \\
&+ \Bigg(- \sum_{s'\in L_\ell} K(\ell)_{s,s'}b_{-s'}(-\ell) + \mathcal{E}_s(\ell)\Bigg) b_r(\ell)\Bigg) \\
= \sum_{\ell\in\mathbb{Z}^3_*} \sum_{r,s\in L_\ell} A(\ell)_{r,s} \Bigg(& - \sum_{s'\in L_\ell} K(\ell)_{s,s'}b^*_r(\ell)b^*_{-s'}(-\ell) + b^*_r(\ell)\mathcal{E}^*_s(\ell) \\
&- \sum_{s'\in L_\ell} K(\ell)_{r,s'}b_{-s'}(-\ell)b_s(\ell) + \mathcal{E}_r(\ell)b_s(\ell) \\
&- \sum_{s'\in L_\ell} K(\ell)_{r,s'}b^*_s(\ell)b^*_{-s'}(-\ell) + b^*_s(\ell)\mathcal{E}^*_r(\ell) \\
&- \sum_{s'\in L_\ell} K(\ell)_{s,s'}b_{-s'}(-\ell)b_r(\ell) + \mathcal{E}_s(\ell)b_r(\ell)\Bigg)
\end{aligned}$$

$$= -\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s}\Big(K(\ell)_{s,s'}\big(b^*_r(\ell)b^*_{-s'}(-\ell)+b_{-s'}(-\ell)b_r(\ell)\big)$$
$$+K(\ell)_{r,s'}\big(b^*_s(\ell)b^*_{-s'}(-\ell)+b_{-s'}(-\ell)b_s(\ell)\big)\Big)$$
$$+\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(b^*_r(\ell)\mathcal{E}^*_s(\ell)+\mathcal{E}_r(\ell)b_s(\ell)+b^*_s(\ell)\mathcal{E}^*_r(\ell)+\mathcal{E}_s(\ell)b_r(\ell)\Big). \tag{44}$$

Now we represent the second sum in (44) as $E_{Q_1}(A)$. Furthermore, we exchange r and s in first and fourth term of second sum in (44) and we have

$$E_{Q_1}(A) = -\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(b^*_s(\ell)\mathcal{E}^*_r(\ell)+\mathcal{E}_r(\ell)b_s(\ell)+b^*_s(\ell)\mathcal{E}^*_r(\ell)+\mathcal{E}_r(\ell)b_s(\ell)\Big)$$
$$= -2\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(\mathcal{E}_r(\ell)b_s(\ell)+b^*_s(\ell)\mathcal{E}^*_r(\ell)\Big). \tag{45}$$

Continuing with (44) while having the error $E_{Q_1}(A)$.

$$(44) = -\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s}\Big(K(\ell)_{s,s'}\big(b^*_r(\ell)b^*_{-s'}(-\ell)+b_{-s'}(-\ell)b_r(\ell)\big)$$
$$+K(\ell)_{r,s'}\big(b^*_s(\ell)b^*_{-s'}(-\ell)+b_{-s'}(-\ell)b_s(\ell)\big)\Big) - E_{Q_1}(A)$$
$$= -\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s}\Big(K(\ell)_{s,s'}\big(b^*_{-s'}(-\ell)b^*_r(\ell)+b_r(\ell)b_{-s'}(-\ell)\big)$$
$$+K(\ell)_{r,s'}\big(b^*_s(\ell)b^*_{-s'}(-\ell)+b_{-s'}(-\ell)b_s(\ell)\big)\Big) - E_{Q_1}(A) \tag{46}$$

Then we do a sequence of identifications on the second term, first we exchange s and s'

$$-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s'}K(\ell)_{r,s}\big(b^*_{s'}(\ell)b^*_{-s}(-\ell)+b_{-s}(-\ell)b_{s'}(\ell)\big). \tag{47}$$

Next we exchange r and s and arrive at

$$-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{s,s'}K(\ell)_{s,r}\big(b^*_{s'}(\ell)b^*_{-r}(-\ell)+b_{-r}(-\ell)b_{s'}(\ell)\big). \tag{48}$$

Finally we reflect all the momenta (i.e. $\ell\to-\ell, r\to-r, s\to-s, s'\to-s'$) and it gives us

$$-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{s,s'}K(\ell)_{s,r}\big(b^*_{-s'}(-\ell)b^*_r(\ell)+b_r(\ell)b_{-s'}(-\ell)\big). \tag{49}$$

Then substituting (49) in (46) and interpreting the two terms as a matrix product, we arrive at

$$(46) = -\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell}\big\{A(\ell),K(\ell)\big\}_{r,s}\big(b_r(\ell)b_{-s}(-\ell)+b^*_r(\ell)b^*_{-s}(-\ell)\big) - E_{Q_1}(A) \tag{50}$$
$$= -Q_2\left(\big\{A,K\big\}\right) - E_{Q_1}(A). \tag{51}$$

□

Lemma 2.11 (Commutator between $\mathcal{K}$ and Q_2)**.** *We consider the above defined Bogoliubov kernel $\mathcal{K}$ and the quadratic operator $Q_2(A)$, with $A(\ell)_{s,r} = A(\ell)_{r,s} = A(-\ell)_{-r,-s\cdot}$, which implies the relation,*

$$[Q_2(A),\mathcal{K}] = -Q_1\left(\big\{A,K\big\}\right) - \sum_{\ell\in\mathbb{Z}^3_*}\sum_{r\in L_\ell}\big\{A(\ell),K(\ell)\big\}_{r,r} + E_{Q_2}(A) \tag{52}$$

where,

$$E_{Q_2}(A)=\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell}\Big(A(\ell)_{r,s}\big(\{\mathcal{E}^*_r(\ell),b_{-s}(-\ell)\}+\{b^*_{-s}(-\ell),\mathcal{E}_r(\ell)\}\big)-\{A(\ell),K(\ell)\}_{r,s}\epsilon_{r,s}(\ell,\ell)\Big). \tag{53}$$

Proof. We begin with $[Q_2(A),\mathcal{K}]$.

$$\begin{aligned}[Q_2(A),\mathcal{K}]&=\left[\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell}A(\ell)_{r,s}\Big(b_r(\ell)b_{-s}(-\ell)+b^*_{-s}(-\ell)b^*_r(\ell)\Big),\mathcal{K}\right]\\&=\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell}A(\ell)_{r,s}\left[\Big(b_r(\ell)b_{-s}(-\ell)+b^*_{-s}(-\ell)b^*_r(\ell)\Big),\mathcal{K}\right]\\&=\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell}A(\ell)_{r,s}\Big(b_r(\ell)[b_{-s}(-\ell),\mathcal{K}]+[b_r(\ell),\mathcal{K}]b_{-s}(-\ell)\\&\qquad+b^*_{-s}(-\ell)[b^*_r(\ell),\mathcal{K}]+[b^*_{-s}(-\ell),\mathcal{K}]b^*_r(\ell)\Big)\end{aligned} \tag{54}$$

Now we use the commutation relation (30) and (31) to get

$$\begin{aligned}(54)=\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell}A(\ell)_{r,s}\Bigg(&b_r(\ell)\Bigg(-\sum_{s'\in L_{-\ell}}K(-\ell)_{-s,s'}b^*_{-s'}(\ell)+\mathcal{E}^*_{-s}(-\ell)\Bigg)\\&+\Bigg(-\sum_{s'\in L_\ell}K(\ell)_{r,s'}b^*_{-s'}(-\ell)+\mathcal{E}^*_r(\ell)\Bigg)b_{-s}(-\ell)\\&+b^*_{-s}(-\ell)\Bigg(-\sum_{s'\in L_\ell}K(\ell)_{r,s'}b_{-s'}(-\ell)+\mathcal{E}_r(\ell)\Bigg)\\&+\Bigg(-\sum_{s'\in L_{-\ell}}K(-\ell)_{-s,s'}b_{-s'}(\ell)+\mathcal{E}_{-s}(-\ell)\Bigg)b^*_r(\ell)\Bigg)\end{aligned} \tag{55}$$

Next we do the identification $s'\to -s'$ and then use the symmetry $K(\ell)_{r,s}=K(-\ell)_{-r,-s}$ in the first and fourth term (excluding the error terms) in order to bring all the sum over the new index s' to the same lune

$$\begin{aligned}(55)=\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell}A(\ell)_{r,s}\Bigg(&b_r(\ell)\Big(-\sum_{s'\in L_\ell}K(\ell)_{s,s'}b^*_{s'}(\ell)+\mathcal{E}^*_{-s}(-\ell)\Big)\\&+\Big(-\sum_{s'\in L_\ell}K(\ell)_{r,s'}b^*_{-s'}(-\ell)+\mathcal{E}^*_r(\ell)\Big)b_{-s}(-\ell)\\&+b^*_{-s}(-\ell)\Big(-\sum_{s'\in L_\ell}K(\ell)_{r,s'}b_{-s'}(-\ell)+\mathcal{E}_r(\ell)\Big)\\&\Big(-\sum_{s'\in L_\ell}K(\ell)_{s,s'}b_{s'}(\ell)+\mathcal{E}_{-s}(-\ell)\Big)b^*_r(\ell)\Bigg)\end{aligned}$$

$$
\begin{aligned}
=&-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(\sum_{s'\in L_\ell} K(\ell)_{s,s'}b_r(\ell)b^*_{s'}(\ell)-b_r(\ell)\mathcal{E}^*_{-s}(-\ell)\\
&\qquad+\sum_{s'\in L_\ell} K(\ell)_{r,s'}b^*_{-s'}(-\ell)b_{-s}(-\ell)-\mathcal{E}^*_r(\ell)b_{-s}(-\ell)\\
&\qquad+\sum_{s'\in L_\ell} K(\ell)_{r,s'}b^*_{-s}(-\ell)b_{-s'}(-\ell)-b^*_{-s}(-\ell)\mathcal{E}_r(\ell)\\
&\qquad+\sum_{s'\in L_\ell} K(\ell)_{s,s'}b_{s'}(\ell)b^*_r(\ell)-\mathcal{E}_{-s}(-\ell)b^*_r(\ell)\Big)\\
=&-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s}\Big(K(\ell)_{s,s'}b_r(\ell)b^*_{s'}(\ell)+K(\ell)_{r,s'}b^*_{-s'}(-\ell)b_{-s}(-\ell)\\
&\qquad+K(\ell)_{r,s'}b^*_{-s}(-\ell)b_{-s'}(-\ell)+K(\ell)_{s,s'}b_{s'}(\ell)b^*_r(\ell)\Big)\\
&+\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(b_r(\ell)\mathcal{E}^*_{-s}(-\ell)+\mathcal{E}^*_r(\ell)b_{-s}(-\ell)+b^*_{-s}(-\ell)\mathcal{E}_r(\ell)+\mathcal{E}_{-s}(-\ell)b^*_r(\ell)\Big). \qquad (56)
\end{aligned}
$$

Here we represent the second sum (in (56)) as $\tilde{E}_{Q_2}(A)$, the commutation error, which can be further written as

$$
\tilde{E}_{Q_2}(A)=\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(\mathcal{E}^*_r(\ell)b_{-s}(-\ell)+b^*_{-s}(-\ell)\mathcal{E}_r(\ell)+b_r(\ell)\mathcal{E}^*_{-s}(-\ell)+\mathcal{E}_{-s}(-\ell)b^*_r(\ell)\Big) \qquad (57)
$$

Then in the last two terms, we exchange the indices r and s and reflect all the momenta (i.e. $\ell\to-\ell, r\to -r, s\to -s$) to get

$$
\begin{aligned}
(57)&=\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(\mathcal{E}^*_r(\ell)b_{-s}(-\ell)+b^*_{-s}(-\ell)\mathcal{E}_r(\ell)+b_{-s}(-\ell)\mathcal{E}^*_r(\ell)+\mathcal{E}_r(\ell)b^*_{-s}(-\ell)\Big)\\
&=\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s\in L_\ell} A(\ell)_{r,s}\Big(\big\{\mathcal{E}^*_r(\ell),b_{-s}(-\ell)\big\}+\big\{\mathcal{E}_r(l),b^*_{-s}(-l)\big\}\Big). \qquad (58)
\end{aligned}
$$

Now we substitute this $\tilde{E}_{Q_2}(A)$ in (56) to have

$$
\begin{aligned}
(56)=&-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s}K(\ell)_{s,s'}\big(b_r(\ell)b^*_{s'}(\ell)+b_{s'}(\ell)b^*_r(\ell)\big)\\
&-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s}K(\ell)_{r,s'}\big(b^*_{-s'}(-\ell)b_{-s}(-\ell)+b^*_{-s}(-\ell)b_{-s'}(-\ell)\big)+E_{Q_2}(A). \qquad (59)
\end{aligned}
$$

Next we reflect all the momenta (i.e. $\ell\to-\ell, r\to-r, s\to-s, s'\to-s'$) in the second sum of (59) to have

$$
-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s}K(\ell)_{r,s'}\big(b^*_{s'}(\ell)b_s(\ell)+b^*_s(\ell)b_{s'}(\ell)\big). \qquad (60)
$$

Then we do a sequence of identifications on the second term, first we exchange s and s'

$$
-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{r,s'}K(\ell)_{r,s}\big(b^*_s(\ell)b_{s'}(\ell)+b^*_{s'}(\ell)b_s(\ell)\big). \qquad (61)
$$

Next, we exchange s and r to arrive at

$$
-\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r,s,s'\in L_\ell} A(\ell)_{s,s'}K(\ell)_{r,s}\big(b^*_r(\ell)b_{s'}(\ell)+b^*_{s'}(\ell)b_r(\ell)\big). \qquad (62)
$$

Then substituting (62) in (59) to arrive at

$$
\begin{aligned}
(59) = &- \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s,s' \in L_\ell} A(\ell)_{r,s} K(\ell)_{s,s'} b_r(\ell) b^*_{s'}(\ell) + \underbrace{A(\ell)_{r,s} K(\ell)_{s,s'} b_{s'}(\ell) b^*_r(\ell)}_{\text{(a)}} \\
&- \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s,s' \in L_\ell} A(\ell)_{s,s'} K(\ell)_{r,s} b^*_r(\ell) b_{s'}(\ell) + \underbrace{A(\ell)_{s,s'} K(\ell)_{r,s} b^*_{s'}(\ell) b_r(\ell)}_{\text{(b)}} + \tilde{E}_{Q_2}(A).
\end{aligned} \tag{63}
$$

And finally to interpret the terms as a matrix product, we exchange r and s' in terms (a) and (b) above to have

$$
\begin{aligned}
(63) &= - \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s \in L_\ell} \Big\{ A(\ell), K(\ell) \Big\}_{r,s} \big(b^*_r(\ell) b_s(\ell) + b_r(\ell) b^*_s(\ell) \big) + \tilde{E}_{Q_2}(A) \\
&= - \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s \in L_\ell} \Big\{ A(\ell), K(\ell) \Big\}_{r,s} \big(b^*_r(\ell) b_s(\ell) + b^*_s(\ell) b_r(\ell) + \delta_{r,s}\delta_{\ell,\ell} + \epsilon_{r,s}(\ell,\ell) \big) + \tilde{E}_{Q_2}(A) && (64) \\
&= - Q_1 \left(\Big\{ A, K \Big\} \right) - \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r \in L_\ell} \Big\{ A(\ell), K(\ell) \Big\}_{r,r} - \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s \in L_\ell} \Big\{ A(\ell), K(\ell) \Big\}_{r,s} \epsilon_{r,s}(\ell,\ell) + \tilde{E}_{Q_2}(A). && (65)
\end{aligned}
$$

And we define $E_{Q_2}(A(\ell))$ as the total error from the commutation, which can be succinctly written as

$$
E_{Q_2}(A) = \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s \in L_\ell} \Big(A(\ell)_{r,s} \big(\{ \mathcal{E}^*_r(\ell), b_{-s}(-\ell) \} + \{ b^*_{-s}(-\ell), \mathcal{E}_r(\ell) \} \big) - \big\{ A(\ell), K(\ell) \big\}_{r,s} \epsilon_{r,s}(\ell,\ell) \Big) . \tag{66}
$$

Then, we have

$$
(65) = -Q_1 \left(\Big\{ A, K \Big\} \right) - \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r \in L_\ell} \Big\{ A(\ell), K(\ell) \Big\}_{r,r} + E_{Q_2}(A) \tag{67}
$$

□

Before we begin the evaluation, we define

Reflection transformation: A reflection transformation is a unitary transformation $\mathfrak{R} : \mathcal{F} \to \mathcal{F}$ defined by its action as

$$
\mathfrak{R} : a^*_{k_1} \dots a^*_{k_n} \Omega \mapsto a^*_{-k_1} \dots a^*_{-k_n} \Omega \tag{68}
$$

while leaving the vacuum state invariant.

Lemma 2.12. *For the symmetry transformation $\mathfrak{R}$ and the almost bosonic Bogoliubov transformation T, we have*

$$
\mathfrak{R} T \Omega = T \Omega \tag{69}
$$

From this lemma we observe that

$$
\big\langle T\Omega, a^*_q a_q T\Omega \big\rangle = \big\langle T\Omega, a^*_{-q} a_{-q} T\Omega \big\rangle \tag{70}
$$

And hence motivated by Lemma 2.12, we evaluate $\frac{1}{2} \langle \Omega, T^*_1 (n_q + n_{-q}) T_1 \Omega \rangle$.

2.1 Bogoliubov transformation and the expectation value

Before we start the evaluation of the expectation value, we first study the effect of the Bogoliubov transformation defined above on the relevant operators.

2.1.1 Transformation of the number operator

Lemma 2.13. *For $q \in B^c_{\mathrm{F}}$, we define a rank 2 operator, projecting to momentum q and $-q$: $P^q = \frac{1}{2}(|q\rangle\langle q| + |-q\rangle\langle -q|) \in \ell^2(L_k) \otimes \ell^2(L_k)$, for $k \in \mathbb{Z}^3_*$ with an explicit matrix representation as*

$$
(P^q)_{r,s} := \frac{1}{2} \delta_{r,s} (\delta_{r,q} + \delta_{r,-q}) \tag{71}
$$

and we get

$$T_1^* \left(n_q + n_{-q}\right) T_1 = \left(n_q + n_{-q}\right) - \int_0^1 \mathrm{d}\lambda\, T_\lambda^* Q_2\Big(\big\{K(\ell), P^q\big\}\Big) T_\lambda \tag{72}$$

Proof. We start by applying Duhamel's formula to $T_1^* \left(n_q + n_{-q}\right) T_1$ and we have

$$\begin{aligned}
&\left(n_q + n_{-q}\right) + \int_0^1 \mathrm{d}\lambda \frac{d}{d\lambda} \left(T_\lambda^* \left(n_q + n_{-q}\right) T_\lambda\right) \\
&= \left(n_q + n_{-q}\right) + \int_0^1 \mathrm{d}\lambda \left\langle \Omega, T_\lambda^* (-\mathcal{K}) \left(n_q + n_{-q}\right) T_\lambda + T_\lambda^* \left(n_q + n_{-q}\right) \mathcal{K} T_\lambda \Omega \right\rangle \\
&= \left(n_q + n_{-q}\right) + \int_0^1 \mathrm{d}\lambda \left\langle \Omega, T_\lambda^* [\left(n_q + n_{-q}\right), \mathcal{K}] T_\lambda \Omega \right\rangle .
\end{aligned} \tag{73}$$

Next using the definition of $\mathcal{K}$, we write the expression for the commutator.

$$\begin{aligned}
[n_q, \mathcal{K}] &= \frac{1}{2} \sum_{\ell \in \mathbb{Z}_*^3} \sum_{r,s \in L_\ell} K(\ell)_{r,s} \left[a_q^* a_q, \left(b_r(\ell) b_{-s}(-\ell) - b_{-s}^*(-\ell) b_r^*(\ell)\right)\right] \\
&= \frac{1}{2} \sum_{\ell \in \mathbb{Z}_*^3} \sum_{r,s \in L_\ell} K(\ell)_{r,s} \Big(\left[a_q^* a_q, b_r(\ell)\right] b_{-s}(-\ell) + b_r(\ell) \left[a_q^* a_q, b_{-s}(-\ell)\right] \\
&\qquad - \left[a_q^* a_q, b_{-s}^*(-\ell)\right] b_r^*(\ell) - b_{-s}^*(-\ell) \left[a_q^* a_q, b_r^*(\ell)\right] \Big) \\
&= \frac{1}{2} \sum_{\ell \in \mathbb{Z}_*^3} \sum_{r,s \in L_\ell} K(\ell)_{r,s} \Big((-1) \left(\delta_{q,r} + \delta_{q,r-\ell} + \delta_{q,-s} + \delta_{q,-s+\ell}\right) \left(b_r(\ell) b_{-s}(-\ell) + b_{-s}^*(-\ell) b_r^*(\ell)\right) \Big)
\end{aligned} \tag{74}$$

Now, since $q \in B_{\mathrm{F}}^c$, $\delta_{q,r-\ell} = \delta_{q,-s+\ell} = 0$, hence we have

$$[n_q, \mathcal{K}] = -\sum_{\ell \in \mathbb{Z}_*^3} \sum_{r,s \in L_\ell} K(\ell)_{r,s} \frac{1}{2} \left(\delta_{q,r} + \delta_{q,-s}\right) \left(b_r(\ell) b_{-s}(-\ell) + b_{-s}^*(-\ell) b_r^*(\ell)\right) . \tag{75}$$

Similarly for $[n_{-q}, \mathcal{K}]$, we have

$$[n_{-q}, \mathcal{K}] = -\sum_{\ell \in \mathbb{Z}_*^3} \sum_{r,s \in L_\ell} K(\ell)_{r,s} \; \frac{1}{2} \left(\delta_{-q,r} + \delta_{-q,-s}\right) \left(b_r(\ell) b_{-s}(-\ell) + b_{-s}^*(-\ell) b_r^*(\ell)\right) . \tag{76}$$

Next we substitute commutators (75) and (76) in (73),

$$\begin{aligned}
(73) = \left(n_q + n_{-q}\right) - \int_0^1 \mathrm{d}\lambda\, T_\lambda^* \Bigg(\sum_{\ell \in \mathbb{Z}_*^3} \sum_{r,s \in L_\ell} \frac{1}{2} \Big(&\underbrace{K(\ell)_{r,s} \left(\delta_{q,r} + \delta_{q,-s} + \delta_{-q,r} + \delta_{-q,-s}\right)}_{\text{interpret as matrix product}} \\
&\times \left(b_r(\ell) b_{-s}(-\ell) + b_{-s}^*(-\ell) b_r^*(\ell)\right) \Big) \Bigg) T_\lambda
\end{aligned}$$

$$= (n_q + n_{-q}) - \int_0^1 \mathrm{d}\lambda\, T_\lambda^* \Big(\sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s,m \in L_\ell} \Big(K(\ell)_{r,m}\ \frac{1}{2} \underbrace{(\delta_{m,q}\delta_{m,s} + \delta_{m,-q}\delta_{m,s})}_{\text{(a)}} \tag{77}$$

$$+ \ \frac{1}{2} \underbrace{(\delta_{r,q}\delta_{r,m} + \delta_{r,-q}\delta_{r,m})}_{\text{(b)}} K(\ell)_{m,s} \Big) (b_r(\ell) b_{-s}(-\ell) + b^*_{-s}(-\ell) b^*_r(\ell)) \Big) T_\lambda$$

Next, we observe that (a) and (b) are projections of a momentum (r or $s \in L_\ell$) to momentum q or $-q$. We then arrive at

$$(77) = (n_q + n_{-q}) - \int_0^1 \mathrm{d}\lambda\, T_\lambda^* \Big(\sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s,m \in L_\ell} \Big(K(\ell)_{r,m} P^q_{m,s} + P^q_{r,m} K(\ell)_{m,s} \Big)$$

$$\times (b_r(\ell) b_{-s}(-\ell) + b^*_{-s}(-\ell) b^*_r(\ell)) \Big) T_\lambda$$

$$= (n_q + n_{-q}) - \int_0^1 \mathrm{d}\lambda\, T_\lambda^* \Big(\sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s \in L_\ell} \Big\{ K(\ell), P^q \Big\}_{r,s} \times (b_r(\ell) b_{-s}(-\ell) + b^*_{-s}(-\ell) b^*_r(\ell)) \Big) T_\lambda \tag{78}$$

Using the definition of Q_2, (40), we arrive at

$$(78) = (n_q + n_{-q}) - \int_0^1 \mathrm{d}\lambda\, T_\lambda^* Q_2 \Big(\big\{ K(\ell), P^q \big\} \Big) T_\lambda \tag{79}$$

which is the claimed equality. □

2.1.2 Transformation of quadratic operators

Lemma 2.14 (Operator expansion for the Quadratic Operators)**.** *For $\lambda \in [0,1]$, we have $T_\lambda = e^{\lambda \mathcal{K}}$. Let Q_1 and Q_2 be the quadratic operators defined above for symmetric $A : \ell^2(L_\ell) \to \ell^2(L_\ell)$ where $\ell \in \mathbb{Z}^3_*$, then*

$$T_\lambda^* Q_1(A) T_\lambda = Q_1(A) - \int_0^\lambda \mathrm{d}\lambda' \big(T^*_{\lambda'} (Q_2(\{K, A\})) T_{\lambda'} \big) - \int_0^\lambda \mathrm{d}\lambda' \big(T^*_{\lambda'} E_{Q_1}(A) T_{\lambda'} \big) \tag{80}$$

$$T_\lambda^* Q_2(A) T_\lambda = Q_2(A) - \int_0^\lambda \mathrm{d}\lambda' \big(T^*_{\lambda'} (Q_1(\{K, A\})) T_{\lambda'} \big) + \int_0^\lambda \mathrm{d}\lambda' \big(T^*_{\lambda'} E_{Q_2}(A) T_{\lambda'} \big)$$

$$- \int_0^\lambda \mathrm{d}\lambda' T^*_{\lambda'} \left(\sum_{\ell \in \mathbb{Z}^3_*} \sum_{r \in L_\ell} \big\{ K(\ell), A(\ell) \big\}_{r,r} \right) T_{\lambda'} \tag{81}$$

Proof. We begin with $T_1^* Q_2(A) T_1$ and apply Duhamel's formula,

$$T_\lambda^* Q_2(A) T_\lambda = Q_2(A) + \int_0^\lambda \mathrm{d}\lambda' \Big(\frac{d}{\mathrm{d}\lambda'} (T^*_{\lambda'} Q_2(A) T_{\lambda'}) \Big)$$

$$= Q_2(A) + \int_0^\lambda \mathrm{d}\lambda' \Big(T^*_{\lambda'} (-\mathcal{K}) Q_2(A) T_{\lambda'} + T^*_{\lambda'} Q_2(A) (\mathcal{K}) T_{\lambda'} \Big)$$

$$= Q_2(A) + \int_0^\lambda \mathrm{d}\lambda' T^*_{\lambda'}[Q_2(A), \mathcal{K}]T_{\lambda'}. \tag{82}$$

Then from Lemma 2.11, we get

$$\begin{aligned}
(82) &= Q_2(A) + \int_0^\lambda \mathrm{d}\lambda' \Big(T^*_{\lambda'}\big(- Q_1(\{K, A\}) + E_{Q_2}(A) - \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r \in L_\ell} \{K(\ell), A(\ell)\}_{r,r} \big) T_{\lambda'} \Big) \\
&= Q_2(A) - \int_0^\lambda \mathrm{d}\lambda' \Big(T^*_{\lambda'} Q_1(\{K, A\}) T_{\lambda'} \Big) + \int_0^\lambda \mathrm{d}\lambda' \Big(T^*_{\lambda'} E_{Q_1}(A) T_{\lambda'} \Big) \\
&\quad - \int_0^\lambda \mathrm{d}\lambda' T^*_{\lambda'} \left(\sum_{\ell \in \mathbb{Z}^3_*} \sum_{r \in L_\ell} \{K(\ell), A(\ell)\}_{r,r} \right) T_{\lambda'}
\end{aligned} \tag{83}$$

Similarly, we can prove the operator identity for $Q_1(A)$ using Duhamel's formula and Lemma 2.10. $\square$

For our convenience, we introduce the following notation for writing the nested anti-commutators

$$\Theta^n_K(A) = \underbrace{\{K, \{\dots, \{K}_{\text{n times}}, A\}\dots\}\} \qquad \text{with,} \quad \Theta^0_K(A) = A\,. \tag{84}$$

And we denote the simplex integral as

$$\int_{\Delta^m_1} \mathrm{d}^m \underline{\lambda} = \int_0^1 \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \cdots \int_0^{\lambda_{m-1}} \mathrm{d}\lambda_m \quad , \tag{85}$$

with

$$\int_{\Delta^m_\lambda} \mathrm{d}^m \underline{\lambda} = \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \cdots \int_0^{\lambda_{m-1}} \mathrm{d}\lambda_m \quad , \tag{86}$$

while following

$$\int_{\Delta^m_1} \mathrm{d}^m \underline{\lambda} = \int_0^1 \mathrm{d}\lambda \int_{\Delta^m_\lambda} \mathrm{d}^m \underline{\lambda} \tag{87}$$

Lemma 2.15 (Action of T_λ on $Q_2(A)$)**.** *For $\lambda \in [0, 1]$ and let Q_2 be the quadratic operator defined above for symmetric $A : \ell^2(L_\ell) \to \ell^2(L_\ell)$ where $\ell \in \mathbb{Z}^3_*$, then*

$$\begin{aligned}
T^*_\lambda Q_2(A) T_\lambda &= \sum_{m=1}^n (-1)^{m-1} \frac{\lambda^{m-1}}{(m-1)!} Q_{\sigma(m-1)}(\Theta^{m-1}_K(A)) - \sum_{m=1}^{\lfloor (n+1)/2 \rfloor} \frac{\lambda^{(2m-1)}}{(2m-1)!} \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r \in L_\ell} (\Theta^{(2m-1)}_K(A))_{r,r} \\
&\quad + \sum_{m=1}^n \int_{\Delta^m_\lambda} \mathrm{d}^m \underline{\lambda} \big(T^*_{\lambda_m} E_{Q_{\sigma(m-1)}}(\Theta^{m-1}_K(A)) T_{\lambda_m} \big) + \int_{\Delta^n_\lambda} \mathrm{d}^n \underline{\lambda} (-1)^n \Big(T^*_{\lambda_n} (Q_{\sigma(n)}(\Theta^n_K(A)) T_{\lambda_n} \Big)
\end{aligned} \tag{88}$$

where $\sigma(m) = \begin{cases} 1 & \text{for } m \text{ odd} \\ 2 & \text{for } m \text{ even} \end{cases}$, Θ^n_K and the simplex integral are defined as above and, E_{Q_1} and E_{Q_2} are defined as in (42) and (53) respectively.

Proof. Write the proof with induction
We begin with $T^*_\lambda Q_2(A)T_\lambda$ and from (81) we have

$$T^*_\lambda Q_2(A)T_\lambda = Q_2(A) - \int_0^\lambda \mathrm{d}\lambda_1 \big(T^*_{\lambda_1} Q_1(\{K,A\})T_{\lambda_1}\big) + \int_0^\lambda \mathrm{d}\lambda_1 \big(T^*_{\lambda_1} E_{Q_2}(A)T_{\lambda_1}\big)$$
$$- \int_0^\lambda \mathrm{d}\lambda_1 \left(T^*_{\lambda_1}\left(\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r\in L_\ell}\{K,A\}_{r,r}\right)T_{\lambda_1}\right). \tag{89}$$

Next, we use (80) from Lemma 2.14 to arrive at

$$(89) = Q_2(A) + \int_0^\lambda \mathrm{d}\lambda_1 \big(T^*_{\lambda_1} E_{Q_2}(A)T_{\lambda_1}\big) - \int_0^\lambda \mathrm{d}\lambda_1 \left(T^*_{\lambda_1}\left(\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r\in L_\ell}\{K,A\}_{r,r}\right)T_{\lambda_1}\right)$$
$$- \int_0^\lambda \mathrm{d}\lambda_1 \big(Q_1(\{K,A\}) + \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \big(T^*_{\lambda_2} Q_2(\{K,\{K,A\}\})T_{\lambda_2}\big)$$
$$+ \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \big(T^*_{\lambda_2} E_{Q_1}(\{K,A\})T_{\lambda_2}\big). \tag{90}$$

Again we use (81) from Lemma 2.14

$$(90) = Q_2(A) + \int_0^\lambda \mathrm{d}\lambda_1 \big(T^*_{\lambda_1} E_{Q_2}(A)T_{\lambda_1}\big) - \int_0^\lambda \mathrm{d}\lambda_1 \left(T^*_{\lambda_1}\left(\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r\in L_\ell}\{K,A\}_{r,r}\right)T_{\lambda_1}\right)$$
$$+ \int_0^\lambda \mathrm{d}\lambda_1 \big(Q_1(\{K,A\}) + \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \big(T^*_{\lambda_2} E_{Q_1}(\{K,A\})T_{\lambda_2}\big)$$
$$+ \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \big(Q_2(\{K,\{K,A\}\})\big)$$
$$+ \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \int_0^{\lambda_2} \mathrm{d}\lambda_3 \big(T^*_{\lambda_3} E_{Q_2}(\{K,\{K,A\}\})T_{\lambda_3}\big)$$
$$- \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \int_0^{\lambda_2} \mathrm{d}\lambda_3 \left(T^*_{\lambda_3}\left(\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r\in L_\ell}\{K,\{K,\{K,A\}\}\}_{r,r}\right)T_{\lambda_3}\right) \tag{91}$$
$$- \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \int_0^{\lambda_2} \mathrm{d}\lambda_3 \big(T^*_{\lambda_3} Q_1(\{K,\{K,\{K,A\}\}\})T_{\lambda_3}\big). \tag{92}$$

Then after multiple interations we arrive at

$$(92) = Q_2(\Theta^0_K(A)) - \frac{\lambda}{1!}Q_1(\Theta^1_K(A)) + \frac{\lambda^2}{2!}Q_2(\Theta^2_K(A) - \frac{\lambda^3}{3!}Q_1(\Theta^3_K(A) + \cdots$$

$$
\begin{aligned}
&-\frac{\lambda}{1!}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r\in L_\ell}\{K,A\}_{r,r}-\frac{\lambda^3}{3!}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r\in L_\ell}\{K,\{K,\{K),A\}\}_{r,r}-\cdots\\
&+\int_0^\lambda \mathrm{d}\lambda_1\big(T^*_{\lambda_1}E_{Q_2}(\Theta^0_K(A))T_{\lambda_1}\big)+\int_0^\lambda\int_0^{\lambda_1}\mathrm{d}\lambda_1\mathrm{d}\lambda_2\big(T^*_{\lambda_2}E_{Q_1}(\Theta^1_K(A))T_{\lambda_2}\big)\\
&+\int_0^\lambda\int_0^{\lambda_1}\int_0^{\lambda_2}\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\lambda_3\big(T^*_{\lambda_3}E_{Q_2}(\Theta^2_K(A))T_{\lambda_3}\big)+\cdots\\
&+\int_0^\lambda\int_0^{\lambda_1}\cdots\int_0^{\lambda_{n-1}}\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_n(-1)^n\Big(T^*_{\lambda_n}(Q_{\sigma(n)}(\Theta^n_K(A))T_{\lambda_n}\Big)
\end{aligned}\tag{93}
$$

which when written as sums gives us the required operator expansion. □

Proposition 2.16 (Final Operator Identity)**.** *For $q\in B^c_{\mathrm{F}}$, we have*

$$
\begin{aligned}
T^*_1\,(n_q+n_{-q})\,T_1&=(n_q+n_{-q})+\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{m=1}^{\lfloor(n+1)/2\rfloor}\frac{\operatorname{Tr}\Theta^{(2m)}_K(P^q)}{(2m)!}+\sum_{m=1}^{n}E_m(P^q)+Q_1\left(\sum_{m=1}^{\lfloor n/2\rfloor}\frac{\Theta^{2m}_K(P^q)}{(2m)!}\right)\\
&-Q_2\left(\sum_{m=1}^{\lfloor(n+1)/2\rfloor}\frac{\Theta^{2m-1}_K(P^q)}{(2m-1)!}\right)+\int_{\Delta^n_1}\mathrm{d}^n\underline{\lambda}(-1)^{n+1}\Big(T^*_{\lambda_n}Q_{\sigma(n)}(\Theta^{n+1}_K(P^q))T_{\lambda_n}\Big)
\end{aligned}\tag{94}
$$

where $E_m(P^q)$ is defined as

$$
E_m(P^q):=-\int_{\Delta^m_1}\mathrm{d}^m\underline{\lambda}T^*_{\lambda_m}E_{Q_{\sigma(m-1)}}\left(\Theta^m_K(P^q)\right)T_{\lambda_m}.\tag{95}
$$

with E_{Q_1} and E_{Q_2} defined above and, Θ^n_K, the simplex integral and $\sigma(n)$ are defined above.

Remark 2.17. In the infinite n limit, the terms Q_1 and Q_2 converge, respectively, to a cosh and sinh series in their arguments.

Proof. From Lemma 2.13, we have the equality

$$
T^*_1\,(n_q+n_{-q})\,T_1=(n_q+n_{-q})-\int_0^1\mathrm{d}\lambda\,T^*_\lambda Q_2\Big(\big\{K(\ell),P^q\big\}\Big)T_\lambda\tag{96}
$$

Then we use Lemma 2.15 with $A(\ell)=\big\{K(\ell),P^q\big\}$ to arrive at

$$
\begin{aligned}
=(n_q+n_{-q})-\int_0^1\mathrm{d}\lambda\Bigg(&\sum_{m=1}^{n}(-1)^{m-1}\frac{\lambda^{m-1}}{(m-1)!}Q_{\sigma(m-1)}(\Theta^{m-1}_K\{K(\ell),P^q\})\\
-&\sum_{m=1}^{\lfloor(n+1)/2\rfloor}\frac{\lambda^{(2m-1)}}{(2m-1)!}\sum_{\ell\in\mathbb{Z}^3_*}\sum_{r\in L_\ell}(\Theta^{(2m-1)}_K\{K(\ell),P^q\})_{r,r}\\
+&\sum_{m=1}^{n}\int_{\Delta^m_\lambda}\mathrm{d}^m\underline{\lambda}\,T^*_{\lambda_m}E_{Q_{\sigma(m-1)}}(\Theta^{m-1}_K\{K(\ell),P^q\})T_{\lambda_m}+\int_{\Delta^n_\lambda}\mathrm{d}^n\underline{\lambda}(-1)^n\,T^*_{\lambda_n}Q_{\sigma(n)}(\Theta^n_K\{K(\ell),P^q\})T_{\lambda_n}\Bigg)
\end{aligned}
$$

$$= (n_q + n_{-q}) + \sum_{m=1}^{n} \frac{(-1)^m}{(m)!} Q_{\sigma(m-1)}(\Theta_K^m(P^q)) + \sum_{m=1}^{\lfloor (n+1)/2 \rfloor} \frac{1}{(2m)!} \sum_{\ell \in \mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \Theta_K^{(2m)}(P^q)_{q,q}$$

$$- \sum_{m=1}^{n} \int_{\Delta_1^m} \mathrm{d}^m \underline{\lambda} (T^*_{\lambda_m} E_{Q_{\sigma(m-1)}} \Theta_K^m(P^q) T_{\lambda_m}) + \int_{\Delta_1^n} \mathrm{d}^n \underline{\lambda} (-1)^{(n+1)} \left(T^*_{\lambda_n} Q_{\sigma(n)}(\Theta_K^{n+1}(P^q)) T_{\lambda_n} \right) \tag{97}$$

In the second term, we separate the odd and even terms which results in sums of Q_2 and Q_1 operators. Since the quadratic operators are linear in their argument, we can interpret them as cosh and sinh series of $\Theta_K(P^q)$ operator in the infinite n limit as mentioned in the remark. In the third term, again using the linearity of the sum over all momenta transfer ℓ and the trace we recover the trace term. And for the fourth term, we just use the definition of $E_M(A)$. Doing these identifications we arrive at the desired operator identity. □

2.1.3 Evaluation of the expectation value

Lemma 2.18. *For the quadratic operators $Q_1(A)$ and $Q_2(B)$ for symmetric $A, B : \ell^2(L_\ell) \to \ell^2(L_\ell)$, we have*

$$\langle \Omega, Q_1(A) \Omega \rangle = 0, \tag{98}$$
$$\langle \Omega, Q_2(B) \Omega \rangle = 0. \tag{99}$$

Proof. The proof follows by plugging in the definitions of the operators Q_1 and Q_2 and observing the fact that both Q_1 and Q_2 are normal ordered in the fermionic creation and annihilation operators. □

Proposition 2.19 (Final Expectation)**.** *For $q \in B_{\mathrm{F}}^c$ and the vacuum state $\Omega \in \mathcal{H}_N$, we have*

$$\left\langle \Omega, T_1^* \frac{1}{2} (n_q + n_{-q}) T_1 \Omega \right\rangle = \frac{1}{2} \sum_{\ell \in \mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{m=1}^{\lfloor (n+1)/2 \rfloor} \frac{\operatorname{Tr} \Theta_K^{(2m)}(P^q)}{(2m)!} - \frac{1}{2} \left\langle \Omega, \sum_{m=1}^{n} E_m(P^q)) \Omega \right\rangle$$
$$- \frac{1}{2} \int_{\Delta_1^n} \mathrm{d}^n \underline{\lambda} (-1)^n \left\langle \Omega, T^*_{\lambda_n} Q_{\sigma(n)}(\Theta_K^{n+1}(P^q) T_{\lambda_n} \Omega \right\rangle \tag{100}$$

Proof. The proof follows from Proposition 2.16 and Lemma 2.18. □

Section on matrix element bounds

3 Error Bounds

Before we start with the estimates we introduce some definitions.
Definition 3.1 (Norms)**.** For any $k \in \mathbb{Z}^3_*$ and $A(k) \in \ell^2(L_k) \otimes \ell^2(L_k)$, we define

$$\|A(k)\|_{\max} := \sup_{p,q \in L_k} |A(k)_{p,q}| \tag{101}$$

and

$$\|A(k)\|_{\max,2} := \sup_{q \in L_k} \left(\sum_{p \in L_k} |A(k)_{p,q}|^2 \right)^{\frac{1}{2}}. \tag{102}$$

We bound the head term next, and to begin we start by establishing certain necessary bounds.
Lemma 3.2 (Bounds on Pair Operators)**.** *Let $k \in \mathbb{Z}^3_*$ and $p \in L_k$, then*

$$\sum_{p \in L_k} \|b_p(k)\psi\|^2 \leq \langle \psi, \mathcal{N} \psi \rangle \qquad \forall \psi \in \mathcal{H}_N \, . \tag{103}$$

Furthermore, for $f \in \ell^2(L_k)$ and for all $\psi \in \mathcal{H}_N$, we have

$$\left\| \sum_{p \in L_k} f_p(k) b_p(k) \psi \right\| \leq \left(\sum_{p \in L_k} |f_p(k)|^2 \right)^{\frac{1}{2}} \left\| \mathcal{N}^{\frac{1}{2}} \psi \right\| \tag{104}$$

$$\left\| \sum_{p \in L_k} f_p(k) b_p^*(k) \psi \right\| \leq \left(\sum_{p \in L_k} |f_p(k)|^2 \right)^{\frac{1}{2}} \left\| (\mathcal{N}+1)^{\frac{1}{2}} \psi \right\|. \tag{105}$$

Proof. For the first estimate, we begin with

$$\sum_{p \in L_k} \|b_p(k)\psi\|^2 = \sum_{p \in L_k} \langle b_p(k)\psi, b_p(k)\psi \rangle = \sum_{p \in L_k} \left\langle \psi, a_p^* a_{p-k}^* a_{p-k} a_p \psi \right\rangle. \tag{106}$$

We use $a_{p-k}^* a_{p-k} \leq \mathbb{1}$ to get

$$\leq \sum_{p \in L_k} \left\langle \psi, a_p^* a_p \psi \right\rangle \leq \left\langle \psi, \sum_{p \in \mathbb{Z}_*^3} a_p^* a_p \psi \right\rangle = \langle \psi, \mathcal{N} \psi \rangle \tag{107}$$

This proves the estimate (103). For the estimate in (104) we begin with

$$\begin{aligned} \left\| \sum_{p \in L_k} f_p(k) b_p(k) \psi \right\|^2 &= \left\langle \sum_{p \in L_k} f_p(k) b_p(k) \psi, \sum_{p' \in L_k} f_{p'}(k) b_{p'}(k) \psi \right\rangle \\ &= \sum_{p,p' \in L_k} \overline{f_p(k)} f_{p'}(k) \left\langle \psi, b_p^*(k) b_{p'}(k) \psi \right\rangle. \end{aligned}$$

and, we use the Cauchy-Schwarz inequality and $a_{p'-k}^* a_{p'-k} \leq \mathbb{1}$ to arrive at

$$\begin{aligned} &\leq \sum_{p \in L_k} |f_p(k)|^2 \sum_{p' \in L_k} \left\langle \psi, a_{p'}^* a_{p'} \psi \right\rangle \leq \sum_{p \in L_k} |f_p(k)|^2 \left\langle \psi, \sum_{p' \in L_k} a_{p'}^* a_{p'} \psi \right\rangle \\ &\leq \sum_{p \in L_k} |f_p(k)|^2 \left\langle \psi, \sum_{p' \in Z_*^3} a_{p'}^* a_{p'} \psi \right\rangle = \sum_{p \in L_k} |f_p(k)|^2 \langle \psi, \mathcal{N} \psi \rangle \end{aligned} \tag{108}$$

For the next inequality, we use Lemma 2.2 and (104). We begin with

$$\begin{aligned} \left\| \sum_{p \in L_k} f_p(k) b_p^*(k) \psi \right\|^2 &= \left\langle \sum_{p \in L_k} f_p(k) b_p^*(k) \psi, \sum_{q \in L_k} f_q(k) b_q^*(k) \psi \right\rangle \\ &= \sum_{p,q \in L_k} \overline{f_p(k)} f_q(k) \left(\left\langle \psi, b_p^*(k) b_q(k) \psi \right\rangle + \left\langle \psi, \left[b_p(k), b_q^*(k) \right] \psi \right\rangle \right) \\ &= \sum_{p,q \in L_k} \overline{f_p(k)} f_q(k) \left(\left\langle \psi, b_p^*(k) b_q(k) \psi \right\rangle + \langle \psi, (\delta_{p,q} + \epsilon_{p,q}(k,k)) \psi \rangle \right) \end{aligned} \tag{109}$$

Then we know that $\epsilon_{p,q}(k,k) \leq 0$ and we have

$$\begin{aligned} &\leq \sum_{p,q \in L_k} \overline{f_p(k)} f_q(k) \left\langle \psi, b_p^*(k) b_q(q) \psi \right\rangle + \sum_{p,q \in L_k} \overline{f_p(k)} f_q(k) \langle \psi, \delta_{p,q} \psi \rangle \\ &= \left\| \sum_{p \in L_k} f_p(k) b_p(k) \psi \right\|^2 + \sum_{p \in L_k} |f_p(k)|^2 \langle \psi, \psi \rangle \end{aligned}$$

$$\leq \sum_{p\in L_k} |f_p(k)|^2 \langle \psi, \mathcal{N}\psi\rangle + \sum_{p\in L_k} |f_p(k)|^2 \langle \psi, \psi\rangle \tag{110}$$

and we have the second estimate. □

Lemma 3.3. *Let $\ell \in \mathbb{Z}^3_*$, then we have*

$$|\langle \Psi, Q_1(A)\Psi\rangle| \leq 2\sum_{\ell\in\mathbb{Z}^3_*} \|A(\ell)\|_{\mathrm{HS}} \langle \Psi, \mathcal{N}\Psi\rangle \tag{111}$$

$$|\langle \Psi, Q_2(A)\Psi\rangle| \leq 2\sum_{\ell\in\mathbb{Z}^3_*} \|A(\ell)\|_{\mathrm{HS}} \langle \Psi, (\mathcal{N}+1)\Psi\rangle \tag{112}$$

for all $\Psi \in \mathcal{H}_N$.

Proof. We begin with the quantity we want to bound and use the definition of the Q_2 operator.

$$\begin{aligned}
|\langle \Psi, Q_2(A)\Psi\rangle| &= \left|\left\langle \Psi, \sum_{\ell\in\mathbb{Z}^3_*}\sum_{p,q\in L_\ell} A(\ell)_{p,q}\big(b^*_{-q}(-\ell)b^*_p(\ell) + \mathrm{h.c.}\big)\Psi\right\rangle\right| \\
&\leq \sum_{\ell\in\mathbb{Z}^3_*}\left|\left\langle \Psi, \sum_{p,q\in L_\ell} A(\ell)_{p,q}\big(b^*_{-q}(-\ell)b^*_p(\ell) + \mathrm{h.c.}\big)\Psi\right\rangle\right| \\
&\leq 2\sum_{\ell\in\mathbb{Z}^3_*}\left|\left\langle \Psi, \sum_{p,q\in L_\ell} A(\ell)_{p,q}\big(b^*_{-q}(-\ell)b^*_p(\ell)\big)\Psi\right\rangle\right| \\
&= 2\sum_{\ell\in\mathbb{Z}^3_*}\left|\left\langle \Psi, \sum_{q\in L_\ell} b^*_{-q}(-\ell)\Big(\sum_{p\in L_\ell} A(\ell)_{p,q}b^*_p(\ell)\Big)\Psi\right\rangle\right| \\
&= 2\sum_{\ell\in\mathbb{Z}^3_*}\sum_{q\in L_\ell}\left|\left\langle b_{-q}(-\ell)\Psi, \sum_{p\in L_\ell} A(\ell)_{p,q}b^*_p(\ell)\Psi\right\rangle\right|
\end{aligned} \tag{113}$$

Then we use Cauchy-Schwarz inequality to get

$$\leq 2\sum_{\ell\in\mathbb{Z}^3_*}\sum_{q\in L_\ell} \|b_{-q}(-\ell)\Psi\| \left\|\sum_{p\in L_\ell} A(\ell)_{p,q}b^*_p(\ell)\Psi\right\| \tag{114}$$

Then we use the estimates from Lemma 3.2 to have

$$\begin{aligned}
&\leq 2\sum_{\ell\in\mathbb{Z}^3_*}\left(\sum_{q\in L_\ell}\|b_{-q}(-\ell)\Psi\|^2\right)^{\frac{1}{2}}\Big(\sum_{p,q\in L_\ell}|A(\ell)_{p,q}|^2\Big)^{\frac{1}{2}}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\Psi\right\| \\
&\leq 2\sum_{\ell\in\mathbb{Z}^3_*}\Big(\sum_{p,q\in L_\ell}|A(\ell)_{p,q}|^2\Big)^{\frac{1}{2}}\left\|\mathcal{N}^{\frac{1}{2}}\Psi\right\|\left\|(\mathcal{N}+1)^{\frac{1}{2}}\Psi\right\| \\
&= 2\sum_{\ell\in\mathbb{Z}^3_*}\|A(\ell)\|_{\mathrm{HS}}\left\|\mathcal{N}^{\frac{1}{2}}\Psi\right\|\left\|(\mathcal{N}+1)^{\frac{1}{2}}\Psi\right\| \\
&\leq 2\sum_{\ell\in\mathbb{Z}^3_*}\|A(\ell)\|_{\mathrm{HS}}\langle \Psi, (\mathcal{N}+1)\Psi\rangle
\end{aligned} \tag{115}$$

Hence, we have the required estimate and we can similarly prove (111). □

Lemma 3.4 (Grönwall Bound). *Let $\lambda \in [0,1]$, then we have the following operator inequality*

$$T_\lambda^*(\mathcal{N}+1)T_\lambda \leq e^C(\mathcal{N}+1)\,, \tag{116}$$

where $C = \exp(4\sum_{l\in\mathbb{Z}_*^3}\|K(\ell)\|_{\mathrm{HS}})$,

Similarly

$$T_\lambda^*(\mathcal{N}+1)^\alpha T_\lambda \leq e^{C_\alpha}(\mathcal{N}+1)^\alpha\,, \tag{117}$$

[2].

Proof. For a given $\Psi \in \mathcal{H}_N$, we start with taking a derivative of the expectation of the LHS of (116) above.

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\langle\Psi,(T_\lambda^*(\mathcal{N}+1)T_\lambda)\Psi\rangle\right| &= |\langle\Psi,(T_\lambda^*\left[\mathcal{K},\mathcal{N}\right]T_\lambda)\Psi\rangle| \\
&= \left|4\mathrm{Re}\left\langle T_\lambda\Psi,\sum_{\ell\in\mathbb{Z}_*^3}\sum_{r,s\in L_\ell}K(\ell)_{r,s}b_{-s}^*(-\ell)b_r^*(\ell)T_\lambda\Psi\right\rangle\right| \\
&\leq 4\sum_{\ell\in\mathbb{Z}_*^3}\left|\left\langle\sum_{s\in L_\ell}b_{-s}(-\ell)T_\lambda\Psi,\sum_{r\in L_\ell}K(\ell)_{r,s}b_r^*(\ell)T_\lambda\Psi\right\rangle\right|
\end{aligned} \tag{118}$$

Then using Cauchy-Schwarz inequality and the estimates from Lemma 3.2, we get

$$\begin{aligned}
(118) &\leq 4\sum_{\ell\in\mathbb{Z}_*^3}\sum_{s\in L_\ell}\|b_{-s}(-\ell)T_\lambda\Psi\|\left\|\sum_{r\in L_\ell}K(\ell)_{r,s}b_r^*(\ell)T_\lambda\Psi\right\| \\
&\leq 4\sum_{\ell\in\mathbb{Z}_*^3}\|K(\ell)\|_{\mathrm{HS}}\left\|\mathcal{N}^{\frac{1}{2}}T_\lambda\Psi\right\|\left\|(\mathcal{N}+1)^{\frac{1}{2}}T_\lambda\Psi\right\| \\
&\leq 4\sum_{\ell\in\mathbb{Z}_*^3}\|K(\ell)\|_{\mathrm{HS}}\langle\Psi,T_\lambda^*(\mathcal{N}+1)T_\lambda\Psi\rangle
\end{aligned} \tag{119}$$

Then using Grönwall's lemma, we have

$$\langle\Psi,T_\lambda^*(\mathcal{N}+1)T_\lambda\Psi\rangle \leq \exp(4\sum_{l\in\mathbb{Z}_*^3}\|K(\ell)\|_{\mathrm{HS}})\langle\Psi,(\mathcal{N}+1)\Psi\rangle \tag{120}$$

And this proves the estimate. We proceed next with proving (117) write the proof for the general Grönwall estimate ☐

Lemma 3.5 (HS Norm bound on the K). *For $\ell \in \mathbb{Z}_*^3$, we have*

$$\|K(\ell)\|_{\mathrm{HS}} \leq C\hat{V}(\ell)\min\{1,k_F^2|\ell|^{-2}\} \tag{121}$$

Proof. to be filled ☐

Next, we compile some bounds on the correlation structure $K(\ell)$.

Lemma 3.6 (Bounds on K). *Let $\ell \in \mathbb{Z}_*^3$, $m \in \mathbb{N}$ and $r,s \in L_\ell$. We then have the pointwise estimate*

$$|(K(\ell)^m)_{r,s}| \leq \frac{(C\hat{V}(\ell))^m k_{\mathrm{F}}^{-1}}{\lambda_{\ell,r}+\lambda_{\ell,s}}\,. \tag{122}$$

[SL: Make sure that $\lambda_{\ell,r}$ has been defined] Further, we have the bounds

$$\|K(\ell)^m\|_{\max} \leq (C\hat{V}(\ell))^m k_{\mathrm{F}}^{-1}\,, \qquad \|K(\ell)^m\|_{\max,2} \leq (C\hat{V}(\ell))^m|k|^{\frac{11}{6}}k_{\mathrm{F}}^{-\frac{2}{3}}\log(k_{\mathrm{F}})^{\frac{1}{3}}\,. \tag{123}$$

[2]The value of C can be different for every new appearance, unless explicitly stated.

Proof. From [CHN22, Prop. 7.10] we readily retrieve (122) for $m = 1$:

$$|K(\ell)_{r,s}| \leq C \frac{\hat{V}(\ell) k_{\mathrm{F}}^{-1}}{\lambda_{\ell,r} + \lambda_{\ell,s}} \,. \tag{124}$$

For $m \geq 2$, we proceed by induction: Suppose, (122) was shown to hold until $m-1$. Then, using $\lambda_{\ell,r} > 0$ and [CHN22, Prop. A.2] $\sum_{r \in L_\ell} \lambda_{\ell,r}^{-1} \leq C k_{\mathrm{F}}$, we get

$$\begin{aligned} |(K(\ell)^m)_{r,s}| &\leq \sum_{r' \in L_\ell} |(K(\ell)^{m-1})_{r,r'}| \, |K(\ell)_{r',s}| \leq (C\hat{V}(\ell))^m k_{\mathrm{F}}^{-2} \sum_{r' \in L_\ell} \frac{1}{\lambda_{\ell,r} + \lambda_{\ell,r'}} \frac{1}{\lambda_{\ell,r'} + \lambda_{\ell,s}} \\ &\leq (C\hat{V}(\ell))^m k_{\mathrm{F}}^{-2} \sum_{r' \in L_\ell} \frac{1}{\lambda_{\ell,r'}(\lambda_{\ell,r} + \lambda_{\ell,s})} \leq (C\hat{V}(\ell))^m k_{\mathrm{F}}^{-1} \frac{1}{\lambda_{\ell,r} + \lambda_{\ell,s}} \,. \end{aligned} \tag{125}$$

The first bound in (123) then follows immediately noting that $\lambda_{\ell,r} \geq \frac{1}{2}$ uniformly in ℓ, r. For the second bound in (123), we proceed as in [CHN22, Prop. 7.2]: Using that [CHN22, Prop. A.3] for all $\beta < -\frac{4}{3}$,

$$\sum_{r \in L_\ell} \lambda_{\ell,r}^{-\beta} \leq C |k|^{\frac{11}{3}} k_{\mathrm{F}}^{\frac{2}{3}} \log(k_{\mathrm{F}})^{\frac{2}{3}} \,, \tag{126}$$

we get with $\beta = 2$:

$$\|K(\ell)^m\|_{\max,2}^2 = \sup_{q \in \mathbb{Z}^3} \sum_{r \in L_\ell} |(K(\ell)^m)_{r,q}|^2 \leq \sup_{q \in \mathbb{Z}^3} \sum_{r \in L_\ell} (C\hat{V}(\ell))^{2m} k_{\mathrm{F}}^{-2} \lambda_{\ell,r}^{-2} \leq (C\hat{V}(\ell))^{2m} |k|^{\frac{11}{3}} k_{\mathrm{F}}^{-\frac{4}{3}} \log(k_{\mathrm{F}})^{\frac{2}{3}} \,. \tag{127}$$

Taking the square root implies the second bound in (123).

□

Lemma 3.7 (Bound on nested anti-commutator)**.** *For $\ell \in \mathbb{Z}^3_*$, we have for all symmetric $A : \ell^2(L_\ell) \to \ell^2(L_\ell)$,*

$$\sum_{\ell \in \mathbb{Z}^3_*} \|\Theta_K^n(A)(\ell)\|_{\mathrm{HS}} \leq \sum_{\ell \in \mathbb{Z}^3_*} 2^n \|K(\ell)\|_{\mathrm{op}}^n \|A(\ell)\|_{\mathrm{HS}} \tag{128}$$

with Θ_K^n defined in (84).

Proof. We begin with

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}^3_*} \|\Theta_K^n(A)(\ell)\|_{\mathrm{HS}} = \sum_{\ell \in \mathbb{Z}^3_*} \left\| \{K(\ell), \Theta_K^{n-1}(A)(\ell)\} \right\|_{\mathrm{HS}} &= \sum_{\ell \in \mathbb{Z}^3_*} \left\| K(\ell)\Theta_K^{n-1}(A)(\ell) + \Theta_K^{n-1}(A)(\ell)K(\ell) \right\|_{\mathrm{HS}} \\ &\leq 2 \sum_{\ell \in \mathbb{Z}^3_*} \left\| K(\ell)\Theta_K^{n-1}(A)(\ell) \right\|_{\mathrm{HS}} \end{aligned} \tag{129}$$

Then using the inequality $\|AB\|_{\mathrm{HS}} \leq \|A\|_{\mathrm{op}} \|B\|_{\mathrm{HS}}$, we get

$$\leq 2 \sum_{\ell \in \mathbb{Z}^3_*} \|K(\ell)\|_{\mathrm{op}} \left\| \Theta_K^{n-1}(A)(\ell) \right\|_{\mathrm{HS}} \leq 2^n \sum_{\ell \in \mathbb{Z}^3_*} \|K(\ell)\|_{\mathrm{op}}^n \|A(\ell)\|_{\mathrm{HS}} \tag{130}$$

□

Proposition 3.8 (The head term)**.** *For $q \in B_{\mathrm{F}}^c$, we have the following bound*

$$\left| \int_{\Delta_1^m} \mathrm{d}^n \underline{\lambda} \left\langle \Omega, \left(T^*_{\lambda_n} Q_{\sigma(n)}(\Theta_K^{n+1}(P^q)) T_{\lambda_n} \right) \Omega \right\rangle \right| \leq \frac{2^{n+2}}{(n+1)!} \|K(\ell)\|_{\mathrm{HS}}^{n+1} C \left\langle \Omega, \left(\mathcal{N} + 1 \right) \Omega \right\rangle \tag{131}$$

Proof. We first look at the case n even. We begin with the L.H.S. of the above expression and use the the estimate from Lemma 3.3 to get

$$\text{L.H.S. of (131)} \leq \left| 2 \int_{\Delta_1^n} \mathrm{d}^n \underline{\lambda} \left\| \Theta_K^{n+1}(P^q) \right\|_{\mathrm{HS}} \left\langle \Omega, \left(T_{\lambda_n}^* (\mathcal{N}+1) T_{\lambda_n} \right) \Omega \right\rangle \right| \tag{132}$$

Then using Lemma 3.7 we get

$$\leq \left| 2 \int_{\Delta_1^n} \mathrm{d}^n \underline{\lambda}\, 2^{n+1} \| K(\ell) \|_{\mathrm{op}}^{n+1} \| (P^q) \|_{\mathrm{HS}} \left\langle \Omega, \left(T_{\lambda_n}^* (\mathcal{N}+1) T_{\lambda_n} \right) \Omega \right\rangle \right| \tag{133}$$

Here we observe that $\|P^q\|_{\mathrm{HS}} = \frac{1}{\sqrt{2}}$, and then we use the Grönwall estimate from Lemma 3.4 to have

$$\leq \left| 2^{n+2} \int_{\Delta_1^n} \mathrm{d}^n \underline{\lambda} \| K(\ell) \|_{\mathrm{op}}^{n+1}\, C \left\langle \Omega, \left(\mathcal{N}+1 \right) \Omega \right\rangle \right| = \frac{2^{n+2}}{(n+1)!}\, C \, \| K(\ell) \|_{\mathrm{op}}^{n+1} \left\langle \Omega, \left(\mathcal{N}+1 \right) \Omega \right\rangle$$

where $C > 0$ and we have the required bound. As for the case where n is odd, we have the same bound coming from the fact that $\mathcal{N} < (\mathcal{N}+1)$. □

Remark 3.9. The above bound for the head term is not optimal but in the infinite n limit proves to be sufficient. Write why it is not optimal

Lemma 3.10 (The infinite n limit)**.**

$$\lim_{n\to\infty} \left\langle \Omega, T_1^* \frac{1}{2} \left(n_q + n_{-q} \right) T_1 \Omega \right\rangle = \frac{1}{2} \sum_{\ell \in \mathbb{Z}_*^3} \mathbb{1}_{L_\ell}(q) \left(\cosh(2K(\ell)) - 1 \right)_{q,q} - \frac{1}{2} \sum_{m=1}^{\infty} \langle \Omega, E_m(P^q) \Omega \rangle \tag{134}$$

Proof. We take the $n \to \infty$ in Proposition 2.19 and from Proposition 3.8 we see that the last term in the expansions tends to 0 in the limit. Hence we obtain the above expression. □

4 Bosonization Errors and Estimates

In this section we bound the bosonization errors.

Lemma 4.1. *For any $\psi \in \mathcal{H}_N$, $q \in \mathbb{Z}^3$, and $\ell \in \mathbb{Z}_*^3$ we have*

$$\| a_q \psi \| = \left\| n_q^{\frac{1}{2}} \psi \right\| \tag{135}$$

$$\| b_q(\ell) \psi \| \leq \left\| n_q^{\frac{1}{2}} \psi \right\| \tag{136}$$

Proof. For the first estimate, we begin with

$$\| a_q \psi \|^2 = \left\langle \psi, a_q^* a_q \psi \right\rangle = \left\| n_q^{\frac{1}{2}} \psi \right\|^2 \tag{137}$$

For the second estimate

$$\begin{aligned} \| b_q(\ell) \psi \|^2 = \left\langle \psi, b_q^*(\ell) b_q(\ell) \psi \right\rangle &= \left\langle \psi a_q^* a_{q-\ell}^* a_{q-\ell} a_q \psi \right\rangle \\ &\leq \left\langle \psi a_q^* a_q \psi \right\rangle = \left\| n_q^{\frac{1}{2}} \psi \right\|^2 \end{aligned} \tag{138}$$

where we used $a_{q-\ell}^* a_{q-\ell} \leq \mathbb{1}$ □

Definition 4.2 (Bootstrap Quantity). For $q \in \mathbb{Z}^3$ and $\lambda \in [0,1]$, we define

$$\Xi_\lambda(q) \coloneqq \langle T_\lambda \Omega, a_q^* a_q T_\lambda \Omega \rangle = \left\| n_q^{\frac{1}{2}} T_\lambda \Omega \right\|^2 \tag{139}$$

$$\Xi \coloneqq \sup_{\lambda \in [0,1]} \sup_{q \in L_k} \langle T_\lambda \Omega, a_q^* a_q T_\lambda \Omega \rangle = \sup_{q \in \mathbb{Z}^3} \left\| n_q^{\frac{1}{2}} T_\lambda \Omega \right\|^2 . \tag{140}$$

To bound the bosonization error in (134), we start by bounding the expectation value of each of the $E_{Q_{\sigma(m)}}$. We spell out the two different error terms (42) and (53) depending on the iteration step m and for a symmetric operator A.

$$E_{Q_1}(A) = -2 \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s \in L_\ell} A_{r,s}(\ell) \Big(\mathcal{E}_r(\ell) b_s(\ell) + b_s^*(\ell) \mathcal{E}_r^*(\ell) \Big)$$

$$E_{Q_2}(A) = \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s \in L_\ell} A_{r,s}(\ell) \Big(\big\{ \mathcal{E}_r^*(\ell), b_{-s}(-\ell) \big\} + \big\{ b^*_{-s}(-l), \mathcal{E}_r(l) \big\} \Big) - \big\{ A, K \big\}_{r,s}(\ell) \epsilon_{r,s}(\ell,\ell).$$

new notation for the anti-commutator

By substituting the definition of $\mathcal{E}_p(k)$ from (32), we arrive at

$$E_{Q_1}(A) = \sum_{\ell,\ell_1 \in \mathbb{Z}^3_*} \sum_{\substack{r,s \in L_\ell \\ r_1,s_1 \in L_{\ell_1}}} A(\ell)_{r,s} K(\ell_1)_{r_1,s_1} \Big(b_s^*(\ell) \{ \epsilon_{r_1,r}(\ell_1.\ell), b_{-s_1}(-\ell_1) \}^* + \text{h.c.} \Big) \tag{141}$$

$$E_{Q_2}(A) = -\frac{1}{2} \sum_{\ell,\ell_1 \in \mathbb{Z}^3_*} \sum_{\substack{r,s \in L_\ell \\ r_1,s_1 \in L_{\ell_1}}} A(\ell)_{r,s} K(\ell_1)_{r_1,s_1} \Big(\big\{ \{ \epsilon_{r_1,r}(\ell_1.\ell), b_{-s_1}(-\ell_1) \}^*, b_{-s}(-l) \big\} + \text{h.c.} \Big) \tag{142}$$

$$- \sum_{\ell \in \mathbb{Z}^3_*} \sum_{r,s \in L_\ell} \big\{ A(\ell), K(\ell) \big\}_{r,s} \epsilon_{r,s}(\ell,\ell). \tag{143}$$

When we substitute the definition of $\epsilon_{r,s}(\ell,k)$, we have that the error terms are not normal ordered. We then normal order all the fermionic operators appearing in the error terms above. We have two combinations of fermionic operators, i.e., one with only one anti-commutator and the other with two anti-commutators. To begin with the normal ordering, we only consider the first term of $\epsilon_{r,r_1}(\ell,\ell_1)$ which is $a^*_{r_1-\ell_1} a_{r-\ell}$. Also when using these identities, we have to take the deltas associated with the quasi-bosonic commutation error into consideration.

For the error term with one anti-commutator, we normal order it as follows

$$\begin{aligned} b_s^*(\ell) \{ a^*_{r_1-\ell_1} a_{r-\ell}, b^*_{-s_1}(-\ell_1) \} &= b_s^*(\ell) a^*_{r_1-\ell_1} \{ a_{r-\ell}, b^*_{-s_1}(-\ell_1) \} \\ &= b_s^*(\ell) a^*_{r_1-\ell_1} a_{r-\ell} b^*_{-s_1}(-\ell_1) + b_s^*(\ell) a^*_{r_1-\ell_1} b^*_{-s_1}(-\ell_1) a_{r-\ell} \\ &= 2 a^*_{r_1-\ell_1} b_s^*(\ell) b^*_{-s_1}(-\ell_1) a_{r-\ell} + b_s^*(\ell) a^*_{r_1-\ell_1} [b_{-s_1}(-\ell_1), a^*_{r-\ell}]^* . \end{aligned} \tag{144}$$

The normal ordering for the term with two anti-commutators is a bit involved. We begin with

$$\big\{ \{ a^*_{r_1-\ell_1} a_{r-\ell}, b^*_{-s_1}(-\ell_1) \}, b_{-s}(-\ell) \big\} = b_{-s}(-\ell) \{ a^*_{r_1-\ell_1} a_{r-\ell}, b^*_{-s_1}(-\ell_1) \} + \{ a^*_{r_1-\ell_1} a_{r-\ell}, b^*_{-s_1}(-\ell_1) \} b_{-s}(-\ell). \tag{145}$$

For the normal ordering of the second term, we proceed as we did to get (144). We normal order the first term as

$$\begin{aligned} &= b_{-s}(-\ell) a^*_{r_1-\ell_1} \{ a_{r-\ell}, b^*_{-s_1}(-\ell_1) \} \\ &= a^*_{r_1-\ell_1} b_{-s}(-\ell) \{ a_{r-\ell}, b^*_{-s_1}(-\ell_1) \} + [b_{-s}(-\ell), a^*_{r_1-\ell_1}] \{ a_{r-\ell}, b^*_{-s_1}(-\ell_1) \} \\ &= a^*_{r_1-\ell_1} b_{-s}(-\ell) a_{r-\ell} b^*_{-s_1}(-\ell_1) + a^*_{r_1-\ell_1} b_{-s}(-\ell) b^*_{-s_1}(-\ell_1) a_{r-\ell} \\ &\quad + [b_{-s}(-\ell), a^*_{r_1-\ell_1}] a_{r-\ell} b^*_{-s_1}(-\ell_1) + [b_{-s}(-\ell), a^*_{r_1-\ell_1}] b^*_{-s_1}(-\ell_1) a_{r-\ell} \\ &= a^*_{r_1-\ell_1} b_{-s}(-\ell) b^*_{-s_1}(-\ell_1) a_{r-\ell} + a^*_{r_1-\ell_1} b_{-s}(-\ell) [b_{-s_1}(-\ell_1), a^*_{r-\ell}]^* \end{aligned}$$

$$
\begin{aligned}
&+ a^*_{r_1-\ell_1} b^*_{-s_1}(-\ell_1) b_{-s}(-\ell) a_{r-\ell} + a^*_{r_1-\ell_1}[b_{-s}(-\ell), b^*_{-s_1}(-\ell_1)] a_{r-\ell} \\
&+ [b_{-s}(-\ell), a^*_{r_1-\ell_1}] b^*_{-s_1}(-\ell_1) a_{r-\ell} - [b_{-s}(-\ell), a^*_{r_1-\ell_1}][a^*_{r-\ell}, b_{-s_1}(-\ell_1)]^* \\
&+ b^*_{-s_1}(-\ell_1)[b_{-s}(-\ell), a^*_{r_1-\ell_1}] a_{r-\ell} + \big[b_{-s_1}(-\ell_1), [b_{-s}(-\ell), a^*_{r_1-\ell_1}]^*\big]^* a_{r-\ell} \\
= {}& 2a^*_{r_1-\ell_1} b^*_{-s_1}(-\ell_1) b_{-s}(-\ell) a_{r-\ell} + 2a^*_{r_1-\ell_1}[b_{-s}(-\ell), b^*_{-s_1}(-\ell_1)] a_{r-\ell} \\
&+ a^*_{r_1-\ell_1}[b_{-s_1}(-\ell_1), a^*_{r-\ell}]^* b_{-s}(-\ell) + a^*_{r_1-\ell_1}\big[b_{-s}(-\ell), [b_{-s_1}(-\ell_1), a^*_{r-\ell}]^*\big] \\
&+ 2b^*_{-s_1}(-\ell_1)[b_{-s}(-\ell), a^*_{r_1-\ell_1}] a_{r-\ell} + 2\big[b_{-s_1}(-\ell_1), [b_{-s}(-\ell), a^*_{r_1-\ell_1}]^*\big]^* a_{r-\ell} \\
&+ \big\{[b_{-s}(-\ell), a^*_{r_1-\ell_1}], [b_{-s_1}(-\ell_1), a^*_{r-\ell}]^*\big\} - [b_{-s_1}(-\ell_1), a^*_{r-\ell}]^*[b_{-s}(-\ell), a^*_{r_1-\ell_1}].
\end{aligned} \tag{146}
$$

Then we have

$$
\begin{aligned}
&\big\{\{a^*_{r_1-\ell_1} a_{r-\ell}, b^*_{-s_1}(-\ell_1)\}, b_{-s}(-\ell)\big\} \\
= {}& 4a^*_{r_1-\ell_1} b^*_{-s_1}(-\ell_1) b_{-s}(-\ell) a_{r-\ell} + 2a^*_{r_1-\ell_1}[b_{-s}(-\ell), b^*_{-s_1}(-\ell_1)] a_{r-\ell} \\
&+ 2a^*_{r_1-\ell_1}[b_{-s_1}(-\ell_1), a^*_{r-\ell}]^* b_{-s}(-\ell) + a^*_{r_1-\ell_1}\big[b_{-s}(-\ell), [b_{-s_1}(-\ell_1), a^*_{r-\ell}]^*\big] \\
&+ 2b^*_{-s_1}(-\ell_1)[b_{-s}(-\ell), a^*_{r_1-\ell_1}] a_{r-\ell} + 2\big[b_{-s_1}(-\ell_1), [b_{-s}(-\ell), a^*_{r_1-\ell_1}]^*\big]^* a_{r-\ell} \\
&+ \big\{[b_{-s}(-\ell), a^*_{r_1-\ell_1}], [b_{-s_1}(-\ell_1), a^*_{r-\ell}]^*\big\} - [b_{-s_1}(-\ell_1), a^*_{r-\ell}]^*[b_{-s}(-\ell), a^*_{r_1-\ell_1}].
\end{aligned} \tag{147}
$$

Then one can proceed in a similar manner to normal order the second term, i.e. $\delta_{r_1-\ell_1,r-\ell} a^*_{r_1} a_r$, of $\epsilon_{r,r_1}(\ell,\ell_1)$. In all the commutators above we see two different momenta $p,q \in \mathbb{Z}^3_*$ in the fermionic creation and annihilation operators. We can resolve these commutators depending on whether p,q are in B_F or B_F^c as

$$
\begin{aligned}
[b_{-s_1}(-\ell_1), a^*_p]^* = [a_{-s_1+\ell_1} a_{-s_1}, a^*_p]^* &= \big(a_{-s_1+\ell_1}\{a_{-s_1}, a^*_p\} - \{a_{-s_1+\ell_1}, a^*_p\} a_{-s_1}\big)^* \\
&= \begin{cases} -\delta_{-s_1+\ell_1,p} a^*_{-s_1} & \text{for} \quad p \in B_F \\ \delta_{-s_1,p} a^*_{-s_1+\ell_1} & \text{for} \quad p \in B_F^c \end{cases}.
\end{aligned} \tag{148}
$$

Similarly

$$
[b_{-s}(-\ell), a^*_p] = \begin{cases} -\delta_{-s+\ell,p} a_{-s} & \text{for} \quad p \in B_F \\ \delta_{-s,p} a_{-s+\ell} & \text{for} \quad p \in B_F^c \end{cases} \tag{149}
$$

$$
\big[b_{-s}(-\ell), [b_{-s_1}(-\ell_1), a^*_p]^*\big] = \begin{cases} -\delta_{-s_1+\ell_1,p}\delta_{s,s_1} a_{-s+\ell} & \text{for} \quad p \in B_F \\ -\delta_{-s_1,p}\delta_{s-\ell,s_1-\ell_1} a_{-s} & \text{for} \quad p \in B_F^c \end{cases} \tag{150}
$$

$$
\big[b_{-s_1}(-\ell_1), [b_{-s}(-\ell), a^*_p]^*\big]^* = \begin{cases} -\delta_{-s+\ell,p}\delta_{s,s_1} a^*_{-s_1+\ell_1} & \text{for} \quad p \in B_F \\ -\delta_{-s,p}\delta_{-s+\ell,-s_1+\ell_1} a^*_{-s_1} & \text{for} \quad p \in B_F^c \end{cases} \tag{151}
$$

$$
[b_{-s_1}(-\ell_1), a^*_p]^*[b_{-s}(-\ell), a^*_q] = \begin{cases} \delta_{-s_1+\ell_1,p}\delta_{-s+\ell,q} a^*_{-s_1} a_{-s} & \text{for} \quad p,q \in B_F \\ \delta_{-s_1,p}\delta_{-s,q} a^*_{-s_1+\ell_1} a_{-s+\ell} & \text{for} \quad p,q \in B_F^c \end{cases}. \tag{152}
$$

In the last commutation relation both p,q are simultaneously either in B_F or B_F^c.
Now for any iteration step m, we insert the operator $A(\ell) = \Theta^m_K(P^q)(\ell)$. And with the definition of the commutation error we see that each of the error terms are further divided into two terms. We can write the terms with $\delta_{r_1-\ell_1,r-\ell} a^*_{r_1} a_r$, by shifting the momenta over which we are summing, giving us terms which have similar forms as their counterparts. Using (144) and (147), along with (148)-(152) and we get the normal ordered error term as

$$
\begin{aligned}
E_{Q_1}(\Theta^m_K(P^q)) = - \sum_{\ell,\ell_1 \in \mathbb{Z}^3_*} \sum_{\substack{r \in L_\ell \cap L_{\ell_1} \\ s \in L_\ell, s_1 \in L_{\ell_1}}} \Theta^m_K(P^q)(\ell)_{r,s} K(\ell_1)_{r,s_1} \Big(& 2a^*_{r-\ell_1} b^*_s(\ell) b^*_{-s_1}(-\ell_1) a_{r-\ell} \\
& - \delta_{-s_1+\ell_1,r-\ell} b^*_s(\ell) a^*_{r-\ell_1} a^*_{-s_1} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{\substack{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)\\ s\in L_\ell,s_1\in L_{\ell_1}}}\Theta^m_K(P^q)(\ell)_{r+\ell,s}K(\ell_1)_{r+\ell_1,s_1}\Big(2a^*_{r+\ell_1}b^*_s(\ell)b^*_{-s_1}(-\ell_1)a_{r+\ell}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \delta_{-s_1,r+\ell}b^*_s(\ell)a^*_{r+\ell_1}a^*_{-s_1+\ell_1}\Big)\\
&+\text{h.c.} =: \sum_{i=1}^2\sum_{j=1}^2 E^{i,j}_{Q_1}+\text{h.c.}
\end{aligned}
$$

and

$$
\begin{aligned}
2E_{Q_2}(\Theta^m_K(P^q)) = &\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell,s_1\in L_{\ell_1}}}\Theta^m_K(P^q)(\ell)_{r,s}K(\ell_1)_{r,s_1}\Big(4a^*_{r-\ell_1}b^*_{-s_1}(-\ell_1)b_{-s}(-\ell)a_{r-\ell}\\
&+4\delta_{s,s_1}\delta_{\ell,\ell_1}a^*_{r-\ell_1}a_{r-\ell}-2\delta_{s-\ell,s_1-\ell_1}a^*_{r-\ell_1}a^*_{-s_1}a_{-s}a_{r-\ell}\\
&-2\delta_{s,s_1}a^*_{r-\ell_1}a^*_{-s_1+\ell_1}a_{-s+\ell}a_{r-\ell}-2\delta_{-s_1+\ell_1,r-\ell}a^*_{r-\ell_1}a^*_{-s_1}b_{-s}(-\ell)\\
&-\delta_{-s_1+\ell_1,r-\ell}\delta_{s,s_1}a^*_{r-\ell_1}a_{-s+\ell}-2\delta_{-s+\ell,r-\ell_1}b^*_{-s_1}(-\ell_1)a_{-s}a_{r-\ell}\\
&-2\delta_{-s+\ell,r-\ell_1}\delta_{s,s_1}a^*_{-s_1+\ell_1}a_{r-\ell}-\delta_{-s_1+\ell_1,r-\ell}\delta_{-s+\ell,r-\ell_1}a^*_{-s_1}a_{-s}\\
&+\delta_{-s+\ell,r-\ell_1}\delta_{-s_1+\ell_1,r-\ell}\delta_{s,s_1}\Big)\\
+&\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{\substack{r\in(L_\ell-\ell)\\ \cap\\ (L_{\ell_1}-\ell_1)\\ s\in L_\ell,s_1\in L_{\ell_1}}}\Theta^m_K(P^q)(\ell)_{r+\ell,s}K(\ell_1)_{r+\ell_1,s_1}\Big(4a^*_{r+\ell_1}b^*_{-s_1}(-\ell_1)b_{-s}(-\ell)a_{r+\ell}\\
&+4\delta_{s,s_1}\delta_{\ell,\ell_1}a^*_{r+\ell_1}a_{r+\ell}-2\delta_{s-\ell,s_1-\ell_1}a^*_{r+\ell_1}a^*_{-s_1}a_{-s}a_{r+\ell}\\
&-2\delta_{s,s_1}a^*_{r+\ell_1}a^*_{-s_1+\ell_1}a_{-s+\ell}a_{r+\ell}+2\delta_{-s_1,r+\ell}a^*_{r+\ell_1}a^*_{-s_1+\ell_1}b_{-s}(-\ell)\\
&-\delta_{-s_1,r+\ell}\delta_{s-\ell,s_1-\ell_1}a^*_{r+\ell_1}a_{-s}+2\delta_{-s,r+\ell_1}b^*_{-s_1}(-\ell_1)a_{-s+\ell}a_{r+\ell}\\
&-2\delta_{-s,r+\ell_1}\delta_{-s+\ell,-s_1+\ell_1}a^*_{-s_1}a_{r+\ell}-\delta_{-s_1,r+\ell}\delta_{-s,r+\ell_1}a^*_{-s_1+\ell_1}a_{-s+\ell}\\
&+\delta_{-s,r+\ell_1}\delta_{-s_1,r+\ell}\delta_{-s+\ell,-s_1+\ell_1}\Big)\\
+&\text{h.c.} =: \sum_{i=1}^2\sum_{j=1}^{10} E^{i,j}_{Q_2}+\text{h.c.}\quad .
\end{aligned}
$$

Here, the first superscript refers to the momentum r,s,s_1 being summed over different sets and the second superscript refers to the different terms within, i.e., 2 terms for every ith sum in E_{Q_1} and 10 terms for every ith sum in E_{Q_2}. These terms either have six, four, two or no fermionic operators.
Preliminary to bounding these errors, we identify certain terms in $E^{i,j}_{Q_1}, E^{i,j}_{Q_2}$ with each other to further reduce the number of terms to be dealt with while writing the error estimates.

Lemma 4.3. *1.* $E^{1,2}_{Q_1}=E^{2,2}_{Q_1}$ *2.* $E^{1,3}_{Q_2}=E^{2,4}_{Q_2}$ *3.* $E^{1,5}_{Q_2}=E^{2,5}_{Q_2}$
4. $E^{1,7}_{Q_2}=E^{2,7}_{Q_2}$ *5.* $E^{1,8}_{Q_2}=2E^{1,6}_{Q_2}=2E^{2,9}_{Q_2}$ *6.* $E^{2,8}_{Q_2}=2E^{1,9}_{Q_2}=2E^{2,6}_{Q_2}$
7. $E^{1,10}_{Q_2}=E^{2,10}_{Q_2}$

Proof. We begin with writing the terms explicitly and do the necessary identification in order to see that they are exactly the same. We start by proving $E^{1,2}_{Q_1}=E^{2,2}_{Q_1}$.

$$
\begin{aligned}
E^{1,2}_{Q_1} &= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell,s_1\in L_{\ell_1}}}\Theta^m_K(P^q)(\ell)_{r,s}K_{r,s_1}(\ell_1)\Big(\delta_{-s_1+\ell_1,r-\ell}b^*_s(\ell)a^*_{r-\ell_1}a^*_{-s_1}\Big)\\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\cap(-L_{\ell_1}+\ell_1+\ell)\\ s\in L_\ell}}\Theta^m_K(P^q)(\ell)_{r,s}K_{r,-r+\ell_1+\ell}(\ell_1)\Big(b^*_s(\ell)a^*_{r-\ell_1}a^*_{r-\ell_1-\ell}\Big)
\end{aligned}\tag{153}
$$

$$
\begin{aligned}
E^{2,2}_{Q_1} &= -\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{\substack{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)\\ s\in L_\ell,s_1\in L_{\ell_1}}}\Theta^m_K(P^q)(\ell)_{r+\ell,s}K_{r+\ell_1,s_1}(\ell_1)\Big(\delta_{-s_1,r+\ell}b^*_q(\ell)a^*_{r+\ell_1}a^*_{-s_1+\ell_1}\Big)\\
&= -\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{\substack{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)\cap(-L_{\ell_1}-\ell)\\ s\in L_\ell}}\Theta^m_K(P^q)(\ell)_{r+\ell,s}K_{r+\ell_1,-r-\ell}(\ell_1)\Big(b^*_s(\ell)a^*_{r+\ell_1}a^*_{r+\ell+\ell_1}\Big)
\end{aligned}\tag{154}
$$

Next, we substitute $r = r' - \ell$, which also changes the summed over set, which gives us

$$= - \sum_{\ell,\ell_1 \in \mathbb{Z}^3_*} \sum_{\substack{r' \in L_\ell \cap (-L_{\ell_1}) \cap (L_{\ell_1} - \ell_1 + \ell) \\ s \in L_\ell}} \Theta^m_K(P^q)(\ell)_{r',s} K_{r'+\ell_1-\ell,-r'}(\ell_1) \Big(b^*_s(\ell) a^*_{r'+\ell_1-\ell} a^*_{r'+\ell_1} \Big) \tag{155}$$

Then we flip the ℓ_1 momenta, i.e., $\ell_1 = -\ell_1$ and use the symmetry $K(\ell)_{p,q} = K(-\ell)_{-p,-q}$ to have

$$\begin{aligned}
&= - \sum_{\ell,\ell_1 \in \mathbb{Z}^3_*} \sum_{\substack{r' \in L_\ell \cap (L_{\ell_1}) \cap (-L_{\ell_1} + \ell_1 + \ell) \\ s \in L_\ell}} \Theta^m_K(P^q)(\ell)_{r',s} K_{r'-\ell_1-\ell,-r'}(-\ell_1) \Big(b^*_s(\ell) a^*_{r'-\ell_1-\ell} a^*_{r'-\ell_1} \Big) \\
&= - \sum_{\ell,\ell_1 \in \mathbb{Z}^3_*} \sum_{\substack{r' \in L_\ell \cap (L_{\ell_1}) \cap (-L_{\ell_1} + \ell_1 + \ell) \\ s \in L_\ell}} \Theta^m_K(P^q)(\ell)_{r',s} K_{-r'+\ell_1+\ell,r'}(\ell_1) \Big(b^*_s(\ell) a^*_{r'-\ell_1-\ell} a^*_{r'-\ell_1} \Big) \\
&= \sum_{\ell,\ell_1 \in \mathbb{Z}^3_*} \sum_{\substack{r' \in L_\ell \cap (L_{\ell_1}) \cap (-L_{\ell_1} + \ell_1 + \ell) \\ s \in L_\ell}} \Theta^m_K(P^q)(\ell)_{r',s} K_{r',-r'+\ell_1+\ell}(\ell_1) \Big(b^*_s(\ell) a^*_{r'-\ell_1} a^*_{r'-\ell_1-\ell} \Big) \qquad (156)
\end{aligned}$$

where in the last equality we used the CAR to exchange the two creation operators and $K(\ell)_{p,q} = K(\ell)_{q,p}$. One can similarly prove $E^{1,5}_{Q_2} = E^{2,5}_{Q_2}$, $E^{1,6}_{Q_2} = E^{2,9}_{Q_2}$, $E^{1,10}_{Q_2} = E^{2,10}_{Q_2}$ and $E^{1,9}_{Q_2} = E^{2,6}_{Q_2}$ using the same identifications as above. For $E^{1,7}_{Q_2} = E^{2,7}_{Q_2}$, we substitute $r = r' - \ell_1$ and then follow the same steps as above. For $2E^{i,6}_{Q_2} = E^{i,8}_{Q_2}$ for $i = \{1,2\}$, we just interchange ℓ and ℓ_1. For $E^{1,3}_{Q_2} = E^{2,4}_{Q_2}$, we use the CAR relation twice and interchange the r and s indices to get the desired result. □

Finally, we have the error terms as

$$E_{Q_1}\left(\Theta^m_K(P^q)\right) = E^{1,1}_{Q_1} + E^{2,1}_{Q_1} + 2E^{1,2}_{Q_1} + \text{h.c.} \tag{157}$$

$$\begin{aligned}
2E_{Q_2}\left(\Theta^m_K(P^q)\right) = {} & E^{1,1}_{Q_2} + E^{2,1}_{Q_2} + E^{1,2}_{Q_2} + E^{2,2}_{Q_2} + 2E^{1,3}_{Q_2} + E^{2,3}_{Q_2} + E^{1,4}_{Q_2} \\
& + 2E^{1,5}_{Q_2} + 2E^{1,7}_{Q_2} + 2E^{1,8}_{Q_2} + 2E^{2,8}_{Q_2} + 2E^{1,10}_{Q_2} + \text{h.c.} \qquad (158)
\end{aligned}$$

Remark 4.4. We can decompose the nested m-fold anti commutator, $\Theta^m_K(P^q)(\ell)$, as

$$\Theta^m_K(P^q)(\ell)_{r,s} = \left(K^m \cdot P^q\right)(\ell)_{r,s} + \left(\sum_{j=1}^{m-1} \binom{m}{j} K^{m-j} \cdot P^q \cdot K^j \right)(\ell)_{r,s} + \left(P^q \cdot K^m\right)(\ell)_{r,s} \tag{159}$$

When we explicitly put P^q, we get the decomposition of the error terms corresponding to momentum fixing at the very right, at all the intermediate positions and the very left, respectively. Each of these momentum fixed terms require slightly different estimates. Also P^q fixes momenta to both q and $-q$ but $q \in \mathbb{Z}^3_*$ is reflection symmetric. Hence we only perform the estimates for momentum fixing to q and multiply it by two to get the contribution for momentum fixing to $-q$. Effectively, we have

$$\Theta^m_K(P^q)(\ell)_{r,s} = 2 \cdot \mathbb{1}_{L_\ell}(q) \frac{1}{2} \left(K^m(\ell)_{r,q} \delta_{q,s} + \sum_{j=1}^{m-1} \binom{m}{j} K^{m-j}(\ell)_{r,q} K^j(\ell)_{q,s} + K^m(\ell)_{q,s} \delta_{q,r} \right), \text{ for } r,s \in L_\ell \tag{160}$$

Next we bound these decomposed error terms and in order to do so we have the following estimates.

4.1 E_{Q_1} Estimates

Lemma 4.5 ($E^{1,1}_{Q_1}$)**.** *For any $\psi \in \mathcal{H}_N$ and $q \in \mathbb{Z}^3_*$, we have*

$$\left| \left\langle \psi, \left(E^{1,1}_{Q_1} + \text{h.c.} \right) \psi \right\rangle \right| \le C\, \Xi^{\frac{1}{2}} \left(\sum_{\ell \in \mathbb{Z}^3_*} \|K^m(\ell)\|_{\max} + \sum_{j=0}^{m} \sum_{\ell \in \mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max} \right) \times$$

$$\times\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \tag{161}$$

Proof. We start with the L.H.S. of (161).

$$\begin{aligned}
&\left|\left\langle\psi,\left(E^{1,1}_{Q_1}+\text{h.c.}\right)\psi\right\rangle\right| = \left|\left\langle\psi, 2\mathrm{Re}\left(E^{1,1}_{Q_1}\right)\psi\right\rangle\right| = 2\left|\left\langle\psi, E^{1,1}_{Q_1}\psi\right\rangle\right| \\
= \ & 4\left|\left\langle\psi,\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell, s_1\in L_{\ell_1}}}\Theta^m_K(P^q)(\ell)_{r,s}K(\ell_1)_{r,s_1}a^*_{r-\ell_1}b^*_s(\ell)b^*_{-s_1}(-\ell_1)a_{r-\ell}\psi\right\rangle\right| \\
\leq \ & 4\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s_1\in L_{\ell_1}}}\left|\left\langle\psi, K^m(\ell)_{r,q}K(\ell_1)_{r,s_1}a^*_{r-\ell_1}b^*_q(\ell)b^*_{-s_1}(-\ell_1)a_{r-\ell}\psi\right\rangle\right| \qquad (162)\\
& +4\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left|\left\langle\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell_1)_{r,s_1}a^*_{r-\ell_1}b^*_s(\ell)b^*_{-s_1}(-\ell_1)a_{r-\ell}\psi\right\rangle\right| \\
& \qquad (163)\\
& +4\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{s\in L_\ell, s_1\in L_{\ell_1}}\left|\left\langle\psi, K^m(\ell)_{q,s}K(\ell_1)_{q,s_1}a^*_{q-\ell_1}b^*_s(\ell)b^*_{-s_1}(-\ell_1)a_{q-\ell}\psi\right\rangle\right| \qquad (164)
\end{aligned}$$

where the last inequality is implied by Remark 4.4 and we used (160). For (162)-(164), we start by using resolution of the identity $I=(\mathcal{N}+1)^{\alpha}(\mathcal{N}+1)^{-\alpha}$ for some $\alpha\in\mathbb{R}$. Then we use the Cauchy-Schwarz inequality, Lemma 2.5 and the bounds from Lemma 3.2. We estimate (162) as

(162)

$$\begin{aligned}
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\left|\left\langle\sum_{s_1\in L_{\ell_1}}K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)K^m_{r,q}(\ell)b_q(\ell)a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}(\mathcal{N}+1)^{-\alpha}\psi, a_{r-\ell}\psi\right\rangle\right| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\left|\left\langle\sum_{s_1\in L_{\ell_1}}K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)K^m_{r,q}(\ell)b_q(\ell)a_{r-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\right\rangle\right| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\left\|\sum_{s_1\in L_{\ell_1}}K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)K^m_{r,q}(\ell)b_q(\ell)a_{r-\ell_1}(\mathcal{N}+5)^{-\alpha}\psi\right\|\|a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\left\|\left(K^m_{r,q}(\ell)b_q(\ell)\right)a_{r-\ell_1}(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K(\ell_1)\|_{\max,2}\sum_{r\in L_\ell\cap L_{\ell_1}}\left\|a_{r-\ell_1}\left(K^m_{r,q}(\ell)b_q(\ell)\right)(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max,2}\|K(\ell_1)\|_{\max}\sum_{r\in L_\ell\cap L_{\ell_1}}\left\|a_{r-\ell_1}b_q(\ell)(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max,2}\|K(\ell_1)\|_{\max}\left(\sum_{r\in L_{\ell_1}}\left\|a_{r-\ell_1}b_q(\ell)(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac{1}{2}}\left(\sum_{r\in L_\ell}\|a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\|^2\right)^{\frac{1}{2}} \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max,2}\|K(\ell_1)\|_{\max}\left\|b_q(\ell)(\mathcal{N}+1)(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+5)^{\alpha}\psi\right\|
\end{aligned}$$

wherein we used $\mathcal{N} < (\mathcal{N}+2) < (\mathcal{N}+5)$ and $\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+5) \le C(\mathcal{N}+1)^{\frac{3}{2}}$. Then for $\alpha = 1$, we have

$$\begin{aligned}
&\le C \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\|K(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|b_q(\ell)\psi\|\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \\
&\le C \sup_{q\in L_\ell}\left\|n_q^{\frac{1}{2}}\psi\right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \\
(162) &\le C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\|.
\end{aligned} \tag{165}$$

We estimate (163) as

$$\begin{aligned}
&(163) \\
&= \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left|\left\langle K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)b_s(\ell)a_{r-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\right\rangle\right| \\
&\le \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)b_s(\ell)a_{r-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\| \\
&\le \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max}\sum_{r\in L_\ell\cap L_{\ell_1}}\left\|a_{r-\ell_1}(\mathcal{N}+1)(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{r-\ell}(\mathcal{N}+5)^{\alpha}\psi\|
\end{aligned}$$

For $\alpha = 1$, we have

$$\begin{aligned}
&\le \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max}\sum_{r\in L_\ell\cap L_{\ell_1}}\|a_{r-\ell_1}\psi\|\|a_{r-\ell}(\mathcal{N}+5)\psi\| \\
&\le C\sup_{r\in\mathbb{Z}^3_*}\left\|n_r^{\frac{1}{2}}\psi\right\|\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \\
(163) &\le C\,\Xi^{\frac{1}{2}}\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\|
\end{aligned} \tag{166}$$

We estimate (164) as

$$\begin{aligned}
&(164) \\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_{\ell_1}}(q)\mathbb{1}_{L_\ell}(q)\sum_{s\in L_\ell,s_1\in L_{\ell_1}}\left|\left\langle K^m(\ell)_{q,s}K(\ell_1)_{q,s_1}b_{-s_1}(-\ell_1)b_s(\ell)a_{q-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, a_{q-\ell}(\mathcal{N}+5)^{\alpha}\psi\right\rangle\right| \\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_{\ell_1}}(q)\mathbb{1}_{L_\ell}(q)\sum_{s\in L_\ell,s_1\in L_{\ell_1}}\left\|K^m(\ell)_{q,s}K(\ell_1)_{q,s_1}b_{-s_1}(-\ell_1)b_s(\ell)a_{q-\ell_1}(\mathcal{N}+5)^{-\alpha}\psi\right\|\|a_{q-\ell}(\mathcal{N}+5)^{\alpha}\psi\| \\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|a_{q-\ell_1}(\mathcal{N}+1)(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{q-\ell}(\mathcal{N}+5)^{\alpha}\psi\|
\end{aligned}$$

For $\alpha = 1$

$$\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|a_{q-\ell_1}\psi\|\|a_{q-\ell}(\mathcal{N}+5)\psi\|$$

Then we use $\|a_q\psi\| \leq \left\|\mathcal{N}^{\frac{1}{2}}\psi\right\|$ for $q \in B_F$ or $q \in B_F^c$ and we arrive at

$$
\begin{aligned}
&\leq C \sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac{1}{2}}\psi\right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \\
\text{(164)} &\leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \qquad (167)
\end{aligned}
$$

Then adding (165),(166) and (167), we arrive at the bound above (161). $\square$

Lemma 4.6 ($E^{2,1}_{Q_1}$)**.** *For any $\psi \in \mathcal{H}_N$ and $q \in \mathbb{Z}^3_*$, we have*

$$
\begin{aligned}
\left|\left\langle\psi, \left(E^{2,1}_{Q_1} + \text{h.c.}\right)\psi\right\rangle\right| \leq C\,\Xi^{\frac{1}{2}}&\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max} + \sum_{j=0}^{m}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\times \\
&\times\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \qquad (168)
\end{aligned}
$$

Proof. We start with the L.H.S. of (168).

$$
\begin{aligned}
&\left|\left\langle\psi, \left(E^{2,1}_{Q_1} + \text{h.c.}\right)\psi\right\rangle\right| = \left|\left\langle\psi, 2\mathrm{Re}\left(E^{2,1}_{Q_1}\right)\psi\right\rangle\right| = 2\left|\left\langle\psi, E^{2,1}_{Q_1}\psi\right\rangle\right| \\
&\leq 4\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)\\ s_1\in L_{\ell_1}}}\left|\left\langle\psi, K^m(\ell)_{r+\ell,q}K(\ell_1)_{r+\ell_1,s_1}a^*_{r+\ell_1}b^*_q(\ell)b^*_{-s_1}(-\ell_1)a_{r+\ell}\psi\right\rangle\right| \qquad (169) \\
&+4\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left|\left\langle\psi, K^{m-j}(\ell)_{r+\ell,q}K^j(\ell)_{q,s}K(\ell_1)_{r+\ell_1,s_1}a^*_{r+\ell_1}b^*_s(\ell)b^*_{-s_1}(-\ell_1)a_{r+\ell}\psi\right\rangle\right| \\
&\qquad (170) \\
&+4\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q)\sum_{s\in L_\ell, s_1\in L_{\ell_1}}\left|\left\langle\psi, K^m(\ell)_{q+\ell,s}K(\ell_1)_{q+\ell_1,s_1}a^*_{q+\ell_1}b^*_s(\ell)b^*_{-s_1}(-\ell_1)a_{q+\ell}\psi\right\rangle\right| \\
&\qquad (171)
\end{aligned}
$$

where the last inequality is implied by Remark 4.4 and we used (160). For (169)-(171), we start by using resolution of the identity $I = (\mathcal{N}+1)^{\alpha}(\mathcal{N}+1)^{-\alpha}$ for some $\alpha \in \mathbb{R}$. Then we use the Cauchy-Schwarz inequality, Lemma 2.5 and the bounds from Lemma 3.2. We estimate (169) as

$$
\begin{aligned}
&(169) \\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)\\ s_1\in L_{\ell_1}}}\left|\left\langle K(\ell_1)_{r+\ell_1,s_1}b_{-s_1}(-\ell_1)K^m_{r+\ell,q}(\ell)b_q(\ell)a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi, a_{r+\ell}(\mathcal{N}+5)^{-\alpha}\psi\right\rangle\right| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)}}\sum_{s_1\in L_{\ell_1}}\left\|K(\ell_1)_{r+\ell_1,s_1}b_{-s_1}(-\ell_1)K^m_{r+\ell,q}(\ell)b_q(\ell)a_{r+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|a_{r+\ell}(\mathcal{N}+5)^{\alpha}\psi\right\| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K(\ell_1)\|_{\max}\|K^m(\ell)\|_{\max}\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)}}\left\|a_{r+\ell_1}b_q(\ell)(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|a_{r+\ell}(\mathcal{N}+5)^{\alpha}\psi\right\|
\end{aligned}
$$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max} \left(\sum_{r\in(L_{\ell_1}-\ell_1)} \left\|a_{r+\ell_1}b_q(\ell)(\mathcal{N}+1)^{\frac12}(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac12}\times$$
$$\times\left(\sum_{r\in(L_\ell-\ell)} \|a_{r+\ell}(\mathcal{N}+5)^{\alpha}\psi\|^2\right)^{\frac12}$$

wherein we used $\sum_{p\in\mathbb{Z}^3_*}\|a_p\psi\|^2 \leq \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+2)\psi\|^2$, $\sum_{p\in L_\ell}\|b_p(\ell)\psi\|^2 \leq \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+5)\psi\|^2$ and $\mathcal{N}^{\frac12}(\mathcal{N}+5) \leq C(\mathcal{N}+1)^{\frac32}$. Then for $\alpha=1$, we have

$$\leq C\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\|K(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|b_q(\ell)\psi\|\left\|(\mathcal{N}+1)^{\frac32}\psi\right\|$$
$$(169) \leq C\,\Xi^{\frac12}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac32}\psi\right\|. \tag{172}$$

We estimate (170) as

(170)
$$= \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left|\left\langle K^{m-j}(\ell)_{r+\ell,q}K^j(\ell)_{q,s}K(\ell_1)_{r+\ell_1,s_1}b_{-s_1}(-\ell_1)b_s(\ell)a_{r+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, a_{r+\ell}(\mathcal{N}+5)^{\alpha}\psi\right\rangle\right|$$
$$\leq \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left\|K^{m-j}(\ell)_{r+\ell,q}K^j(\ell)_{q,s}K(\ell_1)_{r+\ell_1,s_1}b_{-s_1}(-\ell_1)b_s(\ell)a_{r+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{r+\ell}(\mathcal{N}+5)^{\alpha}\psi\|$$
$$\leq \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max}\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)}}\left\|a_{r+\ell_1}(\mathcal{N}+1)(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{r+\ell}(\mathcal{N}+5)^{\alpha}\psi\|$$

For $\alpha=1$, we have

$$\leq \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max}\sum_{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)}\|a_{r+\ell_1}\psi\|\|a_{r+\ell}(\mathcal{N}+5)\psi\|$$
$$\leq C\sup_{r\in\mathbb{Z}^3_*}\left\|n_r^{\frac12}\psi\right\|\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac32}\psi\right\|$$
$$(170) \leq C\,\Xi^{\frac12}\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac32}\psi\right\| \tag{173}$$

We estimate (171) as

(171)
$$= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q)\sum_{s\in L_\ell, s_1\in L_{\ell_1}}\left|\left\langle K^m(\ell)_{q+\ell,s}K(\ell_1)_{q+\ell_1,s_1}b_{-s_1}(-\ell_1)b_s(\ell)a_{q+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, a_{q+\ell}(\mathcal{N}+5)^{\alpha}\psi\right\rangle\right|$$
$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q)\sum_{s\in L_\ell, s_1\in L_{\ell_1}}\left\|K^m(\ell)_{q+\ell,s}K(\ell_1)_{q+\ell_1,s_1}b_{-s_1}(-\ell_1)b_s(\ell)a_{q+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{q+\ell}(\mathcal{N}+5)^{\alpha}\psi\|$$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|a_{q+\ell_1}(\mathcal{N}+1)(\mathcal{N}+1)^{-\alpha}\psi\right\|\|a_{q+\ell}(\mathcal{N}+5)^{\alpha}\psi\|$$

For $\alpha = 1$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|a_{q+\ell_1}\psi\|\|a_{q+\ell}(\mathcal{N}+5)\psi\|$$

Then we use $\|a_q\psi\| \leq \left\|\mathcal{N}^{\frac{1}{2}}\psi\right\|$ for $q \in B_F$ or $q \in B_F^c$ and we arrive at

$$\leq C \sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac{1}{2}}\psi\right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\|$$

$$(171) \leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \tag{174}$$

Then adding (172),(173) and (174), we arrive at the bound above (168). □

Lemma 4.7 ($E^{1,2}_{Q_1}$)**.** *For any $\psi \in \mathcal{H}_N$, we have*

$$\begin{aligned}
&2\left|\left\langle\psi,\left(E^{1,2}_{Q_1}+\text{h.c.}\right)\psi\right\rangle\right| \\
&\leq \quad C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \\
&\quad + C\,\Xi^{\frac{1}{2}}\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max,2}\right)\|(\mathcal{N}+1)\psi\|
\end{aligned} \tag{175}$$

Proof. We start with the L.H.S. of (175).

$$2\left|\left\langle\psi,\left(E^{1,2}_{Q_1}+\text{h.c.}\right)\psi\right\rangle\right| = 2\left|\left\langle\psi,2\mathrm{Re}\left(E^{1,2}_{Q_1}\right)\psi\right\rangle\right| = 4\left|\left\langle\psi,E^{1,2}_{Q_1}\psi\right\rangle\right|$$

$$\leq 4\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}|\langle\psi,K^m(\ell)_{r,q}K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell_1}b_q(\ell)\psi\rangle| \tag{176}$$

$$+4\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell}}\left|\left\langle\psi,K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell_1}b_s(\ell)\psi\right\rangle\right| \tag{177}$$

$$+4\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_{\ell_1}+\ell+\ell_1)}(q)\sum_{s\in L_\ell}|\langle\psi,K^m(\ell)_{q,s}K(\ell_1)_{q,-q+\ell+\ell_1}a_{q-\ell-\ell_1}a_{q-\ell_1}b_s(\ell)\psi\rangle| \tag{178}$$

where the last inequality is implied by Remark 4.4 and we used (160). For (176)-(178), we start by using resolution of the identity $I = (\mathcal{N}+5)^{-\alpha}(\mathcal{N}+5)^{\alpha}$ for some $\alpha \in \mathbb{R}$. Then we use the Cauchy-Schwarz inequality, Lemma 2.5 and the bounds from Lemma 3.2. We estimate (176) as

$$\begin{aligned}
(176) &= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\left|\left\langle(\mathcal{N}+5)^{\alpha}\psi,K^m(\ell)_{r,q}K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell_1}b_q(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\rangle\right| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\|(\mathcal{N}+5)^{\alpha}\psi\|\left\|K^m(\ell)_{r,q}K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell_1}b_q(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|(\mathcal{N}+5)^{\alpha}\psi\|\left\|b_q(\ell)(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|
\end{aligned}$$

wherein we used $\|a_p\| \leq \mathbb{1}$, $\sum_{p\in\mathbb{Z}^3_*} \|a_p\psi\|^2 \leq \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+2)\psi\|^2$ and $(\mathcal{N}+5) \leq C(\mathcal{N}+1)$. Then for $\alpha = \frac{1}{2}$, we have

$$(176) \leq C\,\Xi^{\frac{1}{2}} \left(\sum_{\ell\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \tag{179}$$

We estimate (177) as

$$\begin{aligned}
&(177)\\
&= \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell}} \left|\left\langle K(\ell_1)_{r,-r+\ell+\ell_1} a^*_{r-\ell-\ell_1}(\mathcal{N}+5)^\alpha\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}a_{r-\ell_1}b_s(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\rangle\right|\\
&\leq \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max} \sum_{r\in L_\ell\cap L_{\ell_1}} \left\|K(\ell_1)_{r,-r+\ell+\ell_1}a^*_{r-\ell-\ell_1}(\mathcal{N}+5)^\alpha\psi\right\|\times\\
&\qquad\qquad\times\left\|a_{r-\ell_1}(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|
\end{aligned}$$

Then for $\alpha = \frac{1}{2}$, we have

$$\begin{aligned}
&\leq \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max} \sum_{r\in L_\ell\cap L_{\ell_1}} \left\|K(\ell_1)_{r,-r+\ell+\ell_1}a^*_{r-\ell-\ell_1}(\mathcal{N}+5)^{\frac{1}{2}}\psi\right\|\times\\
&\qquad\qquad\times\|a_{r-\ell_1}\psi\|
\end{aligned}$$

$$(177) \leq C\,\Xi^{\frac{1}{2}} \left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max,2}\right)\|(\mathcal{N}+1)\psi\| \tag{180}$$

We estimate (178) as

$$\begin{aligned}
(178) &= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_{\ell_1}+\ell+\ell_1)}(q) \sum_{s\in L_\ell} \left|\left\langle(\mathcal{N}+5)^\alpha\psi, K^m(\ell)_{q,s}K(\ell_1)_{q,-q+\ell+\ell_1}a_{q-\ell-\ell_1}a_{q-\ell_1}b_s(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\rangle\right|\\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_{\ell_1}+\ell+\ell_1)}(q) \sum_{s\in L_\ell} \|(\mathcal{N}+5)^\alpha\psi\|\left\|K^m(\ell)_{q,s}K(\ell_1)_{q,-q+\ell+\ell_1}a_{q-\ell-\ell_1}a_{q-\ell_1}b_s(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\|
\end{aligned}$$

For $\alpha = \frac{1}{2}$, we have

$$\leq C \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_{\ell_1}+\ell+\ell_1)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|a_{q-\ell_1}\psi\|\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|$$

$$(178) \leq C\,\Xi^{\frac{1}{2}} \left(\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \tag{181}$$

Then adding (179),(180) and (181), we arrive at the bound above (175). □

Lemma 4.8 (E_{Q_1})**.** *For any $\psi \in \mathcal{H}_N$, we have*

$$\begin{aligned}
&|\langle\psi, E_{Q_1}\psi\rangle|\\
&\leq C
\end{aligned} \tag{182}$$

Proof. We start by □

4.2 E_{Q_2} Estimates

Lemma 4.9 ($E^{1,1}_{Q_2}$)**.** *For any $\psi \in \mathcal{H}_N$, we have*

$$\left|\left\langle \psi, \left(E^{1,1}_{Q_2} + \text{h.c.}\right)\psi\right\rangle\right| \le C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max} + \sum_{j=0}^{m}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\times$$
$$\times\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \tag{183}$$

Proof. We start with the L.H.S. of (183).

$$\left|\left\langle \psi, \left(E^{1,1}_{Q_2} + \text{h.c.}\right)\psi\right\rangle\right| = \left|\left\langle \psi, 2\text{Re}\left(E^{1,1}_{Q_2}\right)\psi\right\rangle\right| = 2\left|\left\langle \psi, E^{1,1}_{Q_2}\psi\right\rangle\right|$$
$$\le\ 8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s_1\in L_{\ell_1}}}\left|\left\langle\psi, K^m(\ell)_{r,q}K(\ell_1)_{r,s_1}a^*_{r-\ell_1}b^*_q(\ell)b_{-s_1}(-\ell_1)a_{r-\ell}\psi\right\rangle\right| \tag{184}$$
$$+8\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left|\left\langle\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell_1)_{r,s_1}a^*_{r-\ell_1}b^*_s(\ell)b_{-s_1}(-\ell_1)a_{r-\ell}\psi\right\rangle\right| \tag{185}$$
$$+8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{s\in L_\ell, s_1\in L_{\ell_1}}\left|\left\langle\psi, K^m(\ell)_{q,s}K(\ell_1)_{q,s_1}a^*_{q-\ell_1}b^*_s(\ell)b_{-s_1}(-\ell_1)a_{q-\ell}\psi\right\rangle\right| \tag{186}$$

where the last inequality is implied by Remark 4.4 and we used (160). For (184)-(186), we start by using resolution of the identity $I = (\mathcal{N}+1)^{\alpha}(\mathcal{N}+1)^{-\alpha}$ for some $\alpha\in\mathbb{R}$. Then we use the Cauchy-Schwarz inequality, Lemma 2.5 and the bounds from Lemma 3.2.We estimate (184) as

(184)
$$= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s_1\in L_{\ell_1}}}\left|\left\langle K^m_{r,q}(\ell)b_q(\ell)a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}(\mathcal{N}+1)^{-\alpha}\psi, K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)a_{r-\ell}\psi\right\rangle\right|$$
$$\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s_1\in L_{\ell_1}}}\left|\left\langle K^m_{r,q}(\ell)b_q(\ell)a_{r-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)a_{r-\ell}(\mathcal{N}+1)^{\alpha}\psi\right\rangle\right|$$
$$\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\sum_{s_1\in L_{\ell_1}}\left\|K^m_{r,q}(\ell)b_q(\ell)a_{r-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)a_{r-\ell}(\mathcal{N}+1)^{\alpha}\psi\right\|$$
$$\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\sum_{r\in L_\ell\cap L_{\ell_1}}\left\|a_{r-\ell_1}b_q(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|a_{r-\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\alpha}\psi\right\|$$
$$\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left(\sum_{r\in L_{\ell_1}}\left\|a_{r-\ell_1}b_q(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac{1}{2}}\left(\sum_{r\in L_\ell}\left\|a_{r-\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\alpha}\psi\right\|^2\right)^{\frac{1}{2}}$$
$$\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|b_q(\ell)(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|\mathcal{N}(\mathcal{N}+1)^{\alpha}\psi\right\|$$

wherein we used $\sum_{p\in\mathbb{Z}^3_*}\|a_p\psi\|^2 \le \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+2)\psi\|^2$, $\sum_{p\in L_\ell}\|b_p(\ell)\psi\|^2 \le \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+1)\psi\|^2$ and $\mathcal{N}(\mathcal{N}+1)^{\frac{1}{2}} \le C(\mathcal{N}+1)^{\frac{3}{2}}$. Then for $\alpha = \frac{1}{2}$, we have

$$\le C\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|b_q(\ell)\psi\|\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\|$$

$$\leq C \sup_{q\in L_\ell} \left\| n_q^{\frac{1}{2}}\psi \right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \|K(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \left\| (\mathcal{N}+1)^{\frac{3}{2}}\psi \right\|$$

$$(184) \leq C\,\Xi^{\frac{1}{2}} \left(\sum_{\ell\in\mathbb{Z}^3_*} \|K(\ell)\|_{\max} \right) \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max} \right) \left\| (\mathcal{N}+1)^{\frac{3}{2}}\psi \right\|. \tag{187}$$

We estimate (185) as

$$\begin{aligned}
&(185)\\
&= \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell, s_1\in L_{\ell_1}}} \left|\left\langle K^{m-j}(\ell)_{r,q} K^j(\ell)_{q,s} b_s(\ell) a_{r-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, K(\ell_1)_{r,s_1} b_{-s_1}(-\ell_1) a_{r-\ell}(\mathcal{N}+1)^{\alpha}\psi \right\rangle\right|\\
&\leq \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in L_\ell, s_1\in L_{\ell_1}}} \left\| K^{m-j}(\ell)_{r,q} K^j(\ell)_{q,s} b_s(\ell) a_{r-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi \right\| \|K(\ell_1)_{r,s_1} b_{-s_1}(-\ell_1) a_{r-\ell}(\mathcal{N}+1)^{\alpha}\psi\|\\
&\leq \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max} \|K(\ell_1)\|_{\max} \sum_{r\in L_\ell\cap L_{\ell_1}} \left\| a_{r-\ell_1}(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi \right\| \times\\
&\qquad \times \left\| a_{r-\ell}(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{\alpha}\psi \right\|
\end{aligned}$$

For $\alpha = \frac{1}{2}$, we have

$$\begin{aligned}
&\leq \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max} \|K(\ell_1)\|_{\max} \sum_{r\in L_\ell\cap L_{\ell_1}} \|a_{r-\ell_1}\psi\| \|a_{r-\ell}(\mathcal{N}+1)\psi\|\\
&\leq C \sup_{r\in\mathbb{Z}^3_*} \left\| n_r^{\frac{1}{2}}\psi \right\| \left(\sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell\in\mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max} \right) \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max} \right) \left\| (\mathcal{N}+1)^{\frac{3}{2}}\psi \right\|\\
(185) &\leq C\,\Xi^{\frac{1}{2}} \left(\sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell\in\mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max} \right) \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max} \right) \left\| (\mathcal{N}+1)^{\frac{3}{2}}\psi \right\|
\end{aligned} \tag{188}$$

We estimate (186) as

$$\begin{aligned}
&(186)\\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_{\ell_1}}(q)\mathbb{1}_{L_\ell}(q) \sum_{s\in L_\ell, s_1\in L_{\ell_1}} \left|\left\langle K^m(\ell)_{q,s} b_s(\ell) a_{q-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, K(\ell_1)_{q,s_1} b_{-s_1}(-\ell_1) a_{q-\ell}(\mathcal{N}+1)^{\alpha}\psi \right\rangle\right|\\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_{\ell_1}}(q)\mathbb{1}_{L_\ell}(q) \sum_{s\in L_\ell, s_1\in L_{\ell_1}} \left\| K^m(\ell)_{q,s} b_s(\ell) a_{q-\ell_1}(\mathcal{N}+1)^{-\alpha}\psi \right\| \|K(\ell_1)_{q,s_1} b_{-s_1}(-\ell_1) a_{q-\ell}(\mathcal{N}+1)^{\alpha}\psi\|\\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q) \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \left\| a_{q-\ell_1}(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi \right\| \left\| a_{q-\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\alpha}\psi \right\|
\end{aligned}$$

For $\alpha = \frac{1}{2}$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q) \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \|a_{q-\ell_1}\psi\| \left\| a_{q-\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\frac{1}{2}}\psi \right\|$$

Then we use $\|a_q\psi\| \leq \left\|\mathcal{N}^{\frac{1}{2}}\psi\right\|$ for $q \in B_F$ or $q \in B_F^c$ and we arrive at

$$\leq C \sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac{1}{2}}\psi\right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\|$$

$$(186) \leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \tag{189}$$

Then adding (187),(188) and (189), we arrive at the bound above (183). □

Lemma 4.10 ($E^{2,1}_{Q_2}$)**.** *For any $\psi \in \mathcal{H}_N$, we have*

$$\left|\left\langle\psi,\left(E^{2,1}_{Q_2}+\text{h.c.}\right)\psi\right\rangle\right| \leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}+\sum_{j=0}^{m}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\times$$
$$\times\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \tag{190}$$

Proof. We start with the L.H.S. of (190).

$$\left|\left\langle\psi,\left(E^{2,1}_{Q_2}+\text{h.c.}\right)\psi\right\rangle\right| = \left|\left\langle\psi,2\mathrm{Re}\left(E^{2,1}_{Q_2}\right)\psi\right\rangle\right| = 2\left|\left\langle\psi,E^{2,1}_{Q_2}\psi\right\rangle\right|$$
$$\leq 8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)\\ s_1\in L_{\ell_1}}}\left|\left\langle\psi,K^m(\ell)_{r+\ell,q}K(\ell_1)_{r+\ell_1,s_1}a^*_{r+\ell_1}b^*_q(\ell)b_{-s_1}(-\ell_1)a_{r+\ell}\psi\right\rangle\right| \tag{191}$$

$$+8\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)\\ s\in L_\ell,s_1\in L_{\ell_1}}}\left|\left\langle\psi,K^{m-j}(\ell)_{r+\ell,q}K^j(\ell)_{q,s}K(\ell_1)_{r+\ell_1,s_1}a^*_{r+\ell_1}b^*_s(\ell)b_{-s_1}(-\ell_1)a_{r+\ell}\psi\right\rangle\right| \tag{192}$$

$$+8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q)\sum_{s\in L_\ell,s_1\in L_{\ell_1}}\left|\left\langle\psi,K^m(\ell)_{q+\ell,s}K(\ell_1)_{q+\ell_1,s_1}a^*_{q+\ell_1}b^*_s(\ell)b_{-s_1}(-\ell_1)a_{q+\ell}\psi\right\rangle\right| \tag{193}$$

where the last inequality is implied by Remark 4.4 and we used (160). For (191)-(193), we start by using resolution of the identity $I = (\mathcal{N}+1)^\alpha(\mathcal{N}+1)^{-\alpha}$ for some $\alpha \in \mathbb{R}$. Then we use the Cauchy-Schwarz inequality, Lemma 2.5 and the bounds from Lemma 3.2. We estimate (191) as

(191)

$$= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)\\ s_1\in L_{\ell_1}}}\left|\left\langle K^m_{r+\ell,q}(\ell)b_q(\ell)a_{r+\ell_1}(\mathcal{N}+1)^\alpha\psi,K(\ell_1)_{r+\ell_1,s_1}b_{-s_1}(-\ell_1)a_{r+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\rangle\right|$$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)}}\sum_{s_1\in L_{\ell_1}}\left\|K^m_{r+\ell,q}(\ell)b_q(\ell)a_{r+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|K(\ell_1)_{r+\ell_1,s_1}b_{-s_1}(-\ell_1)a_{r+\ell}(\mathcal{N}+1)^{\alpha}\psi\right\|$$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K(\ell_1)\|_{\max}\|K^m(\ell)\|_{\max}\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)}}\left\|a_{r+\ell_1}b_q(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|a_{r+\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^\alpha\psi\right\|$$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left(\sum_{r\in(L_{\ell_1}-\ell_1)}\left\|a_{r+\ell_1}b_q(\ell)(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac{1}{2}}\times$$
$$\times\left(\sum_{r\in(L_\ell-\ell)}\left\|a_{r+\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\alpha}\psi\right\|^2\right)^{\frac{1}{2}}$$

wherein we used $\sum_{p\in\mathbb{Z}^3_*}\|a_p\psi\|^2 \leq \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+2)\psi\|^2$, $\sum_{p\in L_\ell}\|b_p(\ell)\psi\|^2 \leq \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+1)\psi\|^2$ and $\mathcal{N}(\mathcal{N}+1)^{\frac{1}{2}} \leq C(\mathcal{N}+1)^{\frac{3}{2}}$. Then for $\alpha=1$, we have

$$\leq C\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|b_q(\ell)\psi\|\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\|$$
$$(191) \leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\|. \tag{194}$$

We estimate (192) as

(192)
$$= \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left|\left\langle K^{m-j}(\ell)_{r+\ell,q}K^j(\ell)_{q,s}b_s(\ell)a_{r+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, K(\ell_1)_{r+\ell_1,s_1}b_{-s_1}(-\ell_1)a_{r+\ell}(\mathcal{N}+1)^{\alpha}\psi\right\rangle\right|$$
$$\leq \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)\\ s\in L_\ell, s_1\in L_{\ell_1}}}\left\|K^{m-j}(\ell)_{r+\ell,q}K^j(\ell)_{q,s}b_s(\ell)a_{r+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi\right\|\left\|K(\ell_1)_{r+\ell_1,s_1}b_{-s_1}(-\ell_1)a_{r+\ell}(\mathcal{N}+1)^{\alpha}\psi\right\|$$
$$\leq \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max}\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)}}\left\|a_{r+\ell_1}(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi\right\|\times$$
$$\times\left\|a_{r+\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\alpha}\psi\right\|$$

For $\alpha=\frac{1}{2}$, we have

$$\leq \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max}\sum_{\substack{r\in(L_\ell-\ell)\\ \cap(L_{\ell_1}-\ell_1)}}\|a_{r+\ell_1}\psi\|\left\|a_{r+\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|$$
$$\leq C\sup_{r\in\mathbb{Z}^3_*}\left\|n_r^{\frac{1}{2}}\psi\right\|\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\|$$
$$(192) \leq C\,\Xi^{\frac{1}{2}}\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \tag{195}$$

We estimate (193) as

(193)
$$= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q)\sum_{s\in L_\ell, s_1\in L_{\ell_1}}\left|\left\langle K^m(\ell)_{q+\ell,s}b_s(\ell)a_{q+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi, K(\ell_1)_{q+\ell_1,s_1}b_{-s_1}(-\ell_1)a_{q+\ell}(\mathcal{N}+1)^{\alpha}\psi\right\rangle\right|$$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q) \sum_{s\in L_\ell, s_1\in L_{\ell_1}} \left\| K^m(\ell)_{q+\ell,s} b_s(\ell) a_{q+\ell_1}(\mathcal{N}+1)^{-\alpha}\psi \right\| \| K(\ell_1)_{q+\ell_1,s_1} b_{-s_1}(-\ell_1) a_{q+\ell}(\mathcal{N}+1)^{\alpha}\psi \|$$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q) \|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max} \left\| a_{q+\ell_1}(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^{-\alpha}\psi \right\| \left\| a_{q+\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\alpha}\psi \right\|$$

For $\alpha = \frac{1}{2}$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q) \|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max} \|a_{q+\ell_1}\psi\| \left\| a_{q+\ell}\mathcal{N}^{\frac{1}{2}}(\mathcal{N}+1)^{\frac{1}{2}}\psi \right\|$$

Then we use $\|a_q\psi\| \leq \left\|\mathcal{N}^{\frac{1}{2}}\psi\right\|$ for $q\in B_F$ or $q\in B_F^c$ and we arrive at

$$\begin{aligned} &\leq C \sup_{q\in\mathbb{Z}^3_*}\left\| n_q^{\frac{1}{2}}\psi\right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \\ (193) &\leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{3}{2}}\psi\right\| \end{aligned} \tag{196}$$

Then adding (194),(195) and (196), we arrive at the bound above (190). ☐

Lemma 4.11 ($E^{1,2}_{Q_2}$)**.** *For any $\psi\in\mathcal{H}_N$, we have*

$$\left|\left\langle \psi, \left(E^{1,2}_{Q_2}+\text{h.c.}\right)\psi\right\rangle\right| \leq C\,\Xi^{\frac{1}{2}}\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\left(\|K^m(\ell)\|_{\max}+\sum_{j=1}^{m-1}\binom{m}{j}\|K^{m-j}(\ell)\|_{\max}\|K^j(\ell)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \tag{197}$$

Proof. We start with the L.H.S. of (197).

$$\begin{aligned} \left|\left\langle \psi, \left(E^{1,2}_{Q_2}+\text{h.c.}\right)\psi\right\rangle\right| &= \left|\left\langle \psi, 2\mathrm{Re}\left(E^{1,2}_{Q_2}\right)\psi\right\rangle\right| = 2\left|\left\langle \psi, E^{1,2}_{Q_2}\psi\right\rangle\right| \\ \leq\ 8\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell}&\left|\left\langle\psi, K^m(\ell)_{r,q}K(\ell)_{r,q}a^*_{r-\ell}a_{r-\ell}\psi\right\rangle\right| && (198)\\ +8\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r,s\in L_\ell}&\left|\left\langle\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell)_{r,s}a^*_{r-\ell}a_{r-\ell}\psi\right\rangle\right| && (199)\\ +8\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{s\in L_\ell}&\left|\left\langle\psi, K^m(\ell)_{q,s}K(\ell)_{q,s}a^*_{q-\ell}a_{q-\ell}\psi\right\rangle\right| && (200) \end{aligned}$$

where the last inequality is implied by Remark 4.4 and we used (160). For (198)-(200), we start by using resolution of the identity $I=(\mathcal{N}+1)^{\alpha}(\mathcal{N}+1)^{-\alpha}$ for some $\alpha\in\mathbb{R}$. Then we use the Cauchy-Schwarz inequality, Lemma 2.5 and the bounds from Lemma 3.2. We estimate (198) as

$$\begin{aligned} (198) &= \sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell}|\langle K(\ell)_{r,q}a_{r-\ell}\psi, K^m(\ell)_{r,q}a_{r-\ell}\psi\rangle| \\ &\leq \sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell}\|K(\ell)_{r,q}a_{r-\ell}\psi\|\|K^m(\ell)_{r,q}a_{r-\ell}\psi\| \\ &\leq C\sup_{r\in\mathbb{Z}^3_*}\left\|n_r^{\frac{1}{2}}\psi\right\|\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\|K^m(\ell)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \\ &\leq C\,\Xi^{\frac{1}{2}}\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\|K^m(\ell)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \end{aligned} \tag{201}$$

where we used $\sum_{q\in\mathbb{Z}^3}\|a_q\| \le \left\|\mathcal{N}^{\frac{1}{2}}\psi\right\|$ We estimate (199) as

$$
\begin{aligned}
(199) &= \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r,s\in L_\ell}\left|\left\langle K(\ell)_{r,s}a_{r-\ell}\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}a_{r-\ell}\psi\right\rangle\right| \\
&\le \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r,s\in L_\ell}\|K(\ell)_{r,s}a_{r-\ell}\psi\|\left\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}a_{r-\ell}\psi\right\| \\
&\le C\,\Xi^{\frac{1}{2}}\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \qquad (202)
\end{aligned}
$$

We estimate (200) as

$$
\begin{aligned}
(200) &= \sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{s\in L_\ell}\left|\left\langle K(\ell)_{q,s}a^*_{q-\ell}\psi, K^m(\ell)_{q,s}a_{q-\ell}\psi\right\rangle\right| \\
&\le \sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{s\in L_\ell}\left\|K(\ell)_{q,s}a^*_{q-\ell}\psi\right\|\|K^m(\ell)_{q,s}a_{q-\ell}\psi\| \\
&\le C\,\Xi^{\frac{1}{2}}\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\|K^m(\ell)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \qquad (203)
\end{aligned}
$$

Then adding (201),(202) and (203), we arrive at the bound above (197). □

Lemma 4.12 ($E^{2,2}_{Q_2}$)**.** *For any $\psi\in\mathcal{H}_N$, we have*

$$
\left|\left\langle\psi,\left(E^{2,2}_{Q_2}+\mathrm{h.c.}\right)\psi\right\rangle\right| \le C\,\Xi^{\frac{1}{2}}\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\left(\|K^m(\ell)\|_{\max}+\sum_{j=1}^{m-1}\binom{m}{j}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \qquad (204)
$$

Proof. We start with the L.H.S. of (204).

$$
\begin{aligned}
&\left|\left\langle\psi,\left(E^{2,2}_{Q_2}+\mathrm{h.c.}\right)\psi\right\rangle\right| = \left|\left\langle\psi, 2\mathrm{Re}\left(E^{2,2}_{Q_2}\right)\psi\right\rangle\right| = 2\left|\left\langle\psi, E^{2,2}_{Q_2}\psi\right\rangle\right| \\
\le\;& 8\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell}|\langle\psi, K^m(\ell)_{r,q}K(\ell)_{r,q}a^*_r a_r\psi\rangle| \qquad (205) \\
&+8\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r,s\in L_\ell}\left|\left\langle\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell)_{r,s}a^*_r a_r\psi\right\rangle\right| \qquad (206) \\
&+8\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{s\in L_\ell}\left|\left\langle\psi, K^m(\ell)_{q,s}K(\ell)_{q,s}a^*_q a_q\psi\right\rangle\right| \qquad (207)
\end{aligned}
$$

where the last inequality is implied by Remark 4.4 and we used (160). Then we use the Cauchy-Schwarz inequality and We estimate (205) as

$$
\begin{aligned}
(205) &= \sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell}|\langle K(\ell)_{r,q}a_r\psi, K^m(\ell)_{r,q}a_r\psi\rangle| \\
&\le \sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r\in L_\ell}\|K(\ell)_{r,q}a_r\psi\|\|K^m(\ell)_{r,q}a_r\psi\| \\
&\le C\sup_{r\in\mathbb{Z}^3_*}\left\|n_r^{\frac{1}{2}}\psi\right\|\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\|K^m(\ell)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|
\end{aligned}
$$

$$\leq C\,\Xi^{\frac{1}{2}} \sum_{\ell\in\mathbb{Z}^3_*} \|K(\ell)\|_{\max}\|K^m(\ell)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \tag{208}$$

where we used $\sum_{q\in\mathbb{Z}^3}\|a_q\| \leq \left\|\mathcal{N}^{\frac{1}{2}}\psi\right\|$ We estimate (206) as

$$\begin{aligned}
(206) &= \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r,s\in L_\ell}\left|\left\langle K(\ell)_{r,s}a_r\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}a_r\psi\right\rangle\right| \\
&\leq \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{r,s\in L_\ell}\|K(\ell)_{r,s}a_r\psi\|\left\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}a_r\psi\right\| \\
&\leq C\,\Xi^{\frac{1}{2}}\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|
\end{aligned} \tag{209}$$

We estimate (200) as

$$\begin{aligned}
(207) &= \sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{s\in L_\ell}\left|\left\langle K(\ell)_{q,s}a_q^*\psi, K^m(\ell)_{q,s}a_q\psi\right\rangle\right| \\
&\leq \sum_{\ell\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{s\in L_\ell}\left\|K(\ell)_{q,s}a_q^*\psi\right\|\|K^m(\ell)_{q,s}a_q\psi\| \\
&\leq C\,\Xi^{\frac{1}{2}}\sum_{\ell\in\mathbb{Z}^3_*}\|K(\ell)\|_{\max}\|K^m(\ell)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|
\end{aligned} \tag{210}$$

Then adding (208),(209) and (210), we arrive at the bound above (204). $\square$

Lemma 4.13 ($E^{1,3}_{Q_2}$). *For any $\psi\in\mathcal{H}_N$, we have*

$$\begin{aligned}
2\left|\left\langle\psi,\left(E^{1,3}_{Q_2}+\text{h.c.}\right)\psi\right\rangle\right| &\leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}+\sum_{j=0}^{m}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\right)\times \\
&\quad\times\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\|(\mathcal{N}+1)\psi\|
\end{aligned} \tag{211}$$

Proof. We start with the L.H.S. of (211).

$$2\left|\left\langle\psi,\left(E^{1,3}_{Q_2}+\text{h.c.}\right)\psi\right\rangle\right| = 2\left|\left\langle\psi, 2\text{Re}\left(E^{1,3}_{Q_2}\right)\psi\right\rangle\right| = 4\left|\left\langle\psi, E^{1,3}_{Q_2}\psi\right\rangle\right|$$

$$\leq\ 8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{(-L_{\ell_1}+\ell+\ell_1)}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\left|\left\langle\psi, K^m(\ell)_{r,q}K(\ell_1)_{r,q-\ell+\ell_1}a^*_{r-\ell_1}a^*_{-q+\ell-\ell_1}a_{-q}a_{r-\ell}\psi\right\rangle\right| \tag{212}$$

$$+8\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)}}\left|\left\langle\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s+\ell}K(\ell_1)_{r,s+\ell_1}a^*_{r-\ell_1}a^*_{-s-\ell_1}a_{-s-\ell}a_{r-\ell}\psi\right\rangle\right| \tag{213}$$

$$+8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)}\left|\left\langle\psi, K^m\ell_{q,s+\ell}K(\ell_1)_{q,s+\ell_1}a^*_{q-\ell_1}a^*_{-s-\ell_1}a_{-s-\ell}a_{q-\ell}\psi\right\rangle\right| \tag{214}$$

where the last inequality is implied by Remark 4.4 and we used (160). For (212)-(214), we start by using resolution of the identity $I=(\mathcal{N}+1)^{-\alpha}(\mathcal{N}+1)^{\alpha}$ for some $\alpha\in\mathbb{R}$. Then we use the Cauchy-Schwarz inequality and the bounds from Lemma 3.2. We estimate (212) as

(212)

$$
\begin{aligned}
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(-L_{\ell_1}+\ell+\ell_1)}(q) \sum_{r\in L_\ell\cap L_{\ell_1}} \left|\left\langle K(\ell_1)_{r,q-\ell+\ell_1} a_{-q+\ell-\ell_1} a_{r-\ell_1}(\mathcal{N}+1)^\alpha\psi, K^m(\ell)_{r,q} a_{-q} a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\rangle\right| \\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(-L_{\ell_1}+\ell+\ell_1)}(q) \sum_{r\in L_\ell\cap L_{\ell_1}} \|K(\ell_1)_{r,q-\ell+\ell_1} a_{-q+\ell-\ell_1} a_{r-\ell_1}(\mathcal{N}+1)^\alpha\psi\| \left\|K^m(\ell)_{r,q} a_{-q} a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\| \\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(-L_{\ell_1}+\ell+\ell_1)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max} \sum_{r\in L_\ell\cap L_{\ell_1}} \|a_{r-\ell_1}(\mathcal{N}+1)^\alpha\psi\| \left\|a_{r-\ell}a_{-q}(\mathcal{N}+1)^{-\alpha}\psi\right\| \\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(-L_{\ell_1}+\ell+\ell_1)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max} \left(\sum_{r\in L_{\ell_1}} \|a_{r-\ell_1}(\mathcal{N}+1)^\alpha\psi\|^2\right)^{\frac{1}{2}} \left(\sum_{r\in L_\ell} \left\|a_{r-\ell}a_{-q}(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac{1}{2}} \\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(-L_{\ell_1}+\ell+\ell_1)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max} \left\|(\mathcal{N}+1)^{\frac{1}{2}}(\mathcal{N}+1)^\alpha\psi\right\| \left\|(\mathcal{N}+2)^{\frac{1}{2}} a_{-q}(\mathcal{N}+1)^{-\alpha}\psi\right\|
\end{aligned}
$$

wherein we used $\sum_{p\in\mathbb{Z}^3_*}\|a_p\psi\|^2 \le \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+2)\psi\|^2$. Then for $\alpha = \frac{1}{2}$, we have

$$
\begin{aligned}
&\le C \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(-L_{\ell_1}+\ell+\ell_1)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|a_{-q}\psi\|\|(\mathcal{N}+1)\psi\| \\
&\le C \sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac{1}{2}}\psi\right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|(\mathcal{N}+1)\psi\| \\
(212) &\le C\,\Xi^{\frac{1}{2}} \left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\|(\mathcal{N}+1)\psi\|
\end{aligned}
\tag{215}
$$

We estimate (213) as

(213)

$$
\begin{aligned}
&= \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)}} \Big|\Big\langle K(\ell_1)_{r,s+\ell_1} a_{-s-\ell_1} a_{r-\ell_1}(\mathcal{N}+1)^\alpha\psi, \\
&\qquad\qquad K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s+\ell} a_{-s-\ell}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\Big\rangle\Big| \\
&\le \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)}} \|K(\ell_1)_{r,s+\ell_1} a_{-s-\ell_1} a_{r-\ell_1}(\mathcal{N}+1)^\alpha\psi\|\times \\
&\qquad\qquad \times\left\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s+\ell} a_{-s-\ell}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\| \\
&\le \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{r\in L_\ell\cap L_{\ell_1}} \left(\sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \|K(\ell_1)_{r,s+\ell_1} a_{-s-\ell_1} a_{r-\ell_1}(\mathcal{N}+1)^\alpha\psi\|^2\right)^{\frac{1}{2}}\times \\
&\qquad\qquad \times\left(\sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s+\ell} a_{-s-\ell}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac{1}{2}} \\
&\le \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max} \sum_{r\in L_\ell\cap L_{\ell_1}} \left\|(\mathcal{N}+1)^{\frac{1}{2}} a_{r-\ell_1}(\mathcal{N}+1)^\alpha\psi\right\|\times \\
&\qquad\qquad \times\left\|(\mathcal{N}+2)^{\frac{1}{2}} a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|
\end{aligned}
$$

Then for $\alpha = \frac{1}{2}$, we have

$$
\le C \sup_{r\in\mathbb{Z}^3_*}\left\|n_r^{\frac{1}{2}}\psi\right\| \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max}\|(\mathcal{N}+1)\psi\|
$$

$$(213) \leq C\,\Xi^{\frac{1}{2}} \left(\sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell\in\mathbb{Z}_*^3} \big\|K^{m-j}(\ell)\big\|_{\max} \big\|K^{j}(\ell)\big\|_{\max} \right) \left(\sum_{\ell_1\in\mathbb{Z}_*^3} \|K(\ell_1)\|_{\max} \right) \|(\mathcal{N}+1)\psi\| \tag{216}$$

We estimate (214) as

$$\begin{aligned}
&(214)\\
&= \sum_{\ell,\ell_1\in\mathbb{Z}_*^3} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q) \sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left|\left\langle K(\ell_1)_{q,s+\ell_1} a_{-s-\ell_1} a_{q-\ell_1}(\mathcal{N}+1)^{\alpha}\psi, K^m\ell_{q,s+\ell} a_{-s-\ell} a_{q-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\rangle\right|\\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}_*^3} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q) \sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \|K(\ell_1)_{q,s+\ell_1} a_{-s-\ell_1} a_{q-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\| \big\|K^m\ell_{q,s+\ell} a_{-s-\ell} a_{q-\ell}(\mathcal{N}+1)^{-\alpha}\psi\big\|\\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}_*^3} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q) \left(\sum_{s\in(L_{\ell_1}-\ell_1)} \|K(\ell_1)_{q,s+\ell_1} a_{-s-\ell_1} a_{q-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|^2 \right)^{\frac{1}{2}} \times\\
&\qquad\qquad \times \left(\sum_{s\in(L_\ell-\ell)} \big\|K^m\ell_{q,s+\ell} a_{-s-\ell} a_{q-\ell}(\mathcal{N}+1)^{-\alpha}\psi\big\|^2 \right)^{\frac{1}{2}}\\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}_*^3} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q) \|K^m\ell\|_{\max} \|K(\ell_1)\|_{\max} \Big\|(\mathcal{N}+1)^{\frac{1}{2}} a_{q-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\Big\| \Big\|(\mathcal{N}+2)^{\frac{1}{2}} a_{q-\ell}(\mathcal{N}+1)^{-\alpha}\psi\Big\|
\end{aligned}$$

Then for $\alpha = \frac{1}{2}$, we have

$$\leq \sup_{q\in\mathbb{Z}_*^3} \Big\|n_q^{\frac{1}{2}}\psi\Big\| \sum_{\ell,\ell_1\in\mathbb{Z}_*^3} \|K^m\ell\|_{\max} \|K(\ell_1)\|_{\max} \Big\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\Big\|$$

$$(214) \leq C\,\Xi^{\frac{1}{2}} \left(\sum_{\ell\in\mathbb{Z}_*^3} \|K^m(\ell)\|_{\max} \right) \left(\sum_{\ell_1\in\mathbb{Z}_*^3} \|K(\ell_1)\|_{\max} \right) \|(\mathcal{N}+1)\psi\| \tag{217}$$

Then adding (215),(216) and (217), we arrive at the bound above (211). ☐

Lemma 4.14 ($E_{Q_2}^{2,3}$). *For any $\psi \in \mathcal{H}_N$, we have*

$$\begin{aligned}
\left|\left\langle \psi, \left(E_{Q_2}^{2,3} + \text{h.c.}\right)\psi\right\rangle\right| \leq C\,\Xi^{\frac{1}{2}} &\left(\sum_{\ell\in\mathbb{Z}_*^3} \|K^m(\ell)\|_{\max} + \sum_{j=0}^{m} \sum_{\ell\in\mathbb{Z}_*^3} \big\|K^{m-j}(\ell)\big\|_{\max} \big\|K^{j}(\ell)\big\|_{\max} \right) \times\\
&\times \left(\sum_{\ell_1\in\mathbb{Z}_*^3} \|K(\ell_1)\|_{\max} \right) \|(\mathcal{N}+1)\psi\|
\end{aligned} \tag{218}$$

Proof. We start with the L.H.S. of (218).

$$\left|\left\langle \psi, \left(E_{Q_2}^{2,4} + \text{h.c.}\right)\psi\right\rangle\right| = \left|\left\langle \psi, 2\mathrm{Re}\left(E_{Q_2}^{2,4}\right)\psi\right\rangle\right| = 2\left|\left\langle \psi, E_{Q_2}^{2,4}\psi\right\rangle\right|$$

$$\leq 4 \sum_{\ell,\ell_1\in\mathbb{Z}_*^3} \mathbb{1}_{L_\ell}(q) \sum_{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left|\left\langle \psi, K^m(\ell)_{r+\ell,q} K(\ell_1)_{r+\ell_1,q-\ell+\ell_1} a^*_{r+\ell_1} a^*_{-q+\ell-\ell_1} a_{-q} a_{r+\ell}\psi\right\rangle\right| \tag{219}$$

$$+4 \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}_*^3} \mathbb{1}_{L_\ell}(q) \sum_{r,s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left|\left\langle \psi, K^{m-j}(\ell)_{r+\ell,q} K^{j}(\ell)_{q,s+\ell} K(\ell_1)_{r+\ell_1,s+\ell_1} a^*_{r+\ell_1} a^*_{-s-\ell_1} a_{-s-\ell} a_{r+\ell}\psi\right\rangle\right| \tag{220}$$

$$+4 \sum_{\ell,\ell_1\in\mathbb{Z}_*^3} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{(L_{\ell_1}-\ell_1+\ell)}(q) \sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left|\left\langle \psi, K^m(\ell)_{q,s+\ell} K(\ell_1)_{q-\ell+\ell_1,s+\ell_1} a^*_{q-\ell+\ell_1} a^*_{-s-\ell_1} a_{-s-\ell} a_q\psi\right\rangle\right| \tag{221}$$

where the last inequality is implied by Remark 4.4 and we used (160). For (219)-(221), we start by using resolution of the identity $I = (\mathcal{N}+1)^{-\alpha}(\mathcal{N}+1)^{\alpha}$ for some $\alpha \in \mathbb{R}$. Then we use the Cauchy-Schwarz inequality and the bounds from Lemma 3.2. We estimate (219) as

$$
\begin{aligned}
&(219)\\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left|\left\langle K(\ell_1)_{r+\ell_1,q-\ell+\ell_1} a_{-q+\ell-\ell_1} a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi, K^m(\ell)_{r+\ell,q} a_{-q} a_{r+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\rangle\right|\\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \|K(\ell_1)_{r+\ell_1,q-\ell+\ell_1} a_{-q+\ell-\ell_1} a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi\| \left\|K^m(\ell)_{r+\ell,q} a_{-q} a_{r+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|\\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \sum_{(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \|a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi\| \left\|a_{r+\ell} a_{-q}(\mathcal{N}+1)^{-\alpha}\psi\right\|\\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \left(\sum_{r\in(L_{\ell_1}-\ell_1)} \|a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|^2\right)^{\frac12} \left(\sum_{r\in(L_\ell-\ell)} \left\|a_{r+\ell} a_{-q}(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac12}\\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \left\|(\mathcal{N}+1)^{\frac12}(\mathcal{N}+1)^{\alpha}\psi\right\| \left\|(\mathcal{N}+2)^{\frac12} a_{-q}(\mathcal{N}+1)^{-\alpha}\psi\right\|
\end{aligned}
$$

wherein we used $\sum_{p\in\mathbb{Z}^3_*} \|a_p\psi\|^2 \le \|\mathcal{N}\psi\|^2 < \|(\mathcal{N}+2)\psi\|^2$. Then for $\alpha = \frac12$, we have

$$
\begin{aligned}
&\le C \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \mathbb{1}_{(-L_{\ell_1}+\ell+\ell_1)}(q) \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \|a_{-q}\psi\| \|(\mathcal{N}+1)\psi\|\\
&\le C \sup_{q\in\mathbb{Z}^3_*} \left\|n_q^{\frac12}\psi\right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \|(\mathcal{N}+1)\psi\|\\
(219) &\le C\,\Xi^{\frac12} \left(\sum_{\ell\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max}\right) \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max}\right) \|(\mathcal{N}+1)\psi\| \qquad (222)
\end{aligned}
$$

We estimate (220) as

$$
\begin{aligned}
&(220)\\
&= \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{r,s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \Big|\Big\langle K(\ell_1)_{r+\ell_1,s+\ell_1} a_{-s-\ell_1} a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi,\\
&\qquad\qquad K^{m-j}(\ell)_{r+\ell,q} K^j(\ell)_{q,s+\ell} a_{-s-\ell} a_{r+\ell}(\mathcal{N}+1)^{-\alpha}\psi\Big\rangle\Big|\\
&\le \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{r,s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \|K(\ell_1)_{r+\ell_1,s+\ell_1} a_{-s-\ell_1} a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi\| \times\\
&\qquad\qquad \times \left\|K^{m-j}(\ell)_{r+\ell,q} K^j(\ell)_{q,s+\ell} a_{-s-\ell} a_{r+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|\\
&\le \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left(\sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \|K(\ell_1)_{r+\ell_1,s+\ell_1} a_{-s-\ell_1} a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|^2\right)^{\frac12} \times\\
&\qquad\qquad \times \left(\sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left\|K^{m-j}(\ell)_{r+\ell,q} K^j(\ell)_{q,s+\ell} a_{-s-\ell} a_{r+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac12}\\
&\le \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max} \|K(\ell_1)\|_{\max} \sum_{r\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left\|(\mathcal{N}+1)^{\frac12} a_{r+\ell_1}(\mathcal{N}+1)^{\alpha}\psi\right\| \times\\
&\qquad\qquad \times \left\|(\mathcal{N}+2)^{\frac12} a_{r+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|
\end{aligned}
$$

Then for $\alpha = \frac{1}{2}$, we have

$$
\begin{aligned}
&\leq C \sup_{r\in\mathbb{Z}^3_*} \left\| n_r^{\frac{1}{2}} \psi \right\| \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \left\| K^{m-j}(\ell) \right\|_{\max} \left\| K^j(\ell) \right\|_{\max} \| K(\ell_1)\|_{\max} \|(\mathcal{N}+1)\psi\| \\
(220) &\leq C\,\Xi^{\frac{1}{2}} \left(\sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell\in\mathbb{Z}^3_*} \left\| K^{m-j}(\ell) \right\|_{\max} \left\| K^j(\ell) \right\|_{\max} \right) \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \| K(\ell_1)\|_{\max} \right) \|(\mathcal{N}+1)\psi\| \qquad (223)
\end{aligned}
$$

We estimate (221) as

$$
\begin{aligned}
&(221) \\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \mathbb{1}_{(L_{\ell_1}+\ell-\ell_1)}(q) \sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \left| \left\langle K(\ell_1)_{q-\ell+\ell_1,s+\ell_1} a_{-s-\ell_1} a_{q-\ell+\ell_1} (\mathcal{N}+1)^{\alpha}\psi, K^m \ell_{q,s+\ell} a_{-s-\ell} a_q (\mathcal{N}+1)^{-\alpha}\psi \right.\right. \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \mathbb{1}_{(L_{\ell_1}+\ell-\ell_1)}(q) \sum_{s\in(L_\ell-\ell)\cap(L_{\ell_1}-\ell_1)} \| K(\ell_1)_{q-\ell+\ell_1,s+\ell_1} a_{-s-\ell_1} a_{q-\ell+\ell_1} (\mathcal{N}+1)^{\alpha}\psi \| \left\| K^m \ell_{q,s+\ell} a_{-s-\ell} a_q (\mathcal{N}+1)^{-\alpha}\psi \right. \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \mathbb{1}_{(L_{\ell_1}+\ell-\ell_1)}(q) \left(\sum_{s\in(L_{\ell_1}-\ell_1)} \| K(\ell_1)_{q-\ell+\ell_1,s+\ell_1} a_{-s-\ell_1} a_{q-\ell+\ell_1} (\mathcal{N}+1)^{\alpha}\psi \|^2 \right)^{\frac{1}{2}} \times \\
&\qquad \times \left(\sum_{s\in(L_\ell-\ell)} \left\| K^m \ell_{q,s+\ell} a_{-s-\ell} a_q (\mathcal{N}+1)^{-\alpha}\psi \right\|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \mathbb{1}_{(L_{\ell_1}+\ell-\ell_1)}(q) \| K^m \ell \|_{\max} \| K(\ell_1)\|_{\max} \left\| (\mathcal{N}+1)^{\frac{1}{2}} a_{q-\ell+\ell_1} (\mathcal{N}+1)^{\alpha}\psi \right\| \left\| (\mathcal{N}+2)^{\frac{1}{2}} a_q (\mathcal{N}+1)^{-\alpha}\psi \right\|
\end{aligned}
$$

Then for $\alpha = \frac{1}{2}$, we have

$$
\begin{aligned}
&\leq \sup_{q\in\mathbb{Z}^3_*} \left\| n_q^{\frac{1}{2}} \psi \right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \| K^m \ell \|_{\max} \| K(\ell_1)\|_{\max} \left\| (\mathcal{N}+1)^{\frac{1}{2}} \psi \right\| \\
(221) &\leq C\,\Xi^{\frac{1}{2}} \left(\sum_{\ell\in\mathbb{Z}^3_*} \| K^m(\ell)\|_{\max} \right) \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \| K(\ell_1)\|_{\max} \right) \|(\mathcal{N}+1)\psi\| \qquad (224)
\end{aligned}
$$

Then adding (222),(223) and (224), we arrive at the bound above (218). ◻

Lemma 4.15 ($E^{1,4}_{Q_2}$)**.** *For any $\psi \in \mathcal{H}_N$, we have*

$$
\begin{aligned}
\left| \left\langle \psi, \left(E^{1,4}_{Q_2} + \text{h.c.} \right) \psi \right\rangle \right| &\leq C\,\Xi^{\frac{1}{2}} \left(\sum_{\ell\in\mathbb{Z}^3_*} \| K^m(\ell)\|_{\max} + \sum_{j=0}^{m} \sum_{\ell\in\mathbb{Z}^3_*} \left\| K^{m-j}(\ell) \right\|_{\max} \left\| K^j(\ell) \right\|_{\max} \right) \times \\
&\qquad \times \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \| K(\ell_1)\|_{\max} \right) \|(\mathcal{N}+1)\psi\| \qquad (225)
\end{aligned}
$$

Proof. We start with the L.H.S. of (225).

$$
\begin{aligned}
&\left| \left\langle \psi, \left(E^{1,4}_{Q_2} + \text{h.c.} \right) \psi \right\rangle \right| = \left| \left\langle \psi, 2\mathrm{Re} \left(E^{1,4}_{Q_2} \right) \psi \right\rangle \right| = 2 \left| \left\langle \psi, E^{1,4}_{Q_2} \psi \right\rangle \right| \\
&\leq 4 \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \mathbb{1}_{L_{\ell_1}}(q) \sum_{r\in L_\ell\cap L_{\ell_1}} \left| \left\langle \psi, K^m(\ell)_{r,q} K(\ell_1)_{r,q} a^*_{r-\ell_1} a^*_{-q+\ell_1} a_{-q+\ell} a_{r-\ell} \psi \right\rangle \right| \qquad (226)
\end{aligned}
$$

$$+4\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{r,s\in L_\ell\cap L_{\ell_1}}\left|\left\langle\psi, K^{m-j}(\ell)_{r,q}K_j(\ell)_{q,s}K(\ell_1)_{r,s}a^*_{r-\ell_1}a^*_{-s+\ell_1}a_{-s+\ell}a_{r-\ell}\psi\right\rangle\right| \tag{227}$$

$$+4\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{s\in L_\ell\cap L_{\ell_1}}\left|\left\langle\psi, K(\ell)_{q,s}K(\ell_1)_{q,s}a^*_{q-\ell_1}a^*_{-s+\ell_1}a_{-s+\ell}a_{q-\ell}\psi\right\rangle\right| \tag{228}$$

where the last inequality is implied by Remark 4.4 and we used (160). For (226)-(228), we start by using resolution of the identity $I=(\mathcal{N}+1)^{-\alpha}(\mathcal{N}+1)^{\alpha}$ for some $\alpha\in\mathbb{R}$. Then we use the Cauchy-Schwarz inequality and the bounds from Lemma 3.2. We estimate (226) as

$$\begin{aligned}
&(226)\\
&=\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\left|\left\langle K(\ell_1)_{r,q}a_{-q+\ell_1}a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}\psi, K^m(\ell)_{r,q}a_{-q+\ell}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\rangle\right|\\
&\le\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\|K(\ell_1)_{r,q}a_{-q+\ell_1}a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|\left\|K^m(\ell)_{r,q}a_{-q+\ell}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|\\
&\le\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\sum_{r\in L_\ell\cap L_{\ell_1}}\|a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|\left\|a_{r-\ell}a_{-q+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|\\
&\le\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left(\sum_{r\in L_{\ell_1}}\|a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|^2\right)^{\frac12}\left(\sum_{r\in L_\ell}\left\|a_{r-\ell}a_{-q+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac12}\\
&\le\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac12}(\mathcal{N}+1)^{\alpha}\psi\right\|\left\|(\mathcal{N}+2)^{\frac12}a_{-q+\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|
\end{aligned}$$

wherein we used $\sum_{p\in\mathbb{Z}^3_*}\|a_p\psi\|^2\le\|\mathcal{N}\psi\|^2<\|(\mathcal{N}+2)\psi\|^2$. Then for $\alpha=\frac12$, we have

$$\begin{aligned}
&\le C\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|a_{-q+\ell}\psi\|\|(\mathcal{N}+1)\psi\|\\
&\le C\sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac12}\psi\right\|\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\|(\mathcal{N}+1)\psi\|\\
(226)&\le C\,\Xi^{\frac12}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\|(\mathcal{N}+1)\psi\|
\end{aligned}\tag{229}$$

We estimate (227) as

$$\begin{aligned}
&(227)\\
&=\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{r,s\in L_\ell\cap L_{\ell_1}}\Big|\Big\langle K(\ell_1)_{r,s}a_{-s+\ell_1}a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}\psi,\\
&\qquad\qquad K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}a_{-s+\ell}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\Big\rangle\Big|\\
&\le\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{r,s\in L_\ell\cap L_{\ell_1}}\|K(\ell_1)_{r,s}a_{-s+\ell_1}a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|\times\\
&\qquad\qquad\times\left\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}a_{-s+\ell}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|\\
&\le\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{r\in L_\ell\cap L_{\ell_1}}\left(\sum_{s\in L_\ell\cap L_{\ell_1}}\|K(\ell_1)_{r,s}a_{-s+\ell_1}a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|^2\right)^{\frac12}\times\\
&\qquad\qquad\times\left(\sum_{s\in L_\ell\cap L_{\ell_1}}\left\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}a_{-s+\ell}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\right\|^2\right)^{\frac12}
\end{aligned}$$

$$\leq \sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\big\|K^{m-j}(\ell)\big\|_{\max}\big\|K^j(\ell)\big\|_{\max}\|K(\ell_1)\|_{\max}\sum_{r\in L_\ell\cap L_{\ell_1}}\Big\|(\mathcal{N}+1)^{\frac12}a_{r-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\Big\|\times \times\Big\|(\mathcal{N}+2)^{\frac12}a_{r-\ell}(\mathcal{N}+1)^{-\alpha}\psi\Big\|$$

Then for $\alpha=\frac12$, we have

$$\leq C\sup_{r\in\mathbb{Z}^3_*}\Big\|n_r^{\frac12}\psi\Big\|\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\big\|K^{m-j}(\ell)\big\|_{\max}\big\|K^j(\ell)\big\|_{\max}\|K(\ell_1)\|_{\max}\|(\mathcal{N}+1)\psi\|$$

$$(227)\leq C\,\Xi^{\frac12}\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\big\|K^{m-j}(\ell)\big\|_{\max}\big\|K^j(\ell)\big\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\|(\mathcal{N}+1)\psi\| \tag{230}$$

We estimate (228) as

$$\begin{aligned}
&(228)\\
&=\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{s\in L_\ell\cap L_{\ell_1}}\big|\big\langle K(\ell_1)_{q,s}a_{-s+\ell_1}a_{q-\ell_1}(\mathcal{N}+1)^{\alpha}\psi, K^m\ell_{q,s}a_{-s+\ell}a_{q-\ell}(\mathcal{N}+1)^{-\alpha}\psi\big\rangle\big|\\
&\leq\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\sum_{s\in L_\ell\cap L_{\ell_1}}\|K(\ell_1)_{q,s}a_{-s+\ell_1}a_{q-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|\big\|K^m\ell_{q,s}a_{-s+\ell}a_{q-\ell}(\mathcal{N}+1)^{-\alpha}\psi\big\|\\
&\leq\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\left(\sum_{s\in L_{\ell_1}}\|K(\ell_1)_{q,s}a_{-s+\ell_1}a_{q-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\|^2\right)^{\frac12}\times\\
&\qquad\qquad\times\left(\sum_{s\in L_\ell}\big\|K^m\ell_{q,s+\ell}a_{-s-\ell}a_{q-\ell}(\mathcal{N}+1)^{-\alpha}\psi\big\|^2\right)^{\frac12}\\
&\leq\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}}(q)\|K^m\ell\|_{\max}\|K(\ell_1)\|_{\max}\Big\|(\mathcal{N}+1)^{\frac12}a_{q-\ell_1}(\mathcal{N}+1)^{\alpha}\psi\Big\|\Big\|(\mathcal{N}+2)^{\frac12}a_{q-\ell}(\mathcal{N}+1)^{-\alpha}\psi\Big\|
\end{aligned}$$

Then for $\alpha=\frac12$, we have

$$\leq\sup_{q\in\mathbb{Z}^3_*}\Big\|n_q^{\frac12}\psi\Big\|\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m\ell\|_{\max}\|K(\ell_1)\|_{\max}\Big\|(\mathcal{N}+1)^{\frac12}\psi\Big\|$$

$$(228)\leq C\,\Xi^{\frac12}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\|(\mathcal{N}+1)\psi\| \tag{231}$$

Then adding (229),(230) and (231), we arrive at the bound above (225). □

Lemma 4.16 ($E^{1,5}_{Q_2}$)**.** *For any $\psi\in\mathcal{H}_N$, we have*

$$2\Big|\Big\langle\psi,\Big(E^{1,5}_{Q_2}+\text{h.c.}\Big)\psi\Big\rangle\Big|\leq C\,\Xi^{\frac12}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}+\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\big\|K^{m-j}(\ell)\big\|_{\max}\big\|K^j(\ell)\big\|_{\max,2}\right)\times \times\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\Big\|(\mathcal{N}+1)^{\frac12}\psi\Big\| \tag{232}$$

Proof. We start with the L.H.S. of (232).

$$2\Big|\Big\langle\psi,\Big(E^{1,5}_{Q_2}+\text{h.c.}\Big)\psi\Big\rangle\Big|=2\Big|\Big\langle\psi,2\text{Re}\Big(E^{1,5}_{Q_2}\Big)\psi\Big\rangle\Big|=4\Big|\Big\langle\psi,E^{1,5}_{Q_2}\psi\Big\rangle\Big|$$

$$\leq 8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\left|\left\langle\psi,K^m(\ell)_{r,q}K(\ell_1)_{r,-r+\ell+\ell_1}a^*_{r-\ell_1}a^*_{r-\ell-\ell_1}b_{-q}(-\ell)\psi\right\rangle\right| \tag{233}$$

$$+8\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_{\ell_1}+\ell+\ell_1)\\ s\in L_\ell}}\left|\left\langle\psi,K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell_1)_{r,-r+\ell+\ell_1}a^*_{r-\ell_1}a^*_{r-\ell-\ell_1}b_{-s}(-\ell)\psi\right\rangle\right| \tag{234}$$

$$+8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_{\ell_1}+\ell+\ell_1)}(q)\sum_{s\in L_\ell}\left|\left\langle\psi,K^m(\ell)_{q,s}K(\ell_1)_{q,-q+\ell+\ell_1}a^*_{q-\ell_1}a^*_{q-\ell-\ell_1}b_{-s}(-\ell)\psi\right\rangle\right| \tag{235}$$

where the last inequality is implied by Remark 4.4 and we used (160). Then we use the Cauchy-Schwarz inequality and the bounds from Lemma 3.2. We estimate (233) as

$$\begin{aligned}
&(233)\\
&=\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\left|\left\langle K^m(\ell)_{r,q}K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell_1}\psi,b_{-q}(-\ell)\psi\right\rangle\right|\\
&\leq\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\|K^m(\ell)_{r,q}K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell_1}\psi\|\|b_{-q}(-\ell)\psi\|\\
&\leq\sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac12}\psi\right\|\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\|a_{r-\ell_1}\psi\|\\
&\leq C\,\Xi^{\frac12}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac12}\psi\right\|
\end{aligned} \tag{236}$$

wherein we used $\|a_q\|\leq\mathbb{1}$ and $\sum_{p\in\mathbb{Z}^3_*}\|a_p\psi\|^2\leq\|\mathcal{N}\psi\|^2<\|(\mathcal{N}+1)\psi\|^2$. We estimate (234) as

$$\begin{aligned}
&(234)\\
&=\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_{\ell_1}+\ell+\ell_1)\\ s\in L_\ell}}\left|\left\langle\psi,K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,s}K(\ell_1)_{r,-r+\ell+\ell_1}a^*_{r-\ell_1}a^*_{r-\ell-\ell_1}b_{-s}(-\ell)\psi\right\rangle\right|\\
&\leq\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_{\ell_1}+\ell+\ell_1)\\ s\in L_\ell}}\left\|K^{m-j}(\ell)_{r,q}K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell_1}\psi\right\|\left\|K^j(\ell)_{q,s}b_{-s}(-\ell)\psi\right\|\\
&\leq\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\|K(\ell_1)\|_{\max}\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_{\ell_1}+\ell+\ell_1)\\ s\in L_\ell}}\|a_{r-\ell_1}\psi\|\left\|K^j(\ell)_{q,s}b_{-s}(-\ell)\psi\right\|\\
&\leq C\sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac12}\psi\right\|\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\sum_{j=1}^{m-1}\binom{m}{j}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max,2}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac12}\psi\right\|\\
&\leq C\,\Xi^{\frac12}\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max,2}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac12}\psi\right\|
\end{aligned} \tag{237}$$

We estimate (235) as

$$
\begin{aligned}
&(235)\\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_{\ell_1}+\ell+\ell_1)}(q) \sum_{s\in L_\ell} |\langle K(\ell_1)_{q,-q+\ell+\ell_1} a_{q-\ell-\ell_1} a_{q-\ell_1}\psi, K^m(\ell)_{q,s} b_{-s}(-\ell)\psi\rangle|\\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_{\ell_1}+\ell+\ell_1)}(q) \sum_{s\in L_\ell} \|K(\ell_1)_{q,-q+\ell+\ell_1} a_{q-\ell-\ell_1} a_{q-\ell_1}\psi\| \|K^m(\ell)_{q,s} b_{-s}(-\ell)\psi\|\\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_{\ell_1}+\ell+\ell_1)}(q) \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \|a_{q-\ell_1}\psi\| \sum_{s\in L_\ell} \|b_{-s}(-\ell)\psi\|\\
&\le C \sup_{q\in\mathbb{Z}^3_*} \left\| n_q^{\frac12}\psi\right\| \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max} \|K(\ell_1)\|_{\max} \left\|(\mathcal{N}+1)^{\frac12}\psi\right\|\\
&\le C\,\Xi^{\frac12} \left(\sum_{\ell\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max}\right) \left\|(\mathcal{N}+1)^{\frac12}\psi\right\| \qquad (238)
\end{aligned}
$$

Then adding (236),(237) and (238), we arrive at the bound above (232). □

Lemma 4.17 ($E^{1,7}_{Q_2}$)**.** *For any $\psi\in\mathcal{H}_N$, we have*

$$
\begin{aligned}
2\left|\left\langle\psi, \left(E^{1,7}_{Q_2} + \text{h.c.}\right)\psi\right\rangle\right| &\le C\,\Xi^{\frac12} \left(\sum_{\ell\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max} + \sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell\in\mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max,2}\right)\times\\
&\quad\times \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac12}\psi\right\| \qquad (239)
\end{aligned}
$$

Proof. We start with the L.H.S. of (239).

$$
\begin{aligned}
&2\left|\left\langle\psi, \left(E^{1,7}_{Q_2} + \text{h.c.}\right)\psi\right\rangle\right| = 2\left|\left\langle\psi, 2\mathrm{Re}\left(E^{1,7}_{Q_2}\right)\psi\right\rangle\right| = 4\left|\left\langle\psi, E^{1,7}_{Q_2}\psi\right\rangle\right|\\
\le\ & 8 \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell\cap(-L_{\ell_1}+\ell+\ell_1)\cap(-L_\ell+\ell+\ell_1)}(q) \sum_{s_1\in L_{\ell_1}} |\langle\psi, K^m(\ell)_{q,-q+\ell+\ell_1} K(\ell_1)_{-q_\ell+\ell_1,s_1} b^*_{-s_1}(-\ell_1) a_{-q} a_{-q+\ell_1}\psi\rangle| \qquad (240)\\
&+8\sum_{j=1}^{m-1}\binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q) \sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ s_1\in L_{\ell_1}}} \left|\left\langle\psi, K^{m-j}(\ell)_{r,q} K^j(\ell)_{q,-r+\ell+\ell_1} K(\ell_1)_{r,s_1} b^*_{-s_1}(-\ell_1) a_{r-\ell-\ell_1} a_{r-\ell}\psi\right\rangle\right| \qquad (241)\\
&+8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_\ell+\ell+\ell_1)}(q) \sum_{s_1\in L_{\ell_1}} \left|\left\langle\psi, K^m(\ell)_{q,-q+\ell+\ell_1} K(\ell_1)_{q,s_1} b^*_{-s_1}(-\ell_1) a_{q-\ell-\ell_1} a_{q-\ell}\psi\right\rangle\right| \qquad (242)
\end{aligned}
$$

where the last inequality is implied by Remark 4.4 and we used (160). Then we use the Cauchy-Schwarz inequality and the bounds from Lemma 3.2. We estimate (240) as

$$
\begin{aligned}
&(240)\\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell\cap(-L_{\ell_1}+\ell+\ell_1)\cap(-L_\ell+\ell+\ell_1)}(q) \sum_{s_1\in L_{\ell_1}} |\langle K(\ell_1)_{-q_\ell+\ell_1,s_1} b_{-s_1}(-\ell_1)\psi, K^m(\ell)_{q,-q+\ell+\ell_1} a_{-q} a_{-q+\ell_1}\psi\rangle|\\
&\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell\cap(-L_{\ell_1}+\ell+\ell_1)\cap(-L_\ell+\ell+\ell_1)}(q) \sum_{s_1\in L_{\ell_1}} \|K(\ell_1)_{-q_\ell+\ell_1,s_1} b_{-s_1}(-\ell_1)\psi\| \|K^m(\ell)_{q,-q+\ell+\ell_1} a_{-q} a_{-q+\ell_1}\psi\|
\end{aligned}
$$

$$\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{L_\ell\cap(-L_{\ell_1}+\ell+\ell_1)\cap(-L_\ell+\ell+\ell_1)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\sum_{s_1\in L_{\ell_1}}\|b_{-s_1}(-\ell_1)\psi\|\|a_{-q}a_{-q+\ell_1}\psi\|$$
$$\leq C\sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac{1}{2}}\psi\right\|\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|$$
$$\leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \tag{243}$$

wherein we used $\|a_q\|\leq \mathbb{1}$ and $\sum_{p\in L_\ell}\|b_p(\ell)\psi\|^2\leq\|\mathcal{N}\psi\|^2<\|(\mathcal{N}+1)\psi\|^2$. We estimate (241) as

$$(241)$$
$$=\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ s_1\in L_{\ell_1}}}\left|\left\langle K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell}\psi\right\rangle\right|$$
$$\leq\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ s_1\in L_{\ell_1}}}\|K(\ell_1)_{r,s_1}b_{-s_1}(-\ell_1)\psi\|\left\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,-r+\ell+\ell_1}a_{r-\ell-\ell_1}a_{r-\ell}\psi\right\|$$
$$\leq\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\|K(\ell_1)\|_{\max}\left\|K^{m-j}(\ell)\right\|_{\max}\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)}}\left\|K^j(\ell)_{q,-r+\ell+\ell_1}a_{r-\ell}\psi\right\|\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|$$
$$\leq C\sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac{1}{2}}\psi\right\|\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max,2}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|$$
$$\leq C\,\Xi^{\frac{1}{2}}\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\left\|K^{m-j}(\ell)\right\|_{\max}\left\|K^j(\ell)\right\|_{\max,2}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \tag{244}$$

We estimate (242) as

$$(242)$$
$$=\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_\ell+\ell+\ell_1)}(q)\sum_{s_1\in L_{\ell_1}}\left|\langle K(\ell_1)_{q,s_1}b_{-s_1}(-\ell_1)\psi, K^m(\ell)_{q,-q+\ell+\ell_1}a_{q-\ell-\ell_1}a_{q-\ell}\psi\rangle\right|$$
$$\leq\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_\ell+\ell+\ell_1)}(q)\sum_{s_1\in L_{\ell_1}}\|K(\ell_1)_{q,s_1}b_{-s_1}(-\ell_1)\psi\|\|K^m(\ell)_{q,-q+\ell+\ell_1}a_{q-\ell-\ell_1}a_{q-\ell}\psi\|$$
$$\leq\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\mathbb{1}_{L_{\ell_1}\cap(-L_\ell+\ell+\ell_1)}(q)\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\sum_{s_1\in L_{\ell_1}}\|b_{-s_1}(-\ell_1)\psi\|\|a_{q-\ell-\ell_1}a_{q-\ell}\psi\|$$
$$\leq C\sup_{q\in\mathbb{Z}^3_*}\left\|n_q^{\frac{1}{2}}\psi\right\|\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\|K(\ell_1)\|_{\max}\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\|$$
$$\leq C\,\Xi^{\frac{1}{2}}\left(\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\left\|(\mathcal{N}+1)^{\frac{1}{2}}\psi\right\| \tag{245}$$

Then adding (243),(244) and (245), we arrive at the bound above (239). □

Lemma 4.18 ($E^{1,8}_{Q_2}$)**.** *For any $\psi\in\mathcal{H}_N$, we have*

$$2\left|\left\langle\psi,\left(E^{1,8}_{Q_2}+\text{h.c.}\right)\psi\right\rangle\right|\leq C \tag{246}$$

Proof. We start with the L.H.S. of (246).

$$
\begin{aligned}
&2\Big|\Big\langle \psi, \Big(E^{1,8}_{Q_2}+\text{h.c.}\Big)\psi\Big\rangle\Big| = 2\Big|\Big\langle \psi, 2\mathrm{Re}\Big(E^{1,8}_{Q_2}\Big)\psi\Big\rangle\Big| = 4\Big|\Big\langle \psi, E^{1,8}_{Q_2}\psi\Big\rangle\Big| \\
&\leq\; 8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\big|\big\langle \psi, K^m(\ell)_{q,-q+\ell+\ell_1}K(\ell_1)_{q,-q+\ell+\ell_1}a^*_{-q+\ell}a_{-q+\ell}\psi\big\rangle\big| \qquad (247)\\
&+8\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\big|\big\langle \psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,-r+\ell+\ell_1}K(\ell_1)_{r,-r+\ell+\ell_1}a^*_{r-\ell_1}a_{r-\ell_1}\psi\big\rangle\big| \\
&\qquad (248)\\
&+8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\big|\big\langle \psi, K^m(\ell)_{q,-q+\ell+\ell_1}K(\ell_1)_{q,-q+\ell+\ell_1}a^*_{q-\ell_1}a_{q-\ell_1}\psi\big\rangle\big| \qquad (249)
\end{aligned}
$$

where the last inequality is implied by Remark 4.4 and we used (160). Then we use the Cauchy-Schwarz inequality and the bounds from Lemma 3.2. We estimate (247) as

$$
\begin{aligned}
(247) &= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\big|\big\langle K(\ell_1)_{q,-q+\ell+\ell_1}a_{-q+\ell}\psi, K^m(\ell)_{q,-q+\ell+\ell_1}a_{-q+\ell}\psi\big\rangle\big| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\|K(\ell_1)_{q,-q+\ell+\ell_1}a_{-q+\ell}\psi\|\|K^m(\ell)_{q,-q+\ell+\ell_1}a_{-q+\ell}\psi\| \\
&\leq C\,\Xi^{\frac12}\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\Big\|(\mathcal{N}+1)^{\frac12}\psi\Big\| \qquad (250)
\end{aligned}
$$

We estimate (248) as

$$
\begin{aligned}
&(248)\\
&=\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\big|\big\langle K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell_1}\psi, K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,-r+\ell+\ell_1}a_{r-\ell_1}\psi\big\rangle\big| \\
&\leq\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\|K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell_1}\psi\|\big\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,-r+\ell+\ell_1}a_{r-\ell_1}\psi\big\| \\
&\leq C\sup_{r\in\mathbb{Z}^3_*}\Big\|n_r^{\frac12}\psi\Big\|\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\big\|K^{m-j}(\ell)\big\|_{\max}\big\|K^j(\ell)\big\|_{\max}\|K(\ell_1)\|_{\max}\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\|a_{r-\ell_1}\psi\| \\
&\leq C\,\Xi^{\frac12}\left(\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell\in\mathbb{Z}^3_*}\big\|K^{m-j}(\ell)\big\|_{\max}\big\|K^j(\ell)\big\|_{\max}\right)\left(\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\right)\Big\|(\mathcal{N}+1)^{\frac12}\psi\Big\| \qquad (251)
\end{aligned}
$$

We estimate (249) as

$$
\begin{aligned}
&(249)\\
&=\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\big|\big\langle K(\ell_1)_{q,-q+\ell+\ell_1}a^*_{q-\ell_1}\psi, \Theta^m_K(P^q)_{q,-q+\ell+\ell_1}a_{q-\ell_1}\psi\big\rangle\big|
\end{aligned}
$$

$$\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\big\|K(\ell_1)_{q,-q+\ell+\ell_1}a^*_{q-\ell_1}\psi\big\|\|\Theta^m_K(P^q)_{q,-q+\ell+\ell_1}a_{q-\ell_1}\psi\|$$

$$\le C\,\Xi^{\frac12}\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\Big\|(\mathcal{N}+1)^{\frac12}\psi\Big\| \tag{252}$$

Then adding (250),(251) and (252), we arrive at the bound above (246). □

Lemma 4.19 ($E^{2,8}_{Q_2}$)**.** *For any $\psi\in\mathcal{H}_N$, we have*

$$2\Big|\Big\langle\psi,\Big(E^{2,8}_{Q_2}+\text{h.c.}\Big)\psi\Big\rangle\Big|\le C \tag{253}$$

Proof. We start with the L.H.S. of (253).

$$2\Big|\Big\langle\psi,\Big(E^{2,8}_{Q_2}+\text{h.c.}\Big)\psi\Big\rangle\Big| = 2\Big|\Big\langle\psi,2\text{Re}\Big(E^{2,8}_{Q_2}\Big)\psi\Big\rangle\Big| = 4\Big|\Big\langle\psi,E^{2,8}_{Q_2}\psi\Big\rangle\Big|$$

$$\le\ 8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\big|\big\langle\psi,K^m(\ell)_{q,-q+\ell+\ell_1}K(\ell_1)_{q,-q+\ell+\ell_1}a^*_{-q}a_{-q}\psi\big\rangle\big| \tag{254}$$

$$+8\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\big|\big\langle\psi,K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,-r+\ell+\ell_1}K(\ell_1)_{r,-r+\ell+\ell_1}a^*_{r-\ell-\ell_1}a_{r-\ell-\ell_1}\psi\big\rangle\big| \tag{255}$$

$$+8\sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\big|\big\langle\psi,K^m(\ell)_{q,-q+\ell+\ell_1}K(\ell_1)_{q,-q+\ell+\ell_1}a^*_{q-\ell-\ell_1}a_{q-\ell-\ell_1}\psi\big\rangle\big| \tag{256}$$

where the last inequality is implied by Remark 4.4 and we used (160). Then we use the Cauchy-Schwarz inequality and the bounds from Lemma 3.2. We estimate (254) as

$$(254) = \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)|\langle K(\ell_1)_{q,-q+\ell+\ell_1}a_{-q}\psi,K^m(\ell)_{q,-q+\ell+\ell_1}a_{-q}\psi\rangle|$$

$$\le \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q)\|K(\ell_1)_{q,-q+\ell+\ell_1}a_{-q}\psi\|\|K^m(\ell)_{q,-q+\ell+\ell_1}a_{-q}\psi\|$$

$$\le C\,\Xi^{\frac12}\sum_{\ell\in\mathbb{Z}^3_*}\|K^m(\ell)\|_{\max}\sum_{\ell_1\in\mathbb{Z}^3_*}\|K(\ell_1)\|_{\max}\Big\|(\mathcal{N}+1)^{\frac12}\psi\Big\| \tag{257}$$

We estimate (255) as

$$(255)$$

$$=\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\big|\big\langle K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}\psi,K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,-r+\ell+\ell_1}a_{r-\ell-\ell_1}\psi\big\rangle\big|$$

$$\le\sum_{j=1}^{m-1}\binom{m}{j}\sum_{\ell,\ell_1\in\mathbb{Z}^3_*}\mathbb{1}_{L_\ell}(q)\sum_{\substack{r\in L_\ell\cap L_{\ell_1}\\ \cap(-L_\ell+\ell+\ell_1)\\ \cap(-L_{\ell_1}+\ell+\ell_1)}}\|K(\ell_1)_{r,-r+\ell+\ell_1}a_{r-\ell-\ell_1}\psi\|\big\|K^{m-j}(\ell)_{r,q}K^j(\ell)_{q,-r+\ell+\ell_1}a_{r-\ell-\ell_1}\psi\big\|$$

$$\leq C \sup_{r\in\mathbb{Z}^3_*} \left\| n_r^{\frac{1}{2}\psi} \right\| \sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max} \|K(\ell_1)\|_{\max} \sum_{\substack{r\in L_\ell\cap L_{\ell_1} \\ \cap(-L_\ell+\ell+\ell_1) \\ \cap(-L_{\ell_1}+\ell+\ell_1)}} \|a_{r-\ell-\ell_1}\psi\|$$

$$\leq C\,\Xi^{\frac{1}{2}} \left(\sum_{j=1}^{m-1} \binom{m}{j} \sum_{\ell\in\mathbb{Z}^3_*} \left\|K^{m-j}(\ell)\right\|_{\max} \left\|K^j(\ell)\right\|_{\max} \right) \left(\sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max} \right) \left\| (\mathcal{N}+1)^{\frac{1}{2}}\psi \right\| \tag{258}$$

We estimate (256) as

$$\begin{aligned}
&(256) \\
&= \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1} \\ \cap(-L_\ell+\ell+\ell_1) \\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q) \left| \left\langle K(\ell_1)_{q,-q+\ell+\ell_1} a^*_{q-\ell-\ell_1}\psi, \Theta^m_K(P^q)_{q,-q+\ell+\ell_1} a_{q-\ell-\ell_1}\psi \right\rangle \right| \\
&\leq \sum_{\ell,\ell_1\in\mathbb{Z}^3_*} \mathbb{1}_{\substack{L_\ell\cap L_{\ell_1} \\ \cap(-L_\ell+\ell+\ell_1) \\ \cap(-L_{\ell_1}+\ell+\ell_1)}}(q) \left\| K(\ell_1)_{q,-q+\ell+\ell_1} a^*_{q-\ell-\ell_1}\psi \right\| \left\| \Theta^m_K(P^q)_{q,-q+\ell+\ell_1} a_{q-\ell-\ell_1}\psi \right\| \\
&\leq C\,\Xi^{\frac{1}{2}} \sum_{\ell\in\mathbb{Z}^3_*} \|K^m(\ell)\|_{\max} \sum_{\ell_1\in\mathbb{Z}^3_*} \|K(\ell_1)\|_{\max} \left\| (\mathcal{N}+1)^{\frac{1}{2}}\psi \right\|
\end{aligned} \tag{259}$$

Then adding (257),(258) and (259), we arrive at the bound above (253). □

Lemma 4.20 (E_{Q_2})**.** *For any $\psi\in\mathcal{H}_N$, we have*

$$\begin{aligned} &\langle \psi, E_{Q_2}\psi \rangle \\ &\leq \end{aligned} \tag{260}$$

Proof. We begin with expanding the error as

□

References