Saddle Point Integration on the Real Axis

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1 Core Principle

For integrals of the form

$$I(N) = \int_a^b g(x) \mathrm{e}^{Nf(x)} \, \mathrm{d}x, \quad N \to +\infty, \tag{1}$$

the dominant contribution comes from neighborhoods of critical points where $f(x)$ is maximized.

2 Key Steps

2.1 Step 1: Locate the Critical Point

Solve the equation:

$$f'(x_0) = 0, \tag{2}$$

and verify the second derivative:

$$f''(x_0) < 0 \quad \text{(local maximum)}. \tag{3}$$

2.2 Step 2: Local Expansion

Expand $f(x)$ and $g(x)$ around x_0:

$$f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2, \tag{4}$$

$$g(x) \approx g(x_0). \tag{5}$$

2.3 Step 3: Gaussian Integral Approximation

Substitute into $I(N)$:

$$I(N) \approx g(x_0) \mathrm{e}^{Nf(x_0)} \int_{-\infty}^{\infty} \mathrm{e}^{N \frac{f''(x_0)}{2} (x - x_0)^2} \mathrm{d}x. \tag{6}$$

Let $t = x - x_0$, and use $\int_{-\infty}^{\infty} \mathrm{e}^{-at^2} \mathrm{d}t = \sqrt{\pi/a}$:

$$I(N) \sim g(x_0) \mathrm{e}^{Nf(x_0)} \sqrt{\frac{2\pi}{N|f''(x_0)|}}. \tag{7}$$

3 Important Modifications

3.1 Endpoint Contributions

If the critical point x_0 coincides with an endpoint a or b, expand $f(x)$ as:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2. \tag{8}$$

3.2 Higher-Order Corrections

Include higher-order terms for better precision:

$$I(N) \sim \mathrm{e}^{Nf(x_0)}\sqrt{\frac{2\pi}{N|f''(x_0)|}}\left[g(x_0) + \frac{1}{N}\left(\frac{g''(x_0)}{2|f''(x_0)|} - \frac{g(x_0)f''''(x_0)}{8|f''(x_0)|^2}\right)\right]. \tag{9}$$

4 Example: Stirling's Formula

The Gamma function $\Gamma(N+1) = \int_0^\infty x^N \mathrm{e}^{-x}\mathrm{d}x$ can be rewritten as:

$$\Gamma(N+1) = \int_0^\infty \mathrm{e}^{N\ln x - x}\mathrm{d}x. \tag{10}$$

(i) Find the critical point:

$$f(x) = \ln x - \frac{x}{N} \implies x_0 = N.$$

(ii) Expand around x_0:

$$f(x) \approx \ln N - 1 - \frac{(x-N)^2}{2N^2}. \tag{11}$$

(iii) Compute the integral:

$$\Gamma(N+1) \sim \mathrm{e}^{N\ln N - N}\sqrt{2\pi N} = \sqrt{2\pi N}\left(\frac{N}{\mathrm{e}}\right)^N. \tag{12}$$

5 Cautions

- **Convergence**: Ensure the Gaussian tail dominates neglected regions.
- **Boundary effects**: Modify expansion if x_0 is near endpoints.
- **Multiple critical points**: Compare contributions from all maxima.