Method for Calculating the Electric Field Induced by a Moving Electron in a Homogeneous Dielectric

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Consider a free electron moving at a constant velocity $\vec{v}$ in a homogeneous dielectric, we can write the charge density and current density as

$$\rho_f\left(\vec{r},t\right) = -e\delta\left(\vec{r}-\vec{v}t\right); \quad \vec{j}_f = -e\vec{v}\delta\left(\vec{r}-\vec{v}t\right) \tag{1}$$

where $\vec{r}$ is the position vector of the electron, e is the unit charge, and the subscript f represents the free electron.

For a homogeneous dielectric, we have

$$\vec{D}\left(\vec{r},t\right) = \varepsilon\varepsilon_0\vec{E}\left(\vec{r},t\right); \quad \vec{B}\left(\vec{r},t\right) = \mu_0\vec{H}\left(\vec{r},t\right) \tag{2}$$

where ε is the dielectric function of the homogeneous dielectric, which can depend on both the wave-vector of the electron $\vec{q}$ (see The correlation between dielectric function and wave-vector), and frequency ω, i.e.

$$\varepsilon = \varepsilon\left(\vec{q},\omega\right) \tag{3}$$

but the wave-vector dependence is ignored in this passage (see Neglecting the correlation between dielectric function and wave-vector), i.e.

$$\varepsilon = \varepsilon\left(\omega\right) \tag{4}$$

We also have Maxwell equations in the homogeneous dielectrics

$$\begin{cases} \nabla\cdot\vec{D}\left(\vec{r},t\right) = \rho_f\left(\vec{r},t\right) \\ \nabla\times\vec{E}\left(\vec{r},t\right) = -\frac{\partial}{\partial t}\vec{B}\left(\vec{r},t\right) \\ \nabla\cdot\vec{B}\left(\vec{r},t\right) = 0 \\ \nabla\times\vec{H}\left(\vec{r},t\right) = \vec{j}_f\left(\vec{r},t\right) + \frac{\partial}{\partial t}\vec{D}\left(\vec{r},t\right) \end{cases} \tag{5}$$

Now, we use Fourier transform to transform the charge density, current density, and Maxwell equations into $\vec{q}-\omega$ domain (the method of Fourier transform for Maxwell equations is shown in Method)

$$\begin{aligned} \rho_f\left(\vec{q},\omega\right) &= -e\int \mathrm{d}\vec{r}\mathrm{d}t\delta\left(\vec{r}-\vec{v}t\right)\mathrm{e}^{-\mathrm{i}\left(\vec{q}\cdot\vec{r}-\omega t\right)} \\ &= -e\int \mathrm{d}t\mathrm{e}^{-\mathrm{i}\left(\vec{q}\cdot\vec{v}-\omega\right)t} \\ &= -2\pi e\delta\left(\omega-\vec{q}\cdot\vec{v}\right) \\ \vec{j}_f\left(\vec{q},\omega\right) &= -2\pi e\vec{v}\delta\left(\omega-\vec{q}\cdot\vec{v}\right) \end{aligned} \tag{6}$$

$$
\begin{aligned}
\mathrm{i}\vec{q}\cdot\vec{D}\left(\vec{q},\omega\right) &= \rho_f\left(\vec{q},\omega\right) &(7)\\
\mathrm{i}\vec{q}\times\vec{E}\left(\vec{q},\omega\right) &= i\omega\vec{B}\left(\vec{q},\omega\right) &(8)\\
\mathrm{i}\vec{q}\cdot\vec{B}\left(\vec{q},\omega\right) &= 0 &(9)\\
\mathrm{i}\vec{q}\times\vec{H}\left(\vec{q},\omega\right) &= \vec{j}_f\left(\vec{q},\omega\right)-i\omega\vec{D}\left(\vec{q},\omega\right) &(10)
\end{aligned}
$$

By combining Eq. (2), Eq. (8), and Eq. (10), we can obtain

$$
\begin{aligned}
&\mathrm{i}\vec{q}\times\left(\vec{q}\times\vec{E}\left(\vec{q},\omega\right)\right)=\omega\mu_0\vec{j}_f\left(\vec{q},\omega\right)-\mathrm{i}\omega^2\varepsilon\varepsilon_0\mu_0\vec{E}\left(\vec{q},\omega\right)\\
&\Rightarrow\vec{q}\times\left(\vec{q}\times\vec{E}\left(\vec{q},\omega\right)\right)=-\mathrm{i}\omega\mu_0\vec{j}_f\left(\vec{q},\omega\right)-k^2\varepsilon\vec{E}\left(\vec{q},\omega\right)
\end{aligned}
\tag{11}
$$

where $k=\frac{\omega}{c}$ is the light wave vector in free space, noting that $\frac{1}{c^2}=\varepsilon_0\mu_0$.

Combining Eq. (2), Eq. (7), and Eq. (11) and noting that

$$
\vec{q}\times\left(\vec{q}\times\vec{E}\left(\vec{q},\omega\right)\right)=\left(\vec{q}\cdot\vec{E}\left(\vec{q},\omega\right)\right)\vec{q}-q^2\vec{E}\left(\vec{q},\omega\right) \tag{12}
$$

we can obtain

$$
-\mathrm{i}\frac{\rho_f\left(\vec{q},\omega\right)}{\varepsilon\varepsilon_0}\vec{q}+\mathrm{i}\omega\mu_0\vec{j}_f\left(\vec{q},\omega\right)=\left(q^2-k^2\varepsilon\right)\vec{E}\left(\vec{q},\omega\right) \tag{13}
$$

Considering formula Eq. (6), we can rewrite Eq. (13) as

$$
\begin{aligned}
\vec{E}\left(\vec{q},\omega\right)&=\mathrm{i}\left(\frac{\vec{q}}{\varepsilon\varepsilon_0}-\omega\mu_0\vec{v}\right)\frac{2\pi e\delta\left(\omega-\vec{q}\cdot\vec{v}\right)}{q^2-k^2\varepsilon}\\
&=\frac{2\pi\mathrm{i}e}{\varepsilon_0}\frac{\vec{q}/\varepsilon-k\vec{v}/c}{q^2-k^2\varepsilon}\delta\left(\omega-\vec{q}\cdot\vec{v}\right)
\end{aligned}
\tag{14}
$$

Performing inverse Fourier transform on $\vec{q}$ in Eq. (14) yields

$$
\begin{aligned}
\vec{E}\left(\vec{r},\omega\right)&=\frac{1}{\left(2\pi\right)^3}\frac{2\pi\mathrm{i}e}{\varepsilon_0}\int\mathrm{d}\vec{q}\frac{\vec{q}/\varepsilon-k\vec{v}/c}{q^2-k^2\varepsilon}\delta\left(\omega-\vec{q}\cdot\vec{v}\right)\mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{r}}\\
&=\frac{\mathrm{i}e}{4\pi^2\varepsilon_0}\int\mathrm{d}\vec{q}\frac{\vec{q}/\varepsilon-k\vec{v}/c}{q^2-k^2\varepsilon}\delta\left(\omega-\vec{q}\cdot\vec{v}\right)\mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{r}}
\end{aligned}
\tag{15}
$$

By using Gaussian units, we can obtain a equation in the same form as the original text [1]

$$
\vec{E}\left(\vec{r},\omega\right)=\frac{\mathrm{i}e}{\pi}\int\mathrm{d}\vec{q}\frac{\vec{q}/\varepsilon-k\vec{v}/c}{q^2-k^2\varepsilon}\mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{r}}\delta\left(\omega-\vec{q}\cdot\vec{v}\right) \tag{16}
$$

To complete this integral, we simplify Eq. (16) as

$$
\vec{E}\left(\vec{r},\omega\right)=\frac{\mathrm{i}e}{\pi}\int\mathrm{d}q_z\mathrm{d}\vec{q}_\perp\frac{\left(q_z\hat{z}+\vec{q}_\perp\right)/\varepsilon-\omega v/c^2\hat{z}}{q_z^2+q_\perp^2-\omega^2\varepsilon/c^2}\mathrm{e}^{\mathrm{i}\left(q_z z+\vec{q}_\perp\cdot\vec{R}\right)}\delta\left(\omega-q_z v\right) \tag{17}
$$

where we have $\vec{r}=\vec{R}+z\hat{z}=R\hat{R}+z\hat{z}$, $\vec{q}=q_z\hat{z}+\vec{q}_\perp$ and noting that $\vec{v}=v\hat{z}$.

By integrating q_z, we can obtain

$$
\vec{E}\left(\vec{r},\omega\right)=\frac{\mathrm{i}e}{\pi v}\int\mathrm{d}\vec{q}_\perp\frac{\frac{\vec{q}_\perp}{\varepsilon}+\frac{\omega}{v\varepsilon}\left(1-\frac{\varepsilon v^2}{c^2}\right)\hat{z}}{\left(\frac{\omega}{v}\right)^2\left(1-\frac{\varepsilon v^2}{c^2}\right)+q_\perp^2}\mathrm{e}^{\mathrm{i}\left(\omega z/v+\vec{q}_\perp\cdot\vec{R}\right)} \tag{18}
$$

let $\gamma_\varepsilon = 1/\sqrt{1-\varepsilon v^2/c^2}$, we can obtain

$$\vec{E}\left(\vec{r},\omega\right)=\frac{\mathrm{i}e\omega}{\pi\varepsilon v^2\gamma_\varepsilon}\mathrm{e}^{\mathrm{i}(\omega z/v)}\int\mathrm{d}\vec{q}_\perp\left(\frac{1}{\gamma_\varepsilon}\frac{\hat{z}}{\left(\frac{\omega}{v\gamma_\varepsilon}\right)^2+q_\perp^2}+\frac{\frac{v\gamma_\varepsilon}{\omega}\vec{q}_\perp}{\left(\frac{\omega}{v\gamma_\varepsilon}\right)^2+q_\perp^2}\right)\mathrm{e}^{\mathrm{i}\left(\vec{q}_\perp\cdot\vec{R}\right)} \tag{19}$$

then

$$\vec{E}\left(\vec{r},\omega\right)=\frac{\mathrm{i}e\omega}{\pi\varepsilon v^2\gamma_\varepsilon}\mathrm{e}^{\mathrm{i}(\omega z/v)}\int_0^{+\infty}\int_0^{2\pi}q_\perp\mathrm{d}q_\perp\mathrm{d}\theta\left(\frac{1}{\gamma_\varepsilon}\frac{\hat{z}}{\left(\frac{\omega}{v\gamma_\varepsilon}\right)^2+q_\perp^2}+\frac{\frac{v\gamma_\varepsilon}{\omega}q_\perp\cos\theta\hat{R}}{\left(\frac{\omega}{v\gamma_\varepsilon}\right)^2+q_\perp^2}\right)\mathrm{e}^{\mathrm{i}q_\perp R\cos\theta} \tag{20}$$

Noting that

$$\begin{aligned}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}q_\perp R\cos\theta}\mathrm{d}\theta&=2\pi\mathrm{J}_0\left(q_\perp R\right)\\ \int_0^{2\pi}\cos\theta\mathrm{e}^{\mathrm{i}q_\perp R\cos\theta}\mathrm{d}\theta&=2\pi\mathrm{i}\mathrm{J}_1\left(q_\perp R\right)\end{aligned} \tag{21}$$

where $\mathrm{J}_n\left(z\right)$ is the first type of n-th order Bessel function, we can obtain

$$\vec{E}\left(\vec{r},\omega\right)=\frac{2e\omega}{\varepsilon v^2\gamma_\varepsilon}\mathrm{e}^{\mathrm{i}(\omega z/v)}\int_0^{+\infty}q_\perp\mathrm{d}q_\perp\left(\frac{\mathrm{i}}{\gamma_\varepsilon}\frac{\mathrm{J}_0\left(q_\perp R\right)}{\left(\frac{\omega}{v\gamma_\varepsilon}\right)^2+q_\perp^2}\hat{z}-\frac{\frac{v\gamma_\varepsilon}{\omega}q_\perp\mathrm{J}_1\left(q_\perp R\right)}{\left(\frac{\omega}{v\gamma_\varepsilon}\right)^2+q_\perp^2}\hat{R}\right) \tag{22}$$

By completing the integral, we can obtain

$$\vec{E}\left(\vec{r},\omega\right)=\frac{2e\omega}{\varepsilon v^2\gamma_\varepsilon}\vec{g}\left(\vec{r}\right) \tag{23}$$

where

$$\vec{g}\left(\vec{r}\right)=\mathrm{e}^{\mathrm{i}(\omega z/v)}\left[\frac{\mathrm{i}}{\gamma_\varepsilon}\mathrm{K}_0\left(\frac{\omega R}{v\gamma_\varepsilon}\right)\hat{z}-\mathrm{K}_1\left(\frac{\omega R}{v\gamma_\varepsilon}\right)\hat{R}\right] \tag{24}$$

where $\mathrm{K}_n\left(z\right)$ is the second type of n-th order modified Bessel function.

Eq. (23) and Eq. (24) are the same as the equations in the original text [1].

Method

Consider the forward (inverse) Fourier transform between vector field $\vec{A}\left(\vec{r},t\right)$ and vector field $\vec{A}\left(\vec{q},\omega\right)$

$$\vec{A}\left(\vec{q},\omega\right)=\int\vec{A}\left(\vec{r},t\right)e^{-\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)}\mathrm{d}\vec{r}\mathrm{d}t \tag{25}$$

$$\vec{A}\left(\vec{r},t\right)=\frac{1}{\left(2\pi\right)^4}\int\vec{A}\left(\vec{q},\omega\right)e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)}\mathrm{d}\vec{q}\mathrm{d}\omega \tag{26}$$

Applying the divergence, curl operator, and time partial differential to the both sides of the Eq. (26), respectively, we can obtain

$$\begin{aligned}\nabla\cdot\vec{A}\left(\vec{r},t\right)&=\frac{1}{\left(2\pi\right)^4}\nabla\cdot\int\vec{A}\left(\vec{q},\omega\right)e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)}\mathrm{d}\vec{q}\mathrm{d}\omega\\ &=\frac{1}{\left(2\pi\right)^4}\int\nabla\cdot\left(\vec{A}\left(\vec{q},\omega\right)e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)}\right)\mathrm{d}\vec{q}\mathrm{d}\omega\\ &=\frac{1}{\left(2\pi\right)^4}\int\left(\nabla e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)}\right)\cdot\vec{A}\left(\vec{q},\omega\right)\mathrm{d}\vec{q}\mathrm{d}\omega\\ &=\frac{1}{\left(2\pi\right)^4}\int\left(\mathrm{i}\vec{q}\cdot\vec{A}\left(\vec{q},\omega\right)\right)e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)}\mathrm{d}\vec{q}\mathrm{d}\omega\end{aligned} \tag{27}$$

$$
\begin{aligned}
\nabla \times \vec{A}(\vec{r},t) &= \frac{1}{(2\pi)^4} \nabla \times \int \vec{A}(\vec{q},\omega)\, e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)} \mathrm{d}\vec{q}\mathrm{d}\omega \\
&= \frac{1}{(2\pi)^4} \int \nabla \times \left(\vec{A}(\vec{q},\omega)\, e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)} \right) \mathrm{d}\vec{q}\mathrm{d}\omega \\
&= \frac{1}{(2\pi)^4} \int \left(\nabla e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)} \right) \times \vec{A}(\vec{q},\omega)\, \mathrm{d}\vec{q}\mathrm{d}\omega \\
&= \frac{1}{(2\pi)^4} \int \left(\mathrm{i}\vec{q} \times \vec{A}(\vec{q},\omega) \right) e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)} \mathrm{d}\vec{q}\mathrm{d}\omega
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t} \vec{A}(\vec{r},t) &= \frac{1}{(2\pi)^4} \frac{\partial}{\partial t} \int \vec{A}(\vec{q},\omega)\, e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)} \mathrm{d}\vec{q}\mathrm{d}\omega \\
&= \frac{1}{(2\pi)^4} \int \frac{\partial}{\partial t} \left(\vec{A}(\vec{q},\omega)\, e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)} \right) \mathrm{d}\vec{q}\mathrm{d}\omega \\
&= \frac{1}{(2\pi)^4} \int \frac{\partial}{\partial t} \left(e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)} \right) \vec{A}(\vec{q},\omega)\, \mathrm{d}\vec{q}\mathrm{d}\omega \\
&= \frac{1}{(2\pi)^4} \int \left(-\mathrm{i}\omega \vec{A}(\vec{q},\omega) \right) e^{\mathrm{i}(\vec{q}\cdot\vec{r}-\omega t)} \mathrm{d}\vec{q}\mathrm{d}\omega
\end{aligned}
\tag{29}
$$

So the Fourier transform of

$$
\begin{cases}
\nabla \cdot \vec{A}(\vec{r},t) \\
\nabla \times \vec{A}(\vec{r},t) \\
\frac{\partial}{\partial t} \vec{A}(\vec{r},t)
\end{cases}
\tag{30}
$$

are

$$
\begin{cases}
\mathrm{i}\vec{q} \cdot \vec{A}(\vec{q},\omega) \\
\mathrm{i}\vec{q} \times \vec{A}(\vec{q},\omega) \\
-\mathrm{i}\omega \vec{A}(\vec{q},\omega)
\end{cases}
\tag{31}
$$

respectively.

The correlation between dielectric function and wave-vector

Spatial Dispersion: In a homogeneous dielectric medium, the dielectric function is generally not uniform across all wave-vectors. This phenomenon is known as "spatial dispersion". The response of the medium to an external disturbance depends on the spatial characteristics of that disturbance, represented by the wave-vector $\vec{q}$. Thus, ε can vary with $\vec{q}$, as different spatial frequencies (wave-vectors) interact with the medium differently.

Nonlocal Response: The dependence of ε on $\vec{q}$ reflects the "nonlocality" of the material's response. In a nonlocal medium, the polarization at a point depends not just on the electric field at that point, but also on the fields in nearby regions. This leads to a dielectric function that varies with $\vec{q}$, encapsulating the spatial extent of the medium's response.

One is often interested in opening the collection angle to increase the inelastic signal and to reduce sample damage. Nonlocal effects are then apparent. Under such circumstances, we need to include spatial dispersion in the dielectric function, which has different forms for longitudinal and transverse fields, $\varepsilon_{lon}(\vec{q},\omega)$, $\varepsilon_{tr}(\vec{q},\omega)$, respectively.

Neglecting the correlation between dielectric function and wavevector

This is the local response approximation, which applies when low enough momentum transfers are collected below a certain cutoff

$$\hbar q_c \approx \sqrt{\left(m_e v \varphi_{out}\right)^2 + \left(\hbar\omega/v\right)^2} \tag{32}$$

References

[1] F. J. García de Abajo, *Optical excitations in electron microscopy*, Rev. Mod. Phys. **82** (2010Feb), 209–275.