Derivation of the Volume and Surface Area of an N-Dimensional Sphere via Gaussian Integrals

Dixuan Wu

May 25, 2025

1 Problem Definition

Consider an N-dimensional sphere in Euclidean space, whose volume $V_N(R)$ and surface area $S_N(R)$ are defined as:

$$\begin{cases} V_N(R) = \displaystyle\int_{x_1^2+\cdots+x_N^2 \leq R^2} \mathrm{d}x_1 \cdots \mathrm{d}x_N \\ S_N(R) = \dfrac{\mathrm{d}}{\mathrm{d}R} V_N(R) \end{cases} \tag{1}$$

The goal is to derive explicit expressions using Gaussian integrals.

2 Core Derivation

2.1 Two Representations of the Gaussian Integral

First, compute the N-dimensional Gaussian integral:

$$I = \int_{\mathbb{R}^N} \mathrm{e}^{-\boldsymbol{x}\cdot\boldsymbol{x}} \mathrm{d}V, \quad \boldsymbol{x}\cdot\boldsymbol{x} = \sum_{i=1}^{N} x_i^2 \tag{2}$$

2.1.1 Cartesian Coordinate Method

Using separability of the integral:

$$I = \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \mathrm{e}^{-x_i^2} \mathrm{d}x_i = (\sqrt{\pi})^N \tag{3}$$

2.1.2 Spherical Coordinate Method

Introduce spherical coordinates with $\mathrm{d}V = S_N(1) r^{N-1} \mathrm{d}r$, where $S_N(1)$ is the surface area of the unit sphere:

$$I = S_N(1) \int_0^{+\infty} \mathrm{e}^{-r^2} r^{N-1} \mathrm{d}r \tag{4}$$

2.2 Derivation of the Unit Sphere Surface Area

Equating the two results:

$$S_N(1) = \frac{(\sqrt{\pi})^N}{\int_0^{+\infty} \mathrm{e}^{-r^2} r^{N-1} \mathrm{d}r} \tag{5}$$

Using the substitution $t = r^2$, we obtain:

$$\int_0^{+\infty} \mathrm{e}^{-r^2} r^{N-1} \mathrm{d}r = \frac{1}{2} \int_0^{+\infty} \mathrm{e}^{-t} t^{\frac{N}{2}-1} \mathrm{d}t = \frac{1}{2}\Gamma\left(\frac{N}{2}\right) \tag{6}$$

Thus, the surface area of the unit sphere is:

$$\boxed{S_N(1) = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)}} \tag{7}$$

2.3 Derivation of the Unit Sphere Volume

The volume and surface area satisfy the differential relation:

$$V_N(1) = \int_0^1 S_N(r)\mathrm{d}r = S_N(1) \int_0^1 r^{N-1}\mathrm{d}r = \frac{S_N(1)}{N} \tag{8}$$

Substituting the expression for $S_N(1)$ and using the Gamma function property $\Gamma(z+1) = z\Gamma(z)$, we obtain:

$$\boxed{V_N(1) = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}+1\right)}} \tag{9}$$

3 Generalization to Arbitrary Radius

Dimensional analysis yields:

$$\begin{cases} S_N(R) = S_N(1)R^{N-1} \\ V_N(R) = V_N(1)R^N \end{cases} \tag{10}$$

The final results are:

$$\boxed{\begin{aligned} V_N(R) &= \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}+1\right)} R^N \\ S_N(R) &= \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} R^{N-1} \end{aligned}} \tag{11}$$