Markov's Inequality in Probability Theory

ChatGPT 4o

May 31, 2025

1 Formal Statement of Markov's Inequality

Let X be a **non-negative** random variable, i.e., $X \geq 0$ almost surely, and suppose the expectation $\mathbb{E}[X]$ exists and is finite. Then, for any $a > 0$, the following inequality holds:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a} \tag{1.0.1}$$

This inequality provides an upper bound on the probability that X exceeds a positive threshold a, based only on the expectation of X.

2 Mathematical Proof

2.1 Assumptions

- $X \geq 0$
- $\mathbb{E}[X] < \infty$
- $a > 0$

2.2 Step 1: Use Indicator Function

Note that for all outcomes,

$$X \geq X \cdot \mathbf{1}_{\{X \geq a\}} \geq a \cdot \mathbf{1}_{\{X \geq a\}} \tag{2.2.1}$$

This holds because:

- If $X \geq a$, then $\mathbf{1}_{\{X \geq a\}} = 1$, so $X \geq a$
- If $X < a$, then $\mathbf{1}_{\{X \geq a\}} = 0$, and both sides are zero or positive

Taking expectations:

$$\mathbb{E}[X] \geq \mathbb{E}\left[a \cdot \mathbf{1}_{\{X \geq a\}}\right] = a \cdot \mathbb{P}(X \geq a) \tag{2.2.2}$$

2.3 Step 2: Rearranging

Rearranging the inequality gives:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a} \tag{2.3.1}$$

Q.E.D.

3 Intuitive Interpretation

Markov's inequality is a very general result. It says:

> If you only know the <u>mean</u> of a non-negative random variable, then this inequality gives an upper bound on the probability that the variable exceeds a certain multiple of its mean.

Example: If $\mathbb{E}[X] = 10$, then

$$\mathbb{P}(X \geq 50) \leq \frac{10}{50} = 0.2$$

Even without knowing the distribution of X, we can bound how likely it is to take large values.

4 Remarks and Applications

- The inequality is **tight**: there exist random variables for which equality holds.
- The assumption that $X \geq 0$ is essential; Markov's inequality does not hold for arbitrary (possibly negative) random variables.
- It is a building block for more advanced results, such as:
  - **Chebyshev's inequality**, derived by applying Markov's inequality to $(X - \mu)^2$
  - **Concentration inequalities** in probability theory and theoretical computer science
- It is commonly used in the analysis of **randomized algorithms** and **tail bounds**.