Legendre Transform

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1 Initial Setup

Consider a binary function $F(x, y)$, whose total differential is:

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y = u(x, y)\mathrm{d}x + v(x, y)\mathrm{d}y \tag{1.1}$$

where:

$$u(x, y) = \frac{\partial F}{\partial x}, \quad v(x, y) = \frac{\partial F}{\partial y} \tag{1.2}$$

Here, x and y are independent variables, and u and v are functions of x and y.

2 Objective of Variable Transformation

We wish to transform the independent variables from (x, y) to (x, v). That is, to express y as a function of v. To achieve this, we need to construct a new function $G(x, v)$ whose total differential contains only $\mathrm{d}x$ and $\mathrm{d}v$.

3 Construction of the New Function

Observing that $\mathrm{d}F = u\mathrm{d}x + v\mathrm{d}y$, we aim to eliminate $\mathrm{d}y$ and introduce $\mathrm{d}v$. To do this, we subtract $\mathrm{d}(yv)$ from both sides:

$$\mathrm{d}(F - yv) = \mathrm{d}F - \mathrm{d}(yv) \tag{3.1}$$

Computing $\mathrm{d}(yv)$:

$$\mathrm{d}(yv) = y\mathrm{d}v + v\mathrm{d}y \tag{3.2}$$

Therefore:

$$\mathrm{d}(F - yv) = u\mathrm{d}x + v\mathrm{d}y - (y\mathrm{d}v + v\mathrm{d}y) = u\mathrm{d}x - y\mathrm{d}v \tag{3.3}$$

Now, the right-hand side contains only $\mathrm{d}x$ and $\mathrm{d}v$, which is exactly what we need.

Define the new function $G(x, v)$ as:

$$G(x, v) = F(x, y) - yv \tag{3.4}$$

Its total differential is:

$$\mathrm{d}G = u\mathrm{d}x - y\mathrm{d}v \tag{3.5}$$

4 Partial Derivatives of the New Function

From $\mathrm{d}G = u\mathrm{d}x - y\mathrm{d}v$, we can see:

$$\frac{\partial G}{\partial x} = u, \quad \frac{\partial G}{\partial v} = -y \tag{4.1}$$

This indicates:

$$u \text{ is the partial derivative of the new function } G \text{ with respect to } x \tag{4.2}$$

$$y \text{ is the negative of the partial derivative of } G \text{ with respect to } v \tag{4.3}$$

5 Transformation of Variable Relationships

The original variable y can now be expressed as a function of v. Specifically:

$$\text{From } v = \frac{\partial F}{\partial y}, \text{ we can solve for } y = y(x, v) \tag{5.1}$$

Substituting y into $G(x, v) = F(x, y) - yv$ completes the variable transformation.

6 Geometric Interpretation of the Legendre Transform

The Legendre transform can be understood as re-describing the function through the slopes of its tangent lines (i.e., the partial derivatives). The differential information of the original function $F(x, y)$ is fully preserved in the new function $G(x, v)$, with only the independent variable changed from y to v.

7 Summary

The specific steps of the Legendre transform are as follows:

1.Start with the original function $F(x, y)$ and compute its total differential $\mathrm{d}F = u\mathrm{d}x + v\mathrm{d}y$ (7.1)

2.Define the new function $G(x, v) = F(x, y) - yv$ (7.2)

3.Compute $\mathrm{d}G = u\mathrm{d}x - y\mathrm{d}v$ to obtain the partial derivative relationships of the new function (7.3)

4.Express y as a function of v using $v = \frac{\partial F}{\partial y}$ (7.4)

Thus, we successfully transform the independent variables from (x, y) to (x, v) while retaining all the information of the system.

8 Additional Remarks

If the goal is to transform x instead of y, the method is completely analogous: define $G(u, y) = F(x, y) - xu$.

The Legendre transform is widely used in physics, for example, in the transition from Lagrangian mechanics to Hamiltonian mechanics (transforming generalized velocities to generalized momenta).