Module - 4 Optimization

Constraint Maxima and Minima

In a typical unconstraint optimization problem to find the local minima of and maxima of a function $f(x,y)$ by setting its partial derivatives to zero However, in constraint optimization find these extreme values subject to some constraints.

CONSTRAINTS

Constraints can be in the form of equalities or inequalities.

EQUALITY CONSTRAINTS

$g(x,y,z \ldots) = 0$

Inequality Constraint :

$g(x,y,z \ldots) \leq 0$

$g(x,y,z \ldots) \geq 0$

Method of Lagrange's Multiplier

This method of lagrange multiplier common method for solving a constraint optimization problem. It converts a constraint problem into a form that can be solved as an unconstraint problem by introducing additional variables called Lagrange Multiplier.

One constraint Form

Given a function $f(x,y)$ to be maximized or minimized subject to a constraint $g(x,y) = 0$.

1. Lagrangian function :

→ Construct the Lagrangian function,

$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$, where λ is lagrangian multiplier.

2. System of Eqns :-

→ Find the critical points by solving the system of eqns.

$\frac{\partial L}{\partial x} = 0$

$\frac{\partial L}{\partial \lambda} = 0$

$\frac{\partial L}{\partial y} = 0$

This yields
$$\begin{cases} \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \\ g(x,y) = 0 \end{cases}$$

Q1. Maximize $f(x,y) = x^2+y^2$ subject to the constraint $g(x,y) = x+y-1 = 0$

$f(x,y) = x^2+y^2$

$g(x,y) = x+y-1 = 0$

Construct the lagrange's function.

$L(x,y,\lambda) = x^2+y^2 + \lambda(x+y-1)$

System of eqn.

$\frac{\partial L}{\partial x} = 2x + \lambda = 0 \rightarrow (1)$

$\frac{\partial L}{\partial y} = 2y + \lambda = 0 \rightarrow (2)$

$\frac{\partial L}{\partial \lambda} = x+y-1 = 0 \rightarrow (3)$

$(1) \Rightarrow 2x + \lambda = 0,$
$\lambda = -2x$

$(2) \Rightarrow 2y + \lambda = 0$
$\lambda = -2y$

$\therefore -2x = -2y$

$x = y$ sub in (3)

$\Rightarrow x + x - 1 = 0$

$2x - 1 = 0$

$2x = 1$

$x = 1/2, \; y = 1/2$

$\therefore (1/2, 1/2)$ are the critical points

Then, $f(1/2, 1/2) = 1/2^2 + 1/2^2 = 1/4 + 1/4$

$= 2/4 = 1/2$

2. Maximize $f(x,y) = xy$ Subject to $x^2+y^2 = 1$

$f(x,y) = xy$

$g(x,y) = x^2+y^2-1 = 0$

Construct the Lagrange's function

$L(x,y,\lambda) = xy + \lambda(x^2+y^2-1)$

System of eqn.

$\frac{\partial L}{\partial x} = y + 2x\lambda = 0 \rightarrow (1)$

$\frac{\partial L}{\partial y} = x + 2y\lambda = 0 \rightarrow (2)$

$\frac{\partial L}{\partial \lambda} = x^2+y^2-1 = 0 \rightarrow (3)$

$(1) \Rightarrow y + 2x\lambda = 0$

$2x\lambda = -y$

$\lambda = \frac{-y}{2x}$

$(2) \Rightarrow x + 2y\lambda = 0$

$2y\lambda = -x$

$\lambda = -x/2y$

$\therefore \frac{-y}{2x} = \frac{-x}{2y}$

$\cancel{-}\cancel{2}y^2 = \cancel{-}\cancel{2}x^2$

$y^2 = x^2$ sub in (3)

$(3) \Rightarrow x^2 + x^2 - 1 = 0$

$2x^2 = 1$

$x^2 = 1/2 \quad \Rightarrow x = \sqrt{1/2}$

$x = \pm \frac{1}{\sqrt{2}} \Rightarrow y = \pm \cancel{} \frac{1}{\sqrt{2}}$

$\therefore (1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, \frac{-1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ are the critical points.

$f(1/\sqrt{2}, 1/\sqrt{2}) = 1/\sqrt{2} \times \frac{1}{\sqrt{2}} = 1/\cancel{\sqrt{2}}2$

$f(-1/\sqrt{2}, -1/\sqrt{2}) = -1/\sqrt{2} + \frac{-1}{\sqrt{2}} = 1/2$

$f(-1/\sqrt{2}, 1/\sqrt{2}) = -1/\sqrt{2} \times \frac{1}{\sqrt{2}} = -1/2$

$f(\cancel{+}1/\sqrt{2}, -1/\sqrt{2}) = \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = -1/2$

$\therefore$ The max. value is $1/2$ at $(1/\sqrt{2}, 1/\sqrt{2}) \cup (-1/\sqrt{2}, -1/\sqrt{2})$

3. Maximize $f(x,y) = x+y$ subject to the constraint $x^2+4y^2=4$.

$f(x,y) = x+y$

$g(x,y) = x^2+4y^2-4 = 0$

Construct the Lagrange's fn

$L(x,y,\lambda) = x+y+\lambda(x^2+4y^2-4)$

~~Sy~~ $\frac{\partial L}{\partial x} = 1+2x\lambda = 0 \rightarrow (1)$

$\frac{\partial L}{\partial y} = 1+8y\lambda = 0 \rightarrow (2)$

$\frac{\partial L}{\partial \lambda} = \cancel{x^2+y}\ x^2+4y^2-4 = 0 \rightarrow (3)$

$(1) \Rightarrow 1+2x\lambda = 0$

$2x\lambda = -1$

$\lambda = -1/2x$

$(2) \Rightarrow 1+8y\lambda = 0$

$8y\lambda = -1$

$\lambda = -1/8y$

$\therefore \frac{-1}{2x} = -1/8y$

$-8y = -2x$

$y = \frac{-2x}{-8} \Rightarrow y = \frac{x}{4}$ or $x = 4y$

Sub in (3)

$(4y)^2 + 4y^2 - 4 = 0$

$16y^2 + 4y^2 = 4$

$20y^2 = 4$

$y^2 = \frac{4}{20} = 1/5$

$y = \pm \frac{1}{\sqrt{5}}$

when $y = 1/\sqrt{5}$, $x = 4/\sqrt{5}$

when $y = -1/\sqrt{5}$, $x = -4/\sqrt{5}$

$\therefore (4/\sqrt{5}, 1/\sqrt{5})$ and $(-4/\sqrt{5}, -1/\sqrt{5})$ are the critical pts

$f(4/\sqrt{5}, 1/\sqrt{5}) = 4/\sqrt{5} + 1/\sqrt{5} = 5/\sqrt{5} = \frac{\sqrt{5}\times\sqrt{5}}{\sqrt{5}} = \sqrt{5}$

$f(-4/\sqrt{5}, -1/\sqrt{5}) = -4/\sqrt{5} - \frac{1}{\sqrt{5}} = \frac{-5}{\sqrt{5}} = -\sqrt{5}$

$\therefore$ The max. value $= \sqrt{5}$ at $f(4/\sqrt{5}, 1/\sqrt{5})$

4. Find the extreme values of the fn. $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2+y^2-110$

$f(x,y) = xy$

$g(x,y) = x^2+y^2-10 = 0$

Construct the lagrange's function,

$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

$= xy + \lambda(x^2+y^2-10)$

System of eqn

$\frac{\partial L}{\partial x} = y + 2x\lambda = 0 \rightarrow (1)$

$\frac{\partial L}{\partial y} = x + 2y\lambda = 0 \rightarrow (2)$

$$\frac{\partial L}{\partial x} = x^2 + y^2 - 10 \rightarrow (3)$$

$(1) \Rightarrow y + 2x\lambda = 0$

$2x\lambda = -y$

$\lambda = \frac{-y}{2x}$

$(2) \Rightarrow x + 2y\lambda = 0$

$2y\lambda = -x$

$\lambda = \frac{-x}{2y}$

$\therefore \frac{-y}{2x} = \frac{-x}{2y}$

$-2y^2 = -2x^2$

$x^2 = y^2$

Sub in (3)

$\Rightarrow x^2 + x^2 - 10 = 0$

$2x^2 = 10$

$x^2 = \frac{10}{2}$

$x = \pm\sqrt{5} \Rightarrow y = \pm\sqrt{5}$

$\therefore (\sqrt{5}, \sqrt{5}), (-\sqrt{5}, -\sqrt{5}), (\sqrt{5}, -\sqrt{5}), (-\sqrt{5}, \sqrt{5})$ are the critical pts.

$f(\sqrt{5}, \sqrt{5}) = \sqrt{5} \times \sqrt{5} = 5$

$f(-\sqrt{5}, \sqrt{5}) = -\sqrt{5} \times -\sqrt{5} = 5$

$f(-\sqrt{5}, \sqrt{5}) = -\sqrt{5} \times \sqrt{5} = -5$

$f(\sqrt{5}, -\sqrt{5}) = \sqrt{5} \times -\sqrt{5} = -5$

$\therefore$ The max. value is 5 at $(\sqrt{5}, \sqrt{5})$ and $(-\sqrt{5}, -\sqrt{5})$

$\therefore$ The max. value is -5 at $(-\sqrt{5}, \sqrt{5})$ & $(\sqrt{5}, -\sqrt{5})$

Q. Find the point $P(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to the origin.

Here, the problem asked us to find the min. value of fn. ~~d=~~ op.

$$d = op = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

$d = \sqrt{x^2+y^2+z^2}$, d is a minimum, then d^2 is also equivalent to the minimum

$\therefore d^2 = x^2+y^2+z^2$ $\qquad g(x,y,z) = 2x+y-z-5=0$

we can minimize $f(x,y,z) = x^2+y^2+z^2$ subject to $2x+y-z-5=0$

$$h(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) = (x^2+y^2+z^2) + \lambda(2x+y-z-5) = 0$$

$$= \cancel{x^2+y^2+z^2+\lambda(2x+y-z-5)=0}$$

$$\frac{\partial L}{\partial x} = 2x+2\lambda = 0 \rightarrow (1)$$

$$\frac{\partial L}{\partial y} = 2y+\lambda = 0 \rightarrow (2)$$

$$\frac{\partial L}{\partial z} = 2z-\lambda = 0 \rightarrow (3)$$

$$\frac{\partial L}{\partial \lambda} = 2x+y-z-5=0 \rightarrow (4)$$

$(1) \Rightarrow 2x+2\lambda = 0$

$\lambda = -x \Rightarrow x = -\lambda$

$(2) \Rightarrow 2y+\lambda = 0$

$\lambda = -2y \Rightarrow y = \frac{-\lambda}{2}$

$(3) \Rightarrow 2z-\lambda = 0$

$\lambda = 2z \Rightarrow z = \lambda/2$

Sub in (4)

$$2(-\lambda) + \frac{-\lambda}{2} - \frac{\lambda}{2} - 5 = 0 \qquad \Rightarrow -6\lambda = 10$$

$$-2\lambda - \frac{\lambda}{2} - \frac{\lambda}{2} = 5 \qquad \lambda = \frac{-10}{6}$$

$$\frac{-4\lambda - \lambda - \lambda}{2} = 5 \qquad \lambda = -5/3$$

when $\lambda = -5/3$

$x = 5/3$

$y = 5/6$

$z = \frac{-5/3}{2} = \frac{-5}{6}$

$(5/3, 5/6, -5/6)$ are the point closest to the origin

Q Find the max and min values of the fn $f(x,y) = 3x+4y$ on the circle $x^2+y^2=1$ Subject to the constraint

$f(x,y) = 3x+4y$

$g(x,y) = x^2+y^2-1 = 0$

$L(x,y) = f(x,y) + \lambda g(x,y)$

$= 3x+4y+\lambda(x^2+y^2-1) = 0$

$\frac{\partial L}{\partial x} = 3+2\lambda x = 0 \rightarrow (1) \Rightarrow 2\lambda x = -3 \Rightarrow \lambda = \frac{-3}{2x} \Rightarrow x = \frac{-3}{2\lambda}$

$\frac{\partial L}{\partial y} = 4+2\lambda y = 0 \rightarrow (2) \Rightarrow 2\lambda y = -4 \Rightarrow \lambda = \frac{-4}{2y} \Rightarrow y = \frac{-4}{2\lambda}$

$\frac{\partial L}{\partial \lambda} = x^2+y^2-1 = 0 \rightarrow (3) \Rightarrow$ Sub in (3)

$\Rightarrow \left(\frac{-3}{2\lambda}\right)^2 + \left(\frac{-4}{2\lambda}\right)^2 - 1 = 0$

$\Rightarrow \frac{9}{4\lambda^2} + \frac{16}{4\lambda^2} - 1 = 0$

$\Rightarrow \frac{25}{4\lambda^2} - 1 = 0$

$\Rightarrow \frac{25}{4\lambda^2} = 1$

$\Rightarrow 25 = 4\lambda^2$

$\Rightarrow \frac{25}{4} = \lambda^2$

$\Rightarrow \sqrt{25/4} = \lambda$

$\Rightarrow \lambda = \pm 5/2$

when $\lambda = 5/2$

$x = \frac{-3}{2\times\frac{5}{2}} = \frac{-3}{5}$

$y = \frac{-4}{2\times\frac{5}{2}} = \frac{-4}{5}$

$(-3/5, -4/5)$ are the critical point.

When $\lambda = -5/2$

$x = \frac{-3}{2\times\frac{-5}{2}} = \frac{3}{5}$

$y = \frac{-4}{2\times\frac{-5}{2}} = \frac{4}{5}$

$(3/5, 4/5)$ are the critical point.

at $(-3/5, -4/5)$

$$f(x,y) = 3\times\frac{-3}{5} + 4\times\frac{-4}{5}$$

$$= \frac{-9}{5} - \frac{16}{5} = \frac{-25}{5} = \underline{\underline{-5}}$$

at $(3/5, 4/5)$

$$f(x,y) = 3\times\frac{3}{5} + 4\times\frac{4}{5}$$

$$= \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = \underline{\underline{5}}$$

∴ The max. value is 5 at $(3/5, 4/5)$

The min. value is -5 at $(-3/5, -4/5)$

Q. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

~~f(x,y,z)~~ ⇒ $x+y+z=9 \Rightarrow g(x,y,z)$.

$f(x,y,z) = x^2+y^2+z^2$. [minimize ~~fun, the~~ fn subject to g(x)]

$$h(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$= x^2+y^2+z^2+\lambda(x+y+z-9)$$

$\frac{\partial L}{\partial x} = 2x+\lambda = 0 \rightarrow (1) \Rightarrow x = \frac{-\lambda}{2}$

$\frac{\partial L}{\partial y} = 2y+\lambda = 0 \rightarrow (2) \Rightarrow y = \frac{-\lambda}{2}$

$\frac{\partial L}{\partial z} = 2z+\lambda = 0 \rightarrow (3) \Rightarrow z = \frac{-\lambda}{2}$

$\frac{\partial L}{\partial \lambda} = x+y+z-9 = 0 \rightarrow (4)$ sub in (4)

$$\frac{-\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 9$$

$$-\frac{3\lambda}{2} = 9$$

$$-3\lambda = 18$$

$$\lambda = \underline{\underline{-6}}$$

when $\lambda = -6$

$x = \frac{-\lambda}{2} = 3$

$y = 3$

$z = 3$

$(3,3,3)$ are the critical points.

At $(3,3,3)$

$f(x,y,z) = 3^2 + 3^2 + 3^2 = 9+9+9 = \underline{27}$

Two Constraints

for a function $f(x,y,z)$ subject to 2 constraint $g(x,y,z)=0$ and $h(x,y,z)=0$

1) Lagrangient function

2) Construct the lagrangient fn.

$$L(x,y,z,\mu) = f(x,y,z) + \lambda g(x,y,z) + \mu h(x,y,z)$$

where λ and μ are the Lagrangient multiplieers.

2) System of Equations.

Find the critical points by solving the following system of eqns.

$\frac{\partial L}{\partial x} = 0$ $\quad$ $\frac{\partial L}{\partial z} = 0$ $\quad$ $\frac{\partial L}{\partial \mu} = 0$

$\frac{\partial L}{\partial y} = 0$ $\quad$ $\frac{\partial L}{\partial \lambda} = 0$

Q. Find the max. and min. values of $f(x,y) = x^2+y^2$ subject to the constraint $x+y=1$ and $x-y=1$.

$f(x,y) = x^2+y^2$

$g(x,y) = x+y-1 = 0$

$h(x,y) = x-y-1 = 0$

$L(x,y,\lambda,\mu) = x^2+y^2 + \lambda(x+y-1) + \mu(x-y-1)$

$\frac{\partial L}{\partial x} = 2x + \lambda + \mu = 0 \rightarrow (1)$ $\quad$ $\frac{\partial L}{\partial \lambda} = x+y-1 = 0 \rightarrow (3)$

$\frac{\partial L}{\partial y} = 2y + \lambda - \mu = 0 \rightarrow (2)$ $\quad$ $\frac{\partial L}{\partial \mu} = x-y-1 = 0 \rightarrow (4)$

$\cancel{(3)+(4) \Rightarrow} \; 2x =$

Only one critical pt → minimum

$(1)+(2) \Rightarrow 2x + 2y + 2\lambda = 0 \to (5)$

$(3) - (4) \Rightarrow \cancel{2y + 2 = 0}$ $\quad 2x - 2 = 0$

$\cancel{2y = -2}$ $\quad 2x = 2$

$\cancel{y = -}$ $\quad x = 1$

Sub in (3) $\Rightarrow 1 + y - 1 = 0$

$y = 1 - 1$

$y = 0$

$(1, 0)$ is the critical point.

At $(1,0)$ $f(x,y) = 1^2 + 0^2$

$= 1$ minimum value

$2 \times 1 + 2 \times 0 + 2\lambda = 0$

$2\lambda = -2$

$\lambda = \underline{\underline{-1}}$

$2 \times 1 - 1 + \mu = 0$

$2 - 1 + \mu = 0$

$\mu = \underline{\underline{-1}}$

2. Find the max. and min values of $f(x,y,z) = x^2+y^2+z^2$ subject to the constraints $x+y+z=3$ and $x-y+z=1$.

$f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = x+y+z-3 = 0$

$h(x,y,z) = x-y+z-1 = 0$

$h(x,y,z,\lambda,\mu) = x^2+y^2+z^2 + \lambda(x+y+z-3) + \mu(x-y+z-1)$

$\frac{\partial L}{\partial x} = 2x + \lambda + \mu = 0 \to (1)$ $\qquad \frac{\partial L}{\partial z} = 2z + \lambda + \mu = 0 \to (3)$

$\frac{\partial L}{\partial y} = 2y + \lambda - \mu = 0 \to (2)$ $\qquad \frac{\partial L}{\partial \lambda} = x+y+z-3 = 0 \to (4)$

$\frac{\partial L}{\partial \mu} = x-y+z-1 = 0 \to (5)$

$\begin{aligned} x + y + z - 3 \\ + x - y + z - 1 \\ \hline 2x + 2z - 4 = 0 \end{aligned}$

$(4) - (5) \Rightarrow 2y - 2 = 0$

$2y = 2$

$y = 1$

$x = 3 - y - z$

$x = 3 - 1 - z$

$x = 2 - z$

$x + 2 = z$

$(1) \Rightarrow 2x + \lambda + \mu = 0$

$\lambda + \mu = -2x$

$(3) \Rightarrow 2z + \lambda + \mu = 0$

$\lambda + \mu = -2z$

$\Rightarrow -2x = -2z$

$x = z$

$(4) \Rightarrow x + y + x = 3$

$x + 1 + x = 3$

$2x = 2$

$x = 1$

$\therefore (1,1,1)$ is the critical pt

3. Find the extrema of $f(x,y,z) = x^2+y^2+z^2$ subject to the constraint $x+y+z=1$ and $x-y=0$.

$f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = x+y+z-1=0$

$h(x,y) = x-y = 0$

$L(x,y,z,\lambda,\mu) = x^2+y^2+z^2 + \lambda(x+y+z-1) + \mu(x-y)$

$\frac{\partial L}{\partial x} = 2x + \lambda + \mu = 0 \rightarrow (1)$

$\frac{\partial L}{\partial y} = 2y + \lambda - \mu = 0 \rightarrow (2)$

$\frac{\partial L}{\partial z} = 2z + \lambda = 0 \rightarrow (3)$

$\frac{\partial L}{\partial \lambda} = x+y+z-1 = 0 \rightarrow (4)$

$\frac{\partial L}{\partial \mu} = x-y = 0 \rightarrow (5)$

From $(5) \Rightarrow x-y=0$

$x = y$

$(2) \Rightarrow 2x + \lambda - \mu = 0$

$(1) \Rightarrow 2x + \lambda + \mu = 0$

$(2)-(1) = -2\mu = 0$

$\mu = 0$

$2x + \lambda = 0 \rightarrow (6)$

$2z + \lambda = 0 \rightarrow (3)$

$(6)-(3) \Rightarrow 2x - 2z = 0$

$2x = 2z$

$x = z$

$x + x + x - 1 = 0$

$3x - 1 = 0$

$3x = 1$

$x = 1/3$, $y = 1/3$, $z = 1/3$

$(1/3, 1/3, 1/3)$ is the critical point

minimum value $= 1/3$

4. For a rectangle whose perimeter is 20m use the lagrangiant multiplier method to find the dimension that will maximize the area

Let x and y be length and breadth.

$\cancel{2(x+y) = 20}$

$2(x+y) = 20 \Rightarrow x+y = 10 \Rightarrow g(x,y) = x+y-10$

$f(x,y) = xy$

$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

$= xy + \lambda(x+y-10)$

$\frac{\partial L}{\partial x} = y + \lambda = 0 \Rightarrow \lambda = -y,\ x = -\lambda$

$\frac{\partial L}{\partial y} = x + \lambda = 0 \Rightarrow \lambda = -x,\ y = -\lambda$

$\frac{\partial L}{\partial \lambda} = x+y-10 = 0 \rightarrow$ sub in (3)

$\Rightarrow \cancel{x-x-10=0}$

$\Rightarrow -\lambda - \lambda - 10 = 0$

$-2\lambda = 10$

$\lambda = -5$

$x = 5$ (length) $\quad y = 5$ (breadth)

Linear Programming

Key concepts in Linear programming:

Decision Variables are the variables that represent the choices or decisions to be made in the problem. These variables are what you are trying to determine in order to optimize the objective function. They are the unknowns that need to be solved. They must be non-negative in most LPP.

Example :

In manufacturing problem where the objective is to maximize profit from producing 2 products. the decision variables might be x_1

x_1 : The no. of units of product 1 to produce.

x_2 : The no. of units of product 2 to produce.

Objective function

It is the fn. that needs to be optimized (maximized / minimized) based on the decision variables.

Constraints

They are the restrictions or limitations imposed on the decision variables. Constraints are typically expressed as linear eqns or inequalities involving the decision variables. They define the feasible region of the soln space.

They are in the form

$a_{11}x_1 + a_{12}x_2 \leq b_1$

$a_{21}x_1 + a_{12}x_2 \geq b_2$

$a_{11}x_1 + a_{12}x_2 = b_1$

Linear Programming Problem (LPP)

A linear programming problem involves optimizing a linear objective fn. subject to a set of linear inequality / equality constraints. The general form of an LPP is objective function

maximize / minimize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to the constraint

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$

Basic Feasible Solution

A solution to the LPP is feasible if it satisfies all the constraints including non-negativity constraints. A basic feasible soln is obtained by setting n-m variables to 0 (where n is the no. of variables and m is the no. of constraints). and solving the resulting system of eqns.

Optimal Solution

An optimal soln is a feasible soln that maximises or minizes the objective function. The graphical method is a common algorithm used to find the optimal soln.

atleast 'n' upto 'n'

Formulation Of LPP

Q. Manufacturing Problem

A factory manufactures 2 products P_1 and P_2 (var) each product requires different amounts of resources. The objective is to maximize profit (objective). Profit from P_1 = ₹3 per unit. Profit from P_2 : ₹5 per unit. Resource constraints:

Resource 1 - 4 units available

Resources 2 = 6 units available

P_1 requires 1 unit of resource 1 and 2 unit of resource 2.

P_2 requires 2 units of resource 1 and 1 " "

Ans: Let x_1 and x_2 be the no. of units produced by the manufacturing factory.

Objective fn : Maximize $Z = 3x_1 + 5x_2$

Subject to : $x_1 + 2x_2 \le 4$

$2x_1 + x_2 \le 6$

$x_1 \ge 0, \ x_2 \ge 0$

(diet problem)

Q. A hospital wants to create a nutritious diet for patients, that needs atleast certain nutritional requirements at the minimum cost. The diet must include 2000 cal., 50 g of protein and 300 g of carbohydrates per day. There are 2 food items available with the following nutritional values and cost

Food 1 : cost ₹3 per unit provides 500 cal. 20 g of protein & ~~50~~ 30 g of carbs.

Food 2 : cost ₹4 per unit " 600 cal., 30 g of protein & 40 g of carbs.

Let x_1 and x_2 be the units.

minimize $z = 3x_1 + 4x_2$

Subject to the Constraints:

$500x_1 + 600x_2 \geq 2000$

$20x_1 + 30x_2 \geq 50$

$30x_1 + 40x_2 \geq 300$

$x_1 \geq 0, \quad x_2 \geq 0$

or

$x_1, x_2 \geq 0.$ non-negative fn.

Q3. A company needs to schedule employs to ensure enough staff is available throughout the day. The company requires 5 employs from 8 A.M to 12 P.M 3 employs from 12 P.M to 4 P.M. 4 employs from 4 P.M to 8 P.M. Employs work 8 hour shift and can't start at 8 A.M, 12 P.M or 4 P.M. The goal is to minimize the no. of employees needed.

Let x_1, x_2, x_3 be the ~~units~~ shifts

~~minimize~~ $\cancel{z = 5x_1 + 5x_1 + 3x_2 + 4x_3}$

~~Subject to constraints.~~

Let $x_1 \rightarrow$ 8 AM – 12 PM

$x_2 \rightarrow$ 12 PM – 4 PM

$x_3 \rightarrow$ 4 PM – 8 PM

Minimize $z = x_1 + x_2 + x_3$

$x_1 \geq 5$

$x_1 + x_2 \geq 3$

$x_2 + x_3 \geq 4$

$x_1, x_2, x_3 \geq 0.$

Graphical Method for Solving LPP

Q. Solve the LPP maximize $z = 3x_1+4x_2$ subject to $4x_1+2x_2 \leq 80$, $2x_1+5x_2 \leq 180$, $x_1, x_2 \leq 0$.

$4x_1+2x_2 = 80$

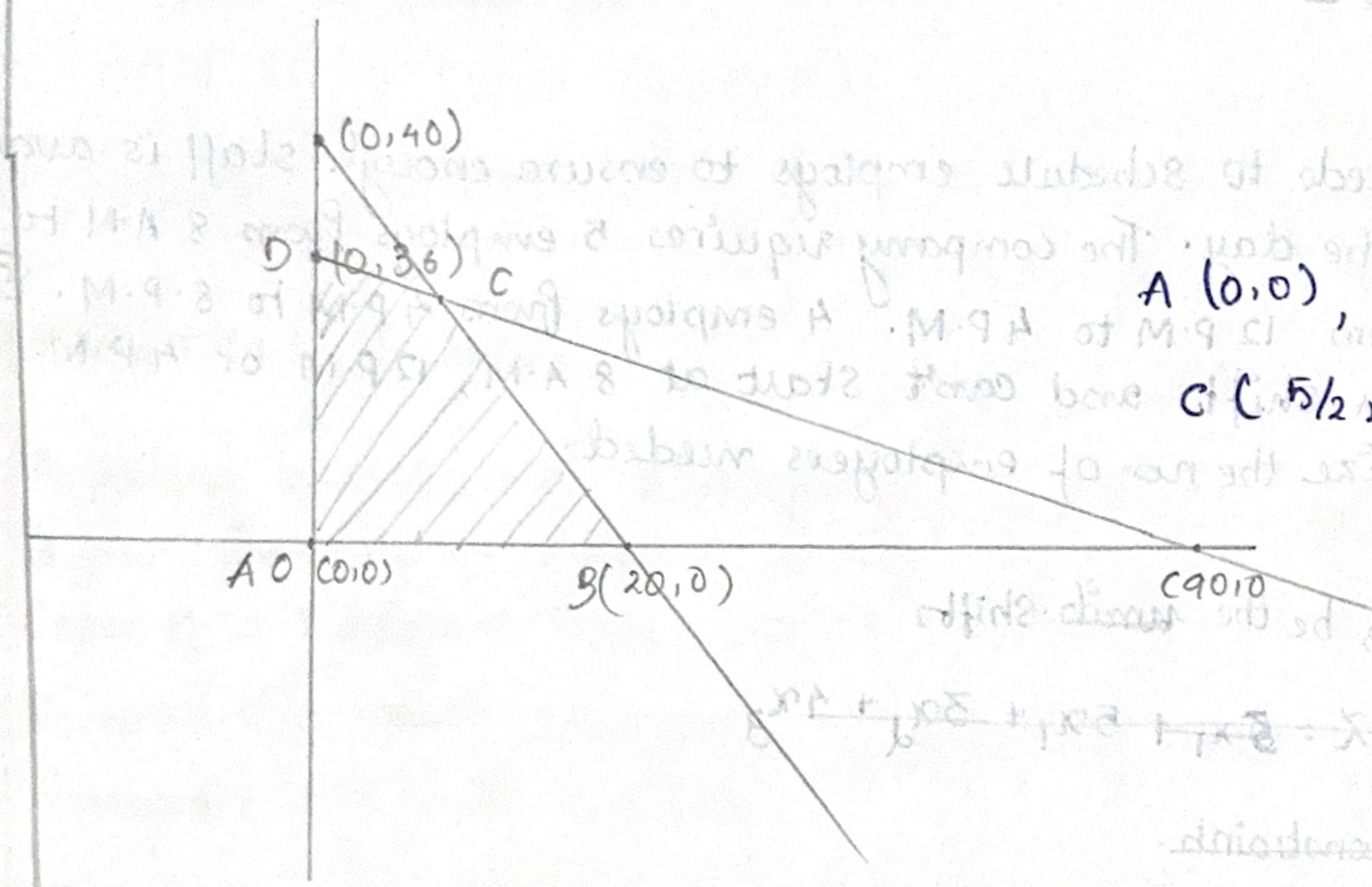
$x_1 = 0 \Rightarrow x_2 = 40 \quad (0,40)$

$x_2 = 0 \Rightarrow x_1 = 20 \quad \cancel{(0,20)}\,(20,0)$

$2x_1+5x_2 = 180$

$x_1 = 0 \Rightarrow x_2 = 36 \quad (0,36)$

$x_2 = 0 \Rightarrow x_1 = 90 \quad \cancel{(0,90)}\,(90,0)$



A (0,0), B (20,0), D (0,36)

C (5/2, 35).

For C, solve the 2 eqns.

$\Rightarrow 4x_1+2x_2 = 80 \rightarrow (1)$

$2x_1+5x_2 = 180 \rightarrow (2)$

x by 2 with (2) $\Rightarrow 4x_1+10x_2 = \cancel{180}\,360 \rightarrow (3)$

$4x_1+2x_2 = 80 \rightarrow (1)$

$(1)-(3) \Rightarrow 8x_2 = 280$

$x_2 = \frac{280}{8}$ 35

$x_2 = 35$

sub $x_2 = 35$ in (1)

$4x_1 + 2\times 35 = 80$

$4x_1 = 80-70$

$x_1 = \frac{10}{4}$

$x_1 = 5/2$

$z = 3x_1 + 4x_2$

$(0,0) \Rightarrow 3\times0 + 4\times0 = \underline{0}$

$(20,0) \Rightarrow 3\times20 + 0 = \underline{60}$

$(5/2, 35) \Rightarrow \cancel{4\times\frac{5}{2} + 4\times35 = 10+80 = 180}$ $3\times\frac{5}{2} + 4\times35 = \frac{15}{2} + 140.$

$= \frac{15+160}{2} = \frac{210}{2} = \frac{175}{2} = 147.5$

160
15
175

(

$(0,36) = 3\times0 + 4\times36 = \underline{144}$

Q.2 Solve the LPP graphically.

maximize $z = 15x_1 + 10x_2$

Subject to $4x_1 + 6x_2 \le 360$

$3x_1 \le 180$

$5x_2 \le 200$

$x_1, x_2 \ge 0$

$4x_1 + 6x_2 \le 360$

$4x_1 + 6x_2 = 360$

$x_1 = 0 \Rightarrow x_2 = 60 \quad (0,60)$

$x_2 = 0 \Rightarrow x_1 = 90 \quad (90,0)$

$3x_1 \le 180$

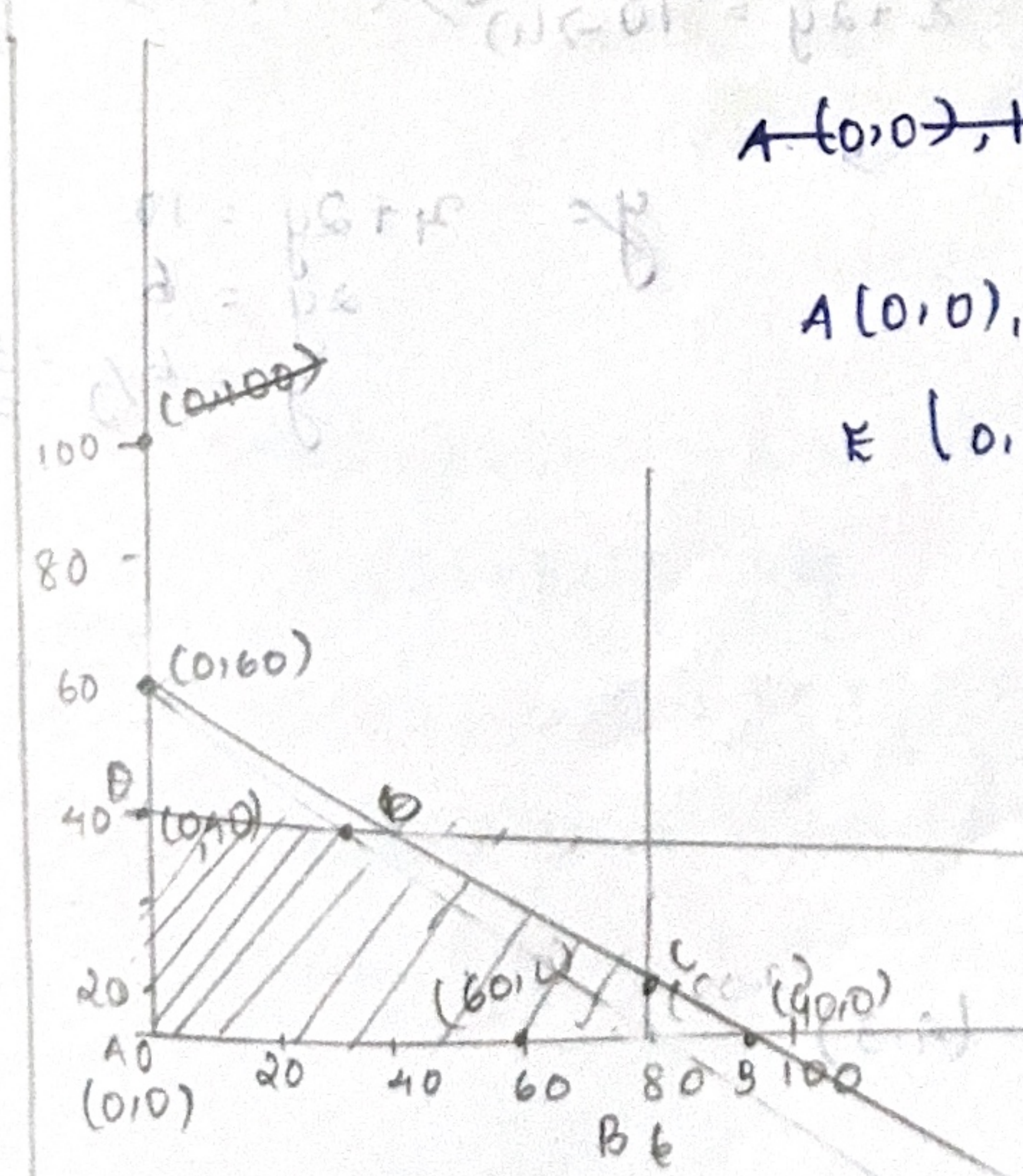
$3x_1 = 180$

$x_1 = 60 \quad (60,0)$

$5x_2 \le 500$

$5x_2 = 200$

$x_2 = \cancel{100}\ 40 \quad (0, \cancel{100}\ 40)$



~~A (0,0), B (80,0)~~, C (→), D ~~(0,40)~~

A (0,0), B (90,0), C (60,20) D (40,30)

E (0,40)

$4\times60 + 6x_2 = 360$

$6x_2 = 360 - 240$

$6x_2 = 120$

$x_2 = \underline{20}$

$4x_1 + 6\times40 = 360$

$4x_1 = 360 - 240$

$4x_1 = 120$

$x_1 = 30$

$z = 15x_1 + 10x_2$

$(0,0) \Rightarrow z = \underline{0}$

$(90,0) \Rightarrow z = 15\times 90 + 0 = \underline{1350}$

$(60,20) \Rightarrow z = 15\times 60 + 10\times 20 = 900 + 200 = \underline{900}$

$(40,30) \Rightarrow z = 15\times 40 + 10\times 30 = 600 + 300 = \underline{900}$

$(0,40) \Rightarrow z = 0 + 10\times 40 = \underline{400}$

Q3 $z = 3x + 2y$ [maximize]

Subject to the constraint $x + 2y \le 10$

$3x + y \le 15$

$x, y \ge 0$

$x + 2y = 10$

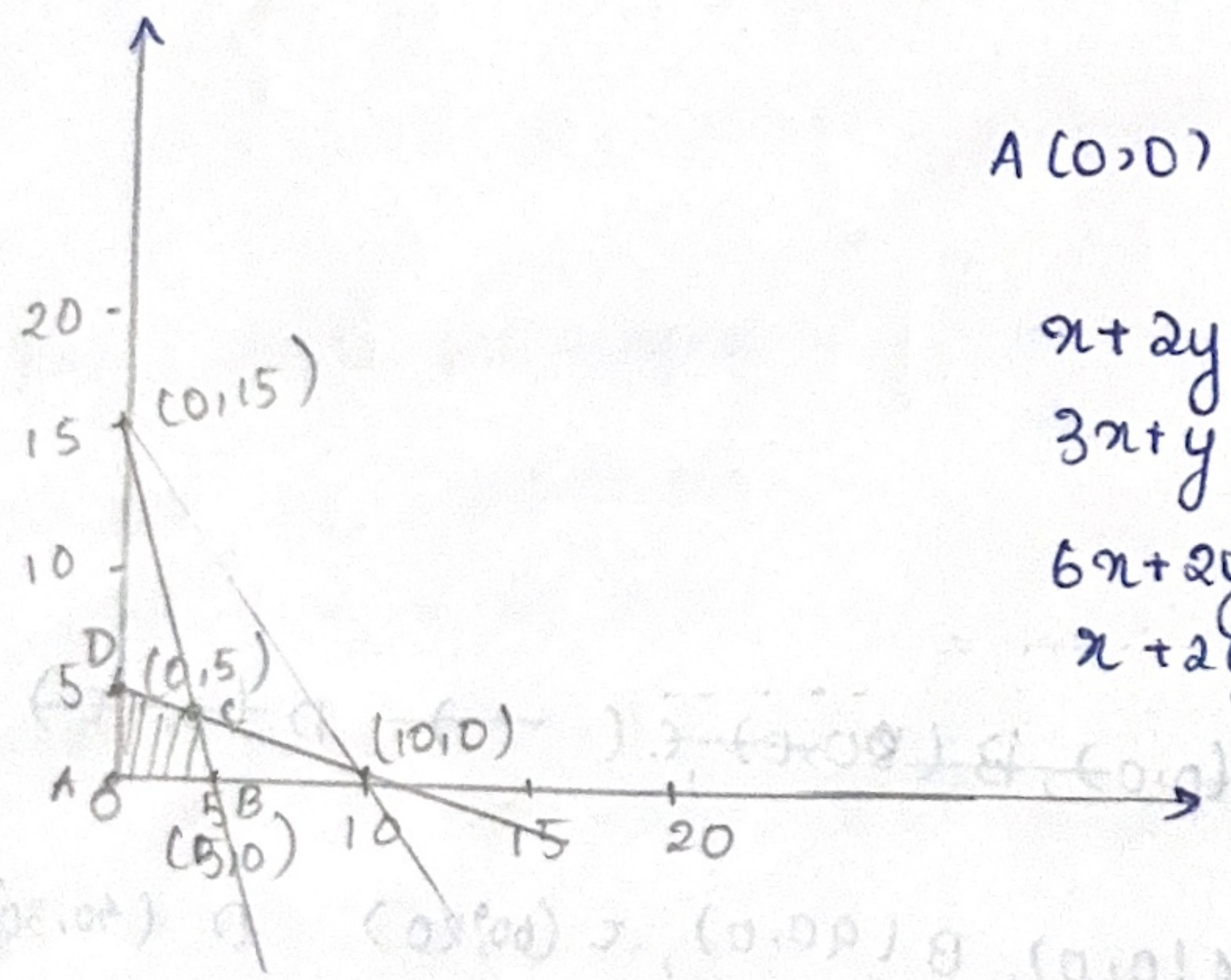
$x = 0 \Rightarrow y = 5 \quad (0,5)$

$y = 0 \Rightarrow x = 10 \quad (10,0)$

$3x + y = 15$

$x \Rightarrow 0 \Rightarrow y = 15 \quad (0,15)$

$y \Rightarrow 0 \Rightarrow x = 5 \quad (5,0)$



A(0,0), B(5,0), C(4,3), D(0,5)

$x + 2y = 10$

$3x + y = 15$

$6x + 2y = 30 \rightarrow (3)$

$x + 2y = 10 \rightarrow (1) \Rightarrow (1)-(3) \Rightarrow 5x = 20$

$x = \underline{4}$

$y = \quad 4 + 2y = 10$

$2y = 6$

$y = 6/2 = \underline{3}$

~~$z = 3x$~~

$(0,0) \Rightarrow z = 3\times 0 + 2\times 0 = 0$

$(5,0) \Rightarrow z = 3\times 5 = 15$

$(4,3) \Rightarrow z = 12 + 6 = 18$

$(0,5) \Rightarrow z = 2\times 5 = 10$

Maximum at value of $z = 18$ at $(4,3)$

Q. Maximize $z = 2x + 3y$

subject to the constraint $x + y \leq 30$

$y \geq 3$

$0 \leq y \leq 12 \rightarrow y \geq 0, \; y \leq 12$

$x - y \geq 0$

$0 \leq x \leq 20 \rightarrow x \geq 0, \; x \leq 20$

$x + y = 30$

$x = 0 \Rightarrow y = 30 \quad (0,30)$

$y = 0 \Rightarrow x = 30 \quad (30,0)$

$y \leq 12$

$y = 12 \quad (0,12)$

$y \geq 3$

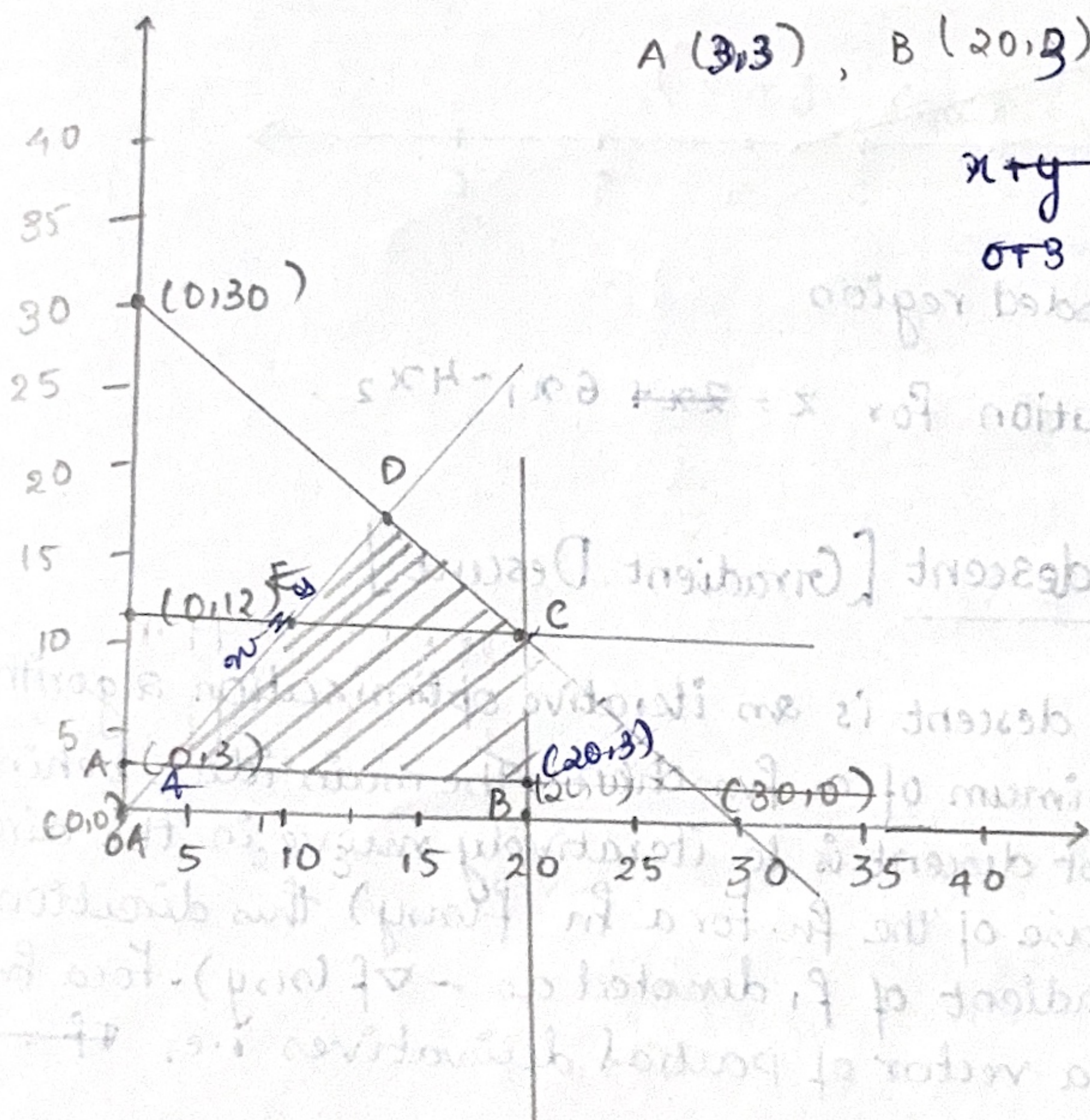
$y = 3 \quad (0,3)$

$x - y = 0 \Rightarrow x = y$

~~$x = 0 \Rightarrow y = 0 \quad (0,0)$~~

$y = 0 \Rightarrow x = 0 \quad (0,0)$

$x = 20 \quad (20,0)$



A (3,3), B (20,3), C (20,10), D (18,12), E (12,12)

~~$x + y = 30$~~

$0 + 3$

$Z = 2 \times 3 + 3 \times 3 = 6 + 9 = 15$

$Z = 2 \times 20 + 3 \times 3 = 49$

$Z = 2 \times 20 + 3 \times 10 = 40 + 30 = 70$

$Z = 2 \times 18 + 3 \times 12 = 36 + 36 = \cancel{102} \; 72$

$Z = 2 \times 12 + 3 \times 12 = 24 + 36 = \cancel{48} \; 60$

Q_5 Use the graphical method to solve the LPP.

Maximize ~~z=6x+~~ $z = 6x_1 - 4x_2$

Subject to the constrainb $2x_1 + 4x_2 \leq 4$

$4x_1 + 8x_2 \geq 15$

$x_1, x_2 \geq 0$

$2x_1 + 4x_2 = 4$

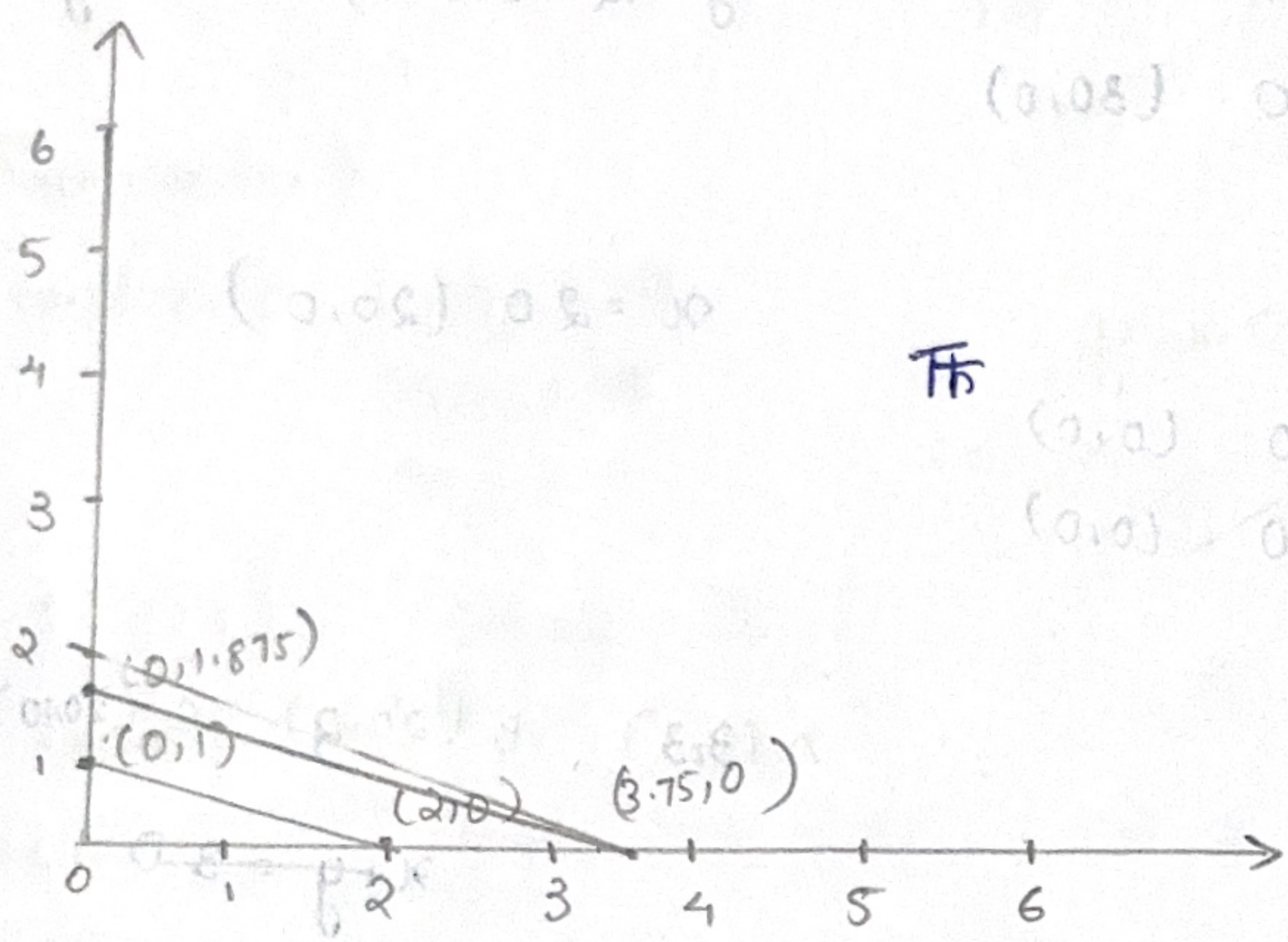
$x_1 = 0 \Rightarrow x_2 = 1$, $(0, 1)$

$x_2 = 0 \Rightarrow x_1 = 2$, $(2, 0)$

$4x_1 + 8x_2 = 15$

$x_1 = 0 \Rightarrow x_2 = 15/8$ $(0, 15/8)$ $(0, 1.875)$

$x_2 = 0 \Rightarrow x_1 = 15/4$ $(15/4, 0)$ $(3.75, 0)$



There is no bounded region

$\therefore$ No ~~soluti~~ Solution for $z = \cancel{2x+}\ 6x_1 - 4x_2$.

★ Method Of Steepest descent [Gradient Descent]

The method of steepest descent is an iterative optimization algorithm used to find the minimum of a function. The main idea behind the method of steepest descent is to iteratively move in the direction of the steepest decrease of the fn. For a fn $f(x,y)$ this direction is given by the -ve gradient of f, denoted as $-\nabla f(x,y)$. For a fn $f(x,y)$ the gradient ∇f is a vector of partial derivatives i.e, ~~$\nabla f =$~~

$$\nabla f = i\frac{df}{dx} + j\frac{df}{dy} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

The direction of steepest descent is given by $-\nabla f(x,y)$, the algorithm updates the current point (x_k, y_k) to the new point (x_{k+1}, y_{k+1}) using the update rule $(x_{k+1}, y_{k+1}) = (x_k, y_k) - \alpha \nabla f(x_k, y_k)$, where α is the stepsize or learning rate.

Q. Find the minimum of $f(x,y) = x^2 + 3y^2$, starting from the point $(x_0, y_0) = (6,3)$ where $\alpha = 0.1$ using the method steepest descent.

Choose alpha 0.1 when it's not given.

~~$\nabla f(x,y)$~~

The gradient of f is given

$$\nabla f(x,y) = i\frac{\partial}{\partial x}(x^2+3y^2) + j\left(\frac{\partial}{\partial y}(x^2+3y^2)\right)$$

$$= 2xi + 6yj = (2x, 6y) \Rightarrow \nabla f(6,3) = (2x, 6y) = (12, 18)$$

The update rule for steepest descent is:

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - \alpha \nabla f(x_k, y_k)$$

where α is the steepest size or learning rate

$\alpha = 0.1$

Iteration 0 (initial point)

$(x_0, y_0) = (6, 3)$

Iteration 1 (

$(x_1, y_1) = (x_0, y_0) - \alpha \nabla f(x_0, y_0)$

$= (6,3) - 0.1(12, 18) = (6,3) - (1.2, 1.8)$

$= (4.8, 1.2)$

$\nabla f(4.8, 1.2) = (2x, 6y) = (9.6, 7.2)$

$(x_2, y_2) = (x_1, y_1) - \alpha \nabla f(x_1, y_1)$

$= (4.8, 1.2) - 0.1(9.6, 7.2)$

$= (4.8, 1.2) - (0.96, 0.72)$

$= (3.84, 0.48)$

$$\begin{array}{r} 4.8 \\ - 0.96 \\ \hline 3.96 \end{array}$$

$\nabla f(3.84, 0.48) = (7.68, 2.88)$

$(x_3, y_3) = (x_2, y_2) - \alpha \nabla f(x_2, y_2)$

$= (3.84, 0.48) - 0.1(7.68, 2.88)$

$= (3.84, 0.48) - (0.768, 0.288)$

$= (3.072, 0.192)$

$\nabla f(3.072, 0.192) = (6.144, 1.152)$

$(x_4, y_4) = (x_3, y_3) - \alpha \nabla f(x_3, y_3)$

$= (3.072, 0.192) - 0.1(6.144, 1.152)$

$= (3.072, 0.192) - (0.6144, 0.1152)$

$= (2.4576, 0.0768)$ //

$(x_5, y_5) = (x_4, y_4) - \alpha \nabla f(x_4, y_4)$

$= (2.4576, 0.0768) - 0.1(4.9152, 0.4608)$

$= (2.4576, 0.0768) - (0.49152, 0.04608)$

$= (1.96608, 0.03072)$

After 5 iterations using the method of steepest descent with a step size of $\alpha = 0.1$, the point has been updated from $(6, 3)$ to app. $(1.96608, 0.03072)$.

Q. Minimize quadratic function $f(x, y) = 3x^2 + 4y^2$ starting from the point $(x_0, y_0) = (1, 1)$ using the method of steepest descent with a fixed step size $\alpha = 0.01$ iterate 5 steps.

$\alpha = 0.01$

$\nabla f(x, y) = 6xi + 8yj$

i.e.

$(x_0, y_0) = (1, 1)$

Iteration 1 : $(x_0, y_0) - \alpha \nabla f(x_0, y_0)$

$= (1, 1) - 0.01(6x\hat{i} + 8y\hat{j})$

$= (1, 1) - (0.06\hat{i} + 0.08\hat{j})$

$(x_1, y_1) = (0.94, 0.92)$

$(x_2, y_2) = (x_1, y_1) - \alpha \nabla f(x_1, y_1)$

$= (0.94, 0.92) - 0.01(5.64, 7.36)$

$= (0.94, 0.92) - (0.0564, 0.0736)$

$= (0.8836, 0.8464)$

$(x_3, y_3) = (0.8836, 0.8464) - 0.01(5.3016, 6.7712)$

$= (0.8836, 0.8464) - (0.053016, 0.067712)$

$= (0.830584, 0.778688)$

$(x_4, y_4) = (0.830584, 0.778688) - 0.01(4.983504, \cancel{4.672128}\ 6.229504\ \cancel{4.672128})$

$= (0.78074896, 0.73196672)$

$(x_5, y_5) = (0.7807, 0.7319) - 0.01(4.6842, 5.8552)$

$= (0.7338, 0.6733)$

Q. Minimize the function $f(x,y) = x^2 + y^2$ using steepest descent.

Let initial value, $(x_0, y_0) = (1, 1)$

Let step size $\alpha = 0.1$

$\nabla f(x,y) = 2x\hat{i} + 2y\hat{j}$

$(x_1, y_1) = (1,1) - 0.1(2, 2)$

$= (0.8, 0.8)$

$(x_2, y_2) = \cancel{(0.8, 0.2) - 0.1(0.}\ (0.64, 0.64)$

$(x_3, y_3) = (0.512, 0.512)$

Q. Minimize the fn $f(x,y) = (x-2)^2 + (y+3)^2$ using steepest descent method to find the minimum value for the fn. starting from the point $(x_0, y_0) = (0, 0)$ and the learning rate is 0.1.

$\nabla f(x,y) = (2x-4)\hat{i} + (2y+6)\hat{j}$

$(x_1, y_1) = (0,0) - 0.1(2x-4, 2y+6)$

$= (0,0) - 0.1(-4, 6)$

$= (0,0) - (-0.4, 0.6)$

$= (0.4, -0.6)$

$(x_2, y_2) = \cancel{(0.4, -0.6) - 0.1(}\ (0.72, -1.08)$

Q. Maximize $f = 5x_1 + 25x_2$, subject to the constraints $x_1 - 3x_2 \geq -6$, $x_1 + x_2 \leq 6$, $x_1, x_2 \geq 0$. Solve this eqn graphically.

$f = 5x_1 + 25x_2$

$x_1 - 3x_2 \geq -6$

$x_1 - 3x_2 + 6 = 0$

$x_1 = 0 \Rightarrow -3x_2 = 6$ $\quad (0, 2)$

$x_2 = 2$

$x_2 = 0 \Rightarrow x_1 = 6$ $\quad (6, 0)$

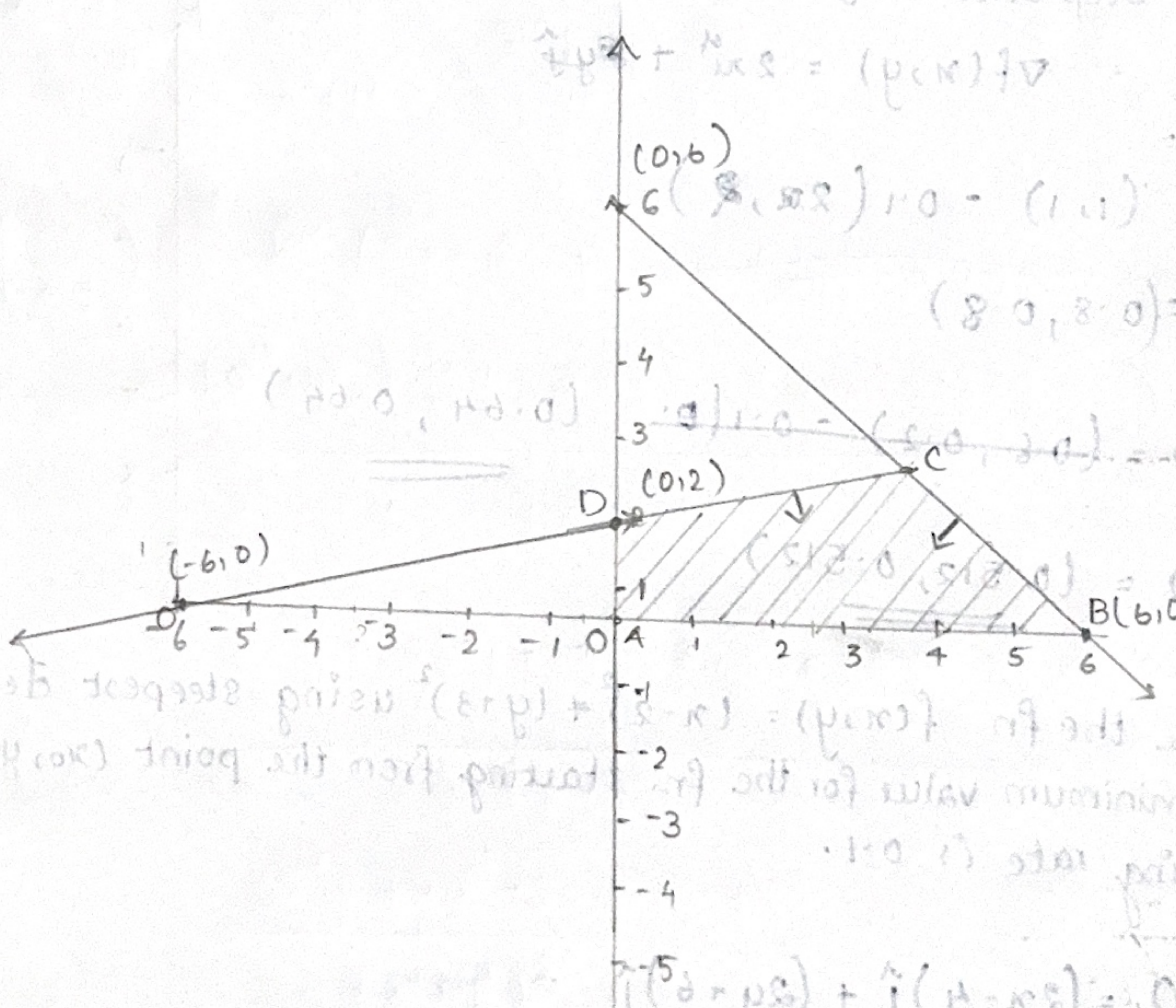
$x_1 + x_2 \leq 6$

$x_1 + x_2 - 6 = 0$

$x_1 = 0 \Rightarrow x_2 = 6$ $\quad (0, 6)$

$x_2 = 0 \Rightarrow x_1 = 6$ $\quad (6, 0)$

In $\leq$ case: for the bounding region direction, upwards: both x and y should be +ve if one of the cordinate is negative should put values and check if its true and direction will be downward



$A(0,0)$, $B(6,0)$, $C(3,3)$, $D(0,2)$

$f = 5x_1 + 25x_2$.

$(0,0) \Rightarrow f = 0$.

$(6,0) \Rightarrow f = 30$

$(3,3) \Rightarrow f = 15 + 75 = 90$

$(0,2) \Rightarrow f = 50$

Maximum value $= 90$ at x $(3,3)$

A furniture manufacturer makes two product chairs and table with the help of 2 machines A and B. A chair requires 2 hours on machine A, 6 hours on machine B. A table requires 5 hrs of ~~mac~~ A and 1 hrs on B. there are 16 hrs per day available on machine A and 30 hrs on machine B. Profit gain by the manufacturer from a chair and a table is ₹2 and ₹10 respectively. formulate this ~~prob~~ prob. as a linear prgraming prob to maximize the total profit of the manufacturer also using graphical method to solve this LPP.

Let x_1 and x_2 be the units

~~maximize~~

maximize, $Z = 2x_1 + 10x_2$

Subject constraint to:

$2x_1 + 5x_2 \leq 16$ → machine A

$6x_1 + x_2 \leq 30$ → machine B

$x_1, x_2 \geq 0$.

x_1	x_2
0	16/5
8	0

x_1	x_2
0	30
5	0

$2x_1 + 5x_2 = 16$

$6x_1 + x_2 = 30$

$30x_1 + 5x_2 = 150$

$2x_1 + 5x_2 = 16$

$28x_1 = 134$

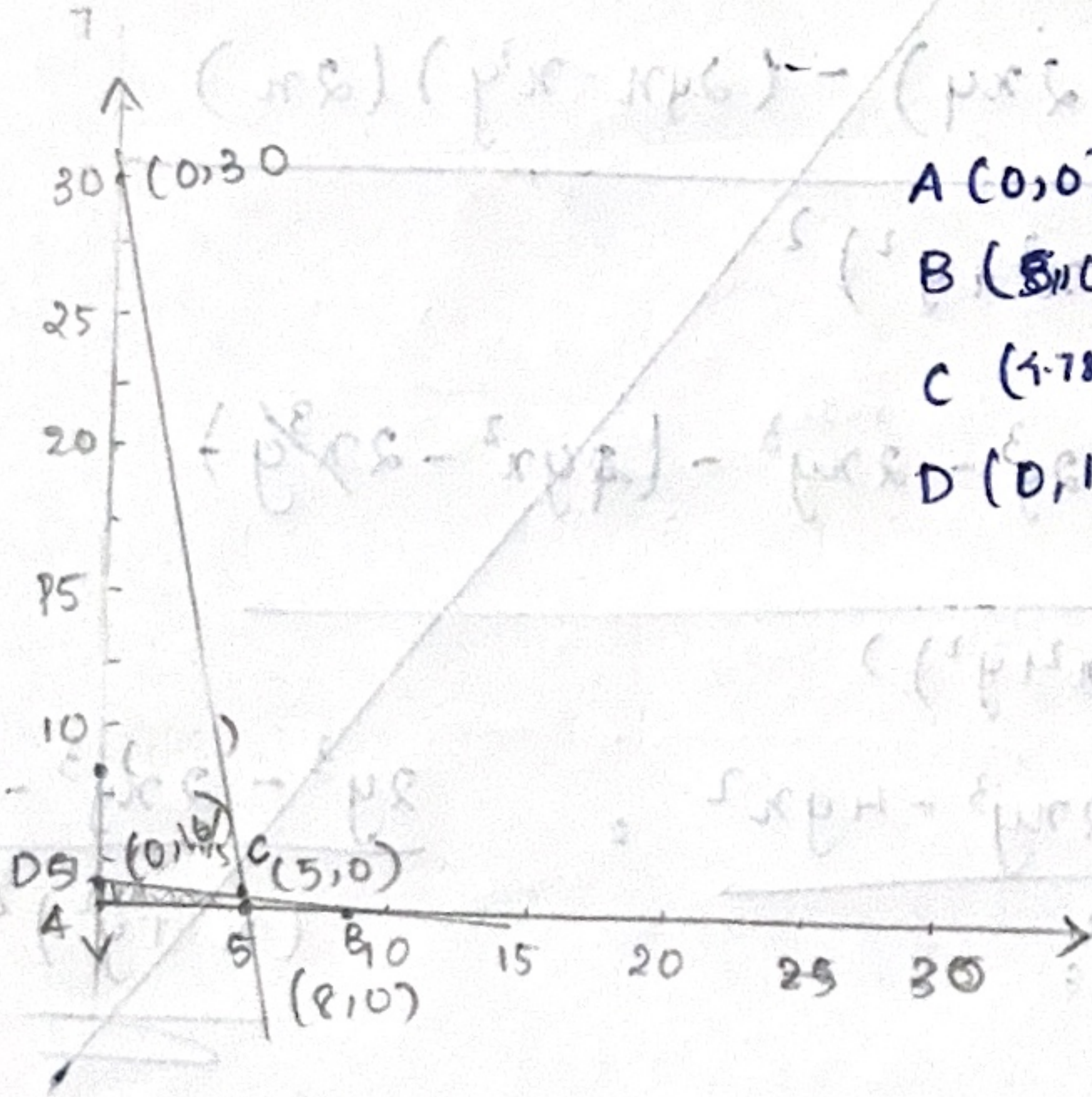
$x_1 = 4.78$

$x_2 \Rightarrow 2x_1 + 5x_2 = 16$

$5x_2 = 16 - 9.56$

$x_2 = \frac{6.44}{5}$

$x_2 = 1.2885$



A (0,0)

B (8,0)

C (4.78, 1.288)

D (0, 16/5)

$Z = 2x_1 + 10x_2$

A(0,0) = 0

B(8,0) = ~~16~~ 10

C (4.78, 1.28) = 2×4.78 + 10×1.28

= 9.56 + 12.8

= 22.36

D(0, 16/5) = $\cancel{10}^{2} \times \frac{16}{\cancel{5}}$ = 32

Max. value is 32 at (0, 16/5)

$$\begin{array}{r} 12.8 \\ 9.56 \\ \hline 22.36 \end{array}$$