B. Tech S_1 Examination December 2021

Linear Algebra and Calculus

Part A

1. Find the rank of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$

soln: $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix}$ $R_1 \longleftrightarrow R_2$

$\sim \begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 4R_2$

$\therefore$ Rank $= 2$

2. Find the Eigen values of the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ what are the eigen values of A^2, A^{-1} without using it characteristic equation.

soln: trace of $A = \lambda_1 + \lambda_2$

$\therefore \lambda_1 + \lambda_2 = 4$ —— (1)

$|A| = \lambda_1 \lambda_2 = 7$ $\lambda_1 \lambda_2 = 3$ —— (2)

From eqn (1) and (2) $\lambda_1 = 1$, $\lambda_2 = 3$

Eigen values of A are 1, 3

Eigen values of A^2 are $1^2, 3^2$ ie 1, 9

Eigen values of A^{-1} are 1, 1/3

Part B

11 a) Solve the following linear system of equations using Gauss elimination method $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$

Soln:

$$\tilde{A} = \begin{bmatrix} 1 & 2 & -1 & : 3 \\ 3 & -1 & 2 & : 1 \\ 2 & -2 & 3 & : 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \end{bmatrix} \quad R_3 \to R_3 - \frac{6}{7} R_2$$

rank A = rank $\tilde{A}$ = no. of unknowns = 3

$\therefore$ System of equation have unique solution

By Back substitution, $\frac{5}{7} z = \frac{20}{7} \Rightarrow z = 4$

$-7y + 5z = -8 \Rightarrow 5z = 7y - 8 \Rightarrow y = 4$

$x + 2y - z = 3 \Rightarrow x + 2\times 4 - 4 = 3$

$\Rightarrow x = -1$

$\therefore$ Solution is $x = -1$, $y = 4$, $z = 4$

11 b) Find the Eigen values and Eigen vectors of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Soln: Characteristic equation is $\lambda^3 - \text{trace } A \lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$

$$\lambda^3 - 18\lambda^2 + [5+20+20]\lambda - 0 = 0$$

Eigen values of are $\lambda = 0, 3, 15$

Eigen vector corresponding to $\lambda = 0$

Consider $(A - \lambda I).x = 0$

$$A - 0I = A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 20/8 & \frac{-10}{4} \\ 0 & \frac{-10}{4} & \frac{5}{2} \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + \frac{6}{8} R_1 \\ R_3 \to R_3 - \frac{2}{8} R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 20/8 & -10/4 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

By Back substitution $\frac{20}{8} x_2 - \frac{10}{4} x_3 = 0$

$$x_2 = x_3$$

Let $x_2 = 2 \Rightarrow x_3 = 2$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$8x_1 = 6x_2 - 2x_3 = \frac{6x_2 - 2x_3}{8} \Rightarrow x_1 = \cancel{x_2}\ 1$$

Eigen vector corresponding to $\lambda = 0$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 3$.

Consider $(A-3I)X = 0$

$$A - 3I = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16/5 & -8/5 \\ 0 & -8/5 & -4/5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - \frac{-6}{5} R_1 \\ R_3 \to R_3 - \frac{2}{5} R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16/5 & -8/5 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - \frac{1}{2} R_2$$

By Back substitution $-\frac{16}{5} x_2 - \frac{8}{5} x_3 = 0$

$-2x_2 = x_3$

Let $x_2 = 1$

$x_3 = -2$

$5x_1 - 6x_2 + 2x_3 = 0$

$5x_1 = 6x_2 - 2x_3 = 6 \times 1 - 2 \times -2 = 10$

$x_1 = 2$

Eigen vector corresponding to $\lambda = 3$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 15$:

Consider $(A - 15I)x = 0$

$$A - 15I = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & -6 & +2 \\ 0 & \frac{-20}{7} & \frac{-40}{7} \\ 0 & \frac{-40}{7} & \frac{-80}{7} \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - \frac{6}{7} R_1 \\ R_3 \to R_3 + \frac{2}{7} R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ 0 & -20/7 & -40/7 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

By Back substitution,

$$-\frac{20}{7} x_2 - \frac{40}{7} x_3 = 0$$

$$-x_2 - 2x_3 = 0 \Rightarrow x_2 = -2x_3$$

let $x_3 = 1$

$$x_2 = -2$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$7x_1 = -6x_2 + 2x_3 = -6 \times -2 + 2 \times 1 = 14$$

$$x_1 = 2$$

Eigen vector corresponding to $\lambda = 15$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ Matrix of transformation that diagonlalize the matrix

$$P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Diagonal matrix $D = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

12 a) Solve the following linear system of equations using Gauss elimination $2x-2y+4z=0$, $-3x+3y-6z+5w=15$, $x-y+2z=0$

$$\tilde{A} = \begin{bmatrix} 2 & -2 & 4 & 0 & 0 \\ -3 & 3 & -6 & 5 & 15 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 15 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - \frac{-3}{2} R_1 \\ R_3 \to R_3 - \frac{1}{2} R_1 \end{matrix}$$

$\therefore$ Rank A = Rank $\tilde{A} = 2 <$ no. of variables

$\therefore$ System of equations have infinite no. of solutions.

By Back substitution,

$5w = 15 \Rightarrow w = 3$

$2x - 2y + 4z = 0$

let $y = t_1$, $z = t_2$

$2x = 2y - 4z = 2t_1 - 4t_2$

$x = t_1 - 2t_2$

$\therefore$ Solutions are $x = t_1 - 2t_2$, $y = t_1$, $z = t_2$, $w = 3$

12 b) Find the matrix of transformation that diagonalize the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. Also write the diagonal matrix.

Soln: Characteristic equation is $\lambda^2 - \text{trace } A\lambda + |A| = 0$

$$\lambda^2 - -7\lambda + 6 = 0$$

Eigen values are $\lambda = -1, -6$

Eigen vector corresponding to $\lambda = -1$

consider $(A - \lambda I)x = 0$

$$A + I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - \frac{2}{-4} R_1$$

By Back substitution, $-4x_1 + 2x_2 = 0$

$4x_1 = 2x_2 \Rightarrow 2x_1 = x_2$

Let $x_1 = 1 \Rightarrow x_2 = 2$

Eigen vector corresponding to $\lambda = -1$

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = -6$

Consider $(A - -6I)X = 0$

$$A + 6I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

By Back substitution,

$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

Let $x_2 = 1$ $\quad x_1 = -2$

Eigen vector corresponding to $\lambda = -6$

$$X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The matrix of transformation that diagonalize the matrix

$$P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$$

③ If $z = \dfrac{xy}{x^2+y^2}$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

$$z_x = \frac{y^3 - x^2 y}{(x^2+y^2)^2}.$$

Module 2

$$\frac{\partial z}{\partial y} = \frac{x^3 - y^2 x}{(x^2+y^2)^2}.$$

4 Show that the eqn $u(x,t) = \sin(x-ct)$ Satisfies wave eqn $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial^2 u}{\partial t^2} = u_{tt}$$

$$u_t = \cos(x-ct) \times -c$$

$$u_{tt} = -c^2 \times -\sin(x-ct)$$

$$= c^2 \sin(x-ct) \longrightarrow (1)$$

$$u_x = \cos(x-ct) \times 1$$

$$u_{xx} = -\sin(x-ct) \longrightarrow (2)$$

From eqn (1) and (2)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

13. a) The length and width of a rectangle are measured with errors of at most 3% and 4% respectively. Use differentials to approximate the max. % error in the calculated area.

b) Find the local linear approximation L of $f(x,y,z) = xyz$ at the point $P(1,2,3)$. Compute the error in approximation f by L

at the point $Q(1.001, 2.002, 3.003)$.

a) Area

$$A = xy$$

$$A_0 = x_0 y_0$$

$$dA = A_x(x_0 y_0)\,dx + A_y(x_0 y_0)\,dy$$

$$A = xy$$

$$A_x = y \quad , \quad A_y = x$$

$$A_x(x_0 y_0) = y_0$$

$$A_y(x_0 y_0) = x_0$$

$$dA = y_0\,dx + x_0\,dy$$

If $dx = \Delta x$ and $dy = \Delta y$

$$\Delta A \approx dA$$

$$\Delta x = x - x_0$$

$$\Delta y = y - y_0$$

$$\left|\frac{\Delta x}{x_0}\right| \leq 0.03$$

$$\left|\frac{\Delta y}{y_0}\right| \leq 0.04$$

$$\frac{\Delta A}{A_0} = \frac{y_0 \Delta x + x_0 \Delta y}{x_0 y_0}$$

$$= \frac{\Delta x}{x_0} + \frac{\Delta y}{y_0}$$

$$\left|\frac{dA}{A_0}\right| \le \left|\frac{\Delta x}{x_0}\right| + \left|\frac{\Delta y}{y_0}\right|$$

$$= 0.03 + 0.04$$

$$= 0.07$$

$$= 7\%$$

$\therefore$ The max. % error is area

$$= 7\%$$

b) $L(x,y,z) = f(x_0, y_0 z_0) + f_x(x_0 y_0 z_0)(x - x_0) + f_y(x_0 y_0 z_0)$
$(y - y_0) + f_z(x_0 y_0 z_0)(z - z_0)$

$(x_0 y_0 z_0) = 1, 2, 3$

$f(1,2,3) = 6$

$f_x = yz = 6$

$f_y = xz$
$= 3$

$f_z = xy = 2$

$L(x,y,z) = 6 + 6(1.001 - 1) + 3(2.002 - 2) + 2(3.003 - 3)$

$= 6.018$

Error $= 0.018$

distance b/w ~~the~~ p and Q

$$= \sqrt{(1.001-1)^2+(2.002-2)^2+(3.003-3)^2}$$

$$= \sqrt{0.01}$$

$\therefore$ Error is less than distance b/w the points

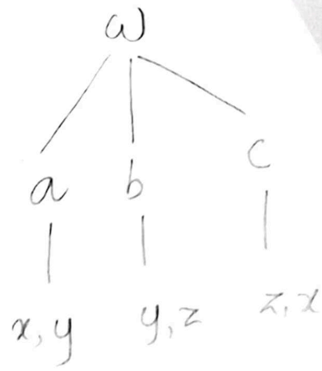
14 a) Let f be a differentiable function of three variables and suppose that

$w = f(x-y, y-z, z-x)$, show that $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$

b) Locate all relative extrema of $f(x,y)$ =

$f(x,y) = 4xy - y^4 - x^4$

a) Let $(x-y) = a$, $(y-z) = b$, $(z-x) = c$



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial a} \times \frac{\partial a}{\partial x} + \frac{\partial w}{\partial c} \cdot \frac{\partial c}{\partial x}$$

$$= \frac{\partial w}{\partial a} \times 1 + \frac{\partial w}{\partial c} \times -1$$

$$= \frac{\partial w}{\partial a} - \frac{\partial w}{\partial c} \longrightarrow (1)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial a} \times \frac{\partial a}{\partial y} + \frac{\partial w}{\partial b} \times \frac{\partial b}{\partial y}$$

$$= \frac{\partial w}{\partial a} \times -1 + \frac{\partial w}{\partial b} \times 1$$

$$= -\frac{\partial w}{\partial a} + \frac{\partial w}{\partial b} \longrightarrow (2)$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial b} \times \frac{\partial b}{\partial z} + \frac{\partial w}{\partial c} \times \frac{\partial c}{\partial z}$$

$$= \frac{\partial w}{\partial b} \times -1 + \frac{\partial w}{\partial c} \times 1$$

$$= -\frac{\partial w}{\partial b} + \frac{\partial w}{\partial c} \longrightarrow (3)$$

From eqn (1), (2) and (3)

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

b) $f(x,y) = 4xy - y^4 - x^4$

$f_x = 4y - 4x^3$

$f_y = 4x - 4y^3$

Critical point is obtained by solving

$f_x = 0$ and $f_y = 0$

$f_x = 0$

$\Rightarrow 4y - 4x^3 = 0$

$y = x^3 \longrightarrow (1)$

$f_y = 0$

$\Rightarrow 4x - 4y^3 = 0$

$x = y^3 \longrightarrow (2)$

Substitute eqn (1) in (2)

$x = (x^3)^3$

$x = x^9$

$x - x^9 = 0$

$x(1 - x^8) = 0$

$x = 0$

$1 - x^8 = 0$

$x^8 = 1 \Rightarrow x = 1$ or -1

$y = x^3$

If $x = 0$, $y = 0$

$x = 1$, $y = 1$

$x = -1$, $y = -1$

$\therefore$ The critical points are

$(0,0)$ $(1,1)$ $(-1,-1)$

$D = f_{xx}(x_0 y_0) f_{yy}(x_0 y_0) - f_{xy}^2(x_0 y_0)$

$f_{xx} = -12x^2$

$f_{yy} = -12y^2$

$f_{xy} = 4$

<u>at $(0,0)$</u>

$D = (-12 \times 0)(-12 \times 0) - 4^2$

$= -16$

$D < 0$

$\therefore$ $(0,0)$ is a saddle point.

at (1,1)

$D = (-12 \times 1^2) \times (-12 \times 1^2) - 4^2$

$= (-12 \times -12) - 4^2$

$= 144 - 16$

$= 128$

$D > 0$

$\therefore f_x = -12$

$f_x < 0$

$\therefore$ ~~(x,y)~~ f has relative maximum at (1,1)

at (-1,-1)

$D = -12 \times (-1)^2 \times -12 \times (-1)^2 - 4^2$

$= 144 - 16$

$= 128$

$D > 0$

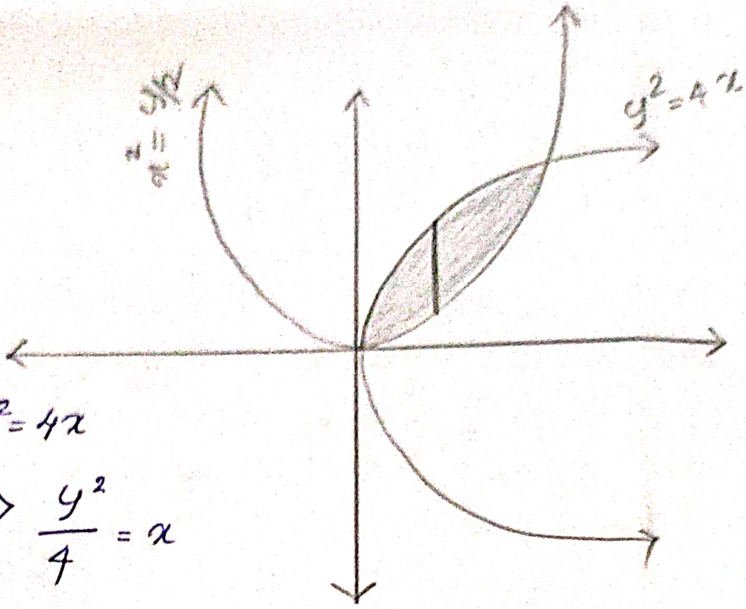
$f_x = -12 \quad , f_x < 0$

$\therefore$ f has relative maximum at (-1,-1)

Module - 3

15. a) Find the area bounded by the parabolas $y^2 = 4x$ and $x^2 = \frac{y}{2}$

b) Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy$ by using polar co-ordinates.



ans:- a) The limits of y are,

$y = 2x^2, \quad y = 2\sqrt{x}$

Given $x^2 = \frac{y}{2}$, and $y^2 = 4x$

$\Rightarrow \frac{y^2}{4} = x$

$$\frac{y^4}{16} = \frac{y}{2}$$

$\Rightarrow y^3 = 8 \Rightarrow y = 2$

$x^2 = \frac{y}{2} \Rightarrow x^2 = 1 \Rightarrow x = 1$

$\therefore$ limits of x are $x = 0$ and $x = 1$.

$$\text{Area} = \iint_R dA = \int_0^1\int_{2x^2}^{2\sqrt{x}} dy\,dx = \int_0^1 [y]_{2x^2}^{2\sqrt{x}}\,dx$$

$$= \int_0^1 (2\sqrt{x} - 2x^2)\,dx$$

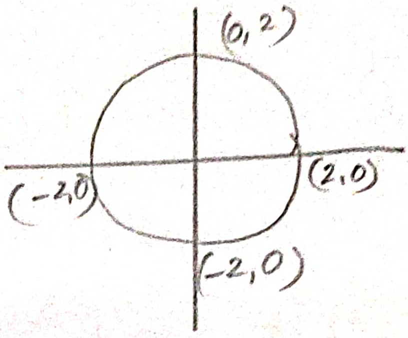
$$= 2\left[\frac{2}{3}\left[x^{3/2}\right]_0^1 - \frac{1}{3}\left[x^3\right]_0^1\right]$$

$$= 2\left[\frac{2}{3} - \frac{1}{3}\right] = \underline{\underline{\frac{2}{3}}}$$

b) To change into polar co-ordinates,

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$, $dy\,dx = r\,dr\,d\theta$.

$$\int_0^{2\pi}\int_0^{2} e^{-r^2} r\,dr\,d\theta$$



$$= \int_0^{2\pi}\int_0^{2} \frac{e^{-t}}{2}\,dt\,d\theta$$

put
$t = r^2$
$\frac{dt}{dr} = 2r$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{e^{-t}}{-1}\right]_0^2 d\theta$$

$$= \frac{-1}{2}\int_0^{2\pi} \left[e^{-r^2}\right]_0^2 d\theta \quad = \frac{-1}{2}\int_0^{2\pi}\left[e^{-4}-e^{0}\right]d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}(e^{-4}-1)\,d\theta$$

$$= -\frac{1}{2}(e^{-4}-1)\,[\theta]_0^{2\pi}$$

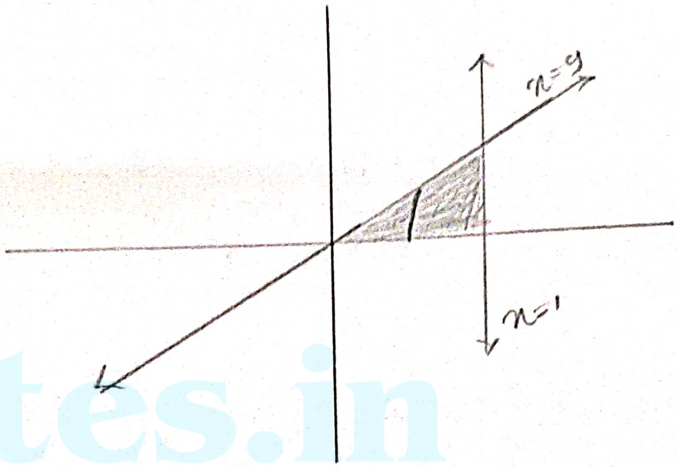
$$= \underline{\underline{\pi(1-e^{-4})}}$$

16. a) Evaluate $\int_0^1\int_y^1 \frac{x}{x^2+y^2}\,dx\,dy$ by reversing the order of integration

b) Use triple integral to find the volume of the solid within the cylinder $x^2+y^2=9$ and between the planes $z=1$ and $x+z=5$.

b) polar co-ordinates

ans:- a) $\int_0^1 \int_y^1 \frac{x}{x^2+y^2}\, dx\, dy$



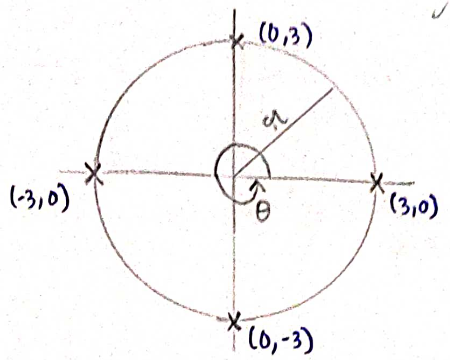
$= \int_0^1 \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$

$= \int_0^1 x \left[\frac{1}{x} \tan^{-1} \frac{y}{x}\right]_0^x dx$

$= \int_0^1 \frac{x}{x} \left[\tan^{-1} 1 - \tan^{-1} 0\right] dx$

$= \int_0^1 \frac{\pi}{4}\, dx \quad = \frac{\pi}{4} [x]_0^1 \quad = \underline{\underline{\frac{\pi}{4}}}$

b) $z \to 1$ to $5-x$



$x^2+y^2=9$

r varies from 0 to 3

θ varies from 0 to 2π

Using triple integrals

$$\int_0^{2\pi}\int_0^3\int_1^{5-x} dz\, r\,dr\,d\theta$$

$$\Rightarrow \int_0^{2\pi}\int_0^3 [z]_1^{5-x}\, r\,dr\,d\theta \Rightarrow \int_0^{2\pi}\int_0^3 5-x-1\; r\,dr\,d\theta$$

$$\Rightarrow \int_0^{2\pi}\int_0^3 (5-r\cos\theta-1)\,r\,dr\,d\theta \Rightarrow \int_0^{2\pi}\int_0^3 (5r-r^2\cos\theta-r)\,dr\,d\theta$$

$$\Rightarrow \int_0^{2\pi}\left[5\frac{r^2}{2}\right]_0^3-\left[\cos\theta\,\frac{r^3}{3}\right]_0^3-\left[\frac{r^2}{2}\right]_0^3 d\theta \Rightarrow \int_0^{2\pi}\left(\frac{45}{2}-9\cos\theta-\frac{9}{2}\right)d\theta$$

$$\Rightarrow \int_0^{2\pi} 18-9\cos\theta\; d\theta \Rightarrow \int_0^{2\pi} 18\,d\theta - \int_0^{2\pi} 9\cos\theta\,d\theta$$

$$\Rightarrow \Big[18\,[\theta]_0^{2\pi} - 9\,[\sin\theta]_0^{2\pi}\Big]_0^{2\pi} \Rightarrow 36\pi$$

5. $\int_0^3 \int_0^2 \int_0^1 xyz \, dx\, dy\, dz = \int_0^3 \int_0^2 yz \left[\frac{x^2}{2}\right]_0^1 dy\, dz$

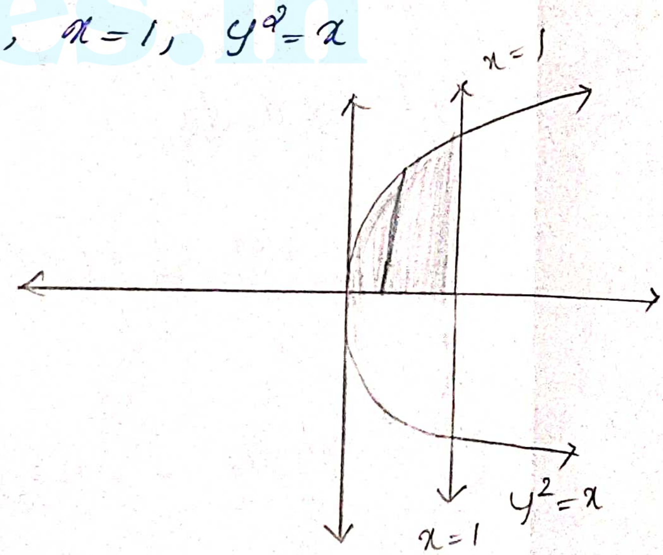
$$= \frac{1}{2} \int_0^3 z \cdot \left[\frac{y^2}{2}\right]_0^2 dz$$

$$= \frac{1}{2} \times \frac{4}{2} \int_0^3 z \, dz$$

$$= \frac{1}{2} \times 2 \times \left[\frac{z^2}{2}\right]_0^3 = 1 \times \frac{9}{2}$$

$$= \frac{9}{2}$$

6. Find the mass of the lamina with density $\delta(x,y) = x+2y$ is bounded by the x-axis, $x=1$, $y^2 = x$



ans:- $\int_0^1 \int_0^{\sqrt{x}} (x+2y)\, dy\, dx$

$$= \int_0^1 \left[x[y]_0^{\sqrt{x}} + [y^2]_0^{\sqrt{x}} \right] dx$$

$$= \int_0^1 \left(x^{3/2} + x \right) dx$$

$$= \frac{2}{5}\left[x^{5/2}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 = \frac{2}{5} + \frac{1}{2}$$

$$= \frac{9}{10}$$

MODULE - IV Dec-2021

PART-A

7. Find the rational number represented by the repeating decimal $5.373737....$

Ans: $5.373737... = 5 + .37 + .0037 + .00037 + ...$

$$.37 + .0037 + .00037 + ... = \frac{37}{99}$$

$$\therefore 5.373737... = 5 + \frac{37}{99} = \underline{\underline{\frac{532}{99}}}$$

8. Examine the convergence of $\sum_{k=1}^{\infty} \frac{k^2}{2k^2+3}$.

Ans: Consider $\sum \frac{k^2}{k^2} = \sum 1$ which is divergent.

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^2}{2k^2+3} = \lim_{k\to\infty} \frac{k^2}{k^2\left(2+\frac{3}{k^2}\right)}$$

$$= 1 > 0 \text{ \& finite.}$$

by limit comparison test, since $\sum 1$ is divergent

$\sum_{k=1}^{\infty} \frac{k^2}{2k^2+3}$ is also divergent.

Dec-2021

Module - 4

17a) Test the convergence of (i) $\sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$

(ii) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

Soln: i) Let $a_k = \frac{3k^3-2k^2+4}{k^7-k^3+2}$

$$b_k = \frac{3}{k^4}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2} \cdot \frac{k^4}{3}$$

$$= \lim_{k\to\infty} \frac{3k^7-2k^6+4k^4}{3(k^7-k^3+2)}$$

$$= \lim_{k\to\infty} \frac{3k^7\left(1-\frac{2k^6}{3k^7}+\frac{4k^4}{3k^7}\right)}{3k^7\left(1-\frac{k^3}{k^7}+\frac{2}{k^7}\right)} = 1$$

By limit comparison test $\sum a_k$ converges since $\sum b_k$ is a convergent series by p-test.

ii) $a_k = \frac{k^k}{k!}$

$$a_{k+1} = \frac{(k+1)^{k+1}}{(k+1)!}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \frac{k!}{k^k} = \lim_{k\to\infty} \frac{(k+1)^k \cdot (k+1)}{k^k \cdot (k+1)}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e > 1$$

By ratio test, the series diverges.

17 b) Test whether the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k+1}}$ is absolutely convergent or conditionally convergent.

Ans:

$$\text{Let } a_k = \frac{1}{\sqrt{k+1}}$$

$$\underset{k\to\infty}{Lt}\ a_k = \underset{k\to\infty}{Lt} \frac{1}{\sqrt{k+1}} = 0$$

$$\frac{a_{k+1}}{a_k} = \frac{1}{\sqrt{k+1}+1} \cdot \sqrt{k+1} < 1$$

$\therefore$ by Leibnitz test, the series converges.

$$u_k = (-1)^{k+1} \frac{1}{\sqrt{k+1}}$$

$$|u_k| = \frac{1}{\sqrt{k+1}}$$

$$\therefore\ v_k = \frac{1}{\sqrt{k}}$$

$$|u_k| < v_k$$

by limit comparison test, $\Sigma |u_k|$ diverges

$\Rightarrow \Sigma u_k$ diverges absolutely.

$\Rightarrow$ conditionally convergent.

18.a) Test the convergence of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

Ans:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

$$\text{Let } a_k = \frac{1}{k(k+1)}, \quad b_k = \frac{1}{k^2}$$

$$\underset{k\to\infty}{Lt} \frac{a_k}{b_k} = \underset{k\to\infty}{Lt} \frac{k^2}{k^2+1} = 1$$

$\therefore$ by limit comparison test, Σa_k converges since Σb_k is a convergent p-series.

(8b) Test the convergence of i) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$ (ii) $\sum_{k=1}^{\infty} \frac{7^k}{k!}$

Ans: i) Let $a_k = \left(\frac{k}{k+1}\right)^{k^2}$

by root test $\underset{k\to\infty}{Lt} (a_k)^{1/k} = \underset{k\to\infty}{Lt} \left[\left(\frac{k}{k+1}\right)^{k^2}\right]^{1/k}$

$$= \underset{k\to\infty}{Lt} \left(\frac{k}{k+1}\right)^{k} = \underset{k\to\infty}{Lt} \frac{k^k}{(k+1)^k}$$

$$= \underset{k\to\infty}{Lt} \frac{k^k}{k^k\left(1+\frac{1}{k}\right)^k} = \frac{1}{e} < 1$$

$\therefore$ the series converges.

ii) $a_k = \frac{7^k}{k!}$

$a_{k+1} = \frac{7^{k+1}}{(k+1)!}$

by ratio test, $\underset{k\to\infty}{Lt} \frac{a_{k+1}}{a_k} = \underset{k\to\infty}{Lt} \cancel{\frac{(k+1)!}{7^{k+1}} \cdot \frac{7^k}{k!}}$

$$= \underset{k\to\infty}{Lt} \frac{7^{k+1}}{(k+1)!} \cdot \frac{k!}{7^k}$$

$$= \underset{k\to\infty}{Lt} \frac{7}{k+1} = 0 < 1$$

$\therefore \Sigma a_k$ converges.

PART A

9. Find the Taylor series expansion of $f(x) = \sin \pi x$ about $x = \frac{1}{2}$

Soln: Taylor series expansion of $f(x)$ about $x = x_0$ is $f(x) = f(x_0) + \frac{x - x_0}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots$

$f(x) = \sin \pi x$

$f(1/2) = \sin \pi/2 = 1$

$f'(x) = \cos \pi x \times \pi$

$f'(1/2) = \pi \times \cos \pi/2$

$= 0$

$f''(x) = \pi \times \bar{\ } \sin \pi x \times \pi$

$= -\pi^2 \sin \pi x$

$f''(1/2) = -\pi^2 \sin \pi/2$

$= -\pi^2$

$f'''(x) = -\pi^3 \cos \pi x$

$f'''(1/2) = -\pi^3 \times \cos \pi/2$

$= 0$

$f^{IV}(x) = +\pi^4 \sin \pi x$

$f^{IV}(1/2) = \pi^4 \sin \pi/2$

$= \pi^4$

$\therefore f(x) = 1 + \frac{x - 1/2}{1!} \times 0 + \frac{(x-1/2)^2}{2!} \times -\pi^2 + \frac{(x-1/2)^3}{3!} \times 0 + \frac{(x-1/2)^4}{4!} \times \pi^4 + \ldots$

$$f(x) = 1 - \pi^2 \frac{(x-1/2)^2}{2!} + \pi^4 \frac{(x-1/2)^4}{4!} - \cdots$$

10. If $f(x)$ is a periodic function with period 2π defined in $[-\pi, \pi]$. Write the Euler's formula a_0, a_n, b_n for $f(x)$.

$$a_0 = \frac{1}{2L} \int_{\alpha}^{\alpha+2L} f(x)\, dx$$

$$\therefore \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$\therefore \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\therefore \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

19 a) Find the fourier series expansion of

$f(x) = x - x^2$ in the range $(-1, 1)$.

Soln: Fourier series expansion of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{2l} \int_{\alpha}^{\alpha+2l} f(x)\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} x - x^2 \, dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{3} - \left(\frac{1}{2} - \frac{-1}{3}\right)\right]$$

$$= \frac{1}{2} \times \frac{-2}{3} = \frac{-1}{3}$$

$$a_n = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_{-1}^{1} (x - x^2) \cos(n\pi x)\, dx$$

$$= (x - x^2) \frac{\sin(n\pi x)}{n\pi} - \left[(1 - 2x) \cdot \frac{-\cos(n\pi x)}{n^2\pi^2}\right.$$

$$\left. + (-2) \cdot \left(\frac{-\sin(n\pi x)}{n^3\pi^3}\right)\right]_{-1}^{1}$$

$$= \Big[(1-1)\sin\left(\frac{n\pi x}{n\pi}\right) - \left(\frac{-1-1}{n\pi}\right)\sin(-n\pi) + \frac{1}{n^2\pi^2}\big((1-2)\cos n\pi$$

$$- (1+2)\cos(-n\pi)\big) + \frac{2}{n^3\pi^3}(\sin n\pi - \sin(-n\pi)) \Big]$$

$$= 0 - \frac{4}{n^2\pi^2}\cos n\pi$$

$$= \frac{-4}{n^2\pi^2}(-1)^n$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_{-1}^{1}(x-x^2)\sin(n\pi x)\,dx$$

$$= (x-x^2)\cdot\frac{-\cos n\pi x}{n\pi} - (1-2x)\times\frac{-\sin(n\pi x)}{n^2\pi^2}$$

$$+ (-2)\cdot\frac{\cos(n\pi x)}{n^3\pi^3}\Bigg]_{-1}^{1}$$

$$= \frac{-1}{n\pi}\left[0 - -2\times\cos n\pi\right] - 0 - \frac{2}{n^3\pi^3}\left[\cos n\pi - \cos n\pi\right]$$

$$= \frac{-2}{n\pi}\cos n\pi$$

$$= \frac{-2}{n\pi}(-1)^n$$

$$\therefore f(x) = \frac{-1}{3} + \sum_{n=1}^{\infty}\frac{-4}{n^2\pi^2}(-1)^n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty}\frac{-2}{n\pi}(-1)^n\sin\left(\frac{n\pi x}{l}\right)$$

b) Obtain the half range Fourier cosine series of $f(x) = e^{-x}$ in $0 < x < 2$.

Soln Half range fourier cosine series of

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 e^{-x}dx = \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_0^2 = \frac{-1}{2}\left[e^{-2} - e^{0}\right]$$

$$= \frac{-1}{2}\left[e^{-2} - 1\right]$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{2}\int_0^2 e^{-x}\cdot\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{e^{-x}}{(-1)^2 + \frac{n^2\pi^2}{4}}\left[\cancel{\frac{n\pi}{2}} - \cos\left(\frac{n\pi x}{2}\right) + \frac{n\pi}{2}\cdot\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{e^{-2}}{1+\frac{n^2\pi^2}{4}}\left[-\cos(n\pi) + \frac{n\pi}{2}\times\sin(n\pi)\right] - \left[\left(\frac{e^{-0}}{1+\frac{n^2\pi^2}{4}}\right)(\cos 0 + 0)\right]$$

$$= \frac{e^{-2}}{1+\frac{n^2\pi^2}{4}}\left[(-1)^n\right] + \left(\frac{1}{1+\frac{n^2\pi^2}{4}}\right)$$

$$= \frac{1 - e^{-2}(-1)^n}{1 + \frac{n^2\pi^2}{4}}$$

$$\therefore f(x) = \frac{-1}{2}\left[e^{-2} - 1\right] + \sum_{n=1}^{\infty}\frac{1 - e^{-2}(-1)^n}{1 + \frac{n^2\pi^2}{4}}\cos\left(\frac{n\pi x}{2}\right)$$

20 a) Find the fourier series expansion of $f(x) = x^2$ in the interval $-\pi < x < \pi$. Hence show that $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

Soln: Since $f(x) = x^2$ is an even function, $b_n = 0$.

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$$

$$= \frac{2}{2l} \int_0^l f(x)\, dx \qquad (\because f(x) \text{ is even})$$

$$= \frac{1}{l} \int_0^l f(x) dx$$

$$= \frac{1}{\pi} \int_0^\pi x^2\, dx = \frac{1}{\pi} \left[\frac{x^3}{3}\right]_0^\pi = \frac{1}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l x^2 \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \int_0^\pi x^2 \cos(nx)\, dx$$

$$= \frac{2}{\pi} \left[x^2 \times \frac{\sin(nx)}{n} - 2x \times \frac{-\cos(nx)}{n^2} + 2 \times \frac{-\sin nx}{n^3} \right]_0^\pi$$

$$= \frac{2}{\pi}\left\{\left[\pi^2 \cdot \frac{\sin n\pi}{n} - 0\right] + \frac{2\pi}{n^2}(\cos n\pi - 0) - \frac{2}{n^3}(\sin n\pi - \sin 0)\right\}$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi}{n^2}(-1)^n - 0\right]$$

$$= \frac{4\pi}{\pi n^2}(-1)^n$$

$$= \frac{4}{n^2}(-1)^n$$

$$\therefore\ f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} \ldots\right] \rightarrow (1)$$

Put $x = \pi$, in (1)

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

When $x = 0$

$$0 = \frac{\pi^2}{3} + 4\left[-1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{3} = 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

20 b) Obtain the half range Fourier sine series of

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$$

Soln: Fourier sine series for $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left\{ x\cdot\frac{-\cos nx}{n} - 1\left[\frac{-\sin nx}{n^2}\right]_0^{\pi/2} + \left.(\pi-x)\cdot\frac{-\cos nx}{n} - (-1)\cdot\frac{-\sin nx}{n^2}\right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{-\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right)}{n} + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right] - \left[0 - 0 - \frac{\frac{\pi}{2}\cdot\cos\frac{n\pi}{2}}{n} - \left(\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right)\right]\right\}$$

$$= \frac{2}{\pi}\,\frac{2}{n^2}\sin\frac{n\pi}{2}$$

$$= \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi}\sin\frac{n\pi}{2}\sin nx$$

B. Tech S_1 (Special Improvement) Examination January 2021

Linear Algebra & Calculus.

Part A

1. Determine the rank of the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 5 & 3 & -2 \end{bmatrix}$

Soln:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 5 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 8 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - \frac{8}{4} R_2$$

$\therefore$ Rank of $A = 2$

2. What kind of conic section is represented by the quadratic form $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$. Transform it into canonical form.

Soln:

$$A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}$$

Characteristic equation is $\lambda^2 - \text{trace } A \lambda + |A| = 0$

$$\lambda^2 - 14\lambda + 40 = 0$$

$$(\lambda - 10)(\lambda - 4) = 0 \Rightarrow \lambda = 10, 4$$

$\therefore$ Eigen values are $\lambda = 4, 10$

Canonical form is $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$$Q = 4y_1^2 + 10y_2^2 = 200$$

$$\frac{y_1^2}{\left(\frac{200}{4}\right)} + \frac{y_2^2}{\left(200/10\right)} = 1, \text{ Ellipse.}$$

Part B

11 a) Test for consistency and solve the system of equations

$x+2y-z=3, \quad 3x-y+2z=1, \quad 2x-2y+3z=2, \quad x-y+z=-1$

Soln:

$$\tilde{A} = \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - \frac{6}{7}R_2 \\ R_4 \to R_4 - \frac{3}{7}R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & -4/7 \end{array}\right] \quad R_4 \to R_4 - \frac{-1}{5}R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Rank A = Rank $\tilde{A}$ = no. of unknowns $=3$

$\therefore$ System of equations have unique solution.

By Back substitution,

$\frac{5}{7}z = \frac{20}{7} \Rightarrow z = 4$

$-7y+5z = -8 \Rightarrow 7y-8 = 5z = 20 \Rightarrow y = 4$

$x+2y-z = 3 \Rightarrow x+2\times4-4 = 3$

$\Rightarrow x+4 = 3 \Rightarrow x = -1$

$\therefore$ solution is $x=-1, \; y=4, \; z=4$

11 b) Find the Eigen values and Eigen vectors of $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

Characteristic equation is $\lambda^3 - \text{trace} A \lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A|$

$- |A| = 0$

$\lambda^3 - 6\lambda^2 + (5+3+3)\lambda - 6 = 0$

$\lambda = 1, 2, 3.$

Eigen vector corresponding to $\lambda = 1$

Consider $(A - \lambda I)x = 0$

$(A - I)x = 0$

$A - I = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

rank = 2

$\therefore$ no. of eigen vectors corresponding to $\lambda = 1$

is no. of variables − rank = 3-2 = 1

By Back substitution $x_2 + 2x_3 = 0$

let $x_3 = 1$ $x_2 = -2$

$-x_1 + x_2 + x_3 = 0$

$x_1 = x_2 + x_3 = -2+1 = -1$

Eigen vector corresponding to $\lambda = 1$, $X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = 2$

Consider $(A-2I)x = 0$

$$A-2I = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 \rightarrow R_2 - R_1$$

$$\sim \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2$$

Rank $= 2$

no. of Eigen vectors $= 3-2 = 1$

By Back substitution,

$-x_2 - x_3 = 0 \Rightarrow x_2 = -x_3$

Let $x_3 = 1 \quad \therefore x_2 = -1$.

$-x_1 + x_2 + 2x_3 = 0$

$x_1 = x_2 + 2x_3 = -1 + 2 = 1$

Eigen vector corresponding to $\lambda = 2$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 3$

Consider $(A-3I)x = 0$

$$A-3I = \begin{bmatrix} -2 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & -3/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad R_2 \to R_2 - \tfrac{1}{2} R_1$$

$$\sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + \tfrac{2}{3} R_2$$

Rank = 2

No. of eigen vectors $= 3-2 = 1$

By Back substitution,

$-\frac{3}{2} x_2 = 0 \implies x_2 = 0$

$-2x_1 + x_2 + 2x_3 = 0$

Let $x_3 = 1$

$2x_1 = x_2 + 2x_3 = 0 + 2 = 2$

$x_1 = 1$

Eigen vector corresponding to $\lambda = 3$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

12 a) For what value of a and b the system of equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+az=b$ have 1) no solution 2) unique solution 3) more than one solution.

Soln: $\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & : 6 \\ 1 & 2 & 3 & : 10 \\ 1 & 2 & a & : b \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & : 6 \\ 0 & 1 & 2 & : 4 \\ 0 & 1 & a-1 & b-6 \end{bmatrix}$ $\quad R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & : 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{bmatrix}$ $\quad R_3 \to R_3 - R_2$

No solution when Rank $A \neq$ Rank $\tilde{A}$

when $a = 3$ and $b \neq 10$, Rank $A = 2$, Rank $\tilde{A} = 3$

$\therefore$ No solution when $a = 3$, $b \neq 10$

Unique solution when Rank A = Rank $\tilde{A} = 3$

ie when $a \neq 3$ and b may have any value

Infinite no. of solutions when Rank A = Rank $\tilde{A} \neq 3$

ie when $a = 3$ and $b = 10$

$\therefore$ Rank A = Rank $\tilde{A} = 2$.

12 b) Find the matrix of transformation that diagonalize the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. Also write the diagonal matrix.

Soln: Characteristic equation is $\lambda^3 - \text{trace}\,A\,\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$

$$\lambda^3 - 18\lambda^2 + [5+20+20]\lambda - 0 = 0$$

Eigen values of are $\lambda = 0, 3, 15$

Eigen vector corresponding to $\lambda = 0$

Consider $(A - \lambda I).x = 0$

$$A - 0I = A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 20/8 & \frac{-10}{4} \\ 0 & \frac{-10}{4} & \frac{5}{2} \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 \to R_2 + \frac{6}{8}R_1 \\ R_3 \to R_3 - \frac{2}{8}R_1 \end{array}$$

$$\sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 20/8 & -10/4 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

By Back substitution $\frac{20}{8}x_2 - \frac{10}{4}x_3 = 0$

$$x_2 = x_3$$

Let $x_2 = 2 \Rightarrow x_3 = 2$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$8x_1 = 6x_2 - 2x_3 = \cancel{6x_2 - 2x_3} \Rightarrow x_1 = \cancel{x_2}\ 1$$

Eigen vector corresponding to $\lambda = 0$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 3$.

Consider $(A - 3I)X = 0$

$$A - 3I = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16/5 & -8/5 \\ 0 & -8/5 & -4/5 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - \frac{-6}{5} R_1 \\ R_3 \to R_3 - \frac{2}{5} R_1 \end{array}$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16/5 & -8/5 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - \frac{1}{2} R_2$$

By Back substitution $-\frac{16}{5} x_2 - \frac{8}{5} x_3 = 0$

$-2x_2 = x_3$

let $x_2 = 1$

$x_3 = -2$

$5x_1 - 6x_2 + 2x_3 = 0$

$5x_1 = 6x_2 - 2x_3 = 6 \times 1 - 2 \times -2 = 10$

$x_1 = 2$

Eigen vector corresponding to $\lambda = 3$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Eigen vector corresponding to $\lambda = 15$:

Consider $(A - 15I)x = 0$

$$A - 15I = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} -7 & -6 & +2 \\ 0 & \frac{-20}{7} & \frac{-40}{7} \\ 0 & \frac{-40}{7} & \frac{-80}{7} \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - \frac{6}{7} R_1 \\ R_3 \to R_3 + \frac{2}{7} R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ 0 & -20/7 & -40/7 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

By Back substitution,

$$-\frac{20}{7} x_2 - \frac{40}{7} x_3 = 0$$

$$-x_2 - 2x_3 = 0 \implies x_2 = -2x_3$$

let $x_3 = 1$

$$x_2 = -2$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$7x_1 = -6x_2 + 2x_3 = -6 \times -2 + 2 \times 1 = 14$$

$$x_1 = 2$$

Eigen vector corresponding to $\lambda = 15$

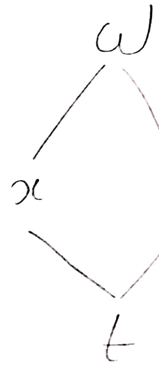
$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ Matrix of transformation that diagonalize the matrix

$$P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Diagonal matrix $D = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

3. Find the derivative of $\omega = x^2 + y^2$ w.r.to t along the path $x = at^2$, $y = 2at$.



$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial x} \times \frac{dx}{dt} + \frac{\partial \omega}{\partial y} \times \frac{dy}{dt}$$

$$= 2x \times 2at + 2y \times 2a$$

$$= 2at^2 \times 2at + 2 \times 2at + 2a$$

$$= \underline{\underline{4a^2t^3 + 8a^2t}}$$

Module 2

4. Let $f(x,y) = \sqrt{3x+2y}$, find the slope of the surface $z = f(x,y)$ in the y-direction at the point $(2,5)$.

Ans:
$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{3x+2y}} \times 2 = \frac{1}{\sqrt{3x+2y}}$$

$$\left(\frac{\partial z}{\partial y}\right)_{(2,5)} = \frac{1}{\sqrt{3(2)+2(5)}} = \frac{1}{\sqrt{6+10}}$$

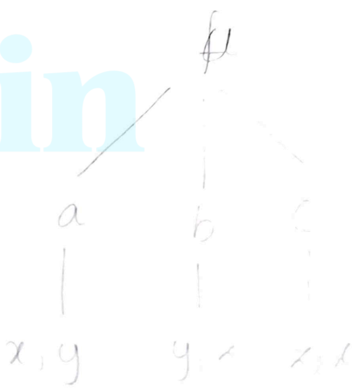
$$= \frac{1}{\sqrt{16}} = \frac{1}{4} //$$

Slope of the surface in y-direction @ $(2,5)$.

13. a) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ find the value of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

a) Let $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} \times \frac{\partial a}{\partial x} + \frac{\partial u}{\partial c} \times \frac{\partial c}{\partial x}$$

$$= \frac{\partial u}{\partial a} \times \frac{1}{y} + \frac{\partial u}{\partial c} \times \frac{-z}{x^2}$$

$$= \frac{\partial u}{\partial a} \times \frac{1}{y} - \frac{\partial u}{\partial c} \times \frac{z}{x^2} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} \times \frac{\partial a}{\partial y} + \frac{\partial u}{\partial b} \times \frac{\partial b}{\partial y}$$

$$= \frac{\partial u}{\partial a} \times \frac{-x}{y^2} + \frac{\partial u}{\partial b} \times \frac{1}{z}$$

$$= \frac{-\partial u}{\partial a}\frac{x}{y^2} + \frac{\partial u}{\partial b} \times \frac{1}{z} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial b} \times \frac{\partial b}{\partial z} + \frac{\partial u}{\partial c} \times \frac{\partial c}{\partial z}$$

$$= \frac{-y}{z^2}\frac{\partial u}{\partial b} + \frac{\partial u}{\partial c} \times \frac{1}{x} \longrightarrow (3)$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z}$$

$$= 0$$

ie, $\left[x \frac{\partial u}{\partial a} \times \frac{1}{y} - \frac{\partial u}{\partial c} \times \frac{z}{x}\right] + \left[\frac{-\partial u}{\partial a}\frac{x}{y} + \frac{\partial u}{\partial b}\frac{y}{z}\right]$

$+ \left[-\frac{y}{z}\frac{\partial u}{\partial b} + \frac{\partial u}{\partial c}\frac{z}{x}\right]$

$$= \underline{\underline{0}}$$

13 b) If the local linear approximation of a function $f(x,y,z) = xy + z^2$ at a point P is $L(x,y,z) = y + 2z - x$, find the point P.

Ans:

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$f(x_0,y_0,z_0) = x_0 y_0 + z_0^2$$

$f_x = y$; $f_x(x_0,y_0,z_0) = y_0$

$f_y = x$; $f_y(x_0,y_0,z_0) = x_0$

$f_z = 2z$; $f_z(x_0,y_0,z_0) = 2z_0$

$$L(x,y,z) = x_0 y_0 + z_0^2 + y_0(x-x_0) + x_0(y-y_0) + 2z_0(z-z_0)$$

$$y + 2z - x = x_0 y_0 + z_0^2 + y_0 x - \cancel{y_0 x_0} + x_0 y - \cancel{x_0 y_0} + 2z_0 z - 2z_0^2$$

$$y + 2z - x = x_0 y + 2z_0 z + y_0 x + x_0 y_0 + z_0^2 - 2z_0^2$$

Comparing coefficients of x, y, z.

$y = x_0 y \Rightarrow x_0 = 1$

$2z = 2z_0 z \Rightarrow z_0 = 1$

$-x = y_0 x \Rightarrow y_0 = -1$

$\therefore P(x_0,y_0,z_0) = (1,-1,1)$

14. a) If $z = e^{xy}$, $x = 2u+v$, $y = \frac{u}{v}$ find $\frac{\partial z}{\partial u}$

b) Locate all relative extrema of $f(x,y) = 3x^2 - 2xy + y^2 - 8y$.

a) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$

$= \left[e^{xy} \times y\right] \times 2 + \left[e^{xy} \times x\right] \times \frac{1}{v}$

$= 2ye^{xy} + \frac{x}{v} e^{xy}$

$= 2 \times \frac{u}{v} e^{((2u+v) \times \frac{u}{v})} + \frac{2u+v}{v} \times e^{((2u+v) \times \frac{u}{v})}$

$= e^{(2u+v) \times \frac{u}{v}} \left[\frac{2u}{v} + \frac{2u+v}{v}\right]$

b) Critical points

$f_x = 6x - 2y$

$f_y = -2x + 2y - 8$

$f_x = 0$

$\Rightarrow 6x - 2y = 0 \longrightarrow (1)$

$f_y = 0$

$\Rightarrow -2x + 2y - 8 = 0 \longrightarrow (2)$

$(1) + (2)$

$= 4x - 8 = 0$

$\therefore x = 2$

$y = 6$

The critical point is

$= (2,6)$

$D = f_{xx}(x_0 y_0) f_{yy}(x_0 y_0) - f_{xy}^2(x_0 y_0)$

$f_{xx} = 6$

$f_{yy} = 2$

$f_{xy} = -2$

$\therefore D = (6 \times 2) - 4$

$= 8$

$D > 0$

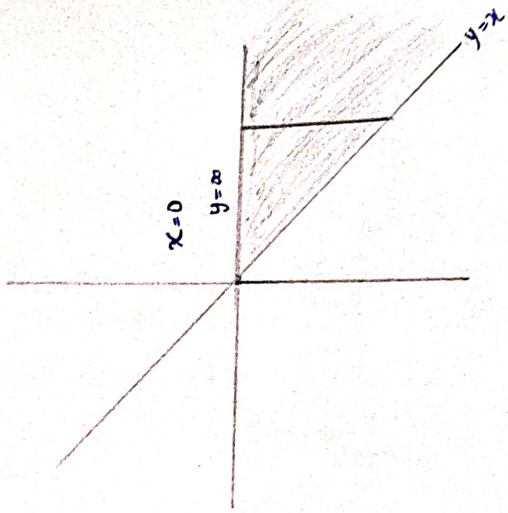
$f_{xx} = 6 > 0$

$\therefore$ F has relative minimum at $(2,6)$

Module - 3

15. (a) Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ by reversing order of integration

A) $$\int_0^\infty \int_0^\infty \frac{e^{-y}}{y}\, dy\, dx$$



Given $y = x$; $y = \infty$

$x \longrightarrow 0$ to y

$y \longrightarrow 0$ to ∞

$\therefore$ Order of Integration now is

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$\left[\int_b^a e^{-y} = \left[e^{-y}\right]_b^a \right]$$

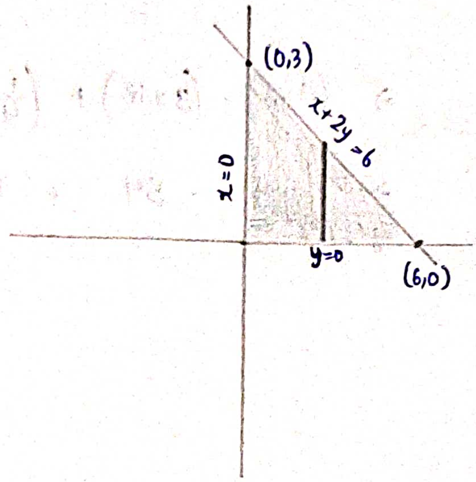
$$= \int_0^\infty \frac{e^{-y}}{y} \left[x\right]_0^y dy \Rightarrow \int_0^\infty \frac{e^{-y}}{\cancel{y}} \left[\cancel{y}\right] dy$$

$$\Rightarrow \int_0^\infty e^{-y}\, dy \Rightarrow \left[e^{-y}\right]_0^\infty \Rightarrow \left[e^{-\infty} - e^{-0}\right]$$

$$\Rightarrow \left[0 + 1\right] = \underline{1}.$$

15. b) Find the volume of the solid in the first octant bounded by the coordinance planes and the plane $x + 2y + z = 6$

ans)



Volume $= \iiint_G dV$

$z \longrightarrow 0$ to $6 - x - 2y$

$x + 2y = 6$

x	0	6
y	3	0

The points are (0,3) (6,0)

$y \longrightarrow 0$ to $3 - x/2$; $x \longrightarrow 0$ to 6

$$\int_0^6 \int_0^{3-\frac{x}{2}} \int_0^{6-2y-x} dz\,dy\,dx$$

$$\Rightarrow \int_0^6 \int_0^{3-\frac{x}{2}} \left[6 - 2y - x\right] dy\,dx$$

$$\Rightarrow \int_0^6 \left[6\left[y\right]_0^{3-\frac{x}{2}} - 2\left[y^2/2\right]_0^{3-\frac{x}{2}} - x\left[y\right]_0^{3-\frac{x}{2}}\right] dx$$

$$\Rightarrow \int_0^6 \left[6\left(3-\frac{x}{2}\right) - x\left(3-\frac{x}{2}\right) - \frac{2}{2}\left[9 - \frac{6x}{2} + \frac{x^2}{4}\right]\right] dx$$

$$\Rightarrow \int_0^6 \left[18 - \frac{6x}{2} - \cancel{3x} + \cancel{\frac{x^2}{2}} - 9 + \cancel{3x} - \frac{x^2}{4}\right] dx$$

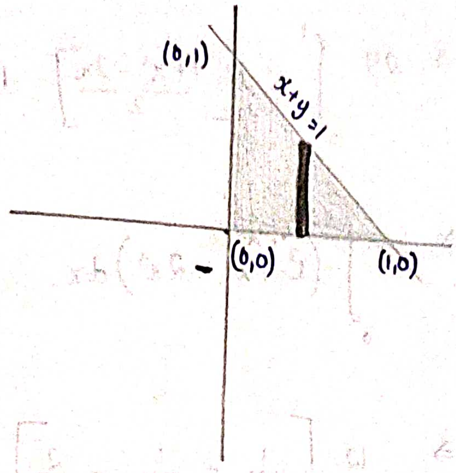
$$\Rightarrow \int_0^6 9 - 3x + \frac{x^2}{4}\, dx$$

$$\Rightarrow \left[9\left[x\right]_0^6 - 3\left[\frac{x^2}{2}\right]_0^6 + \frac{1}{4}\left[\frac{x^3}{3}\right]_0^6 \right]$$

$$\Rightarrow (9\times6) - (3\times18) + \left(\frac{1}{4}\times72\right)$$

$$\Rightarrow \cancel{54} - \cancel{54} + 18 \qquad \Rightarrow \underline{18}.$$

16. ⓐ Find the mass and centre of gravity of the triangular lamina with vertices (0,0), (0,1), (1,0) with density function $\delta(x,y) = xy$.



A) $M = \iint_R \delta(x,y)\, dA$

We have $\delta(x,y) = xy$

$y \longrightarrow 0$ to $1x$

$x \longrightarrow 0$ to 1

$$M = \int_0^1 \int_0^{1-x} xy \, dy\, dx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_0^{1-x} dx \Rightarrow \int_0^1 x \left[\frac{1+x^2-2x}{2}\right] dx$$

$$\Rightarrow \int_0^1 \frac{1}{2} x \left[1 + x^2 - 2x\right] dx \Rightarrow \frac{1}{2}\int_0^1 x + x^3 - 2x^2 \, dx$$

$$\Rightarrow \frac{1}{2}\left[\left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1 - 2\left[\frac{x^3}{3}\right]_0^1\right] \Rightarrow \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{12}\right] \Rightarrow \frac{1}{24}$$

Centre of gravity $= \bar{x} = \frac{1}{M} \cdot \iint_R x\, \delta(x,y)\, dA$

We have $M = \frac{1}{24}$,

$$\Rightarrow \frac{1}{\frac{1}{24}} \iint_R x\,(xy)\,dA$$

$$\Rightarrow 24 \int_0^1 \int_0^{1-x} x^2 y \, dy\, dx \Rightarrow 24 \int_0^1 x^2 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$\Rightarrow 24 \int_0^1 x^2 \left[\frac{1+x^2-2x}{2}\right] dx \Rightarrow \frac{24}{2} \int_0^1 x^2 (1+x^2-2x)\, dx$$

$$\Rightarrow 12 \int_0^1 (x^2+x^4-2x^3)\, dx \Rightarrow 12 \left[\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1 - 2\left[\frac{x^4}{4}\right]_0^1\right]$$

$$\Rightarrow 12 \left[\frac{1}{3} + \frac{1}{5} - \frac{2}{4}\right] \Rightarrow 12 \times \left[\frac{1}{30}\right]$$

$$\Rightarrow \frac{12}{30} = \underline{\underline{\frac{2}{5}}}.$$

Centre of gravity $\bar{y} = \frac{1}{M} \iint_R y\, \delta(x,y)\, dA$

$$= 24 \int_0^1 \int_0^{1-x} x y^2 \, dy\, dx \Rightarrow 24 \int_0^1 x \left[\frac{y^3}{3}\right]_0^{1-x} dx$$

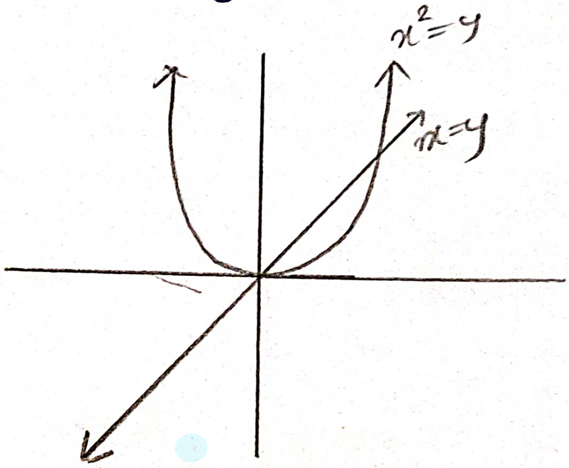
$$\Rightarrow 24 \int_0^1 x \left[\frac{1-3x+3x^2-x^3}{3}\right] dx$$

$$\Rightarrow \frac{24}{3} \int_0^1 x\,(1-3x+3x^2-x^3)\, dx \Rightarrow 8 \int_0^1 (x-3x^2+3x^3-x^4)\, dx$$

$$\Rightarrow 8 \left[\left[\frac{x^2}{2}\right]_0^1 - 3\left[\frac{x^3}{3}\right]_0^1 + 3\left[\frac{x^4}{4}\right]_0^1 - \left[\frac{x^5}{5}\right]_0^1\right] \Rightarrow 8 \left[\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5}\right]$$

$$\Rightarrow 8 \times {}^{1}/_{20} = \underline{\underline{{}^{2}/_{5}}}.$$

16. b) Evaluate $\iint_R x^2\,dy\,dx$ where R is the region between $y = x$ and $y = x^2$.



ans:- $\int_0^1\int_{x^2}^{x} x^2\,dy\,dx$

$= \int_0^1 x^2\,[y]_{x^2}^{x}\,dx$

$= \int_0^1 x^2 (x - x^2)\,dx$

$= \left[\frac{x^4}{4}\right]_0^1 - \left[\frac{x^5}{5}\right]_0^1$

$= \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

5. Evaluate $\int_0^a \int_0^a \int_0^a (yz + xz + xy)\, dx\, dy\, dz$

ans:- $\int_0^a \int_0^a \int_0^a (yz+xz+xy)\, dx\, dy\, dz = \int_0^a \int_0^a \left[yz[x]_0^a + \left[\frac{x^2}{2}\right]_0^a z + \left[\frac{x^2}{2}\right]_0^a y \right] dy\, dz$

$= \int_0^a \left[a.z \left[\frac{y^2}{2}\right]_0^a + \frac{a^2}{2}.z.[y]_0^a + \frac{a^2}{2}.\left[\frac{y^2}{2}\right]_0^a \right] dz$

$= \left[a.\frac{a^2}{2}.\left[\frac{z^2}{2}\right]_0^a\right] + \left[a.\frac{a^2}{2}.\left[\frac{z^2}{2}\right]_0^a\right] + \left[\frac{a^2}{2}.\frac{a^2}{2}.[z]_0^a\right]$

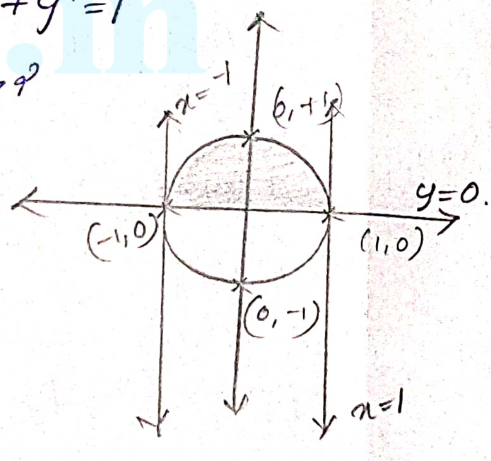
$= a.\frac{a^2}{2}.\frac{a^2}{2} + a.\frac{a^2}{2}.\frac{a^2}{2} + a.\frac{a^2}{2}.\frac{a^2}{2} = \underline{\underline{\frac{3a^5}{4}}}$

6. Use Polar co-ordinates to evaluate $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$

ans:- $x=-1,\ x=1,\ y=0,\ y=\sqrt{1-x^2} \Rightarrow x^2+y^2=1$

substitute $x = r\cos\theta,\ y = r\sin\theta,\ x^2+y^2 = r^2$

$dy\,dx = r\,dr\,d\theta.$



$\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2} = \int_0^1 \int_0^{\pi} (r^2)^{3/2} . r\, d\theta\, dr$

$r: 0 \to 1$

$\theta: 0 \to \pi$

$= \int_0^1 \int_0^{\pi} r^4\, d\theta\, dr = \int_0^1 r^4 [\theta]_0^{\pi}\, dr$

$= \pi . \left[\frac{r^5}{5}\right]_0^1 = \underline{\underline{\frac{\pi}{5}}}$

Spec. Imp.
Jan 2021

PART-A

7. Test the convergence of $\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$

Ans: Consider the series $\sum b_k = \sum \frac{1}{k^{17}}$ which is a convergent p-series.

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{(2k+3)^{17}} \cdot k^{17}$$

$$= \lim_{k\to\infty} \frac{k^{17}}{k^{17}\left(2+\frac{3}{k}\right)^{17}}$$

$$= \frac{1}{2} > 0 \text{ \& finite.}$$

by limit comparison test, since $\sum_{k=1}^{\infty} \frac{1}{k^{17}}$ converges, $\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$ also converges.

8. Examine whether the series converges or not.

$$\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}.$$

Ans: By root test, $\rho = \lim_{k\to\infty} (a_k)^{1/k}$

$$= \lim_{k\to\infty} \left(\frac{1}{(\ln(k+1))^k}\right)^{1/k}$$

$$= \lim_{k\to\infty} \frac{1}{\ln(k+1)} = 0 < 1$$

$\therefore$ the series converges.

Spec. Imp - 2021 Jan

Module - IV

17a) Discuss the convergence of the series

i) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ ii) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

Ans:

$$a_k = \frac{k!}{k^k}, \quad a_{k+1} = \frac{(k+1)!}{(k+1)^{k+1}}$$

$$\underset{k\to\infty}{Lt} \frac{a_{k+1}}{a_k} = \underset{k\to\infty}{Lt} \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!}$$

$$= \underset{k\to\infty}{Lt} \frac{(k+1) \cdot k^k}{(k+1)^k \cdot (k+1)} = \underset{k\to\infty}{Lt} \left(\frac{k}{k+1}\right)^k$$

$$= \underset{k\to\infty}{Lt} \left(\frac{k}{k(1+\frac{1}{k})}\right)^k = \underset{k\to\infty}{Lt} \frac{1}{(1+\frac{1}{k})^k} = \frac{1}{e} < 1$$

$\therefore$ by ratio test Σa_k converges.

ii) Let $a_k = \left(\frac{k}{k+1}\right)^{k^2}$

$$\underset{k\to\infty}{Lt} a_k = \underset{k\to\infty}{Lt} \left[\left(\frac{k}{k+1}\right)^{k^2}\right]^{1/k} = \underset{k\to\infty}{Lt} \left(\frac{k}{k+1}\right)^k$$

$$= \underset{k\to\infty}{Lt} \left(\frac{1}{1+1/k}\right)^k = \frac{1}{e} < 1$$

$\therefore$ by root test the series converges.

17 b) Examine the convergence and divergence of the series $\frac{x}{1\cdot 3}+\frac{x^2}{3\cdot 5}+\frac{x^3}{5\cdot 7}+\cdots$

Ans: Let $a_k = \frac{x^k}{(2k-1)(2k+1)}$

$$a_{k+1} = \frac{x^{k+1}}{(2(k+1)-1)(2(k+1)+1)} = \frac{x^{k+1}}{(2k+1)\cdot(2k+3)}$$

$$\underset{k\to\infty}{Lt} \frac{a_{k+1}}{a_k} = \underset{k\to\infty}{Lt} \frac{x^{k+1}}{(2k+1)(2k+3)} \cdot \frac{(2k-1)(2k+1)}{x^k}$$

$$= \underset{k\to\infty}{Lt} \frac{x(2k-1)}{(2k+3)} = x \underset{k\to\infty}{Lt} \frac{2k\left(1-\frac{1}{2k}\right)}{2k\left(1+\frac{3}{2k}\right)}$$

$$= x$$

If $x<1$, the series converges, and $x>1$ the series diverges.

If $x=1$, the series becomes, $\sum \frac{1}{(2k-1)(2k+1)}$

Let $a_k = \frac{1}{(2k-1)(2k+1)} = \frac{1}{4k^2-1}$

Let $b_k = \frac{1}{k^2}$

$$\underset{k\to\infty}{Lt} \frac{a_k}{b_k} = \underset{k\to\infty}{Lt} \frac{k^2}{4k^2-1} = \frac{1}{4} > 0 \text{ \& finite}$$

$\therefore$ by limit comparison test, since $\sum \frac{1}{k^2}$ is a convergent p-series $\sum a_k$ is also convergent.

18a) Test the convergence of $1 + \frac{1.3}{3!} + \frac{1.3.5}{5!} + \frac{1.3.5.7}{7!} + \cdots$

Ans: Let $a_k = \frac{1.3.5.7.\cdots\cdots.2k-1}{(2k-1)!}$

$$a_{k+1} = \frac{1.3.5.7.\cdots.2(k+1)-1}{(2(k+1)-1)!}$$

$$\underset{k\to\infty}{Lt} \frac{a_{k+1}}{a_k} = \underset{k\to\infty}{Lt} \frac{1.3.5\cdots2k+1}{(2k+1)!} \cdot \frac{(2k-1)!}{1.3.5\cdots(2k-1)}$$

$$= \underset{k\to\infty}{Lt} \frac{(2k+1)}{2k(2k+1)} = 0 < 1$$

$\therefore$ the series converges by ratio test.

18 b) Prove that the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k-2)}{k(k+1)}$ is conditionally convergent.

Ans Let $a_k = \frac{k-2}{k(k+1)}$

$$a_{k+1} = \frac{(k+1)-2}{(k+1)(k+2)} = \frac{k-1}{(k+1)(k+2)}$$

$\therefore a_k > a_{k+1}$

$$\underset{k\to\infty}{Lt} a_k = \underset{k\to\infty}{Lt} \frac{k-2}{k^2(1+\frac{1}{k})} = 0$$

$\therefore$ by Leibnitz test the series converges.

Consider $|a_k| = \dfrac{k-2}{k(k+1)}$

Let $b_k = \dfrac{1}{k}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k-2}{k^2+k} \cdot k = \lim_{k\to\infty} \frac{k^2-2k}{k^2+k}$$

$$= \lim_{k\to\infty} \frac{k^2\left(1-\frac{2k}{k^2}\right)}{k^2\left(1+\frac{k}{k^2}\right)}$$

$= 1 > 0$ & finite.

Since Σb_k is a divergent p-series, by limit comparison test $\Sigma|a_k|$ is also divergent.

~~Σa_k converges & $\Sigma|a_k|$ diverges~~

$\Sigma|a_k|$ diverges $\Rightarrow$ Σa_k diverges absolutely.

Since Σa_k converges & diverges absolutely

$\Rightarrow$ Σa_k is conditionally convergent.

Module V [KTU Dec: 2021 Special Improvement]

PART A

9. Find the maclaurin series of $\frac{1}{x+1}$ upto third degree term.

Soln Maclaurin series of $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

Given $f(x) = \frac{1}{x+1}$

$f(0) = \frac{1}{0+1} = 1$

$f'(x) = \frac{-1}{(1+x)^2}$

$f'(0) = \frac{-1}{1^2} = -1$

$f''(x) = \frac{+2}{(1+x)^3}$

$f''(0) = \frac{+2}{1^3} = +2$

$f'''(x) = +2 \times -3 \times (1+x)^{-4}$

$= \frac{-6}{(1+x)^4}$

$f'''(0) = -6$

$\therefore f(x) = 1 + x \times -1 + \frac{x^2}{2} \times 2 + \frac{x^3}{6} \times -6 + \cdots$

$= 1 - x + x^2 - x^3 + \cdots$

10. Find the half range sine series of $f(x) = x$ in $0 < x < \pi$

Soln: Half range sine series is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin(n\pi x/\pi)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx$$

$$= \frac{2}{\pi}\left[x \times \frac{-\cos(nx)}{n} - 1 \times \frac{-1}{n}\frac{\sin(nx)}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - 0\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{n} \times (-1)^n\right]$$

$$= \frac{-2}{n}(-1)^n$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n \sin(nx)$$

Module V (KTU Dec: 2021 Special Improvement)

19 a) Obtain the fourier series for the function

$f(x) = |\sin x|$ in $-\pi < x < \pi$

Soln: Since $f(x)$ is even, $b_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi} \times 2 \int_0^{\pi} |\sin x|\,dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi} \left[-\cos x\right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\cos\pi - -\cos 0\right]$$

$$= \frac{1}{\pi} \left[1+1\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \cos(nx)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cos(nx)\,dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(1+n)x + \sin(1-n)x\,dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(n+1)x - \sin(n-1)x\,dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \left[\frac{-\cos 0}{n+1} - \frac{\cos 0}{n-1}\right]\right]$$

$$\boxed{\cos(n-1)\pi = (-1)^{n-1}}$$

$$= \frac{1}{\pi} \times \frac{2}{n^2-1}\left[(-1)^{n-1} - 1\right] \quad \text{if } n \neq 1$$

When $n=1$, $a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x \, dx$

$$= \frac{2}{\pi}\int_0^\pi \frac{\sin 2x}{2} dx$$

$$= \frac{2}{\pi} \times -\frac{\cos 2x}{4}\Big]_0^\pi$$

$$= -\frac{\cos 2x}{\pi 2}\Big]_0^\pi$$

$$= \frac{-1}{2\pi}\left[\cos 2\pi - \cos 0\right]$$

$$= \frac{-1}{2\pi}\left[1 - 1\right]$$

$$= \underline{\underline{0}}$$

19 b) If $f(x) = \begin{cases} kx, & 0 < x < \pi/2 \\ k(\pi - x), & \pi/2 < x < \pi \end{cases}$ then show that

$$f(x) = \frac{4k}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$$

Soln: From the question we see that $f(x)$ has to be expanded as a half range fourier sine series in $(0, \pi)$.

Fourier sine series of $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} kx \sin nx \, dx + \int_{\pi/2}^{\pi} k(\pi - x) \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left\{\left[kx\left[-\frac{\cos nx}{n}\right] - k\left[-\frac{\sin nx}{n^2}\right]\right]_0^{\pi/2} + \left[k(\pi - x)\cdot\left(-\frac{\cos nx}{n}\right) - (-k)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2k}{\pi}\left\{\left[\frac{-\frac{\pi}{2}\cos\frac{n\pi}{2}}{n} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] - \left[0 - 0 - \frac{\frac{\pi}{2}\cos\frac{n\pi}{2}}{n} - \frac{\sin\frac{n\pi}{2}}{n^2}\right]\right\}$$

$$= \frac{2k}{\pi} \cdot \frac{2}{n^2} \sin \frac{n\pi}{2}$$

$$= \frac{4k}{\pi n^2} \sin \frac{n\pi}{2}$$

$\therefore$ Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4k}{\pi n^2} \sin \frac{n\pi}{2} \sin nx$$

$$= \frac{4k}{\pi} \left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots \right]$$

20 a) Find the fourier cosine series of $f(x) = x^2$ in $(0, \pi)$. Hence show that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

Fourier cosine series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l} \int_0^l f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^\pi x^2\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^\pi = \frac{1}{\pi} \times \frac{\pi^3}{3} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{\pi} \int_0^\pi x^2 \cdot \cos(nx)\, dx$$

$$= \frac{2}{\pi}\left[x^2 \times \frac{\sin(nx)}{n} - 2x \times \frac{-\cos(nx)}{n^2} + 2 \times \frac{-1}{n^3} \times \sin nx\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi\right] = \frac{4}{n^2}(-1)^n$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$f(x) = \frac{\pi^2}{3} + \frac{4}{1^2} \times -1 \cdot \cos x + \frac{4}{2^2}\cos 2x + \frac{4}{3^2} \times -1 \times \cos 3x + \cdots$$

$$x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

When $x = \pi$,

$$\pi^2 = \frac{\pi^2}{3} + 4\left[-\cos\pi + \frac{1}{2^2} \times \cos 2\pi - \frac{1}{3^2} \times \cos 3\pi + \cdots\right]$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{4}\left[\pi^2 - \frac{\pi^2}{3}\right]$$

$$= \frac{1}{4}\left[\frac{2\pi^2}{3}\right]$$

$$\therefore 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

20 b) Find the fourier series for the function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 1-x, & 1 < x < 2 \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{2l}\int_{\alpha}^{\alpha+2l} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 1-x\,dx\right]$$

$$= \frac{1}{2}\left[\left[\frac{x^2}{2}\right]_0^1 + \left[x-\frac{x^2}{2}\right]_1^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + 2 - \frac{4}{2} - \left(1-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} - 1 + \frac{1}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^1 x\cos(n\pi x)\,dx + \int_1^2 (1-x)\cos(n\pi x)\,dx$$

$$= \left. x \times \frac{\sin(n\pi x)}{n\pi} - 1 \times \frac{1}{n^2\pi^2} \times -\cos(n\pi x)\right]_0^1 + (1-x).\frac{\sin(n\pi x)}{n\pi}$$

$$\left. - -1 \times \frac{-1}{n^2\pi^2}\cos(n\pi x)\right]_1^2$$

$$= \frac{1}{n\pi} \times \sin n\pi + \frac{1}{n^2\pi^2}\cos n\pi - \left(\frac{1}{n^2\pi^2}\right) + \frac{-1}{n^2\pi^2} \times 1 - \left[\left(-\frac{1}{n^2\pi^2} \times (-1)^n\right)\right]$$

$$= \frac{1}{n^2\pi^2}(-1)^n - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{1}{n^2\pi^2}(-1)^n$$

$$= 2 \times \frac{1}{n^2\pi^2}(-1)^n - 2 \times \frac{1}{n^2\pi^2}$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l}\int_{\alpha}^{\alpha+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^1 x\sin(n\pi x)\,dx + \int_1^2 (1-x)\sin(n\pi x)\,dx$$

$$= \left. x \times \frac{-\cos(n\pi x)}{n\pi} - 1 \times \frac{-1}{n^2\pi^2}\sin(n\pi x)\right]_0^1$$

$$+ \left. (1-x)\cdot\frac{-\cos(n\pi x)}{n\pi} - -1 \times \frac{-1}{n^2\pi^2}\cdot\sin(n\pi x)\right]_1^2$$

$$= \frac{-1}{n\pi}(-1)^n + \frac{1}{n\pi}\times 1$$

$$= \frac{1}{n\pi}\left[1 - (-1)^n\right]$$

$$f(x) = \frac{2}{n^2\pi^2}(-1)^n - 1 \quad \cos(n\pi x) + \frac{1}{n\pi} \quad 1-(-1)^n \sin(n\pi x)$$


KTU
NOTES

Reg No.:_______________ Name:_______________________

APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

FIRST SEMESTER B.TECH DEGREE EXAMINATION(2019 SCHEME), DECEMBER 2019

Course Code: MAT101

Course Name: LINEAR ALGEBRA AND CALCULUS

(2019-Scheme)

Max. Marks: 100 Duration: 3 Hours

PART A

Answer all questions, each carries 3 marks.

1 Determine the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}$ (3)

2 If 2 is an eigen value of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, without using its characteristic equation, find the other eigen values. (3)

3 If $f(x,y) = xe^{-y} + 5y$ find the slope of f (x,y) in the x-direction at (4,0). (3)

4 Show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, where $z = e^x \sin y + e^y \cos x$ (3)

5 Find the mass of the square lamina with vertices (0,0) (1,0) (1,1) and (0,1) and density function $x^2 y$ (3)

6 Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$ by changing to polar coordinates. (3)

7 Test the convergence of the series $\sum_{k=1}^{\infty} \frac{k}{2k+1}$ (3)

8 Check the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^{k/2}}$ (3)

9 Find the Taylors series for $f(x) = \cos x$ about $x = \frac{\pi}{2}$ up to third degree terms. (3)

10 Find the Fourier half range sine series of $f(x) = e^x$ in 0 < x < 1 (3)

PART B

Answer one full question from each module, each question carries 14 marks

Module-I

11 a) Solve the system of equations by Gauss elimination method. (7)

$$x + 2y + 3z = 1$$
$$2x + 3y + 2z = 2$$
$$3x + 3y + 4z = 1$$

b) Find the eigenvalues and eigenvectors of (7)

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

12 a) Find the values of λ and μ for which the system of equations (7)

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

has (i) no solution (ii) a unique solution and (iii) infinite solution

b) Find the matrix of transformation that diagonalize the matrix (7)

$A=\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$. Also write the diagonal matrix.

Module-II

13 a) Let f be a differentiable function of three variables and suppose that $w = f(x-y, y-z, z-x)$,show that $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$ (7)

b) Locate all relative extrema of $f(x,y) = 4xy - y^4 - x^4$ (7)

14 a) Find the local linear approximation L to the function $f(x,y) = \sqrt{x^2 + y^2}$ at the point P(3,4).Compare the error in approximating f by L at the point Q(3.04,3.98) with the distance PQ. (7)

b) The radius and height of a right circular cone are measured with errors of at most 1% and 4%, respectively. Use differentials to approximate the maximum percentage error in the calculated volume. (7)

Module-III

15 a) Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$. (7)

b) Use double integral to find the area of the region enclosed between the parabola $y = \dfrac{x^2}{2}$ and the line $y = 2x$. (7)

16 a) Evaluate $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy$ by reversing the order of integration (7)

b) Use triple integrals to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$. (7)

Module-IV

17 a) Find the general term of the series $1 + \dfrac{1\cdot2}{1\cdot3} + \dfrac{1\cdot2\cdot3}{1\cdot3\cdot5} + \dfrac{1\cdot2\cdot3\cdot4}{1\cdot3\cdot5\cdot7} + \ldots\ldots$ and use the ratio test to show that the series converges. (7)

b) Test whether the following series is absolutely convergent or conditionally convergent $\sum_{k=1}^{\infty} \dfrac{(-1)^k}{\sqrt{k(k+1)}}$ (7)

18 a) Test the convergence of $\dfrac{x}{1.2} + \dfrac{x^2}{2.3} + \dfrac{x^3}{3.4} + \ldots\ldots\ldots \dfrac{x^k}{k(k+1)} + \text{--------}$ (7)

b) Test the convergence of the series $\sum_{k=1}^{\infty} \dfrac{(k+1)!}{4!\,k!\,4^k}$ (7)

Module-V

19 a) Find the Fourier series of periodic function with period 2 which is given below $f(x) = \begin{cases} -x\,; -1 \le x \le 0 \\ x\,;\ 0 \le x \le 1 \end{cases}$. Hence prove that $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + --- = \dfrac{\pi^2}{8}$ (7)

b) Find the half range cosine series for $f(x) = \begin{cases} kx & 0 \le x \le L/2 \\ k(L-x) & L/2 \le x \le L \end{cases}$ (7)

20

a) Find the Fourier series of $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases}$ (7)

b) Obtain the Fourier series expansion for $f(x) = x^2, \quad -\pi < x < \pi$. (7)

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Final Scheme/ Answer Key for Valuation

Scheme of evaluation (marks in brackets) and answers of problems/key

APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

FIRST SEMESTER B.TECH DEGREE EXAMINATION, DECEMBER 2019

Course Code: MAT101

Course Name: LINEAR ALGEBRA AND CALCULUS

(2019-Scheme)

Max. Marks: 100 Duration: 3 Hours

PART A

Answer all questions, each carries 3 marks.

1 $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ ....(2) , Rank of $A = 3$......(1) (3)

2 $2+\lambda_2+\lambda_3=11$.......(1) $2\lambda_2\lambda_3=36$.........(1) $\lambda_2=3$ and $\lambda_3=6$...........(1) (3)

3 Slope in the x direction $f_x = e^{-y}$ ....(2) slope at (4,0) =1....(1) (3)

4 $\frac{\partial z}{\partial x} = e^x \sin y - e^y \sin x,\quad \frac{\partial^2 z}{\partial x^2} = e^x \sin y - e^y \cos x$ ....(1) (3)

$\frac{\partial z}{\partial y} = e^x \cos y + e^y \cos x,\quad \frac{\partial^2 z}{\partial y^2} = -e^x \sin y + e^y \cos x$ ....(1)

Conclusion........(1)

5 $Mass = \iint \delta(x,y)\,dx\,dy$ ....(1) $\int_0^1\int_0^1 x^2 y\,dx\,dy$ .... (1) 1/6.......(1) (3)

6 $\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$ ....(1) $-\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[e^{-t}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta$ ....(1) $\frac{\pi}{4}$ ....(1) (3)

7 $\underset{k\to\infty}{Lt}\ U_k = \underset{k\to\infty}{Lt}\ \frac{1}{(2+1/k)} = 1/2 \neq 0$........(2) Divergent ....(1) (3)

8 $\underset{k\to\infty}{Lt}\ (U_k)^{1/k} = \underset{k\to\infty}{Lt}\ \frac{1}{k^{1/2}} = 0 < 1$........(2) Convergent .... (1) (3)

9 $f(x)=\cos x\ \ f'(x)=-\sin x, f''(x)=-\cos x\ , f'''(x)=\sin x\ \ f''''(x)=\cos x$ ....(1) (3)

$f(\pi/2)=0,\ \ f'(x)=-1\ \ f''(\pi/2)=0\ f'''(\pi/2)=1\ f''''(\pi/2)=0$ .... (1)

$f(x)=\frac{(x-\pi/2)}{1!}\left(-1\right)+\frac{(x-\pi/2)^3}{3!}+$.........(1)

OR Alternate method

10

Formula $b_n = \frac{2n\pi}{1+\pi^2 n^2}\left(1-(-1)^n e\right)$.....(1+1) $f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+\pi^2 n^2}\left(1-(-1)^n e\right)\sin n\pi x$ ......(1) (3)

PART B

Answer one full question from each module, each question carries 14 marks

Module-I

11 a) $[A:B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix}$ .... (1) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 7 & -2 \end{bmatrix}$ ....(3) (7)

x = - 3/7, y = 8/7, z = - 2/7 ------(1+1+1)

b) $\lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$ ..... (2) $\lambda = 1,4,7$....(2) eigen vectors $[2 \;\; -1 \;\; 2]^T$, $[-1 \;\; 2 \;\; 2]^T, [-2 \; -2 \;\; 1]^T$....(3) (7)

12 a)

Augmented Matrix .... (1)Reducing $[A:B] \sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$ .... (3)No (7)

solution when $\lambda=5$ and $\mu \neq 9$ ....(1)

unique solution when $\lambda \neq 5$ and μ may have any value....... (1)

infinite number of solutions when $\lambda =5$ and $\mu =9$ ......(1)

b) characteristic equation λ^3 -12λ-16=0 .... (1) Getting λ= -2, -2, 4 .... (1) (7)

Eigen Vectors [1,0,-1] [1,1,0] [1,1,2] .... (1+1+1)

$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$ .... (1) Diagonal matrix $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ .... (1)

Module-II

13 a) Put $r = x-y, s = y-z, t = z-x$ ...(1) $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} - \frac{\partial w}{\partial t}$ .... (2) $\frac{\partial w}{\partial y} = -\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}$ ......(2) (7)

$\frac{\partial w}{\partial z} = -\frac{\partial w}{\partial s} + \frac{\partial w}{\partial t}$ .... (2)

b) $f_x = 4y - 4x^3, f_y = 4x - 4y^3$..........(1) (7)

$f_{xx} = -12x^2, \;\; f_{yy} = -12y^2, \;\; f_{xy} = 4$..................(1)

$f_x = f_y = 0$. Critical points (0,0), (1,1), (−1, −1).......................(2)

(0,0) saddle point,.......(1) (1,1), (−1, −1) point of maxima.........................(1+1)

14 a) $L(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$ ---------- (3) (7)

$L(Q) = 5.008$ .... (1) $L(Q) - f(Q) = 0.00019,$ ....(1)

$|PQ| = 0.045$ .... (1) $Error = \frac{|L(Q)-f(Q)|}{|PQ|} = 0.0042$ .... (1)

b) $V = \frac{1}{3}\pi r^2 h$..........(2) $\log V = \log\frac{1}{3}\pi + 2\log r + \log h$..........(2) (7)

$\frac{dV}{V}\times 100 = 2\frac{dr}{r}\times 100 + \frac{dh}{h}\times 100$.........(2) Ans= 6% .... (1)

Module-III

15 a) Region of integration------(1) $\iint_R y\,dx\,dy = \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} y\,dx\,dy$ ... (2) (7)

$\int_0^4 (2y^{\frac{3}{2}} - \frac{y^3}{4})\,dy$.............(3) $= \frac{48}{5}$............(1)

OR

Region of integration------(1) $\iint_R y\,dx\,dy = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\,dx$ ... (2)

$\int_0^4 (4x - \frac{x^4}{16})\,dy$.............(3) $= \frac{48}{5}$............(1)

b) Region of integration........(1) (7)

$\text{Area} = \iint_R dx\,dy$......(1) $\int_0^4 \int_{x^2/2}^{2x} dy\,dx$......(2) $\int_0^4 \left(2x - \frac{x^2}{2}\right)dx$.....(1) $= \frac{16}{3}$.....(2)

OR

Region of integration........(1)

$\text{Area} = \iint_R dx\,dy$......(1) $\int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy$......(2) $\int_0^8 \left(\sqrt{2y} - \frac{y}{2}\right)dy$.....(1) $= \frac{16}{3}$.....(2)

16 a) Region of integration .... (1) (7)

$\int_0^2\int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy = \int_0^1\int_0^{2x} e^{x^2}\,dx\,dy$ ....(2) $\int_0^1 e^{x^2}\,2x\,dx$ ......(2) $e-1$...... (2)

b) $V = \iiint_G dV$ ... (1) $= \iint_R \int_1^{5-x} dz\,dy\,dx$.....(2) (7)

$\int_0^{2\pi}\int_0^3 (4 - r\cos\theta)\,r\,dr\,d\theta$ .... (2) $\int_0^{2\pi}(18 - 9\cos\theta)\,d\theta$ ... (1) $=$ 36π..............(1)

OR

$V = \iiint_G dV$ ... (1) $= \iint_R \int_1^{5-x} dz\,dy\,dx$.....(2)

$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(4-x)\,dy\,dx \ldots\ldots\ldots(2) = 36\pi \ldots\ldots\ldots(2)$

Module-IV

17 a) (7)

$u_k = \dfrac{1.2.3\ldots\ldots k}{1\cdot3\cdot5\cdot\ldots(2k-1)}$ ------ (2)

$u_{k+1} = \dfrac{(k+1)!}{1\cdot3\cdot5\cdot\ldots(2k-1)(2k+1)}$ -----(1) $\rho = \lim\limits_{k\to\infty}\dfrac{k+1}{2k+1} = \dfrac{1}{2} < 1$---(3)

Hence converges------(1)

b) (7)

$|U_k| = \left|\dfrac{1}{\sqrt{k(1+k)}}\right|$.......(1) $\sum\limits_{k=1}^{\infty}V_k = \sum\limits_{k=1}^{\infty}\dfrac{1}{k}$ *Divergent*........(1) $\underset{k\to\infty}{Lt}\dfrac{U_k}{V_k} = 1$.....(1)

Not Absolute Convergent ---(1)

U1>U2>......(1) $\underset{k\to\infty}{Lt}\, U_k = 0$---(1) conditionally convergent---(1)

18 a) (7)

$U_{k+1} = \dfrac{x^{k+1}}{(k+1)(k+2)}$.....(1) $\underset{k\to\infty}{Lt}\dfrac{U_{k+1}}{U_k} = x$ ....(1)

x<1 Convergent, x > 1 Divergent x = 1 test fails .... (1)

If x=1 $U_k = \dfrac{1}{(k)(k+1)}$.......(1) $\sum\limits_{k=1}^{\infty}V_k = \sum\limits_{k=1}^{\infty}\dfrac{1}{k^2}$ *convergent*......(1) $\underset{k\to\infty}{Lt}\dfrac{U_k}{V_k} = 1 \neq 0$.......(1)

Convergent ------(1)

OR

$\underset{k\to\infty}{Lt}\left|\dfrac{U_{k+1}}{U_k}\right| = |x|$ ....(1) $-1 < x < 1$ series convergences.........(1)

$x = -1$, U_k *decreases* & $\lim U_k = 0$, *so it converge at* $x = -1$---------- (1)

If x=1 $U_k = \dfrac{1}{(k)(k+1)}$.......(1) $\sum\limits_{k=1}^{\infty}V_k = \sum\limits_{k=1}^{\infty}\dfrac{1}{k^2}$ *convergent*......(1) $\underset{k\to\infty}{Lt}\dfrac{U_k}{V_k} = 1 \neq 0$.......(1)

Convergent ------(1)

b) (7)

$U_{k+1} = \dfrac{(k+2)!}{4!(k+1)!4^{k+1}}$.......(2) $\underset{k\to\infty}{Lt}\dfrac{U_{k+1}}{U_k} = 1/4 < 1$......(4)Convergent....... (1)

Module-V

19 a) (If the answer is correct without writing the formula give full mark) (7)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x \text{ ....(1)}$$

Formula $a_0=2\int_0^1 x\,dx=1....(1+1)$ *Formula* $a_n=2\int_0^1 x\cos n\pi x\,dx=\frac{2\left((-1)^n-1\right)}{n^2\pi^2}$ ....(1+1)

$b_n=0$........(1)

Deduction----(1)

OR

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x \text{ .... (1)}$$

Formula $a_0=\int_0^1 x\,dx=\frac{1}{2}....(1+1)$ *Formula* $a_n=2\int_0^1 x\cos n\pi x\,dx=\frac{2\left((-1)^n-1\right)}{n^2\pi^2}$ ....(1+1)

$b_n=0$........(1)

Deduction----(1)

b) (If the answer is correct without writing the formula give full mark.) (7)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \text{ ....(2)}$$

Formula, $a_0=\frac{2}{L}\int_0^L f(x)\,dx=\frac{kL}{2}...(1+1)$

Formula $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx=\frac{2kL}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}-1-(-1)^n\right).....(1+2)$

OR

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \text{ ....(2)}$$

Formula, $a_0=\frac{1}{L}\int_0^L f(x)\,dx=\frac{kL}{4}...(1+1)$

Formula $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx=\frac{2kL}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}-1-(-1)^n\right).....(1+2)$

20 a) (If the answer is correct without writing the formula give full mark) (7)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\,x + b_n \sin n\,x \;\ldots\ldots\ldots(1)$$

$$Formula, a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{\pi}\int_{0}^{\pi} x^2\,dx=\frac{\pi^2}{3}\;\ldots\ldots\ldots(1+1)$$

$$Formula, a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx=\frac{1}{\pi}\int_{0}^{\pi} x^2 \cos nx dx=\frac{2(-1)^n}{n^2}\;\ldots\ldots\ldots(1+1)$$

$$Formula,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_{0}^{\pi} x^2 \sin nx dx=\frac{2\left((-1)^n-1\right)}{\pi n^3}+\frac{\pi(-1)^{n+1}}{n}$$

....(1+1)

OR

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x \;\ldots\ldots\ldots(1)$$

$$Formula, a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{2\pi}\int_{0}^{\pi} x^2\,dx=\frac{\pi^2}{6}\;\ldots\ldots\ldots(1+1)$$

$$Formula, a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx=\frac{1}{\pi}\int_{0}^{\pi} x^2 \cos nx dx=\frac{2(-1)^n}{n^2}\;\ldots\ldots\ldots(1+1)$$

$$Formula,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_{0}^{\pi} x^2 \sin nx dx=\frac{2\left((-1)^n-1\right)}{\pi n^3}+\frac{\pi(-1)^{n+1}}{n}$$

....

(7)

b)

(If the answer is correct without writing the formula give full mark)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\,x + b_n \sin n\,x \;\ldots\ldots\ldots(2)$$

2

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$$Formula, a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2\pi^2}{3} \text{..........}(1+1)$$

$$Formula, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{4(-1)^n}{n^2} \text{...........}(1+1)$$

$$Formula,\ b_n = 0 \text{..............}(1)$$

OR

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,x + b_n \sin n\,x \text{ .... } (2)$$

$$Formula, a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{\pi^2}{3} \text{..........}(1+1)$$

$$Formula, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{4(-1)^n}{n^2} \text{...........}(1+1)$$

$$Formula,\ b_n = 0 \text{..............}(1)$$

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December 2020 (MODULE-1)

1 5. Determine the rank of the matrix, $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_1$

$\sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{bmatrix}$ $R_3 \rightarrow R_3 - R_2$ $\sim \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Rank of $A = \underline{\underline{2}}$

11) Q. Solve the following linear system of equations using Gauss elimination method

a) $x+y+z=6$

$x+2y-3z=-4$

$-x-4y+9z=18$

$$A=\begin{bmatrix}1&1&1\\1&2&-3\\-1&-4&9\end{bmatrix}\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}\quad B=\begin{bmatrix}6\\-4\\18\end{bmatrix}$$

$$A:B=\left[\begin{array}{ccc:c}1&1&1&6\\1&2&-3&-4\\-1&-4&9&18\end{array}\right]\quad \begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3+R_1\end{array}$$

$$\left[\begin{array}{ccc:c}1&1&1&6\\0&1&-4&-10\\0&-3&10&24\end{array}\right]\quad R_3\to R_3+3R_2$$

$$\sim\left[\begin{array}{ccc:c}1&1&1&6\\0&1&-4&-10\\0&0&-2&-6\end{array}\right]$$

$\rho(A:B)=\rho(A)=n=3.$
So, unique solution.

$x+y+z=6$

$y-4z=-10$

$-2z=-6 \Rightarrow z=3$

$y-12=-10 \Rightarrow y=-10+12=2$

$x+2+3=6 \Rightarrow x=1$

$$X=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

b) Find eigen values and eigen vectors of the matrix.

$$A=\begin{bmatrix}1&1&2\\-1&2&1\\0&1&3\end{bmatrix}$$

So, the Characteristic Equation is,

$$\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$$

$$\lambda^3 - 6\lambda^2 + (5+3+3)\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$\therefore$ Eigen values are, $\lambda = 1, 3, 2$

To find eigen vector corresponding to $\lambda = 1$, Put $\lambda = 1$ in eq ①.

$$\begin{bmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_2 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A_1) = 2 \quad n = 3 \quad \therefore n - r = 3 - 2 = \underline{1}$

$$-x_1 + x_2 + x_3 = 0$$

$$x_2 + 2x_3 = 0$$

Put $x_3 = k \implies x_2 + 2k = 0 \implies x_2 = \underline{-2k}$

$-x_1 - 2k + k = 0 \implies -x_1 = k \implies x_1 = -k$

$$X = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}_{k=1}$$

To find eigen vector corresponding to $\lambda = 3$, Put $\lambda = 3$ in eq ①.

$$A_2 = \begin{bmatrix} -2 & 1 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_2 \to 2R_2 - R_1$$

$$\sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad R_3 \to 3R_3 + R_2 \quad \sim \begin{bmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2 \;\; n = 3 \;\; \therefore n - r = 1$$

$-2x_1 + x_2 + 2x_3 = 0$

$-3x_2 = 0 \Rightarrow x_2 = 0$

$-2x_1 + 2x_3 = 0$

Put $x_3 = k \Rightarrow -2x_1 = -2k$

$x_1 = k$

$$X_2 = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{k=1}$$

To find eigen vector corresponding to $\lambda = 2$, Put $\lambda = 2$ in eq ①.

$$A_3 = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad \sim \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2, \quad n = 3, \quad n - r = 1$

$-x_1 + x_2 + 2x_3 = 0$

$-x_2 - x_3 = 0$

Put $x_2 = k$

$-k = x_3$

$-x_1 + k - 2k = 0 \Rightarrow -x_1 = k$

$x_1 = -k$

$$X_3 = \begin{bmatrix} -k \\ k \\ -k \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}_{k=1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

2. Show that the quadratic form $4x^2 + 12xy + 13y^2$ is Positive definite.

$$A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, the characteristic equation is,

$\lambda^2 - tr(A)\lambda + |A| = 0$

$\lambda^2 - 17\lambda + 16 = 0$

$\therefore$ Values of $\lambda = 16, 1$

all values are Positive.

$\therefore$ the quadratic form is Positive definite.

12 a. Show that the equation,

$x+y+z = a, \quad 3x+4y+5z = b, \quad 2x+3y+4z = c$

(i) have no solution, if $a = b = c = 1$

(ii) have many solutions if $a = b/2 = c = 1$

Ans:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$A:B = \left[\begin{array}{ccc:c} 1 & 1 & 1 & a \\ 3 & 4 & 5 & b \\ 2 & 3 & 4 & c \end{array}\right]$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-3a \\ 0 & 1 & 2 & c-2a \end{array}\right]$ $R_3 \to R_3 - R_2$ $\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-3a \\ 0 & 0 & 0 & a-b+c \end{array}\right]$

(i) $a = b = c = 1$

$(A:B) \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\rho(A:B) = 3$ $\rho(A) = 2$

$\therefore \rho(A:B) \neq \rho(A)$

no solution

(ii) $a = b/2 = c = 1$ or $a=1$ $b=2$, $c=1$

$(A:B) \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\rho(A) = 2 < n = 3$

b) Find the matrix of transformation that diagonalize the matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Ans- So the characteristic equation is,

$$\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$$

$$\lambda^3 - 12\lambda^2 + (8+14+14)\lambda - 32 = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\therefore \lambda = 8, 2, 2$$

Consider, $(A-\lambda I)x \Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$.

To find eigen vector corresponding to $\lambda = 8$, Put $\lambda = 8$ in eq ①.

$$A_1 = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + R_1 \end{matrix}$$

$$A_1 = \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2, n = 3, n - r = \underline{1}$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$-3x_2 - 3x_3 = 0 \quad \text{Put } x_2 = \underline{k}$$

$$-3k = 3x_3 \Rightarrow x_3 = \underline{-k}$$

$$-2x_1 - 2k - 2k = 0 \Rightarrow -2x_1 = 4k \Rightarrow x_1 = \underline{-2k}$$

$$X_1 = \begin{bmatrix} -2k \\ k \\ -k \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}_{k=1} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

To find eigen vector corresponding to $\lambda = 2$, Put $\lambda = 2$ in eq ①

$$A_2 = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$R_3 \to 2R_3 - R_1$

$$\sim \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 1$, $n = 3$, $n - r = \underline{\underline{2}}$

$$4x_1 - 2x_2 + 2x_3 = 0$$

Put $x_2 = k_1$ & $x_3 = k_2$

$$4x_1 - 2k_1 + 2k_2 = 0$$

$$4x_1 = 2k_1 - 2k_2$$

$$x_1 = \frac{k_1}{2} - \frac{k_2}{2} = \frac{k_1 - k_2}{2}$$

$$X_2 = \begin{bmatrix} \frac{k_1 - k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 & 1/2 & -1/2 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$X^{-1} A X = D$$

$$\begin{bmatrix} -2 & 1/2 & -1/2 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1/2 & -1/2 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \underline{\underline{D}}$$

3 If $z = \sin(y^2 - 4x)$. Find the rate of change of z with respect to x at the point $(3,1)$ with y held fixed.

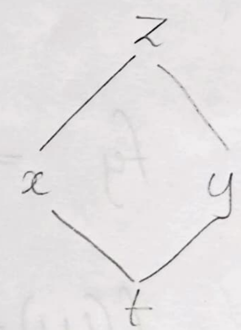
$$\frac{\partial z}{\partial x} = \cos(y^2 - 4x) \times -4$$

$$= -4\cos(y^2 - 4x)$$

$$\left.\frac{\partial z}{\partial x}\right|_{(3,1)} = -4\cos(1^2 - (4 \times 3))$$

$$= -4\cos(11)$$

4 Find $\frac{dz}{dt}$ by chain rule, where $z = 3x^2y^2$, $x = t^4$, $y = t^3$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt}$$

$$= 6xy^2 \times 4t^3 + 6x^2y \times 3t^2$$

$$= 6 \times t^4 \times t^6 \times 4t^3 + 6 \times t^8 \times t^3 \times 3t^2$$

$$= 24t^{13} + 18t^{13}$$

$$= \underline{\underline{42t^{13}}}$$

13 a) Find the local linear approximation of $\frac{4y}{x+z}$ at $(1,1,1)$

b) Find the absolute extrema of the fn $f(x,y) = x^2 - 3y^2 - 2x + 6y$ over the square region with vertices $(0,0)$ $(0,2)$ $(2,2)$ and $(2,0)$

a) $L(x,y,z) = f(x_0 y_0 z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0 y_0 z_0)$
$\times (y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$

$$f = \frac{4y}{x+z}$$

$$f_x = \frac{-4y}{(x+z)^2}$$

$$f_y = \frac{4}{x+z}$$

$$f_x(1,1,1) = \frac{-4 \times 1}{2^2}$$

$$= \underline{\underline{-1}}$$

$$f_y(1,1,1) = 2$$

$$f_z = \frac{-4y}{(x+z)^2}$$

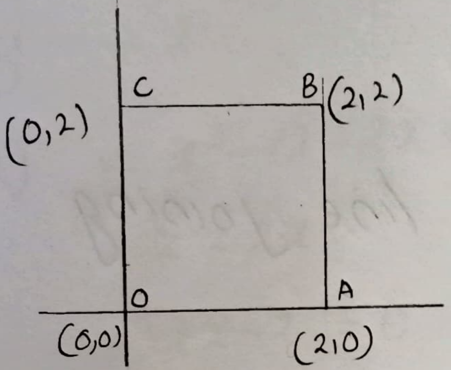
$$f_z(1,1,1) = -1$$

$$L(x,y,z) = 2 + (-1)(x-1) + 2(y-1) + -1(z-1)$$

$$= 2 - x + 1 + 2y - 2 - z + 1$$

$$= 2 - x + 2y - z$$

b)



The critical points of $f(x,y)$ in the interior of R are obtained by solving $f_x = 0$, $f_y = 0$

$$f_x = 2x - 2 = 0$$

$$\Rightarrow \quad 2x = 2$$

$$x = 1$$

$$f_y = -6y + 6 = 0$$

$$\Rightarrow \quad 6y = 6$$

$$y = 1$$

$\therefore$ Critical point is $(1,1)$

Consider the boundary of the region. It is the combination of 4 line segments.

line joining (0,0) and (2,0)

$$y = 0$$

$$f(x,y) = x^2 - 2x$$

$$f' = 2x - 2 = 0$$

$$\Rightarrow \quad 2x = 2$$

$$x = 1$$

The critical point on the line joining (0,0) (2,0) is (1,0)

$\therefore$ absolute extrema can occur on the boundary points (0,0) (2,0) and the critical point (1,0)

line joining (2,0) and (2,2)

$$x = 2$$

$$f(x,y) = f(2,y)$$

$$= \cancel{4} - 3y^2 - \cancel{4} + 6y$$

$$= -3y^2 + 6y$$

$$f' = -6y + 6 = 0$$

$$\Rightarrow \quad y = 1$$

$\therefore$ the critical point on this line is (2,1)

$\therefore$ abs. extrema can occur on the boundary points $(2,0)$ $(2,2)$ and the critical point $(2,1)$

line joining $(2,2)$ $(0,2)$

$y = 2$

$f(x,2) = x^2 - 12 - 2x + 12$

$= x^2 - 2x$

$f' = 2x - 2 = 0$

$\Rightarrow x = 1$

$\therefore$ The critical point on this line is $(1,2)$

$\therefore$ abs. extrema can occur on the boundary points $(2,2)$ $(0,2)$ and the critical point $(1,2)$

line joining $(0,2)$ and $(0,0)$

$x = 0$

$f(0,y) = -3y^2 + 6y$

$f' = -6y + 6 = 0$

$\Rightarrow y = 1$

$\therefore$ critical point is $(0,1)$

$\therefore$ abs. extrema can occur on $(0,2)$ $(0,0)$ and the critical point $(0,1)$

(x,y)	$(0,0)$	$(2,0)$	$(2,2)$	$(0,2)$	$(1,0)$	$(2,1)$	$(1,1)$	$(1,2)$	$(0,1)$
$f(x,y)$	0	0	0	0	-1	3	2	-1	3

$\therefore$ abs. minimum value is -1

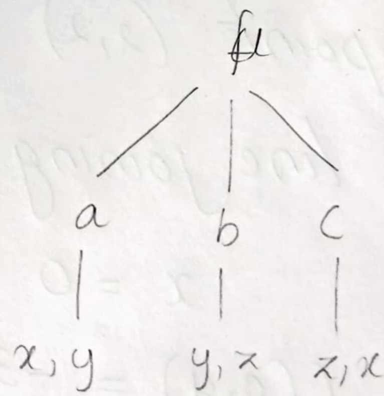
abs. maximum value is 3.

12. a) If $u = f\left(\frac{x}{y}, \frac{y}{x}, \frac{z}{x}\right)$ find the value of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

b) Locate all relative extrema of $f(x,y) = 2xy - x^3 - y^2$

a) Let $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} \times \frac{\partial a}{\partial x} + \frac{\partial u}{\partial c} \times \frac{\partial c}{\partial x}$$

$$= \frac{\partial u}{\partial a} \times \frac{1}{y} + \frac{\partial u}{\partial c} \times \frac{-z}{x^2}$$

$$= \frac{\partial u}{\partial a} \times \frac{1}{y} - \frac{\partial u}{\partial c} \times \frac{z}{x^2} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} \times \frac{\partial a}{\partial y} + \frac{\partial u}{\partial b} \times \frac{\partial b}{\partial y}$$

$$= \frac{\partial u}{\partial a} \times \frac{-x}{y^2} + \frac{\partial u}{\partial b} \times \frac{1}{z}$$

$$= \frac{-\partial u}{\partial a}\frac{x}{y^2} + \frac{\partial u}{\partial b} \times \frac{1}{z} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial b} \times \frac{\partial b}{\partial z} + \frac{\partial u}{\partial c} \times \frac{\partial c}{\partial z}$$

$$= \frac{-y}{z^2}\frac{\partial u}{\partial b} + \frac{\partial u}{\partial c} \times \frac{1}{x} \longrightarrow (3)$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z}$$

$$= 0$$

ie, $\left[x \frac{\partial u}{\partial a} \times \frac{1}{y} - \frac{\partial u}{\partial c} \times \frac{z}{x}\right] + \left[\frac{-\partial u}{\partial a}\frac{x}{y} + \frac{\partial u}{\partial b}\frac{y}{z}\right]$

$$+ \left[-\frac{y}{z}\frac{\partial u}{\partial b} + \frac{\partial u}{\partial c}\frac{z}{x}\right]$$

$$= \underline{\underline{0}}$$

b) Critical points

$f_x = 2y - 3x^2 = 0$ ie, $2y - 3x^2 = 0 \longrightarrow (1)$

$f_y = 2x - 2y$

$2x - 2y = 0$

$x = y \longrightarrow (2)$

Sub. eqn (2) in (1)

$2x - 3x^2 = 0$

$x(2 - 3x) = 0$

$\Rightarrow \quad x = 0 \quad or \quad 2 - 3x = 0$

$2 - 3x = 0$

$3x = 2$

$x = \frac{2}{3}$

$\therefore$ Critical points are $(0,0)$ $(2/3, 2/3)$

$D = f_{xx}(x_0 y_0)\, f_{yy}(x_0 y_0) - f_{xy}^2(x_0 y_0)$

$f_{xx} = -6x$

$f_{yy} = -2$

$f_{xy} = 2$

$\therefore$ at $(0,0)$

$D = (-6 \times 0) \times (-2) - 2^2$

$= -4$

$D < 0 \quad \therefore (0,0)$ is a saddle point.

at $(2/3, 2/3)$

$$D = \left(-6 \times \frac{2}{3}\right) \times -2 - 2^2$$

$$= \underline{\underline{4}}$$

$D > 0$

$f_{xx} = -4 < 0$

$\therefore$ f has relative maximum at $(2/3, 2/3)$

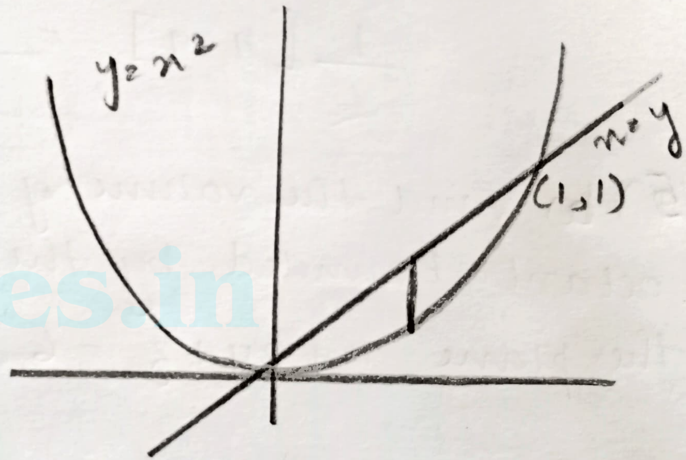
December 2020

5) Find the mass of the lamina with density function x^2 which is bounded by $y = x$ and $y = x^2$ (MODULE-III)

ans: $\int(x, y) = x^2$

mass, $M = \iint_R s(x, y)\, dA$

$y = x^2$ $\quad$ $y = x$



$y: x^2 \longrightarrow x$

$x: 0 \longrightarrow 1$

$x = x^2$

$x - x^2 = 0$

$x(1-x) = 0$

$x = 0 \qquad x = 1$

$y = 0 \qquad y = 1$

$$M = \int_0^1 \int_{x^2}^{x} x^2\, dy\, dx$$

$$= \int_0^1 x^2 [y]_{x^2}^{x} \, dx$$

$$= \int_0^1 x^2 [x - x^2] \, dx = \int_0^1 x^3 - x^4 \, dx$$

$$= \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \left[\frac{1}{4} - \frac{1}{5}\right] - 0$$

$$= 1/20$$

6.) Evaluate $\iint_R y^2 x \, dA$ over the region $R = \{(x, y) - 3 \leq x \leq 2, 0 \leq y \leq 1\}$.

ans:

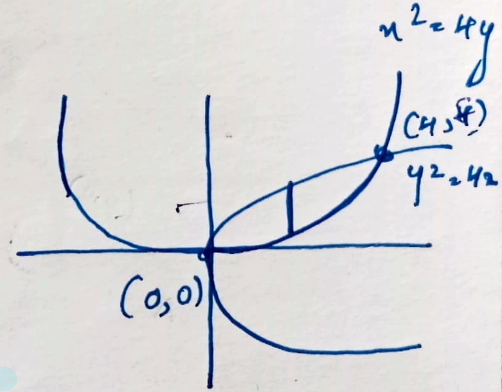
$$\iint_R y^2 x \, dA = \int_{-3}^{2} \int_0^1 y^2 x \, dy \, dx$$

$$= \int_{-3}^{2} \left[\frac{y^3}{3}\right]_0^1 x \, dx = \frac{1}{3} \int_{-3}^{2} x \, dx = \frac{1}{3} \left[\frac{x^2}{2}\right]_3^2$$

$$= \frac{1}{6} [4 - 9] = \frac{1}{6} \times -5 = \frac{-5}{6}$$

15 a) use double integral to find the area of the region enclosed b/w the parabola, $y = \frac{x^2}{2}$ and the line $y = 2x$.

ans: $\int_0^4 \int_{x^2/2}^{2x} dy\, dx = \int_0^4 [y]_{\frac{x^2}{2}}^{2x} dx.$



$= \int_0^4 \left[2x - \frac{x^2}{2}\right] dx.$

$= \left[\frac{2x^2}{2} - \frac{x^3}{6}\right]_0^4 = 4^2 - \frac{4^3}{6}.$

$= 16 - \frac{32}{3}$

$= \frac{16}{3}$

$x^2 = 4y.$

$x^2 = 4 \cdot 2\sqrt{x}.$

$x^2 = 8\sqrt{x}$

$x^{3/2} = 8$

$x^{1/2} = 2$

$x = 4 \Rightarrow y = 4$

$y : \frac{x^2}{4} \rightarrow \sqrt{4x}$

$x : 0 \rightarrow 4.$

15) b) Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $x + 2y + z = 6$

ans: $x + 2y + z = 6$

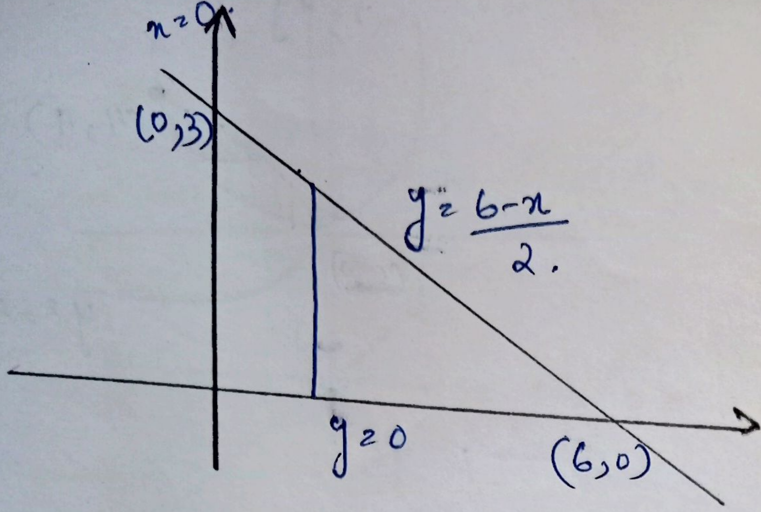
$\therefore z = 6 - x - 2y$

$\therefore z : 0 \longrightarrow 6 - x - 2y$.

put $z = 0$

$x + 2y = 6$

x	$y = 6 - x/2$
0	3
1	5/2
6	0



$y: 0 \longrightarrow \frac{6-x}{2}$

$x: 0 \longrightarrow 6.$

$$Vol = \iiint_R dA = \int_0^6 \int_0^{\frac{6-x}{2}} \int_0^{6-x-2y} dz\,dy\,dx$$

$$\int_0^6 \int_0^{\frac{6-x}{2}} [z]_0^{6-x-2y} dy\,dx = \int_0^6 \int_0^{\frac{6-x}{2}} 6-x-2y\,dy\,dx$$

$$= \int_0^6 \left[6y - xy\ \frac{2y^2}{2}\right]_0^{\frac{6-x}{2}} dx$$

$$= \int_0^6 \frac{1}{2}\left[36 - 6x - 6x + x^2 - \frac{36}{2} + 6x - \frac{x^2}{2}\right] dx.$$

$$= \frac{1}{2}\int_0^6 \left[18 - 6x + \frac{x^2}{2}\right] dx = \frac{1}{2}\left[18x - \frac{\cancel{6}^{3} x^2}{\cancel{2}} + \frac{x^3}{6}\right]_0^6$$

$$= \frac{1}{2}\left[18\times 6 - 3\times 6^2 + \frac{6^3}{6}\right] = \frac{1}{2}[36] = 18 //$$

16) a) change the order of integration and hence evaluate $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\,dx$.

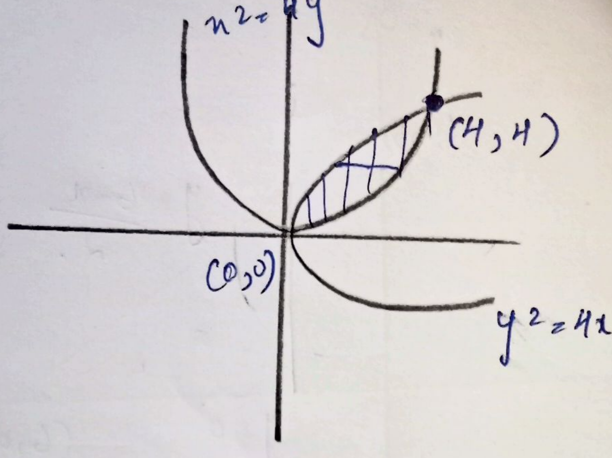
ans: $y: \frac{x^2}{4} \longrightarrow 2\sqrt{x}$

$y = \frac{x^2}{4} = x^2 = 4y$

$y = 2\sqrt{x}$

$\frac{y}{2} = \sqrt{x} \Rightarrow x = \frac{y^2}{4}$

$y^2 = 4x$



$x : 0 \longrightarrow 4$

$\frac{x^2}{4} = 2\sqrt{x}$

$x^2 = 8\sqrt{x} = x^4 = 64x$

$x^4 - 64x = 0 \longrightarrow x(x^3 - 64) = 0$

$x = 0$ or $x^3 = 64$ $\therefore$ $x = 4$, $y = 4$ //

$$= \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\,dy$$

$$\int_0^4 [x]_{y^2/4}^{2\sqrt{y}} dy = \int_0^4 [2\sqrt{y} - y^2/4]\,dy = \frac{1}{4}\int_0^4 8\sqrt{y} - y^2 dy$$

$$= \frac{1}{4}\left[8 \times \frac{y^{3/2}}{3/2} - \frac{y^3}{3}\right]_0^4 = \frac{1}{4}\left[\frac{16y^{3/2}}{3} - \frac{y^3}{5}\right]_0^4$$

$$= \frac{1}{12}\left[16 \times 2^3 - y^2\right]_0^4 = \frac{1}{12}\left[16 \times 8 - 64\right]$$

$$= \frac{1}{12} \times 64 = \frac{16}{3} //$$

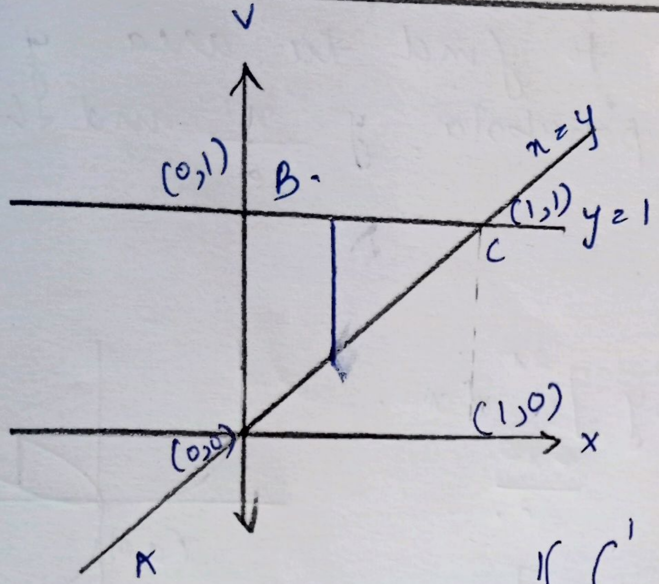
b) Evaluate $\iiint z\,dv$ where G is the in the 1st octant cut off from the cylindrical solid $y^2 + z^2 \leq 1$ and the planes $y = x$ and $x = 0$

ans: $y = x$ $\quad y^2 + z^2 \leq 1$ $\quad x = 0$

put $z = 0$ $\qquad z^2 \leq 1 - y^2$

$y^2 \leq 1$ $\qquad z \leq \sqrt{1 - y^2}$

$y \leq 1$ $\qquad z : 0 \longrightarrow \sqrt{1 - y^2}$



$y : x \to 1$

$x : 0 \to 1$

$$\int_0^1 \int_x^1 \int_0^{\sqrt{1-y^2}} z \, dz \, dy \, dx$$

ans : ~~gee x ee y²~~ $\cancel{y^2 + z^2} \le 1$

$$= \int_0^1 \int_x^1 \left[\frac{z^2}{2}\right]_0^{\sqrt{1-y^2}} dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_x^1 [1-y^2] \, dy \, dx = \frac{1}{2} \int_0^1 \left[y - \frac{y^3}{3}\right]_x^1 dx$$

$$= \frac{1}{2} \int_0^1 \left[1 - \frac{1}{3}\right] - \left[x - \frac{x^3}{3}\right] dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{2}{3} - x + \frac{x^3}{3}\right] dx.$$

$$= \frac{1}{2} \left[\frac{2}{3} x - \frac{x^2}{2} + \frac{x^4}{12}\right]_0^1$$

$$= \frac{1}{2} \left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12}\right]$$

$$= \frac{1}{2} \left[\frac{8 - 6 + 1}{12}\right] = \frac{3}{24}$$

$$= 1/8 .$$

Decmber 2020

MODULE - IV

7. Test the convergence of the series $\sum_{k=1}^{\infty} \left(\frac{k}{100}\right)^k$

Soln:

$$a_k = \left(\frac{k}{100}\right)^k$$

$$\rho = \lim_{k\to\infty} a_k^{1/k} = \lim_{k\to\infty} \left(\left(\frac{k}{100}\right)^k\right)^{1/k} = \lim_{k\to\infty} \frac{k}{100}$$

$$= \infty$$

$\therefore$ series diverges by Root test.

8. Does the series $\sum_{k=1}^{\infty} \left(\frac{-3}{4}\right)^k$ converge? If so find its sum.

Soln. It is a geometric series with $a = -\frac{3}{4}$

$$r = \frac{-3}{4}$$

$|r| = \frac{3}{4} < 1$ $\quad\therefore$ series converges.

$$\text{sum} = \frac{a}{1-r} = \frac{-3/4}{1 - -3/4} = \frac{-3/4}{1+3/4} = \frac{-3/4}{7/4}$$

$$= -3/7$$

17 a) Test the convergence of the series $1 + \frac{1.3}{3!} + \frac{1.3.5}{5!} + \dots$

Soln:

$$a_k = \frac{1.3.\dots(2k-1)}{(2k-1)!}$$

$$a_{k+1} = \frac{1.3\dots(2k-1)(2k+1)}{(2k+1)!}$$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \left[\frac{1.3\dots(2k-1)(2k+1)}{(2k+1)!}\right]\frac{(2k-1)!}{1.3\dots(2k-1)}$$

$$= \lim_{k\to\infty} \frac{(2k-1)!\,(2k+1)}{(2k-1)!\,2k\,(2k+1)} = \lim_{k\to\infty} \frac{1}{2k}$$

$$= 0 < 1$$

$\therefore$ By Ratio test, the series converges.

17 b) Find the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

Soln:

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{(k+1)}$$

$$S_n = a_0 + a_1 + \dots + a_n = \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{(n+1)}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

$\therefore$ Sum $= 1$

18 a) Test the convergence of 1) $\sum_{k=1}^{\infty} \frac{k!}{3!(k-1)!\,3^k}$

2) $\sum_{k=1}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$.

Soln:

$$a_k = \frac{k!}{3!(k-1)!\,3^k}$$

$$a_{k+1} = \frac{(k+1)!}{3!\,k!\,3^{k+1}}$$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)!}{3!\,k!\,3^{k+1}} \cdot \frac{3!(k-1)!\,3^k}{k!}$$

$$= \lim_{k\to\infty} \frac{(k+1)}{3k} = \frac{1}{3} \lim_{k\to\infty} \frac{1+1/k}{1}$$

$$= \frac{1}{3} < 1$$

$\therefore$ series converges by Ratio test.

2) $$a_k = \left(\frac{4k-5}{2k+1}\right)^k$$

$$\rho = \lim_{k\to\infty} a_k^{1/k} = \lim_{k\to\infty} \left(\left(\frac{4k-5}{2k+1}\right)^k\right)^{1/k}$$

$$= \lim_{k\to\infty} \frac{4k-5}{2k+1} = \lim_{k\to\infty} \frac{4-5/k}{2+1/k} = \frac{4}{2} = 2 > 1$$

$\therefore$ series diverges by Root test.

18 b) Test the absolute or conditional convergence of

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{(k^3+1)}$$

Soln:

$$a_k = \frac{k^2}{k^3+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^3+1} = \lim_{k\to\infty} \frac{1/k}{1+1/k^3} = 0$$

$$a_{k+1} = \frac{(k+1)^2}{(k+1)^3+1}$$

$$a_k > a_{k+1}$$

$\therefore$ By alternating series test, $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k^3+1}$ converges

Consider the series of absolute values

ie $\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$ $\quad \therefore a_k = \frac{k^2}{k^3+1}$

By informal principle $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{k^2}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k}$,

harmonic series $\therefore$ diverges.

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{k^2}{k^3+1}}{1/k} = \lim_{k\to\infty} \frac{k(k^2)}{k^3+1}$$

$$= \lim_{k\to\infty} \frac{1}{1+1/k^3} = 1 > 0 \text{ and finite}$$

$\therefore$

By Limit comparison test, since $\sum \frac{1}{k}$ diverges

$$\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$$ also diverges.

$$\therefore \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k^3+1}$$ diverges absolutely.

$$\therefore \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k+1}$$ conditionally convergent.

MODULE-V KTU Dec 2020

PART A

9. Find the binomial series for $f(x) = (1+x)^{1/3}$ upto third degree term.

Binomial series for $f(x)$ is,

$$f(x) = (1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$

$$(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \cdots$$

$$= 1 + \frac{1}{3}x + \frac{\frac{1}{3} \times \frac{-2}{3}}{2}x^2 + \cdots$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \cdots$$

10. Find the Maclaurin's series of $f(x) = \log(1+x)$ upto third degree term.

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

$f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = \frac{-1}{(1+x)^2}$

$f'''(x) = \frac{2}{(1+x)^3}$

$f(0) = \log 1 = 0$

$f'(0) = 1$

$f''(0) = -1$

$f'''(0) = \frac{2}{1^3} = 2$

$$\therefore f(x) = 0 + x \times 1 + \frac{x^2}{2!} \times -1 + \frac{x^3}{3} \times 2 + \cdots$$

$$= x - \frac{x^2}{2!} + \frac{2x^3}{3} - \cdots$$

19 a) Expand into fourier series, $f(x) = e^{-x}$, $0 < x < 2\pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$2l = 2\pi$
$l = \pi$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi} = \frac{-1}{2\pi}\left[e^{-2\pi} - e^0\right]$$

$$= \frac{-1}{2\pi}\left[e^{-2\pi} - 1\right] = \frac{1}{2\pi}\left[1 - e^{-2\pi}\right]$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cdot\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-1\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\frac{e^{-2\pi}}{1+n^2}\left[-\cos n2\pi + 2\pi\sin n2\pi\right] - \left(\frac{e^0}{1+n^2}\left(-\cos 0 + n\sin 0\right)\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{e^{-2\pi}}{1+n^2}\left[-\cos 2n\pi\right] - \frac{1}{1+n^2}\left[-1\right]\right\}$$

$$= \frac{1}{\pi(1+n^2)}\left[1 - e^{-2\pi}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x} \sin(nx)\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}(-n) - \frac{1}{1+n^2}(-n)\right]$$

$$= \frac{n}{\pi(n^2+1)}\left[1 - e^{-2\pi}\right]$$

$$\therefore f(x) = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi(n^2+1)}\left(1-e^{-2\pi}\right)\cos nx +$$

$$\sum_{n=1}^{\infty} \frac{n}{\pi(n^2+1)}\left(1-e^{-2\pi}\right)\sin nx$$

b) Find the half range cosine series for $f(x) = (x-1)^2$ in $0 \le x \le 1$

Half range cosine series of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$a_0 = \frac{1}{1}\int_0^1 (x-1)^2\, dx$$

$$= \left.\frac{(x-1)^3}{3}\right]_0^1 = \frac{0}{3} - \frac{(-1)^3}{3} = \underline{\underline{\frac{1}{3}}}$$

$$a_n = \frac{2}{1}\int_0^1 (x-1)^2 \cos(n\pi x)\,dx$$

$$= 2\left\{ (x-1)^2 \times \frac{\sin(n\pi x)}{n\pi} - 2(x-1).\frac{1}{n\pi} \times -\frac{\cos(n\pi x)}{n\pi} + 2 \times -\frac{1}{n^2\pi^2} \times \frac{\sin(n\pi x)}{n\pi} \Big]_0^1 \right\}$$

$$= 2\left\{ 0 - \left[\frac{2}{n^2\pi^2} \times -1 \times \cos 0 \right] \right\}$$

$$= \frac{2 \times 2}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}$$

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos(n\pi x)$$

20 a) Find the fourier series of the function $f(x) = |x|$ in $-1 \le x \le 1$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} |x|\,dx = \frac{1}{2} \times 2\int_0^1 x\,dx$$

$$= \frac{x^2}{2}\Big]_0^1 = \frac{1}{2}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} |x| \cdot \cos(n\pi x)\, dx$$

$$= 2\int_0^1 x \cos n\pi x \, dx$$

$$= 2\left[x \times \frac{\sin(n\pi x)}{n\pi} - 1 \times \frac{1}{n\pi} \times \frac{-\cos(n\pi x)}{n\pi} \right]_0^1$$

$$= 2\left[\frac{1}{n\pi}\sin(n\pi) + \frac{1}{n^2\pi^2}\cos(n\pi) - \left(0 + \frac{1}{n^2\pi^2} \times \cos 0\right) \right]$$

$$= 2\left[\frac{1}{n^2\pi^2}(-1)^n - \frac{1}{n^2\pi^2} \right]$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1 \right]$$

$$b_n = \frac{1}{1}\int_{-1}^{1} |x| \sin(n\pi x)\, dx \qquad \left(\because \int_{-a}^{a} f(x)\, dx = 0 \text{ if } f(x) \text{ is odd}\right)$$

$$= 0$$

$$\therefore f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right) \cos(n\pi x)$$

20 b) Find the fourier sine series of $f(x) = x\cos x$ in $0 < x < \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\boxed{\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}}$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos x \sin(nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \left[\frac{\sin(n+1)x + \sin(n-1)x}{2} \right] dx$$

$$= \frac{1}{\pi}\left[\int_0^\pi x\left[\sin(n+1)x + \sin(n-1)x\right]\cdot\right] dx$$

$$= \frac{1}{\pi}\left[\int_0^\pi x \sin(n+1)x\, dx + \int_0^\pi x.\sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left\{\left[x \times \frac{-\cos(n+1)x}{n+1} - 1 \times \frac{-1}{n+1}\cdot\frac{\sin(n+1)x}{n+1}\right]_0^\pi\right.$$

$$\left. + x \times \frac{-\cos(n-1)x}{n-1} - 1\times\frac{-1}{n-1}\cdot\frac{\sin(n-1)x}{n-1}\right]_0^\pi \Bigg\}$$

$$= \frac{1}{\pi}\left\{\pi \times \frac{-1}{n+1}\cos(n+1)\pi + 0 + \pi \times \frac{-1}{n-1}\cos(n-1)\pi\right\}$$

$$= \frac{1}{\pi}\times\pi\left[\frac{-1}{n+1}(-1)^{n+1} - \frac{1}{n-1}(-1)^{n-1}\right]$$

$$= (-1)^{n-1}\left[\frac{-1}{n+1} - \frac{1}{n-1}\right] = (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{2n}{n^2-1}(-1)^n, \quad n \neq 1$$

When $n=1$, $b_1 = \frac{2}{\pi}\int_0^\pi x \sin x \cos x\, dx$

$$= \frac{2}{\pi}\int_0^\pi \frac{x}{2}\sin 2x\, dx$$

$$= \frac{1}{\pi}\int_0^\pi x \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[x \times \frac{-\cos 2x}{2} - 1 \times \frac{-1}{2}\times\frac{\sin 2x}{2}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\times\cos 2\pi\right] - 0$$

$$= \frac{1}{\pi}\times\frac{-\pi}{2}\times 1$$

$$= \frac{-1}{2}$$

$$\therefore f(x) = \frac{-1}{2}\sin x + \sum_{n=2}^{\infty}\frac{2n}{n^2-1}(-1)^n \sin nx$$