Module :- IV

Optimization

Constrained maxima and minima

Lagrange multiplier method

Q) Maximise $f(x,y) = x^2+y^2$ subject to $g(x,y) = \underbrace{x+y+\frac{-1}{2}=0}_{\text{constrain}}$

$$L(x,y,\lambda) = x^2+y^2+\lambda(x+y-1)$$

$$\frac{\partial L}{\partial x} = 2x+\lambda = 0$$

$$x = \frac{-\lambda}{2}$$

$$\frac{\partial L}{\partial y} = 2y+\lambda = 0$$

$$y = \frac{-\lambda}{2}$$

$$\frac{\partial L}{\partial \lambda} = x+y-1=0$$

$$= \frac{-\lambda}{2} + \frac{-\lambda}{2} - 1 = 0$$

$$= \frac{-2\lambda}{2} - 1 = 0$$

$$= -\lambda - 1 = 0$$

$$= -\lambda = 1, \quad \lambda = -1$$

$$x = \frac{+1}{2}, \quad y = \frac{+1}{2}$$

$$= (1/2, 1/2)$$

max value $= f(1/2, 1/2)$.

max value $= x^2+y^2$

$$= (1/2)^2 + (1/2)^2$$

$$= 1/4 + 1/4$$

$$= \frac{2}{4}$$

$$= 1/2 \text{ at } (1/2, 1/2)$$

2) Find the point x, y, z on the plane $2x+y-z-5=0$ that is closest to the origin.
(minimum)

$(0,0)$ (x,y,z)

distance minimum

$= \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$

$= \sqrt{x^2+y^2+z^2}$

$f(x,y,z) = x^2+y^2+z^2$ is minimum

minimise $x^2+y^2+z^2$ subjected to $2x+y-z-5=0$

$L(x,y,\lambda) = x^2+y^2+z^2+\lambda(2x+y-z-5)$

$\frac{\partial L}{\partial x} = 2x+2\lambda = 0$

$= x = -\lambda$

$\frac{\partial L}{\partial y} = 2y+\lambda = 0$

$= y = \frac{-\lambda}{2}$

$\frac{\partial L}{\partial z} = 2z+\lambda = 0$

$= z = \frac{\lambda}{2}$

$\frac{\partial L}{\partial \lambda} = 2x+y-z-5 = 0$

$\Rightarrow 2x-\lambda+\frac{-\lambda}{2} - \frac{\lambda}{2} -5 = 0$

$= -2\lambda - \frac{\lambda}{2} - \frac{\lambda}{2} - 5 = 0$

$= -2\lambda - \lambda - 5 = 0$

$= -3\lambda - 5 = 0$

$= -3\lambda = +5$

$\lambda = \frac{-5}{3}$

$x = \frac{-5}{3}$

$y = \frac{-5/3}{2} = \frac{-5}{3} \times \frac{1}{2}$

$= \frac{-5}{6}$

$z = \frac{-5/3}{2} = \frac{-5}{6}$

$P(x,y,z) = P\left(\frac{-5}{3}, \frac{-5}{6}, \frac{-5}{6}\right)$

Two constraints

2) Find the minimum value of $f(x,y,z) = x^2+y^2+z^2$ subject to the constraints $x+y+z=3$, $x-y+z=1$

$L(x,y,z,\lambda,\mu) = x^2+y^2+z^2+\lambda(x+y+z-3)+\mu(x-y+z-1)$

$\frac{\partial L}{\partial x} = 2x+\lambda+\mu = 0$

$2x = -\lambda-\mu$

$x = \frac{-\lambda-\mu}{2}$

$\frac{\partial L}{\partial y} = 2y+\lambda-\mu=0$

$y = \frac{-\lambda+\mu}{2}$

$\frac{\partial L}{\partial z} = 2z+\lambda+\mu=0$

$= 2z = -\lambda-\mu$

$z = \frac{-\lambda-\mu}{2}$

$\frac{\partial L}{\partial \lambda} = (x+y+z-3) = 0$

$\frac{\partial L}{\partial \mu} = x-y+z-1 = 0$

$\frac{\partial L}{\partial \lambda} = \frac{-\lambda-\mu}{2}+\frac{-\lambda+\mu}{2}+\frac{-\lambda-\mu}{2} = 3$

$= -3\lambda-\mu = 6 \rightarrow (1)$

$= \cancel{\lambda-\mu} = \cancel{\frac{6}{-3}}\ 2$

$\frac{\partial L}{\partial \mu} = \frac{-\lambda-\mu}{2}+\frac{-\lambda+\mu}{2}-\frac{\lambda-\mu}{2} = 1$

$= -\lambda-3\mu = 2 \rightarrow (2)$

$\cancel{(1)-(2)}$ $(2)-(1)$

$-\lambda - 3\mu = 2 \rightarrow (2)$

$\cancel{\lambda} - 3\mu$

$-3\lambda - \mu = 6 \rightarrow (1)$

$\lambda = -2 \quad \mu = 0$

$x = \frac{-\lambda - \mu}{2}$

$= \frac{--2-0}{2} = \frac{2}{2} = 1$

$y = \frac{-\lambda + \mu}{2} = \frac{-(-2)+0}{2} = +1$

$z = \frac{-\lambda - \mu}{2} = \frac{--2-0}{2} = 1$

$(x, y, z) = (1, 1, 1)$

min. ~~max~~ value $= x^2 + y^2 + z^2$

$= 1^2 + 1^2 + 1^2 = 3$

2) Find a point closest to the orgin of the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$

distance minimum $= \sqrt{x^2 + y^2 + z^2}$ is minimum

$\Rightarrow x^2 + y^2 + z^2$ is minimum.

$L(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 + \lambda(y + 2z - 12) + \mu(x + y - 6) = 0$

$\frac{\partial L}{\partial x} = 2x + \mu = 0$

$= 2x = -\mu$

$x = \frac{-\mu}{2}$

$\frac{\partial L}{\partial y} = 2y + \lambda + \mu = 0$

$= 2y = -\lambda - \mu$

$y = \frac{-\lambda - \mu}{2}$

$$\frac{\partial L}{\partial z} = 2z + 2\lambda = 0$$
$$= 2z = -2\lambda$$
$$z = -\lambda$$

$$\frac{\partial L}{\partial \lambda} = y + 2z - 12 = 0$$
$$= \frac{-\lambda-\mu}{2} + 2\cdot\lambda = 12$$
$$= \frac{-\lambda-\mu-4\lambda}{2} = 12$$
$$= -\lambda - \mu - 4\lambda = 24$$
$$= -5\lambda - \mu = 24 \rightarrow (1)$$

$$\frac{\partial L}{\partial \mu} = x + y - 6 = 0$$
$$= \frac{-\mu}{2} + \frac{-\lambda-\mu}{2} = 6$$
$$= -\mu - \lambda - \mu = 12$$
$$= -2\mu - \lambda = 12 \rightarrow (2)$$

(2) - (1)
$$= -2\mu - \lambda = 12$$
$$= -\mu - 5\lambda = 24$$

$$\mu = -4 \quad \lambda = -4$$

$$x = \frac{-\mu}{2} = \frac{-(-4)}{2} = 2 \qquad = (x, y, z) = (2, 4, 4).$$

$$y = \frac{-\lambda-\mu}{2} = \frac{-(-4)-(-4)}{2}$$
$$= \frac{8}{2} = 4$$

$$z = -\lambda = -(-4) = 4$$

Q) Find the maximum and minimum values (extremum value) of $f(x,y) = x^2+y^2$ subject to the constraints $x+y=1$ and $x-y=1$

$$L(x,y,\lambda,\mu) = x^2+y^2+\lambda(x+y-1)+\mu(x-y-1)$$

$f''(x)<0 \Rightarrow$ maximum

$f''(x)>0 =$ minimum

$$\frac{\partial L}{\partial x} = 2x+\lambda+\mu = 0$$

$$= x = \frac{-\lambda-\mu}{2}$$

$$\frac{\partial L}{\partial y} = 2y+\lambda-\mu = 0$$

$$= 2y = -\lambda+\mu$$

$$y = \frac{-\lambda+\mu}{2}$$

$$\frac{\partial L}{\partial \lambda} = x+y-1 = 0$$

$$= \frac{-\lambda-\mu}{2} + \frac{-\lambda+\mu}{2} = 1$$

$$= -\lambda-\mu+-\lambda+\mu = 2$$

$$= -2\lambda = 2$$

$$\lambda = -1$$

$$\frac{\partial L}{\partial \mu} = x-y-1 = 0$$

$$= \frac{-\lambda-\mu}{2} - \frac{(-\lambda+\mu)}{2} = 1$$

$$= -\lambda-\mu+\lambda-\mu = 2$$

$$= -2\mu = 2$$

$$\mu = -1$$

$$x = \frac{-\lambda-\mu}{2}$$

$$= \frac{-(-1)+1}{2} = 1$$

$$y = \frac{-\lambda+\mu}{2}$$

$$= \frac{--1+-1}{2}$$

$$= \frac{1+-1}{2} = 0$$

$x = 1$

$y = 0$

$h(x,y) = x + y = 1$

$y = 1 - x$

$f(x,y) = f(x, 1-x)$

$= x^2 + y^2$

$= x^2 + (1-x)^2$

$= x^2 + 1 - 2x + x^2$

$= 2x^2 - 2x + 1$

$f'(x, 1-x) = 4x - 2$

$f''(x, 1-x) = 4 > 0$

$f''(x,y)$ is minimum.

f is minimum at $(1,0)$

minimum value $= x^2 + y^2$

$= 1^2 + 0^2$

$= 1$

Method of steepest descent (Gradient descent).

The method steepest descent is an iterative optimisation algorithm used to find the minimum of a function.

Iterative update rule :-

$$\{(x_{k+1}, y_{k+1}) = (x_k, y_k) - \alpha \nabla f(x_k, y_k)\}$$

where α is the stepsize ~~and~~

continuing the iterations would bring the point even closer to the minimum

$\boxed{\nabla f = \langle f_x , f_y \rangle}$

Q) Minimise the function $f(x,y) = 2x^2 + y^2 - 4x + 4y$ starting from $(x_0, y_0) = (0,0)$ using steepest descent algorithm (perform 3 iterations, choose $\alpha = 0.1$)

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - \alpha \nabla f(x_k, y_k)$$

(x_0, y_0):

$$\nabla f = \langle f_x, f_y \rangle$$
$$= \langle 4x-4 , 2y+4 \rangle \qquad \alpha = 0.1$$
$$(x_0, y_0) = (0, 0)$$

1st iteration

$$(x_1, y_1) = (x_0, y_0) - \alpha \nabla f (x_0, y_0)$$
$$= (0,0) - 0.1 \, \nabla f (0,0)$$
$$= (0,0) - 0.1 \, (-4, 4)$$
$$= (0.0) - (-.4, .4)$$
$$= (.4, -0.4)$$

2nd iteration

$$(x_2, y_2) = (x_1, y_1) - \alpha \nabla f (x_1, y_1)$$
$$= (.4, -.4) - 0.1 \, \nabla f (.4, -.4)$$
$$= (.4, -.4) - 0.1 \, (4 \times .4 - 4, \; 2 \times -.4 + 1)$$
$$= (.4, -.4) - 0.1 \, (-2.4, \; 3.2)$$

$(0.4, -0.7) - (-0.24, 3.2)$

$(0.64, -0.72)$

IIIrd iteration

$(x_3, y_2) = (x_2, y_2) - \alpha \nabla f(x_2, y_2)$

$= (0.64)(-.72) - 0.1 \nabla f(0.64, -0.72)$

$= (0.64)(-.72) - 0.1(-1.44, 2.56)$

$= (0.64)(-.702) - (-0.144, 0.256)$

$= (0.784, -0.976)$

) Minimise the function $f(x,y,z) = x^2+2y^2+3z^2$ using the method of steepest distance descent starting from the point $(x_0, y_0, z_0) = (1,1,1)$ with a step size $\alpha = 0.01$ upto 3 iteration

$\nabla f = \langle f_x, f_y, f_z \rangle$

$= (2x, 4y, 6z)$

$(x_0, y_0, z_0) = (1,1,1) \quad \alpha = 0.01$

Ist

$(x_1, y_1, z_1) = (x_0, y_0, z_0) - \alpha \nabla f(x_0, y_0, z_0)$

$= (1,1,1) - 0.01 \nabla f(1,1,1)$

$= (1,1,1) - 0.01(2,4,6)$

$= (1,1,1) - (.02, 0.04, 0.06)$

$= (0.98, 0.96, 0.94)$

2nd

$(x_2, y_2, z_2) = (x_1, y_1, z_1) - \alpha \nabla f(x_1, y_1, z_1)$

$= (0.98, .96, .94) - 0.01 \nabla f(.98, .96, .94)$

$= (.98, .96, .94) - 0.01 (1.96, 3.84, 5.64)$

$= (.98, .96, .94) - (0.0196, 0.0384, 0.0564)$

$= (.9604, .9216, 0.8836)$

3rd

$(x_3, y_3, z_3) = (x_2, y_2, z_2) - \alpha \nabla f(x_2, y_2, z_2)$

$= \cancel{(.98, .96.}$

$= (.9604, .9216, .8836) - 0.01 (\cancel{3.8416}\ 1.9208, 3.6864, 5.3016)$

$= (.9604, .9216, .8836) - (0.0192, 0.0368, 0.0530)$

$= (0.9412, 0.8848, 0.8306)$

Linear Programing Problem

Graphical method :

Q) Solve the LPP graphically maximise $Z = 3x + 2y$ subject to $x + 2y \leq 10$, $3x + 2y \leq 15$, $x, y \geq 0$.

$Z = 3x + 2y$ } objective function.

$x + 2y \leq 10$
$3x + y \leq 15$ } constraints

$x, y \geq 0$ & no-ve constant.

$x+2y = 10$

when $x=0$ $2y=10$
$y=5$

$(0,5)$

when $y=0$ $x=10$

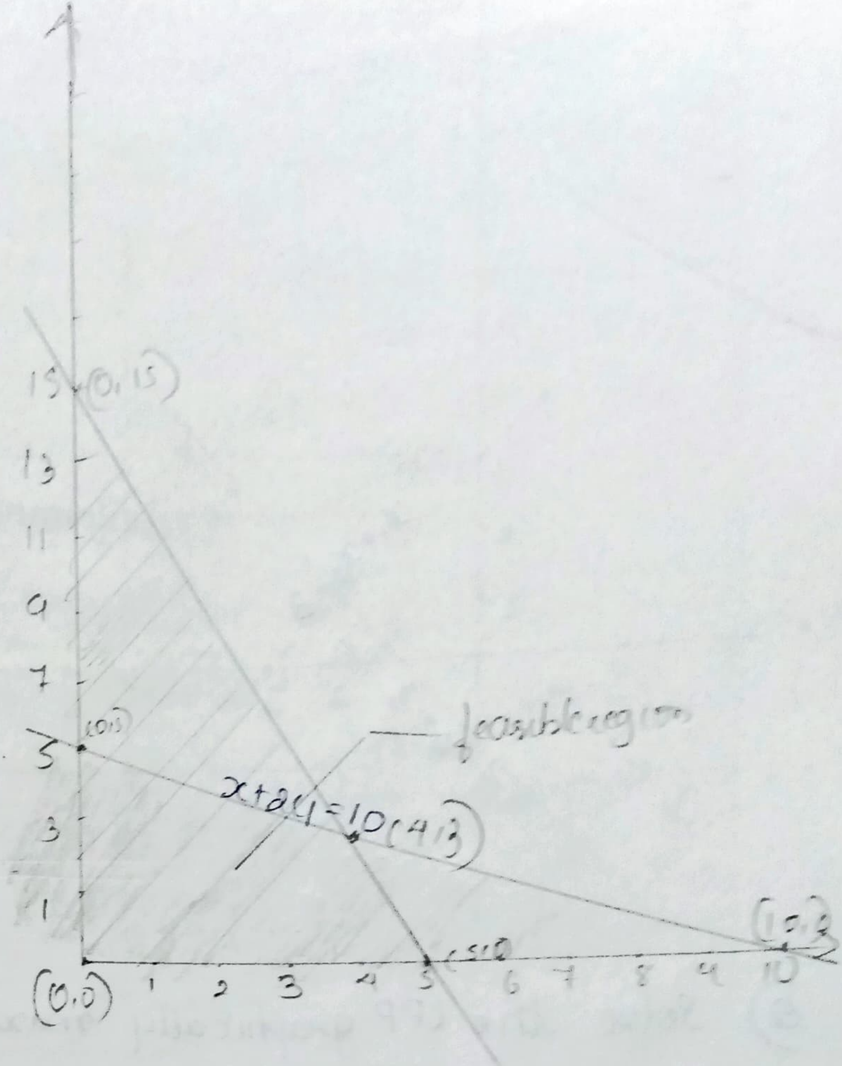
$(10,0)$.

$3x+2y = 15$

$x=0$ $y=15$

$y=0$ $x=5$

$(0,15)$, $(5,0)$



$x+2y = 10$
$3x+y = 15$

$x=4$ $y=3$

$(0,5) = 10$

$(10,0) = 0$

~~(0,15)~~ =

$(5,0) = 15$

$(4,3) = 18$

Maximum value $Z=18$ at $(4,3)$.

2) Solve the LPP maximise $Z=3x_1+2x_2$ subject to $x_1+x_2 \geq 3$, $x_1-x_2 \geq 1$, $x_1, x_2 \geq 0$

$x_1+x_2 \leq 3$

$x_1=0$ $x_2=3$ $(0,3)$

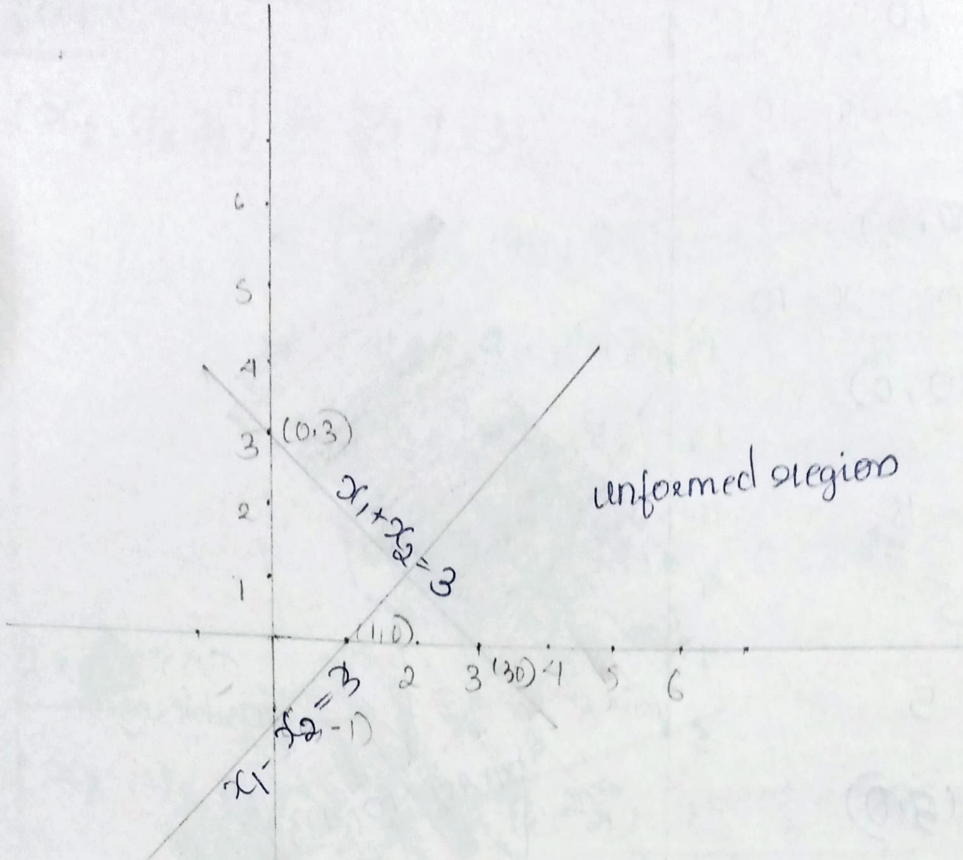
$x_2=0$ $x_1=3$ $(3,0)$

$x_1-x_2=1$

$x_1=0$ $x_2=-1$ $(0,-1)$

$x_2=0$ $x_1=1$ $(1,0)$.

minimum
2770



(Q) Solve the LPP graphically maximise $z = 2x + 3y$ subject to

$x + y \geq 30$, $y \leq 32$, $0 \leq y \leq 12$

$x - y \geq 0$, $0 \leq x \leq 20$

1) $x + y = 30$

$x = 0 \quad y = 30 \quad (0, 30)$

$y = 0 \quad x = 30 \quad (30, 0)$

2) $y = 32$ (draw ‖ to x axis)

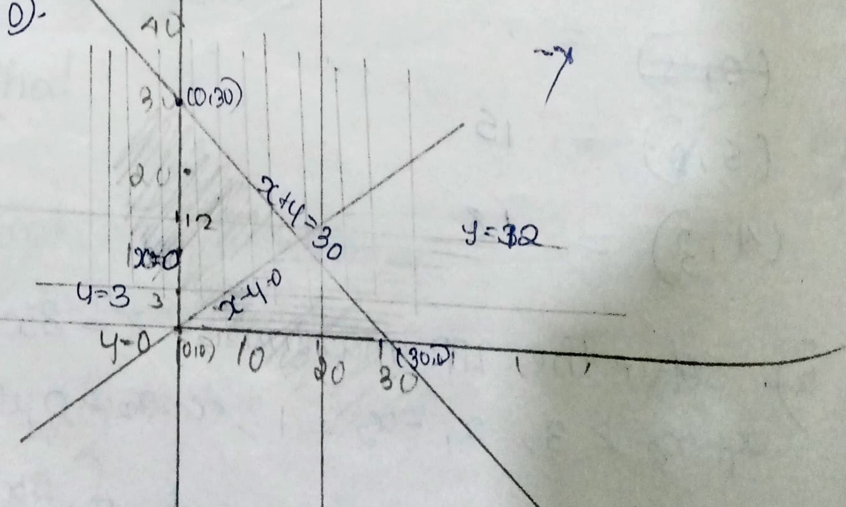
3) $y = 0$, $y = 12$

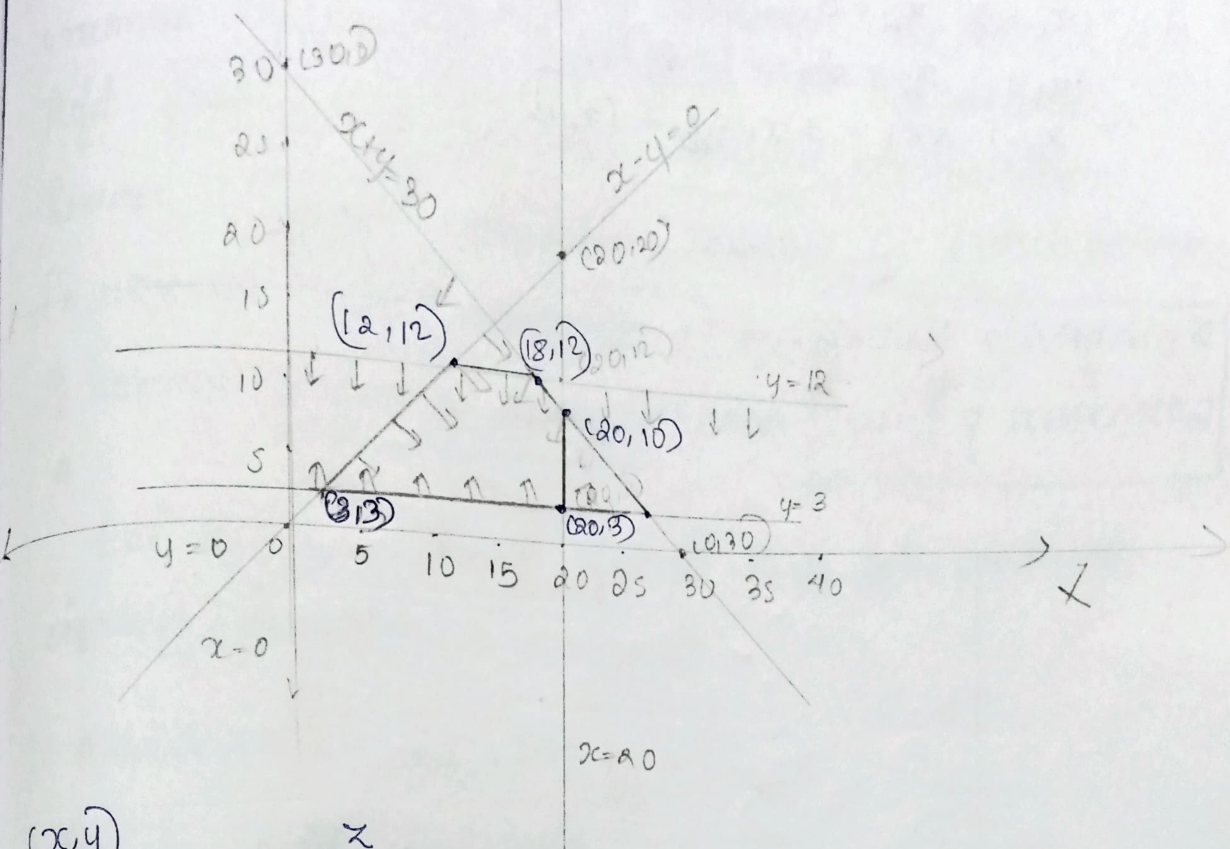
4) $x - y = 0$

$x = y$

$(0,0)\ (1,1)\ (2,2)\ \ldots$

5) $x = 0 \quad x = 20$





(x,y)		z
(3,3)	=	15
(12,12)	=	60
(18,12)	=	72
(20,10)	=	70
(20,3)	=	49

maximum value :- 72 at (18,12).

2) Solve the LPP graphically maximize $z = 2x_1 + 3x_2$ subject to $4x_1 + 3x_2 \geq 12$, $x_1 - x_2 \geq -3$, $x_2 \leq 6$, $2x_1 - 3x_2 \leq 0$

$4x_1 + 8x_2 = 12$

$x_1 = 0$. $x_2 = 4$ (0,4)

$x_2 = 0$ $x_1 = 3$ (3,0)

$x_2 \leq 6$

$x_2 = 6$

$y = 6$

$2x_1 - 3x_2 = 0$

$x_1 = 0$ $x_2 = 0$ (0,0)

$x_2 = 0$ $x_1 = 0$ (0,0)

= (0,0), (1,2), (2,0)

$x_1 - x_2 \geq -3$

$x_1 = 0$, $x_2 = 3$ (0,3)

$x_2 = 0$ $x_1 = -3$ = (-3,0) = (0,0) (0,1) (0,2)

$2x_1 - 3x_2 \leq 0$

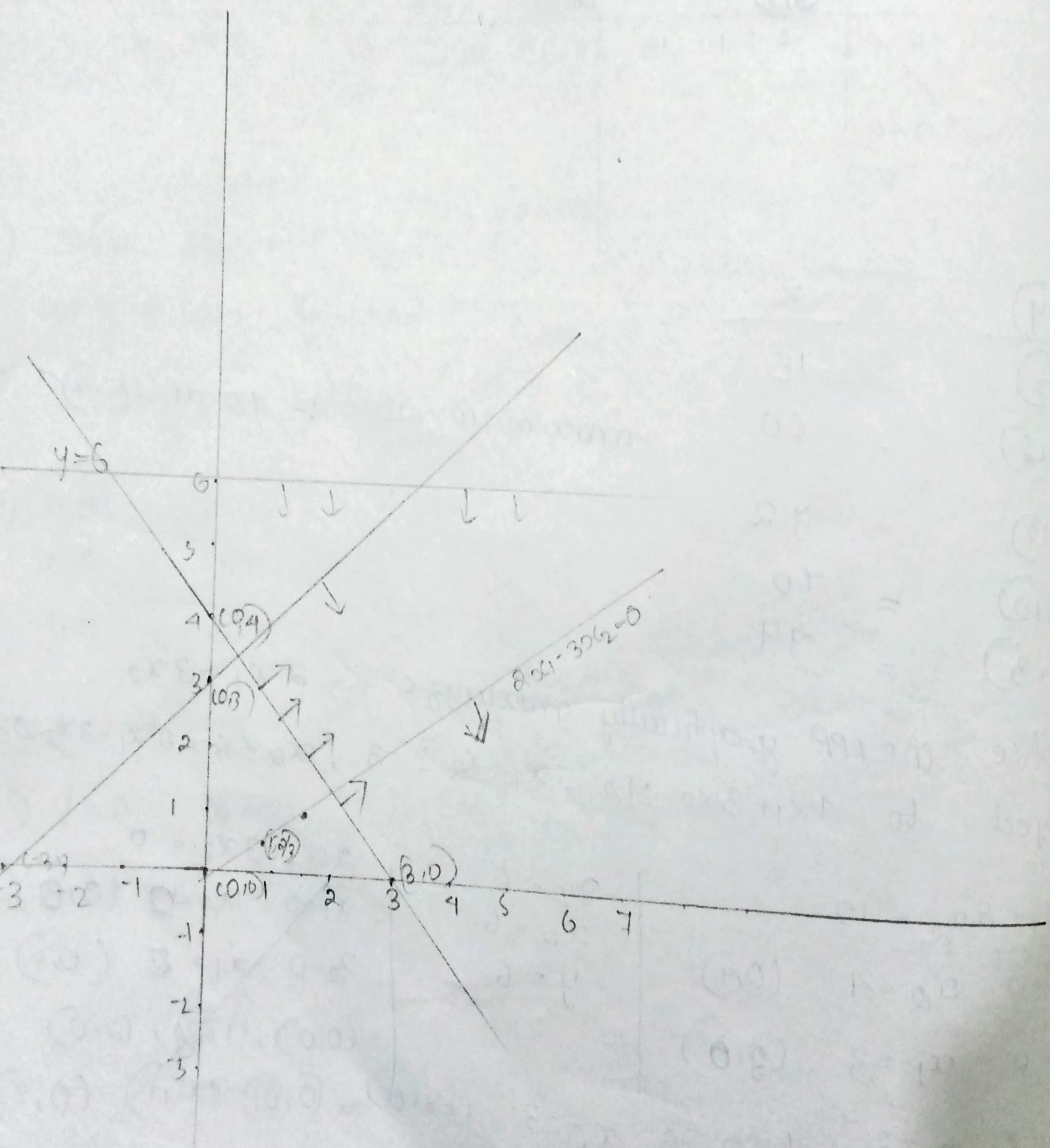
$2x_1 - 3x_2 = 0$

$2x_1 = 3x_2$

$x_1 = 0 \quad x_2 = 0 \quad (0,0)$

$x_1 = 1 \quad x_2 = 2/3 = (1, 2/3)$

$x_2 = 1 \quad 2x_1 = 3 \quad x_1 = 3/2 = (3/2, 1)$.



Formulation of LPP

1) A factory manufactures two products P_1 and P_2. Each product requires different amount of resources. The objective is to maximise the profit. Profit from P_1 - ₹ 3 per unit; Profit from P_2 - ₹ 5 per unit. Resource constraints :-

Resource 1 :- 4 units available, Resource 2 :- 6 units available.

P_1 requires $\boxed{1 \text{ unit of resource } 1}$ and $\boxed{2 \text{ units of resource } 2}$

P_2 requires $\boxed{2 \text{ unit of resource } 1}$ and $\boxed{1 \text{ unit of resource } 2}$

Let x_1, x_2 be the no. of units, P_1, P_2 to be produced.

objective function :

Maximise $Z = 3x_1 + 5x_2$

Subject to the constraints.

$1x_1 + 2x_2 \leq 4$ (Resource 1 constraint)

$2x_1 + x_2 \leq 6$ (Resource 2 ")

$x_1, x_2 \geq 0$ (non -ve constraints)

2) Diet Problem

A hospital wants to create a nutritious diet for patients that meets certain nutritional requirements at the minimum cost. The diet must include (atleast) 2000 calories, 500 g of proteins and 800 g of carbohydrates per day. There are 2 food items available with the following nutritional value and cost.

food1 :- cost $ 3/units provides 500 calories, 20g of proteins and 30 g of carbohydrates.

food 2 : cost $ 4/unit provides 600 calories, 30g of proteins and 40g of carbohydrates.

Objective function.

minimise $Z = 3x_1 + 4x_2$

Subject to the constraints.

$500x_1 + 600x_2 \geq 2000$

$20x_1 + 30x_2 \geq 500$

$30x_1 + 40x_2 \geq 300$

(2) A farmer has 480 acres to cultivate wheat, Barley, and oats. wheat requires 2 acres of land, 4 hours of labours and dollar 10 of fertilizer. Barley requires 1 acre of land, 3 hours of labour and dollar 8 of fertilizer. oats requires 3 acres of land, 2 hours of labour and requires dollar 12 of fertilizer. Labour availability is 1440 hours and fertilizer is $ 4800. Profit per acre is $ 50 for wheat, $ 40 for Barley and $ 30 for oats. formulate an LPP to maximise the total profit.

maximize profit $Z = 50x_1 + 40x_2 + 30x_3$

subject to the constraints

$2x_1 + x_2 + 3x_3 \leq 480$ (land)

$4x_1 + 3x_2 + 2x_3 \leq 1440$ (hours)

$10x_1 + 8x_2 + 12x_3 \leq 1800$ (fertilizer).

$x_1, x_2, x_3 \geq 0$ (non-ve constraints)

Module (2)

Domain and Range

1) find the domain and range of the function $f(x,y) = e^{x+y}$.

Domain (all real numbers in R^2).

Range $(0, \infty)$

$e^0 = 1$, $e^{-1} = \frac{1}{e}$ — eg any value of x and y is given to e^{x+y} is range

2) find the domain and range of $f(x,y) = \frac{x+y}{x-y}$

Domain $= R^2 - \{(x,y) | x = y\}$. or $R^2 / \{(x,y) / x = y\}$

Range $= R$

eg: $= \frac{x+y}{x-y}$

$= \frac{-1+(-1)}{-1-(-1)}$ (not possible)

Levelcurve :-

A level curve of a function $f(x,y)$ is a curve along which the function has a constant value. i.e.

$\boxed{f(x,y) = k}$

Q) find the level curves of the function $f(x,y) = \sqrt{9-x^2-y^2}$ for $c = 0, 1, 2, 3$. hence sketch the graph of level curve.

$f(x,y) = c$

$\sqrt{9-x^2-y^2} = c$

$9 - x^2 - y^2 = c^2$

$-x^2 - y^2 = c^2 - 9$

multiply by a (-)

$x^2 + y^2 = 9 - c^2$

when $c = 0$

$x^2 + y^2 = 9 \Rightarrow r = 3$

$c = 1$

$x^2 + y^2 = 8 \quad r = 2\sqrt{2}$