Module 2 : Partial Derivatives

Limits for Functions Of Two Variables

A function $f(x,y)$ approaches the limit L, as ~~ordered po~~ (x,y) app (x_0, y_0) and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Q. ~~Find limit ord~~

1. Find $\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3}$

$$= \frac{\lim_{(x,y)\to(0,1)}(x-xy+3)}{\lim_{(x,y)\to(0,1)}(x^2y+5xy-y^3)} = \frac{0-0\times1+3}{0^2\times1+5\times0\times1-1^3}$$

$$= \frac{3}{-1} = -3$$

2. $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$

$$= \sqrt{\lim_{(x,y)\to(3,-4)}(x^2+y^2)} = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

3. $\lim_{(x,y)\to(\pi/2,0)} \left(\frac{x}{\sin x} - \frac{\sin y}{y}\right)$

$$= \frac{\lim_{(x,y)\to(\pi/2,0)} x}{\lim_{(x,y)\to(\pi/2,0)} \sin x} - \frac{\lim_{(x,y)\to(\pi/2,0)} \sin y}{\lim_{(x,y)\to(\pi/2,0)} y}$$

$$= \frac{\pi/2}{\sin \pi/2} - \frac{\sin 0}{0} = \frac{\frac{\pi}{2}}{1} - \frac{0}{0} = \frac{\pi}{2} = \pi/2 - 1$$

Q. Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \frac{0^2 - 0\times 0}{\sqrt{0} - \sqrt{0}} = 0$$

$$= \cancel{x^2 - xy} \lim_{(x,y)\to(0,0)} \frac{x^2 - xy\,(\sqrt{x} + \sqrt{y})}{(\sqrt{x})^2 - (\sqrt{y})^2} = \frac{\cancel{x^2\sqrt{x} + x^2\sqrt{y}} - \cancel{xy}\sqrt{x} - xy\sqrt{y}}{x - y}$$

$$= 0 \quad \cancel{x^2\sqrt{x} + x^2\sqrt{y} - xy\sqrt{x} - xy}$$

$$\frac{x(\cancel{x-y})(\sqrt{x} + \sqrt{y})}{\cancel{x-y}} = 0\,(0+0) = 0$$

Q. Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

$$\frac{\lim_{(x,y)\to(0,0)} x^2 - y^2}{\lim_{(x,y)\to(0,0)} x^2 + y^2}$$

Let $(x,y) \to (0,0)$ along the path $y = mx$, then.

$$\lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{x^2(1 - m^2)}{x^2(1 + m^2)}$$

$$= \frac{1 - m^2}{1 + m^2}$$

~~Finally which~~ this depends on the value of m

$\therefore$ the limit does not exist.

Q. Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist limit.

Let $(x,y) \to (0,0)$ along the path $y = mx$ then,

$$\lim_{x\to 0} \frac{x \cdot mx}{x^2 + m^2x^2} = \frac{x^2 m}{x^2(1 + m^2)} = \frac{m}{1 + m^2}$$

~~the~~ which depends on the value of m

$\therefore$ the limit does not exist.

Continuity

A function $f(x,y)$ is continuous at (x_0, y_0) if f.

1. $f(x_0, y_0)$ is defined.
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ is exist
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

A function is continuous if it is continuous at every point of its domain.

Q1. Show that $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$ is continuous at every point except the Origin.

$f(0,0) = 0$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} \frac{2xy}{x^2+y^2} = 0/0$$

Let $(x,y) \to (0,0)$ along the path $y = mx$.

$$= \lim_{(x,y)\to(0,0)} \frac{2x\,mx}{x^2+m^2x^2} = \lim_{(x,y)\to(0,0)} \frac{2x^2 m}{x^2(1+m^2)} = \lim_{(x,y)\to(0,0)} \frac{2m}{(1+m^2)}$$

$$= \frac{2m}{(1+m^2)}$$

$\Rightarrow$ the function depends on the value

$\therefore$ The function $f(x,y)$ is not continuous at the origin.

Q2. Show that $f(x,y) = \begin{cases} y/\sqrt{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$ is discontinuous at $(0,0)$.

$f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}} = 0/0$$

Let $(x,y) \to (0,0)$ along the path $y = mx$.

$$= \lim_{(x,y)\to(0,0)} \frac{mx}{\sqrt{x^2+m^2x^2}} = \lim_{(x,y)\to(0,0)} \frac{mx}{\sqrt{x^2(1+m^2)}} = \lim_{(x,y)\to(0,0)} \frac{mx}{x\sqrt{1+m^2}}$$

$$= \frac{m}{\sqrt{1+m^2}}$$

The function is not continuous at $(0,0)$

Partial Derivatives

The partial derivative of ~~for~~ $z = f(x,y)$ with respect to x is $\frac{\partial z}{\partial x} =$

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

The paritial derivative of f $z = f(x,y)$ with respect to y is

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Q/Su **NOTE:**

Let $z = f(x,y)$, the 1st order partial derivatives are

1) $\frac{\partial z}{\partial x}$ (diff. z partially with x ~~con~~ keeping y as is constant)

2) $\frac{\partial z}{\partial y}$ (diff z partially with respect to y, keeping x as is constant.

Q: Suppose we have the fun. $f(x,y) = x^2y + 3xy^2$. Find the derivatives with respect to x and y.

$$f(x,y) = x^2y + 3xy^2$$

$$\frac{\partial f}{\partial x} = y \cdot 2x + 3y^2 \cdot 1$$

$$= 2xy + 3y^2$$

$$\frac{\partial f}{\partial y} = x^2 \cdot 1 + 3x \cdot 2y$$

$$= x^2 + 6xy$$

Q2. Let $f(x,y) = e^{x^2y^3}\sin(x^2-y^2)$ find f_x and f_y.

$$\partial f_x = \cancel{y^3 e^{x}}\; e^{x^2y^3}\cdot\cos(x^2-y^2)\cdot y^2\cdot 2x + \sin(x^2-y^2)\cdot e^{x^3y^3}\cdot y^3(2x)$$

$$= \cancel{2xy^2y}\; 2x\, e^{x^2y^3}\cos(x^2-y^2) + 2xy^3\sin(x^2-y^2)\cdot e^{x^2y^3}$$

$$f_y = \cancel{e^{x^2y^3}\sin f}\; e^{x^2y^3}\cos(x^2-y^2)\cdot -2y + \sin(x^2-y^2)\cdot e^{x^2y^3}\cdot x^2\cdot 3y^2$$

$$= -2y\, e^{x^2y^3}\cos(x^2-y^2) + 3y^2x^2\sin(x^2-y^2)e^{x^2y^3}.$$

Q3. Let $f(x,y) = \dfrac{x-y}{x^2+y^2}$ find f_x and f_y.

$$f_x = \frac{(x^2+y^2)\cdot 1 - ((x-y)(\cdot 2x+0))}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-(2x^2-2xy)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2+2xy}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+2xy}{(x^2+y^2)^2}$$

$$f_y = \frac{(x^2+y^2)(0-1) - ((x-y)(\cancel{0}+2y))}{(x^2+y^2)^2}$$

$$= \frac{\cancel{(x^2+y^2)} - x^2 - y^2 - (2xy - 2y^2)}{(x^2+y^2)^2} = \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

Q4. Find the 1st order partial derivatives of the following functions.

$f(x,y) = \sqrt{x^2+y^2}$.

$$f_x = \frac{1}{2\sqrt{x^2+y^2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y = \frac{y}{\sqrt{x^2+y^2}}$$

Q5. Find the 1^{st} order partial derivatives of the following:

1) $f(x,y) = 2x^2 - 3y - 4$.

$f(x,y) = (x+1)(y-2) \Rightarrow f_x = (y-2).1 = (y-2)$, $f_y = (x+1).1 = (x+1)$

1) $f(x,y) = 2x^2 - 3y - 4$

$f_x = 4x - 0 - 0 = 4x$

$f_y = 0 - 3 - 0 = -3$

Q6 Find the 1^{st} order partial derivative of $f(x,y) = \log(x^2+y^2)$

$f(x,y) = \log(x^2+y^2)$

$$f(x) = \frac{\partial f}{\partial x} \log(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$$

$$f_y = \frac{\partial f}{\partial y}(\log(x^2+y^2))$$

$$= \frac{1}{x^2+y^2} \cdot (2y) = 2y/x^2+y^2$$

Q7. $f(x,y) = e^{ax}\sin by$

$$f_x = \frac{\partial}{\partial x}(e^{ax}\sin by)$$

$$= \cancel{e^{ax}\cos by} \cdot \sin by \cdot \frac{\partial}{\partial x}(e^{ax})$$

$$= \sin by \cdot e^{ax} \cdot a = ae^{ax}\sin by$$

$$f_y = \frac{\partial}{\partial y}(e^{ax}\sin by)$$

$$= e^{ax} \cdot \cos by \cdot b$$

$$= be^{ax} \cdot \cos by$$

Q. 8 $z = \sqrt{3x+4y}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{3x+4y}\right) = \frac{1}{2\sqrt{3x+4y}} \cdot 3 = \frac{3}{2\sqrt{3x+4y}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{3x+4y}\right) = \frac{1}{\cancel{2}\sqrt{3x+4y}} \cdot \cancel{4}\,2 = \frac{2}{\sqrt{3x+4y}}$$

Q9. $f(x,y) = xe^{-y} + 5y$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^{-y} + 5y) = e^{-y} \cdot 1 + 0 = e^{-y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(xe^{-y} + 5y) = x \cdot e^{-y} \cdot -1 + 5 = 5 - xe^{-y}$$

Slope Of the Curve

Let $z = f(x,y)$ be any curve, the slope of this curve in x-direction is $\frac{\partial z}{\partial x}$. Slope of this curve in y-direction is $\frac{\partial z}{\partial y}$. $\frac{\partial z}{\partial x}$ is also called the rate of change of z ~~wit~~ in x-direction, $\frac{\partial z}{\partial y}$ is also called the rate of change of z in y-direction.

Q1 Find the $\frac{dy}{dx}$ slope of the surface $z = 10 - 4x^2 - 2y^2$ at the point $(1,2)$ in the x-direction and the y-direction.

$$\frac{\partial z}{\partial x} = 0 - 8x - 0 = -8x \implies \left.\frac{\partial z}{\partial x}\right|_{(1,2)} = -8 \times 1 = -8$$

$$\frac{\partial z}{\partial y} = 0 - 0 - 4y = -4y \implies \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = -4 \times 2 = -8$$

Q2. Find the rate of change of $z = \sin(y^2 - 4x)$ with respect to y at the point $(3,1)$.

$$\frac{\partial z}{\partial y} = \cos(y^2 - 4x) \cdot (2y - 0)$$
$$= 2y\cos(y^2 - 4x)$$

$$\left.\frac{\partial z}{\partial y}\right|_{(3,1)} = 2 \times 1 \cos(1 - 4 \times 3)$$
$$= 2\cos(-11) = 2\cos 11$$

Second Order Partial Derivatives

Let $z = f(x,y)$ the 1^{st} order partial derivatives are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Again differentiate these 1^{st} order with repect. to x and y we get 2^{nd} order partial derivatives.

1) $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}$

2) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$

3) $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$ f_{yx}

4) $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$

mixed fraction

1) Find all the 2^{nd} order partial derivatives for the function $f(x,y) = x^3y + 2xy^2$

$\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3y + 2xy^2)$

$= y \cdot 3x^2 + 2y^2$

$= \underline{3x^2y + 2y^2}$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y + 2xy^2)$

$= x^3 \cdot 1 + 2x^2 \cdot 1$

$= \underline{x^3 + \cancel{2x^2} 4xy}$

$\frac{\partial^2 f}{\partial x^2} = 6xy + 0$

$= \underline{6xy}$

$\Rightarrow \frac{\partial^2 f}{\partial y^2} = \cancel{0 + 0 = 0}\ 0 + 4x$

$= \underline{4x}$

$\frac{\partial^2 f}{\partial y \partial x} = \cancel{6x + 4y}$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2y + 2y^2)$

$= \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial y}(2y^2)$

$= \underline{3x^2 + 4y}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}(x^3 + 4xy)$

$= \underline{3x^2 + 4y}$

2) If $f(x,y) = 2x^5y^3 + 5x + 7y$. find $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$

$$\frac{\partial f}{\partial x} = 10x^4y^3 + 5 + 0 = \underline{10x^4y^3 + 5}$$

$$\frac{\partial f}{\partial y} = 6x^5y^2 + 0 + 7 = \underline{6x^5y^2 + 7}$$

$$\frac{\partial^2 f}{\partial x^2} = \underline{40x^3y^3}$$

$$\frac{\partial^2 f}{\partial y^2} = \underline{12x^5y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \partial\, \underline{30x^4y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \underline{30x^4y^2}$$

3) if $f(x,y) = x^3y + e^{xy^2}$ show that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

$$\frac{\partial f}{\partial x} = \cancel{6}\,3x^2y + e^{xy^2}\cdot y^2 = \underline{3x^2y + y^2e^{xy^2}}$$

$$\frac{\partial f}{\partial y} = x^3 + e^{xy^2}\cdot 2yx = \underline{x^3 + 2yxe^{xy^2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2y + y^2e^{xy^2}\right)$$

$$= 0 + 2y\cdot e^{xy^2} + y^2\cdot e^{xy^2}\cdot x2y$$

$$= \underline{2ye^{xy^2} + 2xy^3e^{xy^2} + 3x^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^3 + 2yxe^{xy^2}\right)$$

$$= \cancel{0 + 2yx\,e^{xy^2}}\cdot 3x^2 + 2yx1e^{xy^2}\cdot y^2 + e^{xy^2}\cdot 2y$$

$$= \underline{3x^2 + 2y^3xe^{xy^2} + 2ye^{xy^2}}$$

4) If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 y}{\partial z^2} = 0$

Let $u = f(x,y,z)$ diff u partially with respect to x keeping y and z as constants.

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\sqrt{x^2+y^2+z^2}\cdot 1 - \left(x\cdot\frac{1}{2\sqrt{x^2+y^2+z^2}}\right)}{\left(\sqrt{x^2+y^2+z^2}\right)^2}$$

$$= \frac{\sqrt{x^2+y^2+z^2} - \frac{x}{2\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2} \qquad = \frac{\frac{2(x^2+y^2+z^2)-x}{2\sqrt{x^2+y^2+z^2}}}{\left(\sqrt{x^2+y^2+z^2}\right)^2} \qquad \frac{1}{\frac{1}{\sqrt{x^2}}}$$

$$= \frac{x^2+y^2+z^2-x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial y} = $$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)^{-1/2} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{+1/2} \cdot 2x$$

$$= -x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$-\frac{3-2}{2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(-x\left(x^2+y^2+z^2\right)^{-3/2}\right)$$

$$= -x \cdot -\frac{3}{2}\left(x^2+y^2+z^2\right)^{-5/2} \cdot 2x + \left(x^2+y^2+z^2\right)^{-3/2} \cdot -1$$

$$= 3x^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-1/2} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2} \cdot 2y = -y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial^2 u}{\partial y^2} = -y \cdot -\frac{3}{2} \cdot \left(x^2+y^2+z^2\right)^{-5/2} \cdot 2y + \left(x^2+y^2+z^2\right)^{-3/2} \cdot -1$$

$$= 3y^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-1/2} = -z\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial^2 u}{\partial z^2} = -z \cdot -\frac{3}{2}\left(x^2+y^2+z^2\right)^{-5/2} \cdot 2z + \left(x^2+y^2+z^2\right)^{-3/2} \cdot -1$$

$$= 3z^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial^2 u}{} = 3x^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2} + 3y^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2}$$

$$+ 3z^2\left(x^2+y^2+z^2\right)^{-5/2} - \left(x^2+y^2+z^2\right)^{-3/2}$$

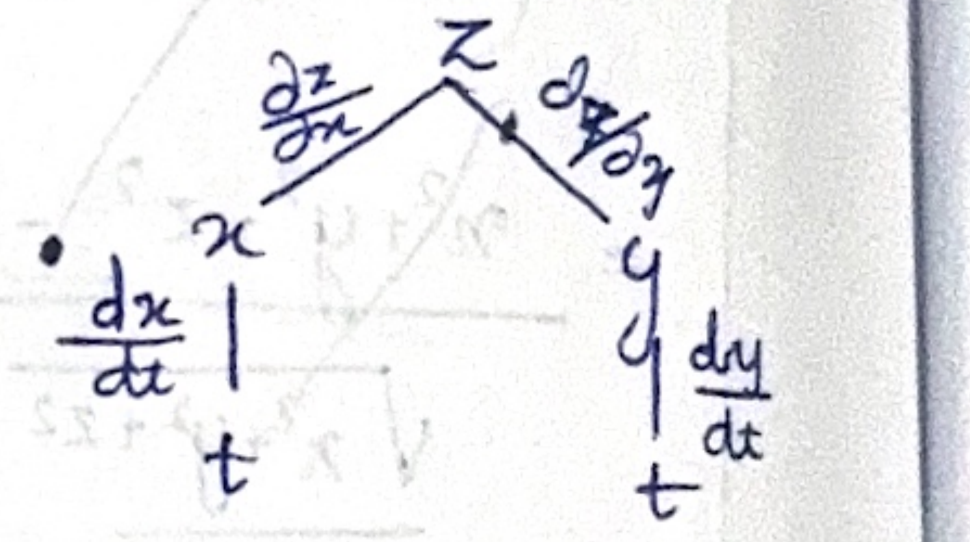
$$= \left(3x^2+3y^2+3z^2\right)\left(x^2+y^2+z^2\right)^{-5/2} - 3\left(x^2+y^2+z^2\right)^{-3/2}$$

$$= 3\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-5/2} - 3\left(x^2+y^2+z^2\right)^{-3/2} = 0$$

Chain Rule for Two Variable

Let $z = f(x, y)$ be a function in 2 variable x and y and $x = f(t)$ and $y = g(t)$ then z is a function of t alone.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$



Q1. Suppose that $z = 5x^3y^2$, $x = t^3$, $y = t^5$. Use the chain rule to find $\frac{dz}{dt}$ and check the result by expressing z as a function of t and diff. directly.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= 15x^2y^2\cdot 3t^2 + 10x^3y\cdot 5t^4$$

$$= 45(t^3)^2(t^5)^2\cdot t^2 + 10(t^3)^3\cdot t^5\cdot 5\times t^4$$

$$= 45t^6t^{10}t^2 + 50t^9\cdot t^5\cdot t^4$$

$$= 45t^{18} + 50t^{18} = 95t^{18}$$

$$\frac{\partial z}{\partial x} = 5\times 3x^2\cdot y^2 = 15x^2y^2$$

$$\frac{\partial z}{\partial y} = 5x^3\cdot 2y = 10x^3y$$

$$\frac{dx}{dt} = 3t^2$$

$$\frac{dy}{dt} = 5t^4$$

$\frac{dz}{dt}$ $z = 5x^3y^2$

Sub $x = t^3$ and $y = t^5$ in $z = 5x^3y^2$

$$\Rightarrow z = 5(t^3)^3\cdot(t^5)^2$$

$$= 5t^9\cdot t^{10} = 5t^{19}$$

$$\frac{dz}{dt} = 19\times 5\,t^{18} = 95t^{18}$$

Q2. Suppose that $z = \log(3x^2 + y)$, $x = \sqrt{t}$, $y = t^{1/3}$. Use chain rule to find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{6x}{3x^2+y}\cdot\frac{1}{2\sqrt{t}} + \frac{1}{3x^2+y}\cdot\frac{1}{3}t^{-2/3}$$

$$= \frac{6\times\sqrt{t}}{3(\sqrt{t})^2 + t^{1/3}}\cdot\frac{1}{2\sqrt{t}} + \frac{1}{3(\sqrt{t})^2 + t^{1/3}}\cdot\frac{1}{3}t^{-2/3}$$

$$= \frac{6\sqrt{t}}{3t + t^{1/3}}\cdot\frac{1}{2\sqrt{t}} + \frac{1}{3t + t^{1/3}}\cdot\frac{1}{3}t^{-2/3}$$

$$\frac{\partial z}{\partial x} = \frac{1}{3x^2+y}\cdot 6x = \frac{6x}{3x^2+y}$$

$$\frac{\partial z}{\partial y} = \frac{1}{3x^2+y}\cdot 1 = \frac{1}{3x^2+y}$$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt} = \frac{1}{3}\cdot t^{-2/3}$$

$\frac{1}{3} - 1$ $\frac{1-3}{3}$

$$\frac{6\sqrt{t}}{2\sqrt{t}(3t+t^{1/3})} + \frac{1}{3}t^{-2/3}\left(\frac{1}{3t+t^{1/3}}\right)$$

$$= \frac{1}{3t+t^{1/3}}\left(\frac{3\cancel{6}\cancel{\sqrt{t}}}{\cancel{2}\cancel{\sqrt{t}}} + \frac{1}{3}t^{-2/3}\right)$$

$$= \frac{1}{3t+t^{1/3}}\left(3+\frac{1}{3}t^{-2/3}\right) = \frac{3}{3t+t^{1/3}} + \frac{t^{-2/3}}{3(3t+t^{1/3})}$$

Q_3. The length and width of a rectangle are increasing at the rat eof 1.5 cm/sec and 0.5 cm s^{-1} respectively. Find the rate at which the area is increasing at the instant when the length is 40 cm and width is 30 cm respectively.

Let x and y be the length and width of a rectangle.

$\frac{dx}{dt} = 1.5$ cm/sec $\qquad \frac{dy}{dt} = 0.5$ cms^{-1}

$A = xy$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial A}{\partial y}\cdot\frac{dy}{dt}$$

$$= y\cdot 1.5 + x\cdot 0.5$$

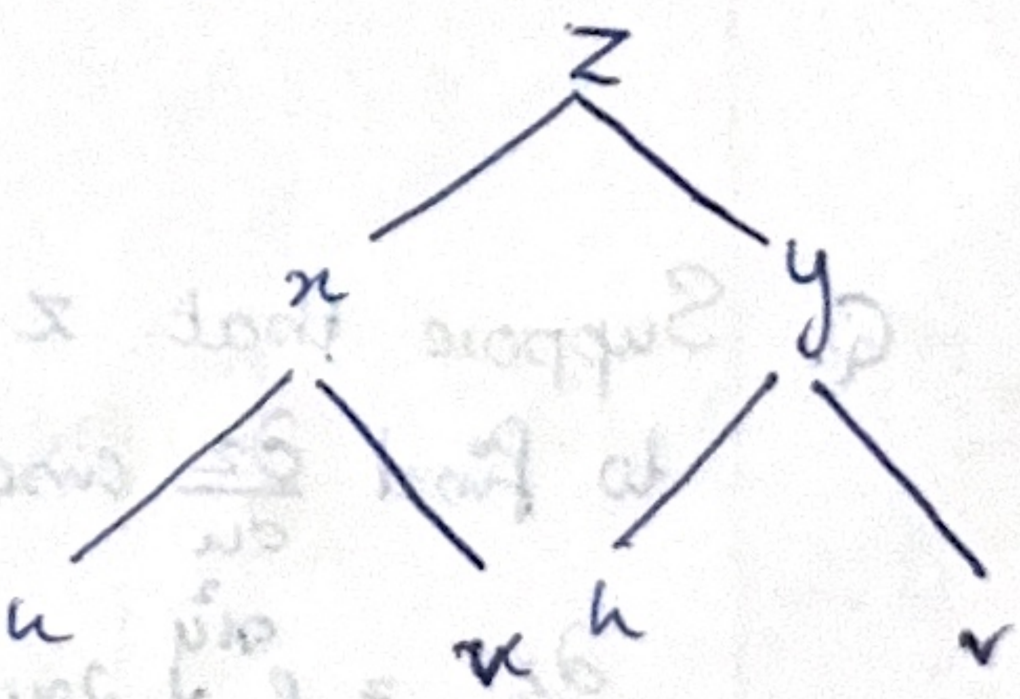
$$= 30\cdot 1.5 + 40\cdot 0.5 = 45 + 20 = 65 \text{ cm}s^{-1}$$

Change Rule for Partial Derivatives

Let $z = f(x,y)$ be a function in 2 variables and $x = x(u,v)$ and $y = y(u,v)$ are functions in u and v therefore z is a function of u and v.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$



Q. Suppose that $z = x^2 - y\tan x$, $x = \frac{u}{v}$, $y = uv$. Use appropriate form of chain rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

$$\frac{\partial z}{\partial x} = 2x - y\sec^2 x \qquad \frac{\partial x}{\partial u} = \frac{v\cdot 1 - u\cdot 0}{v^2} = \frac{v}{v^2} = \frac{1}{v}$$

$$\frac{\partial z}{\partial y} = -\tan x \qquad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = \frac{v\cdot 0 - u\cdot 1}{v^2} = \frac{-u}{v^2} \qquad \frac{\partial y}{\partial v} = u$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= 2x - y\sec^2 x \cdot \frac{1}{v} + -\tan x \cdot v$$

$$= \frac{2x - y\sec^2 x}{v} - v\tan x = \frac{2\times\frac{u}{v} - uv\sec^2\frac{u}{v}}{v} - v\tan\frac{u}{v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= (2x - y\sec^2 x)\cdot\frac{-u}{v^2} + -\tan x \cdot u = -\left(2\times\frac{u}{v} - uv\sec^2\left(\frac{u}{v}\right)\right)\frac{u}{v^2} - u\tan\left(\frac{u}{v}\right)$$

Q. Suppose that $z = e^{x^2 y}$, $x = \sqrt{uv}$, $y = \frac{1}{v}$. Use appropriate form of chain rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial x} = e^{x^2 y}\cdot 2xy = 2xye^{x^2 y} \qquad \frac{\partial z}{\partial y} = e^{x^2 y}\cdot x^2 = x^2 e^{x^2 y}$$

$$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{uv}}\cdot v = \frac{v}{2\sqrt{uv}} \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = \frac{1}{2\sqrt{uv}} \cdot u = \frac{u}{2\sqrt{uv}}$$

$$\frac{\partial y}{\partial v} = \frac{-1}{v^2} \cdot 1 = \frac{-1}{v^2}$$

$$\frac{\partial z}{\partial u} = 2xye^{x^2y} \cdot \frac{v}{2\sqrt{uv}} + x^2e^{x^2y} \cdot 0$$

$$= 2xye^{x^2y} \cdot \frac{v}{2\sqrt{uv}} = 2\times\sqrt{uv} \cdot \frac{1}{v} \cdot e^{(\sqrt{uv})^2 \times \frac{1}{v}} \times \frac{v}{2\sqrt{uv}} = \frac{e^{(\sqrt{uv})^2/v}}{v}$$

$$= \frac{e^{uv/v}}{v} = \frac{e^u}{v}$$

$$\frac{\partial z}{\partial v} = 2xye^{x^2y} \cdot \frac{u}{2\sqrt{uv}} + x^2e^{x^2y} \cdot \frac{-1}{v^2}$$

$$= 2\times\sqrt{uv}\times\frac{1}{v} e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} \cdot \frac{u}{2\sqrt{uv}} + (\sqrt{uv})^2 \cdot e^{(\sqrt{uv})^2 \cdot 1/v} \cdot \frac{-1}{v^2}$$

$$= \frac{u}{v} e^{(\sqrt{uv})^2 \cdot \frac{1}{v}} - \frac{uv\left(e^{(\sqrt{uv})^2/v}\right)}{v^2}$$

$$= \frac{u}{v}e^u - \frac{uv \cdot e^u}{v^2} = \frac{ue^u}{v} - \frac{ue^u}{v}$$

$$= 0$$

Q. Use chain rule to find the derivative $\omega = x^2+y^2$ along the path $x = at^2, y = 2at$

$$\frac{\partial \omega}{\partial x} = 2x \qquad \frac{\partial \omega}{\partial y} = 2y$$

$$\frac{\partial x}{\partial t} = 2at \qquad \frac{\partial y}{\partial t} = 2a$$

$$\frac{d\omega}{dt} = 2x \times 2at + 2y \times 2a$$

$$= 2\times at^2\times 2at + 2y\times 2at\times 2a$$

$$= 4a^2t^3 + 8a^2t$$

$$2xye^{x^2y} \cdot \frac{v}{2\sqrt{uv}} + x^2e^{xy} \cdot 0 = 2xye^{xy} \frac{v}{2\sqrt{uv}} = xye^{xy}\frac{v}{\sqrt{uv}} = \sqrt{ux} \cdot \frac{1}{x} \cdot e^{xy} \cdot \frac{x}{\sqrt{uv}}$$

$$= e^{(\sqrt{uv})^2 \frac{1}{y}} = e^{uv/v} = e^{\frac{u}{v}}$$

Implicit Differentiation

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Q1 Using partial derivatives find dy/dx if $x^3+y^3 = \cancel{3xy}\ 3axy$. And verify the result using implicit differentiation.

$$\frac{dy}{dx} = \cancel{\frac{\partial f}{\partial x}} \qquad x^3+y^3 = 3axy.$$

$$f(x,y) = C$$

$$f(x,y) = x^3+y^3-3axy$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3+y^3-3axy)$$

$$= 3x^2-3ay.$$

$$\frac{\partial f}{\partial y} = 3y^2-3ax$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{-3x^2-3ay}{3y^2-3ax}$$

$$x^3+y^3 = 3axy$$

$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3) = 3a\frac{d}{dx}(xy)$$

$$3x^2+3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx}+y\right]$$

$$3x^2+3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx}+3ay.$$

$$\cancel{3x^2}$$

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay-3x^2$$

$$\frac{dy}{dx}\left[3y^2-3ax\right] = 3ay-3x^2$$

$$\frac{dy}{dx} = \frac{3ay-3x^2}{3y^2-3ax}$$

Q_2: Find $\frac{du}{dx}$ if $u = x^2+y^2-a^2$

$f(x,y) = x^2+y^2-a^2$

$\frac{\partial u}{\partial x} = 2x$ $\quad$ $\frac{\partial u}{\partial y} = 2y$

$\frac{du}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

Q_3: If $f(x,y,z) = e^{xyz}$, find $\frac{\partial^3 f}{\partial x \partial y \partial z}$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right)$$

$\frac{\partial f}{\partial z} = e^{xyz} \cdot xy = xye^{xyz}$

$\frac{\partial f}{\partial y} = e^{xyz} \cdot xz = xze^{xyz}$

$\frac{\partial f}{\partial x} = e^{xyz} \cdot yz = yze^{xyz}$

$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}($

$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(xye^{xyz}\right)$

$= xy \cdot e^{xyz} \cdot xz + e^{xyz} \cdot x$

$= xy \cdot e^{xyz} xz + xe^{xyz}$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial z}\right) = \frac{\partial}{\partial x}\left(x^2yz e^{xyz} + xe^{xyz}\right)$$

$= x^2yz \cdot e^{xyz} \cdot yz + e^{xyz} \cdot 2xyz + x \cdot e^{xyz} \cdot yz + e^{xyz}$

$= x^2y^2z^2 e^{xyz} + 2xyz e^{xyz} + xyz e^{xyz} + e^{xyz}$

$= e^{xyz}\left[3xyz + x^2y^2+z^2+1\right]$

Q_4: If $u = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$.

$$\frac{\partial u}{\partial y} = \left[x^2 \cdot \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} + \tan^{-1}(y/x)\cdot 0\right] - \left[y^2 \frac{1}{1+(x/y)^2} \cdot \frac{-x}{y^2} + \tan^{-1}(x/y)\cdot 2y\right]$$

$$= \frac{x^2}{\frac{x^2+y^2}{x^2}} \cdot \frac{1}{x} - \frac{y^2 x}{\frac{y^2+x^2}{y^2}} + \tan^{-1}(x/y)\cdot 2y$$

$$\frac{\left[3x^2+y^2\right]-\left[x^3+xy^2\right]\cdot 2x]}{x^2+y^2} - \frac{2y^2}{y^2+x^2}$$

$$\frac{\left[3x^2+y^2-2x^4-2x^2y^2\right]}{x^2+y^2} - \frac{2y^2}{y^2+x^2} = \frac{x^2\left[3+\cancel{y^2}-2x^2-\cancel{2y^2}\right]-2y^2}{x^2+y^2}$$

$$= \frac{x^2\left[3-\cancel{y^2}-2x^2\right]-2y^2}{x^2+y^2} \quad = \frac{\cancel{x^2\left[3-3y^2-2x^2\right]}}{x^2+y^2}$$

$$1 - \frac{2y^2}{y^2+x^2} = \frac{x^2+y^2-2y^2}{y^2+x^2} = \frac{x^2-y^2}{x^2+y^2}$$

The altitude of right ~~angle~~ circular cone is increasing at the rate of $0.2\,cm\,s^{-1}$, the radius of its base is decreasing at the rate $0.3\,cm\,s^{-1}$. Find the rate at which its volume is changing at the instant, when the altitude is 15 cm and radius is 10 cm.

Let x be the radius, y be the altitude and v is the volume.

$\frac{dx}{dt} = -0.3\,cm\,s^{-1}$ $\quad$ V of cone $= \frac{1}{3}\pi r^2 h$.

$\frac{dy}{dt} = 0.2\,cm\,s^{-1}$

$V = \frac{1}{3}\pi x^2 y$ $\quad$ $\frac{\partial v}{\partial x} = \frac{1}{3}\pi 2xy = \frac{2\pi xy}{3}$

$\frac{\partial v}{\partial y} = \frac{1}{3}\pi x^2.$

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{2\pi xy}{3}(-0.3) + \frac{1}{3}\pi x^2\cdot(0.2)$$

$$= \frac{0.2\pi x^2}{3} - \frac{0.6\,2\pi xy}{3} = \frac{2\pi}{3}\left(0.1\,x^2 - 0.6xy\right)$$

$$= \frac{2\pi}{3}\left(0.2\times10^2 - 0.6\times10\times15\right) = \cancel{(0.20-90)}\frac{2\pi}{3} \quad \frac{-140\pi}{3} \quad \frac{-70\pi}{3}\,cm^3 s^{-1}$$

Given that $z = 2x^2y^2 - 3x^2y$ and x is increasing at the rate of $2\,cm s^{-1}$ find the rate at which y changes at the instant when $x = 3cm$ and $y = 1$ if z remains constant.

$\frac{dx}{dt} = 2\,cm s^{-1}$

$\frac{\partial z}{\partial x} = 2y^2 - 6xy$

$\frac{\partial z}{\partial y} = 4xy - 3x^2$

$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$0 = (2y^2 - 6xy) \cdot 2 + (4xy - 3x^2) \cdot \frac{dy}{dt}$

$0 = 4y^2 - 12xy + (4xy - 3x^2)\frac{dy}{dt}$

$(4xy - 3x^2)\frac{dy}{dt} = 4y^2 - 12xy$

$(4 \times 3 \times 1 - 3 \times 3^2)\frac{dy}{dt} = -(4 \times 1^2 - 12 \times 3 \times 1)$

$(12 - 27)\frac{dy}{dt} = -(4 - 36)$

$\frac{dy}{dt} = \frac{-(32)}{+15} = \frac{32}{15}$

Find the derivative of $y = \frac{(3x^3 + 2x)(3x + 5)}{x^4}$.

$\frac{dy}{dx} = \frac{x^4\left((3x^3 + 2x \cdot 3) + (3x+5)(9x^2+2)\right) - (3x^3+2x)(3x+5) \cdot 4x^3}{(x^4)^2}$

$= \frac{x^4(9x^3 + 6x + 27x^3 + 6x + 45x^2 + 10) - (3x^3+2x)(3x+5)4x^3}{(x^4)^2}$

$= \frac{x^4(36x^3 + 12x + 45x^2 + 10) - 4x^3(3x^3+2x)(3x+5)}{x^8}$

~~$\frac{d^2y}{dx^2} = x^4 \cdot (108x^2 + 12 + 90x + 0) + 4(36x^3 + 12x + 45x^2 + 10) \cdot 4x^3 - 4x^3 \cdot$~~

$= \frac{x^4(36x^3 + 12x + 45x^2 + 10) - 4x^3(9x^4 + 15x^3 + 6x^2 + 10x)}{x^8}$

$= \dfrac{\cancel{36x^{12}} + 12x^5 + 45x^8 + 10x^4 - \cancel{36x^{12}} - 60x^3 - 24x^6 - 40x^4}{x^8}$

$= 12x^5 + 45x^8 - 60x^3 - 30x^4 / x^8$

$= \dfrac{x^3(12x^2 + 45x^5 - 60x - 30x)}{x^8} = \dfrac{12x^2 + 45x^5 - 60 - 30x}{x^5}$

$= \dfrac{\cancel{36x^7} + 12x^5 + 45x^6 + 10x^4 - \cancel{36x^7} - 60x^6 - 24x^5 - 40x^4}{x^8}$

$= \dfrac{-12x^5 - 15x^6 - 30x^4}{x^8} = \dfrac{x^3(-12x^2 + 15x^3 - 30x)}{x^8} = \dfrac{-12x^2 - 15x^3 - 30x}{x^5}$

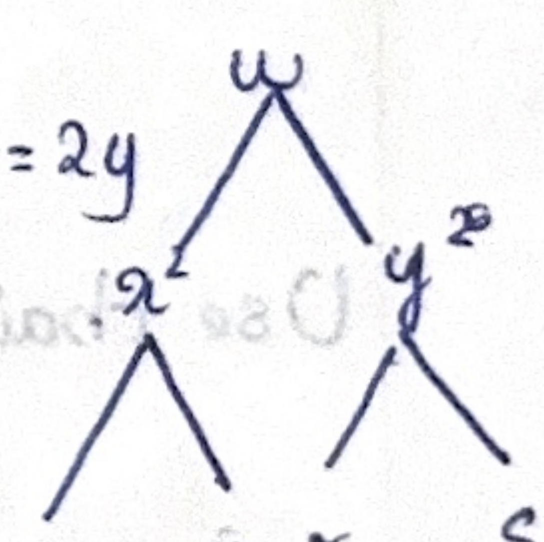
$\dfrac{d^2y}{dx^2} = \dfrac{x^5 \cdot (-24x - 45x^2 - 30) - (-12x^2 - 15x^3 - 30x) \cdot 5x^4}{x^{10}}$

$= \dfrac{-24x^6 - 45x^7 - 30x^5 + 60x^6 + 75x^7 + 150x^5}{x^{10}}$

$= \dfrac{36x^6 + 30x^7 + 120x^5}{x^{10}} = \dfrac{x^5(36x + 30x^2 + 120)}{x^{10}} = \dfrac{36x + 30x^2 + 120}{x^5}$

Express $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ in terms of r and s if $w = x^2 + y^2$, $x = r - s$ $y = r + s$

$\dfrac{\partial w}{\partial x} = 2x \quad \dfrac{\partial w}{\partial y} = 2y$



$\dfrac{\partial x}{\partial r} = 1 \quad \dfrac{\partial y}{\partial r} = 1$

$\dfrac{\partial x}{\partial s} = -1 \quad \dfrac{\partial y}{\partial s} = 1$

$\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{\partial r}$

$= 2x \cdot 1 + 2y \cdot 1$

$= 2x + 2y = 2(r - s) + 2(r + s)$

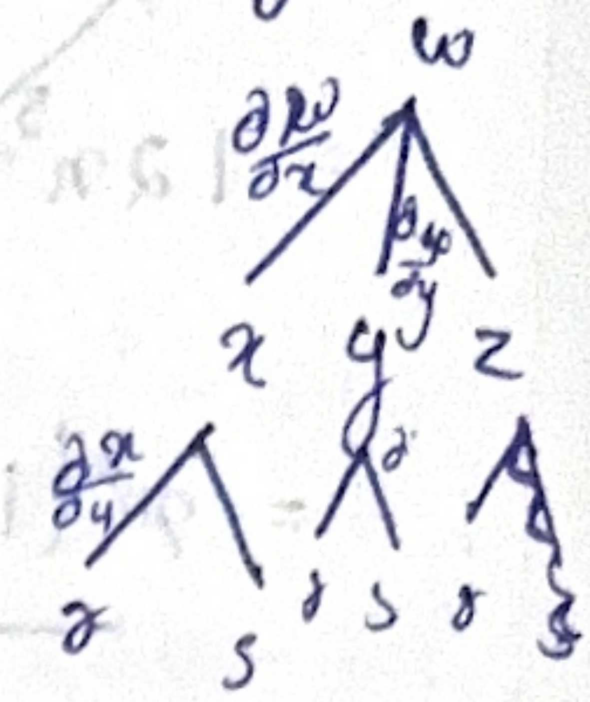
$= 2r - \cancel{2s} + 2r + \cancel{2s} = 4r$

$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{\partial s}$

$= 2x \cdot -1 + 2y \cdot 1$

$= 2y - 2x = 2(r + s) - 2(r - s) = \cancel{2r} + 2s - \cancel{2r} + 2s = 4s$

Find the 1^{st} order partial derivatives of the function $\omega = x+2y+z^2$, $x = r/s$, $y = r^2 + \ln s$, $z = 2r$.



$\frac{\partial \omega}{\partial x} = 1$ $\quad \frac{\partial \omega}{\partial y} = 2$ $\quad \frac{\partial \omega}{\partial z} = 2z$

$\frac{\partial x}{\partial r} = \frac{s \cdot 1 - r \cdot 0}{s^2} = \frac{s}{s^2} = 1/s$

$\frac{\partial y}{\partial r} = 2r$ $\quad \frac{\partial z}{\partial r} = 2$

$\frac{\partial x}{\partial s} = \frac{s \cdot 0 - r \cdot 1}{s^2} = \frac{-r}{s^2}$ $\quad \frac{\partial y}{\partial s} = \frac{1}{s}$

$\frac{\partial \omega}{\partial r} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial \omega}{\partial z} \cdot \frac{\partial z}{\partial r}$

$= 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2 = \frac{1}{s} + 4r + 4z$

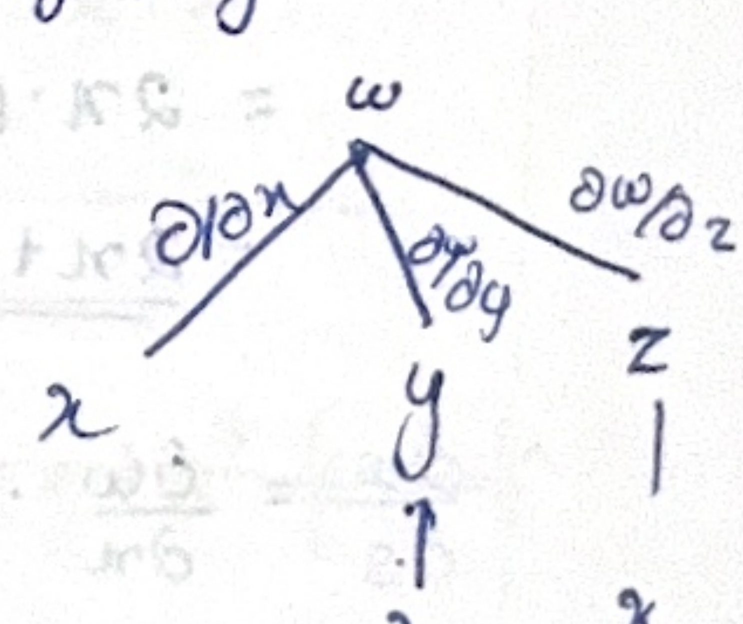
$= \frac{1}{s} - 4r + 4 \times 2r = \frac{1}{s} + 4r + 8r$

$= \frac{1}{s} + 12r$

$\frac{\partial \omega}{\partial s} = 1 \cdot \frac{-r}{s^2} + 2 \times \frac{1}{s} \cancel{+2z}$

$= -r/s^2 + 2/s + \cancel{2 \times 2r \times 0} = -r/s^2 + 2/s$

Use chain rule to find $\frac{d\omega}{\partial x}$ at $(0,1,2)$ for $\omega = xy + yz$; $y = \sin x$, $z = e^x$



$\frac{\partial \omega}{\partial x} = y + 0 = y$

$\frac{\partial \omega}{\partial y} = x + z$ $\quad \frac{\partial \omega}{\partial z} = 0 + y = y$

$\frac{d\omega}{dx} = \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial \omega}{\partial z} \cdot \frac{\partial z}{\partial x}$

$\frac{d\omega}{dx}\Big|_{(0,1,2)} = y + (x+z) \cdot \cos x + e^x \cos x + \sin x \cdot e^x$

$= \sin 0 + 0\cos 0 + e^0 \cos 0 + \sin 0 \cdot e^0$

$= \underline{\underline{1}}$

Use chain rule to find $\frac{dw}{dt}$ if $w = \ln(x^2+y^2+z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$, $t=3$.

$\frac{\partial w}{\partial x} = \frac{1}{x^2+y^2+z^2} \cdot 2x = \frac{2x}{x^2+y^2+z^2}$; $\frac{\partial x}{\partial t} = -\sin t$

$\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2}$; $\frac{\partial y}{\partial t} = \cos t$

$\frac{\partial z}{\partial t} = 4 \cdot \frac{1}{2\sqrt{t}} \cdot 1 = \frac{2}{\sqrt{t}}$

$\frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}$

$\frac{dw}{dt} = \frac{2x}{x^2+y^2+z^2} \cdot -\sin t + \frac{2y}{x^2+y^2+z^2} \cdot \cos t + \frac{2z}{x^2+y^2+z^2} \cdot 2/\sqrt{t}$

$= \frac{2(\cos t) \cdot -\sin t}{\cos^2 t + \sin^2 t + 4\sqrt{t}} + \frac{2(\sin t) \cdot \cos t}{\cos^2 t + \sin^2 t + 4\sqrt{t}} + \frac{2(4\sqrt{t})}{\cos^2 t + \sin^2 t + 4\sqrt{t}} \cdot 2/\sqrt{t}$

$= \frac{-2\cancel{\cos t \sin t} + 2\cancel{\sin t \cos t} + 16}{1 + 4\sqrt{t}} \cdot \frac{16}{1+(4(\sqrt{t}))^2} = \frac{16}{1+16t}$

$\frac{dw}{dt}\Big|_{t=3} = \frac{16}{1+(4(\sqrt{3}))^2} = \frac{16}{1+16\times 3} = \frac{16}{49}$

1) Find the domain and range of a function $f(x,y) = x^2 - y^2$.

The domain is all real ~~real~~ numbers R^2, because there are no restrictions on x and y. The range is all real number R because $x^2 - y^2$ can take any real number.

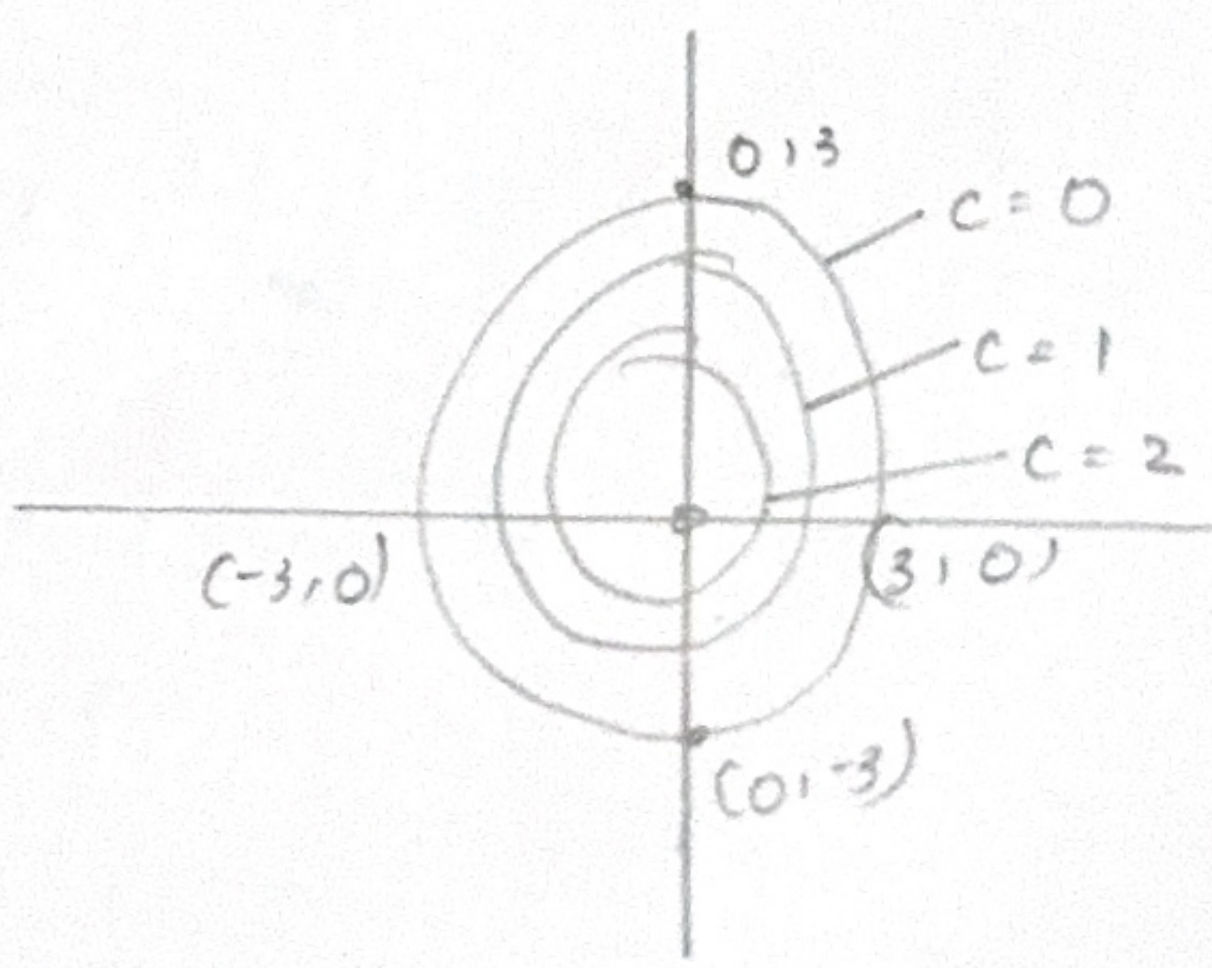
2) Find the domain and range of the function $f(x,y) = \frac{x+y}{x-y}$.

The domain is $R^2 - \{(x,y) / x = y\}$ because the denominator cannot be zero. The range is all real numbers R because the function can take any real value depending on the values of x and y.

Level Curves

A level curve of a function $f(x,y)$ is a curve along with a function as a constant value. For a given constant k, the level curve is defined by the eqn $f(x,y) = k$.

1. Find the graph, and the level curves of the fn. $f(x,y) = \sqrt{9 - x^2 - y^2}$ for $c = 0, 1, 2,$ and 3.



$$f(x,y) = \sqrt{9 - x^2 - y^2}$$

$$f(x,y) = c$$

$$\sqrt{9 - x^2 - y^2} = c$$

$$9 - x^2 - y^2 = c^2$$

$$x^2 + y^2 = 9 - c^2$$

when $c = 0 \Rightarrow x^2 + y^2 = 9$

when $c = 1 \Rightarrow x^2 + y^2 = 8$

when $c = 2 \Rightarrow x^2 + y^2 = 5$

" $c = 3 \Rightarrow x^2 + y^2 = 0 \rightarrow$ origin.

2. Find the level curve for the fn $f(x,y) = x^2 + 2y$ for $k = -2, 0, 2$.

$$f(x,y) = x^2 + 2y$$

$$f(x,y) = k$$

$$x^2 + 2y = k$$

for $k = -2 \Rightarrow x^2 + 2y = -2 \Rightarrow x^2 = -2 - 2y$

$k = 0 \Rightarrow x^2 + 2y = 0 \Rightarrow x^2 = -2y$

$k = 2 \Rightarrow x^2 + 2y = 2 \Rightarrow x^2 = 2 - 2y$.

because $-4ay$

$x^2 = +4ay$

2nd sketch

3) Find the level curves of the fn $f(x,y) = x^2+y^2$ passing through the point $(2,0)$

$f(x,y) = x^2+y^2 = 2^2+0^2 = 4$