Module 1 : Limits Of Function Values

If $f(x)$ is arbitarily close to the no. L, for all x's sufficiently close to c other the c itself, then we say that f approaches the limit L as x approaches to c.

$$\lim_{x \to c} f(x) = L$$

The limit of a function is the value of the function as the input get close or approaches the same number.

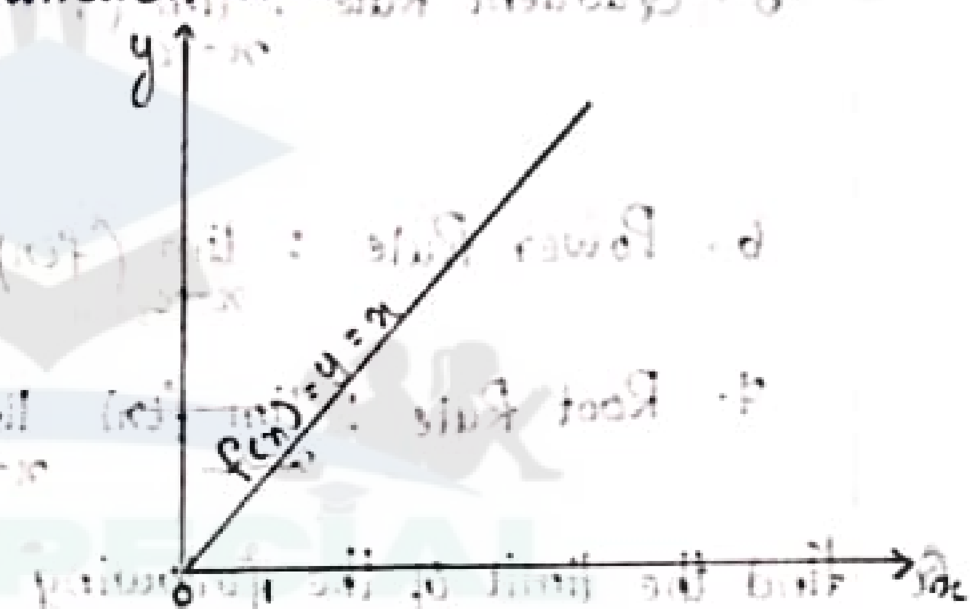
Q: Find the limit of the function $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2-1}{x-1}$$

$$= \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= 1+1 = \underline{\underline{2}}$$

Q. Find the limits of the identity function and constant function as x approaches to c.

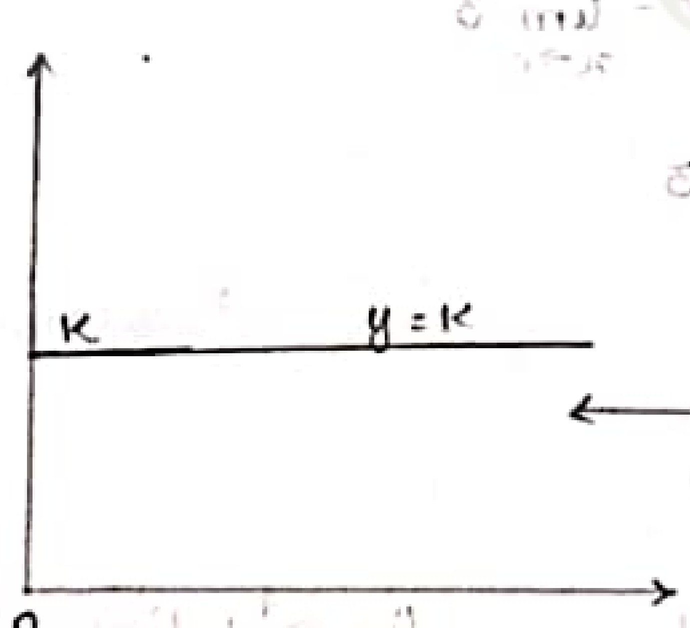


Identity functions :

$f(x) = x$

$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$

Constant Function

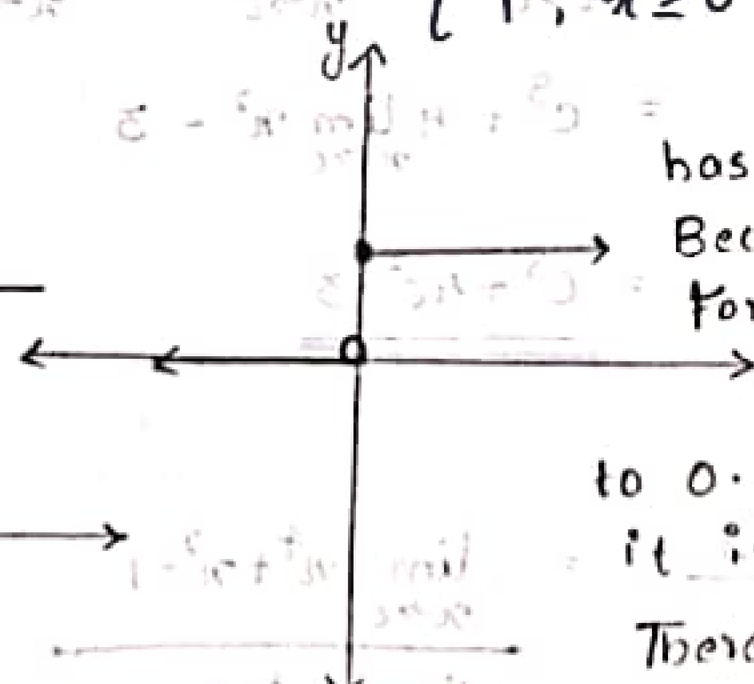


$y = f(x) = K$

$\lim_{x \to c} f(x) = \lim_{x \to c} K = K$.

Discuss the behaviour of the following (unit-step function).

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$



The unit step fn. $u(x)$ has no limit as $x \to 0$. Because it jumps at $x = 0$. For -ve values of x, it is arbitarily close to 0. For +ve values of x, it is arbitarily close to 1. There is no single value L as x approaches to 0.

Limit Laws / Limit Rules

If: L, M, c and k are real numbers and limit extends to c

$$\lim_{x \to c} f(x) = L$$

$$\lim_{x \to c} g(x) = M$$

1. Sum Rule : $\lim_{x \to c} \left(f(x) + g(x) \right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

$= L + M$.

2. Difference Rule : $\lim_{x \to c} \left(f(x) - g(x) \right) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$

3. Constant Multiplication Rule :

$$\lim_{x \to c} (k f(x)) = k \lim_{x \to c} f(x) = kL.$$

4. Product Rule : $\lim_{x \to c} \left(f(x) * g(x) \right) = \lim_{x \to c} f(x) \times \lim_{x \to c} g(x) = L \times M$

5. Quotient Rule : $\lim_{x \to c} \left(f(x) / g(x) \right) = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = L/M$

6. Power Rule : $\lim_{x \to c} \left(f(x) \right)^n = \left(\lim_{x \to c} f(x) \right)^n = (L)^n$.

7. Root Rule : $\cancel{\lim_{x \to c} f(x)}$ $\lim_{x \to c} \sqrt[n]{f(x)} = \left(\lim_{x \to c} f(x) \right)^{1/n} = (L)^{1/n}$ or $\sqrt[n]{L}$

Q. Find the limit of the following

1) $\lim_{x \to c} (x^3 + 4x^2 - 3)$

$$= \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$

$$= c^3 + 4 \lim_{x \to c} x^2 - 3$$

$$= \underline{\underline{c^3 + 4c^2 - 3}}$$

2) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5} = \dfrac{\lim_{x \to c} x^4 + x^2 - 1}{\lim_{x \to c} x^2 + 5} = \dfrac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

3) $\lim_{x \to 2} \sqrt{4x^2+3}$

$$= \left(\lim_{x \to 2} \{4x^2+3\}\right)^{1/2} = \left(\lim_{x \to 2} 4x^2\right)^{1/2} + \left\{\lim_{x \to 2} 3\right\}^{1/2}$$

$$= \lim_{x \to 2} (4 \times 4)^{1/2} + (3)^{1/2}$$

$$= \sqrt{16} + \sqrt{3} = \sqrt{8} \quad \sqrt{16+3} = \sqrt{19}$$

Evaluating limits of Polynomials and Rational functions.

Q. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_0$ be a polynomial of degree n

$\lim_{x \to c} p(x) = P(c)$.

Let $f(x) = \frac{P(x)}{q(x)}$ is a quotient polynomial or rational polynomial

$$\lim_{x \to c} f(x) = \frac{\lim_{x \to c} P(x)}{\lim_{x \to c} q(x)} = \frac{P(c)}{q(c)}, \quad q(c) \neq 0.$$

Q Find $\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2+5}$

$$= \frac{\lim_{x \to -1} x^3 + 4x^2 - 3}{\lim_{x \to -1} x^2 + 5} = \frac{-1 + 4 - 3}{1+5} = 0/6 = 0$$

Q. Evaluate $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$

$$= \frac{\lim_{x \to 1} x^2 + x - 2}{\lim_{x \to 1} x^2 - x} = \frac{1 + 1 - 2}{1 - 1} = 0/0, \text{ not defined}$$

$$\lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}} \quad \frac{1+2}{1} = 3/1 = \underline{\underline{3}}$$

Sum : -a 1
prodult : 2

Q. Evaluate $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$

taking conjugative of radical

$$\lim_{x\to 0} \frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)} = \lim_{x\to 0} \frac{(\sqrt{x^2+100})^2-10^2}{x^2(\sqrt{x^2+100}+10)}$$

$$= \lim_{x\to 0} \frac{x^2+\cancel{100}-\cancel{100}}{x^2(\sqrt{x^2+100}+10)} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}(\sqrt{x^2+100}+10)} = \frac{1}{\sqrt{0^2+100}+10}$$

$$= \frac{1}{10+10} = \underline{\underline{1/20}}$$

The Sandwitch theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c let

$$\lim_{x\to c} g(x) = L$$

$$\lim_{x\to c} h(x) = L$$

$$\lim_{x\to c} f(x) = L$$

Sandwitch theorem also known as squeeze theorem, is a calculus theorem that helps to determine the limit of an inequality function by comparing it to two other function with known limits.

Q. Given a function u satisfying the condition $1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2}$, $x \ne 0$ for all x not equal to 0, find $\lim_{x\to 0} u(x)$.

By using Sandwitch theorem

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} 1-\frac{x^2}{4} = \frac{1-0}{4} = 1-0 = 1$$

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} 1+\frac{x^2}{2} = 1+\frac{0}{2} = 1+0 = 1$$

$$\therefore \lim_{x\to 0} u(x) = \underline{\underline{1}}$$

How does the function $g(x) = x^2 \sin(1/x^2)$ behaves near $x = 0$.

By using sandwitch theorem,

Since the range of the sin function lies between -1 and 1, Similarly $\sin 1/x^2$ is also lies b/w -1 and 1

i.e, $-1 \leq \sin(1/x^2) \leq 1$ [multiplying x^2 in both sides]

$\Rightarrow -x^2 \leq x^2 \sin(1/x^2) \leq x^2$

$f(x) \leq g(x) \leq h(x)$ By using Sandwitch theorem

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} -x^2 = 0$$

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} x^2 = 0$$

$$\therefore \lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 \sin(1/x^2) = 0$$

Q Find the limit of the following.

1. $\lim_{x\to 3} \frac{x^2-6x+9}{x-3}$

$\Rightarrow x^2-6x+9 = 0$

$\Rightarrow (x-3)(x-3) = 0$ P = 9, S = -6

$$\Rightarrow \lim_{x\to 3} \frac{x^2-6x+9}{x-3} = \lim_{x\to 3} \frac{(x-3)^{\cancel{2}}}{\cancel{(x-3)}} = x-3 = 3-3 = 0$$

2. $\lim_{x\to -4} \frac{2x+8}{x^2+x-12}$

$$= \lim_{x\to -4} \frac{2\cancel{(x+4)}}{\cancel{(x+4)}(x-3)} = \frac{2}{-4-3} = \frac{2}{-7} = -2/7$$

3. $\lim_{x\to 5} \frac{x^2-3x-10}{x^2-10x+25} = \lim_{x\to 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{(x-5)}(x-5)}$

$\Rightarrow x^2-3x-10 = 0$

P = -10 $(x-5)(x+2) = 0$

S = -3

$\Rightarrow x^2-10x+25 = 0$

P = 25 $(x-5)(x-5) = 0$

S = -10

$$= \lim_{x\to 5} \frac{(x+2)}{(x-5)}$$

it cannot be factorised further

$\therefore$ limit does not exists.

4. $\lim_{x\to -2} (x^3 - 15)$

$= ((-2)^3 - 15) = -8 - 15 = \underline{-23}$

5. $\lim_{x\to 2} (-x^2 + 5x - 2) = \lim_{x\to 2} (-x^2) + \lim_{x\to 2} (5x) - \lim_{x\to 2} (2)$

~~$= -4 + 10 - 2 = 12$~~

$= -4 + 10 - 2 = \underline{4}$

6. $\lim_{x\to 2} \frac{2x+5}{11 - x^3}$

$= \frac{\lim_{x\to 2} 2x+5}{\lim_{x\to 2} 11 - x^3} = \frac{2\times 2+5}{11 - 2^3} = \frac{9}{3} = \underline{3}$

7. $\lim_{x\to 2/3} (8-3x)(2x-1)$

$= \lim_{x\to 2/3} (8-3x) \times \lim_{x\to 2/3} (2x-1)$

$= 8 - 3\times\frac{2}{3} \cdot 2\times\frac{2}{3} - 1$

$= 6 \times \frac{1}{3} = 6/3 = \underline{2}$

8. $\lim_{x\to -1/2} 4x(3x+4)^2$

$= \lim_{x\to -1/2} (4x) \cdot \left(\lim_{x\to -1/2} (3x+4)\right)^2$

$= 4\times -\frac{1}{2} \cdot \left(3\times -\frac{1}{2} + 4\right)^2$

$= -2 \cdot \left(\frac{-3+8}{2}\right)^2 = -2 \cdot \left(\frac{5}{2}\right)^2 = -2 \times \frac{25}{4} = \underline{\frac{-25}{2}}$

9. $\lim_{y\to -3} (5-y)^{4/3}$

$= \left(\lim_{y\to -3} 5-y\right)^{4/3} = (5-(-3))^{4/3} = (8)^{4/3} = (2^3)^{4/3} = (2)^4 = \underline{16}$

10. $\lim_{h \to 0} \frac{3}{\sqrt{3h+1}+1}$

$= \frac{\lim_{h\to 0} 3}{\lim_{h\to 0} \sqrt{3h11}+1} = \frac{3}{\sqrt{1}+1} = 3/2$

11. $\lim_{h \to 1} \frac{\sqrt{6h+10}-5}{h^2}$

$= \frac{\lim_{h\to 1} \sqrt{6h+10}-5}{\lim_{h\to 1} h^2} = \frac{\sqrt{6+10}-5}{1} = \frac{4-5}{1} = -1$

12. $\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2}$

$= \frac{\lim_{x\to 1} x-1}{\lim_{x\to 1} \sqrt{x+3}-2} = \frac{1-1}{\sqrt{1+3}-2} = \frac{0}{0}$ Not defined

$\lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}-2)} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3})^2-(2)^2}$

$= \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{x+3-4} = \lim_{x\to 1} \frac{\cancel{(x-1)}(\sqrt{x+3}-2)}{\cancel{(x-1)}} = \lim_{x\to 1} \sqrt{1+3}+2$

$= 2+2 = 4$

13. $\lim_{x\to 5} \frac{x-5}{x^2-25}$

$\Rightarrow x^2-25 = (x+5)(x-5)$

Subs $(x+5)(x-5)$,

$\lim_{x\to 5} \frac{\cancel{x-5}}{(x+5)\cancel{(x-5)}} = \frac{1}{5+5} = \frac{1}{10}$

14. $\lim_{x\to 3} \frac{x+4}{x^2+7x+14}$

$= \frac{\lim_{x\to 3} x+4}{\lim_{x\to 3}(x^2+7x+14)} = \frac{3+4}{9+21+14} = \frac{7}{44}$

15. $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$

$\Rightarrow x^2+3x-10 = 0$

$(x+5)(x-2) = 0$

$= \lim_{x \to -5} \frac{\cancel{(x+5)}(x-2)}{\cancel{x+5}}$

$= -5-2 = \underline{\underline{-7}}$

16. $\lim_{x \to 2} \frac{x^2-7x+10}{x-2}$

$\Rightarrow x^2-7x+10 = 0$

$(x-5)(x-2) = 0$

$= \lim_{x \to 2} \frac{(x-5)\cancel{(x-2)}}{\cancel{x-2}}$

$= 2-5 = \underline{\underline{-3}}$

17. $\lim_{t \to 2} \frac{t^2+4t+3}{t^2-9}$

$= \frac{\lim_{t \to 2} t^2+4t+3}{\lim_{t \to 2} t^2-9} = \frac{4+8+3}{4-9} = \frac{-15}{5} = \underline{\underline{-3}}$

18. $\lim_{t \to -2} \frac{-2t-4}{t^3+2t^2}$

$= \lim_{t \to -2} \frac{-2\cancel{(t+2)}}{t^2\cancel{(t+2)}} = \frac{-2}{-2^2} = \frac{-2}{4} = \underline{\underline{-1/2}}$

19. $\lim_{x \to 0} (2\sin x - 1)$

$= \lim_{x \to 0} 2\sin x - \lim_{x \to 0} 1$

$= 2 \times \sin 0 - 1$

$= \underline{\underline{-1}}$

20. $\lim_{x \to 0} \frac{1+x+\sin x}{2\cos x}$

$= \frac{\lim_{x \to 0} 1+x+\sin x}{\lim_{x \to 0} 2\cos x}$

$= \frac{1+0+\sin 0}{2\cos 0} = \underline{\underline{\frac{1}{2}}}$

21. $\lim_{x\to\pi} \sqrt{x+2}\,\{\sin(x/2)\}$

$= \sqrt{\pi+2}\times 1 = \underline{\underline{\sqrt{\pi+2}}}$

22. $\lim_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

$= \lim_{x\to-3} \dfrac{(2-\sqrt{x^2-5})(2+\sqrt{x^2-5})}{(x+3)(2+\sqrt{x^2-5})}$

$= \lim_{x\to-3} \dfrac{(2^2-(\sqrt{x^2-5})^2)}{(x+3)(2+\sqrt{x^2+5})} = \lim_{x\to-3} \dfrac{4-x^2+5}{(x+3)(2+\sqrt{x^2-5})}$

$= \lim_{x\to-3} \dfrac{9-x^2}{(x+3)(2+\sqrt{x-5})} = \lim_{x\to-3} \dfrac{(\cancel{3+x})(3-x)}{(\cancel{x+3})(2+\sqrt{x^2-5})}$

$= \dfrac{3+3}{2+\sqrt{9-5}} = \dfrac{6}{2+2} = 6/4 = \underline{\underline{3/2}}$

23. $\lim_{x\to1} \dfrac{x^{-1}-1}{x-1}$

$= \lim_{x\to1} \dfrac{\frac{1}{x}-1}{x-1} = \dfrac{\lim_{x\to1} \frac{1-x}{x}}{\lim_{x\to1} x-1} = \cancel{\lim \frac{1-1}{1-1}} = \cancel{\frac{0}{1}\times 0} = \cancel{0/1} = 0$

$= \lim_{x\to1} \dfrac{-(\cancel{x-1})}{x}\times\dfrac{1}{\cancel{x-1}} = \lim \dfrac{-1}{1} = \underline{\underline{-1}}$

24. Find $\lim_{x\to0} f(x)$ if $\sqrt{9-5x^2} \le f(x) \le \sqrt{x^2+9}$

By using Sandwitch theorem,

$\lim g(x) = \lim_{x\to0} \sqrt{9-5x^2} = \sqrt{9-0} = \sqrt{9} = \underline{\underline{3}}$

$\lim h(x) = \lim_{x\to0} \sqrt{x^2+9} = \sqrt{0+9} = \sqrt{9} = \underline{\underline{3}}$

$\therefore \lim_{x\to0} f(x) = \underline{\underline{3}}$

Q. 25. $\lim_{x\to 0} \frac{2\sin x}{2-2\cos x}$ if $1 - \frac{x^2}{6} \le \frac{2\sin x}{2-2\cos x} \le 1$

$g(x) \le f(x) \le h(x)$

$\lim_{x\to 0} g(x) = \lim_{x\to 0} 1 - \frac{x^2}{6} = 1 - \frac{0}{6} = \underline{\underline{1}}$

$\lim_{x\to 0} h(x) = \lim_{x\to 0} 1 = \underline{\underline{1}}$

$\therefore \lim_{x\to 0} \frac{2\sin x}{2-2\cos x} = 1$

Q. Evaluate $\lim_{x\to\infty} \frac{3x^2+2x+1}{2x^2+x+1}$

($1/\infty = 0$) if infinity is in denominator, $=0$

$$\lim_{x\to\infty} \frac{\frac{3x^2+2x+1}{x^2}}{\frac{2x^2+x+1}{x^2}} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}+\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{x}{x^2}+\frac{1}{x^2}}$$

$$= \frac{3+\frac{2}{x}+\frac{1}{x^2}}{2+\frac{1}{x}+\frac{1}{x^2}} = \frac{3+\frac{2}{\infty}+\frac{1}{\infty}}{2+\frac{1}{\infty}+\frac{1}{\infty}} = \frac{3+0+0}{2+0+0} = \underline{\underline{3/2}}$$

2. $\lim_{x\to\infty} \frac{4x^2-x+2}{2x^2+3x-1}$

$$\lim_{x\to\infty} \frac{\frac{4x^2-x+2}{x^2}}{\frac{2x^2+3x-1}{x^2}} = \frac{\frac{4x^2}{x^2}-\frac{x}{x^2}+2/x^2}{\frac{2x^2}{x^2}+\frac{3x}{x^2}-1/x^2} = \frac{4-1/x+2/x^2}{2+3/x-1/x^2}$$

$$= \frac{4-\frac{1}{\infty}+2/\infty}{2+3/\infty-1/\infty} = \frac{4-0+0}{2+0-0} = 4/2 = \underline{\underline{2}}$$

OR

$$\lim_{x\to\infty} \frac{x^2(4-1/x+2/x^2)}{x^2(2+3/x-1/x^2)} = \frac{4-1/\infty+2/\infty}{2+3/\infty-1/\infty} = \frac{4-0+0}{2+0-0} = 4/2 = \underline{\underline{2}}$$

3. $$\lim_{x\to\infty} \frac{2x^2+3x+1}{x^2-x+4}$$

$$= \lim_{x\to\infty} \frac{x^2\left(2+3/x+1/x^2\right)}{x^2\left(1-1/x+4/x^2\right)} = \frac{2+\frac{3}{\infty}+1/\infty}{1-1/\infty+4/\infty} = \frac{2+0+0}{1-0+0} = \underline{\underline{2}}$$

Continuity At a Point

A function $f(x)$ continuation at $x=a$, if

* $f(a)$ is exists.
* $\lim_{x\to a} f(x)$ is defined
* $\lim_{x\to a} f(x) = f(a)$.

Consider the function $f(x)=x^2$ check whether this function is continuous at $x=3$.

$f(3) = 3^2 = 9$

$\lim_{x\to 3} f(x) = x^2 = 3^2 = 9$

$\lim_{x\to 3} f(x) = f(3)$

$\therefore$ the given function $f(x)=x^2$ is continuous at $x=3$.

Consider the function $f(x)=1/x$ Check the continuity of the function at $x=1$

$f(1) = 1/x = 1/1 = 1$

$\lim_{x\to 1} f(x) = 1/x = 1/1 = 1$

$\lim_{x\to 1} f(x) = f(1)$

$\therefore f(x)=1/x$ is continuous at $x=1$.

Check whether $f(x)=1/x$ is continuous at $x=0$

$f(0) = 1/0$ Not defined.

$\therefore f(0)$ does not exist

$\therefore f(x) = 1/x$ is not continuous at $x=0$

One-Sided Limit

Right-Handed Continuity

A function $f(x)$ is continuous from the right at $x=a$ if

$$\lim_{x\to a^+} f(x) = f(a)$$

Left-Handed Continuity

A function $f(x)$ is said to continuous from the left at $x=a$ if

$$\lim_{x\to a^-} f(x) = f(a)$$

Q. Determine ~~th~~ if the function $f(x)$ is continuous from the left at $x=2$.

$$f(x) = \begin{cases} x^2+3 & ; x<2 \\ 5 & ; x\geq 2 \end{cases}$$

$$\cancel{f(x)}\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} f(x^2+3) = 2^2+3 = 4+3 = \underline{\underline{7}}$$

$$f(2) = 5$$

$$\Rightarrow \lim_{x\to 2^-} f(x) \neq f(2)$$

$\therefore$ ~~f(x)~~ $\cancel{\lim_{x\to 2^-} f(x)}$ It is not ~~cont~~ left-continuous.

Q. Determine if the function $g(x)$ is continuous from the right at $x=-1$.

$$g(x) = \begin{cases} \sin x & ; x\leq -1 \\ x^2 & ; x>-1 \end{cases}$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} f(x^2) = (-1)^2 = \underline{\underline{1}} \qquad \Rightarrow f(-1) = \sin x = -(\sin(1))$$

$$\lim_{x\to -1^+} f(x) \neq f(-1).$$

$\therefore$ the function $g(x)$ is not right continuous.

Q. Determine if the function $f(x) = \begin{cases} 3x-9 & ; x\leq 3 \\ x^2-6x+9 & ; x>3 \end{cases}$ is continuous from the right at $x=3$.

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} f(x^2-6x+9) = \cancel{9+18}\ 9-18+9 = \underline{\underline{0}}$$

$f(3) = 3\times3-9 = \underline{0}$

$\lim_{x\to3^+} f(x) = f(x3)$

$\therefore$ function $f(x)$ is ~~con~~ right continuous at $x=3$.

Q Determine if the function $f(x)$ is continous from the left at $x=1$

$$f(x) = \begin{cases} 2x+1 ; & x<1 \\ 3 ; & x\geq1 \end{cases}$$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} 2x+1 = 2+1 = \underline{3}$

$\lim f(1) = 3$

$\lim_{x\to1^-} f(x) = f(1)$

$\therefore$ The given function is left continuous at $x=1$.

Continuous function

A function is called continuous if it is continuous at every point in domain. The domain of a function is a set of all input values for which the function is defined.

Consider the function $f(x) = x^2$ is continuous at every point in its domain (its domain is the set of all real numbers). Therefore $f(x) = x^2$ is a continuous function.

Consider $f(x) = 1/x$, is not a continuous function, because it is not continuous at $x=0$. i.e, $f(x) = 1/x$ is continuous at everypoint exept $x=0$. $x=0$ is called the point of discontinuity. or the function has a discontinuity at $x=0$.

Q. At what points are the following functions $y=f(x)$ continuous.

1. $y = \frac{1}{x-2} - 3x$

$x-2=0$

$\Rightarrow$ ~~a~~ $x=2$ is the only point of discontinuity

$\therefore$ the function is continuous at everypoint except $x=2$.

2. $y = \frac{x+1}{x^2-4x+3}$

$x^2 - 4x + 3 = 0$ $\rightarrow y \cdot x$

$(x-3)(x-1) = 0$

$x = 3, 1.$

$\therefore$ the function is continuous at every point except at $x = 1$ and $x = 3$.

3. $y = |x+1| + \sin x$

the function is continuous at every points.

4. $y = \frac{1}{|x|+1} - \frac{x^2}{2}$

here, the denominator

$|x|+1 \neq 0$, for all x also the limit exists

$\therefore$ the function is continuous at every point.

5. $y = \frac{\cos x}{x}$

~~the fun~~ Here the denominator vanishes at $x = 0$

$x = 0$ is the only discontinuous point

$\therefore$ the given function is continuous at every point except at $x = 0$.

6. $y = \tan \frac{\pi x}{2}$

when $x = 1$, $y = \tan \frac{\pi}{2}$, the limit does not exist

$\therefore$ the function is not continuous.

when $x = 3$, $y = \tan \frac{3\pi}{2}$, also discontinuous

$\therefore$ the function is discontinuous at all point where $\frac{\pi x}{2}$ is an odd integer multiple of $\pi/2$.

$\therefore$ the function is continuous at all points except the odd integers.

Derivative

→ Slope of a curve

The derivative gives the rate of change of a function $f(x)$ with respect to x at $x = x_0$, denoted by $f'(x_0)$.

The rate of a function or the derivative of a function $f(x)$ at $x = a$ is defined by $\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$. All the following are the interpretations for the limits of the difference quotient or

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \rightarrow 1^{st} \text{ principle of differentiation}$$

1) The slope of the graph $y = f(x)$ at $x = a$
2) The slope of the tangent line to the curve $y = f(x)$ at $x = a$
3) The rate of change of $f(x)$ at $x = a$
4) The derivative $f'(x)$ at $x = a$.

Q1. Find the derivative of $f(x) = x^2$ at $x = 2$ using the 1^{st} principle of differentiation

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h\to 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h}$$

$$= \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= \lim 4 + 0 = \underline{\underline{4}}$$

Q2. Find the derivative of $f(x) = \sqrt{x}$ at $x = 4$ using 1^{st} principle.

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} = \lim_{h\to 0} \frac{\sqrt{4+0} - 2}{0} = \lim_{h\to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)}$$

$$= \lim_{h\to 0} \frac{(\sqrt{4+h})^2 - (2)^2}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{\cancel{4} + \cancel{h} - \cancel{4}}{\cancel{h}(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2}$$

$$= \frac{1}{\sqrt{4+0} + 2} = \underline{\underline{1/4}}$$

Q5 Find the derivative of $f(x) = 3x^2 + 2x$ at $x = 1$ using 1st principle.

$f(1+h) = 3+2h$ $3(1+h)^2+2(1+h)$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{f(3+2h) - f(3\times1^2 + 2\times1)}{h} = \lim_{h\to 0} \frac{f\ 3+2h-5}{h}$$

$$= \lim_{h\to 0} \frac{3(1+h)^2 + 2(1+h) - (3\times1^2 + 2\times1)}{h}$$

$$= \lim_{h\to 0} \frac{3(1+2h+h^2) + 2 + 2h - 5}{h} = \lim_{h\to 0} \frac{3+6h+3h^2+2+2h-5}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{5} + 8h + 3h^2 - \cancel{5}}{h} = \lim_{h\to 0} \frac{\cancel{h}(8+3h)}{\cancel{h}}$$

$$= 8 + 3\times0 = \underline{\underline{8}}$$

Q7 Find the derivative of $f(x) = 1/x$ at $x = 2$ using 1st first principle.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2-(2+h)}{(2+h)2}}{h} = \lim_{h\to 0} \frac{-h}{4+2h} \times h \div h$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{4+2h} \times \frac{1}{\cancel{h}} = \underline{\underline{-1/4}}$$

Q8 Find the derivative of $f(x) = \sin x$ at $x = 0$ using 1st principle.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(0+h) - \sin 0}{h} = \lim_{h\to 0} \frac{\sin(0+h) - 0}{h}$$

$$= \cancel{\frac{\sin 0 - 0}{0}} = 0$$

$$= \lim_{h\to 0} \frac{\sin h}{h}$$

$$\lim_{h\to 0} \frac{\sin h}{h} = \lim_{h\to 0} \frac{\cos h}{1} \quad \text{[by using L-hospital Rule]}$$

$$= \frac{\cos 0}{1} = 1 \quad \left(\lim_{h\to 0} \frac{\sin x}{x} = 1\right)$$

Q. Find the derivative of $f(x) = \ln x$ at $x = 1$ using 1st principle:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$ [Using L-Hospitals Rule]

$$= \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{\ln(1+h) - 0}{h}$$

$$= \lim_{h\to 0} \frac{1/1+h}{1} = \frac{\frac{1}{1+0}}{1} = \frac{1}{1} = 1 = \underline{1}$$

standard result

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

Rules Of Differentiation

Sum Rule : $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) = f'(x) + g'(x)$

Difference Rule : $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$

Constant Rule : $\frac{d}{dx}(c) = 0$

Constant xplication Rule : $\frac{d}{dx}(k f(x)) = k \frac{d}{dx} f(x) = k f'(x)$

Product Rule : $\frac{d}{dx} f(x) \cdot g(x) = \frac{d}{dx} f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$

Quotient Rule : $\frac{d}{dx} f(x)/g(x) = \frac{g(x) f'(x) - f(x) g'(x)}{(g(x))^2}$

Power Rule : $\frac{d}{dx}(x^n) = n x^{n-1}$

$\frac{d}{dx}(\sin x)$.

Q. Find the derivate of $f(x) = 5x^4$

$f' \frac{d}{dx} f(x) = \cancel{4\times 5(x)}\ \underline{20x^3}$

Q. Find d/dx of $f(x) = (2x^3)\sin x$

$f'(x) = 2x^3 \cdot \cos x + \sin x \cdot 6x^2$

$= 2x^3 \cos x + 6x^2 \cdot \sin x$

Q3. Find d/dx of ~~fend~~

1. $f(x) = e^x \cos x$

$f'(x) = e^x(-\sin x) + \cos x \cdot e^x$

$= e^x\cos x - e^x \sin x = e^x(\cos x - \sin x)$

2. $f(x) = e^x/x^2$

$f'(x) = \dfrac{x^2 \cdot e^x - e^x \cdot 2x}{(x^2)^2} = \dfrac{e^x x^2 - e^x \cdot 2x}{x^4} = \dfrac{e^x x^2 - e^x 2x}{x^4}$

$= \dfrac{e^x(x^2 - 2x)}{x^4} = \dfrac{\cancel{x}(e^x x - 2e^x)}{x^{\cancel{4}}}$

$= \dfrac{e^x x - 2e^x}{x^3}$

3. $f(x) = \dfrac{\ln x}{x^3}$

$f'(x) = \dfrac{x^3 \cdot 1/x - \ln x \times 3x^2}{(x^3)^2} = \dfrac{x^2 - 3x^2\ln x}{x^6}$

$= \dfrac{x^2(1 - 3\ln x)}{x^6} = \dfrac{1 - 3\ln x}{x^4}$

4. Find the slope of the curve $y = x^2$ at $x = 3$.

$f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$

$= \lim_{h\to 0} \dfrac{f(3+h)^2 - (3)^2}{h} = \lim_{h\to 0} \dfrac{f(9 + 6h + h^2) - 9}{h}$

$= \lim_{h\to 0} \dfrac{6h + h^2}{h} = \lim_{h\to 0} \dfrac{h(6+h)}{h} = 6 + 0 = 6$

Find the slope of curve $y = x^2$ at x

" $f(x) = x^2/\sin x$

" $y = \sin x$ at $x = \pi/4$

5. Find the derivative of $f(x) = x^2/\sin x$

$f'(x) = \dfrac{2x \sin x - x^2 \cos x}{\sin^2 x} = \dfrac{2x\sin x - x^2\cos x}{\sin^2 x}$

6. Find the slope of the curve $y = \sin x$ at $x = \pi/4$

$\lim \dfrac{dy}{dx} = \cos x$

at $x = \pi/4 \Rightarrow \dfrac{dy}{dx} = \cos \pi/4 = 1/\sqrt{2}$

Higher Order Derivates

The first order derivative of the fn. $y = f(x)$ is denoted by $\frac{dy}{dx}$ or y' or $f'(x)$. Again differentiate this first order with respect to x. We get the second order derivatives $\frac{d^2y}{dx^2}$ or y'' or $f''(x)$, again differentiate the second orde w.r.t x, we get $\frac{d^3y}{dx^3}$ or y''' or $f'''(x)$ and so on.

Q. Find the fourth order derivative of the function.

A.
$$y = 3x^5$$
$$y' = 15x^4$$
$$y'' = 60x^3$$
$$y''' = \cancel{4}\,180x^2$$
$$y'''' = \underline{360x}$$

Q. Find the 1st order and 2nd order derivative of $f(x) = 1/x$.

$$f'(x) = \frac{-1}{x^2}$$
$$f''(x) = \frac{x^2\cdot 0 - (-1)\cdot 2x}{(x^2)^2} = \frac{2x}{x^4} = \underline{\frac{2}{x^3}}$$

Q. Find the third derivative of $f(x) = x^5 - 4x^3 + x$

$$f'(x) = 5x^4 - 12x^2 + 1$$
$$f''(x) = \cancel{5-12}\ 20x^3 - 24x$$
$$f'''(x) = \underline{60x^2 - 24}$$

Q. Find the derivatives of all orders of $y = \frac{x^4}{2} - \frac{3x^2}{2} - x$

$$y' = \frac{4x^3}{2} - \frac{6x}{2} - 1 = 2x^3 - 3x - 1$$
$$y'' = 6x^2 - 3$$
$$y''' = 12x$$
$$y'''' = 12$$
$$y''''' = \underline{0}$$

Q. Find the 4th derivative of $y = e^x \sin x$.

$$\frac{dy}{dx} = e^x \cdot \cos x + \sin x \cdot e^x = e^x(\cos x + \sin x)$$

$$\frac{d^2y}{dx^2} = e^x \cdot (-\sin x + \cos x) + (\cos x + \sin x)e^x$$

$$= -e^x \cancel{\sin x} + e^x \cos x + e^x \cos x + e^x \cancel{\sin x}$$

$$= 2e^x \cos x$$

$$\frac{d^3y}{dx^3} = 2\left[e^x \cdot -\sin x + \cos x \cdot e^x\right]$$

$$= 2\left[e^x \cos x - e^x \sin x\right] = 2e^x\left[\cos x - \sin x\right]$$

$$\frac{d^4y}{dx^4} = 2e^x \left[-\sin x - \cos x\right] + \left[\cos x - \sin x\right] \cdot 2e^x$$

$$= -2e^x \sin x - \cancel{2e^x \cos x} + \cancel{2e^x \cos x} - 2e^x \sin x$$

$$= -4e^x \sin x$$

Q. Find the third derivative of $f(x) = x^3 \ln x$

$$f'(x) = x^3 \frac{1}{x} + \ln x \cdot 3x^2$$

$$= x^2 + 3x^2 \ln x$$

$$f''(x) = 2x + 6x\ 3x^2 \cdot \frac{1}{x} + \ln x \cdot 6x$$

$$= 2x + 3x + 6x \ln x$$

$$f'''(x) = 2 + 3 + 6x \times \frac{1}{x} + \ln x \cdot 6$$

$$= 5 + 6 + 6\ln x = 11 + 6\ln x$$

Q. Find the 1st order derivative $f(x) = \frac{x^3}{\sqrt{x^2+1}}$

$$f'(x) = \frac{\sqrt{x^2+1} \cdot 3x^2 - x^3 \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{3x^2\sqrt{x^2+1} - \frac{2x^4}{2\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2} = \frac{3x^2(x^2+1) - x^4}{\sqrt{x^2+1}\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{3x^4 + 3x^2 - x^4}{\left(\sqrt{x^2+1}\right)^3} = \frac{2x^4 + 3x^2}{\left(\sqrt{x^2+1}\right)^3}$$

Instantaneous Rate Of Change

Q. Find the instantaneous rate of change of $f(x) = x^2$ at $x = 5$

$f'(x) = 2x$

$f'(x)/_{x=5} = f'(5) = 2\times 5 = \underline{\underline{10}}$

Q. Find the rate of change of the function $f(x) = 4x^3 - x^2 + 2x - 7$ at $x = 1$

$f'(x) = 12x^2 - 2x + 2$

$f'(x)/_{x=1} = 12 \times 1^2 - 2\times 1 + 2$

$= 12 - \cancel{2} + \cancel{2} = \underline{\underline{12}}$

Q. A car's position $s(t)$ in metres at time 't' in seconds is given by $s(t) = 4t^2$. Find the car's instantaneous velocity at $t = 5$ seconds.

$s'(t) = 8t$

$s'(t)/_{t=5} = 8\times 5 = \underline{40 \text{ m/s}}$

Q. The stress $\sigma(t)$ in a material is given by $\sigma(t) = 100t - 5t^2$, where 't' is in seconds. Find the rate of change of stress at $t = 8$ s.

$\sigma'(t) = 100 - 10t$

$\sigma'(t)/_{t=8} = 100 - 10\times 8$

$= 100 - 80$

$= \underline{\underline{20 \text{ Pa/s or } Nm^2}}$

Chain-Rule

Q1. If $y = f(u(x))$, differentiate y with respect to x is given by $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

Find the derivative of $y = (3x+2)^3$.

$y = (3x+2)^3$

$\frac{dy}{dx} = 3(3x+2)^2 \frac{d}{dx}(3x+2)$

$= 3(3x+2)^2 \cdot 3$

$= 9(3x+2)^2$

$y = (3x+2)^3$

$f(x) = u^3 \qquad u(x) = (3x$

$\frac{df}{du} = 3u^2 \qquad \frac{du}{dx} = 3$

$\frac{dy}{dx} = \frac{df}{du} \times \frac{du}{dx}$

$= 3u^2 \times 3$

$= 9u^2 = 9(3x+$

Q2. Find the derivative of $y = \sin(4x^3)$.

$y = \sin(4x^3)$

$\frac{dy}{dx} = \cos(4x^3) \cdot \frac{d}{dx}(4x^3)$

$= \cos(4x^3) \cdot 12x^2$

$= 12x^2 \cos(4x^3)$

$y = (\sin(4x^3))$

$f(x) = \sin u \qquad u(x) =$

$\frac{df}{du} = \cos u \qquad \frac{du}{dx} =$

$\frac{dy}{dx} = \frac{df}{du} \times \frac{du}{dx} = \cos u$

$= \cos(4$

Q3. $y = e^{\sin x}$

$\frac{dy}{dx} = e^{\sin x} \cdot \frac{d}{dx}(\sin x)$

$= \cos x \cdot e^{\sin x}$

Q4. $y = \sqrt{5x^2+3x}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{5x^2+3x}} \cdot \frac{d}{dx}(5x^2+3x)$

$= \frac{1}{2\sqrt{5x^2+3x}} \cdot 10x+3 = \frac{10x+3}{2\sqrt{5x^2+3x}}$

Q_5: $y = \tan(2x+1)$

$$\frac{dy}{dx} = \sec^2(2x+1) \cdot \frac{d}{dx}(2x+1)$$

$$= \sec^2(2x+1) \cdot 2 = 2\sec^2(2x+1)$$

Q_6 $y = x\sin^4 x + x\cos^{-2} x$

$$\frac{dy}{dx} = \cancel{(x\cos^4 x + \sin^4 x \cdot 1)} + \cancel{(x \cdot \sin^{-2} x + \cos^{-2} x \cdot 1)}$$

$$= \cancel{x\cos^4 x + \sin^4 x - x\sin^{-2} x + \cos^{-2} x}$$

$$= x\frac{d}{dx}(\sin^4 x) + \sin^4 x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\cos^{-2} x) + \frac{d}{dx}\cos^{-2} x \cdot \frac{d}{dx}(x)$$

$$= x \cdot 4\sin^3 x \cdot \frac{d}{dx}(\sin x) + \sin^4 x \cdot 1 + x \cdot -2\cos^{-3} x \cdot \frac{d}{dx}(\cos x) + \cos^{-2} x \cdot 1$$

$$= x \cdot 4\sin^3 x \cdot \cos x + \sin^4 x + 2x\cos^{-3} x \sin x + \cos^{-2} x$$

$$= x \cdot 4\sin^3 x\cos x + \sin^4 x + \frac{2\sin x}{\cos 3x} + \frac{1}{\cos^2 x}$$

Q_7: $f(\theta) = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2$

$$\cancel{\frac{dy}{d\theta}} = \cancel{\frac{df}{d\theta}}$$

$$\frac{df}{d\theta} = 2\left(\frac{\sin\theta}{1+\cos\theta}\right) \cdot \frac{d}{dx}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= 2\left(\frac{\sin\theta}{1+\cos\theta}\right) \cdot \frac{(1+\cos\theta)\cos\theta - \sin\theta(0 - \sin\theta)}{(1+\cos\theta)^2}$$

$$= 2\left(\frac{\sin\theta}{1+\cos\theta}\right) \frac{(1+\cos\theta)\cos\theta + \sin^2\theta}{(1+\cos\theta)^2} = \frac{2(\sin\theta)}{(1+\cos\theta)} \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2}$$

$$= \frac{2(\sin\theta)}{(1+\cos\theta)} \cdot \frac{\cancel{1+\cos\theta}}{\cancel{(1+\cos\theta)}^2} = \frac{2(\sin\theta)}{(1+\cos\theta)^2}$$

Q. $y = x\tan 2\sqrt{x} + 7$

$$\frac{dy}{dx} = x \cdot \sec^2 2\sqrt{x} \cdot \frac{d}{dx} 2\sqrt{x} + \tan 2\sqrt{x} \cdot \frac{d}{dx}(x) + \frac{d}{dx}(7)$$

$$= x\sec^2 2\sqrt{x} \cdot 2 \times \frac{1}{2\sqrt{x}} + 0$$

$$= x\sec^2 2\sqrt{x} \cdot \frac{1}{\sqrt{x}} = \frac{x\sec^2 2\sqrt{x}}{\sqrt{x}} + \tan 2\sqrt{x}$$

Implicit Differentiation

It is used when a function is given in an implicit form. meaning, it is not solved for 1 variable in terms of the other.

Q. Find the derivate of :

1) $x^2+y^2=r^2$ implicity to find dy/dx

diff. both sides with respect to x.

$\Rightarrow \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(r^2)$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = \frac{d}{dx}(r^2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$x^2+y^2=r^2$ → eqn of circle.

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -x/y$$

2) diff the eqn of the ~~elip~~ ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ implicitly to find dy/dx

diff both sides with respect to x.

$$\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}\frac{x^2}{a^2}+\frac{d}{dx}y^2/b^2 = 0$$

$$\frac{1}{a^2}\frac{d}{dx}(x^2)+\frac{1}{b^2}\frac{d}{dx}(y^2) = 0$$

$$\frac{1}{a^2}2x+\frac{1}{b^2}\cdot 2y\cdot\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = -2x/a^2$$

$$\frac{dy}{dx} = \frac{-2x/a^2}{2y/b^2}$$

$$\frac{dy}{dx} = -\frac{2x}{a^2}\times\frac{b^2}{2y} = \frac{-b^2x}{a^2y}$$

3) diff. ~~x^2+y^2~~ $x^2+xy+y^2=1$ implicitly to find dy/dx.

diff both sides with respect to x

$$\frac{d}{dx}(x^2+xy+y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$2x + x\cdot\frac{dy}{dx} + y\cdot 1 + 2y\cdot\frac{dy}{dx} = 0$$

$$2x + \frac{x\,dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$2x+y + \frac{dy}{dx}(x+2y) = 0$$

$$\frac{dy}{dx}(x+2y) = -2x-y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x+2y}$$

4) $x^2y+y^3 = 7$

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(7)$$

$$x^2\frac{dy}{dx} + y\cdot 2x + 3y^2\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx}(x^2+3y^2) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2+3y^2}$$

5) $\sin(xy) = x+y$

$$\cos(xy)\cdot\frac{d}{dx}(xy) = 1+\frac{dy}{dx}$$

$$\cos(xy)\left[x\frac{dy}{dx}+y\right] = 1+dy/dx$$

$$\cos(xy)\,x\cdot\frac{dy}{dx} - \frac{dy}{dx} = 1-\cancel{\cos(xy)}-y$$

$$dy/dx\left[\cos(xy)\,x-1\right] = 1-\cos(xy)-y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-\cos(xy)-y}{x-1}$$

$$\Rightarrow \frac{dy}{dx} = 1-$$

$\sin(xy) = x+y$

$\cos(xy)\left[x\frac{dy}{dx}+y\right] = 1+\frac{dy}{dx}$

$x\cos(xy)\frac{dy}{dx} + y\cos(xy) = 1+\frac{dy}{dx}$

$x\cos(xy)\frac{dy}{dx} - \frac{dy}{dx} = 1 - y\cos(xy)$

$\frac{dy}{dx}\left[x\cos(xy) - 1\right] = 1 - y\cos(xy)$

$\frac{dy}{dx} = \frac{1-y\cos(xy)}{x\cos(xy)-1}$

6) $x^2(x-y^2) = x^2 - y^2$

$\frac{dy}{dx}\left(x^2(x-y^2)\right) = \frac{d}{dx}(x^2-y^2)$

$x^2\cdot\left(1-2y\frac{dy}{dx}\right) + (x-y^2)\cdot 2x = 2x - 2y\frac{dy}{dx}$

$x^2 - 2x^2y\frac{dy}{dx} + 2x^2 - 2xy^2 = 2x - 2y\frac{dy}{dx}$

$2y\frac{dy}{dx} - 2x^2y\frac{dy}{dx} = 2x - x^2 - 2x^2 + 2xy^2$

$\frac{dy}{dx}(2y - 2x^2y) = x(2 - x - 2x + 2y^2)$

$\frac{dy}{dx} = \frac{2x - x^2 - 2x^2 + 2xy^2}{2y - 2(x^2y)}$

$\frac{dy}{dx} = \frac{2(x - x^2 + xy^2) - x^2}{2(y - x^2y)}$

$x^2 2(x-y)\frac{d}{dx}(x-y) + (x-y)^2 2x = 2x - 2y\frac{dy}{dx}$

$2x^2(x-y)\left[1 - dy/dx\right] + (x-y^2)\,2x = 2x - 2y\frac{dy}{dx}$

$2x^2(x-y) - 2x^2(x-y)\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 2y\frac{dy}{dx}$

$2x^2(x-y) - 2x^2(x-y)\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 2x(x-y^2)$

$\frac{dy}{dx}\left[2y - 2x^2(x-y)\right] = 2x - 2x(x-y)^2 - 2x^2(x-y)$

$\frac{dy}{dx} = \frac{2x - 2x(x-y)^2 - 2x^2(x-y)}{2y - 2x^2(x-y)}$

$$= \frac{2x[1-(x+y)^2 - x[x-y]]}{2[y-x^2[x-y]]}$$

$$= \frac{x[1-(x+y^2)^2 - x(x-y)]}{[y-x^2[x-y]]}$$

Use Implicit diff. to find $\frac{dy}{dx}$

1. $2xy + y^2 = x + y$
2. $y^2 = \frac{x(1+x)}{x+1}$
3. $x^3y + xy^3 = 6$

Find 1st & 2nd order derivatives of the following

1. $r = \frac{1}{3s^2} - \frac{5}{2s}$
2. $y = 4 - 2x - x^{-3}$
3. $y = \left(\frac{1+3z}{3z}\right)(3-z)$
4. $y = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3}$

Find the derivative of the following functions

1. $y = (5-2x^2)(2x^3-3x+4)$
2. $y = (1+x^2)(x^{3/4} - x^{-3})$
3. $y = \frac{\sqrt{45} - 9}{\sqrt{45} + 9}$

a) $2xy + y^2 = x + y$

$$\frac{d}{dx}(2xy + y^2) = \frac{d}{dx}(x+y)$$

$$2\left(x \cdot dy/dx + y\right) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$$

$$\frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$$

$$\frac{dy}{dx} = \frac{1-2y}{2x+2y-1}$$

8) dy/dx where $x^3 = \frac{2x-y}{x+3y}$

$$3x^2 = \frac{(x+3y)\left(2 - \frac{dy}{dx}\right) - (2x-y)\left(1 + 3\frac{dy}{dx}\right)}{(x+3y)^2}$$

$$3x^2 = \frac{\cancel{2x} + 6y - x\frac{dy}{dx} - 3y\frac{dy}{dx} - \left[\cancel{2x} - y + 6x\frac{dy}{dx} - 3y\frac{dy}{dx}\right]}{(x+3y)^2}$$

$$3x^2 = \frac{6y - x\frac{dy}{dx} + y - \frac{6x\,dy}{dx}}{(x+3y)^2}$$

$$3x^2 = \frac{7y - \frac{dy}{dx}(x - 6x)}{(x+3y)^2}$$

$$3x^2(x+3y)^2 = 7y - \frac{dy}{dx}(-5x)$$

$$3x^2(x+3y)^2 - 7y = -\frac{dy}{dx}(-5x)$$

$$\Rightarrow \frac{3x^2(x+3y)^2 - 7y}{-5x} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2(x+3y^2) - 7y}{5x}$$

Tangent line And Normal line

Equation Of Tangent line

The eqn of the tangent line to the curve $y = f(x)$ at the point (x_0, y_0) is given by $\boxed{(y - y_0) = f'(x_0)(x - x_0)}$ where $f'(x)$ is the slope of the curve at x_0.

Equation of Normal line

A normal line to a curve at a given point is the line that is ⊥ar to the tangent line at that point the slope of the normal line is the -ve reciprocal of the slope of the tangent line. And the eqn of the normal line is

$$\boxed{(y - y_0) = \frac{-1}{f'(x_0)}(x - x_0)}$$

Q1. Find the eqn of the tangent and normal lines to the curve $y = x^2$ at the point $(1,1)$.

Eqn of tangent line : $y - y_0 = f'(x_0)(x - x_0)$ $\quad f'(x) = 2x$, $f'(x_0) = f'(1) = 2$

$$y - 1 = 2(x - 1)$$
$$y = 2x - 2 + 1$$
$$\underline{y = 2x - 1}$$

Eqn of normal line : $y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$

$$y - 1 = \frac{-1}{2}.(x - x_0)$$
$$2y - 2 = -x + 1$$
$$2y = -x + 1 + 2$$
$$\underline{y = -\frac{x}{2} + \frac{3}{2}}$$

Q2. Find the eqn of tangent and the normal lines to the curve $y = x^2 + 3x + 2$ at $x = 1$. $(1, 6)$

$\Rightarrow y = 1^2 + 3\times1 + 2 = 6$

$f'(x) = 2x + 3$

$f'(x_0) = 2 + 3 = \underline{5}$

$$y - y_0 = f'(x_0)(\cancel{y-y}\, x - x_0)$$
$$y - 6 = 5(x - 1)$$
$$y = 5x - 5 + 6$$
$$\underline{y = 5x + 1}$$

Eqn of normal line

$y - 6 = -\frac{1}{5}(x-1)$

$5y - 30 = -x + 1$

$y = -\frac{x}{5} + \frac{31}{5}$

Q3. Find the eqn of tangent and normal lines to the curve, $y = e^x$ at $x = 0$

$y = e^x$ at $x = 0$

$= e^0 = 1$

$y' = e^x$

$f(x_0) = e^0 = 1$

Eqn of tangent line

$y - y_0 = f'(x_0)(x - x_0)$

$y - 1 = 1(x - 0)$

$y = x + 1$

Eqn of normal line

$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$

$y - 1 = \frac{-1}{1}(x - 0)$

$y = -x + 1$

Q4. A projectile is launched its height h(t) in meters is given by the functions $h(t) = -5t^2 + 20t + 25$, where t is the time in seconds. Find the eqn of the tangent line and the normal line to the curve at t = 2s.

$h(t) = -5\times4 + 20\times2 + 25 = -20 + 40 + 25 = 45$ (2, 45)

$h'(t) = -10t + 20$

$\Rightarrow -10\times2 + 20 = 0$

$y - 45 = 0(x - 20)$

$y = 45$ $\Rightarrow$ Eqn of tangant line.

$y - 45 = -\frac{1}{0}(x-2)$

Not defined $\Rightarrow$ Eqn of normal line.

The eqn of normal line is not defined (Since the tangant line is horizontal the normal line is vertical) therefore, the eqn of the normal line is t = 2.

Q5. Find eqn of tangent and normal line to the curve $y = \sin x$ at $x = \pi/4$.

$y = \sin\frac{\pi}{4} = 1/\sqrt{2}$ $(\pi/4, 1/\sqrt{2})$

$y'(x) = \cos x \Rightarrow y'(x_0) = \cos\pi/4 = 1/\sqrt{2}$

Eqn of tangent line

$y - y_0 = f'(x_0)(x - x_0)$

$y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x - \pi/4)$

$\sqrt{2}y - 1 = x - \pi/4$

$y = \frac{x - \pi/4 + 1}{\sqrt{2}}$

Eqn of normal line.

$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$

$y - 1/\sqrt{2} = -\sqrt{2}(x - x_0)$

$y = -\sqrt{2}x + \sqrt{2}\frac{\pi}{4} + 1/\sqrt{2}$

Linearization

Linearization is a method is used to appr. the value of a function near a given point using its derivatives. For a function f(x) that is differentiable at $x = a$, the linear apprximation (linearization) of $f(x)$ near $x = a$ is given by
$L(x) = f(a) + f'(a)(x - a)$.

$L(x) \simeq f(x)$ at ~~all~~

Q. Linearise $f(x) = \sqrt{x}$ at $x = 4$.

$L(x) = f(a) + f'(a)(x - a)$

$= 2 + \frac{1}{4}(x - 4)$

$= 2 + \frac{x}{4} - \frac{4}{4}$

$= \frac{x}{4} + 1$

$f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

Q2. Linearise $f(x) = e^x$ at $x = 0$

$f(0) = e^0 = 1$

$f'(x) = e^x$.

$f'(0) = e^0 = 1$

$L(x) = 1 + 1(x - 0)$

$= 1 + x$

Q3. Linearise $f(x) = \ln x$ at $x = 1$

$f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x}$

$f'(1) = \frac{1}{1} = 1$

$L(x) = 0 + 1(x - 1)$

$= x - 1$

Q4. Linearise $f(x) = \tan x$ at $x = \pi/4$

$f'(\pi/4) = \tan\frac{\pi}{4} = 1$

$f'(x) = \sec^2\frac{\pi}{4}$

$f'(\pi/4) = 2$

$L(x) = 1 + 2(x - \pi/4)$

$= 1 + 2x - \frac{2\pi}{4 \cdot 2}$

$= 1 + 2x - \pi/2$

Q5. The resistance R of a resistor is given by $R(T) = e^{0.1T}$ where T is the temperature in °C linearize the function at $T = 25$ to appx. the resistor for small temperature changes around the point.

$R'(T) = e^{0.1T} \cdot 0.1 = 0.1 e^{0.1T}$

$R'(25) = 0.1 e^{0.1 \times 25} = 1.218$

$R(25) = e^{0.1 \times 25} = 12.182$

$L(t) = R(a) + R'(a)(T - a)$

$= 12.182 + 1.218(T - 25)$

$= 12.182 + 1.218T - 30.45$

$= 1.218 - 18.268$

Q6. Suppose a city's population P(t) in ~~100~~ thousands is given by the eqn $p(t) = 100e^{0.02t}$, where 't' is the number of years. Find the linear approx. of the population in at $t = 0$ in year 2020. Use this linear approx. ~~of the~~ to estimate the population in the year 2021.

$P(t) = 100e^{0.02t}$

$P'(t) = 100 \cdot e^{0.02t} \cdot 0.02$

$= 2 \cdot e^{0.02t}$

$P(0) = 100e^{0.02 \times 0} = 100$

$P'(0) = 100 \cdot e^{0.02 \times 0} \cdot 0.02 = 2$

$L(t) = P(a) + P'(a)(t - a)$

$= 100 + 2(t - 0)$

$= 100 + 2t$

Estimated population in the year 2020 = $100 + 2t$.

$t = 1$

$L(1) = 100 + 2 \times 1$

$= 102$

Estimated population in the year 2021 = 102

Q. Linearise $f(x) = \sin x$ at $x = \pi/6$:

$f'(x) = \cos x \Rightarrow f'(a) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

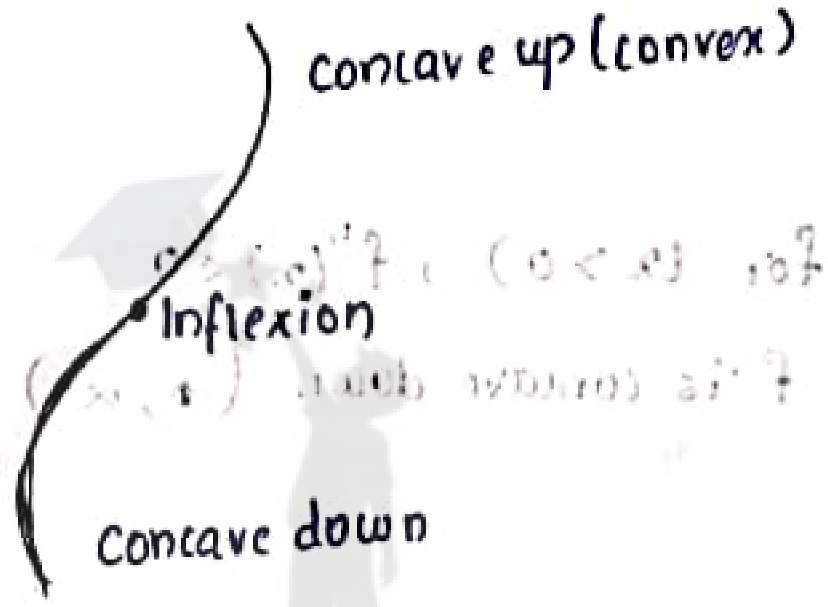
$f(a) = \sin\frac{\pi}{6} = 1/2$

$L(x) = f(a) + f'(a)(x-a)$

$= 1/2 + \sqrt{3}/2(x - \pi/6) = 1/2 + \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}\pi}{12}$

Concavity

Concavity describes the curvature nature of a graph of a function. A function may be concave up or concave down.



The second Derivative Test for Concavity

Let $y = f(x)$ be a twice differentiable function on an interval I,

(i) If $f''(x) > 0$ on I, the graph of f on I is concave up.

(ii) If $f''(x) < 0$ on I, the graph of f over I is concave down.

(iii) If $f''(x) = 0$ or $f''(x)$ is not defined, then this points are known as inflexion points.

Q. Determine the concavity of $f(x) = x^3 - 3x^2 + 4$;

$f(x) = x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

For $x > 1$, $f''(x) > 0$, f is concave up.

For $x < 1$, $f''(x) < 0$, f is concave down.

f is concave up on $(1, \infty)$

f is concave down on $(-\infty, 1)$.

Q. Determine the concavity of $f(x) = x^3 - 6x^2 + 9x + 1$

1) $f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

for $x > 2$, $f''(x) > 0$

for $x < 2$; $f''(x) < 0$

f is concave up on $(2, \infty)$

f is concave down on $(-\infty, 2)$

2) $f(x) = \ln(x)$ for $x > 0$.

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

~~for $x>0$~~, for $(x > 0)$, $f''(x) < 0$

f is concave down $(0, \infty)$

3) $f(x) = e^{-x^2}$

$f'(x) = e^{-x^2} \cdot -2x$

$f''(x) = \cancel{e^{-x^2} \cdot -2x} = \cancel{-4xe^{-x^2}} \quad e^{-x^2} \cdot -2 + -2x \cdot e^{-x^2} \cdot -2x$

$= -2e^{-x^2} + 4x^2 e^{-x^2}$

$= \underline{\underline{2e^{-x^2}(2x^2 - 1)}}$

$\boxed{e^{-x^2} \neq 0}$

for $x^2 > \frac{1}{2}$ $f''(x) > 0$

for $x^2 < \frac{1}{2}$ $f''(x) < 0$

$-\infty \quad -\sqrt{\frac{1}{2}} \quad \sqrt{\frac{1}{2}} \quad \infty$

f is concave up on $(-\infty, -\sqrt{\frac{1}{2}}) \cup (\sqrt{\frac{1}{2}}, \infty)$

f is concave down on $(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$

4) $f(x) = e^x - x^2$

$e^{\ln x} = x$

$e^{\ln 2} = 2$

$f'(x) = e^x - 2x$

$f''(x) = e^x - 2$

$f''(x) = 0$

$e^x - 2 = 0$

$e^x = 2$

$x = \ln 2$

for $x > \ln 2$, $f''(x) > 0$

for $x < \ln 2$, $f''(x) < 0$

f is concave up on $(\ln 2, \infty)$

f is convave up on $(-\infty, \ln 2)$

Q. Find the eqn of the tangent line and normal line to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at $(-1, 0)$

$$f'(x) = 12x + 3\left(\frac{dy}{dx}x + y\right) + 4y\frac{dy}{dx} \cdot 17\frac{dy}{dx}$$

$$= 12x + 3\frac{dy}{dx}x + 3y + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = 0$$

$$3\frac{dy}{dx}x + 4y\frac{dy}{dx} + 17\frac{dy}{dx} = -12x - 3y$$

$$\frac{dy}{dx}(3x + 4y + 17) = -12x - 3y$$

$$\frac{dy}{dx} = \frac{-(12x + 3y)}{\cancel{(2x + 4y)}\ (3x + 4y + 17)}$$

$$f'(x_0, y_0) = \frac{-(12 \times -1 + 3 \times 0)}{3 \times -1 + 4 \times 0 + 17} = \frac{12}{14} = \frac{6}{7}$$

$$y - y_0 = f'(x_0, y_0)(x - x_0)$$

$$y - 0 = \frac{6}{7}(x - (-1))$$

$$7y = 6x + 6$$

$$\underline{\underline{y = \frac{6x}{7} + \frac{6}{7}}}$$

Eqn of normal line

$$y - y_0 = \frac{-1}{f'(x_0, y_0)}(x - x_0)$$

$$y - 0 = \frac{-1}{\frac{6}{7}}(x - (-1))$$

$$6y = -7x - 7$$

$$\underline{\underline{y = \frac{-(7x + 7)}{6}}}$$