What is 3D Calculus?

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1 Introduction to 3D Calculus

Calculus, at its core, is the study of change. Single-variable calculus (often called 2D calculus) deals with functions of one independent variable, like $y = f(x)$, focusing on slopes and areas. 3D calculus, also known as multivariable or vector calculus, extends these concepts to functions with multiple independent variables.

In 3D calculus, we move from the xy-plane to xyz-space, enabling us to model complex phenomena:

- Temperature distribution in a room: $T(x, y, z)$.
- Fluid flow, represented by vector fields.
- Gravitational force, varying with position.
- Volumes of solids and surface areas of curved objects.

3D calculus provides the tools to understand and quantify change in multiple dimensions.

2 Core Concepts of 3D Calculus

2.1 Functions of Multiple Variables

We consider functions like $z = f(x, y)$ or $w = f(x, y, z)$.

- $z = f(x, y)$: A function with two inputs (x, y) and one output (z). Its graph is a surface in 3D space.
- $w = f(x, y, z)$: A function with three inputs and one output. Its graph exists in 4D space, visualized using level surfaces where $f(x, y, z) = c$.

2.2 Partial Derivatives

For $f(x, y)$, partial derivatives describe how f changes with respect to one variable while holding others constant:

- $\frac{\partial f}{\partial x}$: The rate of change of f with respect to x, treating y as constant.
- $\frac{\partial f}{\partial y}$: The rate of change of f with respect to y, treating x as constant.

These represent slopes of the surface in the x and y directions.

2.3 Gradients

The gradient of a scalar function $f(x, y, z)$ is a vector pointing in the direction of the greatest rate of increase of f:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The magnitude $|\nabla f|$ gives the maximum rate of increase.

2.4 Double and Triple Integrals

Double and triple integrals extend the concept of definite integrals to higher dimensions:

- **Double Integrals** ($\iint_R f(x,y)\,dA$): Calculate volumes under surfaces, areas in the xy-plane, or quantities over a 2D region R.
- **Triple Integrals** ($\iiint_E f(x,y,z)\,dV$): Calculate volumes of 3D solids, mass, or average value over a 3D region E.

2.5 Vector Fields

A vector field assigns a vector to each point in space, e.g., $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$. They model forces, fluid flow, and wind patterns.

2.6 Line Integrals and Surface Integrals

- **Line Integrals** ($\int_C \mathbf{F}\cdot d\mathbf{r}$ or $\int_C f(x,y,z)\,ds$): Integrate along a curve C. Applications include work done by a force.
- **Surface Integrals** ($\iint_S \mathbf{F}\cdot d\mathbf{S}$ or $\iint_S f(x,y,z)\,dS$): Integrate over a surface S. Applications include calculating flux.

2.7 Fundamental Theorems of Vector Calculus

These theorems relate integrals and derivatives:

- **Green's Theorem**: Relates a line integral around a closed curve to a double integral over the enclosed region.
- **Stokes' Theorem**: Relates a line integral around a closed curve to a surface integral over a surface bounded by the curve.
- **Divergence Theorem (Gauss's Theorem)**: Relates a surface integral over a closed surface to a triple integral over the enclosed solid.

3 Worked Example: Analyzing a Paraboloid

Consider $f(x,y) = x^2 + y^2$, a paraboloid.

3.1 Visualizing the Surface

The graph of $z = x^2 + y^2$ is shown below.

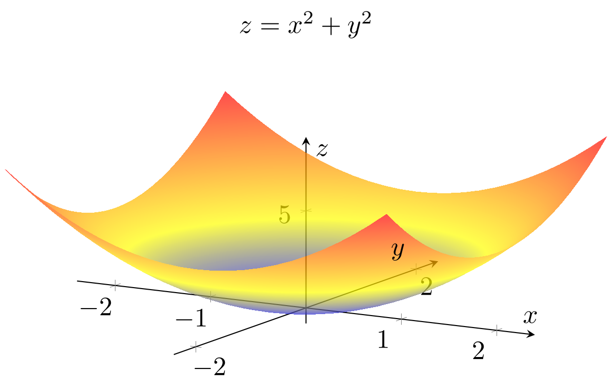

Figure 1: Graph of the paraboloid $z = x^2 + y^2$.

3.2 Partial Derivatives

For $f(x, y) = x^2 + y^2$:

- $\frac{\partial f}{\partial x} = 2x$.
- $\frac{\partial f}{\partial y} = 2y$.

3.3 Gradient

The gradient of $f(x, y) = x^2 + y^2$ is:

$$\nabla f = \langle 2x, 2y \rangle$$

At $(1, 1)$, $\nabla f(1, 1) = \langle 2, 2 \rangle$, pointing in the direction of steepest ascent.

3.4 Double Integral (Volume)

The volume under $z = x^2 + y^2$ over $R = [0, 1] \times [0, 1]$ is:

$$V = \iint_R (x^2 + y^2)\, dA = \int_0^1 \int_0^1 (x^2 + y^2)\, dx\, dy = \frac{2}{3}$$

4 Exercises

Q1. Easy Given $f(x, y) = 3x^2y - 5xy^3 + 7$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution 1. $\frac{\partial f}{\partial x} = 6xy - 5y^3$ $\frac{\partial f}{\partial y} = 3x^2 - 15xy^2$

Q2. Medium Calculate the gradient of $g(x, y, z) = x \sin(yz)$ at $(1, \pi/2, 1)$.

Solution 2. $\frac{\partial g}{\partial x} = \sin(yz)$, $\frac{\partial g}{\partial y} = xz\cos(yz)$, $\frac{\partial g}{\partial z} = xy\cos(yz)$. *At* $(1, \pi/2, 1)$, $\nabla g = \langle \sin(\pi/2), (1)(1)\cos(\pi/2), (1)(\pi/$ $\langle 1, 0, 0 \rangle$.

Q3. Hard A plate occupies the region D bounded by $y = x^2$ and $y = x + 2$. If the density is $\rho(x, y) = x^2y$, set up the double integral for the total mass.

Solution 3. *First, find the intersection points:* $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0$. *So,* $x = -1$ *and* $x = 2$. *The mass is given by* $M = \iint_D \rho(x, y)\, dA = \int_{-1}^{2} \int_{x^2}^{x+2} x^2 y\, dy\, dx$.