Discrete Bayesian Filtering — Complete Notes and Solutions

Robotic Pancake, Markov Chains, Posterior Shortcuts

1 Bayesian Filtering — Core Equations

State X_t, control U_t, observation Z_t.

Prediction (estimation):

$$\overline{bel}(x_t) = \sum_{x_{t-1}} p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})$$

Correction (measurement update):

$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

where η normalizes the belief.

Interpretation:

- Prediction propagates belief through the system model
- Correction weights the prediction by sensor reliability

2 Markov Chain and Transition Matrix

For discrete states, the system evolution is a Markov chain. Let the state distribution be a row vector:

$$\pi_t = [P(x_t = s_1)\ \dots\ P(x_t = s_n)]$$

Transition matrix T:

$$T_{ij} = P(X_{t+1} = s_j \mid X_t = s_i)$$

Prediction in matrix form:

$$\boxed{\pi_{t+1} = \pi_t T}$$

After k steps:

$$\boxed{\pi_{t+k} = \pi_t T^k}$$

Each multiplication by T corresponds to *one time step.*

3 Measurement Update — Matrix Form

Define the observation diagonal matrix:

$$D(z_t) = \operatorname{diag}\big(p(z_t \mid s_1), \dots, p(z_t \mid s_n)\big)$$

Bayes filter (matrix form):

$$\boxed{bel_t = \eta\, (bel_{t-1} T)\, D(z_t)}$$

Whole observation sequence:

$$\boxed{bel_t \propto bel_0\, T D(z_1)\, T D(z_2) \cdots T D(z_t)}$$

4 Robotic Arm Flipping a Pancake

4.1 State Space

$$\Omega = \{U, H, B\}$$

Uncooked, Half-baked, Baked.

4.2 Initial Belief

Initially uncooked:

$$bel(X_0) = [1\ 0\ 0]$$

4.3 Sensor Model

	obsU	obsH	obsB
U	0.7	0.2	0.1
H	0.1	0.6	0.3
B	0	0.2	0.8

4.4 Transition Model

Do Nothing (DN):

$$T_{\text{DN}} = \begin{bmatrix} 0.25 & 0.75 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Flip:

$$T_{\text{Flip}} = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

4.5 t=1: DN, obs = uncooked

Prediction:

$$\overline{bel}_1 = [0.25\ 0.75\ 0]$$

Update:

$$bel_1 \propto [0.25 \cdot 0.7,\ 0.75 \cdot 0.1,\ 0] = [0.175\ 0.075\ 0]$$

Normalize:

$$\boxed{bel_1 = [0.7\ 0.3\ 0]}$$

4.6 t=2: DN, obs = half-baked

Prediction:

$$\overline{bel}_2 = [0.175\ 0.825\ 0]$$

Update:

$$bel_2 \propto [0.035\ 0.495\ 0] \Rightarrow \boxed{bel_2 = [0.066\ 0.934\ 0]}$$

4.7 t=3: Flip, obs = half-baked

Prediction:

$$\overline{bel}_3 = [0.0198\ 0.5132\ 0.4670]$$

Update:

$$bel_3 \propto [0.00396\ 0.3079\ 0.0934]$$

$$\boxed{bel_3 = [0.0098\ 0.7598\ 0.2304]}$$

5 Posterior Probability — Odds Shortcut

5.1 Single Observation

Bayes in odds form:

$$\boxed{\frac{P(H \mid E)}{1 - P(H \mid E)} = \frac{P(H)}{1 - P(H)} \cdot \frac{P(E \mid H)}{P(E \mid \neg H)}}$$

5.2 Multiple Observations

For independent observations $E_1, \dots, E_N$:

$$\boxed{O_N = O_0 \prod_{i=1}^{N} \frac{P(E_i \mid H)}{P(E_i \mid \neg H)}}$$

Recover probability:

$$\boxed{P(H \mid E_{1:N}) = \frac{O_N}{1 + O_N}}$$

Mnemonic:

Posterior odds = Prior odds × Likelihood ratios

6 Faulty Sensor Example

Let F=sensor faulty.

$$P(F) = 0.01, \quad P(E \mid F) = 1, \quad P(E \mid \neg F) = \frac{1}{3}$$

Posterior after N identical observations:

$$\boxed{P(F \mid E^N) = \frac{1}{1 + 99 \cdot 3^{-N}}}$$

Each new observation multiplies the odds by 3.