Numerical Filters — Theory & How to Solve

Embedded Software — Master II Robotics Engineering

1 Digital Filters — Core Concepts

A **digital filter** is a discrete-time system that transforms an input sequence e_n into an output sequence s_n.

Two major families:

- FIR — Finite Impulse Response (non-recursive)
- IIR — Infinite Impulse Response (recursive)

This document focuses on **recursive (IIR-like) structures** and their analysis.

2 Impulse Response — Key Idea

Impulse response is the output of a filter when the input is:

$$e_n = \delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

A filter is **stable** if its impulse response tends to zero as $n \to \infty$.

3 First-Order Recursive Filter

3.1 Difference Equation

A first-order filter is defined by:

$$s_n = \alpha e_n - \beta s_{n-1}$$

Where:

- e_n : input signal
- s_n : output signal
- α : input gain
- β : feedback coefficient

3.2 Block Diagram Logic

This equation corresponds to:

- One delay element (z^{-1})
- One feedback loop with gain $-\beta$
- One feedforward path with gain α

Recursive term ⇒ memory ⇒ possible instability.

3.3 Impulse Response — How to Compute

Set:

$$e_0 = 1, \quad e_n = 0 \; \forall n > 0$$

Assume:

$$s_{-1} = 0$$

Then compute iteratively:

$$\begin{aligned} s_0 &= \alpha \\ s_1 &= -\beta s_0 \\ s_2 &= -\beta s_1 \\ &\vdots \end{aligned}$$

Which gives:

$$s_n = \alpha(-\beta)^n$$

Always compute impulse response **step by step** — marks are awarded per step.

4 Stability Analysis — First Order

A filter is stable if:

$$|s_n| \to 0 \quad \text{as } n \to \infty$$

For:

$$s_n = \alpha(-\beta)^n$$

Stability condition:

$$|\beta| < 1$$

Typical Exam Cases

Case a: $\alpha = 1$, $\beta = -2$

$$|\beta| = 2 > 1 \Rightarrow \text{unstable}$$

Case b: $\alpha = 1 - \beta$, $0 < \beta < 1$

$$|\beta| < 1 \Rightarrow \text{stable}$$

Stability depends on β, not on α.

5 Transfer Function — First Order

Take the Z-transform of:

$$s_n = \alpha e_n - \beta s_{n-1}$$

$$S(z) = \alpha E(z) - \beta z^{-1} S(z)$$

Rearranging:

$$H(z) = \frac{S(z)}{E(z)} = \frac{\alpha}{1 + \beta z^{-1}}$$

Poles determine stability — not zeros.

Pole:

$$z = -\beta \quad \Rightarrow \quad |z| < 1 \Rightarrow \text{stable}$$

6 Second-Order Recursive Filter

6.1 Difference Equation

$$s_n = e_n - a_1 s_{n-1} - a_2 s_{n-2}$$

This is a **second-order recursive filter**.

6.2 Transfer Function Derivation

Applying Z-transform:

$$S(z) = E(z) - a_1 z^{-1} S(z) - a_2 z^{-2} S(z)$$

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

7 Stability Conditions — Second Order

The poles are solutions of:

$$1 + a_1 z^{-1} + a_2 z^{-2} = 0$$

Or:

$$z^2 + a_1 z + a_2 = 0$$

Stability condition:

- All poles must lie **inside the unit circle**
- $|z_i| < 1$

> Second-order stability = **both poles inside unit circle**.

(Exact inequalities are usually provided or deduced graphically in exams.)

8 Frequency Response Concept

From:

$$H(z)$$

Substitute:

$$z = e^{j2\pi f T_e}$$

To obtain:

$$H(f)$$

This gives:

- Gain vs frequency
- Phase vs frequency

> $H(z) \to$ algebraic analysis, $H(f) \to$ frequency behavior.

9 General Exam Methodology

1. Identify the recursion
2. Compute impulse response
3. Check convergence
4. Compute $H(z)$
5. Locate poles
6. Conclude on stability

> Never guess stability — always justify mathematically.