Flight Modeling — review

Dense exam-ready formulas + short explanations

Fundamental Principle of Dynamics (Rigid Body)

Newton–Euler: In an inertial (Galilean) frame, Dynamics of a single rigid body.

$$\sum \boldsymbol{F}_{ext} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}, \qquad \sum \boldsymbol{M}_{A,ext} = \frac{\mathrm{d}\boldsymbol{H}_A}{\mathrm{d}t}.$$

Translation (use G = COM):

$$\boldsymbol{P} = m\boldsymbol{V}_G \Rightarrow \sum \boldsymbol{F}_{ext} = m\boldsymbol{a}_G.$$

Rotation about COM:

$$\sum \boldsymbol{M}_{G,ext} = \frac{\mathrm{d}\boldsymbol{H}_G}{\mathrm{d}t}, \qquad \boldsymbol{H}_G = \mathbf{J}_G\boldsymbol{\omega}.$$

Rigid-body expanded form:

$$\sum \boldsymbol{M}_{G,ext} = \mathbf{J}_G\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{J}_G\boldsymbol{\omega}).$$

Quick check: units N vs $m\,a$, $N \cdot m$ vs $\mathrm{d}H/\,\mathrm{d}t$.

Rotation Matrices (Yaw–Pitch–Roll and Euler)

Always write your convention. Here are the standard ones used in aerospace/mechanics.

(A) Yaw–Pitch–Roll (Tait–Bryan) intrinsic $Z\!-\!Y\!-\!X$

$$P_{YPR} = R_z(\psi)\,R_y(\theta)\,R_x(\phi).$$

Elementary rotations:

$$R_z(\psi) = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix},$$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}.$$

For $\psi = \pi/6, \theta = \pi/4, \phi = \pi/3$ (exact simplified):

$$P_{YPR} = \begin{bmatrix} \frac{\sqrt{6}}{4} & -\frac{1}{4} + \frac{3\sqrt{2}}{8} & \frac{\sqrt{6}}{8} + \frac{\sqrt{3}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{8} + \frac{\sqrt{3}}{4} & -\frac{3}{4} + \frac{\sqrt{2}}{8} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}.$$

(B) Proper Euler angles (commonly $Z\!-\!X\!-\!Z$)

$$P_{Euler} = R_z(\psi)\,R_x(\theta)\,R_z(\phi).$$

For $\psi = \pi/6, \theta = \pi/4, \phi = \pi/3$ (exact simplified):

$$P_{Euler} = \begin{bmatrix} -\frac{\sqrt{6}}{8} + \frac{\sqrt{3}}{4} & -\frac{3}{4} - \frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{4} \\ \frac{1}{4} + \frac{3\sqrt{2}}{8} & -\frac{\sqrt{3}}{4} + \frac{\sqrt{6}}{8} & -\frac{\sqrt{6}}{4} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Checks: $P^TP = I$, $\det(P) = 1$.

Link between P and Angular Velocity $\boldsymbol{\omega}$

Let $P = P_{B,B'}$ be the change-of-basis matrix such that

$$[\boldsymbol{v}]_B = P\,[\boldsymbol{v}]_{B'}.$$

Define the cross-product matrix

$$[\boldsymbol{\omega}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

Key identities:

$\boxed{\dot{P} = [\boldsymbol{\omega}]_\times\,P}$ ($\boldsymbol{\omega}$ components in B), $\boxed{[\boldsymbol{\omega}]_\times = \dot{P}\,P^T}$

Also (components in B'):

$$\boxed{[\boldsymbol{\omega}]_{\times\,B'} = P^T\dot{P}}$$

Why it works (1 line): $P^TP = I \Rightarrow \dot{P}P^T$ is skew-symmetric.

$\boldsymbol{\omega}$ from Euler angles + rates

Convention used (proper Euler $Z\!-\!X\!-\!Z$, intrinsic):

$$P = R_z(\psi)\,R_x(\theta)\,R_z(\phi).$$

Angular velocity decomposition:

$$\boxed{\boldsymbol{\omega} = \dot{\psi}\,\boldsymbol{k} + \dot{\theta}\,\boldsymbol{i}_1 + \dot{\phi}\,\boldsymbol{k}_2}$$

where

$$\boldsymbol{i}_1 = R_z(\psi)\boldsymbol{i}, \qquad \boldsymbol{k}_2 = R_z(\psi)R_x(\theta)\boldsymbol{k}.$$

Compute these axes in inertial basis $B = (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$:

$$\boldsymbol{i}_1 = \begin{bmatrix} \cos\psi \\ \sin\psi \\ 0 \end{bmatrix}, \qquad \boldsymbol{k}_2 = \begin{bmatrix} \sin\psi\sin\theta \\ -\cos\psi\sin\theta \\ \cos\theta \end{bmatrix}.$$

Thus

$$[\boldsymbol{\omega}]_B = \begin{bmatrix} \dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta \\ \dot{\theta}\sin\psi - \dot{\phi}\cos\psi\sin\theta \\ \dot{\psi} + \dot{\phi}\cos\theta \end{bmatrix}.$$

Numerical data: $\psi = \pi/6, \theta = \pi/4, \phi = \pi/3$, $\dot{\psi} = -1, \dot{\theta} = 1, \dot{\phi} = -1$. Using $c_{\pi/6} = \sqrt{3}/2$, $s_{\pi/6} = 1/2$, $s_{\pi/4} = \sqrt{2}/2$, $c_{\pi/4} = \sqrt{2}/2$:

$$\boxed{[\boldsymbol{\omega}]_B = \begin{bmatrix} -\frac{\sqrt{2}}{4} + \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \frac{\sqrt{6}}{4} \\ -1 - \frac{\sqrt{2}}{2} \end{bmatrix}}$$

Basis statement (required): Here $[\boldsymbol{\omega}]_B$ is expressed in the inertial basis B.

3-angle rotation representations

Main difficulty: singularities (loss of one DoF) $\Rightarrow$ **gimbal lock**.
At some angles, mapping between $(\dot{\psi}, \dot{\theta}, \dot{\phi})$ and $\boldsymbol{\omega}$ becomes singular (Jacobian loses rank) $\Rightarrow$ angles not unique + numerically unstable.

Can we remove it?

- **Not globally** with a minimal 3-parameter description of $SO(3)$ (need multiple charts).
- Practical fixes:
  - use **quaternions** (4 params + $\|q\| = 1$) $\Rightarrow$ no gimbal lock;
  - or use **DCM/rotation matrix** $P \in SO(3)$ with orthonormal constraints;
  - or switch Euler sequences / re-parameterize near singularities.

Meaning of an inertial frame

Definition: A frame is **inertial (Galilean)** if Newton's laws hold without fictitious forces:

$$\sum \boldsymbol{F}_{ext} = m\boldsymbol{a} \quad \text{(for any particle/rigid body).}$$

Equivalent characterization: it is non-accelerating and non-rotating relative to a Galilean frame (only constant-velocity translation allowed).

Reminder: In a non-inertial frame, inertial forces appear (centrifugal, Coriolis, Euler, translational).

Inertia matrix of a uniform bar

Problem: Uniform slender bar, length 2ℓ, mass M. Compute the inertia matrix about its center of mass

Chosen basis: Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be an orthonormal body-fixed basis with:

- $\vec{e}_1$ along the bar axis,
- $\vec{e}_2, \vec{e}_3$ perpendicular to the bar,
- origin at G.

Moments of inertia:

$$I_{e_1} \approx 0 \quad \text{(slender bar)}, \qquad I_{e_2} = I_{e_3} = \frac{1}{12} M (2\ell)^2 = \frac{1}{3} M\ell^2.$$

Inertia matrix at G:

$$\boxed{\mathbf{J}_G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}M\ell^2 & 0 \\ 0 & 0 & \frac{1}{3}M\ell^2 \end{bmatrix}_{(\vec{e}_1, \vec{e}_2, \vec{e}_3)}}$$

Check: symmetry $\Rightarrow I_{e_2} = I_{e_3}$, units $M\ell^2$.

Link actions vs given actions

Link actions (actions de liaison): Forces/torques due to *constraints* between bodies of a system.

- joint reactions, contact forces, constraint torques;
- internal to the system;
- cancel in pairs when summing the dynamics of the whole system.

Given actions (actions données): Forces/torques *imposed* on the system.

- gravity, aerodynamic forces, thrust, external torques;
- external to the system;
- do not cancel at system level.

Control actions:

$$\boxed{\text{Control actions are } \textit{given actions.}}$$

They are imposed actuator forces/torques treated as known inputs in the model.

Lie algebra of rigid motions

A rigid velocity field $X \in \mathcal{D}(\mathcal{E})$ has the form:

$$X(M) = v_X + \omega_X \wedge \overrightarrow{OM},$$

with constant $v_X, \omega_X \in \mathbb{R}^3$ (similarly for Y).

Lie bracket definition:

$$[X, Y](M) = \omega_X \wedge Y(M) - \omega_Y \wedge X(M).$$

Computation:

$$[X, Y](M) = (\omega_X \wedge v_Y - \omega_Y \wedge v_X) + \big(\omega_X \wedge (\omega_Y \wedge r) - \omega_Y \wedge (\omega_X \wedge r)\big), \quad r =$$

Using the triple-product identity,

$$\omega_X \wedge (\omega_Y \wedge r) - \omega_Y \wedge (\omega_X \wedge r) = (\omega_X \wedge \omega_Y) \wedge r.$$

Thus:

$$[X, Y](M) = v_{[X,Y]} + (\omega_X \wedge \omega_Y) \wedge r.$$

Result:

$$\boxed{\omega_{[X,Y]} = \omega_X \wedge \omega_Y}.$$

Check: antisymmetry $\Rightarrow \omega_{[Y,X]} = -\omega_{[X,Y]}$.