Control Rigid Bar

Dense exam-ready formulas + short explanations

System Definition

Rigid homogeneous bar S, length 2ℓ, mass m, center of mass G.

Inertial frame:

$$R = (O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$$

Generalized coordinates:

$$q = (x, y, \theta)$$

where (x, y) are the coordinates of G and

$$\theta = (\boldsymbol{i}, \overrightarrow{AB}) = (\boldsymbol{i}, \boldsymbol{i}_s).$$

External actions:

- Driving force at A: $\boldsymbol{F}_A = F_A\,\boldsymbol{i}_A$
- Constant force through G: $\boldsymbol{F} = -F\,\boldsymbol{j}$

Control vector:

$$\boxed{u = (F_A, \alpha)}, \qquad \alpha = (\boldsymbol{i}_s, \boldsymbol{i}_A)$$

Coordinates of A and B

Bar axis unit vector:

$$\boldsymbol{i}_s = \cos\theta\,\boldsymbol{i} + \sin\theta\,\boldsymbol{j}$$

Since G is the midpoint:

$$\overrightarrow{GA} = -\ell\boldsymbol{i}_s, \qquad \overrightarrow{GB} = +\ell\boldsymbol{i}_s$$

$$\boxed{\begin{array}{ll} A: & \begin{cases} x_A = x - \ell\cos\theta \\ y_A = y - \ell\sin\theta \end{cases} \\ \\ B: & \begin{cases} x_B = x + \ell\cos\theta \\ y_B = y + \ell\sin\theta \end{cases} \end{array}}$$

Velocity Field of S

Planar motion $\Rightarrow$ angular velocity:

$$\boldsymbol{\omega} = \dot\theta\,\boldsymbol{k}$$

For any point P such that $\overrightarrow{GP} = s\boldsymbol{i}_s$:

$$\boldsymbol{v}_P = \boldsymbol{v}_G + \boldsymbol{\omega} \times \overrightarrow{GP}$$

$$\boxed{\boldsymbol{v}_P = \dot x\,\boldsymbol{i} + \dot y\,\boldsymbol{j} + s\dot\theta(-\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j})}$$

Special case:

$$\boldsymbol{v}_G = \dot x\,\boldsymbol{i} + \dot y\,\boldsymbol{j}$$

Inertia Matrix Form

Choose body-fixed basis:

$$(\boldsymbol{i}_s, \boldsymbol{j}_s, \boldsymbol{k})$$

Slender bar assumptions:

- negligible radius $\Rightarrow I_{i_s} \approx 0$
- symmetry $\Rightarrow I_{j_s} = I_k$

$$\boxed{\mathbf{J}_G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}}$$

Value of I

Uniform linear density:

$$\lambda = \frac{m}{2\ell}$$

$$I = \int s^2\ \mathrm{d}m = \lambda \int_{-\ell}^{\ell} s^2\,\mathrm{d}s = \frac{m\ell^2}{3}$$

$$\boxed{I = \frac{m\ell^2}{3}}$$

Kinetic Wrench

Linear momentum:

$$\boldsymbol{P} = m\boldsymbol{v}_G$$

Angular momentum at G:

$$\boldsymbol{H}_G = \mathbf{J}_G\boldsymbol{\omega}$$

$$\boxed{\mathcal{H}(S/R) = \left\{ \begin{array}{l} \boldsymbol{R}_H = m(\dot x\,\boldsymbol{i} + \dot y\,\boldsymbol{j}) \\ \boldsymbol{M}_H(G) = I\dot\theta\,\boldsymbol{k} \end{array} \right\}_G}$$

Dynamic Wrench

Definition:

$$\mathcal{K} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}$$

$$\boxed{\mathcal{K}(S/R) = \left\{ \begin{array}{l} \boldsymbol{R}_K = m(\ddot x\,\boldsymbol{i} + \ddot y\,\boldsymbol{j}) \\ \boldsymbol{M}_K(G) = I\ddot\theta\,\boldsymbol{k} \end{array} \right\}_G}$$

Driving Force Reduced at G

Driving direction:

$$\boldsymbol{i}_A = \cos(\theta + \alpha)\boldsymbol{i} + \sin(\theta + \alpha)\boldsymbol{j}$$

$$\boldsymbol{F}_A = F_A\boldsymbol{i}_A$$

Moment at G:

$$\boldsymbol{M}_G(\boldsymbol{F}_A) = \overrightarrow{GA} \times \boldsymbol{F}_A = -\ell F_A \sin\alpha\,\boldsymbol{k}$$

Force Through G

Since $\boldsymbol{F}$ passes through G:

$$\boldsymbol{M}_G(\boldsymbol{F}) = \mathbf{0}$$

$$\boxed{\mathcal{T}_F = \left\{ \begin{array}{l} \boldsymbol{R} = -F\,\boldsymbol{j} \\ \boldsymbol{M}_G = \mathbf{0} \end{array} \right\}_G}$$

Equations of Motion

Newton–Euler at G:

$$\sum \boldsymbol{F}_{ext} = m\boldsymbol{a}_G, \qquad \sum \boldsymbol{M}_G = I\ddot{\theta}\,\boldsymbol{k}$$

$$\boxed{\begin{cases} m\ddot{x} = F_A\cos(\theta+\alpha) \\ m\ddot{y} = F_A\sin(\theta+\alpha) - F \\ I\ddot{\theta} = -\ell F_A \sin\alpha \end{cases}}$$

Controllability Insight

Attitude equation:

$$\ddot{\theta} = -\frac{3}{m\ell}F_A\sin\alpha$$

If $\alpha \in [0,\pi]$ and $F_A \geq 0$:

$$\ddot{\theta} \leq 0 \Rightarrow \text{one torque direction only}$$

$$\boxed{\text{Attitude } \theta \text{ not fully controllable}}$$

Intuitive Explanation — Simple Physical View

This section explains the full problem using simple physical intuition. The goal is to understand what is being modeled and why each step is natural, without relying on heavy mathematics.

Overall picture

The system is a rigid bar moving in a plane. At any time, the bar can:

- move horizontally,
- move vertically,
- rotate.

These three motions are fully described by the position of the center of mass and the orientation of the bar.

Geometry of the bar

The center of mass is exactly in the middle of the bar. Once the position of the center and the orientation of the bar are known, the positions of the two ends are fixed. This step is purely geometric and does not involve dynamics.

Motion of any point

A rigid body moves as a whole. Every point on the bar moves because:

- the center moves,
- the bar rotates around the center.

The velocity of any point is therefore the combination of translation and rotation.

Shape of the inertia matrix

The bar is thin, symmetric, and uniform. Because of this symmetry:

- rotation along the bar axis is negligible,
- rotations perpendicular to the bar are equivalent.

This explains why the inertia matrix is diagonal and simple.

Physical meaning of inertia

Rotational inertia measures how hard it is to rotate the bar. Mass elements farther from the center contribute more. Since the mass is evenly distributed, the inertia depends only on the mass and the square of the bar length.

Kinetic description

The kinetic wrench groups together:

- linear motion of the center,
- rotational motion of the bar.

Evaluating it at the center avoids unnecessary coupling terms.

Dynamic response

The dynamic wrench represents how motion and rotation evolve with time. It is simply Newton's laws applied to a rigid body instead of a point mass.

Effect of the driving force

The driving force is applied at one end of the bar. Because it does not pass through the center, it creates both translation and rotation. The rotational effect depends on how the force direction is oriented relative to the bar.

Force through the center

The second force acts directly at the center of mass. Such a force cannot rotate the bar and only affects its vertical motion.

Equations of motion

Each independent motion has its own equation: horizontal motion, vertical motion, and rotation. Together, these equations fully describe the system's dynamics.

Control interpretation

Rotation is controlled only through the torque generated by the driving force. If this torque can act in only one direction, the bar cannot be freely oriented. This reveals a natural limitation in controllability.