Quadrotor Modeling and Control: A Newton–Euler Approach with Linearization and Control Design

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***Abstract*—This report provides a complete and self-contained solution to a quadrotor modeling and control problem. The nonlinear equations of motion are derived using Newton–Euler rigid body dynamics. The forces and moments generated by the propellers are modeled from first principles. The physical mechanism enabling horizontal translation is explained intuitively. A linearized state-space model is obtained using small disturbance theory around a hovering equilibrium, and a stabilizing control law is designed using a cascade control structure. The report is written to be both mathematically rigorous and pedagogically accessible.**

I. Introduction

Quadrotor unmanned aerial vehicles (UAVs) are widely used due to their mechanical simplicity and maneuverability. A quadrotor uses four fixed-pitch propellers to generate lift and control forces. Despite this apparent simplicity, the vehicle exhibits nonlinear, coupled dynamics due to rigid body motion and the interaction between translation and rotation.

The objective of this report is to derive the quadrotor dynamic model from fundamental physical laws and to design a control law capable of stabilizing and controlling the vehicle near hover. Each section explicitly answers one question of the assignment and includes detailed explanations so that the report can be understood even without prior exposure to quadrotor dynamics.

II. Modeling Assumptions and Coordinate Frames

A. Modeling Assumptions

To make the problem mathematically tractable, the following assumptions are made:

- The quadrotor is modeled as a rigid body.
- The structure is symmetric with respect to the body x and y axes.
- The center of mass coincides with the origin of the body frame.
- Products of inertia are zero: $I_{xy} = I_{xz} = I_{yz} = 0$.
- Aerodynamic drag and motor dynamics are neglected.

These assumptions are standard in introductory quadrotor modeling and are valid near hovering conditions.

B. Reference Frames

Two coordinate frames are used:

- An inertial (world) frame $\mathcal{W} = (X, Y, Z)$ fixed to the Earth.
- A body frame $\mathcal{B} = (x_b, y_b, z_b)$ fixed to the quadrotor center of mass.

Linear velocity in the body frame is denoted:

$$V_b = [u \ v \ w]^T$$

and angular velocity is:

$$\Omega_b = [p \ q \ r]^T$$

III. Answer to Question 1: Nonlinear Quadrotor Dynamics

A. Why Newton–Euler Equations Are Used

The Newton–Euler formalism describes the motion of a rigid body under external forces and moments. It simultaneously captures:

- Translational motion of the center of mass
- Rotational motion about the center of mass

This makes it ideal for modeling aerial vehicles.

B. Newton–Euler Equations

The equations of motion expressed in the body frame are:

$$\begin{bmatrix} mI_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{V}_b \\ \dot{\Omega}_b \end{bmatrix} + \begin{bmatrix} \Omega_b \times (mV_b) \\ \Omega_b \times (I\Omega_b) \end{bmatrix} = \begin{bmatrix} F_b \\ M_b \end{bmatrix} \tag{1}$$

The cross-product terms appear because the equations are written in a rotating (body-fixed) frame.

C. Translational Dynamics

From the translational part:

$$m\dot{V}_b + \Omega_b \times (mV_b) = F_b$$

After dividing by the mass and expanding the cross product, the nonlinear translational dynamics are obtained:

$$\ddot{x}_b = r\dot{y}_b - q\dot{z}_b + \frac{1}{m}F_x \tag{2}$$

$$\ddot{y}_b = p\dot{z}_b - r\dot{x}_b + \frac{1}{m}F_y \tag{3}$$

$$\ddot{z}_b = q\dot{x}_b - p\dot{y}_b + \frac{1}{m}F_z \tag{4}$$

These equations show that translational acceleration depends not only on forces but also on rotational motion.

D. Rotational Dynamics

Assuming a diagonal inertia matrix:

$$I = \mathrm{diag}(I_{xx}, I_{yy}, I_{zz})$$

The rotational equations become:

$$I_{xx}\dot{p} = (I_{yy} - I_{zz})qr + M_x \quad (5)$$
$$I_{yy}\dot{q} = (I_{zz} - I_{xx})pr + M_y \quad (6)$$
$$I_{zz}\dot{r} = (I_{xx} - I_{yy})pq + M_z \quad (7)$$

These nonlinear terms represent gyroscopic coupling between rotational axes.

IV. Answer to Question 2: Forces and Moments Generated by the Motors

Each propeller produces thrust along the body z_b axis. The thrust generated by motor i is:

$$T_i = K_t\omega_i^2$$

The total thrust force in the body frame is therefore:

$$F_b = \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} T_i \end{bmatrix}$$

Weight acts vertically in the inertial frame and is transformed into the body frame through the rotation matrix.

A. Moment Generation

Because the motors are located at a distance L from the center of mass, thrust differences create moments:

$$M_x = L(T_2 - T_4) \quad (8)$$
$$M_y = L(T_3 - T_1) \quad (9)$$

Yaw motion is generated by reaction torques due to propeller drag:

$$M_z = K_m(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2)$$

V. Answer to Question 3: How Horizontal Translation Is Achieved

A quadrotor cannot generate direct thrust along the x or y axes because all propellers are fixed vertically. Instead, horizontal motion is achieved indirectly.

When the quadrotor tilts by changing roll or pitch, the thrust vector is no longer vertical. It is decomposed into:

- A vertical component balancing weight
- A horizontal component accelerating the vehicle

Thus, position control is achieved by attitude control.

VI. Answer to Question 4: Linearized State-Space Model

A. Hovering Equilibrium

Hovering corresponds to:

$$\phi_0 = \theta_0 = p_0 = q_0 = r_0 = 0, \quad \sum T_i = mg$$

B. Small Disturbance Theory

Near hover, angles and angular rates are small. Therefore:

$$\sin\phi \approx \phi, \quad \sin\theta \approx \theta$$

The linearized translational dynamics are:

$$\ddot{x} = g\theta \quad (10)$$
$$\ddot{y} = -g\phi \quad (11)$$
$$\ddot{z} = \frac{1}{m}\Delta T \quad (12)$$

The rotational dynamics become:

$$\dot{p} = \frac{1}{I_{xx}}M_x \quad (13)$$
$$\dot{q} = \frac{1}{I_{yy}}M_y \quad (14)$$
$$\dot{r} = \frac{1}{I_{zz}}M_z \quad (15)$$

At hovering equilibrium, yaw dynamics are decoupled from translational motion and do not affect linearized position dynamics.

VII. Answer to Question 5: Control Law Design

A. Control Architecture

A cascade control structure is adopted:

- Outer loop: position control
- Inner loop: attitude stabilization

This structure exploits the natural time-scale separation between translational and rotational dynamics. The inner attitude loop is assumed to be significantly faster than the outer position loop, allowing independent controller design.

B. Altitude Control

$$\Delta T = m[k_{pz}(z_d - z) + k_{dz}(\dot{z}_d - \dot{z})]$$

C. Position Control

$$\theta_d = \frac{1}{g}[k_{px}(x_d - x) + k_{dx}(\dot{x}_d - \dot{x})] \quad (16)$$
$$\phi_d = -\frac{1}{g}[k_{py}(y_d - y) + k_{dy}(\dot{y}_d - \dot{y})] \quad (17)$$

D. Attitude Control

$$M_x = I_{xx}[k_{p\phi}(\phi_d - \phi) + k_{d\phi}(p_d - p)] \quad (18)$$
$$M_y = I_{yy}[k_{p\theta}(\theta_d - \theta) + k_{d\theta}(q_d - q)] \quad (19)$$
$$M_z = I_{zz}[k_{p\psi}(\psi_d - \psi) + k_{d\psi}(r_d - r)] \quad (20)$$

VIII. Conclusion

This report presented a complete and intuitive solution to the quadrotor modeling and control problem. By starting from fundamental physical principles and progressively simplifying the model, a practical control law suitable for hovering and translational motion was obtained. The explanations provided ensure that the report remains understandable even when revisited without prior knowledge of quadrotor dynamics.