Filter Synthesis — Full Theory Solutions

Embedded Software — Master II Robotics Engineering

1 Exercise 1 — First-Order Low-Pass Filter

1.1 Given Transfer Function

$$H(f) = \frac{G_0}{1 + j\frac{f}{f_c}}$$

1.2 1. Filter Type

This transfer function corresponds to a:

- First-order
- Low-pass filter

Properties:

- DC gain: $H(0) = G_0$
- Cutoff frequency: $f = f_c$
- Slope: -20 dB/decade

A denominator of degree 1 ⇒ first-order filter.

1.3 2. Continuous-Time Differential Equation

Rewrite using Laplace variable $s = j2\pi f$:

$$H(s) = \frac{G_0}{1 + \frac{s}{\omega_c}} \quad \text{with } \omega_c = 2\pi f_c$$

$$(1 + \frac{s}{\omega_c})S(s) = G_0 E(s)$$

Inverse Laplace:

$$\frac{1}{\omega_c}\frac{ds(t)}{dt} + s(t) = G_0 e(t)$$

1.4 3. Discretization (Recurrence Equation)

Using:

$$\frac{ds}{dt} \approx \frac{s(n+1) - s(n)}{T_s}$$

Substitute into ODE:

$$\frac{1}{\omega_c}\frac{s(n+1) - s(n)}{T_s} + s(n) = G_0 e(n)$$

Rearranging:

$$s(n+1) = (1 - \omega_c T_s)\, s(n) + \omega_c T_s G_0 e(n)$$

Always isolate $s(n+1)$ — this is the recurrence.

1.5 4. Stability Condition

Stability requires:

$$|1 - \omega_c T_s| < 1$$

Which gives:

$$0 < \omega_c T_s < 2 \quad \Rightarrow \quad T_s < \frac{2}{\omega_c}$$

Sampling period must be small enough to preserve stability.

2 Exercise 2 — Second-Order Low-Pass Filter

2.1 Given Transfer Function

$$H(f) = \frac{G_0}{1 + 2mj\frac{f}{f_c} + \left(j\frac{f}{f_c}\right)^2}$$

This is a:

- Second-order
- Low-pass filter

Parameter m controls damping.

2.2 1. Continuous-Time Differential Equation

Replace $j\frac{f}{f_c}$ by $\frac{s}{\omega_c}$:

$$\left(1 + 2m\frac{s}{\omega_c} + \frac{s^2}{\omega_c^2}\right) S(s) = G_0 E(s)$$

Inverse Laplace:

$$\frac{1}{\omega_c^2}\frac{d^2 s}{dt^2} + \frac{2m}{\omega_c}\frac{ds}{dt} + s(t) = G_0 e(t)$$

2.3 2. Discretization

Using finite differences:

$$\frac{ds}{dt} \approx \frac{s(n+1) - s(n)}{T_s}$$

$$\frac{d^2 s}{dt^2} \approx \frac{s(n+2) - 2s(n+1) + s(n)}{T_s^2}$$

Substitute into ODE to obtain a second-order recurrence:

$$s(n+2) = a_1 s(n+1) + a_2 s(n) + b e(n)$$

Second derivative ⇒ two delays ⇒ second-order recursion.

2.4 3. Stability Condition

Stability requires:

- Both poles inside the unit circle
- Depends on m, T_s, and ω_c

Second-order stability ⇒ check both poles.

3 Exercise 3 — Bilinear Transform (Theory)

3.1 Purpose

The bilinear transform maps:

$$s \to \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}}$$

Properties:

- Preserves stability
- Maps $j\Omega$ axis to unit circle
- Introduces frequency warping

3.2 Pre-Warping

To preserve a specific frequency f_0:

$$\Omega_0 = \frac{2}{T_s} \tan\left(\pi \frac{f_0}{f_s}\right)$$

> Always pre-warp critical frequencies before bilinear transform.

3.3 Quality Factor

For a band-pass filter:

$$Q = \frac{f_0}{f_2 - f_1}$$

Higher Q:

- Narrower band
- Sharper resonance

4 Exercise 4 — FIR Filter Using Window Method

4.1 Ideal Low-Pass Filter

$$H_{ideal}(f) = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases}$$

This filter is:

- Non-causal
- Infinite impulse response

4.2 Impulse Response

Impulse response:

$$h_i = \frac{1}{f_s} \int_{-f_c}^{f_c} e^{j2\pi i \frac{f}{f_s}} df$$

Which yields a sinc function.

4.3 Windowing

To obtain a finite FIR filter:

$$g_i = h_i \cdot w_i$$

Rectangular window:

$$w_i = \begin{cases} 1 & |i| \leq \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

> Windowing trades frequency sharpness for realizability.

4.4 Causality

To make the filter causal:

$$b_k = g_{k - \frac{N-1}{2}}$$

This introduces:

- Pure delay
- Linear phase

$$H(f) = G(f) e^{-j\pi(N-1)\frac{f}{f_s}}$$

4.5 Filter Type

The resulting FIR filter is:

- Low-pass
- Linear phase
- Stable

> FIR filters are always stable and phase-linear when symmetric.