Bayesian Filtering — Complete Solutions

Coin Toss and Weather Forecast

1 Exercise 1 — Toss a Coin

1.1 State Space

The possible states of the system are:

$$\Omega = \{H, T\}$$

where H denotes *Head* and T denotes *Tail*.

1.2 Initial Belief

The initial state is unknown, therefore we assume a uniform prior:

$$bel(X_0) = \begin{bmatrix} P(H_0) & P(T_0) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

1.3 Sensor Model

$$p(Z_t = \text{obsH} \mid H) = 0.7, \quad p(Z_t = \text{obsT} \mid H) = 0.3$$
$$p(Z_t = \text{obsH} \mid T) = 0.6, \quad p(Z_t = \text{obsT} \mid T) = 0.4$$

1.4 Transition Model

Do Nothing (DN):

$$p(H \mid DN, H) = 0.9, \quad p(T \mid DN, H) = 0.1$$

$$p(H \mid DN, T) = 0.9, \quad p(T \mid DN, T) = 0.1$$

Flip:

$$p(H \mid Flip, H) = 0.1, \quad p(T \mid Flip, H) = 0.9$$

$$p(H \mid Flip, T) = 0.9, \quad p(T \mid Flip, T) = 0.1$$

1.5 Bayes Filter Equations

Prediction:

$$\overline{bel}(x_t) = \sum_{x_{t-1}} p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})$$

Correction:

$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

1.6 Time $t = 1$, Action DN, Observation Tail

Prediction:

$$\overline{bel}(H_1) = 0.9(0.5) + 0.9(0.5) = 0.9$$

$$\overline{bel}(T_1) = 0.1(0.5) + 0.1(0.5) = 0.1$$

Update:

$$bel(H_1) \propto 0.3 \times 0.9 = 0.27$$
$$bel(T_1) \propto 0.4 \times 0.1 = 0.04$$

Normalization:

$$\eta = \frac{1}{0.27 + 0.04} = \frac{1}{0.31}$$

$$bel(X_1) = \begin{bmatrix} 0.87097 & 0.12903 \end{bmatrix}$$

1.7 Time $t = 2$, Action DN, Observation Tail

Prediction:

$$\overline{bel}(X_2) = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$$

Update:

$$bel(X_2) = \begin{bmatrix} 0.87097 & 0.12903 \end{bmatrix}$$

1.8 Time $t = 3$, Action Flip, Observation Tail

Prediction:

$$\overline{bel}(H_3) = 0.1(0.87097) + 0.9(0.12903) = 0.20323$$

$$\overline{bel}(T_3) = 0.9(0.87097) + 0.1(0.12903) = 0.79677$$

Update:

$$bel(H_3) \propto 0.3 \times 0.20323 = 0.06097$$

$$bel(T_3) \propto 0.4 \times 0.79677 = 0.31871$$

$$bel(X_3) = \begin{bmatrix} 0.16058 & 0.83942 \end{bmatrix}$$

1.9 Time $t = 4$, Action DN, Observation Head

Prediction:

$$\overline{bel}(X_4) = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$$

Update:

$$bel(H_4) \propto 0.7 \times 0.9 = 0.63, \quad bel(T_4) \propto 0.6 \times 0.1 = 0.06$$

$$bel(X_4) = \begin{bmatrix} 0.91304 & 0.08696 \end{bmatrix}$$

2 Exercise 2 — Weather Forecast

2.1 State Space

$$\Omega = \{S, C, R\}$$

2.2 Transition Matrix

$$T = \begin{bmatrix} 0 & 0.2 & 0.8 \\ 0.2 & 0.4 & 0.4 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$

2.3 Path Probability

Given:

$$R \to C \to C \to S$$

$$P = 0.6 \times 0.4 \times 0.2 = 0.048$$

2.4 State Propagation Formula

$$\pi_{t+1} = \pi_t T, \quad \pi_{t+k} = \pi_t T^k$$

2.5 Sensor Model

$$O = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.6 Observation Matrix

$$D(z_t) = \mathrm{diag}(p(z_t \mid S), p(z_t \mid C), p(z_t \mid R))$$

2.7 Filtering Equation (Matrix Form)

$$bel_t = \eta\, (bel_{t-1} T) D(z_t)$$

2.8 Whole Sequence

$$bel_t \propto bel_0\, T D(z_1) T D(z_2) \cdots T D(z_t)$$