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Pearson Prentice Hall
Pearson Education, Inc.
Upper Saddle River, NJ 07458

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN 0-13-145777-2

Pearson Education Ltd., *London*
Pearson Education Australia Pty. Ltd., *Sydney*
Pearson Education Singapore, Pte. Ltd.
Pearson Education North Asia Ltd., *Hong Kong*
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INSTRUCTORS'
SOLUTIONS MANUAL

ELEMENTARY DIFFERENTIAL EQUATIONS With BOUNDARY VALUE PROBLEMS

Fifth Edition

C. Henry Edwards ♦ David E. Penney

PRENTICE HALL, Upper Saddle River, NJ 07458

CONTENTS

PREFACE

This is a solutions manual to accompany the textbook **ELEMENTARY DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE PROBLEMS** (5th edition, 2004) by C. Henry Edwards and David E. Penney. We include solutions to most of the problems in the text. The corresponding **Students Solutions Manual** contains solutions to most of the odd-numbered solutions in the text.

Our goal is to support teaching of the subject of elementary differential equations in every way that we can. We therefore invite comments and suggested improvements for future printings of this manual, as well as advice regarding features that might be added to increase its usefulness in subsequent editions. Additional supplementary material can be found at our textbook Web site listed below.

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CHAPTER 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

SECTION 1.1

DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELING

The main purpose of Section 1.1 is simply to introduce the basic notation and terminology of differential equations, and to show the student what is meant by a solution of a differential equation. Also, the use of differential equations in the mathematical modeling of real-world phenomena is outlined.

Problems 1–12 are routine verifications by direct substitution of the suggested solutions into the given differential equations. We include here just some typical examples of such verifications.

3. If $y_1 = \cos 2x$ and $y_2 = \sin 2x$, then $y_1' = -2\sin 2x$ and $y_2' = 2\cos 2x$ so

$$y_1'' = -4\cos 2x = -4y_1 \quad \text{and} \quad y_2'' = -4\sin 2x = -4y_2.$$

Thus $y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$.

4. If $y_1 = e^{3x}$ and $y_2 = e^{-3x}$, then $y_1 = 3e^{3x}$ and $y_2 = -3e^{-3x}$ so

$$y_1'' = 9e^{3x} = 9y_1 \quad \text{and} \quad y_2'' = 9e^{-3x} = 9y_2.$$

5. If $y = e^x - e^{-x}$, then $y' = e^x + e^{-x}$ so $y' - y = \left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) = 2e^{-x}$. Thus $y' = y + 2e^{-x}$.

6. If $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, then $y_1' = -2e^{-2x}$, $y_1'' = 4e^{-2x}$, $y_2' = e^{-2x} - 2xe^{-2x}$, and $y_2'' = -4e^{-2x} + 4xe^{-2x}$. Hence

$$y_1'' + 4y_1' + 4y_1 = \left(4e^{-2x}\right) + 4\left(-2e^{-2x}\right) + 4\left(e^{-2x}\right) = 0$$

and

$$y_2'' + 4y_2' + 4y_2 = \left(-4e^{-2x} + 4xe^{-2x}\right) + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4\left(xe^{-2x}\right) = 0.$$

8. If $y_1 = \cos x - \cos 2x$ and $y_2 = \sin x - \cos 2x$, then $y_1' = -\sin x + 2\sin 2x$, $y_1'' = -\cos x + 4\cos 2x$, and $y_2' = \cos x + 2\sin 2x$, $y_2'' = -\sin x + 4\cos 2x$. Hence

$$y_1'' + y_1 = (-\cos x + 4\cos 2x) + (\cos x - \cos 2x) = 3\cos 2x$$

and

$$y_2'' + y_2 = (-\sin x + 4\cos 2x) + (\sin x - \cos 2x) = 3\cos 2x.$$

11. If $y = y_1 = x^{-2}$ then $y' = -2x^{-3}$ and $y'' = 6x^{-4}$, so

$$x^2 y'' + 5x\,y' + 4y = x^2\left(6x^{-4}\right) + 5x\left(-2x^{-3}\right) + 4\left(x^{-2}\right) = 0.$$

If $y = y_2 = x^{-2}\ln x$ then $y' = x^{-3} - 2x^{-3}\ln x$ and $y'' = -5x^{-4} + 6x^{-4}\ln x$, so

$$\begin{aligned} x^2 y'' + 5x\,y' + 4y &= x^2\left(-5x^{-4} + 6x^{-4}\ln x\right) + 5x\left(x^{-3} - 2x^{-3}\ln x\right) + 4\left(x^{-2}\ln x\right) \\ &= \left(-5x^{-2} + 5x^{-2}\right) + \left(6x^{-2} - 10x^{-2} + 4x^{-2}\right)\ln x = 0. \end{aligned}$$

13. Substitution of $y = e^{rx}$ into $3y' = 2y$ gives the equation $3r\,e^{rx} = 2e^{rx}$ that simplifies to $3r = 2$. Thus $r = 2/3$.

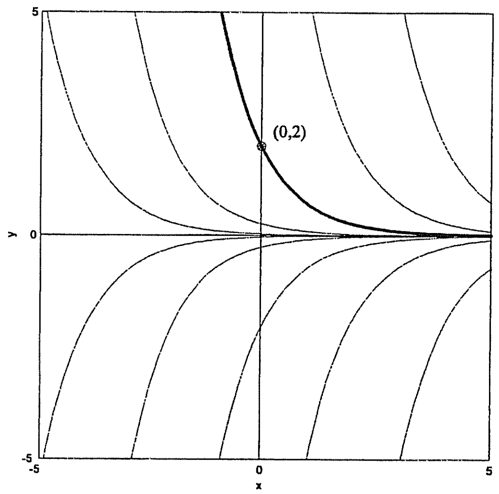
14. Substitution of $y = e^{rx}$ into $4y'' = y$ gives the equation $4r^2 e^{rx} = e^{rx}$ that simplifies to $4r^2 = 1$. Thus $r = \pm 1/2$.

15. Substitution of $y = e^{rx}$ into $y'' + y' - 2y = 0$ gives the equation $r^2 e^{rx} + r\,e^{rx} - 2e^{rx} = 0$ that simplifies to $r^2 + r - 2 = (r+2)(r-1) = 0$. Thus $r = -2$ or $r = 1$.

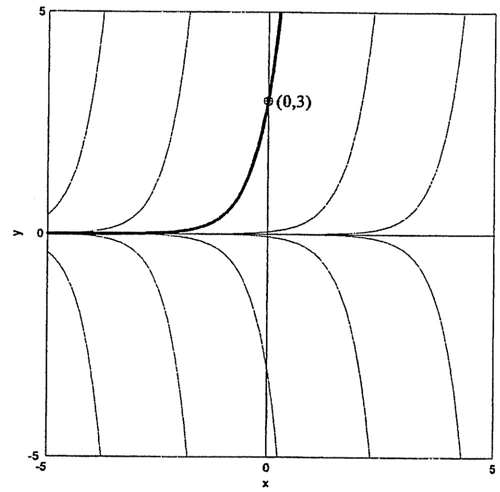
16. Substitution of $y = e^{rx}$ into $3y'' + 3y' - 4y = 0$ gives the equation $3r^2 e^{rx} + 3r\,e^{rx} - 4e^{rx} = 0$ that simplifies to $3r^2 + 3r - 4 = 0$. The quadratic formula then gives the solutions $r = \left(-3 \pm \sqrt{57}\right)/6$.

The verifications of the suggested solutions in Problems 17–26 are similar to those in Problems 1–12. We illustrate the determination of the value of C only in some typical cases. However, we illustrate typical solution curves for each of these problems.

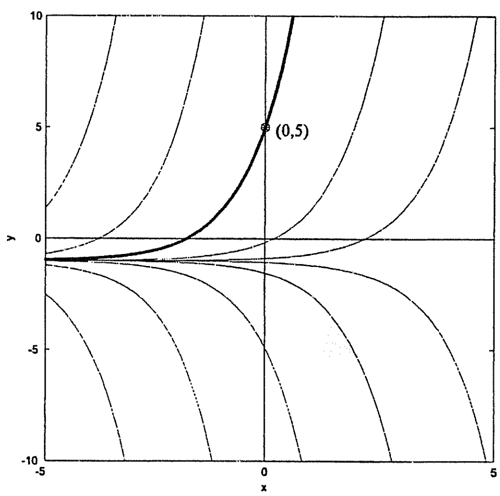
17. $C = 2$

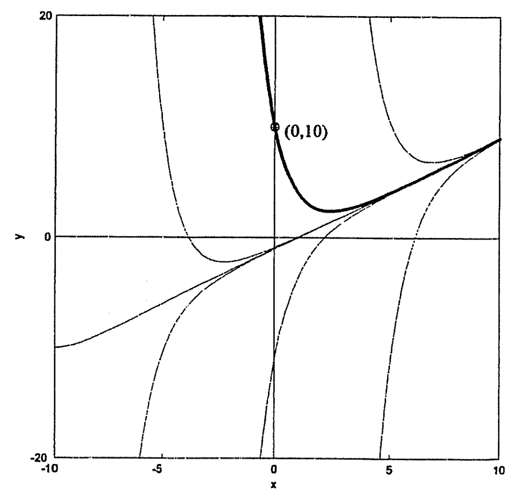


18. $C = 3$



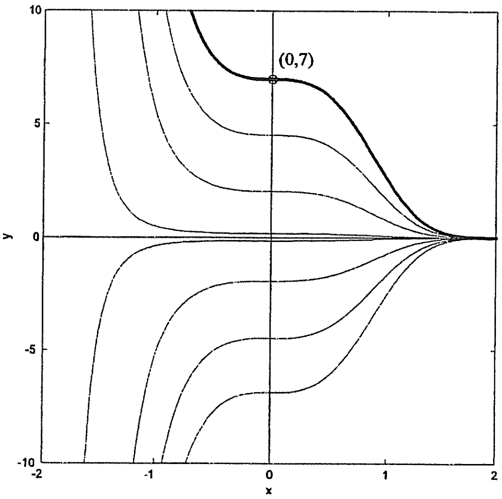
19. If $y(x) = Ce^x - 1$ then $y(0) = 5$ gives $C - 1 = 5$, so $C = 6$. The figure is on the left below.

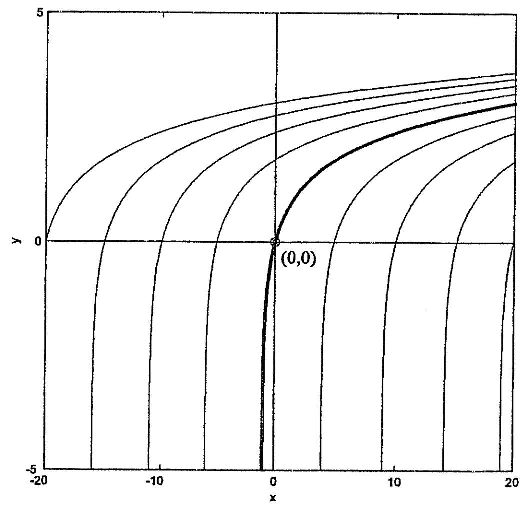




20. If $y(x) = Ce^{-x} + x - 1$ then $y(0) = 10$ gives $C - 1 = 10$, so $C = 11$. The figure is on the right above.

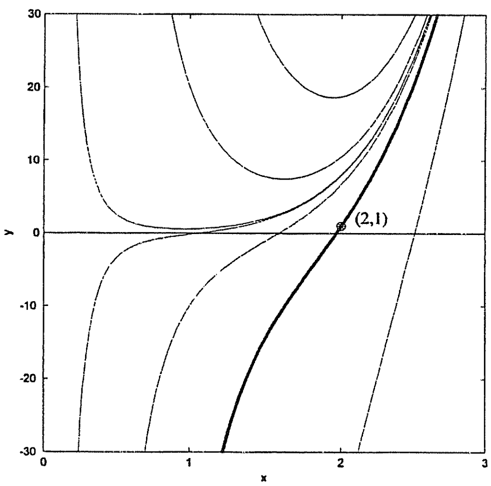
21. $C = 7$. The figure is on the left at the top of the next page.

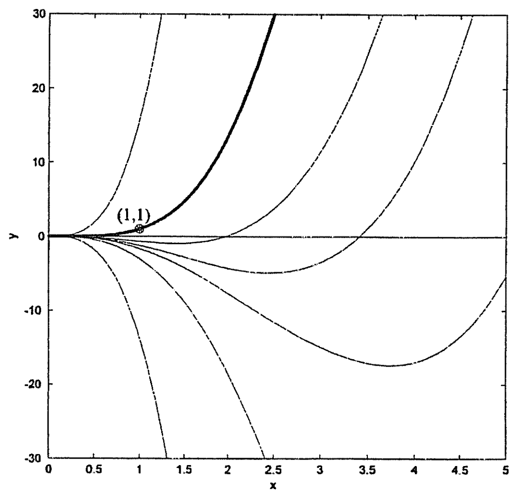




22. If $y(x) = \ln(x+C)$ then $y(0) = 0$ gives $\ln C = 0$, so $C = 1$. The figure is on the right above.

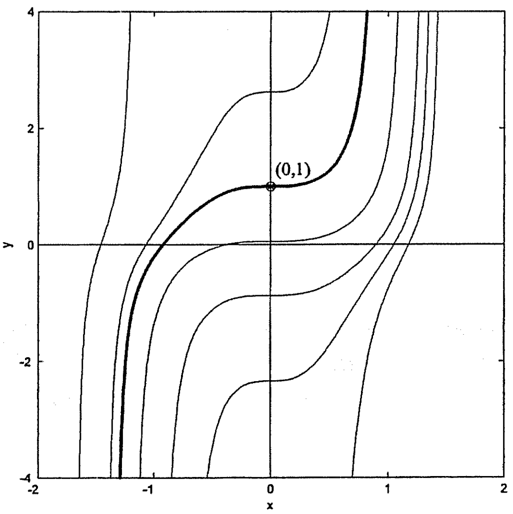
23. If $y(x) = \frac{1}{4}x^5 + Cx^{-2}$ then $y(2) = 1$ gives the equation $\frac{1}{4}\cdot 32 + C\cdot\frac{1}{8} = 1$ with solution $C = -56$. The figure is on the left below.



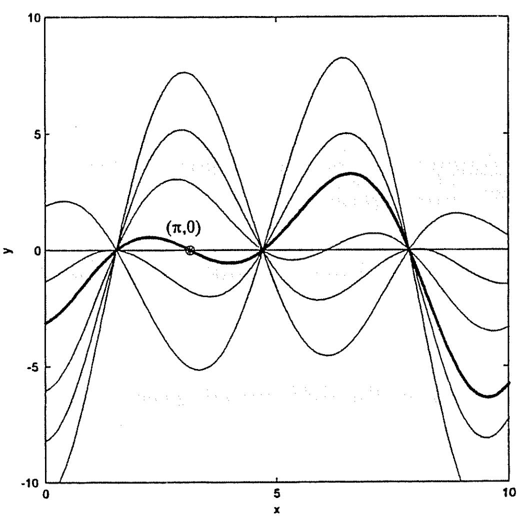


24. $C = 17$. The figure is on the right above.

25. If $y(x) = \tan(x^2 + C)$ then $y(0) = 1$ gives the equation $\tan C = 1$. Hence one value of C is $C = \pi/4$ (as is this value plus any integral multiple of π).



26. Substitution of $x = \pi$ and $y = 0$ into $y = (x + C)\cos x$ yields the equation $0 = (\pi + C)(-1)$, so $C = -\pi$.



27. $y' = x + y$

28. The slope of the line through (x, y) and $(x/2, 0)$ is $y' = (y-0)/(x - x/2) = 2y/x$, so the differential equation is $x\,y' = 2y$.

29. If $m = y'$ is the slope of the tangent line and m' is the slope of the normal line at (x, y), then the relation $m\,m' = -1$ yields $m' = 1/y' = (y-1)/(x-0)$. Solution for y' then gives the differential equation $(1-y)y' = x$.

30. Here $m = y'$ and $m' = D_x(x^2 + k) = 2x$, so the orthogonality relation $m\,m' = -1$ gives the differential equation $2x\,y' = -1$.

31. The slope of the line through (x, y) and $(-y, x)$ is $y' = (x-y)/(-y-x)$, so the differential equation is $(x+y)y' = y-x$.

In Problems 32–36 we get the desired differential equation when we replace the "time rate of change" of the dependent variable with its derivative, the word "is" with the = sign, the phrase "proportional to" with k, and finally translate the remainder of the given sentence into symbols.

32. $dP/dt = k\sqrt{P}$

33. $dv/dt = k\,v^2$

34. $dv/dt = k(250 - v)$

35. $dN/dt = k(P - N)$

36. $dN/dt = k\,N(P - N)$

37. The second derivative of any linear function is zero, so we spot the two solutions $y(x) \equiv 1$ or $y(x) = x$ of the differential equation $y'' = 0$.

38. A function whose derivative equals itself, and hence a solution of the differential equation $y' = y$ is $y(x) = e^x$.

39. We reason that if $y = kx^2$, then each term in the differential equation is a multiple of x^2. The choice $k = 1$ balances the equation, and provides the solution $y(x) = x^2$.

40. If y is a constant, then $y' \equiv 0$ so the differential equation reduces to $y^2 = 1$. This gives the two constant-valued solutions $y(x) \equiv 1$ and $y(x) \equiv -1$.

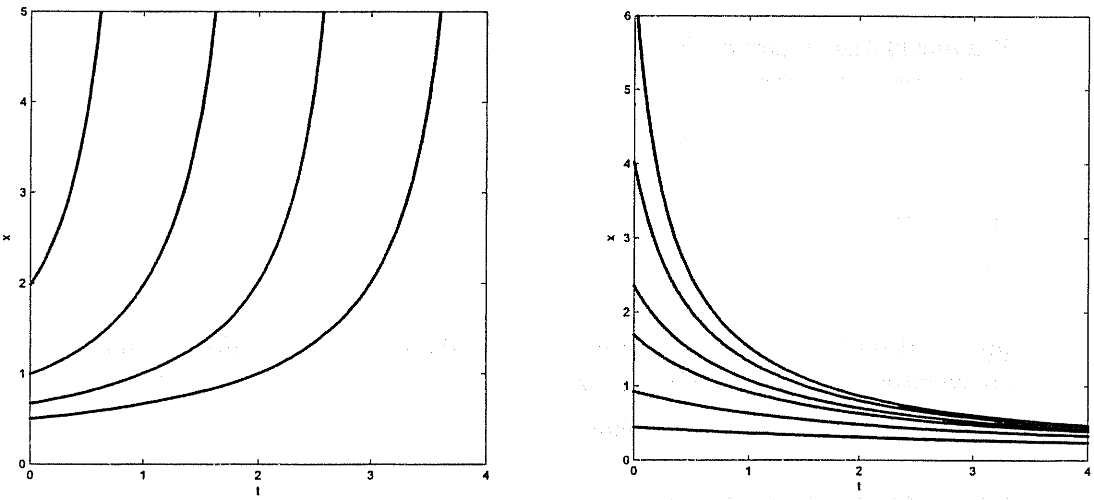
41. We reason that if $y = ke^x$, then each term in the differential equation is a multiple of e^x. The choice $k = \frac{1}{2}$ balances the equation, and provides the solution $y(x) = \frac{1}{2}e^x$.

42. Two functions, each equaling the negative of its own second derivative, are the two solutions $y(x) = \cos x$ and $y(x) = \sin x$ of the differential equation $y'' = -y$.

43. **(a)** We need only substitute $x(t) = 1/(C - kt)$ in both sides of the differential equation $x' = kx^2$ for a routine verification.

(b) The zero-valued function $x(t) \equiv 0$ obviously satisfies the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** The figure on the left below shows typical graphs of solutions of the differential equation $x' = \frac{1}{2}x^2$.



(b) The figure on the right above shows typical graphs of solutions of the differential equation $x' = -\frac{1}{2}x^2$. We see that — whereas the graphs with $k = \frac{1}{2}$ appear to "diverge to infinity" — each solution with $k = -\frac{1}{2}$ appears to approach 0 as $t \to \infty$. Indeed, we see from the Problem 43(a) solution $x(t) = 1/(C - \frac{1}{2}t)$ that $x(t) \to \infty$ as $t \to 2C$. However, with $k = -\frac{1}{2}$ it is clear from the resulting solution $x(t) = 1/(C + \frac{1}{2}t)$ that $x(t)$ remains bounded on any finite interval, but $x(t) \to 0$ as $t \to +\infty$.

45. Substitution of $P' = 1$ and $P = 10$ into the differential equation $P' = kP^2$ gives $k = \frac{1}{100}$, so Problem 43(a) yields a solution of the form $P(t) = 1/(C - t/100)$. The initial condition $P(0) = 2$ now yields $C = \frac{1}{2}$, so we get the solution

$$P(t) = \frac{1}{\frac{1}{2} - \frac{t}{100}} = \frac{100}{50 - t}.$$

We now find readily that $P = 100$ when $t = 49$, and that $P = 1000$ when $t = 49.9$. It appears that P grows without bound (and thus "explodes") as t approaches 50.

46. Substitution of $v' = -1$ and $v = 5$ into the differential equation $v' = kv^2$ gives $k = -\frac{1}{25}$, so Problem 43(a) yields a solution of the form $v(t) = 1/(C + t/25)$. The initial condition $v(0) = 10$ now yields $C = \frac{1}{10}$, so we get the solution

$$v(t) = \frac{1}{\frac{1}{10} + \frac{t}{25}} = \frac{50}{5 + 2t}.$$

We now find readily that $v = 1$ when $t = 22.5$, and that $v = 0.1$ when $t = 247.5$. It appears that v approaches 0 as t increases without bound. Thus the boat gradually slows, but never comes to a "full stop" in a finite period of time.

47. **(a)** $y(10) = 10$ yields $10 = 1/(C - 10)$, so $C = 101/10$.

(b) There is no such value of C, but the constant function $y(x) \equiv 0$ satisfies the conditions $y' = y^2$ and $y(0) = 0$.

(c) It is obvious visually (in Fig. 1.1.8 of the text) that one and only one solution curve passes through each point (a, b) of the xy-plane, so it follows that there exists a unique solution to the initial value problem $y' = y^2$, $y(a) = b$.

48. **(b)** Obviously the functions $u(x) = -x^4$ and $v(x) = +x^4$ both satisfy the differential equation $xy' = 4y$. But their derivatives $u'(x) = -4x^3$ and $v'(x) = +4x^3$ match at $x = 0$, where both are zero. Hence the given piecewise-defined function $y(x)$ is differentiable, and therefore satisfies the differential equation because $u(x)$ and $v(x)$ do so (for $x \leq 0$ and $x \geq 0$, respectively).

(c) If $a \geq 0$ (for instance), choose C_+ fixed so that $C_+ a^4 = b$. Then the function

$$y(x) = \begin{cases} C_- x^4 & \text{if } x \leq 0, \\ C_+ x^4 & \text{if } x \geq 0 \end{cases}$$

satisfies the given differential equation for every real number value of C_-.

SECTION 1.2

INTEGRALS AS GENERAL AND PARTICULAR SOLUTIONS

This section introduces **general solutions** and **particular solutions** in the very simplest situation — a differential equation of the form $y' = f(x)$ — where only direct integration and evaluation of the constant of integration are involved. Students should review carefully the elementary concepts of velocity and acceleration, as well as the fps and mks unit systems.

1. Integration of $y' = 2x+1$ yields $y(x) = \int (2x+1)\,dx = x^2 + x + C$. Then substitution of $x=0$, $y=3$ gives $3 = 0+0+C = C$, so $y(x) = x^2+x+3$.

2. Integration of $y' = (x-2)^2$ yields $y(x) = \int (x-2)^2\,dx = \frac{1}{3}(x-2)^3 + C$. Then substitution of $x=2$, $y=1$ gives $1 = 0 + C = C$, so $y(x) = \frac{1}{3}(x-2)^3$.

3. Integration of $y' = \sqrt{x}$ yields $y(x) = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$. Then substitution of $x=4$, $y=0$ gives $0 = \frac{16}{3} + C$, so $y(x) = \frac{2}{3}(x^{3/2} - 8)$.

4. Integration of $y' = x^{-2}$ yields $y(x) = \int x^{-2}\,dx = -1/x + C$. Then substitution of $x=1$, $y=5$ gives $5 = -1 + C$, so $y(x) = -1/x + 6$.

5. Integration of $y' = (x+2)^{-1/2}$ yields $y(x) = \int (x+2)^{-1/2}\,dx = 2\sqrt{x+2} + C$. Then substitution of $x=2$, $y=-1$ gives $-1 = 2\cdot 2 + C$, so $y(x) = 2\sqrt{x+2} - 5$.

6. Integration of $y' = x(x^2+9)^{1/2}$ yields $y(x) = \int x(x^2+9)^{1/2}\,dx = \frac{1}{3}(x^2+9)^{3/2} + C$. Then substitution of $x=-4$, $y=0$ gives $0 = \frac{1}{3}(5)^3 + C$, so $y(x) = \frac{1}{3}\left[(x^2+9)^{3/2} - 125\right]$.

7. Integration of $y' = 10/(x^2+1)$ yields $y(x) = \int 10/(x^2+1)\,dx = 10\tan^{-1}x + C$. Then substitution of $x=0$, $y=0$ gives $0 = 10\cdot 0 + C$, so $y(x) = 10\tan^{-1}x$.

8. Integration of $y' = \cos 2x$ yields $y(x) = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$. Then substitution of $x=0$, $y=1$ gives $1 = 0 + C$, so $y(x) = \frac{1}{2}\sin 2x + 1$.

9. Integration of $y' = 1/\sqrt{1-x^2}$ yields $y(x) = \int 1/\sqrt{1-x^2}\,dx = \sin^{-1}x + C$. Then substitution of $x=0$, $y=0$ gives $0 = 0 + C$, so $y(x) = \sin^{-1}x$.

10. Integration of $y' = xe^{-x}$ yields

$$y(x) = \int xe^{-x}\,dx = \int ue^{u}\,du = (u-1)e^{u} = -(x+1)e^{-x} + C$$

(when we substitute $u = -x$ and apply Formula #46 inside the back cover of the textbook). Then substitution of $x = 0,\ y = 1$ gives $1 = -1 + C$, so $y(x) = -(x+1)e^{-x} + 2.$

11. If $a(t) = 50$ then $v(t) = \int 50\,dt = 50t + v_0 = 50t + 10$. Hence

$$x(t) = \int (50t+10)\,dt = 25t^2 + 10t + x_0 = 25t^2 + 10t + 20.$$

12. If $a(t) = -20$ then $v(t) = \int (-20)\,dt = -20t + v_0 = -20t - 15$. Hence

$$x(t) = \int (-20t-15)\,dt = -10t^2 - 15t + x_0 = -10t^2 - 15t + 5.$$

13. If $a(t) = 3t$ then $v(t) = \int 3t\,dt = \tfrac{3}{2}t^2 + v_0 = \tfrac{3}{2}t^2 + 5$. Hence

$$x(t) = \int (\tfrac{3}{2}t^2 + 5)\,dt = \tfrac{1}{2}t^3 + 5t + x_0 = \tfrac{1}{2}t^3 + 5t.$$

14. If $a(t) = 2t+1$ then $v(t) = \int (2t+1)\,dt = t^2 + t + v_0 = t^2 + t - 7$. Hence

$$x(t) = \int (t^2 + t - 7)\,dt = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + x_0 = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + 4.$$

15. If $a(t) = 4(t+3)^2$. then $v(t) = \int 4(t+3)^2\,dt = \tfrac{4}{3}(t+3)^3 + C = \tfrac{4}{3}(t+3)^3 - 37$ (taking $C = -37$ so that $v(0) = -1$). Hence

$$x(t) = \int \left[\tfrac{4}{3}(t+3)^3 - 37\right] dt = \tfrac{1}{3}(t+3)^4 - 37t + C = \tfrac{1}{3}(t+3)^4 - 37t - 26.$$

16. If $a(t) = 1/\sqrt{t+4}$ then $v(t) = \int 1/\sqrt{t+4}\ dt = 2\sqrt{t+4} + C = 2\sqrt{t+4} - 5$ (taking $C = -5$ so that $v(0) = -1$). Hence

$$x(t) = \int (2\sqrt{t+4} - 5)\,dt = \tfrac{4}{3}(t+4)^{3/2} - 5t + C = \tfrac{4}{3}(t+4)^{3/2} - 5t - \tfrac{29}{3}$$

(taking $C = -29/3$ so that $x(0) = 1$).

17. If $a(t) = (t+1)^{-3}$ then $v(t) = \int (t+1)^{-3}\,dt = -\frac{1}{2}(t+1)^{-2} + C = -\frac{1}{2}(t+1)^{-2} + \frac{1}{2}$ (taking $C = \frac{1}{2}$ so that $v(0) = 0$). Hence

$$x(t) = \int \left[-\tfrac{1}{2}(t+1)^{-2} + \tfrac{1}{2}\right] dt = \tfrac{1}{2}(t+1)^{-1} + \tfrac{1}{2}t + C = \tfrac{1}{2}\left[(t+1)^{-1} + t - 1\right]$$

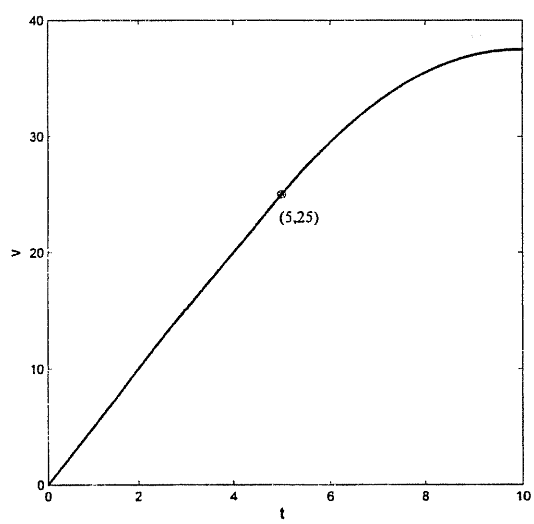
(taking $C = -\frac{1}{2}$ so that $x(0) = 0$).

18. If $a(t) = 50\sin 5t$ then $v(t) = \int 50\sin 5t\,dt = -10\cos 5t + C = -10\cos 5t$ (taking $C = 0$ so that $v(0) = -10$). Hence

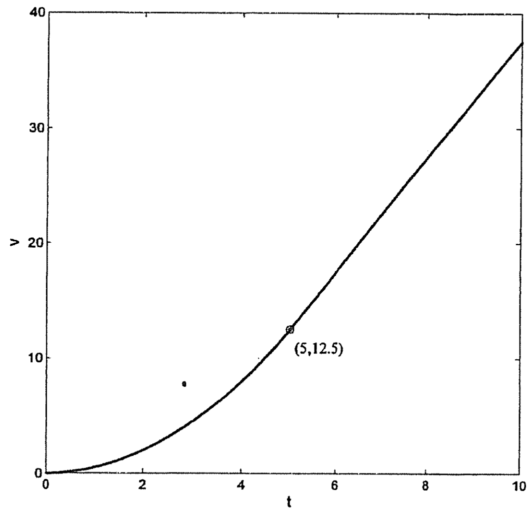
$$x(t) = \int (-10\cos 5t)\,dt = -2\sin 5t + C = -2\sin 5t + 10$$

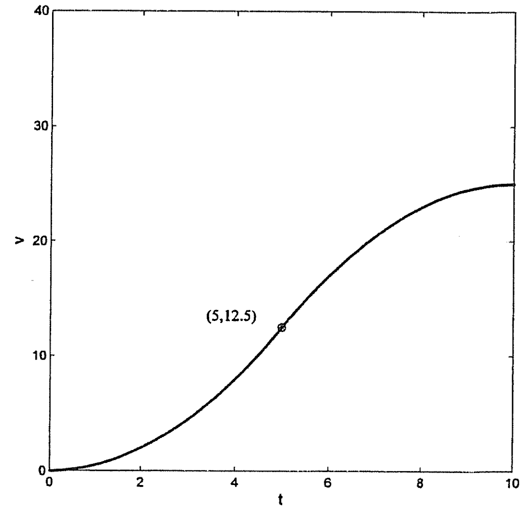
(taking $C = -10$ so that $x(0) = 8$).

19. Note that $v(t) = 5$ for $0 \le t \le 5$ and that $v(t) = 10 - t$ for $5 \le t \le 10$. Hence $x(t) = 5t + C_1$ for $0 \le t \le 5$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ for $5 \le t \le 10$. Now $C_1 = 0$ because $x(0) = 0$, and continuity of $x(t)$ requires that $x(t) = 5t$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ agree when $t = 5$. This implies that $C_2 = -\frac{25}{2}$, and we get the following graph.



20. Note that $v(t) = t$ for $0 \le t \le 5$ and that $v(t) = 5$ for $5 \le t \le 10$. Hence $x(t) = \frac{1}{2}t^2 + C_1$ for $0 \le t \le 5$ and $x(t) = 5t + C_2$ for $5 \le t \le 10$. Now $C_1 = 0$ because $x(0) = 0$, and continuity of $x(t)$ requires that $x(t) = \frac{1}{2}t^2$ and $x(t) = 5t + C_2$ agree when $t = 5$. This implies that $C_2 = -\frac{25}{2}$, and we get the graph on the left below.



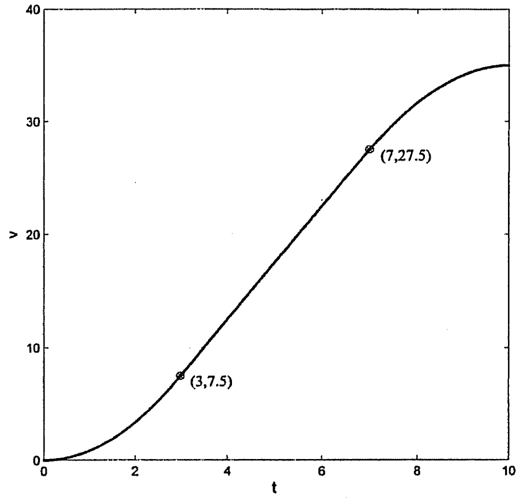


21. Note that $v(t) = t$ for $0 \le t \le 5$ and that $v(t) = 10 - t$ for $5 \le t \le 10$. Hence $x(t) = \frac{1}{2}t^2 + C_1$ for $0 \le t \le 5$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ for $5 \le t \le 10$. Now $C_1 = 0$ because $x(0) = 0$, and continuity of $x(t)$ requires that $x(t) = \frac{1}{2}t^2$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ agree when $t = 5$. This implies that $C_2 = -25$, and we get the graph on the right above.

22. For $0 \le t \le 3$: $v(t) = \frac{5}{3}t$ so $x(t) = \frac{5}{6}t^2 + C_1$. Now $C_1 = 0$ because $x(0) = 0$, so $x(t) = \frac{5}{6}t^2$ on this first interval, and its right endpoint value is $x(3) = 7\frac{1}{2}$.

For $3 \le t \le 7$: $v(t) = 5$ so $x(t) = 5t + C_2$. Now $x(3) = 7\frac{1}{2}$ implies that $C_2 = -7\frac{1}{2}$, so $x(t) = 5t - 7\frac{1}{2}$ on this second interval, where its right endpoint value is $x(7) = 27\frac{1}{2}$.

For $7 \le t \le 10$: $v - 5 = -\frac{5}{3}(t - 7)$, so $v(t) = -\frac{5}{3}t + \frac{50}{3}$. Hence $x(t) = -\frac{5}{6}t^2 + \frac{50}{3}t + C_3$, and $x(7) = 27\frac{1}{2}$ implies that $C_3 = -\frac{290}{6}$. Finally, $x(t) = \frac{1}{6}(-5t^2 + 100t - 290)$ on this third interval, and we get the graph at the top of the next page.



23. $v = -9.8t + 49$, so the ball reaches its maximum height $(v = 0)$ after $t = 5$ seconds. Its maximum height then is $y(5) = -4.9(5)^2 + 49(5) = 122.5$ meters.

24. $v = -32t$ and $y = -16t^2 + 400$, so the ball hits the ground $(y = 0)$ when $t = 5$ sec, and then $v = -32(5) = -160$ ft/sec.

25. $a = -10\text{ m/s}^2$ and $v_0 = 100\text{ km/h} \approx 27.78\text{ m/s}$, so $v = -10t + 27.78$, and hence $x(t) = -5t^2 + 27.78t$. The car stops when $v = 0$, $t \approx 2.78$, and thus the distance traveled before stopping is $x(2.78) \approx 38.59$ meters.

26. $v = -9.8t + 100$ and $y = -4.9t^2 + 100t + 20$.

(a) $v = 0$ when $t = 100/9.8$ so the projectile's maximum height is $y(100/9.8) = -4.9(100/9.8)^2 + 100(100/9.8) + 20 \approx 530$ meters.

(b) It passes the top of the building when $y(t) = -4.9t^2 + 100t + 20 = 20$, and hence after $t = 100/4.9 \approx 20.41$ seconds.

(c) The roots of the quadratic equation $y(t) = -4.9t^2 + 100t + 20 = 0$ are $t = -0.20,\ 20.61$. Hence the projectile is in the air 20.61 seconds.

27. $a = -9.8$ m/s2 so $v = -9.8\,t - 10$ and

$$y = -4.9\,t^2 - 10\,t + y_0.$$

The ball hits the ground when $y = 0$ and

$$v = -9.8\,t - 10 = -60,$$

so $t \approx 5.10$ s. Hence

$$y_0 = 4.9(5.10)^2 + 10(5.10) \approx 178.57\text{ m}.$$

28. $v = -32t - 40$ and $y = -16t^2 - 40t + 555$. The ball hits the ground $(y = 0)$ when $t \approx 4.77$ sec, with velocity $v = v(4.77) \approx -192.64$ ft/sec, an impact speed of about 131 mph.

29. Integration of $dv/dt = 0.12\,t^3 + 0.6\,t$, $v(0) = 0$ gives $v(t) = 0.3\,t^2 + 0.04\,t^3$. Hence $v(10) = 70$. Then integration of $dx/dt = 0.3\,t^2 + 0.04\,t^3$, $x(0) = 0$ gives $x(t) = 0.1\,t^3 + 0.04\,t^4$, so $x(10) = 200$. Thus after 10 seconds the car has gone 200 ft and is traveling at 70 ft/sec.

30. Taking $x_0 = 0$ and $v_0 = 60$ mph $= 88$ ft/sec, we get

$$v = -at + 88,$$

and $v = 0$ yields $t = 88/a$. Substituting this value of t and $x = 176$ in

$$x = -at^2/2 + 88t,$$

we solve for $a = 22$ ft/sec^2. Hence the car skids for $t = 88/22 = 4$ sec.

31. If $a = -20$ m/sec^2 and $x_0 = 0$ then the car's velocity and position at time t are given by

$$v = -20t + v_0, \quad x = -10\,t^2 + v_0 t.$$

It stops when $v = 0$ (so $v_0 = 20t$), and hence when

$$x = 75 = -10\,t^2 + (20t)t = 10\,t^2.$$

Thus $t = \sqrt{7.5}$ sec so

$$v_0 = 20\sqrt{7.5} \approx 54.77 \text{ m/sec} \approx 197 \text{ km/hr}.$$

32. Starting with $x_0 = 0$ and $v_0 = 50$ km/h $= 5\times10^4$ m/h, we find by the method of Problem 24 that the car's deceleration is $a = (25/3)\times10^7$ m/h^2. Then, starting with $x_0 = 0$ and $v_0 = 100$ km/h $= 10^5$ m/h, we substitute $t = v_0/a$ into

$$x = -at^2 + v_0 t$$

and find that $x = 60$ m when $v = 0$. Thus doubling the initial velocity quadruples the distance the car skids.

33. If $v_0 = 0$ and $y_0 = 20$ then

$$v = -at \text{ and } y = -\tfrac{1}{2}at^2 + 20.$$

Substitution of $t = 2$, $y = 0$ yields $a = 10$ ft/sec^2. If $v_0 = 0$ and $y_0 = 200$ then

$$v = -10t \text{ and } y = -5t^2 + 200.$$

Hence $y = 0$ when $t = \sqrt{40} = 2\sqrt{10}$ sec and $v = -20\sqrt{10} \approx -63.25$ ft/sec.

34. **On Earth:** $v = -32t + v_0$, so $t = v_0/32$ at maximum height (when $v = 0$). Substituting this value of t and $y = 144$ in

$$y = -16t^2 + v_0 t,$$

we solve for $v_0 = 96$ ft/sec as the initial speed with which the person can throw a ball straight upward.

On Planet Gzyx: From Problem 27, the surface gravitational acceleration on planet Gzyx is $a = 10$ ft/sec^2, so

$$v = -10t + 96 \quad \text{and} \quad y = -5t^2 + 96t.$$

Therefore $v = 0$ yields $t = 9.6$ sec, and thence $y_{max} = y(9.6) = 460.8$ ft is the height a ball will reach if its initial velocity is 96 ft/sec.

35. If $v_0 = 0$ and $y_0 = h$ then the stone's velocity and height are given by

$$v = -gt, \quad y = -0.5\, gt^2 + h.$$

Hence $y = 0$ when $t = \sqrt{2h/g}$ so

$$v = -g\sqrt{2h/g} = -\sqrt{2gh}.$$

36. The method of solution is precisely the same as that in Problem 30. We find first that, on Earth, the woman must jump straight upward with initial velocity $v_0 = 12$ ft/sec to reach a maximum height of 2.25 ft. Then we find that, on the Moon, this initial velocity yields a maximum height of about 13.58 ft.

37. We use units of miles and hours. If $x_0 = v_0 = 0$ then the car's velocity and position after t hours are given by

$$v = at, \quad x = \tfrac{1}{2}t^2.$$

Since $v = 60$ when $t = 5/6$, the velocity equation yields $a = 72$ mi/hr^2. Hence the distance traveled by 12:50 pm is

$$x = (0.5)(72)(5/6)^2 = 25 \text{ miles.}$$

38. Again we have

$$v = at, \qquad x = \tfrac{1}{2}t^2.$$

But now $v = 60$ when $x = 35$. Substitution of $a = 60/t$ (from the velocity equation) into the position equation yields

$$35 = (0.5)(60/t)(t^2) = 30t,$$

whence $t = 7/6$ hr, that is, 1:10 p.m.

39. Integration of $y' = (9/v_S)(1 - 4x^2)$ yields

$$y = (3/v_S)(3x - 4x^3) + C,$$

and the initial condition $y(-1/2) = 0$ gives $C = 3/v_S$. Hence the swimmer's trajectory is

$$y(x) = (3/v_S)(3x - 4x^3 + 1).$$

Substitution of $y(1/2) = 1$ now gives $v_S = 6$ mph.

40. Integration of $y' = 3(1 - 16x^4)$ yields

$$y = 3x - (48/5)x^5 + C,$$

and the initial condition $y(-1/2) = 0$ gives $C = 6/5$. Hence the swimmer's trajectory is

$$y(x) = (1/5)(15x - 48x^5 + 6),$$

so his downstream drift is $y(1/2) = 2.4$ miles.

41. The bomb equations are $a = -32$, $v = -32$, and $s_B = s = -16t^2 + 800$, with $t = 0$ at the instant the bomb is dropped. The projectile is fired at time $t = 2$, so its corresponding equations are $a = -32$, $v = -32(t-2) + v_0$, and

$$s_P = s = -16(t-2)^2 + v_0(t-2)$$

for $t \geq 2$ (the arbitrary constant vanishing because $s_P(2) = 0$). Now the condition $s_B(t) = -16t^2 + 800 = 400$ gives $t = 5$, and then the requirement that $s_P(5) = 400$ also yields $v_0 = 544/3 \approx 181.33$ ft/s for the projectile's needed initial velocity.

42. Let $x(t)$ be the (positive) altitude (in miles) of the spacecraft at time t (hours), with $t = 0$ corresponding to the time at which the its retrorockets are fired; let $v(t) = x'(t)$ be the velocity of the spacecraft at time t. Then $v_0 = -1000$ and $x_0 = x(0)$ is unknown. But the (constant) acceleration is $a = +20000$, so

$$v(t) = 20000t - 1000 \quad \text{and} \quad x(t) = 10000t^2 - 1000t + x_0.$$

Now $v(t) = 20000t - 1000 = 0$ (soft touchdown) when $t = \frac{1}{20}$ hr (that is, after exactly 3 minutes of descent. Finally, the condition

$$0 = x(\tfrac{1}{20}) = 10000(\tfrac{1}{20})^2 - 1000(\tfrac{1}{20}) + x_0$$

yields $x_0 = 25$ miles for the altitude at which the retrorockets should be fired.

43. The velocity and position functions for the spacecraft are $v_S(t) = 0.0098t$ and $x_S(t) = 0.0049t^2$, and the corresponding functions for the projectile are $v_P(t) = \frac{1}{10}c = 3\times 10^7$ and $x_P(t) = 3\times 10^7 t$. The condition that $x_S = x_P$ when the spacecraft overtakes the projectile gives $0.0049t^2 = 3\times 10^7 t$, whence

$$\begin{aligned} t &= \frac{3\times 10^7}{0.0049} \approx 6.12245\times 10^9 \text{ sec} \\ &\approx \frac{6.12245\times 10^9}{(3600)(24)(365.25)} \approx 194 \text{ years.} \end{aligned}$$

Since the projectile is traveling at $\frac{1}{10}$ the speed of light, it has then traveled a distance of about 19.4 light years, which is about 1.8367×10^{17} meters.

44. Let $a > 0$ denote the constant deceleration of the car when braking, and take $x_0 = 0$ for the cars position at time $t = 0$ when the brakes are applied. In the police experiment with $v_0 = 25$ ft/s, the distance the car travels in t seconds is given by

$$x(t) = -\frac{1}{2}at^2 + \frac{88}{60}\cdot 25t$$

(with the factor $\frac{88}{60}$ used to convert the velocity units from mi/hr to ft/s). When we solve simultaneously the equations $x(t) = 45$ and $x'(t) = 0$ we find that $a = \frac{1210}{81} \approx 14.94$ ft/s^2. With this value of the deceleration and the (as yet) unknown velocity v_0 of the car involved in the accident, it position function is

$$x(t) = -\frac{1}{2}\cdot\frac{1210}{81}t^2 + v_0 t.$$

The simultaneous equations $x(t) = 210$ and $x'(t) = 0$ finally yield $v_0 = \frac{110}{9}\sqrt{42} \approx 79.21$ ft/s, almost exactly 54 miles per hour.

SECTION 1.3

SLOPE FIELDS AND SOLUTION CURVES

The instructor may choose to delay covering Section 1.3 until later in Chapter 1. However, before proceeding to Chapter 2, it is important that students come to grips at some point with the question of the existence of a unique solution of a differential equation — and realize that it makes no sense to look for the solution without knowing in advance that it exists. It may help some students to simplify the statement of the existence-uniqueness theorem as follows:

> Suppose that the function $f(x,y)$ and the partial derivative $\partial f/\partial y$ are both continuous in some neighborhood of the point (a, b). Then the initial value problem
>
> $$\frac{dy}{dx} = f(x,y), \qquad y(a) = b$$
>
> has a unique solution in some neighborhood of the point a.

Slope fields and geometrical solution curves are introduced in this section as a concrete aid in visualizing solutions and existence-uniqueness questions. Instead, we provide some details of the construction of the figure for the Problem 1 answer, and then include without further comment the similarly constructed figures for Problems 2 through 9.

1. The following sequence of *Mathematica* commands generates the slope field and the solution curves through the given points. Begin with the differential equation $dy/dx = f(x,y)$ where

```
f[x_, y_] := -y - Sin[x]
```

Then set up the viewing window

```
a = -3; b = 3; c = -3; d = 3;
```

The components (u,v) of unit vectors corresponding to the short slope field line segments are given by

```
u[x_, y_] := 1/Sqrt[1 + f[x, y]^2]
v[x_, y_] := f[x, y]/Sqrt[1 + f[x, y]^2]
```

The slope field is then constructed by the commands

```
Needs["Graphics`PlotField`"]
dfield = PlotVectorField[{u[x, y], v[x, y]}, {x, a, b}, {y, c, d},
         HeadWidth -> 0, HeadLength -> 0, PlotPoints -> 19,
         PlotRange -> {{a, b}, {c, d}}, Axes -> True, Frame -> True,
         FrameLabel -> {"x", "y"}, AspectRatio -> 1];
```

The original curve shown in Fig. 1.3.12 of the text (and its initial point not shown there) are plotted by the commands

```
x0 = -1.9; y0 = 0;
point0 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{Derivative[1][y][x] == f[x, y[x]], y[x0] == y0},
              y[x], {x, a, b}];
soln[[1,1,2]];
curve0 = Plot[soln[[1,1,2]], {x, a, b},
         PlotStyle -> {Thickness[0.0065], RGBColor[0, 0, 1]}];
```

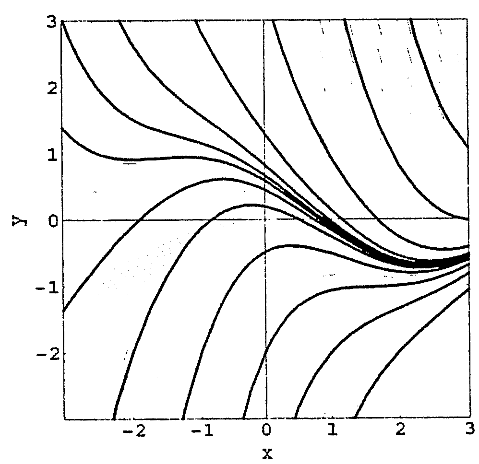
The *Mathematica* **NDSolve** command carries out an approximate numerical solution of the given differential equation. Numerical solution techniques are discussed in Sections 2.4–2.6 of the textbook.

The coordinates of the 12 points are marked in Fig. 1.3.12 in the textbook. For instance the 7th point is (–2.5, 1). It and the corresponding solution curve are plotted by the commands

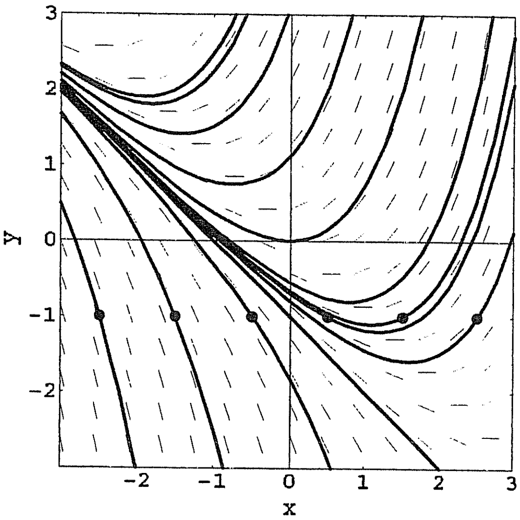
```
x0 = -2.5; y0 = 1;
point7 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{Derivative[1][y][x] == f[x, y[x]], y[x0] == y0},
              y[x], {x, a, b}];
soln[[1,1,2]];
curve7 = Plot[soln[[1,1,2]], {x, a, b},
         PlotStyle -> {Thickness[0.0065], RGBColor[0, 0, 1]}];
```

Finally, the desired figure is assembled by the *Mathematica* command

```
Show[ dfield, point0,curve0,
      point1,curve1, point2,curve2, point3,curve3,
      point4,curve4, point5,curve5, point6,curve6,
      point7,curve7, point8,curve8, point9,curve9,
      point10,curve10, point11,curve11, point12,curve12];
```



2.



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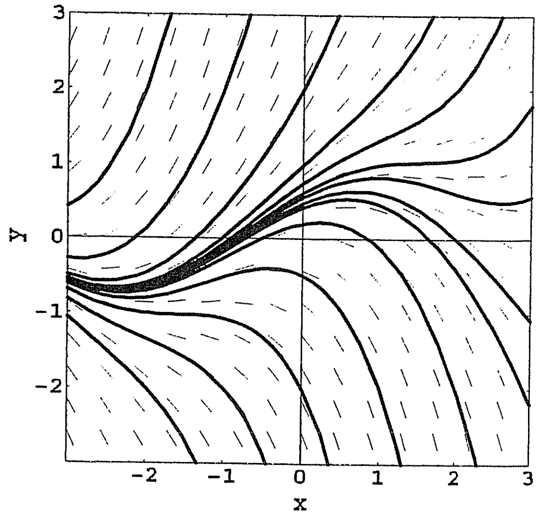
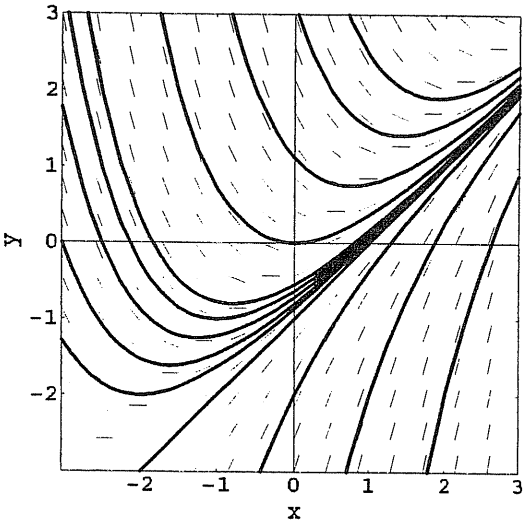


4.



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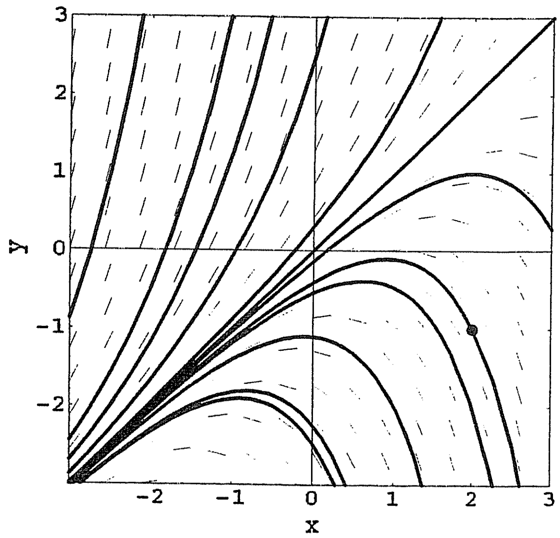

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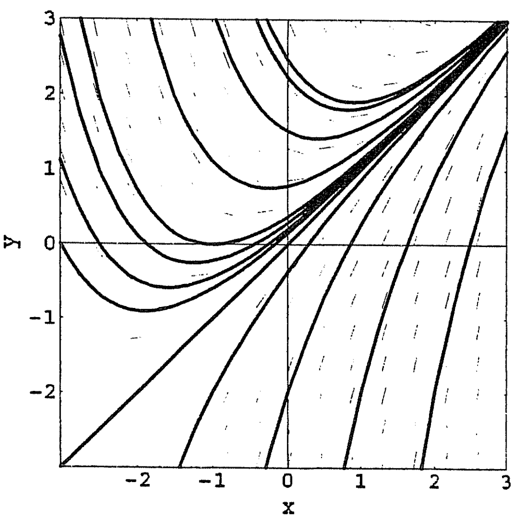

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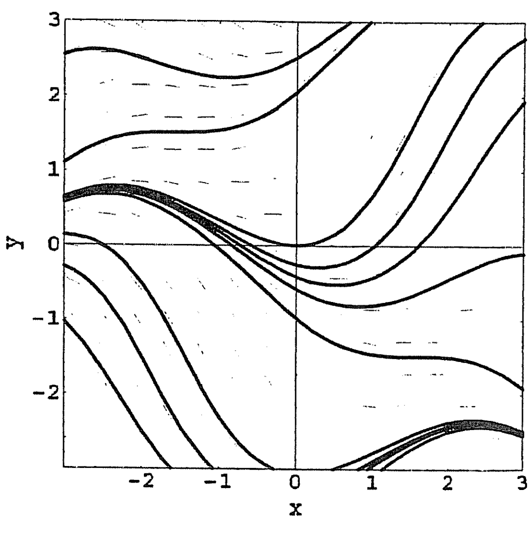

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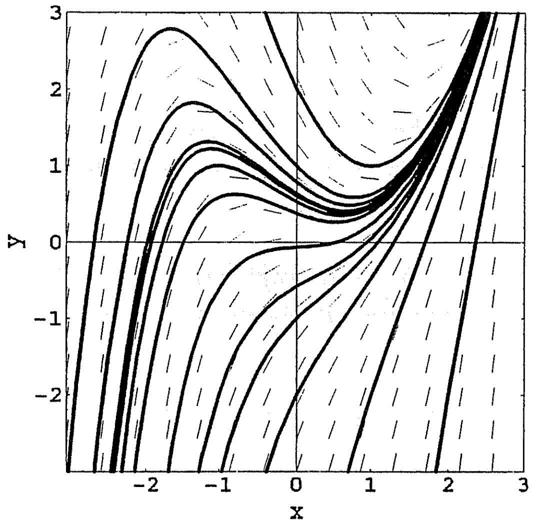
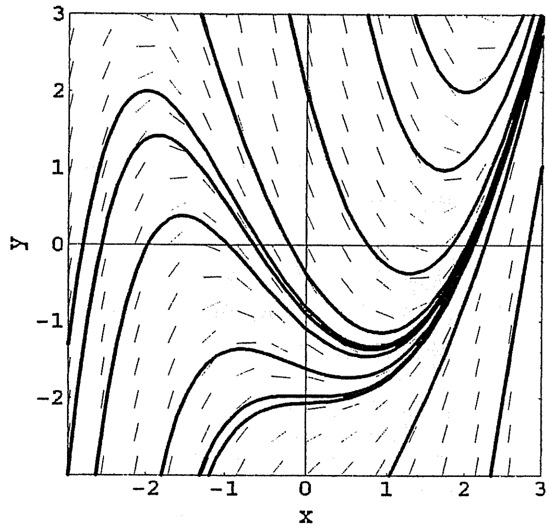
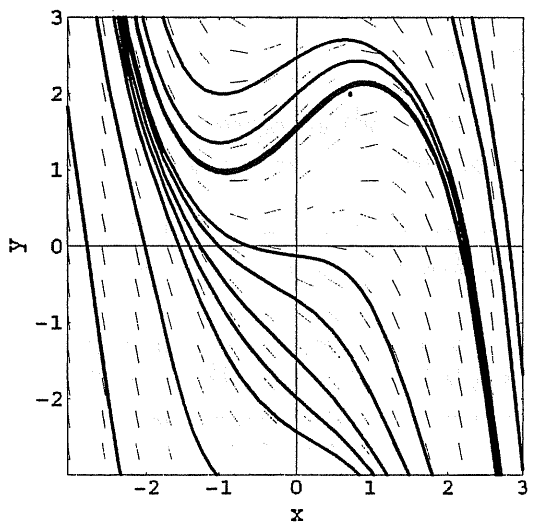


9.



10.



11. Because both $f(x,y) = 2x^2y^2$ and $\partial f/\partial y = 4x^2y$ are continuous everywhere, the existence-uniqueness theorem of Section 1.3 in the textbook guarantees the existence of a unique solution in some neighborhood of $x = 1$.

12. Both $f(x,y) = x\ln y$ and $\partial f/\partial y = x/y$ are continuous in a neighborhood of $(1, 1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 1$.

13. Both $f(x,y) = y^{1/3}$ and $\partial f/\partial y = (1/3)y^{-2/3}$ are continuous near $(0, 1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 0$.

14. $f(x, y) = y^{1/3}$ is continuous in a neighborhood of $(0, 0)$, but $\partial f / \partial y = (1/3)y^{-2/3}$ is not, so the theorem guarantees existence but not uniqueness in some neighborhood of $x = 0$.

15. $f(x, y) = (x - y)^{1/2}$ is not continuous at $(2, 2)$ because it is not even defined if $y > x$. Hence the theorem guarantees neither existence nor uniqueness in *any* neighborhood of the point $x = 2$.

16. $f(x, y) = (x - y)^{1/2}$ and $\partial f / \partial y = -(1/2)(x - y)^{-1/2}$ are continuous in a neighborhood of $(2, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 2$.

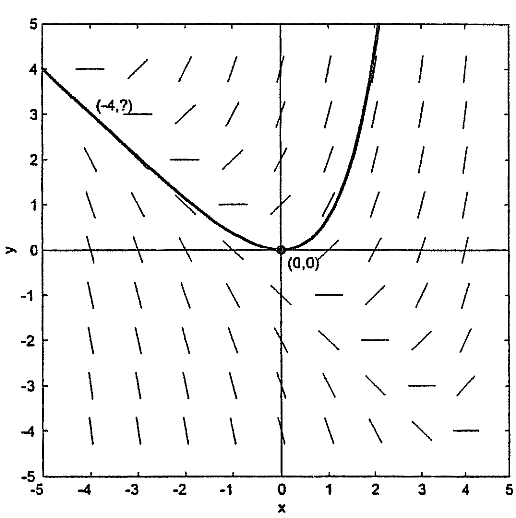
17. Both $f(x, y) = (x - 1/y$ and $\partial f / \partial y = -(x - 1)/y^2$ are continuous near $(0, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 0$.

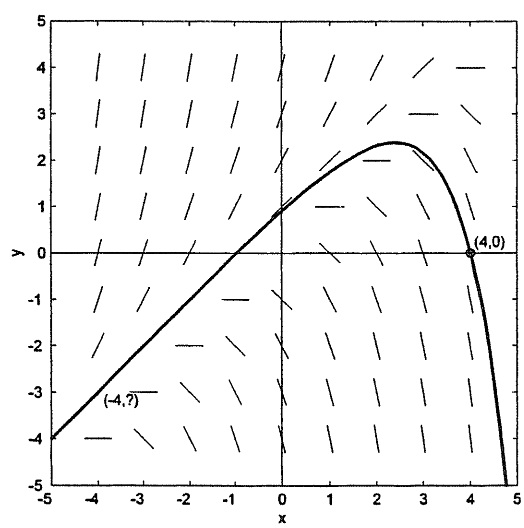
18. Neither $f(x, y) = (x - 1)/y$ nor $\partial f / \partial y = -(x - 1)/y^2$ is continuous near $(1, 0)$, so the existence-uniqueness theorem guarantees nothing.

19. Both $f(x, y) = \ln(1 + y^2)$ and $\partial f / \partial y = 2y/(1 + y^2)$ are continuous near $(0, 0)$, so the theorem guarantees the existence of a unique solution near $x = 0$.

20. Both $f(x, y) = x^2 - y^2$ and $\partial f / \partial y = -2y$ are continuous near $(0, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 0$.

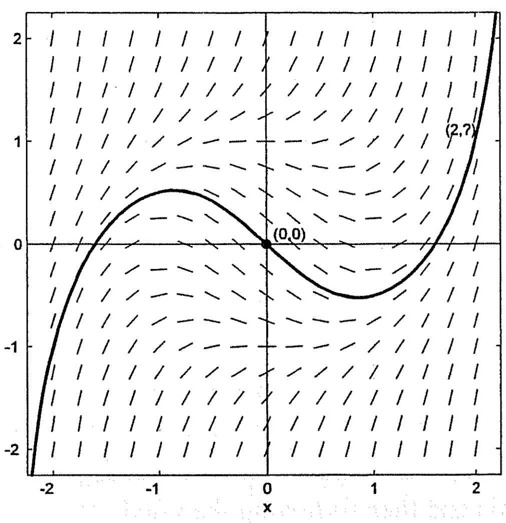
21. The curve in the figure on the left below can be constructed using the commands illustrated in Problem 1 above. Tracing this solution curve, we see that $y(-4) \approx 3$. An exact solution of the differential equation yields the more accurate approximation $y(-4) \approx 3.0183$.

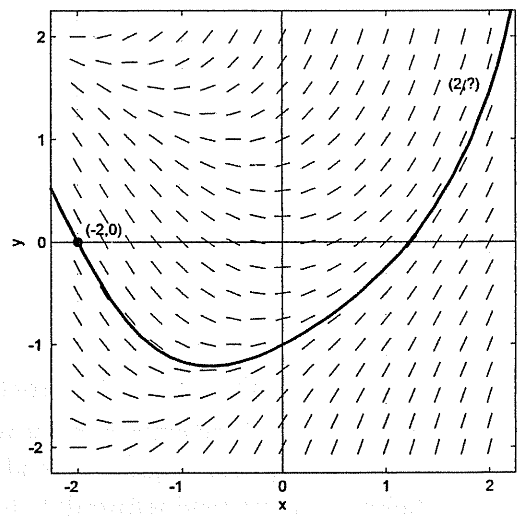




22. Tracing the curve in the figure on the right at the bottom of the preceding page, we see that $y(-4) \approx -3$. An exact solution of the differential equation yields the more accurate approximation $y(-4) \approx -3.0017$.

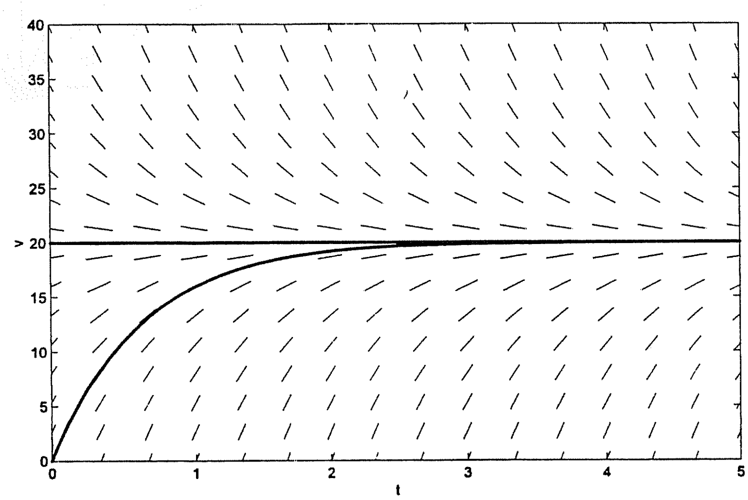
23. Tracing the curve in figure on the left below, we see that $y(2) \approx 1$. A more accurate approximation is $y(2) \approx 1.0044$.



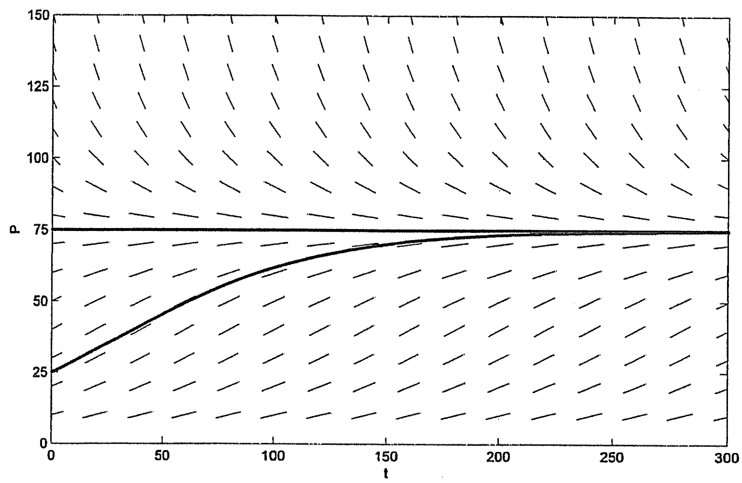


24. Tracing the curve in the figure on the right above, we see that $y(2) \approx 1.5$. A more accurate approximation is $y(2) \approx 1.4633$.

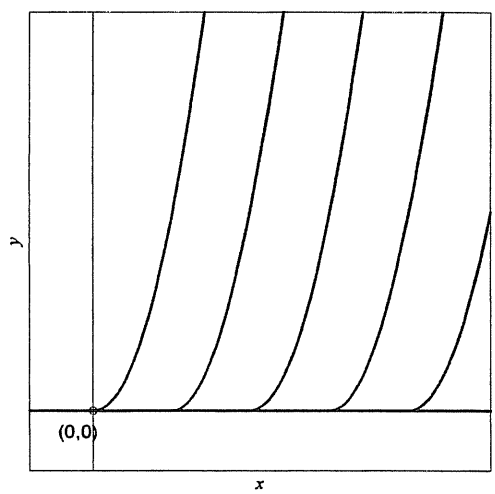
25. The figure below indicates a limiting velocity of 20 ft/sec — about the same as jumping off a $6\frac{1}{4}$-foot wall, and hence quite survivable. Tracing the curve suggests that $v(t) = 19$ ft/sec when t is a bit less than 2 seconds. An exact solution gives $t \approx 1.8723$ then.

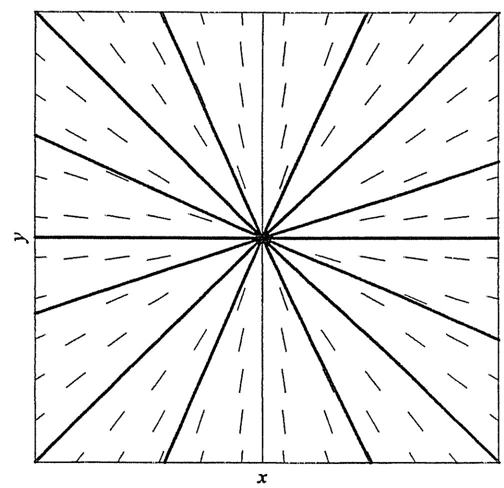


26. The figure below suggests that there are 40 deer after about 60 months; a more accurate value is $t \approx 61.61$. And it's pretty clear that the limiting population is 75 deer.



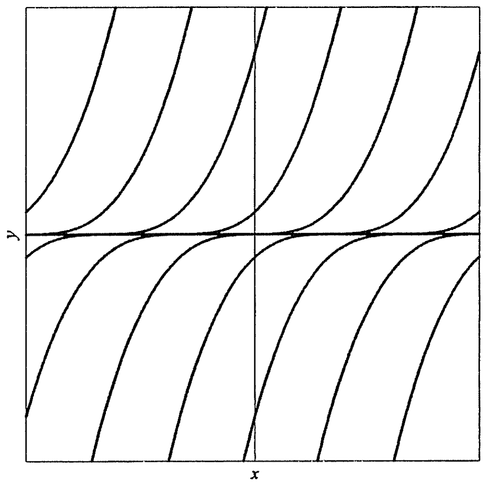
27. If $b < 0$ then the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ has no solution, because the square root of a negative number would be involved. If $b > 0$ we get a unique solution curve through $(0, b)$ defined for all x by following a parabola — in the figure on the left below — down (and leftward) to the x-axis and then following the x-axis to the left. But starting at (0,0) we can follow the positive x-axis to the point $(c, 0)$ and then branching off on the parabola $y = (x - c)^2$. This gives infinitely many different solutions if $b = 0$.

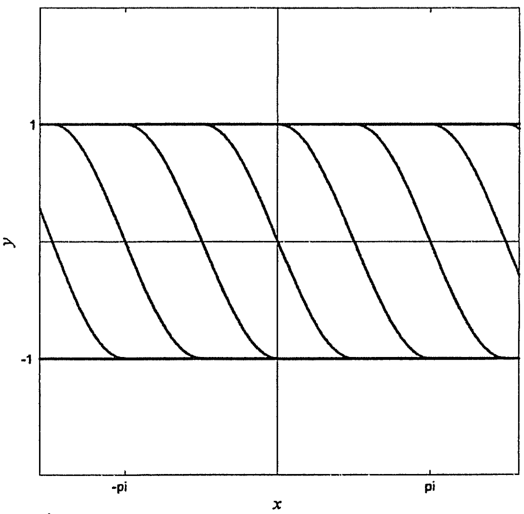




28. The figure on the right above makes it clear initial value problem $xy' = y$, $y(a) = b$ has a unique solution off the y-axis where $a \neq 0$; infinitely many solutions through the origin where $a = b = 0$; no solution if $a = 0$ but $b \neq 0$ (so the point (a, b) lies on the positive or negative y-axis).

29. Looking at the figure on the left below, we see that we can start at the point (a,b) and follow a branch of a cubic up or down to the x-axis, then follow the x-axis an arbitrary distance before branching off (down or up) on another cubic. This gives infinitely many solutions of the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ that are defined for all x. However, if $b \neq 0$ there is only a single cubic $y = (x-c)^3$ passing through (a,b), so the solution is unique near $x = a$.





30. The function $y(x) = \cos(x-c)$, with $y'(x) = -\sin(x-c)$, satisfies the differential equation $y' = -\sqrt{1-y^2}$ on the interval $c < x < c + \pi$ where $\sin(x-c) > 0$, so it follows that

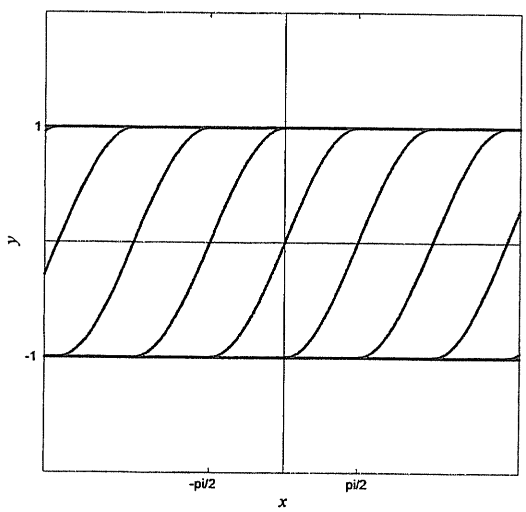
$$-\sqrt{1-y^2} \;=\; -\sqrt{1-\cos^2(x-c)} \;=\; -\sqrt{\sin^2(x-c)} \;=\; -\sin(x-c) \;=\; y.$$

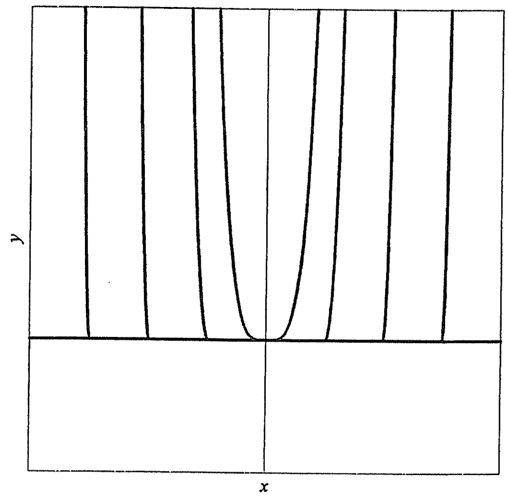
If $|b| > 1$ then the initial value problem $y' = -\sqrt{1-y^2}$, $y(a) = b$ has no solution because the square root of a negative number would be involved. If $|b| < 1$ then there is only one curve of the form $y = \cos(x-c)$ through the point (a,b); this give a unique solution. But if $b = \pm 1$ then we can combine a left ray of the line $y = +1$, a cosine curve from the line $y = +1$ to the line $y = -1$, and then a right ray of the line $y = -1$. Looking at the figure on the right above, we see that this gives infinitely many solutions (defined for all x) through any point of the form $(a, \pm 1)$.

31. The function $y(x) = \sin(x-c)$, with $y'(x) = \cos(x-c)$, satisfies the differential equation $y' = \sqrt{1-y^2}$ on the interval $c - \pi/2 < x < c + \pi/2$ where $\cos(x-c) > 0$, so it follows that

$$\sqrt{1-y^2} \;=\; \sqrt{1-\sin^2(x-c)} \;=\; \sqrt{\cos^2(x-c)} \;=\; -\sin(x-c) \;=\; y.$$

If $|b| > 1$ then the initial value problem $y' = \sqrt{1 - y^2}$, $y(a) = b$ has no solution because the square root of a negative number would be involved. If $|b| < 1$ then there is only one curve of the form $y = \sin(x - c)$ through the point (a, b); this give a unique solution. But if $b = \pm 1$ then we can combine a left ray of the line $y = -1$, a sine curve from the line $y = -1$ to the line $y = +1$, and then a right ray of the line $y = +1$. Looking at the figure on the left below, we see that this gives infinitely many solutions (defined for all x) through any point of the form $(a, \pm 1)$.





32. Looking at the figure on the right above, we see that we can piece together a "left half" of a quartic for x negative, an interval along the x-axis, and a "right half" of a quartic curve for x positive. This makes it clear he initial value problem $y' = 4x\sqrt{y}$, $y(a) = b$ has infinitely many solutions (defined for all x) if $b \geq 0$; there is no solution if $b < 0$ because this would involve the square root of a negative number.

33. Looking at the figure provided in the answers section of the textbook, it suffices to observe that, among the pictured curves $y = x/(cx - 1)$ for all possible values of c,

- there is a unique one of these curves through any point not on either coordinate axis;
- there is no such curve through any point on the y-axis other than the origin; and
- there are infinitely many such curves through the origin (0,0).

But in addition we have the constant-valued solution $y(x) \equiv 0$ that "covers" the x-axis. It follows that the given differential equation has near (a, b)

- a unique solution if $a \neq 0$;
- no solution if $a = 0$ but $b \neq 0$;
- infinitely many different solutions if $a = b = 0$.

SECTION 1.4

SEPARABLE EQUATIONS AND APPLICATIONS

Of course it should be emphasized to students that the possibility of separating the variables is the first one you look for. The general concept of natural growth and decay is important for all differential equations students, but the particular applications in this section are optional. Torricelli's law in the form of Equation (24) in the text leads to some nice concrete examples and problems.

1. $\int \frac{dy}{y} = -\int 2x\,dx; \quad \ln y = -x^2 + c; \quad y(x) = e^{-x^2+c} = Ce^{-x^2}$

2. $\int \frac{dy}{y^2} = -\int 2x\,dx; \quad -\frac{1}{y} = -x^2 - C; \quad y(x) = \frac{1}{x^2+C}$

3. $\int \frac{dy}{y} = \int \sin x\,dx; \quad \ln y = -\cos x + c; \quad y(x) = e^{-\cos x + c} = Ce^{-\cos x}$

4. $\int \frac{dy}{y} = \int \frac{4\,dx}{1+x}; \quad \ln y = 4\ln(1+x) + \ln C; \quad y(x) = C(1+x)^4$

5. $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{2\sqrt{x}}; \quad \sin^{-1} y = \sqrt{x} + C; \quad y(x) = \sin\left(\sqrt{x} + C\right)$

6. $\int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\,dx; \quad 2\sqrt{y} = 2x^{3/2} + 2C; \quad y(x) = \left(x^{3/2} + C\right)^2$

7. $\int \frac{dy}{y^{1/3}} = \int 4x^{1/3}\,dx; \quad \tfrac{3}{2}y^{2/3} = 3x^{4/3} + \tfrac{3}{2}C; \quad y(x) = \left(2x^{4/3} + C\right)^{3/2}$

8. $\int \cos y\,dy = \int 2x\,dx; \quad \sin y = x^2 + C; \quad y(x) = \sin^{-1}\left(x^2 + C\right)$

9. $\int \frac{dy}{y} = \int \frac{2\,dx}{1-x^2} = \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$ (partial fractions)

$\ln y = \ln(1+x) - \ln(1-x) + \ln C; \quad y(x) = C\frac{1+x}{1-x}$

10. $\displaystyle\int \frac{dy}{(1+y)^2} = \int \frac{dx}{(1+x)^2}; \quad -\frac{1}{1+y} = -\frac{1}{1+x} - C = -\frac{1+C(1+x)}{1+x}$

$\displaystyle 1+y = \frac{1+x}{1+C(1+x)}; \quad y(x) = \frac{1+x}{1+C(1+x)} - 1 = \frac{x-C(1+x)}{1+C(1+x)}$

11. $\displaystyle\int \frac{dy}{y^3} = \int x\,dx; \quad -\frac{1}{2y^2} = \frac{x^2}{2} - \frac{C}{2}; \quad y(x) = \left(C - x^2\right)^{-1/2}$

12. $\displaystyle\int \frac{y\,dy}{y^2+1} = \int x\,dx; \quad \tfrac{1}{2}\ln\left(y^2+1\right) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln C; \quad y^2+1 = Ce^{x^2}$

13. $\displaystyle\int \frac{y^3\,dy}{y^4+1} = \int \cos x\,dx; \quad \tfrac{1}{4}\ln\left(y^4+1\right) = \sin x + C$

14. $\displaystyle\int\left(1+\sqrt{y}\right)dy = \int\left(1+\sqrt{x}\right)dx; \quad y + \tfrac{2}{3}y^{3/2} = x + \tfrac{2}{3}x^{3/2} + C$

15. $\displaystyle\int\left(\frac{2}{y^2} - \frac{1}{y^4}\right)dy = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)dx; \quad -\frac{2}{y} + \frac{1}{3y^3} = \ln|x| + \frac{1}{x} + C$

16. $\displaystyle\int \frac{\sin y\,dy}{\cos y} = \int \frac{x\,dx}{1+x^2}; \quad -\ln(\cos x) = \tfrac{1}{2}\ln\left(1+x^2\right) + \ln C$

$\displaystyle \sec y = C\sqrt{1+x^2}; \quad y(x) = \sec^{-1}\left(C\sqrt{1+x^2}\right)$

17. $y' = 1 + x + y + xy = (1+x)(1+y)$

$\displaystyle\int \frac{dy}{1+y} = \int (1+x)\,dx; \quad \ln|1+y| = x + \tfrac{1}{2}x^2 + C$

18. $x^2y' = 1 - x^2 + y^2 - x^2y^2 = (1-x^2)(1+y^2)$

$\displaystyle\int \frac{dy}{1+y^2} = \int\left(\frac{1}{x^2} - 1\right)dx; \quad \tan^{-1} y = -\frac{1}{x} - x + C; \quad y(x) = \tan\left(C - \frac{1}{x} - x\right)$

19. $\displaystyle\int \frac{dy}{y} = \int e^x\,dx; \quad \ln y = e^x + \ln C; \quad y(x) = C\exp(e^x)$

$y(0) = 2e$ implies $C = 2$ so $y(x) = 2\exp(e^x)$.

20. $\displaystyle\int \frac{dy}{1+y^2} = \int 3x^2\,dx; \quad \tan^{-1} y = x^3 + C; \quad y(x) = \tan\left(x^3 + C\right)$

$y(0) = 1$ implies $C = \tan^{-1} 1 = \pi/4$ so $y(x) = \tan\left(x^3 + \pi/4\right)$.

21. $\displaystyle\int 2y\,dy = \int \frac{x\,dx}{\sqrt{x^2-16}}; \quad y^2 = \sqrt{x^2-16} + C$

$y(5) = 2$ implies $C = 1$ so $y^2 = 1 + \sqrt{x^2-16}$.

22. $\displaystyle\int \frac{dy}{y} = \int \left(4x^3 - 1\right) dx; \quad \ln y = x^4 - x + \ln C; \quad y(x) = C\exp(x^4 - x)$

$y(1) = -3$ implies $C = -3$ so $y(x) = -3\exp(x^4 - x)$.

23. $\displaystyle\int \frac{dy}{2y-1} = \int dx; \quad \tfrac{1}{2}\ln(2y-1) = x + \tfrac{1}{2}\ln C; \quad 2y - 1 = Ce^{2x}$

$y(1) = 1$ implies $C = e^{-2}$ so $y(x) = \tfrac{1}{2}\left(1 + e^{2x-2}\right)$.

24. $\displaystyle\int \frac{dy}{y} = \int \frac{\cos x\,dx}{\sin x}; \quad \ln y = \ln(\sin x) + \ln C; \quad y(x) = C\sin x$

$y(\tfrac{\pi}{2}) = \tfrac{\pi}{2}$ implies $C = \tfrac{\pi}{2}$ so $y(x) = \tfrac{\pi}{2}\sin x$.

25. $\displaystyle\int \frac{dy}{y} = \int \left(\frac{1}{x} + 2x\right); \quad \ln y = \ln x + x^2 + \ln C; \quad y(x) = Cx\exp(x^2)$

$y(1) = 1$ implies $C = e^{-1}$ so $y(x) = x\exp(x^2 - 1)$.

26. $\displaystyle\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right); \quad -\frac{1}{y} = x^2 + x^3 + C; \quad y(x) = \frac{-1}{x^2 + x^3 + C}$

$y(1) = -1$ implies $C = -1$ so $y(x) = \dfrac{1}{1 - x^2 - x^3}$.

27. $\displaystyle\int e^y\,dy = \int 6e^{2x}\,dx; \quad e^y = 3e^{2x} + C; \quad y(x) = \ln\left(3e^{2x} + C\right)$

$y(0) = 0$ implies $C = -2$ so $y(x) = \ln\left(3e^{2x} - 2\right)$.

28. $\displaystyle\int \sec^2 y\,dy = \int \frac{dx}{2\sqrt{x}};\quad \tan y = \sqrt{x}+C;\quad y(x) = \tan^{-1}\left(\sqrt{x}+C\right)$

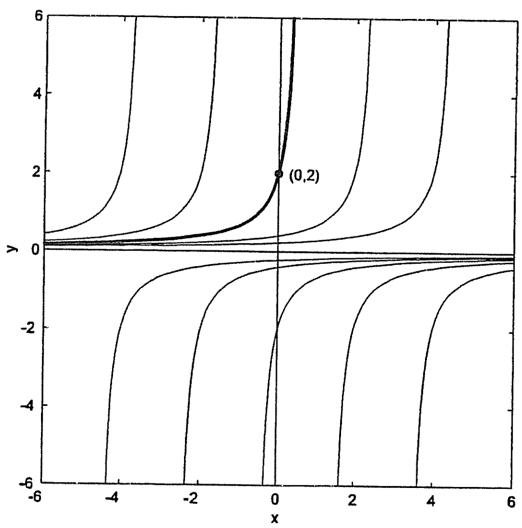
$y(4) = \frac{\pi}{4}$ implies $C = -1$ so $y(x) = \tan^{-1}\left(\sqrt{x}-1\right)$.

29. (a) Separation of variables gives the general solution

$$\int\left(-\frac{1}{y^2}\right)dy = -\int x\,dx;\quad \frac{1}{y} = -x+C;\quad y(x) = -\frac{1}{x-C}.$$

(b) Inspection yields the singular solution $y(x) \equiv 0$ that corresponds to *no* value of the constant C.

(c) In the figure below we see that there is a unique solution curve through every point in the *xy*-plane.



30. When we take square roots on both sides of the differential equation and separate variables, we get

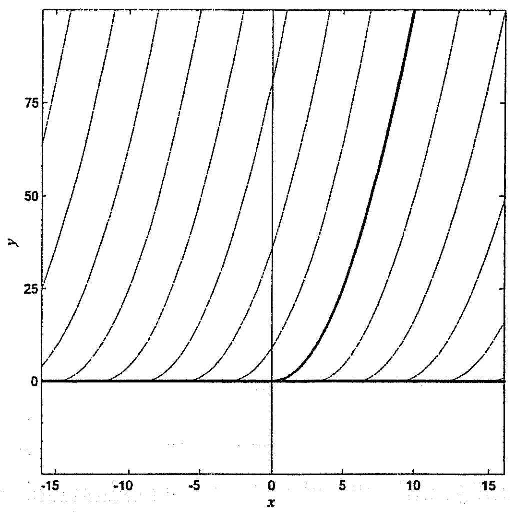
$$\int\frac{dy}{2\sqrt{y}} = \int dx;\quad \sqrt{y} = x-C;\quad y(x) = (x-C)^2.$$

This general solution provides the parabolas illustrated in Fig. 1.4.5 in the textbook. Observe that $y(x)$ is always nonnegative, consistent with both the square root and the original differential equation. We spot also the singular solution $y(x) \equiv 0$ that corresponds to *no* value of the constant C.

(a) Looking at Fig. 1.4.5, we see immediately that the differential equation $(y')^2 = 4y$ has no solution curve through the point (a,b) if $b < 0$.

(b) But if $b \geq 0$ we obviously can combine branches of parabolas with segments along the x-axis to form infinitely many solution curves through (a,b).

(c) Finally, if $b > 0$ then on a interval containing (a,b) there are exactly *two* solution curves through this point, corresponding to the two indicated parabolas through (a,b), one ascending and one descending from left to right.

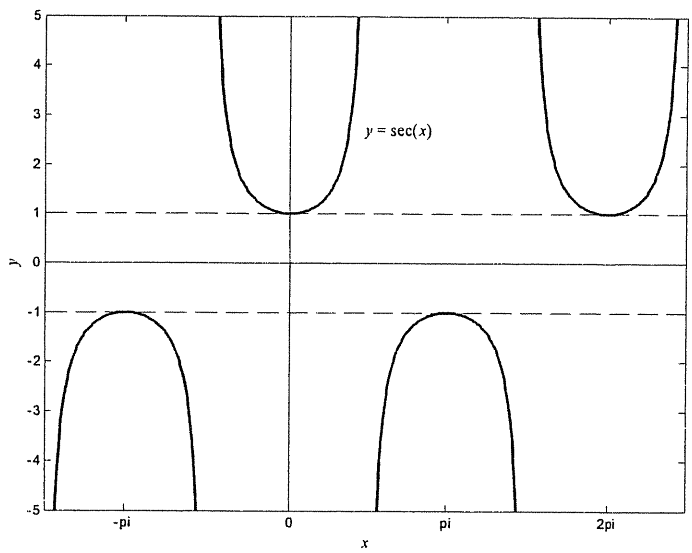


Problem 31 Figure

31. The formal separation-of-variables process is the same as in Problem 30 where, indeed, we started by taking square roots in $(y')^2 = 4y$ to get the differential equation $y' = 2\sqrt{y}$. But whereas y' can be either positive or negative in the original equation, the latter equation requires that y' be *nonnegative*. This means that only the *right half* of each parabola $y = (x-C)^2$ qualifies as a solution curve. Inspecting the figure above, we therefore see that through the point (a,b) there passes

(a) No solution curve if $b < 0$,
(b) A unique solution curve if $b > 0$,
(c) Infinitely many solution curves if $b = 0$, because in this case we can pick any $c > a$ and define the solution $y(x) = 0$ if $x \leq c$, $y(x) = (x-c)^2$ if $x \geq c$.

Problem 32 Figure (a)



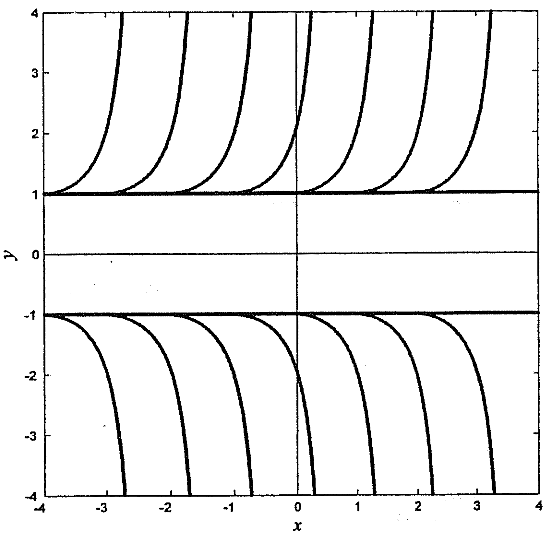
32. Separation of variables gives

$$x = \int \frac{dy}{y\sqrt{y^2-1}} = \sec^{-1}|y| + C$$

if $|y| > 1$, so the general solution has the form $y(x) = \pm\sec(x - C)$. But the original differential equation $y' = y\sqrt{y^2-1}$ implies that $y' > 0$ if $y > 1$, while $y' < 0$ if $y < -1$. Consequently, only the *right halves* of translated branches of the curve $y = \sec x$ (figure above) qualify as general solution curves. This explains the plotted general solution curves we see in the figure at the top of the next page. In addition, we spot the two singular solutions $y(x) \equiv 1$ and $y(x) \equiv -1$. It follows upon inspection of this figure that the initial value problem $y' = y\sqrt{y^2-1}$, $y(a) = b$ has a unique solution if $|b| > 1$ and has no solution if $|b| < 1$. But if $b = 1$ (and similarly if $b = -1$) then we can pick any $c > a$ and define the solution $y(x) = 1$ if $x \le c$, $y(x) = |\sec(x - c)|$ if $c \le x < c + \frac{\pi}{2}$. So we see that if $b = \pm 1$, then the initial value problem $y' = y\sqrt{y^2-1}$, $y(a) = b$ has infinitely many solutions.

33. The population growth rate is $k = \ln(30000/25000)/10 \approx 0.01823$, so the population of the city t years after 1960 is given by $P(t) = 25000e^{0.01823t}$. The expected year 2000 population is then $P(40) = 25000e^{0.01823\times 40} \approx 51840$.

Problem 32 Figure (b)



34. The population growth rate is $k = \ln(6)/10 \approx 0.17918$, so the population after t hours is given by $P(t) = P_0 e^{0.17918t}$. To find how long it takes for the population to double, we therefore need only solve the equation $2P = P_0 e^{0.17918t}$ for $t = (\ln 2)/0.17918 \approx 3.87$ hours.

35. As in the textbook discussion of radioactive decay, the number of ^{14}C atoms after t years is given by $N(t) = N_0 e^{-0.0001216t}$. Hence we need only solve the equation $\frac{1}{6}N_0 = N_0 e^{-0.0001216t}$ for $t = (\ln 6)/0.0001216 \approx 14735$ years to find the age of the skull.

36. As in Problem 35, the number of ^{14}C atoms after t years is given by $N(t) = 5.0 \times 10^{10} e^{-0.0001216t}$. Hence we need only solve the equation $4.6 \times 10^{10} = 5.0 \times 10^{10} e^{-0.0001216t}$ for the age $t = \left(\ln(5.0/4.6)\right)/0.0001216 \approx 686$ years of the relic. Thus it appears not to be a genuine relic of the time of Christ 2000 years ago.

37. The amount in the account after t years is given by $A(t) = 5000e^{0.08t}$. Hence the amount in the account after 18 years is given by $A(18) = 5000e^{0.08\times 18} \approx 21{,}103.48$ dollars.

38. When the book has been overdue for t years, the fine owed is given in dollars by $A(t) = 0.30e^{0.05t}$. Hence the amount owed after 100 years is given by $A(100) = 0.30e^{0.05\times 100} \approx 44.52$ dollars.

39. To find the decay rate of this drug in the dog's blood stream, we solve the equation $\frac{1}{2} = e^{-5k}$ (half-life 5 hours) for $k = (\ln 2)/5 \approx 0.13863$. Thus the amount in the dog's bloodstream after t hours is given by $A(t) = A_0 e^{-0.13863t}$. We therefore solve the equation $A(1) = A_0 e^{-0.13863} = 50 \times 45 = 2250$ for $A_0 \approx 2585$ mg, the amount to anesthetize the dog properly.

40. To find the decay rate of radioactive cobalt, we solve the equation $\frac{1}{2} = e^{-5.27k}$ (half-life 5.27 years) for $k = (\ln 2)/5.27 \approx 0.13153$. Thus the amount of radioactive cobalt left after t years is given by $A(t) = A_0 e^{-0.13153t}$. We therefore solve the equation $A(t) = A_0 e^{-0.13153t} = 0.01 A_0$ for $t = (\ln 100)/0.13153 \approx 35.01$ and find that it will be about 35 years until the region is again inhabitable.

41. Taking $t = 0$ when the body was formed and $t = T$ now, the amount $Q(t)$ of ^{238}U in the body at time t (in years) is given by $Q(t) = Q_0 e^{-kt}$, where $k = (\ln 2)/(4.51\times10^9)$. The given information tells us that

$$\frac{Q(T)}{Q_0 - Q(T)} = 0.9.$$

After substituting $Q(T) = Q_0 e^{-kT}$, we solve readily for $e^{kT} = 19/9$, so $T = (1/k)\ln(19/9) \approx 4.86\times10^9$. Thus the body was formed approximately 4.86 billion years ago.

42. Taking $t = 0$ when the rock contained only potassium and $t = T$ now, the amount $Q(t)$ of potassium in the rock at time t (in years) is given by $Q(t) = Q_0 e^{-kt}$, where $k = (\ln 2)/(1.28\times10^9)$. The given information tells us that the amount $A(t)$ of argon at time t is

$$A(t) = \tfrac{1}{9}[Q_0 - Q(t)]$$

and also that $A(T) = Q(T)$. Thus

$$Q_0 - Q(T) = 9\,Q(T).$$

After substituting $Q(T) = Q_0 e^{-kT}$ we readily solve for

$$T = (\ln 10/\ln 2)(1.28\times10^9) \approx 4.25\times10^9.$$

Thus the age of the rock is about 1.25 billion years.

43. Because $A = 0$ the differential equation reduces to $T' = kT$, so $T(t) = 25e^{-kt}$. The fact that $T(20) = 15$ yields $k = (1/20)\ln(5/3)$, and finally we solve

$$5 = 25e^{-kt} \quad \text{for} \quad t = (\ln 5)/k \approx 63 \text{ min.}$$

44. The amount of sugar remaining undissolved after t minutes is given by $A(t) = A_0e^{-kt}$; we find the value of k by solving the equation $A(1) = A_0e^{-k} = 0.75A_0$ for $k = -\ln 0.75 \approx 0.28768$. To find how long it takes for half the sugar to dissolve, we solve the equation $A(t) = A_0e^{-kt} = \frac{1}{2}A_0$ for $t = (\ln 2)/0.28768 \approx 2.41$ minutes.

45. **(a)** The light intensity at a depth of x meters is given by $I(x) = I_0e^{-1.4x}$. We solve the equation $I(x) = I_0e^{-1.4x} = \frac{1}{2}I_0$ for $x = (\ln 2)/1.4 \approx 0.495$ meters.

(b) At depth 10 meters the intensity is $I(10) = I_0e^{-1.4\times 10} \approx (8.32\times 10^{-7})\,I_0$.

(c) We solve the equation $I(x) = I_0e^{-1.4x} = 0.01I_0$ for $x = (\ln 100)/1.4 \approx 3.29$ meters.

46. **(a)** The pressure at an altitude of x miles is given by $p(x) = 29.92\,e^{-0.2x}$. Hence the pressure at altitude 10000 ft is $p(10000/5280) \approx 20.49$ inches, and the pressure at altitude 30000 ft is $p(30000/5280) \approx 9.60$ inches.

(b) To find the altitude where $p = 15$ in., we solve the equation $29.92\,e^{-0.2x} = 15$ for $x = (\ln 29.92/15)/0.2 \approx 3.452 \text{ miles} \approx 18{,}200 \text{ ft}$.

47. If $N(t)$ denotes the number of people (in thousands) who have heard the rumor after t days, then the initial value problem is

$$N' = k(100 - N), \quad N(0) = 0$$

and we are given that $N(7) = 10$. When we separate variables ($dN/(100-N) = k\,dt$) and integrate, we get $\ln(100-N) = -kt + C$, and the initial condition $N(0) = 0$ gives $C = \ln 100$. Then $100 - N = 100e^{-kt}$, so $N(t) = 100\left(1 - e^{-kt}\right)$. We substitute $t = 7$, $N = 10$ and solve for the value $k = \ln(100/90)/7 \approx 0.01505$. Finally, 50 thousand people have heard the rumor after $t = (\ln 2)/k \approx 46.05$ days.

48. Let $N_8(t)$ and $N_5(t)$ be the numbers of ^{238}U and ^{235}U atoms, respectively, at time t (in billions of years after the creation of the universe). Then $N_8(t) = N_0e^{-kt}$ and $N_5(t) = N_0e^{-ct}$, where N_0 is the initial number of atoms of each isotope. Also, $k = (\ln 2)/4.51$ and $c = (\ln 2)/0.71$ from the given half-lives. We divide the equations for N_8 and N_5 and find that when t has the value corresponding to "now",

$$e^{(c-k)t} = \frac{N_8}{N_5} = 137.7.$$

Finally we solve this last equation for $t = (\ln 137.7)/(c-k) \approx 5.99$. Thus we get an estimate of about 6 billion years for the age of the universe.

49. The cake's temperature will be 100° after 66 min 40 sec; this problem is just like Example 6 in the text.

50. (a) $A(t) = 10e^{kt}$. Also $30 = A(\frac{15}{2}) = 10e^{15k/2}$, so so

$$e^{15k/2} = 3; \quad k = \frac{2}{15}\ln 3 = \ln\left(3^{2/15}\right).$$

Therefore $A(t) = 10(e^k)^t = 10 \cdot 3^{2t/15}$.

(b) After 5 years we have $A(5) = 10 \cdot 3^{2/3} \approx 20.80$ pu.

(c) $A(t) = 100$ when $A(t) = 10 \cdot 3^{2t/15}$; $t = \dfrac{15}{2} \cdot \dfrac{\ln(10)}{\ln(3)} \approx 15.72$ years.

51. (a) $A(t) = 15e^{-kt}$; $10 = A(5) = 15e^{-kt}$, so

$$\frac{3}{2} = e^{kt}; \quad k = \frac{1}{5}\ln\frac{3}{2}.$$

Therefore

$$A(t) = 15\exp\left(-\frac{t}{5}\ln\frac{3}{2}\right) = 15 \cdot \left(\frac{3}{2}\right)^{-t/5} = 15 \cdot \left(\frac{2}{3}\right)^{t/5}.$$

(b) After 8 months we have

$$A(8) = 15 \cdot \left(\frac{2}{3}\right)^{8/5} \approx 7.84 \text{ su.}$$

(c) $A(t) = 1$ when

$$A(t) = 15 \cdot \left(\frac{2}{3}\right)^{t/5} = 1; \quad t = 5 \cdot \frac{\ln(\frac{1}{15})}{\ln(\frac{2}{3})} \approx 33.3944.$$

Thus it will be safe to return after about 33.4 months.

52. If $L(t)$ denotes the number of human language families at time t (in years), then $L(t) = e^{kt}$ for some constant k. The condition that $L(6000) = e^{6000k} = 1.5$ gives

$k = \dfrac{1}{6000}\ln\dfrac{3}{2}$. If "now" corresponds to time $t = T$, then we are given that

$L(T) = e^{kT} = 3300,$ so $T = \dfrac{1}{k}\ln 3300 = \dfrac{6000\ln 3300}{\ln(3/2)} \approx 119887.18$. This result suggests that the original human language was spoken about 120 thousand years ago.

53. If $L(t)$ denotes the number of Native America language families at time t (in years), then $L(t) = e^{kt}$ for some constant k. The condition that $L(6000) = e^{6000k} = 1.5$ gives $k = \dfrac{1}{6000}\ln\dfrac{3}{2}$. If "now" corresponds to time $t = T$, then we are given that

$L(T) = e^{kT} = 150,$ so $T = \dfrac{1}{k}\ln 150 = \dfrac{6000\ln 150}{\ln(3/2)} \approx 74146.48$. This result suggests that the ancestors of today's Native Americans first arrived in the western hemisphere about 74 thousand years ago.

54. With $A(y)$ constant, Equation (19) in the text takes the form

$$\frac{dy}{dt} = k\sqrt{y}$$

We readily solve this equation for $2\sqrt{y} = kt + C$. The condition $y(0) = 9$ yields $C = 6$, and then $y(1) = 4$ yields $k = 2$. Thus the depth at time t (in hours) is $y(t) = (3 - t)^2$, and hence it takes 3 hours for the tank to empty.

55. With $A = \pi(3)^2$ and $a = \pi(1/12)^2$, and taking $g = 32$ ft/sec^2, Equation (20) reduces to $162\,y' = -\sqrt{y}$. The solution such that $y = 9$ when $t = 0$ is given by $324\sqrt{y} = -t + 972$. Hence $y = 0$ when $t = 972$ sec $= 16$ min 12 sec.

56. The radius of the cross-section of the cone at height y is proportional to y, so $A(y)$ is proportional to y^2. Therefore Equation (20) takes the form

$$y^2 y' = -k\sqrt{y},$$

and a general solution is given by

$$2y^{5/2} = -5kt + C.$$

The initial condition $y(0) = 16$ yields $C = 2048$, and then $y(1) = 9$ implies that $5k = 1562$. Hence $y = 0$ when

$$t = C/5k = 2048/1562 \approx 1.31 \text{ hr}.$$

57. The solution of $y' = -k\sqrt{y}$ is given by

$$2\sqrt{y} = -kt + C.$$

The initial condition $y(0) = h$ (the height of the cylinder) yields $C = 2\sqrt{h}$. Then substitution of $t = T,\ y = 0$ gives $k = (2\sqrt{h})/T$. It follows that

$$y = h(1 - t/T)^2.$$

If r denotes the radius of the cylinder, then

$$V(y) = \pi r^2 y = \pi r^2 h(1 - t/T)^2 = V_0(1 - t/T)^2.$$

58. Since $x = y^{3/4}$, the cross-sectional area is $A(y) = \pi x^2 = \pi y^{3/2}$. Hence the general equation $A(y)\,y' = -a\sqrt{2gy}$ reduces to the differential equation $yy' = -k$ with general solution

$$(1/2)y^2 = -kt + C.$$

The initial condition $y(0) = 12$ gives $C = 72$, and then $y(1) = 6$ yields $k = 54$. Upon separating variables and integrating, we find that the the depth at time t is

$$y(t) = \sqrt{144 - 108t}\,y(t).$$

Hence the tank is empty after $t = 144/108$ hr, that is, at 1:20 p.m.

59. **(a)** Since $x^2 = by$, the cross-sectional area is $A(y) = \pi x^2 = \pi by$. Hence the equation $A(y)y' = -a\sqrt{2gy}$ reduces to the differential equation

$$y^{1/2}y' = -k = -(a/\pi b)\sqrt{2g}$$

with the general solution

$$(2/3)y^{3/2} = -kt + C.$$

The initial condition $y(0) = 4$ gives $C = 16/3$, and then $y(1) = 1$ yields $k = 14/3$. It follows that the depth at time t is

$$y(t) = (8 - 7t)^{2/3}.$$

(b) The tank is empty after $t = 8/7$ hr, that is, at 1:08:34 p.m.

(c) We see above that $k = (a/\pi b)\sqrt{2g} = 14/3$. Substitution of $a = \pi r^2,\ b = 1,$

$g = (32)(3600)^2$ ft/hr^2 yields $r = (1/60)\sqrt{7/12}$ ft ≈ 0.15 in for the radius of the bottom-hole.

60. With $g = 32$ ft/sec^2 and $a = \pi(1/12)^2$, Equation (24) simplifies to

$$A(y)\frac{dy}{dt} = -\frac{\pi}{18}\sqrt{y}.$$

If z denotes the distance from the center of the cylinder down to the fluid surface, then $y = 3 - z$ and $A(y) = 10(9 - z^2)^{1/2}$. Hence the equation above becomes

$$10(9-z^2)^{1/2}\frac{dz}{dt} = \frac{\pi}{18}(3-z)^{1/2},$$
$$180(3+z)^{1/2}dz = \pi\, dt,$$

and integration yields

$$120(3+z)^{1/2} = \pi t + C.$$

Now $z = 0$ when $t = 0$, so $C = 120(3)^{3/2}$. The tank is empty when $z = 3$ (that is, when $y = 0$) and thus after

$$t = (120/\pi)(6^{3/2} - 3^{3/2}) \approx 362.90 \text{ sec.}$$

It therefore takes about 6 min 3 sec for the fluid to drain completely.

61. $A(y) = \pi(8y - y^2)$ as in Example 7 in the text, but now $a = \pi/144$ in Equation (24), so the initial value problem is

$$18(8y - y^2)y' = -\sqrt{y}, \qquad y(0) = 8.$$

We seek the value of t when $y = 0$. The answer is $t \approx 869$ sec $=$ 14 min 29 sec.

62. The cross-sectional area function for the tank is $A = \pi(1 - y^2)$ and the area of the bottom-hole is $a = 10^{-4}\pi$, so Eq. (24) in the text gives the initial value problem

$$\pi(1-y^2)\frac{dy}{dt} = -10^{-4}\pi\sqrt{2\times 9.8y}, \quad y(0) = 1.$$

Simplification gives

$$\left(y^{-1/2} - y^{3/2}\right)\frac{dy}{dt} = -1.4\times 10^{-4}\sqrt{10}$$

so integration yields

$$2y^{1/2}-\frac{2}{5}y^{5/2} = -1.4\times 10^{-4}\sqrt{10}\,t+C.$$

The initial condition $y(0)=1$ implies that $C = 2-2/5 = 8/5$, so $y=0$ after $t = (8/5)/(1.4\times 10^{-4}\sqrt{10}) \approx 3614$ seconds. Thus the tank is empty at about 14 seconds after 2 pm.

63. **(a)** As in Example 8, the initial value problem is

$$\pi(8y-y^2)\frac{dy}{dt} = -\pi k\sqrt{y}, \qquad y(0)=4$$

where $k = 0.6r^2\sqrt{2g} = 4.8r^2$. Integrating and applying the initial condition just in the Example 8 solution in the text, we find that

$$\frac{16}{3}y^{3/2}-\frac{2}{5}y^{5/2} = -kt+\frac{448}{15}.$$

When we substitute $y = 2$ (ft) and $t = 1800$ (sec, that is, 30 min), we find that $k \approx 0.009469$. Finally, $y = 0$ when

$$t = \frac{448}{15k} \approx 3154 \text{ sec} = 53 \text{ min } 34 \text{ sec}.$$

Thus the tank is empty at 1:53:34 pm.

(b) The radius of the bottom-hole is

$$r = \sqrt{k/4.8} \approx 0.04442 \text{ ft} \approx 0.53 \text{ in, thus about a half inch.}$$

64. The given rate of fall of the water level is $dy/dt = -4$ in/hr $= -(1/10800)$ ft/sec. With $A = \pi x^2$ and $a = \pi r^2$, Equation (24) is

$$(\pi x^2)(1/10800) = -(\pi r^2)\sqrt{2gy} = -8\pi r^2\sqrt{y}.$$

Hence the curve is of the form $y = kx^4$, and in order that it pass through $(1, 4)$ we must have $k = 4$. Comparing $\sqrt{y} = 2x^2$ with the equation above, we see that

$$(8r^2)(10800) = 1/2,$$

so the radius of the bottom hole is $r = 1/(240\sqrt{3})$ ft $\approx 1/35$ in.

65. Let $t = 0$ at the time of death. Then the solution of the initial value problem

$$T' = k(70 - T), \qquad T(0) = 98.6$$

is

$$T(t) = 70 + 28.6e^{-kt}.$$

If $t = a$ at 12 noon, then we know that

$$T(t) = 70 + 28.6e^{-ka} = 80,$$

$$T(a+1) = 70 + 28.6e^{-k(a+1)} = 75.$$

Hence

$$28.6e^{-ka} = 10 \quad \text{and} \quad 28.6e^{-ka}e^{-k} = 5.$$

It follows that $e^{-k} = 1/2$, so $k = \ln 2$. Finally the first of the previous two equations yields

$$a = (\ln 2.86)/(\ln 2) \approx 1.516 \text{ hr} \approx 1 \text{ hr } 31 \text{ min},$$

so the death occurred at 10:29 a.m.

66. Let $t = 0$ when it began to snow, and $t = t_0$ at 7:00 a.m. Let x denote distance along the road, with $x = 0$ where the snowplow begins at 7:00 a.m. If $y = ct$ is the snow depth at time t, w is the width of the road, and $v = dx/dt$ is the plow's velocity, then "plowing at a constant rate" means that the product wyv is constant. Hence our differential equation is of the form

$$k\frac{dx}{dt} = \frac{1}{t}.$$

The solution with $x = 0$ when $t = t_0$ is

$$t = t_0 e^{kx}.$$

We are given that $x = 2$ when $t = t_0 + 1$ and $x = 4$ when $t = t_0 + 3$, so it follows that

$$t_0 + 1 = t_0 e^{2k} \quad \text{and} \quad t_0 + 3 = t_0 e^{4k}.$$

Elimination of t_0 yields the equation

$$e^{4k} - 3e^{2k} + 2 = (e^{2k} - 1)(e^{2k} - 2) = 0,$$

so it follows (since $k > 0$) that $e^{2k} = 2$. Hence $t_0 + 1 = 2t_0$, so $t_0 = 1$. Thus it began to snow at 6 a.m.

67. We still have $t = t_0 e^{kx}$, but now the given information yields the conditions

$$t_0 + 1 = t_0 e^{4k} \quad \text{and} \quad t_0 + 2 = t_0 e^{7k}$$

at 8 a.m. and 9 a.m., respectively. Elimination of t_0 gives the equation

$$2e^{4k} - e^{7k} - 1 = 0,$$

which we solve numerically for $k = 0.08276$. Using this value, we finally solve one of the preceding pair of equations for $t_0 = 2.5483$ hr $\approx$ 2 hr 33 min. Thus it began to snow at 4:27 a.m.

68. (a) Note first that if θ denotes the angle between the tangent line and the horizontal, then $\alpha = \frac{\pi}{2} - \theta$ so $\cot\alpha = \cot(\frac{\pi}{2} - \theta) = \tan\theta = y'(x)$. It follows that

$$\sin\alpha = \frac{\sin\alpha}{\sqrt{\sin^2\alpha + \cos^2\alpha}} = \frac{1}{\sqrt{1+\cot^2\alpha}} = \frac{1}{\sqrt{1+y'(x)^2}}.$$

Therefore the mechanical condition $(\sin\alpha)/v = \text{constant (positive)}$ with $v = \sqrt{2gy}$ translates to

$$\frac{1}{\sqrt{2gy}\sqrt{1+(y')^2}} = \text{constant, so} \quad y[1+(y')^2] = 2a$$

for some positive constant a. We readily solve the latter equation for the differential equation

$$y' = \frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}.$$

(b) The substitution $y = 2a\sin^2 t,\ dy = 4a\sin t\cos t\, dt$ now gives

$$4a\sin t\cos t\, dt = \sqrt{\frac{2a - 2a\sin^2 t}{2a\sin^2 t}}\, dx = \frac{\cos t}{\sin t}\, dx,$$
$$dx = 4a\sin^2 t\, dt.$$

Integration now gives

$$x = \int 4a\sin^2 t\, dt = 2a\int(1-\cos 2t)\, dt$$
$$= 2a(t - \tfrac{1}{2}\sin 2t) + C = a(2t - \sin 2t) + C,$$

and we recall that $y = 2a\sin^2 t = a(1-\cos 2t)$. The requirement that $x = 0$ when $t = 0$

implies that $C = 0$. Finally, the substitution $\theta = 2t$ (nothing to do with the previously mentioned angle θ of inclination from the horizontal) yields the desired parametric equations

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta)$$

of the cycloid that is generated by a point on the rim of a circular wheel of radius a as it rolls along the x-axis. [See Example 5 in Section 10.4 of Edwards and Penney, *Calculus*, 6th edition (Upper Saddle River, NJ: Prentice Hall, 2002).]

69. Substitution of $v = dy/dx$ in the differential equation for $y = y(x)$ gives

$$a\frac{dv}{dx} = \sqrt{1+v^2},$$

and separation of variables then yields

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{a}; \quad \sinh^{-1} v = \frac{x}{a} + C_1; \quad \frac{dy}{dx} = \sinh\left(\frac{x}{a} + C_1\right).$$

The fact that $y'(0) = 0$ implies that $C_1 = 0$, so it follows that

$$\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right); \qquad y(x) = a\cosh\left(\frac{x}{a}\right) + C.$$

Of course the (vertical) position of the x-axis can be adjusted so that $C = 0$, and the units in which T and ρ are measured may be adjusted so that $a = 1$. In essence, then the shape of the hanging cable is the hyperbolic cosine graph $y = \cosh x$.

SECTION 1.5

LINEAR FIRST-ORDER EQUATIONS

1. $\rho = \exp\left(\int 1\,dx\right) = e^x; \quad D_x\left(y \cdot e^x\right) = 2e^x; \quad y \cdot e^x = 2e^x + C; \quad y(x) = 2 + Ce^{-x}$

$y(0) = 0$ implies $C = -2$ so $y(x) = 2 - 2e^{-x}$

2. $\rho = \exp\left(\int (-2)\,dx\right) = e^{-2x}; \quad D_x\left(y \cdot e^{-2x}\right) = 3; \quad y \cdot e^{-2x} = 3x + C; \quad y(x) = (3x + C)e^{2x}$

$y(0) = 0$ implies $C = 0$ so $y(x) = 3x\,e^{2x}$

3. $\rho = \exp\left(\int 3\,dx\right) = e^{3x}; \quad D_x\left(y \cdot e^{3x}\right) = 2x; \quad y \cdot e^{3x} = x^2 + C; \quad y(x) = (x^2 + C)e^{-3x}$

4. $\rho = \exp\left(\int(-2x)\,dx\right) = e^{-x^2}$; $\quad D_x\left(y \cdot e^{-x^2}\right) = 1$; $\quad y \cdot e^{-x^2} = x + C$; $\quad y(x) = (x+C)e^{x^2}$

5. $\rho = \exp\left(\int(2/x)\,dx\right) = e^{2\ln x} = x^2$; $\quad D_x\left(y \cdot x^2\right) = 3x^2$; $\quad y \cdot x^2 = x^3 + C$

$y(x) = x + C/x^2$; $\quad y(1) = 5$ implies $C = 4$ so $y(x) = x + 4/x^2$

6. $\rho = \exp\left(\int(5/x)\,dx\right) = e^{5\ln x} = x^5$; $\quad D_x\left(y \cdot x^5\right) = 7x^6$; $\quad y \cdot x^5 = x^7 + C$

$y(x) = x^2 + C/x^5$; $\quad y(2) = 5$ implies $C = 32$ so $y(x) = x^2 + 32/x^5$

7. $\rho = \exp\left(\int(1/2x)\,dx\right) = e^{(\ln x)/2} = \sqrt{x}$; $\quad D_x\left(y \cdot \sqrt{x}\right) = 5$; $\quad y \cdot \sqrt{x} = 5x + C$

$y(x) = 5\sqrt{x} + C/\sqrt{x}$

8. $\rho = \exp\left(\int(1/3x)\,dx\right) = e^{(\ln x)/3} = \sqrt[3]{x}$; $\quad D_x\left(y \cdot \sqrt[3]{x}\right) = 4\sqrt[3]{x}$; $\quad y \cdot \sqrt[3]{x} = 3x^{4/3} + C$

$y(x) = 3x + Cx^{-1/3}$

9. $\rho = \exp\left(\int(-1/x)\,dx\right) = e^{-\ln x} = 1/x$; $\quad D_x\left(y \cdot 1/x\right) = 1/x$; $\quad y \cdot 1/x = \ln x + C$

$y(x) = x\ln x + Cx$; $\quad y(1) = 7$ implies $C = 7$ so $y(x) = x\ln x + 7x$

10. $\rho = \exp\left(\int(-3/2x)\,dx\right) = e^{(-3\ln x)/2} = x^{-3/2}$;

$D_x\left(y \cdot x^{-3/2}\right) = 9x^{1/2}/2$; $\quad y \cdot x^{-3/2} = 3x^{3/2} + C$; $\quad y(x) = 3x^3 + Cx^{3/2}$

11. $\rho = \exp\left(\int(1/x - 3)\,dx\right) = e^{\ln x - 3x} = x e^{-3x}$; $\quad D_x\left(y \cdot x e^{-3x}\right) = 0$; $\quad y \cdot x e^{-3x} = C$

$y(x) = C x^{-1} e^{3x}$; $\quad y(1) = 0$ implies $C = 0$ so $y(x) \equiv 0$ (constant)

12. $\rho = \exp\left(\int(3/x)\,dx\right) = e^{3\ln x} = x^3$; $\quad D_x\left(y \cdot x^3\right) = 2x^7$; $\quad y \cdot x^3 = \tfrac{1}{4}x^8 + C$

$y(x) = \tfrac{1}{4}x^5 + Cx^{-3}$; $\quad y(2) = 1$ implies $C = 56$ so $y(x) = \tfrac{1}{4}x^5 + 56x^{-3}$

13. $\rho = \exp\left(\int 1\,dx\right) = e^x$; $\quad D_x\left(y \cdot e^x\right) = e^{2x}$; $\quad y \cdot e^x = \tfrac{1}{2}e^{2x} + C$

$y(x) = \tfrac{1}{2}e^x + Ce^{-x}$; $\quad y(0) = 1$ implies $C = \tfrac{1}{2}$ so $y(x) = \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x}$

14. $\rho = \exp\left(\int(-3/x)\,dx\right) = e^{-3\ln x} = x^{-3}$; $\quad D_x\left(y \cdot x^{-3}\right) = x^{-1}$; $\quad y \cdot x^{-3} = \ln x + C$

$$y(x) = x^3 \ln x + C x^3; \quad y(1) = 10 \text{ implies } C = 10 \text{ so } y(x) = x^3 \ln x + 10x^3$$

15. $\rho = \exp\left(\int 2x\,dx\right) = e^{x^2}; \quad D_x\left(y \cdot e^{x^2}\right) = x e^{x^2}; \quad y \cdot e^{x^2} = \tfrac{1}{2}e^{x^2} + C$

$$y(x) = \tfrac{1}{2} + C e^{-x^2}; \quad y(0) = -2 \text{ implies } C = -\tfrac{5}{2} \text{ so } y(x) = \tfrac{1}{2} - \tfrac{5}{2}e^{-x^2}$$

16. $\rho = \exp\left(\int \cos x\,dx\right) = e^{\sin x}; \quad D_x\left(y \cdot e^{\sin x}\right) = e^{\sin x}\cos x; \quad y \cdot e^{\sin x} = e^{\sin x} + C$

$$y(x) = 1 + C e^{-\sin x}; \quad y(\pi) = 2 \text{ implies } C = 1 \text{ so } y(x) = 1 + e^{-\sin x}$$

17. $\rho = \exp\left(\int 1/(1+x)\,dx\right) = e^{\ln(1+x)} = 1 + x; \quad D_x\left(y \cdot (1+x)\right) = \cos x; \quad y \cdot (1+x) = \sin x + C$

$$y(x) = \frac{C + \sin x}{1+x}; \quad y(0) = 1 \text{ implies } C = 1 \text{ so } y(x) = \frac{1 + \sin x}{1+x}$$

18. $\rho = \exp\left(\int (-2/x)\,dx\right) = e^{-2\ln x} = x^{-2}; \quad D_x\left(y \cdot x^{-2}\right) = \cos x; \quad y \cdot x^{-2} = \sin x + C$

$$y(x) = x^2\left(\sin x + C\right)$$

19. $\rho = \exp\left(\int \cot x\,dx\right) = e^{\ln(\sin x)} = \sin x; \quad D_x\left(y \cdot \sin x\right) = \sin x \cos x$

$$y \cdot \sin x = \tfrac{1}{2}\sin^2 x + C; \quad y(x) = \tfrac{1}{2}\sin x + C\csc x$$

20. $\rho = \exp\left(\int (-1-x)\,dx\right) = e^{-x-x^2/2}; \quad D_x\left(y \cdot e^{-x-x^2/2}\right) = (1+x)e^{-x-x^2/2}$

$$y \cdot e^{-x-x^2/2} = -e^{-x-x^2/2} + C; \quad y(x) = -1 + C e^{x+x^2/2}$$

$$y(0) = 0 \text{ implies } C = 1 \text{ so } y(x) = -1 + e^{x+x^2/2}$$

21. $\rho = \exp\left(\int (-3/x)\,dx\right) = e^{-3\ln x} = x^{-3}; \quad D_x\left(y \cdot x^{-3}\right) = \cos x; \quad y \cdot x^{-3} = \sin x + C$

$$y(x) = x^3 \sin x + C x^3; \quad y(2\pi) = 0 \text{ implies } C = 0 \text{ so } y(x) = x^3 \sin x$$

22. $\rho = \exp\left(\int (-2x)\,dx\right) = e^{-x^2}; \quad D_x\left(y \cdot e^{-x^2}\right) = 3x^2; \quad y \cdot e^{-x^2} = x^3 + C$

$$y(x) = \left(x^3 + C\right)e^{x^2}; \quad y(0) = 5 \text{ implies } C = 5 \text{ so } y(x) = \left(x^3 + 5\right)e^{x^2}$$

23. $\rho = \exp\left(\int (2 - 3/x)\,dx\right) = e^{2x-3\ln x} = x^{-3}e^{2x}; \quad D_x\left(y \cdot x^{-3}e^{2x}\right) = 4e^{2x}$

$$y \cdot x^{-3}e^{2x} = 2e^{2x} + C; \quad y(x) = 2x^3 + C x^3 e^{-2x}$$

24. $\rho = \exp\left(\int 3x/(x^2+4)\,dx\right) = e^{3\ln(x^2+4)/2} = (x^2+4)^{3/2}; \quad D_x\left(y\cdot(x^2+4)^{3/2}\right) = x(x^2+4)^{1/2}$

$y\cdot(x^2+4)^{3/2} = \frac{1}{3}(x^2+4)^{3/2} + C; \quad y(x) = \frac{1}{3} + C(x^2+4)^{-3/2}$

$y(0)=1$ implies $C = \frac{16}{3}$ so $y(x) = \frac{1}{3}\left[1+16(x^2+4)^{-3/2}\right]$

25. First we calculate

$$\int \frac{3x^3\,dx}{x^2+1} = \int\left[3x - \frac{3x}{x^2+1}\right]dx = \frac{3}{2}\left[x^2 - \ln(x^2+1)\right].$$

It follows that $\rho = (x^2+1)^{-3/2}\exp(3x^2/2)$ and thence that

$$\begin{aligned} D_x\left(y\cdot(x^2+1)^{-3/2}\exp(3x^2/2)\right) &= 6x(x^2+4)^{-5/2}, \\ y\cdot(x^2+1)^{-3/2}\exp(3x^2/2) &= -2(x^2+4)^{-3/2}+C, \\ y(x) &= -2\exp(3x^2/2) + C(x^2+1)^{3/2}\exp(-3x^2/2). \end{aligned}$$

Finally, $y(0)=1$ implies that $C=3$ so the desired particular solution is

$$y(x) = -2\exp(3x^2/2) + 3(x^2+1)^{3/2}\exp(-3x^2/2).$$

26. With $x' = dx/dy$, the differential equation is $y^3x' + 4y^2x = 1$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int (4/y)\,dy\right) = e^{4\ln y} = y^4; \quad D_y\left(x\cdot y^4\right) = y$$

$$x\cdot y^4 = \frac{1}{2}y^2 + C; \quad x(y) = \frac{1}{2y^2} + \frac{C}{y^4}$$

27. With $x' = dx/dy$, the differential equation is $x' - x = ye^y$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int (-1)\,dy\right) = e^{-y}; \quad D_y\left(x\cdot e^{-y}\right) = y$$

$$x\cdot e^{-y} = \frac{1}{2}y^2 + C; \quad x(y) = \left(\frac{1}{2}y^2 + C\right)e^y$$

28. With $x' = dx/dy$, the differential equation is $(1+y^2)x' - 2yx = 1$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int(-2y/(1+y^2)dy\right) = e^{-\ln(y^2+1)} = (1+y^2)^{-1}$$

$$D_y\left(x\cdot(1+y^2)^{-1}\right) = (1+y^2)^{-2}$$

An integral table (or trigonometric substitution) now yields

$$\frac{x}{1+y^2} = \int\frac{dy}{\left(1+y^2\right)^2} = \frac{1}{2}\left(\frac{y}{1+y^2}+\tan^{-1}y+C\right)$$

$$x(y) = \tfrac{1}{2}\left[y+\left(1+y^2\right)\left(\tan^{-1}y+C\right)\right]$$

29. $\rho=\exp\left(\int(-2x)dx\right)=e^{-x^2};\quad D_x\left(y\cdot e^{-x^2}\right)=e^{-x^2};\quad y\cdot e^{-x^2}=C+\int_0^x e^{-t^2}\,dt$

$$y(x) = e^{x^2}\left(C+\tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x)\right)$$

30. After division of the given equation by $2x$, multiplication by the integrating factor $\rho = x^{-1/2}$ yields

$$x^{-1/2}y'-\tfrac{1}{2}x^{-3/2}y = x^{-1/2}\cos x,$$
$$D_x\left(x^{-1/2}y\right) = x^{-1/2}\cos x,$$
$$x^{-1/2}y = C+\int_1^x t^{-1/2}\cos t\,dt.$$

The initial condition $y(1) = 0$ implies that $C = 0$, so the desired particular solution is

$$y(x) = x^{1/2}\int_1^x t^{-1/2}\cos t\,dt.$$

31. **(a)** $y_c' = Ce^{-\int P\,dx}(-P) = -P\,y_c$, so $y_c'+P\,y_c = 0.$

(b) $y_p' = (-P)e^{-\int P\,dx}\cdot\left[\int\left(Qe^{\int P\,dx}\right)dx\right]+e^{-\int P\,dx}\cdot Qe^{\int P\,dx} = -Py_p+Q$

32. **(a)** If $y = A\cos x+B\sin x$ then

$$y'+y = (A+B)\cos x+(B-A)\sin x = 2\sin x$$

provided that $A=-1$ and $B=1$. These coefficient values give the particular solution $y_p(x) = \sin x-\cos x$.

(b) The general solution of the equation $y' + y = 0$ is $y(x) = Ce^{-x}$ so addition to the particular solution found in part (a) gives $y(x) = Ce^{-x} + \sin x - \cos x$.

(c) The initial condition $y(0) = 1$ implies that $C = 2$, so the desired particular solution is $y(x) = 2e^{-x} + \sin x - \cos x$.

33. The amount $x(t)$ of salt (in kg) after t seconds satisfies the differential equation $x' = -x/200$, so $x(t) = 100e^{-t/200}$. Hence we need only solve the equation $10 = 100e^{-t/200}$ for $t = 461$ sec $= 7$ min 41 sec (approximately).

34. Let $x(t)$ denote the amount of pollutants in the lake after t days, measured in millions of cubic feet (mft^3). The volume of the lake is 8000 mft^3, and the initial amount $x(0)$ of pollutants is $x_0 = (0.25\%)(8000) = 20$ mft^3. We want to know when $x(t) = (0.10\%)(8000) = 8$ mft^3. We set up the differential equation in infinitesimal form by writing

$$dx = [\text{in}] - [\text{out}] = (0.0005)(500)\,dt - \frac{x}{8000}\cdot 500\,dt,$$

which simplifies to

$$\frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}, \quad \text{or} \quad \frac{dx}{dt} + \frac{1}{16}x = \frac{1}{4}.$$

Using the integrating factor $\rho = e^{t/16}$, we readily derive the solution $x(t) = 4 + 16e^{-t/16}$ for which $x(0) = 20$. Finally, we find that $x = 8$ when $t = 16\ln 4 \approx 22.2$ days.

35. The only difference from the Example 4 solution in the textbook is that $V = 1640\ \text{km}^3$ and $r = 410\ \text{km}^3/\text{yr}$ for Lake Ontario, so the time required is

$$t = \frac{V}{r}\ln 4 = 4\ln 4 \approx 5.5452 \text{ years.}$$

36. **(a)** The volume of brine in the tank after t min is $V(t) = 60 - t$ gal, so the initial value problem is

$$\frac{dx}{dt} = 2 - \frac{3x}{60-t}, \qquad x(0) = 0.$$

The solution is

$$x(t) = (60-t) - \frac{(60-t)^3}{3600}.$$

(b) The maximum amount ever in the tank is $40/\sqrt{3} \approx 23.09$ lb. This occurs after $t = 60 - 20\sqrt{3} \approx 25/36$ min.

37. The volume of brine in the tank after t min is $V(t) = 100 + 2t$ gal, so the initial value problem is

$$\frac{dx}{dt} = 5 - \frac{3x}{100+2t}, \qquad x(0) = 50.$$

The integrating factor $\rho(t) = (100 + 2t)^{3/2}$ leads to the solution

$$x(t) = (100+2t) - \frac{50000}{(100+2t)^{3/2}}.$$

such that $x(0) = 50$. The tank is full after $t = 150$ min, at which time $x(150) = 393.75$ lb.

38. **(a)** $dx/dt = -x/20$ and $x(0) = 50$ so $x(t) = 50e^{-t/20}$.

(b) The solution of the linear differential equation

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200} = \frac{5}{2}e^{-t/20} - \frac{1}{40}y$$

with $y(0) = 50$ is

$$y(t) = 150e^{-t/40} - 100e^{-t/20}.$$

(c) The maximum value of y occurs when

$$y'(t) = -\frac{15}{4}e^{-t/40} + 5e^{-t/20} = -\frac{5}{4}e^{-t/40}\left(3 - 4e^{-t/40}\right) = 0.$$

We find that $y_{\max} = 56.25$ lb when $t = 40\ln(4/3) \approx 11.51$ min.

39. **(a)** The initial value problem

$$\frac{dx}{dt} = -\frac{x}{10}, \qquad x(0) = 100$$

for Tank 1 has solution $x(t) = 100e^{-t/10}$. Then the initial value problem

$$\frac{dy}{dt} = \frac{x}{10} - \frac{y}{10} = 10e^{-t/10} - \frac{y}{10}, \qquad y(0) = 0$$

for Tank 2 has solution $y(t) = 10te^{-t/10}$.

(b) The maximum value of y occurs when

$$y'(t) = 10e^{-t/10} - te^{-t/10} = 0$$

and thus when $t = 10$. We find that $y_{\max} = y(10) = 100e^{-1} \approx 36.79$ gal.

40. **(b)** Assuming inductively that $x_n = t^n e^{-t/2}/(n!2^n)$, the equation for x_{n+1} is

$$\frac{dx_{n+1}}{dt} = \frac{1}{2}x_n - \frac{1}{2}x_{n+1} = \frac{t^n e^{-t/2}}{n!\,2^{n+1}} - \frac{1}{2}x_{n+1}.$$

We easily solve this first-order equation with $x_{n+1}(0) = 0$ and find that

$$x_{n+1} = \frac{t^{n+1}e^{-t/2}}{(n+1)!\,2^{n+1}},$$

thereby completing the proof by induction.

41. **(a)** $A'(t) = 0.06A + 0.12S = 0.06A + 3.6e^{0.05t}$

(b) The solution with $A(0) = 0$ is

$$A(t) = 360(e^{0.06t} - e^{0.05t}),$$

so $A(40) \approx 1308.283$ thousand dollars.

42. The mass of the hailstone at time t is $m = (4/3)\pi r^3 = (4/3)\pi k^3 t^3$. Then the equation $d(mv)/dt = mg$ simplifies to

$$tv' + 3v = gt.$$

The solution satisfying the initial condition $v(0) = 0$ is $v(t) = gt/4$, so $v'(t) = g/4$.

43. The solution of the initial value problem $y' = x - y$, $y(-5) = y_0$ is

$$y(x) = x - 1 + (y_0 + 6)e^{-x-5}.$$

Substituting $x = 5$, we therefore solve the equation $4 + (y_0 + 6)e^{-10} = y_1$ with $y_1 = 3.998,\ 3.999,\ 4,\ 4.001,\ 4.002$ for the desired initial values $y_0 = -50.0529,\ -28.0265,\ -6.0000,\ 16.0265,\ 38.0529$, respectively.

44. The solution of the initial value problem $y' = x + y, \quad y(-5) = y_0$ is

$$y(x) = -x - 1 + (y_0 - 4)e^{x+5}.$$

Substituting $x = 5$, we therefore solve the equation $-6 + (y_0 - 4)e^{10} = y_1$ with $y_1 = -10, -5, 0, 5, 10$ for the desired initial values $y_0 = 3.99982, \ 4.00005, \ 4.00027, \ 4.00050, \ 4.00073$, respectively.

45. With the pollutant measured in millions of liters and the reservoir water in millions of cubic meters, the inflow-outflow rate is $r = \frac{1}{5}$, the pollutant concentration in the inflow is $c_0 = 10$, and the volume of the reservoir is $V = 2$. Substituting these values in the equation $x' = rc_0 - (r/V)x$, we get the equation

$$\frac{dx}{dt} = 2 - \frac{1}{10}x$$

for the amount $x(t)$ of pollutant in the lake after t months. With the aid of the integrating factor $\rho = e^{t/10}$, we readily find that the solution with $x(0) = 0$ is

$$x(t) = 20\left(1 - e^{-t/10}\right).$$

Then we find that $x = 10$ when $t = 10 \ln 2 \approx 6.93$ months, and observe finally that, as expected, $x(t) \to 20$ as $t \to \infty$.

46. With the pollutant measured in millions of liters and the reservoir water in millions of cubic meters, the inflow-outflow rate is $r = \frac{1}{5}$, the pollutant concentration in the inflow is $c_0 = 10(1 + \cos t)$, and the volume of the reservoir is $V = 2$. Substituting these values in the equation $x' = rc_0 - (r/V)x$, we get the equation

$$\frac{dx}{dt} = 2(1 + \cos t) - \frac{1}{10}x, \qquad \text{that is,} \qquad \frac{dx}{dt} + \frac{1}{10}x = 2(1 + \cos t)$$

for the amount $x(t)$ of pollutant in the lake after t months. With the aid of the integrating factor $\rho = e^{t/10}$, we get

$$\begin{aligned} x \cdot e^{t/10} &= \int (2e^{t/10} + 2e^{t/10}\cos t)\,dt \\ &= 20e^{t/10} + 2 \cdot \frac{e^{t/10}}{(\frac{1}{10})^2 + 1^2}\left(\frac{1}{10}\cos t + \sin t\right) + C. \end{aligned}$$

When we impose the condition $x(0) = 0$, we get the desired particular solution

$$x(t) = \frac{20}{101}\left(101 - 102e^{-t/10} + \cos t + 10\sin t\right).$$

In order to determine when $x = 10$, we need to solve numerically. For instance, we can use the *Mathematica* commands

```
x = (20/101)(101 - 102 Exp[-t/10] + Cos[t] + 10 Sin[t]);
FindRoot[ x == 10, {t,7} ]

{t -> 6.474591767017537}
```

and find that this occurs after about 6.47 months. Finally, as $t \to \infty$ we observe that $x(t)$ approaches the function $20 + \frac{20}{101}(\cos t + 10\sin t)$ that does, indeed, oscillate about the equilibrium solution $x(t) \equiv 20$.

SECTION 1.6

SUBSTITUTION METHODS AND EXACT EQUATIONS

It is traditional for every elementary differential equations text to include the particular types of equations that are found in this section. However, no one of them is vitally important solely in its own right. Their main purpose (at this point in the course) is to familiarize students with the technique of transforming a differential equation by substitution. The subsection on airplane flight trajectories (together with Problems 56–59) is included as an application, but is optional material and may be omitted if the instructor desires.

The differential equations in Problems 1–15 are homogeneous, so we make the substitutions

$$v = \frac{y}{x}, \qquad y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

For each problem we give the differential equation in x, $v(x)$, and $v' = dv/dx$ that results, together with the principal steps in its solution.

1. $x(v+1)v' = -(v^2+2v-1)$; $\int \frac{2(v+1)\,dv}{v^2+2v-1} = -\int 2x\,dx$; $\ln(v^2+2v-1) = -2\ln x + \ln C$

$x^2(v^2+2v-1) = C$; $y^2 + 2xy - x^2 = C$

2. $2xvv' = 1$; $\int 2v\,dv = \int \frac{dx}{x}$; $v^2 = \ln x + C$; $y^2 = x^2(\ln x + C)$

3. $xv' = 2\sqrt{v}$; $\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$; $\sqrt{v} = \ln x + C$; $y = x(\ln x + C)^2$

4. $x(v-1)v' = -(v^2+1)$; $\quad \int \frac{2(1-v)\,dv}{v^2+1} = \int \frac{2\,dx}{x}$; $\quad 2\tan^{-1} v - \ln(v^2+1) = 2\ln x + C$

$2\tan^{-1}(y/x) - \ln(y^2/x^2+1) \;=\; 2\ln x + C$

5. $x(v+1)v' = -2v^2$; $\quad \int \left(\frac{1}{v} + \frac{1}{v^2}\right) dv = -\int \frac{2\,dx}{x}$; $\quad \ln v - \frac{1}{v} \;=\; -2\ln x + C$

$\ln y - \ln x - \frac{x}{y} \;=\; -2\ln x + C$; $\quad \ln(xy) \;=\; C + \frac{x}{y}$

6. $x(2v+1)v' = -2v^2$; $\quad \int \left(\frac{2}{v} + \frac{1}{v^2}\right) dv = -\int \frac{2\,dx}{x}$; $\quad \ln v^2 - \frac{1}{v} \;=\; -2\ln x + C$

$2\ln y - 2\ln x - \frac{x}{y} \;=\; -2\ln x + C$; $\quad 2y\ln y \;=\; x + Cy$

7. $x v^2 v' = 1$; $\quad \int 3v^2\,dv = \int \frac{3\,dx}{x}$; $\quad v^3 \;=\; 3\ln x + C$; $\quad y^3 \;=\; x^3(3\ln x + C)$

8. $x v' = e^v$; $\quad -\int e^{-v}\,dv = -\int \frac{dx}{x}$; $\quad e^{-v} \;=\; -\ln x + C$; $\quad -v \;=\; \ln(C - \ln x)$

$y \;=\; -x\,\ln(C - \ln x)$

9. $x v' = v^2$; $\quad -\int \frac{dv}{v^2} = -\int \frac{dx}{x}$; $\quad \frac{1}{v} \;=\; -\ln x + C$; $\quad x \;=\; y(C - \ln x)$

10. $x v v' = 2v^2 + 1$; $\quad \int \frac{4v\,dv}{2v^2+1} = \int \frac{4\,dx}{x}$; $\quad \ln(2v^2+1) \;=\; 4\ln x + \ln C$

$2y^2/x^2 + 1 \;=\; Cx^4$; $\quad x^2 + 2y^2 \;=\; Cx^6$

11. $x(1-v^2)v' = v + v^3$; $\quad \int \frac{1-v^2}{v^3+v}\,dv = \int \frac{dx}{x}$; $\quad \int \left(\frac{1}{v} - \frac{2v}{v^2+1}\right) dv = \int \frac{dx}{x}$

$\ln v - \ln(v^2+1) \;=\; \ln x + \ln C$; $\quad v \;=\; Cx(v^2+1)$; $\quad y \;=\; C(x^2+y^2)$

12. $x v v' = \sqrt{v^2+4}$; $\quad \int \frac{v\,dv}{\sqrt{v^2+4}} = \int \frac{dx}{x}$; $\quad \sqrt{v^2+4} \;=\; \ln x + C$

$v^2 + 4 \;=\; (\ln x + C)^2$; $\quad 4x^2 + y^2 \;=\; x^2(\ln x + C)^2$

13. $x\,v' = \sqrt{v^2+1}; \quad \int \frac{dv}{\sqrt{v^2+1}} = \int \frac{dx}{x}; \quad \ln\left(v+\sqrt{v^2+1}\right) = \ln x + \ln C$

$v+\sqrt{v^2+1} = Cx; \quad y+\sqrt{x^2+y^2} = Cx^2$

14. $x\,v\,v' = \sqrt{1+v^2} - \left(1+v^2\right)$

$$\begin{aligned}\ln x &= \int \frac{v\,dv}{\sqrt{1+v^2}-(1+v^2)} \\ &= \frac{1}{2}\int \frac{du}{\sqrt{u}\left(1-\sqrt{u}\right)} \qquad (u = 1+v^2) \\ &= -\int \frac{dw}{w} = -\ln w + \ln C\end{aligned}$$

with $w = 1-\sqrt{u}$. Back-substitution and simplification finally yields the implicit solution $x - \sqrt{x^2+y^2} = C.$

15. $x(v+1)v' = -2\left(v^2+2v\right); \quad \int \frac{2(v+1)\,dv}{v^2+2v} = -\int \frac{4\,dx}{x}; \quad \ln\left(v^2+2v\right) = -4\ln x + \ln C$

$v^2+2v = C/x^4; \quad x^2y^2+2x^3y = C$

16. The substitution $v = x+y+1$ leads to

$$\begin{aligned}x &= \int \frac{dv}{1+\sqrt{v}} = \int \frac{2u\,du}{1+u} \qquad (v = u^2) \\ &= 2u - 2\ln(1+u) + C \\ x &= 2\sqrt{x+y+1} - 2\ln(1+\sqrt{x+y+1}) + C\end{aligned}$$

17. $v = 4x+y; \quad v' = v^2+4; \quad x = \int \frac{dv}{v^2+4} = \frac{1}{2}\tan^{-1}\frac{v}{2} + \frac{C}{2}$

$v = 2\tan(2x-C); \quad y = 2\tan(2x-C) - 4x$

18. $v = x+y; \quad v\,v' = v+1; \quad x = \int \frac{v\,dv}{v+1} = \int\left(1-\frac{1}{v+1}\right)dv = v - \ln(v+1) - C$

$y = \ln(x+y+1) + C.$

Problems 19–25 are Bernoulli equations. For each, we indicate the appropriate substitution as specified in Equation (10) of this section, the resulting linear differential equation in v, its integrating factor ρ, and finally the resulting solution of the original Bernoulli equation.

19. $v = y^{-2}; \quad v' - 4v/x = -10/x^2; \quad \rho = 1/x^4; \quad y^2 = x/\left(Cx^5 + 2\right)$

20. $v = y^3; \quad v' + 6xv = 18x; \quad \rho = e^{3x^2}; \quad y^3 = 3 + Ce^{-3x^2}$

21. $v = y^{-2}; \quad v' + 2v = -2; \quad \rho = e^{2x}; \quad y^2 = 1/\left(Ce^{-2x} - 1\right)$

22. $v = y^{-3}; \quad v' - 6v/x = -15/x^2; \quad \rho = x^{-6}; \quad y^3 = 7x/\left(7Cx^7 + 15\right)$

23. $v = y^{-1/3}; \quad v' - 2v/x = -1; \quad \rho = x^{-2}; \quad y = \left(x + Cx^2\right)^{-3}$

24. $v = y^{-2}; \quad v' + 2v = e^{-2x}/x; \quad \rho = e^{2x}; \quad y^2 = e^{2x}/(C + \ln x)$

25. $v = y^3; \quad v' + 3v/x = 3/\sqrt{1+x^4}; \quad \rho = x^3; \quad y^3 = \left(C + 3\sqrt{1+x^4}\right)/\left(2x^3\right)$

26. The substitution $v = y^3$ yields the linear equation $v' + v = e^{-x}$ with integrating factor $\rho = e^x$. Solution: $y^3 = e^{-x}(x + C)$

27. The substitution $v = y^3$ yields the linear equation $x\,v' - v = 3x^4$ with integrating factor $\rho = 1/x$. Solution: $y = (x^4 + Cx)^{1/3}$

28. The substitution $v = e^y$ yields the linear equation $x\,v' - 2v = 2x^3e^{2x}$ with integrating factor $\rho = 1/x^2$. Solution: $y = \ln(Cx^2 + x^2e^{2x})$

29. The substitution $v = \sin y$ yields the homogeneous equation $2xv\,v' = 4x^2 + v^2$. Solution: $\sin^2 y = 4x^2 - Cx$

30. First we multiply each side of the given equation by e^y. Then the substitution $v = e^y$ gives the homogeneous equation $(x + v)\,v' = x - v$ of Problem 1 above. Solution: $x^2 - 2x\,e^y - e^{2y} = C$

Each of the differential equations in Problems 31–42 is of the form $M\,dx + N\,dy = 0$, and the exactness condition $\partial M/\partial y = \partial N/\partial x$ is routine to verify. For each problem we give the principal steps in the calculation corresponding to the method of Example 9 in this section.

31. $F = \int (2x + 3y)\,dx = x^2 + 3xy + g(y); \quad F_y = 3x + g'(y) = 3x + 2y = N$

$g'(y) = 2y; \quad g(y) = y^2; \qquad x^2 + 3xy + y^2 = C$

32. $F = \int(4x-y)\,dx = 2x^2 - xy + g(y); \quad F_y = -x + g'(y) = 6y - x = N$

$g'(y) = 6y; \quad g(y) = 3y^2; \quad 2x^2 - xy + 3y^2 = C$

33. $F = \int(3x^2+2y^2)\,dx = x^3 + xy^2 + g(y); \quad F_y = 4xy + g'(y) = 4xy + 6y^2 = N$

$g'(y) = 6y^2; \quad g(y) = 2y^3; \quad x^3 + 2xy^2 + 2y^3 = C$

34. $F = \int(2xy^2+3x^2)\,dx = x^3 + x^2y^2 + g(y); \quad F_y = 2x^2y + g'(y) = 2x^2y + 4y^3 = N$

$g'(y) = 4y^3; \quad g(y) = y^4; \quad x^3 + x^2y^2 + y^4 = C$

35. $F = \int(x^3 + y/x)\,dx = \frac{1}{4}x^4 + y\ln x + g(y); \quad F_y = \ln x + g'(y) = y^2 + \ln x = N$

$g'(y) = y^2; \quad g(y) = \frac{1}{3}y^3; \quad \frac{1}{4}x^3 + \frac{1}{3}y^2 + y\ln x = C$

36. $F = \int(1 + ye^{xy})\,dx = x + e^{xy} + g(y); \quad F_y = xe^{xy} + g'(y) = 2y + xe^{xy} = N$

$g'(y) = 2y; \quad g(y) = y^2; \quad x + e^{xy} + y^2 = C$

37. $F = \int(\cos x + \ln y)\,dx = \sin x + x\ln y + g(y); \quad F_y = x/y + g'(y) = x/y + e^y = N$

$g'(y) = e^y; \quad g(y) = e^y; \quad \sin x + x\ln y + e^y = C$

38. $F = \int(x + \tan^{-1}y)\,dx = \frac{1}{2}x^2 + x\tan^{-1}y + g(y); \quad F_y = \dfrac{x}{1+y^2} + g'(y) = \dfrac{x+y}{1+y^2} = N$

$g'(y) = \dfrac{y}{1+y^2}; \quad g(y) = \frac{1}{2}\ln(1+y^2); \quad \frac{1}{2}x^2 + x\tan^{-1}y + \frac{1}{2}\ln(1+y^2) = C$

39. $F = \int(3x^2y^3 + y^4)\,dx = x^3y^3 + xy^4 + g(y);$

$F_y = 3x^3y^2 + 4xy^3 + g'(y) = 3x^3y^2 + y^4 + 4xy^3 = N$

$g'(y) = y^4; \quad g(y) = \frac{1}{5}y^5; \quad x^3y^3 + xy^4 + \frac{1}{5}y^5 = C$

40. $F = \int(e^x\sin y + \tan y)\,dx = e^x\sin y + x\tan y + g(y);$

$F_y = e^x\cos y + x\sec^2 y + g'(y) = e^x\cos y + x\sec^2 y = N$

$g'(y) = 0; \quad g(y) = 0; \quad e^x\sin y + x\tan y = C$

41. $$F = \int \left(\frac{2x}{y} - \frac{3y^2}{x^4} \right) dx = \frac{x^2}{y} + \frac{y^2}{x^3} + g(y);$$

$$F_y = -\frac{x^2}{y^2} + \frac{2y}{x^3} + g'(y) = -\frac{x^2}{y^2} + \frac{2y}{x^3} + \frac{1}{\sqrt{y}} = N$$

$$g'(y) = \frac{1}{\sqrt{y}}; \quad g(y) = 2\sqrt{y}; \qquad \frac{x^2}{y} + \frac{y^2}{x^3} + 2\sqrt{y} = C$$

42. $$F = \int \left(y^{-2/3} - \frac{3}{2} x^{-5/2} y \right) dx = x\,y^{-2/3} + x^{-3/2} y + g(y);$$

$$F_y = -\frac{2}{3} x\,y^{-5/3} + x^{-3/2} + g'(y) = x^{-3/2} - \frac{2}{3} x\,y^{-5/3} = N$$

$$g'(y) = 0; \quad g(y) = 0; \qquad x\,y^{-2/3} + x^{-3/2} y = C$$

43. The substitution $y' = p,\ y'' = p'$ in $xy'' = y'$ yields

$$xp' = p, \qquad \text{(separable)}$$

$$\int \frac{dp}{p} = \int \frac{dx}{x} \quad \Rightarrow \quad \ln p = \ln x + \ln C,$$

$$y' = p = Cx,$$

$$y(x) = \tfrac{1}{2} C x^2 + B = A x^2 + B.$$

44. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' + (y')^2 = 0$ yields

$$ypp' + p^2 = 0 \quad \Rightarrow \quad yp' = -p, \qquad \text{(separable)}$$

$$\int \frac{dp}{p} = -\int \frac{dy}{y} \quad \Rightarrow \quad \ln p = -\ln y + \ln C,$$

$$p = C/y \quad \Rightarrow \quad x = \int \frac{1}{p} dy = \int \frac{y}{C} dy$$

$$x(y) = \frac{y^2}{2C} + B = Ay^2 + B.$$

45. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' + 4y = 0$ yields

$$pp' + 4y = 0, \qquad \text{(separable)}$$

$$\int p\,dp = -\int 4y\,dy \quad \Rightarrow \quad \tfrac{1}{2} p^2 = -2y^2 + C,$$

$$p^2 = -4y^2 + 2C = 4\left(\tfrac{1}{2}C - y^2\right),$$

$$x = \int \frac{1}{p}dy = \int \frac{dy}{2\sqrt{k^2-y^2}} = \frac{1}{2}\sin^{-1}\frac{y}{k}+D,$$
$$y(x) = k\sin[2x-2D] = k(\sin 2x\cos 2D - \cos 2x \sin 2D),$$
$$y(x) = A\cos 2x + B\sin 2x.$$

46. The substitution $y' = p,\ y'' = p'$ in $xy'' + y' = 4x$ yields

$$xp' + p = 4x, \qquad \text{(linear in } p)$$
$$D_x[x\cdot p] = 4x \quad \Rightarrow \quad x\cdot p = 2x^2 + A,$$
$$p = \frac{dy}{dx} = 2x + \frac{A}{x},$$
$$y(x) = x^2 + A\ln x + B.$$

47. The substitution $y' = p,\ y'' = p'$ in $y'' = (y')^2$ yields

$$p' = p^2, \qquad \text{(separable)}$$
$$\int \frac{dp}{p^2} = \int x\,dx \quad \Rightarrow \quad -\frac{1}{p} = x + B,$$
$$\frac{dy}{dx} = -\frac{1}{x+B},$$
$$y(x) = A - \ln|x+A|.$$

48. The substitution $y' = p,\ y'' = p'$ in $x^2y'' + 3xy' = 2$ yields

$$x^2p' + 3xp = 2 \quad \Rightarrow \quad p' + \frac{3}{p}p = \frac{2}{x^2}, \qquad \text{(linear in } p)$$
$$D_x[x^3\cdot p] = 2x \quad \Rightarrow \quad x^3\cdot p = x^2 + C,$$
$$\frac{dy}{dx} = \frac{1}{x} + \frac{C}{x^3},$$
$$y(x) = \ln x + \frac{A}{x^2} + B.$$

49. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' + (y')^2 = yy'$ yields

$$yp\,p' + p^2 = yp \quad \Rightarrow \quad y\,p' + p = y \qquad \text{(linear in } p),$$
$$D_y[y\cdot p] = y,$$
$$yp = \frac{1}{2}y^2 + \frac{1}{2}C \quad \Rightarrow \quad p = \frac{y^2+C}{2y},$$

$$x = \int \frac{1}{p}\,dy = \int \frac{2y\,dy}{y^2+C} = \ln\left(y^2+C\right) - \ln B,$$

$$y^2 + C = Be^x \quad \Rightarrow \quad y(x) = \pm\left(A + Be^x\right)^{1/2}.$$

50. The substitution $y' = p,\ y'' = p'$ in $y'' = \left(x + y'\right)^2$ gives $p' = (x+p)^2$, and then the substitution $v = x + p,\ p' = v' - 1$ yields

$$v' - 1 = v^2 \quad \Rightarrow \quad \frac{dv}{dx} = 1 + v^2,$$

$$\int \frac{dv}{1+v^2} = \int dx \quad \Rightarrow \tan^{-1} v = x + A,$$

$$v = x + - = \tan(x+A) \quad \Rightarrow \quad \frac{dy}{dx} = \tan(x+A) - x,$$

$$y(x) = \ln\left|\sec(x+A)\right| - \tfrac{1}{2}x^2 + B.$$

51. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' = 2y\left(y'\right)^3$ yields

$$p\,p' = 2yp^3 \quad \Rightarrow \quad \int \frac{dp}{p^2} = \int 2y\,dy \quad \Rightarrow -\frac{1}{p} = y^2 + C,$$

$$x = \int \frac{1}{p}\,dy = -\frac{1}{3}y^3 - Cx + D,$$

$$y^3 + 3x + Ay + B = 0$$

52. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y^3 y'' = 1$ yields

$$y^3 p\,p' = 1 \quad \Rightarrow \quad \int p\,dp = \int \frac{dy}{y^3} \quad \Rightarrow \quad \frac{1}{2}p^2 = -\frac{1}{2y^2} + \frac{A}{2},$$

$$p^2 = \frac{Ay^2 - 1}{y^2} \quad \Rightarrow \quad x = \int \frac{1}{p}\,dy = \int \frac{y\,dy}{\sqrt{Ay^2-1}},$$

$$x = \frac{1}{A}\sqrt{Ay^2 - 1} + C \quad \Rightarrow \quad Ax + B = \sqrt{Ay^2-1},$$

$$Ay^2 - (Ax+B)^2 = 1.$$

53. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $y'' = 2yy'$ yields

$$p\,p' = 2yp \quad \Rightarrow \quad \int dp = \int 2y\,dy \quad \Rightarrow \quad p = y^2 + A^2,$$

$$x = \int \frac{1}{p}\,dy = \int \frac{dy}{y^2 + A^2} = \frac{1}{A}\tan^{-1}\frac{y}{A} + C,$$

$$\tan^{-1}\frac{y}{A} = A(x-C) \quad \Rightarrow \quad \frac{y}{A} = \tan(Ax - AC),$$
$$y(x) = A\tan(Ax+B).$$

54. The substitution $y' = p,\ y'' = p\,p' = p(dp/dy)$ in $yy'' = 3(y')^2$ yields

$$yp\,p' = 3p^2 \quad \Rightarrow \quad \int \frac{dp}{p} = \int \frac{3\,dy}{y}$$
$$\ln p = 3\ln y + \ln C \quad \Rightarrow \quad p = Cy^3,$$
$$x = \int \frac{1}{p}\,dy = \int \frac{dy}{Cy^3} = -\frac{1}{2Cy^2} + B,$$
$$Ay^2(B-x) = 1.$$

55. The substitution $v = ax + by + c,\ y = (v - ax - c)/b$ in $y' = F(ax + by + c)$ yields the separable differential equation $(dv/dx - a)/b = F(v)$, that is, $dv/dx = a + bF(v)$.

56. If $v = y^{1-n}$ then $y = v^{1/(1-n)}$ so $y' = v^{n/(1-n)}v'/(1-n)$. Hence the given Bernoulli equation transforms to

$$\frac{v^{n/(1-n)}}{1-n}\frac{dv}{dx} + P(x)\,v^{1/(1-n)} = Q(x)\,v^{n/(1-n)}.$$

Multiplication by $(1-n)/v^{n/(1-n)}$ then yields the linear differential equation
$v' + (1-n)Pv = (1-n)Q$

$\frac{1}{1-n} - \frac{n}{1-n}$ $\quad 1-n$

57. If $v = \ln y$ then $y = e^v$ so $y' = e^v v'$. Hence the given equation transforms to $e^v v' + P(x)\,e^v = Q(x)\,v\,e^v$. Cancellation of the factor e^v then yields the linear differential equation $v' - Q(x)\,v = P(x)$.

58. The substitution $v = \ln y,\ y = e^v,\ y' = e^v\,v'$ yields the linear equation $x\,v' + 2\,v = 4x^2$ with integrating factor $\rho = x^2$. Solution: $y = \exp(x^2 + C/x^2)$

59. The substitution $x = u - 1,\ y = v - 2$ yields the homogeneous equation

$$\frac{dv}{du} = \frac{u-v}{u+v}.$$

The substitution $v = pu$ leads to

$v' = p'u + p = \frac{u-v}{u+v}$

$p'u + p = \frac{1-p}{1+p}$

$$\ln u = -\int \frac{(p+1)\,dp}{(p^2+2p-1)} = -\frac{1}{2}\left[\ln\left(p^2+2p-1\right) - \ln C\right].$$

We thus obtain the implicit solution

$$\begin{aligned}
u^2\left(p^2+2p-1\right) &= C \\
u^2\left(\frac{v^2}{u^2}+2\frac{v}{u}-1\right) &= v^2+2uv-u^2 = C \\
(y+2)^2+2(x+1)(y+2)-(x+1)^2 &= C \\
y^2+2xy-x^2+2x+6y &= C.
\end{aligned}$$

60. The substitution $x = u-3,\ y = v-2$ yields the homogeneous equation

$$\frac{dv}{du} = \frac{-u+2v}{4u-3v}.$$

The substitution $v = pu$ leads to

$$\begin{aligned}
\ln u &= \int \frac{(4-3p)\,dp}{(3p+1)(p-1)} = \frac{1}{4}\int\left(\frac{1}{p-1}-\frac{15}{3p+1}\right)dp \\
&= \frac{1}{4}\left[\ln(p-1)-5\ln(3p+1)+\ln C\right].
\end{aligned}$$

We thus obtain the implicit solution

$$\begin{aligned}
u^4 &= \frac{C(p-1)}{(3p+1)^5} = \frac{C(v/u-1)}{(3v/u+1)^5} = \frac{Cu^4(v-u)}{(3v+u)^5} \\
(3v+u)^5 &= C(v-u) \\
(x+3y+3)^5 &= C(y-x-5).
\end{aligned}$$

61. The substitution $v = x-y$ yields the separable equation $v' = 1-\sin v$. With the aid of the identity

$$\frac{1}{1-\sin v} = \frac{1+\sin v}{\cos^2 v} = \sec^2 v + \sec v\,\tan v$$

we obtain the solution

$$x = \tan(x-y) + \sec(x-y) + C.$$

62. The substitution $y = vx$ in the given homogeneous differential equation yields the separable equation $x(2v^3 - 1)v' = -(v^4 + v)$ that we solve as follows:

$$\int \frac{2v^3 - 1}{v^4 + v}\,dv = -\int \frac{dx}{x}$$

$$\int \left(\frac{2v-1}{v^2 - v + 1} - \frac{1}{v} + \frac{1}{v+1} \right) dv = -\int \frac{dx}{x} \qquad \text{(partial fractions)}$$

$$\ln(v^2 - v + 1) - \ln v + \ln(v+1) = -\ln x + \ln C$$

$$x(v^2 - v + 1)(v+1) = Cv$$

$$(y^2 - xy + x^2)(x + y) = Cxy$$

$$x^3 + y^3 = Cxy$$

63. If we substitute $y = y_1 + 1/v$, $y' = y_1' - v'/v^2$ (primes denoting differentiation with respect to x) into the Riccati equation $y' = Ay^2 + By + C$ and use the fact that $y_1' = Ay_1^2 + By_1 + C$, then we immediately get the linear differential equation $v' + (B + 2Ay_1)v = -A$.

In Problems 64 and 65 we outline the application of the method of Problem 63 to the given Riccati equation.

64. The substitution $y = x + 1/v$ yields the linear equation $v' - 2xv = 1$ with integrating factor $\rho = e^{-x^2}$. In Problem 29 of Section 1.5 we saw that the general solution of this linear equation is $v(x) = e^{x^2}\left[C + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]$ in terms of the *error function* erf(x) introduced there Hence the general solution of our Riccati equation is given by $y(x) = x + e^{-x^2}\left[C + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right]^{-1}$.

65. The substitution $y = x + 1/v$ yields the trivial linear equation $v' = -1$ with immediate solution $v(x) = C - x$. Hence the general solution of our Riccati equation is given by $y(x) = x + 1/(C - x)$.

66. The substitution $y' = C$ in the Clairaut equation immediately yields the general solution $y = Cx + g(C)$.

67. Clearly the line $y = Cx - C^2/4$ and the tangent line at $(C/2, C^2/4)$ to the parabola $y = x^2$ both have slope C.

68. $\ln\left(v+\sqrt{1+v^2}\right) = -k\ln x + k\ln a = \ln(x/a)^{-k}$

$$v+\sqrt{1+v^2} = (x/a)^{-k}$$

$$\left[(x/a)^{-k} - v\right]^2 = 1+v^2$$

$$(x/a)^{-2k} - 2v(x/a)^{-k} + v^2 = 1+v^2$$

$$v = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-2k} - 1\right] / \left(\frac{x}{a}\right)^{-k} = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right]$$

69. With $a = 100$ and $k = 1/10$, Equation (19) in the text is

$$y = 50[(x/100)^{9/10} - (x/100)^{11/10}].$$

The equation $y'(x) = 0$ then yields

$$(x/100)^{1/10} = (9/11)^{1/2},$$

so it follows that

$$y_{\max} = 50[(9/11)^{9/2} - (9/11)^{11/2}] \approx 3.68 \text{ mi.}$$

70. With $k = w/v_0 = 10/500 = 1/10$, Eq. (16) in the text gives

$$\ln\left(v+\sqrt{1+v^2}\right) = -\frac{1}{10}\ln x + C$$

where $v = y/x$. Substitution of $x = 200, y = 150, v = 3/4$ yields $C = \ln\left(2\cdot 200^{1/10}\right)$, thence

$$\ln\left(\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right) = -\frac{1}{10}\ln x + \ln\left(2\cdot 200^{1/10}\right),$$

which — after exponentiation and then multiplication of the resulting equation by x — simplifies as desired to $y+\sqrt{x^2+y^2} = 2\left(200x^9\right)^{1/10}$. If $x = 0$ then this equation yields $y = 0$, thereby verifying that the airplane reaches the airport at the origin.

71. **(a)** With $a = 100$ and $k = w/v_0 = 2/4 = 1/2$, the solution given by equation (19) in the textbook is $y(x) = 50[(x/100)^{1/2} - (x/100)^{3/2}]$. The fact that $y(0) = 0$ means that this trajectory goes through the origin where the tree is located.

(b) With $k = 4/4 = 1$ the solution is $y(x) = 50[1-(x/100)^2]$ and we see that the swimmer hits the bank at a distance $y(0) = 50$ north of the tree.

(c) With $k = 6/4 = 1$ the solution is $y(x) = 50[(x/100)^{-1/2} - (x/100)^{5/2}]$. This trajectory is asymptotic to the positive x-axis, so we see that the swimmer never reaches the west bank of the river.

72. The substitution $y' = p,\ y'' = p'$ in $ry'' = [1+(y')^2]^{3/2}$ yields

$$rp' = (1+p^2)^{3/2} \quad \Rightarrow \quad \int \frac{rp\,dp}{(1+p^2)^{3/2}} = \int dx.$$

Now integral formula #52 in the back of our favorite calculus textbook gives

$$\frac{rp}{\sqrt{1+p^2}} = x-a \quad \Rightarrow \quad r^2p^2 = (1+p^2)(x-a)^2,$$

and we solve readily for

$$p^2 = \frac{(x-a)^2}{r^2-(x-a)^2} \quad \Rightarrow \quad \frac{dy}{dx} = p = \frac{x-a}{\sqrt{r^2-(x-a)^2}},$$

whence

$$y = \int \frac{(x-a)\,dx}{\sqrt{r^2-(x-a)^2}} = -\sqrt{r^2-(x-a)^2}+b,$$

which finally gives $(x-a)^2+(y-b)^2 = r^2$ as desired.

SECTION 1.7

POPULATION MODELS

Section 1.7 introduces the first of the two major classes of mathematical models studied in the textbook.

In Problems 1–8 we outline the derivation of the desired particular solution, and then sketch some typical solution curves.

1. Noting that $x > 1$ because $x(0) = 2$, we write

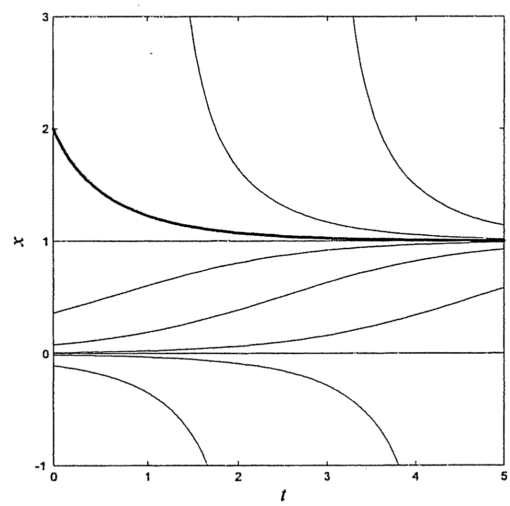
$$\int \frac{dx}{x(1-x)} = \int 1\,dt; \qquad \int\left(\frac{1}{x}-\frac{1}{x-1}\right)dx = \int 1\,dt$$

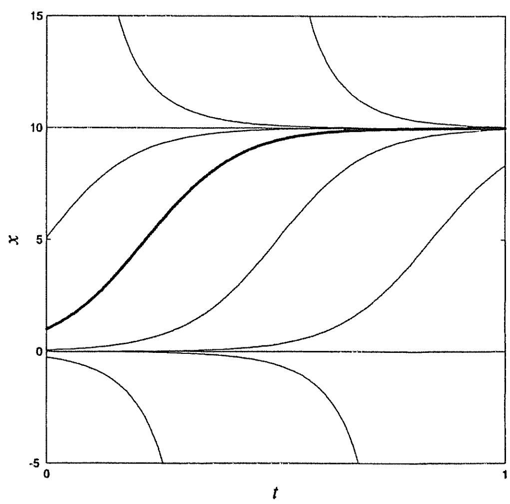
$$\ln x - \ln(x-1) \;=\; t + \ln C; \qquad \frac{x}{x-1} \;=\; C\,e^{t}$$

$$x(0) = 2 \quad \text{implies} \quad C = 2; \qquad x \;=\; 2(x-1)\,e^{t}$$

$$x(t) \;=\; \frac{2\,e^{t}}{2e^{t}-1} \;=\; \frac{2}{2-e^{-t}}.$$

Typical solution curves are shown in the figure on the left below.





2. Noting that $x < 10$ because $x(0) = 1$, we write

$$\int \frac{dx}{x(10-x)} \;=\; \int 1\,dt; \qquad \int\left(\frac{1}{x}+\frac{1}{10-x}\right)dx \;=\; \int 10\,dt$$

$$\ln x - \ln(10-x) \;=\; 10t + \ln C; \qquad \frac{x}{10-x} \;=\; C\,e^{10t}$$

$$x(0) = 1 \quad \text{implies} \quad C = \frac{1}{9}; \qquad 9x \;=\; (10-x)e^{10t}$$

$$x(t) \;=\; \frac{10e^{10t}}{9+e^{10t}} \;=\; \frac{10}{1+9e^{-10t}}.$$

Typical solution curves are shown in the figure on the right above.

3. Noting that $x > 1$ because $x(0) = 3$, we write

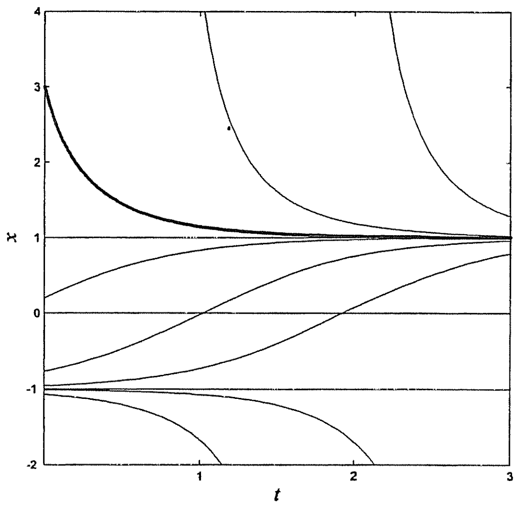
$$\int \frac{dx}{(1+x)(1-x)} \;=\; \int 1\,dt; \qquad \int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx \;=\; \int(-2)\,dt$$

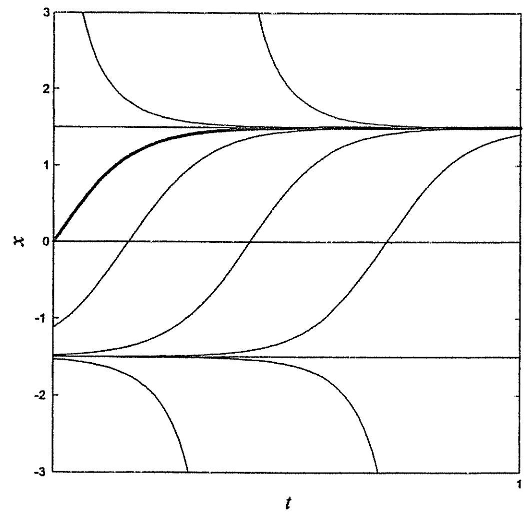
$$\ln(x-1)-\ln(x+1) = -2t+\ln C; \qquad \frac{x-1}{x+1} = Ce^{-2t}$$

$$x(0)=3 \text{ implies } C=\frac{1}{2}; \qquad 2(x-1) = (x+1)e^{-2t}$$

$$x(t) = \frac{2+e^{-2t}}{2-e^{-2t}} = \frac{2e^{2t}+1}{2e^{2t}-1}.$$

Typical solution curves are shown in the figure on the left below.





4. Noting that $|x| < \frac{3}{2}$ because $x(0)=0$, we write

$$\int \frac{dx}{(3+2x)(3-2x)} = \int 1\,dt; \qquad \int\left(\frac{1}{3+2x}+\frac{1}{3-2x}\right)dx = \int 6\,dt$$

$$\frac{1}{2}\ln(3+2x)-\frac{1}{2}\ln(3-2x) = 6t+\frac{1}{2}\ln C; \qquad \frac{3+2x}{3-2x} = Ce^{12t}$$

$$x(0)=0 \text{ implies } C=1; \qquad 3+2x = (3-2x)e^{12t}$$

$$x(t) = \frac{3e^{12t}-3}{2e^{12t}+2} = \frac{3\left(e^{12t}-1\right)}{2\left(e^{12t}+1\right)}.$$

Typical solution curves are shown in the figure on the right above.

5. Noting that $x > 5$ because $x(0)=8$, we write

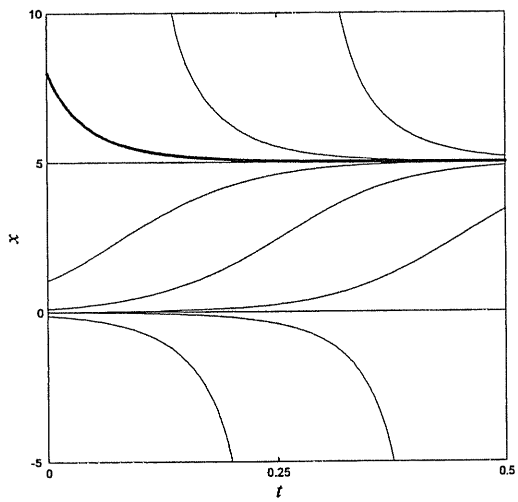
$$\int \frac{dx}{x(x-5)} = \int(-3)\,dt; \qquad \int\left(\frac{1}{x}-\frac{1}{x-5}\right)dx = \int 15\,dt$$

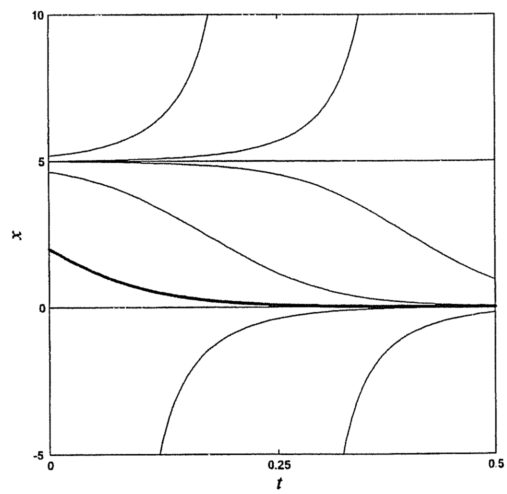
$$\ln x - \ln(x-5) = 15t + \ln C; \qquad \frac{x}{x-5} = Ce^{15t}$$

$$x(0) = 8 \text{ implies } C = 8/3; \qquad 3x = 8(x-5)e^{15t}$$

$$x(t) = \frac{-40e^{15t}}{3-8e^{15t}} = \frac{40}{8-3e^{-15t}}.$$

Typical solution curves are shown in the figure on the left below.





6. Noting that $x < 5$ because $x(0) = 2$, we write

$$\int \frac{dx}{x(5-x)} = \int(-3)\,dt; \qquad \int\left(\frac{1}{x}+\frac{1}{5-x}\right)dx = \int(-15)\,dt$$

$$\ln x - \ln(5-x) = -15t + \ln C; \qquad \frac{x}{5-x} = Ce^{-15t}$$

$$x(0) = 2 \text{ implies } C = 2/3; \qquad 3x = 2(5-x)e^{-15t}$$

$$x(t) = \frac{10e^{-15t}}{3+2e^{-15t}} = \frac{10}{2+3e^{15t}}.$$

Typical solution curves are shown in the figure on the right above.

7. Noting that $x > 7$ because $x(0) = 11$, we write

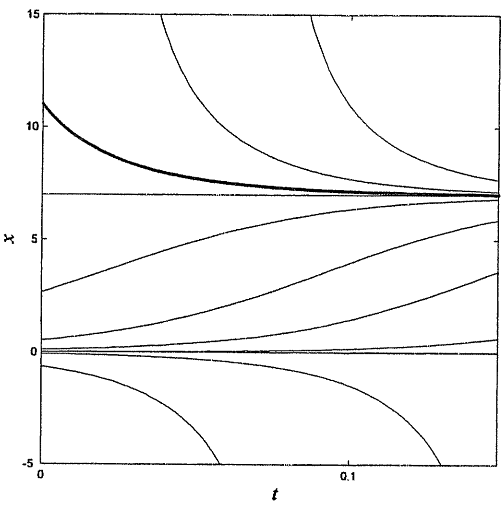
$$\int \frac{dx}{x(x-7)} = \int(-4)\,dt; \qquad \int\left(\frac{1}{x}-\frac{1}{x-7}\right)dx = \int 28\,dt$$

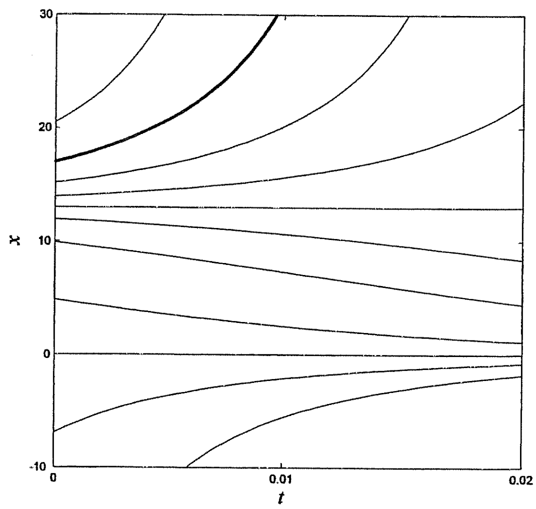
$$\ln x - \ln(x-7) \;=\; 28t + \ln C; \qquad \frac{x}{x-7} \;=\; Ce^{28t}$$

$$x(0) = 11 \quad \text{implies} \quad C = 11/4; \qquad 4x \;=\; 11(x-17)e^{28t}$$

$$x(t) \;=\; \frac{-77e^{28t}}{4-11e^{28t}} \;=\; \frac{77}{11-4e^{-28t}}.$$

Typical solution curves are shown in the figure on the left below.





8. Noting that $x > 13$ because $x(0) = 17$, we write

$$\int \frac{dx}{x(x-13)} \;=\; \int 7\,dt; \qquad \int \left(\frac{1}{x} - \frac{1}{x-13}\right) dx \;=\; \int (-91)\,dt$$

$$\ln x - \ln(x-13) \;=\; -91t + \ln C; \qquad \frac{x}{x-13} \;=\; Ce^{-91t}$$

$$x(0) = 17 \quad \text{implies} \quad C = 17/4; \qquad 4x \;=\; 17(x-13)e^{-91t}$$

$$x(t) \;=\; \frac{-221e^{-91t}}{4-17e^{-91t}} \;=\; \frac{221}{17-4e^{91t}}.$$

Typical solution curves are shown in the figure on the right above.

9. Substitution of $P(0) = 100$ and $P'(0) = 20$ into $P' = k\sqrt{P}$ yields $k = 2$, so the differential equation is $P' = 2\sqrt{P}$. Separation of variables and integration, $\int dP/2\sqrt{P} \;=\; \int dt$, gives $\sqrt{P} \;=\; t + C$. Then $P(0) = 100$ implies $C = 10$, so $P(t) \;=\; (t+10)^2$. Hence the number of rabbits after one year is $P(12) \;=\; 484$.

10. Given $P' = -\delta P = -(k/\sqrt{P})P = -k\sqrt{P}$, separation of variables and integration as in Problem 9 yields $2\sqrt{P} = -kt + C$. The initial condition $P(0) = 900$ gives $C = 60$, and then the condition $P(6) = 441$ implies that $k = 3$. Therefore $2\sqrt{P} = -3t + 60$, so $P = 0$ after $t = 20$ weeks.

11. **(a)** Starting with $dP/dt = k\sqrt{P}$, $dP/dt = k\sqrt{P}$, we separate the variables and integrate to get $P(t) = (kt/2 + C)^2$. Clearly $P(0) = P_0$ implies $C = \sqrt{P_0}$.

(b) If $P(t) = (kt/2 + 10)^2$, then $P(6) = 169$ implies that $k = 1$. Hence $P(t) = (t/2 + 10)^2$, so there are 256 fish after 12 months.

12. Solution of the equation $P' = kP^2$ by separation of variables and integration,

$$\int \frac{dP}{P^2} = \int k\,dt; \quad -\frac{1}{P} = kt - C,$$

gives $P(t) = 1/(C - kt)$. Now $P(0) = 12$ implies that $C = 1/12$, so now $P(t) = 12/(1 - 12kt)$. Then $P(10) = 24$ implies that $k = 1/240$, so finally $P(t) = 240/(20 - t)$. Hence $P = 48$ when $t = 15$, that is, in the year 2003. And obviously $P \to \infty$ as $t \to 20$.

13. **(a)** If the birth and death rates both are proportional to P^2 and $\beta > \delta$, then Eq. (1) in this section gives $P' = kP^2$ with k positive. Separating variables and integrating as in Problem 12, we find that $P(t) = 1/(C - kt)$. The initial condition $P(0) = P_0$ then gives $C = 1/P_0$, so $P(t) = 1/(1/P_0 - kt) = P_0/(1 - kP_0t)$.

(b) If $P_0 = 6$ then $P(t) = 6/(1 - 6kt)$. Now the fact that $P(10) = 9$ implies that $k = 180$, so $P(t) = 6/(1 - t/30) = 180/(30 - t)$. Hence it is clear that $P \to \infty$ as $t \to 30$ (doomsday).

14. Now $dP/dt = -kP^2$ with $k > 0$, and separation of variables yields $P(t) = 1/(kt + C)$. Clearly $C = 1/P_0$ as in Problem 13, so $P(t) = P_0/(1 + kP_0t)$. Therefore it is clear that $P(t) \to 0$ as $t \to \infty$, so the population dies out in the long run.

15. If we write $P' = bP(a/b - P)$ we see that $M = a/b$. Hence

$$\frac{B_0P_0}{D_0} = \frac{(aP_0)P_0}{bP_0^2} = \frac{a}{b} = M.$$

Note also (for Problems 16 and 17) that $a = B_0/P_0$ and $b = D_0/P_0^2 = k$.

16. The relations in Problem 15 give $k = 1/2400$ and $M = 160$. The solution is $P(t) = 19200/(120+40e^{-t/15})$. We find that $P = 0.95M$ after about 27.69 months.

17. The relations in Problem 15 give $k = 1/2400$ and $M = 180$. The solution is $P(t) = 43200/(240-60e^{-3t/80})$. We find that $P = 1.05M$ after about 44.22 months.

18. If we write $P' = aP(P - b/a)$ we see that $M = b/a$. Hence

$$\frac{D_0 P_0}{B_0} = \frac{(bP_0)P_0}{aP_0^2} = \frac{b}{a} = M.$$

Note also (for Problems 19 and 20) that $b = D_0/P_0$ and $a = B_0/P_0^2 = k$.

19. The relations in Problem 18 give $k = 1/1000$ and $M = 90$. The solution is $P(t) = 9000/(100-10e^{9t/100})$. We find that $P = 10M$ after about 24.41 months.

20. The relations in Problem 18 give $k = 1/1100$ and $M = 120$. The solution is $P(t) = 13200/(110+10e^{6t/55})$. We find that $P = 0.1M$ after about 42.12 months.

21. Starting with the differential equation $dP/dt = kP(200-P)$, we separate variables and integrate, noting that $P < 200$ because $P_0 = 100$:

$$\int \frac{dP}{P(200-P)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = \int 200k\,dt;$$

$$\ln \frac{P}{200-P} = 200kt + \ln C \quad \Rightarrow \quad \frac{P}{200-P} = Ce^{200kt}.$$

Now $P(0) = 100$ gives $C = 1$, and $P'(0) = 1$ implies that $1 = k \cdot 100(200-100)$, so we find that $k = 1/10000$. Substitution of these numerical values gives

$$\frac{P}{200-P} = e^{200t/10000} = e^{t/50},$$

and we solve readily for $P(t) = 200/\left(1+e^{-t/50}\right)$. Finally, $P(60) = 200/\left(1+e^{-6/5}\right) \approx 153.7$ million.

22. We work in thousands of persons, so $M = 100$ for the total fixed population. We substitute $M = 100$, $P'(0) = 1$, and $P_0 = 50$ in the logistic equation, and thereby obtain

$$1 = k(50)(100-50), \qquad \text{so} \qquad k = 0.0004.$$

If t denotes the number of days until 80 thousand people have heard the rumor, then Eq. (7) in the text gives

$$80 = \frac{50\times 100}{50+(100-50)e^{-0.04t}},$$

and we solve this equation for $t \approx 34.66$. Thus the rumor will have spread to 80% of the population in a little less than 35 days.

23. **(a)** $x' = 0.8x - 0.004x^2 = 0.004x(200-x)$, so the maximum amount that will dissolve is $M = 200$ g.

(b) With $M = 200$, $P_0 = 50$, and $k = 0.004$, Equation (4) in the text yields the solution

$$x(t) = \frac{10000}{50+150e^{-0.08t}}.$$

Substituting $x = 100$ on the left, we solve for $t = 1.25 \ln 3 \approx 1.37$ sec.

24. The differential equation for $N(t)$ is $N'(t) = kN(15-N)$. When we substitute $N(0) = 5$ (thousands) and $N'(0) = 0.5$ (thousands/day) we find that $k = 0.01$. With N in place of P, this is the logistic equation in Eq. (3) of the text, so its solution is given by Equation (7):

$$N(t) = \frac{15\times 5}{5+10\exp[-(0.01)(15)t]} = \frac{15}{1+2e^{-0.15t}}.$$

Upon substituting $N = 10$ on the left, we solve for $t = (\ln 4)/(0.15) \approx 9.24$ days.

25. Proceeding as in Example 3 in the text, we solve the equations

$$25.00k(M-25.00) = 3/8, \qquad 47.54k(M-47.54) = 1/2$$

for $M = 100$ and $k = 0.0002$. Then Equation (4) gives the population function

$$P(t) = \frac{2500}{25+75e^{-0.02t}}.$$

We find that $P = 75$ when $t = 50 \ln 9 \approx 110$, that is, in 2035 A. D.

26. The differential equation for $P(t)$ is

$$P'(t) = 0.001P^2 - \delta P.$$

When we substitute $P(0) = 100$ and $P'(0) = 8$ we find that $\delta = 0.02$, so

$$\frac{dP}{dt} = 0.001P^2 - 0.02P = 0.001P(P-20).$$

We separate variables and integrate, noting that $P > 20$ because $P_0 = 100$:

$$\int \frac{dP}{P(P-20)} = \int 0.001\, dt \quad \Rightarrow \quad \int \left(\frac{1}{P-20} - \frac{1}{P} \right) dP = \int 0.02\, dt;$$

$$\ln \frac{P-20}{P} = \frac{1}{50} t + \ln C \quad \Rightarrow \quad \frac{P-20}{P} = Ce^{t/50}.$$

Now $P(0) = 100$ gives $C = 4/5$, hence

$$5(P-20) = 4Pe^{t/50} \quad \Rightarrow \quad P(t) = \frac{100}{5 - 4e^{t/50}}.$$

It follows readily that $P = 200$ when $t = 50 \ln(9/8) \approx 5.89$ months.

27. We are given that

$$P' = kP^2 - 0.01P,$$

When we substitute $P(0) = 200$ and $P'(0) = 2$ we find that $k = 0.0001$, so

$$\frac{dP}{dt} = 0.0001P^2 - 0.01P = 0.0001P(P-100).$$

We separate variables and integrate, noting that $P > 100$ because $P_0 = 200$:

$$\int \frac{dP}{P(P-100)} = \int 0.0001\, dt \quad \Rightarrow \quad \int \left(\frac{1}{P-100} - \frac{1}{P} \right) dP = \int 0.01\, dt;$$

$$\ln \frac{P-100}{P} = \frac{1}{100} t + \ln C \quad \Rightarrow \quad \frac{P-100}{P} = Ce^{t/100}.$$

Now $P(0) = 100$ gives $C = 1/2$, hence

$$2(P-100) = Pe^{t/100} \quad \Rightarrow \quad P(t) = \frac{200}{2 - e^{t/100}}.$$

(a) $P = 1000$ when $t = 100 \ln(9/5) \approx 58.78$.

(b) $P \to \infty$ as $t \to 100 \ln 2 \approx 69.31$.

28. Our alligator population satisfies the equation

$$\frac{dP}{dt} = 0.0001x^2 - 0.01x = 0.0001x(x-100).$$

With x in place of P, this is the same differential equation as in Problem 27, but now we use absolute values to allow both possibilities $x < 100$ and $x > 100$:

$$\int \frac{dx}{x(x-100)} = \int 0.0001\,dt \quad \Rightarrow \quad \int \left(\frac{1}{x-100} - \frac{1}{x} \right) dP = \int 0.01\,dt;$$

$$\ln \frac{|x-100|}{x} = \frac{1}{100}t + \ln C \quad \Rightarrow \quad \frac{|x-100|}{x} = Ce^{t/100}. \qquad (*)$$

(a) If $x(0) = 25$ then $x < 100$ and $|x-100| = 100 - x$, so (*) gives $C = 3$ and hence

$$100 - x = 3x\,e^{t/100} \quad \Rightarrow \quad x(t) = \frac{100}{1+3e^{t/100}}.$$

We therefore see that $x(t) \to 0$ as $t \to \infty$.

(b) But if $x(0) = 150$ then $x > 100$ and $|x-100| = x - 100$, so (*) gives $C = 1/3$ and hence

$$3(x-100) = x\,e^{t/100} \quad \Rightarrow \quad x(t) = \frac{300}{3-e^{t/100}}.$$

Now $x(t) \to +\infty$ as $t \to (100 \ln 3)^-$, so doomsday occurs after about 109.86 months.

29. Here we have the logistic equation

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2 = 0.0001489P(210.544 - P)$$

where $k = 0.0001489$ and $P = 210.544$. With $P_0 = 3.9$ also, Eq. (7) in the text gives

$$P(t) = \frac{(210.544)(3.9)}{(3.9) + (210.544 - 3.9)e^{-(0.0001489)(210.544)t}} = \frac{821.122}{3.9 + 206.644e^{-0.03135t}}.$$

(a) This solution gives $P(140) \approx 127.008$, fairly close to the actual 1930 U.S. census population of 123.2 million.

(b) The limiting population as $t \to \infty$ is $821.122/3.9 = 210.544$ million.

(c) Since the actual U.S. population in 200 was about 281 million — already exceeding the maximum population predicted by the logistic equation — we see that that this model did *not* continue to hold throughout the 20th century.

30. The equation is separable, so we have

$$\int \frac{dP}{P} = \int \beta_0 e^{-\alpha t}\,dt, \quad \text{so} \quad \ln P = -\frac{\beta_0}{\alpha}e^{-\alpha t} + C.$$

The initial condition $P(0) = P_0$ gives $C = \ln P_0 + \beta_0/\alpha$, so

$$P(t) = P_0 \exp\left[\frac{\beta_0}{\alpha}\left(1 - e^{-\alpha t}\right)\right].$$

31. If we substitute $P(0) = 10^6$ and $P'(0) = 3\times 10^5$ into the differential equation

$$P'(t) = \beta_0 e^{-\alpha t} P,$$

we find that $\beta_0 = 0.3$. Hence the solution given in Problem 30 is

$$P(t) = P_0 \exp[(0.3/\alpha)(1 - e^{-\alpha t})].$$

The fact that $P(6) = 2P_0$ now yields the equation

$$f(\alpha) = (0.3)(1 - e^{-6\alpha}) - \alpha \ln 2 = 0$$

for α. We apply Newton's iterative formula

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$$

with $f'(\alpha) = 1.8e^{-6\alpha} - \ln 2$ and initial guess $\alpha_0 = 1$, and find that $\alpha \approx 0.3915$. Therefore the limiting cell population as $t \to \infty$ is

$$P_0 \exp(\beta_0/\alpha) = 10^6 \exp(0.3/0.3915) \approx 2.15\times 10^6.$$

Thus the tumor does not grow much further after 6 months.

32. We separate the variables in the logistic equation and use absolute values to allow for both possibilities $P_0 < M$ and $P_0 > M$:

$$\int \frac{dP}{P(M-P)} = \int k\,dt \quad \Rightarrow \quad \int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int kM\,dt;$$

$$\ln\frac{P}{|M-P|} = kMt+\ln C \quad\Rightarrow\quad \frac{P}{|M-P|} = Ce^{kMt}. \qquad (*)$$

If $P_0 < M$ then $P < M$ and $|M-P| = M-P$, so substitution of $t=0,\ P=P_0$ in (*) gives $C = P_0/(M-P_0)$. It follows that

$$\frac{P}{M-P} = \frac{P_0}{M-P_0}e^{kMt}.$$

But if $P_0 > M$ then $P > M$ and $|M-P| = P-M$, so substitution of $t=0,\ P=P_0$ in (*) gives $C = P_0/(P_0-M)$, and it follows that

$$\frac{P}{P-M} = \frac{P_0}{P_0-M}e^{kMt}.$$

We see that the preceding two equations are equivalent, and either yields

$$(M-P_0)P = (M-P)P_0e^{kMt} \quad\Rightarrow\quad P(t) = \frac{MP_0e^{kMt}}{(M-P_0)+P_0e^{kMt}},$$

which gives the desired result upon division of numerator and denominator by e^{kMt}.

33. (a) We separate the variables in the extinction-explosion equation and use absolute values to allow for both possibilities $P_0 < M$ and $P_0 > M$:

$$\int\frac{dP}{P(P-M)} = \int k\,dt \quad\Rightarrow\quad \int\left(\frac{1}{P-M}-\frac{1}{P}\right)dP = \int kM\,dt;$$

$$\ln\frac{|P-M|}{P} = kMt+\ln C \quad\Rightarrow\quad \frac{|P-M|}{P} = Ce^{kMt}. \qquad (*)$$

If $P_0 < M$ then $P < M$ and $|P-M| = M-P$, so substitution of $t=0,\ P=P_0$ in (*) gives $C = (M-P_0)/P_0$. It follows that

$$\frac{M-P}{P} = \frac{M-P_0}{P_0}e^{kMt}.$$

But if $P_0 > M$ then $P > M$ and $|P-M| = P-M$, so substitution of $t=0,\ P=P_0$ in (*) gives $C = (P_0-M)/P_0$, and it follows that

$$\frac{P-M}{P} = \frac{P_0-M}{P_0}e^{kMt}.$$

We see that the preceding two equations are equivalent, and either yields

$$(P-M)P_0 = (P_0-M)Pe^{kMt} \quad \Rightarrow \quad P(t) = \frac{MP_0}{P_0+(M-P_0)e^{kMt}}.$$

(b) If $P_0 < M$ then the coefficient $M-P_0$ is positive and the denominator increases without bound, so $P(t) \to 0$ as $t \to \infty$. But if $P_0 > M$, then the denominator $P_0-(P_0-M)e^{kMt}$ approaches zero — so $P(t) \to +\infty$ — as t approaches the value $(1/kM)\ln([P_0/(P_0-M)] > 0$ from the left.

34. Differentiation of both sides of the logistic equation $P' = kP\cdot(M-P)$ yields

$$\begin{aligned} P'' &= \frac{dP'}{dP}\cdot\frac{dP}{dt} \\ &= [k\cdot(M-P)+kP\cdot(-1)]\cdot kP(M-P) \\ &= k[M-2P]\cdot kP(M-P) = 2k^2P(M-\tfrac{1}{2}P)(M-P) \end{aligned}$$

as desired. The conclusions that $P''>0$ if $0<P<\frac{1}{2}M$, that $P''=0$ if $P=\frac{1}{2}M$, and that $P''<0$ if $\frac{1}{2}M<P<M$ are then immediate. Thus it follows that each of the curves for which $P_0<M$ has an inflection point where it crosses the horizontal line $P=\frac{1}{2}M$.

35. Any way you look at it, you should see that, the larger the parameter $k>0$ is, the faster the logistic population $P(t)$ approaches its limiting population M.

36. With $x = e^{-50kM}$, $P_0 = 5.308$, $P_1 = 23.192$, and $P_2 = 76.212$, Eqs. (7) in the text take the form

$$\frac{P_0M}{P_0+(M-P_0)x} = P_1, \quad \frac{P_0M}{P_0+(M-P_0)x^2} = P_2$$

from which we get

$$P_0+(M-P_0)x = P_0M/P_1, \quad P_0+(M-P_0)x^2 = P_0M/P_2$$

$$x = \frac{P_0(M-P_1)}{P_1(M-P_0)}, \quad x^2 = \frac{P_0(M-P_2)}{P_2(M-P_0)} \qquad \text{(i)}$$

$$\frac{P_0^2(M-P_1)^2}{P_1^2(M-P_0)^2} = \frac{P_0(M-P_2)}{P_2(M-P_0)}$$

$$P_0P_2(M-P_1)^2 = P_1^2(M-P_0)(M-P_2)$$

$$P_0P_2M^2-2P_0P_1P_2M+P_0P_1^2P_2 = P_1^2M^2-P_1^2(P_0+P_2)M+P_0P_1^2P_2$$

We cancel the final terms on the two sides of this last equation and solve for

$$M = \frac{P_1(2P_0P_2 - P_0P_1 - P_1P_2)}{P_0P_2 - P_1^2}. \qquad \text{(ii)}$$

Substitution of the given values $P_0 = 5.308$, $P_1 = 23.192$, and $P_2 = 76.212$ now gives $M = 188.121$. The first equation in (i) and $x = \exp(-kMt_1)$ yield

$$k = -\frac{1}{Mt_1}\ln\frac{P_0(M - P_1)}{P_1(M - P_0)}. \qquad \text{(iii)}$$

Now substitution of $t_1 = 50$ and our numerical values of M, P_0, P_1, P_2 gives $k = 0.000167716$. Finally, substitution of these values of k and M (and P_0) in the logistic solution (4) gives the logistic model of Eq. (8) in the text.

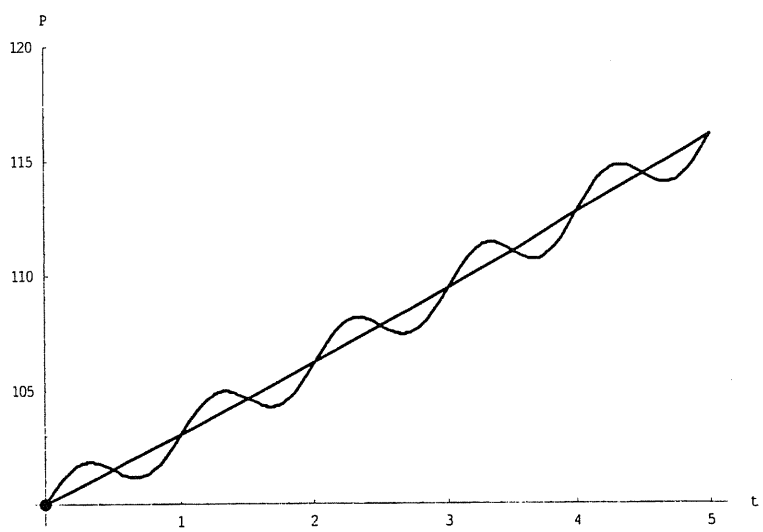
In Problems 37 and 38 we give just the values of k and M calculated using Eqs. (ii) and (iii) in Problem 36 above, the resulting logistic solution, and the predicted year 2000 population.

37. $k = 0.0000668717$ and $M = 338.027$, so $P(t) = \dfrac{25761.7}{76.212 + 261.815e^{-0.0226045t}}$,

predicting $P = 192.525$ in the year 2000.

38. $k = 0.000146679$ and $M = 208.250$, so $P(t) = \dfrac{4829.73}{23.192 + 185.058e^{-0.0305458t}}$,

predicting $P = 248.856$ in the year 2000.



39. We readily separate the variables and integrate:

$$\int \frac{dP}{P} = \int (k + b\cos 2\pi t)\,dt \quad \Rightarrow \quad \ln P = kt + \frac{b}{2\pi}\sin 2\pi t + \ln C.$$

Clearly $C = P_0$, so we find that $P(t) = P_0 \exp\left(kt + \frac{b}{2\pi}\sin 2\pi t\right)$. The colored curve in the figure above shows the graph that results with the typical numerical values $P_0 = 100$, $k = 0.03$, and $b = 0.06$. It oscillates about the black curve which represents natural growth with P_0 and $k = 0.03$. We see that the two agree at the end of each full year.

SECTION 1.8

ACCELERATION-VELOCITY MODELS

This section consists of three essentially independent subsections that can be studied separately: resistance proportional to velocity, resistance proportional to velocity-squared, and inverse-square gravitational acceleration.

1. Equation: $v' = k(250 - v), \quad v(0) = 0, \quad v(10) = 100$

Solution: $\int \frac{(-1)\,dv}{250 - v} = -\int k\,dt; \quad \ln(250 - v) = -kt + \ln C,$

$v(0) = 0$ implies $C = 250; \quad v(t) = 250(1 - e^{-kt})$

$v(10) = 100$ implies $k = \frac{1}{10}\ln(250/150) \approx 0.0511;$

Answer: $v = 200$ when $t = -(\ln 50/250)/k \approx 31.5\ \text{sec}$

2. Equation: $v' = -kv, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $x'(t) = v(t) = v_0 e^{-kt}; \quad x(t) = -(v_0/k)e^{-kt} + C$

$C = x_0 + (v_0/k)e^{-kt}; \quad x(t) = x_0 + (v_0/k)\left(1 - e^{-kt}\right)$

Answer: $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[x_0 + (v_0/k)\left(1 - e^{-kt}\right)\right] = x_0 + (v_0/k)$

3. Equation: $v' = -kv, \quad v(0) = 40; \quad v(10) = 20 \quad x' = v, \quad x(0) = 0$

Solution: $v(t) = 40\,e^{-kt}$ with $k = (1/10)\ln 2$

$x(t) = (40/k)(1 - e^{-kt})$

Answer: $x(\infty) = \lim_{t\to\infty}(40/k)(1-e^{-kt}) = 40/k = 400/\ln 2 \approx 577$ ft

4. Equation: $v' = -kv^2, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $-\int \frac{dv}{v^2} = \int k\,dt; \quad \frac{1}{v} = kt + C; \quad C = \frac{1}{v_0}$

$$x'(t) = v(t) = \frac{v_0}{1+v_0kt}; \quad x(t) = \frac{1}{k}\ln\left(1+v_0kt\right) + x_0$$

$x(t) \to \infty$ as $x(t) \to \infty$

5. Equation: $v' = -kv, \quad v(0) = 40; \quad v(10) = 20 \quad x' = v, \quad x(0) = 0$

Solution: $v = \frac{40}{1+40kt}$ (as in Problem 3)

$v(10) = 20$ implies $40k = 1/10$, so $v(t) = \frac{400}{10+t}$

$x(t) = 400\ln[(10+t)/10]$

Answer: $x(60) = 400\ln 7 \approx 778$ ft

6. Equation: $v' = -kv^{3/2}, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0$

Solution: $-\int \frac{dv}{2v^{3/2}} = \int \frac{k\,dt}{2}; \quad \frac{1}{\sqrt{v}} = \frac{kt}{2} + C; \quad C = \frac{1}{\sqrt{v_0}}$

$$x'(t) = v(t) = \frac{4v_0}{\left(2+kt\sqrt{v_0}\right)^2}; \quad x(t) = -\frac{4\sqrt{v_0}}{k\left(2+kt\sqrt{v_0}\right)} + C$$

$$C = x_0 + \frac{2\sqrt{v_0}}{k}; \quad x(t) = x_0 + \frac{2\sqrt{v_0}}{k}\left(1 - \frac{2}{2+kt\sqrt{v_0}}\right)$$

$x(\infty) = x_0 + 2\sqrt{v_0}/k$

7. Equation: $v' = 10 - 0.1v, \quad x(0) = v(0) = 0$

(a) $\int \frac{-0.1\,dv}{10-0.1v} = \int(-0.1)\,dt; \quad \ln(10-0.1v) = -t/10 + \ln C$

$v(0) = 0$ implies $C = 10; \quad \ln\left[(10-0.1v)/10\right] = -t/10$

$v(t) = 100(1-e^{-t/10}); \quad v(\infty) = 100$ ft/sec (limiting velocity)

(b) $x(t) = 100t - 1000(1 - e^{-t/10})$

$v = 90$ ft/sec when $t = 23.0259$ sec and $x = 1402.59$ ft

8. Equation: $v' = 10 - 0.001v^2, \quad x(0) = v(0) = 0$

(a) $\int \frac{0.01\,dv}{1 - 0.0001v^2} = \int \frac{dt}{10}; \quad \tanh^{-1}\frac{v}{100} = \frac{t}{10} + C$

$v(0) = 0$ implies $C = 0$ so $v(t) = 100\tanh(t/10)$

$$v(\infty) = \lim_{t\to\infty} 100\tanh(t/10) = 100 \lim_{t\to\infty} \frac{e^{t/10} - e^{-t/10}}{e^{t/10} + e^{-t/10}} = 100\text{ ft/sec}$$

(b) $x(t) = 1000\ln(\cosh t/10)$

$v = 90$ ft/sec when $t = 14.7222$ sec and $x = 830.366$ ft

9. The solution of the initial value problem

$$1000\,v' = 5000 - 100\,v, \qquad v(0) = 0$$

is

$$v(\mathrm{t}) = 50(1 - e^{-t/10}).$$

Hence, as $t \to \infty$, we see that $v(t)$ approaches $v_{\max} = 50$ ft/sec ≈ 34 mph.

10. Before opening parachute:

$$v' = -32 - 0.15v, \quad v(0) = 0, \quad y(0) = 10000$$
$$v(t) = 213.333(e^{-0.15t} - 1), \qquad v(20) = -202.712 \text{ ft/sec}$$
$$y(t) = 11422.2 - 1422.22e^{-0.15ty} - 213.333t, \quad y(20) = 7084.75 \text{ ft}$$

After opening parachute:

$$v' = -32 - 1.5v, \quad v(0) = -202.712, \quad y(0) = 7084.75$$
$$v(t) = -21.3333 - 181.379e^{-1.5t}$$
$$y(t) = 6964.83 + 120.919e^{-1.5t} - 21.3333t,$$
$$y = 0 \text{ when } t = 326.476$$

Thus she opens her parachute after 20 sec at a height of 7085 feet, and the total time of descent is $20 + 326.476 = 346.476$ sec, about 5 minutes and 46.5 seconds. Her impact speed is 21.33 ft/sec, about 15 mph.

11. If the paratrooper's terminal velocity was $100 \text{ mph} = 440/3$ ft/sec, then Equation (7) in the text yields $\rho = 12/55$. Then we find by solving Equation (9) numerically with $y_0 = 1200$ and $v_0 = 0$ that $y = 0$ when $t \approx 12.5$ sec. Thus the newspaper account is inaccurate.

12. With $m = 640/32 = 20$ slugs, $W = 640$ lb, $B = (8)(62.5) = 500$ lb, and $F_R = -v$ lb (F_R is upward when $v < 0$), the differential equation is

$$20\, v'(t) = -640 + 500 - v = -140 - v.$$

Its solution with $v(0) = 0$ is

$$v(t) = 140\left(e^{-0.05t} - 1\right),$$

and integration with $y(0) = 0$ yields

$$y(t) = 2800\left(e^{-0.05t} - 1\right) - 140t.$$

Using these equations we find that $t = 20 \ln(28/13) \approx 15.35$ sec when $v = -75$ ft/sec, and that $y(15.35) \approx -648.31$ ft. Thus the maximum safe depth is just under 650 ft.

Given the hints and integrals provided in the text, Problems 13–16 are fairly straightforward (and fairly tedious) integration problems.

17. To solve the initial value problem $v' = -9.8 - 0.0011v^2, \quad v(0) = 49$ we write

$$\int \frac{dv}{9.8 + 0.0011v^2} = -\int dt; \qquad \int \frac{0.010595\,dv}{1 + (0.010595v)^2} = -\int 0.103827\,dt$$

$$\tan^{-1}(0.010595v) = -0.103827t + C; \qquad v(0) = 49 \text{ implies } C = 0.478854$$

$$v(t) = 94.3841\tan(0.478854 - 0.103827t)$$

Integration with $y(0) = 0$ gives

$$y(t) = 108.468 + 909.052\ \ln(\cos(0.478854 - 0.103827t)).$$

We solve $v(0) = 0$ for $t = 4.612$, and then calculate $y(4.612) = 108.468$.

18. We solve the initial value problem $v' = -9.8 + 0.0011v^2, \quad v(0) = 0$ much as in Problem 17, except using hyperbolic rather than ordinary trigonometric functions. We first get

$$v(t) = -94.3841\tanh(0.103827t),$$

and then integration with $y(0) = 108.47$ gives

$$y(t) = 108.47 - 909.052\ \ln(\cosh(0.103827t)).$$

We solve $y(0) = 0$ for $t = \cosh^{-1}(\exp(108.47/909.052))/0.103.827 \approx 4.7992$, and then calculate $v(4.7992) = -43.489$.

19. Equation: $v' = 4 - (1/400)v^2, \quad v(0) = 0$

Solution: $\displaystyle\int \frac{dv}{4-(1/400)v^2} = \int dt; \quad \int \frac{(1/40)\,dv}{1-(v/40)^2} = \int \frac{1}{10}\,dt$

$\tanh^{-1}(v/40) = t/10 + C; \quad C = 0; \quad v(t) = 40\tanh(t/10)$

Answer: $v(10) \approx 30.46$ ft/sec, $\quad v(\infty) = 40$ ft/sec

20. Equation: $v' = -32 - (1/800)v^2, \quad v(0) = 160, \quad y(0) = 0$

Solution: $\displaystyle\int \frac{dv}{32+(1/800)v^2} = -\int dt; \quad \int \frac{(1/160)\,dv}{1+(v/160)^2} = -\int \frac{1}{5}\,dt;$

$\tan^{-1}(v/160) = -t/5 + C; \quad v(0) = 160$ implies $C = \pi/4$

$$v(t) = 160\tan\left(\frac{\pi}{4} - \frac{t}{5}\right)$$

$$y(t) = 800\ln\left(\cos\left(\frac{\pi}{4} - \frac{t}{5}\right)\right) + 400\ln 2$$

We solve $v(t) = 0$ for $t = 3.92699$ and then calculate $y(3.92699) = 277.26$ ft.

21. Equation: $v' = -g - \rho v^2, \quad v(0) = v_0, \quad y(0) = 0$

Solution: $\displaystyle\int \frac{dv}{g+\rho v^2} = -\int dt; \quad \int \frac{\sqrt{\rho/g}\,dv}{1+\left(\sqrt{\rho/g}\,v\right)^2} = -\int \sqrt{g\rho}\,dt;$

$$\tan^{-1}\left(\sqrt{\rho/g}\,v\right) = -\sqrt{g\rho}\,t + C; \qquad v(0) = v_0 \text{ implies } C = \tan^{-1}\left(\sqrt{\rho/g}\,v_0\right)$$

$$v(t) = -\sqrt{\frac{g}{\rho}}\tan\left(t\sqrt{g\rho} - \tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)\right)$$

We solve $v(t) = 0$ for $t = \dfrac{1}{\sqrt{g\rho}}\tan^{-1}\left(v_0\sqrt{\dfrac{\rho}{g}}\right)$ and substitute in Eq. (17) for $y(t)$:

$$\begin{aligned}
y_{\max} &= \frac{1}{\rho}\ln\left|\frac{\cos\left(\tan^{-1}v_0\sqrt{\rho/g} - \tan^{-1}v_0\sqrt{\rho/g}\right)}{\cos\left(\tan^{-1}v_0\sqrt{\rho/g}\right)}\right| \\
&= \frac{1}{\rho}\ln\left(\sec\left(\tan^{-1}v_0\sqrt{\rho/g}\right)\right) = \frac{1}{\rho}\ln\sqrt{1+\frac{\rho v_0^2}{g}} \\
y_{\max} &= \frac{1}{2\rho}\ln\left(1+\frac{\rho v_0^2}{g}\right)
\end{aligned}$$

22. By an integration similar to the one in Problem 19, the solution of the initial value problem $v' = -32 + 0.075v^2$, $v(0) = 0$ is

$$v(t) = -20.666\tanh(1.54919t),$$

so the terminal speed is 20.666 ft/sec. Then a further integration with $y(0) = 0$ gives

$$y(t) = 10000 - 13.333\ln(\cosh(1.54919t)).$$

We solve $y(0) = 0$ for $t = 484.57$. Thus the descent takes about 8 min 5 sec.

23. Before opening parachute:

$$\begin{aligned}
v' &= -32 + 0.00075v^2, \quad v(0) = 0, \quad y(0) = 10000 \\
v(t) &= -206.559\tanh(0.154919t) \quad v(30) = -206.521 \text{ ft/sec} \\
y(t) &= 10000 - 1333.33\ln(\cosh(0.154919t)), \quad y(30) = 4727.30 \text{ ft}
\end{aligned}$$

After opening parachute:

$$\begin{aligned}
v' &= -32 + 0.075v^2, \quad v(0) = -206.521, \quad y(0) = 4727.30 \\
v(t) &= -20.6559\tanh(1.54919t + 0.00519595) \\
y(t) &= 4727.30 - 13.3333\ln(\cosh(1.54919t + 0.00519595)) \\
y &= 0 \text{ when } t = 229.304
\end{aligned}$$

Thus she opens her parachute after 30 sec at a height of 4727 feet, and the total time of descent is $30 + 229.304 = 259.304$ sec, about 4 minutes and 19.3 seconds.

24. Let M denote the mass of the Earth. Then

(a) $\sqrt{2GM/R} = c$ implies $R = 0.884\times10^{-3}$ meters, about 0.88 cm;

(b) $\sqrt{2G(329320M)/R} = c$ implies $R = 2.91\times10^3$ meters, about 2.91 kilometers.

25. (a) The rocket's apex occurs when $v = 0$. We get the desired formula when we set $v = 0$ in Eq. (23),

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r}-\frac{1}{R}\right),$$

and solve for r.

(b) We substitute $v = 0$, $r = R + 10^5$ (100 km = 10^5 m) and the mks values $G = 6.6726\times10^{-11}$, $M = 5.975\times10^{24}$, $R = 6.378\times10^6$ in Eq. (23) and solve for $v_0 = 1389.21$ m/s ≈ 1.389 km/s.

(c) When we substitute $v_0 = (9/10)\sqrt{2GM/R}$ in the formula derived in part (a), we find that $r_{max} = 100R/19$.

26. By an elementary computation (as in Section 1.2) we find that an initial velocity of $v_0 = 16$ ft/sec is required to jump vertically 4 feet high on earth. We must determine whether this initial velocity is adequate for escape from the asteroid. Let r denote the ratio of the radius of the asteroid to the radius $R = 3960$ miles of the earth, so that

$$r = \frac{1.5}{3960} = \frac{1}{2640}.$$

Then the mass and radius of the asteroid are given by

$$M_a = r^3M \quad\text{and}\quad R_a = rR$$

in terms of the mass M and radius R of the earth. Hence the escape velocity from the asteroid's surface is given by

$$v_a = \sqrt{\frac{2GM_a}{R_a}} = \sqrt{\frac{2G\cdot r^3M}{rR_a}} = r\sqrt{\frac{2GM}{R}} = rv_0$$

in terms of the escape velocity v_0 from the earth's surface. Hence $v_a \approx 36680/2640$ ≈ 13.9 ft/sec. Since the escape velocity from this asteroid is thus less than the initial velocity of 16 ft/sec that your legs can provide, you can indeed jump right off this asteroid into space.

27. (a) Substitution of $v_0^2 = 2GM/R = k^2/R$ in Eq. (23) of the textbook gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r}} = \frac{k}{\sqrt{r}}.$$

We separate variables and proceed to integrate:

$$\int \sqrt{r}\,dr = \int k\,dt \quad\Rightarrow\quad \frac{2}{3}r^{3/2} = kt + \frac{2}{3}R^{3/2}$$

(using the fact that $r = R$ when $t = 0$). We solve for $r(t) = \left(\tfrac{3}{2}kt + R^{3/2}\right)^{2/3}$ and note that $r(t) \to \infty$ as $t \to \infty$.

(b) If $v_0 > 2GM/R$ then Eq. (23) gives

$$\frac{dr}{dt} = v = \sqrt{\frac{2GM}{r} + \left(v_0^2 - \frac{2GM}{R}\right)} = \sqrt{\frac{k^2}{r} + \alpha} > \frac{k}{\sqrt{r}}.$$

Therefore, at every instant in its ascent, the upward velocity of the projectile in this part is greater than the velocity at the same instant of the projectile of part (a). It's as though the projectile of part (a) is the fox, and the projectile of this part is a rabbit that runs faster. Since the fox goes to infinity, so does the faster rabbit.

28. **(a)** Integration of gives

$$\frac{1}{2}v^2 = GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

and we solve for

$$\frac{dr}{dt} = v = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

taking the negative square root because $v < 0$ in descent. Hence

$$\begin{aligned} t &= -\sqrt{\frac{r_0}{2GM}}\int \sqrt{\frac{r}{r_0 - r}}\,dr \qquad (r = r_0\cos^2\theta) \\ &= \sqrt{r_0/2GM}\int 2r_0\cos^2\theta\,d\theta \\ &= \frac{r_0^{3/2}}{\sqrt{2GM}}(\theta + \sin\theta\cos\theta) \\ t &= \sqrt{\frac{r_0}{2GM}}\left(\sqrt{rr_0 - r^2} + r_0\cos^{-1}\sqrt{\frac{r}{r_0}}\right) \end{aligned}$$

(b) Substitution of $G = 6.6726\times10^{-11}$, $M = 5.975\times10^{24}$ kg, $r = R = 6.378\times10^{6}$ m, and $r_0 = R + 10^6$ yields $t = 510.504$, that is, about $8\frac{1}{2}$ minutes for the descent to the surface of the earth. (Recall that we are ignoring air resistance.)

(c) Substitution of the same numeral values along with $v_0 = 0$ in the original differential equation of part (a) yields $v = -4116.42$ m/s ≈ -4.116 km/s for the velocity at impact with the earth's surface where $r = R$.

29. Integration of $v\dfrac{dv}{dy} = -\dfrac{GM}{(y+R)^2}$, $y(0)=0$, $v(0)=v_0$ gives

$$\frac{1}{2}v^2 = \frac{GM}{y+R} - \frac{GM}{R} + \frac{1}{2}v_0^2$$

which simplifies to the desired formula for v^2. Then substitution of $G = 6.6726\times 10^{-11}$, $M = 5.975\times 10^{24}$ kg, $R = 6.378\times 10^6$ m $v = 0$, and $v_0 = 1$ yields an equation that we easily solve for $y = 51427.3$, that is, about 51.427 km.

30. When we integrate

$$v\frac{dv}{dr} = -\frac{GM_e}{r^2} + \frac{GM_m}{(S-r)^2}, \quad r(0) = R, \quad r'(0) = v_0$$

in the usual way and solve for v, we get

$$v = \sqrt{\frac{2GM_e}{r} - \frac{2GM_e}{R} - \frac{2GM_m}{r-S} + \frac{2GM_m}{R-S} + v_0^2}.$$

The earth and moon attractions balance at the point where the right-hand side in the acceleration equation vanishes, which is when

$$r = \frac{\sqrt{M_e}\,S}{\sqrt{M_e} - \sqrt{M_m}}.$$

If we substitute this value of r, $M_m = 7.35\times 10^{22}$ kg, $S = 384.4\times 10^6$, and the usual values of the other constants involved, then set $v = 0$ (to just reach the balancing point), we can solve the resulting equation for $v_0 = 11{,}109$ m/s. Note that this is only 71 m/s less than the earth escape velocity of 11,180 m/s, so the moon really doesn't help much.

CHAPTER 1 Review Problems

The main objective of this set of review problems is practice in the identification of the different types of first-order differential equations discussed in this chapter. In each of Problems 1–36 we identify the type of the given equation and indicate an appropriate method of solution.

1. If we write the equation in the form $y' - (3/x)y = x^2$ we see that it is *linear* with integrating factor $\rho = x^{-3}$. The method of Section 1.5 then yields the general solution $y = x^3(C + \ln x)$.

2. We write this equation in the *separable* form $y'/y^2 = (x+3)/x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x/(3 - Cx - x\ln x)$.

3. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = x/(C - \ln x)$.

4. We note that $D_y\left(2xy^3 + e^x\right) = D_x\left(3x^2y^2 + \sin y\right) = 6xy^2$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $x^2y^3 + e^x - \cos y = C$.

5. We write this equation in the *separable* form $y'/y^2 = (2x-3)/x^4$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = C\exp[(1-x)/x^3]$.

6. We write this equation in the *separable* form $y'/y^2 = (1-2x)/x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x/(1 + Cx + 2x\ln x)$.

7. If we write the equation in the form $y' + (2/x)y = 1/x^3$ we see that it is *linear* with integrating factor $\rho = x^2$. The method of Section 1.5 then yields the general solution $y = x^{-2}(C + \ln x)$.

8. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = 3Cx/(C - x^3)$.

9. If we write the equation in the form $y' + (2/x)y = 6x\sqrt{y}$ we see that it is a *Bernoulli equation* with $n = 1/2$. The substitution $v = y^{-1/2}$ of Eq. (10) in Section 1.6 then yields the general solution $y = (x^2 + C/x)^2$.

10. We write this equation in the *separable* form $y'/\left(1+y^2\right) = 1 + x^2$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = \tan(C + x + x^3/3)$.

11. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the general solution $y = x/(C - 3\ln x)$.

12. We note that $D_y\left(6xy^3+2y^4\right) = D_x\left(9x^2y^2+8xy^3\right) = 18xy^2+8y^3$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $3x^2y^3+2xy^4 = C$.

13. We write this equation in the *separable* form $y'/y^2 = 5x^4-4x$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = 1/(C+2x^2-x^5)$.

14. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the implicit general solution $y^2 = x^2/(C+2\ln x)$.

15. This is a *linear* differential equation with integrating factor $\rho = e^{3x}$. The method of Section 1.5 yields the general solution $y = (x^3+C)e^{-3x}$.

16. The substitution $v = y-x$, $y = v+x$, $y' = v'+1$ gives the separable equation $v'+1 = (y-x)^2 = v^2$ in the new dependent variable v. The resulting implicit general solution of the original equation is $y-x-1 = C\,e^{2x}(y-x+1)$.

17. We note that $D_y\left(e^x+ye^{xy}\right) = D_x\left(e^y+xe^{xy}\right) = e^{xy}+xye^{xy}$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $e^x+e^y+e^{xy} = C$.

18. This equation is *homogeneous*. The substitution $y = vx$ of Equation (8) in Section 1.6 leads to the implicit general solution $y^2 = Cx^2(x^2-y^2)$.

19. We write this equation in the *separable* form $y'/y^2 = \left(2-3x^5\right)/x^3$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x^2/(x^5+Cx^2+1)$.

20. If we write the equation in the form $y'+(3/x)y = 3x^{-5/2}$ we see that it is *linear* with integrating factor $\rho = x^3$. The method of Section 1.5 then yields the general solution $y = 2x^{-3/2}+Cx^{-3}$.

21. If we write the equation in the form $y'+\left(1/(x+1)\right)y = 1/(x^2-1)$ we see that it is *linear* with integrating factor $\rho = x+1$. The method of Section then 1.5 yields the general solution $y = [C+\ln(x-1)]/(x+1)$.

22. If we write the equation in the form $y' - (6/x)y = 12x^3y^{2/3}$ we see that it is a *Bernoulli equation* with $n = 1/3$. The substitution $v = y^{-2/3}$ of Eq. (10) in Section 1.6 then yields the general solution $y = (2x^4 + Cx^2)^3$.

23. We note that $D_y\left(e^y + y\cos x\right) = D_x\left(xe^y + \sin x\right) = e^y + \cos x,$ so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $xe^y + y\sin x = C$.

24. We write this equation in the *separable* form $y'/y^2 = \left(1-9x^2\right)/x^{3/2}$. Then separation of variables and integration as in Section 1.4 yields the general solution $y = x^{1/2}/(6x^2 + Cx^{1/2} + 2)$.

25. If we write the equation in the form $y' + \left(2/(x+1)\right)y = 3$ we see that it is *linear* with integrating factor $\rho = \left(x+1\right)^2$. The method of Section 1.5 then yields the general solution $y = x + 1 + C\,(x+1)^{-2}$.

26. We note that $D_y\left(9x^{1/2}y^{4/3} - 12x^{1/5}y^{3/2}\right) = D_x\left(8x^{3/2}y^{1/3} - 15x^{6/5}y^{1/2}\right) =$ $12x^{1/2}y^{1/3} - 18x^{1/5}y^{1/2}$, so the given equation is *exact*. The method of Example 9 in Section 1.6 yields the implicit general solution $6x^{3/2}y^{4/3} - 10x^{6/5}y^{3/2} = C$.

27. If we write the equation in the form $y' + (1/x)y = -x^2y^4/3$ we see that it is a *Bernoulli equation* with $n = 4$. The substitution $v = y^{-3}$ of Eq. (10) in Section 1.6 then yields the general solution $y = x^{-1}(C + \ln x)^{-1/3}$.

28. If we write the equation in the form $y' + (1/x)y = 2e^{2x}/x$ we see that it is *linear* with integrating factor $\rho = x$. The method of Section 1.5 then yields the general solution $y = x^{-1}(C + e^{2x})$.

29. If we write the equation in the form $y' + \left(1/(2x+1)\right)y = (2x+1)^{1/2}$ we see that it is *linear* with integrating factor $\rho = \left(2x+1\right)^{1/2}$. The method of Section 1.5 then yields the general solution $y = (x^2 + x + C)(2x+1)^{-1/2}$.

30. The substitution $v = x + y,\ y = v - x,\ y' = v' - 1$ gives the separable equation $v' - 1 = \sqrt{v}$ in the new dependent variable v. The resulting implicit general solution of the original equation is $x = 2(x+y)^{1/2} - 2\ln[1 + (x+y)^{1/2}] + C$.

31. $dy/(y+7) = 3x^2 dx$ is separable; $y' + 3x^2 y = 21x^2$ is linear.

32. $dy/(y^2-1) = x\,dx$ is separable; $y' + x\,y = x\,y^3$ is a Bernoulli equation with $n = 3$.

33. $(3x^2 + 2y^2)\,dx + 4xy\,dy = 0$ is exact; $y' = -\frac{1}{4}(3x/y + 2y/x)$ is homogeneous.

34. $(x + 3y)\,dx + (3x - y)\,dy = 0$ is exact; $y' = \dfrac{1 + 3y/x}{y/x - 3}$ is homogeneous.

35. $dy/(y+1) = 2x\,dx/\left(x^2+1\right)$ is separable; $y' - \left(2x/(x^2+1)\right)y = 2x/(x^2+1)$ is linear.

36. $dy/\left(\sqrt{y} - y\right) = \cot x\,dx$ is separable; $y' + (\cot x)y = (\cot x)\sqrt{y}$ is a Bernoulli equation with $n = 1/2$.

CHAPTER 2

LINEAR EQUATIONS OF HIGHER ORDER

SECTION 2.1

INTRODUCTION: SECOND-ORDER LINEAR EQUATIONS

In this section the central ideas of the theory of linear differential equations are introduced and illustrated concretely in the context of **second-order** equations. These key concepts include superposition of solutions (Theorem 1), existence and uniqueness of solutions (Theorem 2), linear independence, the Wronskian (Theorem 3), and general solutions (Theorem 4). This discussion of second-order equations serves as preparation for the treatment of *n*th order linear equations in Section 2.2. Although the concepts in this section may seem somewhat abstract to students, the problems set is quite tangible and largely computational.

In each of Problems 1–16 the verification that y_1 and y_2 satisfy the given differential equation is a routine matter. As in Example 2, we then impose the given initial conditions on the general solution $y = c_1y_1 + c_2y_2$. This yields two linear equations that determine the values of the constants c_1 and c_2.

1. Imposition of the initial conditions $y(0) = 0,\ y'(0) = 5$ on the general solution $y(x) = c_1e^x + c_2e^{-x}$ yields the two equations $c_1 + c_2 = 0,\ c_1 - c_2 = 0$ with solution $c_1 = 5/2,\ c_2 = -5/2$. Hence the desired particular solution is $y(x) = 5(e^x - e^{-x})/2$.

2. Imposition of the initial conditions $y(0) = -1,\ y'(0) = 15$ on the general solution $y(x) = c_1e^{3x} + c_2e^{-3x}$ yields the two equations $c_1 + c_2 = -1,\ 3c_1 - 3c_2 = 15$ with solution $c_1 = 2,\ c_2 = 3$. Hence the desired particular solution is $y(x) = 2e^{3x} - 3e^{-3x}$.

3. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 8$ on the general solution $y(x) = c_1\cos 2x + c_2\sin 2x$ yields the two equations $c_1 = 3,\ 2c_2 = 8$ with solution $c_1 = 3,\ c_2 = 4$. Hence the desired particular solution is $y(x) = 3\cos 2x + 4\sin 2x$.

4. Imposition of the initial conditions $y(0) = 10,\ y'(0) = -10$ on the general solution $y(x) = c_1\cos 5x + c_2\sin 5x$ yields the two equations $c_1 = 10,\ 5c_2 = -10$ with solution $c_1 = 3,\ c_2 = 4$. Hence the desired particular solution is $y(x) = 10\cos 5x - 2\sin 5x$.

5. Imposition of the initial conditions $y(0)=1,\ y'(0)=0$ on the general solution $y(x) = c_1e^x + c_2e^{2x}$ yields the two equations $c_1+c_2=1,\ c_1+2c_2=0$ with solution $c_1=2,\ c_2=-1$. Hence the desired particular solution is $y(x) = 2e^x - e^{2x}$.

6. Imposition of the initial conditions $y(0)=7,\ y'(0)=-1$ on the general solution $y(x) = c_1e^{2x} + c_2e^{-3x}$ yields the two equations $c_1+c_2=7,\ 2c_1-3c_2=-1$ with solution $c_1=4,\ c_2=3$. Hence the desired particular solution is $y(x) = 4e^{2x} + 3e^{-3x}$.

7. Imposition of the initial conditions $y(0)=-2,\ y'(0)=8$ on the general solution $y(x) = c_1 + c_2e^{-x}$ yields the two equations $c_1+c_2=-2,\ -c_2=8$ with solution $c_1=6,\ c_2=-8$. Hence the desired particular solution is $y(x) = 6 - 8e^{-x}$.

8. Imposition of the initial conditions $y(0)=4,\ y'(0)=-2$ on the general solution $y(x) = c_1 + c_2e^{3x}$ yields the two equations $c_1+c_2=4,\ 3c_2=-2$ with solution $c_1=14/3,\ c_2=2/3$. Hence the desired particular solution is $y(x) = (14 - 2e^{3x})/3$.

9. Imposition of the initial conditions $y(0)=2,\ y'(0)=-1$ on the general solution $y(x) = c_1e^{-x} + c_2x\,e^{-x}$ yields the two equations $c_1=2,\ -c_1+c_2=-1$ with solution $c_1=2,\ c_2=1$. Hence the desired particular solution is $y(x) = 2e^{-x} + xe^{-x}$.

10. Imposition of the initial conditions $y(0)=3,\ y'(0)=13$ on the general solution $y(x) = c_1e^{5x} + c_2x\,e^{5x}$ yields the two equations $c_1=3,\ 5c_1+c_2=13$ with solution $c_1=3,\ c_2=-2$. Hence the desired particular solution is $y(x) = 3e^{5x} - 2xe^{5x}$.

11. Imposition of the initial conditions $y(0)=0,\ y'(0)=5$ on the general solution $y(x) = c_1e^x\cos x + c_2e^x\sin x$ yields the two equations $c_1=0,\ c_1+c_2=5$ with solution $c_1=0,\ c_2=5$. Hence the desired particular solution is $y(x) = 5e^x\sin x$.

12. Imposition of the initial conditions $y(0)=2,\ y'(0)=0$ on the general solution $y(x) = c_1e^{-3x}\cos 2x + c_2e^{-3x}\sin 2x$ yields the two equations $c_1=2,\ -3c_1+2c_2=5$ with solution $c_1=2,\ c_2=3$. Hence the desired particular solution is $y(x) = e^{-3x}(2\cos 2x + 3\sin 2x)$.

13. Imposition of the initial conditions $y(1)=3,\ y'(1)=1$ on the general solution $y(x) = c_1x + c_2x^2$ yields the two equations $c_1+c_2=3,\ c_1+2c_2=1$ with solution $c_1=5,\ c_2=-2$. Hence the desired particular solution is $y(x) = 5x - 2x^2$.

14. Imposition of the initial conditions $y(2)=10,\ y'(2)=15$ on the general solution $y(x) = c_1x^2 + c_2x^{-3}$ yields the two equations $4c_1 + c_2/8 = 10,\ 4c_1 - 3c_2/16 = 15$ with solution $c_1 = 3,\ c_2 = -16$. Hence the desired particular solution is $y(x) = 3x^2 - 16/x^3$.

15. Imposition of the initial conditions $y(1)=7,\ y'(1)=2$ on the general solution $y(x) = c_1x + c_2x\ln x$ yields the two equations $c_1 = 7,\ c_1 + c_2 = 2$ with solution $c_1 = 7,\ c_2 = -5$. Hence the desired particular solution is $y(x) = 7x - 5x\ln x$.

16. Imposition of the initial conditions $y(1)=2,\ y'(1)=3$ on the general solution $y(x) = c_1\cos(\ln x) + c_2\sin(\ln x)$ yields the two equations $c_1 = 2,\ c_2 = 3$. Hence the desired particular solution is $y(x) = 2\cos(\ln x) + 3\sin(\ln x)$.

17. If $y = c/x$ then $y' + y^2 = -c/x^2 + c^2/x^2 = c(c-1)/x^2 \neq 0$ unless either $c = 0$ or $c = 1$.

18. If $y = cx^3$ then $yy'' = cx^3 \cdot 6cx = 6c^2x^4 \neq 6x^4$ unless $c^2 = 1$.

19. If $y = 1 + \sqrt{x}$ then $yy'' + (y')^2 = (1+\sqrt{x})(-x^{-3/2}/4) + (x^{-1/2}/2)^2 = -x^{-3/2}/4 \neq 0$.

20. Linearly dependent, because

$$f(x) = \pi = \pi(\cos^2 x + \sin^2 x) = \pi g(x)$$

21. Linearly independent, because $x^3 = +x^2|x|$ if $x > 0$, whereas $x^3 = -x^2|x|$ if $x < 0$.

22. Linearly independent, because $1 + x = c(1+|x|)$ would require that $c = 1$ with $x = 0$, but $c = 0$ with $x = -1$. Thus there is no such constant c.

23. Linearly independent, because $f(x) = +g(x)$ if $x > 0$, whereas $f(x) = -g(x)$ if $x < 0$.

24. Linearly dependent, because $g(x) = 2f(x)$.

25. $f(x) = e^x\sin x$ and $g(x) = e^x\cos x$ are linearly independent, because $f(x) = k\,g(x)$ would imply that $\sin x = k\cos x$, whereas $\sin x$ and $\cos x$ are linearly independent.

26. To see that $f(x)$ and $g(x)$ are linearly independent, assume that $f(x) = c\,g(x)$, and then substitute both $x = 0$ and $x = \pi/2$.

27. Let $L[y] = y'' + py' + qy$. Then $L[y_c] = 0$ and $L[y_p] = f$, so

$$L[y_c + y_p] = L[y_c] + [y_p] = 0 + f = f.$$

28. If $y(x) = 1 + c_1\cos x + c_2\sin x$ then $y'(x) = -c_1\sin x + c_2\cos x$, so the initial conditions $y(0) = y'(0) = -1$ yield $c_1 = -2, c_2 = -1$. Hence $y = 1 - 2\cos x - \sin x$.

29. There is no contradiction because if the given differential equation is divided by x^2 to get the form in Equation (8) in the text, then the resulting functions $p(x) = -4/x$ and $q(x) = 6/x^2$ are not continuous at $x = 0$.

30. (a) $y_2 = x^3$ and $y_2 = |x^3|$ are linearly independent because $x^3 = c\,|x^3|$ would require that $c = 1$ with $x = 1$, but $c = -1$ with $x = -1$.

(b) The fact that $W(y_1, y_2) = 0$ everywhere does not contradict Theorem 3, because when the given equation is written in the required form

$$y'' - (3/x)y' + (3/x^2)y = 0,$$

the coefficient functions $p(x) = -3/x$ and $q(x) = 3/x^2$ are not continuous at $x = 0$.

31. $W(y_1, y_2) = -2x$ vanishes at $x = 0$, whereas if y_1 and y_2 were (linearly independent) solutions of an equation $y'' + py' + qy = 0$ with p and q both continuous on an open interval I containing $x = 0$, then Theorem 3 would imply that $W \neq 0$ on I.

32. (a) $W = y_1y_2' - y_1'y_2$, so

$$\begin{aligned} AW' &= A(y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2') \\ &= y_1(Ay_2'') - y_2(Ay_1'') \\ &= y_1(-By_2' - Cy_2) - y_2(-By_1' - Cy_1) \\ &= -B(y_1y_2' - y_1'y_2) \end{aligned}$$

and thus $AW' = -BW$.

(b) Just separate the variables.

(c) Because the exponential factor is never zero.

In Problems 33–42 we give the characteristic equation, its roots, and the corresponding general solution.

33. $r^2 - 3r + 2 = 0$; $r = 1, 2$; $y(x) = c_1e^x + c_2e^{2x}$

34. $r^2 + 2r - 15 = 0$; $r = 3, -5$; $y(x) = c_1e^{-5x} + c_2e^{3x}$

35. $r^2 + 5r = 0$; $r = 0, -5$; $y(x) = c_1 + c_2e^{-5x}$

36. $2r^2 + 3r = 0$; $r = 0, -3/2$; $y(x) = c_1 + c_2 e^{-3x/2}$

37. $2r^2 - r - 2 = 0$; $r = 1, -1/2$; $y(x) = c_1 e^{-x/2} + c_2 e^x$

38. $4r^2 + 8r + 3 = 0$; $r = -1/2, -3/2$; $y(x) = c_1 e^{-x/2} + c_2 e^{-3x/2}$

39. $4r^2 + 4r + 1 = 0$; $r = -1/2, -1/2$; $y(x) = (c_1 + c_2 x)e^{-x/2}$

40. $9r^2 - 12r + 4 = 0$; $r = -2/3, -2/3$; $y(x) = (c_1 + c_2 x)e^{2x/3}$

41. $6r^2 - 7r - 20 = 0$; $r = -4/3, 5/2$; $y(x) = c_1 e^{-4x/3} + c_2 e^{5x/2}$

42. $35r^2 - r - 12 = 0$; $r = -4/7, 3/5$; $y(x) = c_1 e^{-4x/7} + c_2 e^{3x/5}$

In Problems 43–48 we first write and simplify the equation with the indicated characteristic roots, and then write the corresponding differential equation.

43. $(r-0)(r+10) = r^2 + 10r = 0$; $y'' + 10y' = 0$

44. $(r-10)(r+10) = r^2 - 100 = 0$; $y'' - 100y = 0$

45. $(r+10)(r+10) = r^2 + 20r + 100 = 0$; $y'' + 20y' + 100y = 0$

46. $(r-10)(r-100) = r^2 - 110r + 1000 = 0$; $y'' - 110y' + 1000y = 0$

47. $(r-0)(r-0) = r^2 = 0$; $y'' = 0$

48. $(r - 1 - \sqrt{2})(r - 1 + \sqrt{2}) = r^2 - 2r - 1 = 0$; $y'' - 2y' - y = 0$

49. The solution curve with $y(0) = 1$, $y'(0) = 6$ is $y(x) = 8e^{-x} - 7e^{-2x}$. We find that $y'(x) = 0$ when $x = \ln(7/4)$ so $e^{-x} = 4/7$ and $e^{-2x} = 16/49$. It follows that $y(\ln(7/4)) = 16/7$, so the high point on the curve is $(\ln(7/4)), 16/7) \approx (0.56, 2.29)$, which looks consistent with Fig. 2.1.6.

50. The two solution curves with $y(0) = a$ and $y(0) = b$ (as well as $y'(0) = 1$) are

$$y = (2a+1)e^{-x} - (a+1)e^{-2x},$$
$$y = (2b+1)e^{-x} - (b+1)e^{-2x}.$$

Subtraction and then division by $a - b$ gives $2e^{-x} = e^{-2x}$, so it follows that $x = -\ln 2$. Now substitution in either formula gives $y = -2$, so the common point of intersection is $(-\ln 2, -2)$.

51. **(a)** The substitution $v = \ln x$ gives

$$y' = \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dv}$$

Then another differentiation using the chain rule gives

$$\begin{aligned} y'' &= \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dv}\right) \\ &= -\frac{1}{x^2}\cdot\frac{dy}{dv}+\frac{1}{x}\cdot\frac{d}{dv}\left(\frac{dy}{dv}\right)\frac{dv}{dx} = -\frac{1}{x^2}\cdot\frac{dy}{dv}+\frac{1}{x^2}\cdot\frac{d^2y}{dv^2}. \end{aligned}$$

Substitution of these expressions for y' and y'' into Eq. (21) in the text then yields immediately the desired Eq. (22):

$$a\frac{d^2y}{dv^2}+(b-a)\frac{dy}{dv}+cy = 0.$$

(b) If the roots r_1 and r_2 of the characteristic equation of Eq. (22) are real and distinct, then a general solution of the original Euler equation is

$$y(x) = c_1e^{r_1v}+c_2e^{r_2v} = c_1\left(e^v\right)^{r_1}+c_2\left(e^v\right)^{r_2} = c_1x^{r_1}+c_2x^{r_2}.$$

52. The substitution $v = \ln x$ yields the converted equation $d^2y/dv^2 - y = 0$ whose characteristic equation $r^2-1=0$ has roots $r_1 = 1$ and $r_2 = -1$. Because $e^v = x$, the corresponding general solution is

$$y = c_1e^v+c_2e^{-v} = c_1x+\frac{c_2}{x}.$$

53. The substitution $v = \ln x$ yields the converted equation $d^2y/dv^2 + dy/dv - 12y = 0$ whose characteristic equation $r^2+r-12=0$ has roots $r_1 = -4$ and $r_2 = 3$. Because $e^v = x$, the corresponding general solution is

$$y = c_1e^{-4v}+c_2e^{3v} = c_1x^{-4}+c_2x^3.$$

54. The substitution $v = \ln x$ yields the converted equation $4d^2y/dv^2 + 4dy/dv - 3y = 0$ whose characteristic equation $4r^2+4r-3=0$ has roots $r_1 = -3/2$ and $r_2 = 1/2$. Because $e^v = x$, the corresponding general solution is

$$y = c_1 e^{-3v/2} + c_2 e^{v/2} = c_1 x^{-3/2} + c_2 x^{1/2}.$$

55. The substitution $v = \ln x$ yields the converted equation $d^2y/dv^2 = 0$ whose characteristic equation $r^2 = 0$ has repeated roots $r_1, r_2 = 0$. Because $v = \ln x$, the corresponding general solution is

$$y = c_1 + c_2 v = c_1 + c_2 \ln x.$$

56. The substitution $v = \ln x$ yields the converted equation $d^2y/dv^2 - 4\,dy/dv + 4y = 0$ whose characteristic equation $r^2 - 4r + 4 = 0$ has roots $r_1, r_2 = 2$. Because $e^v = x$, the corresponding general solution is

$$y = c_1 e^{2v} + c_2 v e^{2v} = x^2(c_1 + c_2 \ln v).$$

SECTION 2.2

GENERAL SOLUTIONS OF LINEAR EQUATIONS

Students should check each of Theorems 1 through 4 in this section to see that, in the case $n = 2$, it reduces to the corresponding theorem in Section 2.1. Similarly, the computational problems for this section largely parallel those for the previous section. By the end of Section 2.2 students should understand that, although we do not prove the existence-uniqueness theorem now, it provides the basis for everything we do with linear differential equations.

The linear combinations listed in Problems 1–6 were discovered "by inspection" — that is, by trial and error.

1. $(5/2)(2x) + (-8/3)(3x^2) + (-1)(5x - 8x^2) = 0$

2. $(-4)(5) + (5)(2 - 3x^2) + (1)(10 + 15x^2) = 0$

3. $(1)(0) + (0)(\sin x) + (0)(e^x) = 0$

4. $(1)(17) + (-17/2)(2\sin^2 x) + (-17/3)(3\cos^2 x) = 0$, because $\sin^2 x + \cos^2 x = 1$.

5. $(1)(17) + (-34)(\cos^2 x) + (17)(\cos 2x) = 0$, because $2\cos^2 x = 1 + \cos 2x$.

6. $(-1)(e^x) + (1)(\cosh x) + (1)(\sinh x) = 0$, because $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.

7. $W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$ is nonzero everywhere.

8. $W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$ is never zero.

9. $\mathrm{W} = e^x(\cos^2 x + \sin^2 x) = e^x \neq 0$

10. $\mathrm{W} = x^{-7}e^x(x+1)(x+4)$ is nonzero for $x > 0$.

11. $\mathrm{W} = x^3 e^{2x}$ is nonzero if $x \neq 0$.

12. $\mathrm{W} = x^{-2}[2\cos^2(\ln x) + 2\sin^2(\ln x)] = 2x^{-2}$ is nonzero for $x > 0$.

In each of Problems 13–20 we first form the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x),$$

then calculate $y'(x)$ and $y''(x)$, and finally impose the given initial conditions to determine the values of the coefficients c_1, c_2, c_3.

13. Imposition of the initial conditions $y(0) = 1,\ y'(0) = 2,\ y''(0) = 0$ on the general solution $y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad c_1 - c_2 - 2c_3 = 2, \quad c_1 + c_2 + 4c_3 = 0$$

with solution $c_1 = 4/3,\ c_2 = 0,\ c_3 = -1/3$. Hence the desired particular solution is given by $y(x) = (4e^x - e^{-2x})/3$.

14. Imposition of the initial conditions $y(0) = 0,\ y'(0) = 0,\ y''(0) = 3$ on the general solution $y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad c_1 + 2c_2 + 3c_3 = 2, \quad c_1 + 4c_2 + 9c_3 = 0$$

with solution $c_1 = 3/2,\ c_2 = -3,\ c_3 = 3/2$. Hence the desired particular solution is given by $y(x) = (3e^x - 6e^{2x} + 3e^{3x})/2$.

15. Imposition of the initial conditions $y(0) = 2,\ y'(0) = 0,\ y''(0) = 0$ on the general solution $y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^{3x}$ yields the three equations

$$c_1 = 2, \quad c_1 + c_2 = 0, \quad c_1 + 2c_2 + 2c_3 = 0$$

with solution $c_1 = 2,\ c_2 = -2,\ c_3 = 1$. Hence the desired particular solution is given by $y(x) = (2 - 2x + x^2)e^x$.

16. Imposition of the initial conditions $y(0) = 1,\ y'(0) = 4,\ y''(0) = 0$ on the general solution $y(x) = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 + 2c_2 + c_3 = 4, \quad c_1 + 4c_2 + 4c_3 = 0$$

with solution $c_1 = -12,\ c_2 = 13,\ c_3 = -10$. Hence the desired particular solution is given by $y(x) = -12e^x + 13e^{2x} - 10xe^{2x}$.

17. Imposition of the initial conditions $y(0) = 3,\ y'(0) = -1,\ y''(0) = 2$ on the general solution $y(x) = c_1 + c_2 \cos 3x + c_3 \sin 3x$ yields the three equations

$$c_1 + c_2 = 3, \quad 3c_3 = -1, \quad -9c_2 = 2$$

with solution $c_1 = 29/9,\ c_2 = -2/9,\ c_3 = -1/3$. Hence the desired particular solution is given by $y(x) = (29 - 2\cos 3x - 3\sin 3x)/9$.

18. Imposition of the initial conditions $y(0) = 1,\ y'(0) = 0,\ y''(0) = 0$ on the general solution $y(x) = e^x (c_1 + c_2 \cos x + c_3 \sin x)$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 + c_2 + c_3 = 0, \quad c_1 + 2c_3 = 0$$

with solution $c_1 = 2,\ c_2 = -1,\ c_3 = -1$. Hence the desired particular solution is given by $y(x) = e^x(2 - \cos x - \sin x)$.

19. Imposition of the initial conditions $y(1) = 6,\ y'(1) = 14,\ y''(1) = 22$ on the general solution $y(x) = c_1 x + c_2 x^2 + c_3 x^3$ yields the three equations

$$c_1 + c_2 + c_3 = 6, \quad c_1 + 2c_2 + 3c_3 = 14, \quad 2c_2 + 6c_3 = 22$$

with solution $c_1 = 1,\ c_2 = 2,\ c_3 = 3$. Hence the desired particular solution is given by $y(x) = x + 2x^2 + 3x^3$.

20. Imposition of the initial conditions $y(1) = 1,\ y'(1) = 5,\ y''(1) = -11$ on the general solution $y(x) = c_1 x + c_2 x^{-2} + c_3 x^{-2} \ln x$ yields the three equations

$$c_1 + c_2 = 1, \quad c_1 - 2c_2 + c_3 = 5, \quad 6c_2 - 5c_3 = -11$$

with solution $c_1 = 2,\ c_2 = -1,\ c_3 = 1$. Hence the desired particular solution is given by $y(x) = 2x - x^{-2} + x^{-2}\ln x$.

In each of Problems 21–24 we first form the general solution

$$y(x) = y_c(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x),$$

then calculate $y'(x)$, and finally impose the given initial conditions to determine the values of the coefficients c_1 and c_2.

21. Imposition of the initial conditions $y(0) = 2,\ y'(0) = -2$ on the general solution $y(x) = c_1 \cos x + c_2 \sin x + 3x$ yields the two equations $c_1 = 2,\ c_2 + 3 = -2$ with solution $c_1 = 2,\ c_2 = -5$. Hence the desired particular solution is given by $y(x) = 2\cos x - 5\sin x + 3x$.

22. Imposition of the initial conditions $y(0) = 0,\ y'(0) = 10$ on the general solution $y(x) = c_1 e^{2x} + c_2 e^{-2x} - 3$ yields the two equations $c_1 + c_2 - 3 = 0,\ 2c_1 - 2c_2 = 10$ with solution $c_1 = 4,\ c_2 = -1$. Hence the desired particular solution is given by $y(x) = 4e^{2x} - e^{-2x} - 3$.

23. Imposition of the initial conditions $y(0) = 3,\ y'(0) = 11$ on the general solution $y(x) = c_1 e^{-x} + c_2 e^{3x} - 2$ yields the two equations $c_1 + c_2 - 2 = 3,\ -c_1 + 3c_2 = 11$ with solution $c_1 = 1,\ c_2 = 4$. Hence the desired particular solution is given by $y(x) = e^{-x} + 4e^{3x} - 2$.

24. Imposition of the initial conditions $y(0) = 4,\ y'(0) = 8$ on the general solution $y(x) = c_1 e^x \cos x + c_2 e^x \cos x + x + 1$ yields the two equations $c_1 + 1 = 4,\ c_1 + c_2 + 1 = 8$ with solution $c_1 = 3,\ c_2 = 4$. Hence the desired particular solution is given by $y(x) = e^x(3\cos x + 4\sin x) + x + 1$.

25. $L[y] = L[y_1 + y_2] = L[y_1] + L[y_2] = f + g$

26. **(a)** $y_1 = 2$ and $y_2 = 3x$ **(b)** $y = y_1 + y_2 = 2 + 3x$

27. The equations

$$c_1 + c_2 x + c_3 x^2 = 0, \quad c_2 + 2c_3 x + 0, \quad 2c_3 = 0$$

(the latter two obtained by successive differentiation of the first one) evidently imply — by substituting $x = 0$ — that $c_1 = c_2 = c_3 = 0$.

28. If you differentiate the equation $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$ repeatedly, n times in succession, the result is the system

$$
\begin{aligned}
c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n &= 0 \\
c_1 + 2c_2 x + \cdots + n c_n x^{n-1} &= 0 \\
&\vdots \\
(n-1)!\,c_{n-1} + n!\,c_n x &= 0 \\
n!\,c_n &= 0
\end{aligned}
$$

of $n+1$ equations in the $n+1$ coefficients $c_0, c_1, c_2, \cdots, c_n$. Upon substitution of $x = 0$, the $(k+1)$st of these equations reduces to $k!\,c_k = 0$, so it follows that all these coefficients must vanish.

29. If $c_0 e^{rx} + c_1 x e^{rx} + \cdots + c_n x^n e^{rx} = 0$, then division by e^{rx} yields

$$c_0 + c_1 x + \cdots + c_n x^n = 0,$$

so the result of Problem 28 applies.

30. When the equation $x^2 y'' - 2xy' + 2y = 0$ is rewritten in standard form

$$y'' + (-2/x)y' + (2/x^2)y = 0,$$

the coefficient functions $p_1(x) = -2/x$ and $p_2(x) = 2/x^2$ are not continuous at $x = 0$. Thus the hypotheses of Theorem 3 are not satisfied.

31. **(a)** Substitution of $x = a$ in the differential equation gives $y''(a) = -p\,y'(a) - q(a)$.

(b) If $y(0) = 1$ and $y'(0) = 0$, then the equation $y'' - 2y' - 5y = 0$ implies that $y''(0) = 2y'(0) + 5y(0) = 5$.

32. Let the functions $y_1, y_2, \cdots, y_n$ be chosen as indicated. Then evaluation at $x = a$ of the $(k-1)$st derivative of the equation $c_1 y_1 + c_2 y_2 + \cdots c_n y_n = 0$ yields $c_k = 0$. Thus $c_1 = c_2 = \cdots = c_n = 0$, so the functions are linearly independent.

33. This follows from the fact that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-b)(c-a).$$

34. $W(f_1, f_2, \cdots, f_n) = V\exp(r_i x)$, and neither V nor $\exp(r_i x)$ vanishes.

36. If $y = vy_1$ then substitution of the derivatives

$$y' = vy_1' + v'y_1, \qquad y'' = vy_1'' + 2v'y_1' + v''y_1$$

in the differential equation $y'' + py' + qy = 0$ gives

$$\begin{aligned} \left[vy_1'' + 2v'y_1' + v''y_1\right] + p\left[vy_1' + v'y_1\right] + q\left[vy_1\right] &= 0, \\ v\left[y_1'' + py_1' + qy_1\right] + v''y_1 + 2v'y_1' + pv'y_1 &= 0. \end{aligned}$$

But the terms within brackets vanish because y_1 is a solution, and this leaves the equation

$$y_1 v'' + \left(2y_1' + py_1\right)v' = 0$$

that we can solve by writing

$$\begin{aligned} \frac{v''}{v'} &= -2\frac{y_1'}{y_1} - p \quad &&\Rightarrow \quad \ln v' = -2\ln y_1' - \int p(x)\,dx + \ln C, \\ v'(x) &= \frac{C}{y_1^2} e^{-\int p(x)dx} \quad &&\Rightarrow \quad v(x) = C\int \frac{e^{-\int p(x)dx}}{y_1^2}\,dx + K. \end{aligned}$$

With $C = 1$ and $K = 0$ this gives the second solution

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2}\,dx.$$

37. When we substitute $y = vx^3$ in the given differential equation and simplify, we get the separable equation $xv'' + v' = 0$ that we solve by writing

$$\begin{aligned} \frac{v''}{v'} &= -\frac{1}{x} \quad &&\Rightarrow \quad \ln v' = -\ln x + \ln A, \\ v' &= \frac{A}{x} \quad &&\Rightarrow \quad v(x) = A\ln x + B. \end{aligned}$$

With $A = 1$ and $B = 0$ we get $v(x) = \ln x$ and hence $y_2(x) = x^3 \ln x$.

38. When we substitute $y = vx^3$ in the given differential equation and simplify, we get the separable equation $xv'' + 7v' = 0$ that we solve by writing

$$\frac{v''}{v'} = -\frac{7}{x} \quad \Rightarrow \quad \ln v' = -7\ln x + \ln A,$$

$$v' = \frac{A}{x^7} \quad \Rightarrow \quad v(x) = -\frac{A}{6x^6} + B.$$

With $A = -6$ and $B = 0$ we get $v(x) = 1/x^6$ and hence $y_2(x) = 1/x^3$.

39. When we substitute $y = v e^{x/2}$ in the given differential equation and simplify, we eventually get the simple equation $v'' = 0$ with general solution $v(x) = Ax + B$. With $A = 1$ and $B = 0$ we get $v(x) = x$ and hence $y_2(x) = x e^{x/2}$.

40. When we substitute $y = vx$ in the given differential equation and simplify, we get the separable equation $v'' - v' = 0$ that we solve by writing

$$\frac{v''}{v'} = 1 \quad \Rightarrow \quad \ln v' = x + \ln A,$$

$$v' = Ae^x \quad \Rightarrow \quad v(x) = Ae^x + B.$$

With $A = 1$ and $B = 0$ we get $v(x) = e^x$ and hence $y_2(x) = xe^x$.

41. When we substitute $y = v e^x$ in the given differential equation and simplify, we get the separable equation $(1 + x)v'' + x v' = 0$ that we solve by writing

$$\frac{v''}{v'} = -\frac{x}{1+x} = -1 + \frac{1}{1+x} \quad \Rightarrow \quad \ln v' = -x + \ln(1+x) + \ln A,$$

$$v' = A(1+x)e^{-x} \quad \Rightarrow \quad v(x) = A\int(1+x)e^{-x}\,dx = -A(2+x)e^{-x} + B.$$

With $A = -1$ and $B = 0$ we get $v(x) = (2 + x)e^{-x}$ and hence $y_2(x) = 2 + x$.

42. When we substitute $y = v x$ in the given differential equation and simplify, we get the separable equation $x(x^2 - 1)v'' = 2v'$ that we solve by writing

$$\frac{v''}{v'} = \frac{2}{x(x^2-1)} = -\frac{2}{x}+\frac{1}{1+x}-\frac{1}{1-x},$$
$$\ln v' = -2\ln x+\ln(1+x)+\ln(1-x)+\ln A,$$
$$v' = \frac{A(1-x^2)}{x^2} = A\left(\frac{1}{x^2}-1\right) \quad\Rightarrow\quad v(x) = A\left(-\frac{1}{x}-x\right)+B.$$

With $A=-1$ and $B=0$ we get $v(x)=x+1/x$ and hence $y_2(x)=x^2+1$.

43. When we substitute $y = v\,x$ in the given differential equation and simplify, we get the separable equation $x(x^2-1)v''=(2-4x^2)v'$ that we solve by writing

$$\frac{v''}{v'} = \frac{2-4x^2}{x(x^2-1)} = -\frac{2}{x}-\frac{1}{1+x}+\frac{1}{1-x},$$
$$\ln v' = -2\ln x-\ln(1+x)-\ln(1-x)+\ln A,$$
$$v' = \frac{A}{x^2(1-x^2)} = A\left(\frac{1}{x^2}+\frac{1}{2(1+x)}+\frac{1}{2(1-x)}\right),$$
$$v(x) = A\left(-\frac{1}{x}+\frac{1}{2}\ln(1+x)-\frac{1}{2}\ln(1-x)\right)+B.$$

With $A=-1$ and $B=0$ we get

$$v(x)=\frac{1}{x}-\frac{1}{2}\ln(1+x)+\frac{1}{2}\ln(1-x) \quad\Rightarrow\quad y_2(x) = 1-\frac{x}{2}\ln\frac{1+x}{1-x}.$$

44. When we substitute $y = v\,x^{-1/2}\cos x$ in the given differential equation and simplify, we eventually get the separable equation $(\cos x)v''=2(\sin x)v'$ that we solve by writing

$$\frac{v''}{v'} = \frac{2\sin x}{\cos x} \quad\Rightarrow\quad \ln v' = -2\ln\left|\cos x\right|+\ln A = \ln\sec^2 x+\ln A,$$
$$v' = A\sec^2 x \quad\Rightarrow\quad v(x) = A\tan x+B.$$

With $A=1$ and $B=0$ we get $v(x) = \tan x$ and hence

$$y_2(x) = (\tan x)(x^{-1/2}\cos x) = x^{-1/2}\sin x.$$

SECTION 2.3

HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

This is a purely computational section devoted to the single most widely applicable type of higher order differential equations — linear ones with constant coefficients. In Problems 1–20, we write first the characteristic equation and its list of roots, then the corresponding general solution of the given differential equation. Explanatory comments are included only when the solution of the characteristic equation is not routine.

1. $r^2 - 4 = (r-2)(r+2) = 0; \quad r = -2, 2; \quad y(x) = c_1 e^{2x} + c_2 e^{-2x}$

2. $2r^2 - 3r = r(2r-3) = 0; \quad r = 0, 3/2; \quad y(x) = c_1 + c_2 e^{3x/2}$

3. $r^2 + 3r - 10 = (r+5)(r-2) = 0; \quad r = -5, 2; \quad y(x) = c_1 e^{2x} + c_2 e^{-5x}$

4. $2r^2 - 7r + 3 = (2r-1)(r-3) = 0; \quad r = 1/2, 3; \quad y(x) = c_1 e^{x/2} + c_2 e^{3x}$

5. $r^2 + 6r + 9 = (r+3)^2 = 0; \quad r = -3, -3; \quad y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

6. $r^2 + 5r + 5 = 0; \quad r = \left(-5 \pm \sqrt{5}\right)/2$

$y(x) = e^{-5x/2}[c_1 \exp(x\sqrt{5}/2) + c_2 \exp(-x\sqrt{5}/2)]$

7. $4r^2 - 12r + 9 = (2r-3)^2 = 0; \quad r = -3/2, -3/2; \quad y(x) = c_1 e^{3x/2} + c_2 x e^{3x/2}$

8. $r^2 - 6r + 13 = 0; \quad r = \left(6 \pm \sqrt{-16}\right)/2 = 3 \pm 2i; \quad y(x) = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

9. $r^2 + 8r + 25 = 0; \quad r = \left(-8 \pm \sqrt{-36}\right)/2 = -4 \pm 3i; \quad y(x) = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)$

10. $5r^4 + 3r^3 = r^3(5r+3) = 0; \quad r = 0, 0, 0, -3/5; \quad y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-3x/5}$

11. $r^4 - 8r^3 + 16r^2 = r^2(r-4)^2 = 0; \quad r = 0, 0, 4, 4; \quad y(x) = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}$

12. $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3 = 0; \quad r = 0, 1, 1, 1; \quad y(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$

13. $9r^3 + 12r^2 + 4r = r(3r+2)^2 = 0; \quad r = 0, -2/3, -2/3$

$y(x) = c_1 + c_2 e^{-2x/3} + c_3 x e^{-2x/3}$

14. $r^4+3r^2-4 = (r^2-1)(r^2+4) = 0; \quad r = -1, 1, \pm 2i$

$y(x) = c_1e^x + c_2e^{-x} + c_3\cos 2x + c_4\sin 2x$

15. $4r^4-8r^2+16 = (r^2-4)^2 = (r-2)^2(r+2)^2 = 0; \quad r = 2, 2, -2, -2$

$y(x) = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x}$

16. $r^4+18r^2+81 = (r^2+9)^2 = 0; \quad r = \pm 3i, \pm 3i$

$y(x) = (c_1 + c_2x)\cos 3x + (c_3 + c_4x)\sin 3x$

17. $6r^4+11r^2+4 = (2r^2+1)(3r^2+4) = 0; \quad r = \pm i/\sqrt{2}, \pm 2i/\sqrt{3},$

$y(x) = c_1\cos(x/\sqrt{2}) + c_2\sin(x/\sqrt{2}) + c_3\cos(2x/\sqrt{3}) + c_4\sin(2x/\sqrt{3})$

18. $r^4-16 = (r^2-4)(r^2+4) = 0; \quad r = -2, 2, \pm 2i$

$y(x) = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$

19. $r^3+r^2-r-1 = r(r^2-1)+(r^2-1) = (r-1)(r+1)^2 = 0; \quad r = 1, -1, -1;$

$y(x) = c_1e^x + c_2e^{-x} + c_3xe^{-x}$

20. $r^4 + 2r^3 + 3r^2 + 2r + 1 = (r^2 + r + 1)^2 = 0; \quad (-1 \pm \sqrt{3}\,i)/2, \ (-1 \pm \sqrt{3}\,i)/2$

$y = e^{-x/2}(c_1 + c_2x)\cos(x\sqrt{3}/2) + e^{-x/2}(c_3 + c_4x)\sin(x\sqrt{3}/2)$

21. Imposition of the initial conditions $y(0)=7,\ y'(0)=11$ on the general solution $y(x) = c_1e^x + c_2e^{3x}$ yields the two equations $c_1+c_2=7,\ c_1+3c_2=11$ with solution $c_1=5,\ c_2=2$. Hence the desired particular solution is $y(x) = 5e^x + 2e^{3x}$.

22. Imposition of the initial conditions $y(0)=3,\ y'(0)=4$ on the general solution $y(x) = e^{-x/3}\left[c_1\cos(x/\sqrt{3}) + c_2\sin(x/\sqrt{3})\right]$ yields the two equations $c_1=3,\ -c_1/3+c_2/\sqrt{3}=4$ with solution $c_1=3,\ c_2=5\sqrt{3}$. Hence the desired particular solution is $y(x) = e^{-x/3}\left[3\cos(x/\sqrt{3}) + 5\sqrt{3}\sin(x/\sqrt{3})\right]$.

23. Imposition of the initial conditions $y(0)=3,\ y'(0)=1$ on the general solution $y(x) = e^{3x}(c_1\cos 4x + c_2\sin 4x)$ yields the two equations $c_1=3,\ 3c_1+4c_2=1$ with solution $c_1=3,\ c_2=-2$. Hence the desired particular solution is $y(x) = e^{3x}(3\cos 4x - 2\sin 4x)$.

24. Imposition of the initial conditions $y(0)=1,\ y'(0)=-1,\ y''(0)=3$ on the general solution $y(x) = c_1 + c_2 e^{2x} + c_3 e^{-x/2}$ yields the three equations

$$c_1 + c_2 + c_3 = 1, \quad 2c_2 - c_3/2 = -1, \quad 4c_2 + c_3/4 = 3$$

with solution $c_1 = -7/2,\ c_2 = 1/2,\ c_3 = 4$. Hence the desired particular solution is $y(x) = (-7 + e^{2x} + 8e^{-x/2})/2$.

25. Imposition of the initial conditions $y(0)=-1,\ y'(0)=0,\ y''(0)=1$ on the general solution $y(x) = c_1 + c_2 x + c_3 e^{-2x/3}$ yields the three equations

$$c_1 + c_3 = -1, \quad c_2 - 2c_3/3 = 0, \quad 4c_3/9 = 1$$

with solution $c_1 = -13/4,\ c_2 = 3/2,\ c_3 = 9/4$. Hence the desired particular solution is $y(x) = (-13 + 6x + 9e^{-2x/3})/4$.

26. Imposition of the initial conditions $y(0)=1,\ y'(0)=-1,\ y''(0)=3$ on the general solution $y(x) = c_1 + c_2 e^{-5x} + c_3 x e^{-5x}$ yields the three equations

$$c_1 + c_2 = 3, \quad -5c_2 + c_3 = 4, \quad 25c_2 - 10c_3 = 5$$

with solution $c_1 = 24/5,\ c_2 = -9/5,\ c_3 = -5$. Hence the desired particular solution is $y(x) = (24 - 9e^{-5x} - 25xe^{-5x})/5$.

27. First we spot the root $r = 1$. Then long division of the polynomial $r^3 + 3r^2 - 4$ by $r - 1$ yields the quadratic factor $r^2 + 4r + 4 = (r+2)^2$ with roots $r = -2, -2$. Hence the general solution is $y(x) = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$.

28. First we spot the root $r = 2$. Then long division of the polynomial $2r^3 - r^2 - 5r - 2$ by the factor $r - 2$ yields the quadratic factor $2r^2 + 3r + 1 = (2r+1)(r+1)$ with roots $r = -1, -1/2$. Hence the general solution is $y(x) = c_1 e^{2x} + c_2 e^{-x} + c_3 e^{-x/2}$.

29. First we spot the root $r = -3$. Then long division of the polynomial $r^3 + 27$ by $r + 3$ yields the quadratic factor $r^2 - 3r + 9$ with roots $r = 3\left(1 \pm i\sqrt{3}\right)/2$. Hence the general solution is $y(x) = c_1 e^{-3x} + e^{3x/2}[c_2 \cos(3x\sqrt{3}/2) + c_3 \sin(3x\sqrt{3}/2)]$.

30. First we spot the root $r = -1$. Then long division of the polynomial

$$r^4 - r^3 + r^2 - 3r - 6$$

by $r+1$ yields the cubic factor $r^3 - 2r^2 + 3r - 6$. Next we spot the root $r = 2$, and another long division yields the quadratic factor $r^2 + 3$ with roots $r = \pm i\sqrt{3}$. Hence the general solution is $y(x) = c_1e^{-x} + c_2e^{2x} + c_3\cos x\sqrt{3} + c_4\sin x\sqrt{3}$.

31. The characteristic equation $r^3 + 3r^2 + 4r - 8 = 0$ has the evident root $r = 1$, and long division then yields the quadratic factor $r^2 + 4r + 8 = (r+2)^2 + 4$ corresponding to the complex conjugate roots $-2 \pm 2\,i$. Hence the general solution is

$$y(x) = c_1e^x + e^{-2x}(c_2\cos 2x + c_3\sin 2x).$$

32. The characteristic equation $r^4 + r^3 - 3r^2 - 5r - 2 = 0$ has root $r = 2$ that is readily found by trial and error, and long division then yields the factorization

$$(r-2)(r+1)^3 = 0.$$

Thus we obtain the general solution $y(x) = c_1e^{2x} + (c_2 + c_3x + c_4x^2)e^{-x}$.

33. Knowing that $y = e^{3x}$ is one solution, we divide the characteristic polynomial $r^3 + 3r^2 - 54$ by $r - 3$ and get the quadratic factor

$$r^2 + 6r + 18 = (r+3)^2 + 9.$$

Hence the general solution is $y(x) = c_1e^{3x} + e^{-3x}(c_2\cos 3x + c_3\sin 3x)$.

34. Knowing that $y = e^{2x/3}$ is one solution, we divide the characteristic polynomial $3r^3 - 2r^2 + 12r - 8$ by $3r - 2$ and get the quadratic factor $r^2 + 4$. Hence the general solution is

$$y(x) = c_1e^{2x/3} + c_2\cos 2x + c_3\sin 2x.$$

35. The fact that $y = \cos 2x$ is one solution tells us that $r^2 + 4$ is a factor of the characteristic polynomial

$$6r^4 + 5r^3 + 25r^2 + 20r + 4.$$

Then long division yields the quadratic factor $6r^2 + 5r + 1 = (3r+1)(2r+1)$ with roots $r = -1/2, -1/3$. Hence the general solution is

$$y(x) = c_1e^{-x/2} + c_2e^{-x/3} + c_3\cos 2x + c_4\sin 2x$$

36. The fact that $y = e^{-x}\sin x$ is one solution tells us that $(r+1)^2 + 1 = r^2 + 2r + 2$ is a factor of the characteristic polynomial

$$9r^3 + 11r^2 + 4r - 14.$$

Then long division yields the linear factor $9r - 7$. Hence the general solution is

$$y(x) = c_1 e^{7x/9} + e^{-x}(c_2 \cos x + c_3 \sin x).$$

37. The characteristic equation is $r^4 - r^3 = r^3(r-1) = 0$, so the general solution is $y(x) = A + Bx + Cx^2 + De^x$. Imposition of the given initial conditions yields the equations

$$A + D = 18, \quad B + D = 12, \quad 2C + D = 13, \quad D = 7$$

with solution $A = 11,\ B = 5,\ C = 3,\ D = 7$. Hence the desired particular solution is

$$y(x) = 11 + 5x + 3x^2 + 7e^x.$$

38. Given that $r = 5$ is one characteristic root, we divide $(r-5)$ into the characteristic polynomial $r^3 - 5r^2 + 100r - 500$ and get the remaining factor $r^2 + 100$. Thus the general solution is

$$y(x) = Ae^{5x} + B\cos 10x + C\sin 10x.$$

Imposition of the given initial conditions yields the equations

$$A + B = 0, \quad 5A + 10C = 10, \quad 25A - 100B = 250$$

with solution $A = 2,\ B = -2,\ C = 0$. Hence the desired particular solution is $y(x) = 2e^{5x} - 2\cos 10x$.

39. $(r-2)^3 = r^3 - 6r^2 + 12r - 8$, so the differential equation is

$$y''' - 6y'' + 12y' - 8y = 0.$$

40. $(r-2)(r^2+4) = r^3 - 2r^2 + 4r - 8$, so the differential equation is

$$y''' - 2y'' + 4y' - 8y = 0.$$

41. $(r^2+4)(r^2-4) = r^4 - 16$, so the differential equation is $y^{(4)} - 16y = 0$.

42. $(r^2+4)^3 = r^6 + 12r^4 + 48r^2 + 64$, so the differential equation is

$$y^{(6)} + 12y^{(4)} + 48y'' + 64y = 0.$$

44. **(a)** $x = i, -2i$ **(b)** $x = -i, 3i$

45. The characteristic polynomial is the quadratic polynomial of Problem 44(b). Hence the general solution is

$$y(x) = c_1 e^{-ix} + c_2 e^{3ix} = c_1(\cos x - i\sin x) + c_2(\cos 3x + i\sin 3x).$$

46. The characteristic polynomial is $r^2 - ir + 6 = (r+2i)(r-3i)$ so the general solution is

$$y(x) = c_1 e^{3ix} + c_2 e^{-2ix} = c_1(\cos 3x + i\sin 3x) + c_2(\cos 2x - i\sin 2x).$$

47. The characteristic roots are $r = \pm\sqrt{-2+2i\sqrt{3}} = \pm(1+i\sqrt{3})$ so the general solution is

$$y(x) = c_1 e^{(1+i\sqrt{3})x} + c_2 e^{-(1+i\sqrt{3})x} = c_1 e^{x}\left(\cos\sqrt{3}\,x + i\sin\sqrt{3}\,x\right) + c_2 e^{-x}\left(\cos\sqrt{3}\,x - i\sin\sqrt{3}\,x\right)$$

48. The general solution is $y(x) = Ae^{x} + Be^{\alpha x} + Ce^{\beta x}$ where $\alpha = (-1 + i\sqrt{3})/2$ and $\beta = (-1 - i\sqrt{3})/2$. Imposition of the given initial conditions yields the equations

$$\begin{aligned} A + \quad B + \quad C &= 1 \\ A + \alpha B + \beta C &= 0 \\ A + \alpha^2 B + \beta^2 C &= 0 \end{aligned}$$

that we solve for $A = B = C = 1/3$. Thus the desired particular solution is given by $y(x) = \frac{1}{3}\left(e^{x} + e^{(-1+i\sqrt{3})x/2} + e^{(-1-\sqrt{3})x/2}\right)$, which (using Euler's relation) reduces to the given real-valued solution.

49. The general solution is $y = Ae^{2x} + Be^{-x} + C\cos x + D\sin x$. Imposition of the given initial conditions yields the equations

$$\begin{aligned} A + B + C \quad\;\; &= 0 \\ 2A - B \quad + D &= 0 \\ 4A + B - C \quad\;\; &= 0 \\ 8A - B \quad - D &= 30 \end{aligned}$$

that we solve for $A = 2$, $B = -5$, $C = 3$, and $D = -9$. Thus

$$y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x.$$

50. If $x > 0$ then the differential equation is $y'' + y = 0$ with general solution $y = A\cos x + B\sin x$.. But if $x < 0$ it is $y'' - y = 0$ with general solution $y = C\cosh x + D\sin x$. To satisfy the initial conditions $y_1(0) = 1,\ y_1'(0) = 0$ we choose

$A = C = 1$ and $B = D = 0$. But to satisfy the initial conditions $y_2(0) = 0$, $y_2'(0) = 1$ we choose $A = C = 0$ and $B = D = 1$. The corresponding solutions are defined by

$$y_1(x) = \begin{cases} \cos x & \text{if } x \geq 0, \\ \cosh x & \text{if } x \leq 0; \end{cases} \qquad y_2(x) = \begin{cases} \sin x & \text{if } x \geq 0, \\ \sinh x & \text{if } x \leq 0. \end{cases}$$

51. In the solution of Problem 51 in Section 2.1 we showed that the substitution $v = \ln x$ gives

$$y' = \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dv} \quad \text{and} \quad y'' = \frac{d^2y}{dx^2} = -\frac{1}{x^2}\cdot\frac{dy}{dv} + \frac{1}{x^2}\cdot\frac{d^2y}{dv^2}.$$

A further differentiation using the chain rule gives

$$y''' = \frac{d^3y}{dx^3} = \frac{2}{x^3}\cdot\frac{dy}{dv} - \frac{3}{x^3}\cdot\frac{d^2y}{dv^2} + \frac{1}{x^3}\cdot\frac{d^3y}{dv^3}.$$

Substitution of these expressions for y', y'', and y''' into the third-order Euler equation $ax^3y''' + bx^2y'' + cx\,y' + d\,y = 0$ and collection of coefficients quickly yields the desired constant-coefficient equation

$$a\frac{d^3y}{dv^3} + (b - 3a)\frac{d^2y}{dv^2} + (c - b + 2a)\frac{dy}{dv} + d\,y = 0.$$

In Problems 52 through 58 we list first the transformed constant-coefficient equation, then its characteristic equation and roots, and finally the corresponding general solution with $v = \ln x$ and $e^v = x$.

52. $\dfrac{d^2y}{dv^2} + 9y = 0$; $\quad r^2 + 9 = 0$; $\quad r = \pm 3i$

$y(x) = c_1\cos(3v) + c_2\sin(3v) = c_1\cos(3\ln x) + c_2\sin(3\ln x)$

53. $\dfrac{d^2y}{dv^2} + 6\dfrac{dy}{dv} + 25y = 0$; $\quad r^2 + 6r + 25 = 0$; $\quad r = -3 \pm 4i$

$y(x) = e^{-3v}\left[c_1\cos(4v) + c_2\sin(4v)\right] = x^{-3}\left[c_1\cos(4\ln x) + c_2\sin(4\ln x)\right]$

54. $\dfrac{d^3y}{dv^3} + 3\dfrac{d^2y}{dv^2} = 0$; $\quad r^3 + 3r^2 = 0$; $\quad r = 0,\ 0,\ -3$

$y(x) = c_1 + c_2 v + c_3 e^{-3v} = c_1 + c_2\ln x + c_3 x^{-3}$

55. $\dfrac{d^3y}{dv^3} - 4\dfrac{d^2y}{dv^2} + 4\dfrac{dy}{dv} = 0$; $\quad r^3 - 4r^2 + 4r = 0$; $\quad r = 0,\ 2,\ 2$

$y(x) = c_1 + c_2 e^{2v} + c_3 v e^{2v} = c_1 + x^2\left(c_2 + c_3\ln x\right)$

56. $\dfrac{d^3y}{dv^3} = 0; \quad r^3 = 0; \quad r = 0,\ 0,\ 0$

$y(x) = c_1 + c_2 v + c_3 v^2 = c_1 + c_2 \ln x + c_3 (\ln x)^2$

57. $\dfrac{d^3y}{dv^3} - 5\dfrac{d^2y}{dv^2} + 5\dfrac{dy}{dv} = 0; \quad r^3 - 4r^2 + 4r = 0; \quad r = 0,\ 3 \pm \sqrt{3}$

$y(x) = c_1 + c_2 e^{(3-\sqrt{3})v} + c_3 v e^{(3+\sqrt{3})v} = c_1 + x^3\left(c_2 x^{-\sqrt{3}} + c_3 x^{+\sqrt{3}}\right)$

58. $\dfrac{d^3y}{dv^3} + 3\dfrac{d^2y}{dv^2} + 3\dfrac{dy}{dv} + y = 0; \quad r^3 + 3r^2 + 3r + 1 = 0; \quad r = -1, -1, -1$

$y(x) = c_1 e^{-v} + c_2 v e^{-v} + c_3 v^2 e^{-v} = x^{-1}\left[c_1 + c_2 \ln x + c_3 (\ln x)^2\right]$

SECTION 2.4

Mechanical Vibrations

In this section we discuss four types of free motion of a mass on a spring — undamped, underdamped, critically damped, and overdamped. However, the undamped and underdamped cases — in which actual oscillations occur — are emphasized because they are both the most interesting and the most important cases for applications.

1. Frequency: $\omega_0 = \sqrt{k/m} = \sqrt{16/4} = 2 \text{ rad/sec} = 1/\pi \text{ Hz}$
   Period: $P = 2\pi/\omega_0 = 2\pi/2 = \pi \text{ sec}$

2. Frequency $\omega_0 = \sqrt{k/m} = \sqrt{48/0.75} = 8 \text{ rad/sec} = 4/\pi \text{ Hz}$
   Period: $P = 2\pi/\omega_0 = 2\pi/8 = \pi/4 \text{ sec}$

3. The spring constant is $k = 15$ N/0.20 m $= 75$ N/m. The solution of $3x'' + 75x = 0$ with $x(0) = 0$ and $x'(0) = -10$ is $x(t) = -2 \sin 5t$. Thus the amplitude is 2 m; the frequency is $\omega_0 = \sqrt{k/m} = \sqrt{75/3} = 5 \text{ rad/sec} = 2.5/\pi \text{ Hz}$; and the period is $2\pi/5$ sec.

4. **(a)** With $m = 1/4$ kg and $k = (9 \text{ N})/(0.25 \text{ m}) = 36$ N/m we find that $\omega_0 = 12$ rad/sec. The solution of $x'' + 144x = 0$ with $x(0) = 1$ and $x'(0) = -5$ is

$$\begin{aligned} x(t) &= \cos 12t - (5/12)\sin 12t \\ &= (13/12)[(12/13)\cos 12t - (5/13)\sin 12t] \\ x(t) &= (13/12)\cos(12t - \alpha) \end{aligned}$$

where $\alpha = 2\pi - \tan^{-1}(5/12) \approx 5.8884$.

(b) $C = 13/12 \approx 1.0833$ m and $T = 2\pi/12 \approx 0.5236$ sec.

5. The gravitational acceleration at distance R from the center of the earth is $g = GM/R^2$. According to Equation (6) in the text the (circular) frequency ω of a pendulum is given by $\omega^2 = g/L = GM/R^2L$, so its period is $p = 2\pi/\omega = 2\pi R\sqrt{L/GM}$.

6. If the pendulum in the clock executes n cycles per day (86400 sec) at Paris, then its period is $p_1 = 86400/n$ sec. At the equatorial location it takes 24 hr 2 min 40 sec = 86560 sec for the same number of cycles, so its period there is $p_2 = 86560/n$ sec. Now let $R_1 = 3956$ mi be the Earth's "radius" at Paris, and R_2 its "radius" at the equator. Then substitution in the equation $p_1/p_2 = R_1/R_2$ of Problem 5 (with $L_1 = L_2$) yields $R_2 = 3963.33$ mi. Thus this (rather simplistic) calculation gives 7.33 mi as the thickness of the Earth's equatorial bulge.

7. The period equation $p = 3960\sqrt{100.10} = (3960 + x)\sqrt{100}$ yields $x \approx 1.9795$ mi $\approx$ 10,450 ft for the altitude of the mountain.

8. Let n be the number of cycles required for a correct clock with unknown pendulum length L_1 and period p_1 to register 24 hrs = 86400 sec, so $np_1 = 86400$. The given clock with length $L_2 = 30$ in and period p_2 loses 10 min = 600 sec per day, so $np_2 = 87000$. Then the formula of Problem 5 yields

$$\sqrt{\frac{L_1}{L_2}} = \frac{p_1}{p_2} = \frac{np_1}{np_2} = \frac{86400}{87000},$$

so $L_1 = (30)(86400/87000)^2 \approx 29.59$ in.

10. The $F = ma$ equation $\rho\pi r^2hx'' = \rho\pi r^2hg - \pi r^2xg$ simplifies to

$$x'' + (g/\rho h)x = g.$$

The solution of this equation with $x(0) = x'(0) = 0$ is

$$x(t) = \rho h(1 - \cos\omega_0 t)$$

where $\omega_0 = \sqrt{g/\rho h}$. With the given numerical values of ρ, h, and g, the amplitude of oscillation is $\rho h = 100$ cm and the period is $p = 2\pi\sqrt{\rho h/g} \approx 2.01$ sec.

11. The fact that the buoy weighs 100 lb means that $mg = 100$ so $m = 100/32$ slugs. The weight of water is 62.4 lb/ft^3, so the $F = ma$ equation of Problem 10 is

$$(100/32)x'' = 100 - 62.4\pi r^2 x.$$

It follows that the buoy's circular frequency ω is given by

$$\omega^2 = (32)(62.4\pi)r^2/100.$$

But the fact that the buoy's period is $p = 2.5$ sec means that $\omega = 2\pi/2.5$. Equating these two results yields $r \approx 0.3173$ ft ≈ 3.8 in.

12. **(a)** Substitution of $M_r = (r/R)^3 M$ in $F_r = -GM_r m/r^2$ yields

$$F_r = -(GMm/R^3)r.$$

(b) Because $GM/R^3 = g/R$, the equation $mr'' = F_r$ yields the differential equation

$$r'' + (g/R)r = 0.$$

(c) The solution of this equation with $r(0) = R$ and $r'(0) = 0$ is $r(t) = R\cos\omega_0 t$ where $\omega_0 = \sqrt{g/R}$. Hence, with $g = 32.2$ ft/sec^2 and $R = (3960)(5280)$ ft, we find that the period of the particle's simple harmonic motion is

$$p = 2\pi/\omega_0 = 2\pi\sqrt{R/g} \approx 5063.10 \text{ sec} \approx 84.38 \text{ min}.$$

13. **(a)** The characteristic equation $10r^2 + 9r + 2 = (5r+2)(2r+1) = 0$ has roots $r = -2/5, -1/2$. When we impose the initial conditions $x(0) = 0,\ x'(0) = 5$ on the general solution $x(t) = c_1 e^{-2t/5} + c_2 e^{-t/2}$ we get the particular solution $x(t) = 50\left(e^{-2t/5} - e^{-t/2}\right)$.

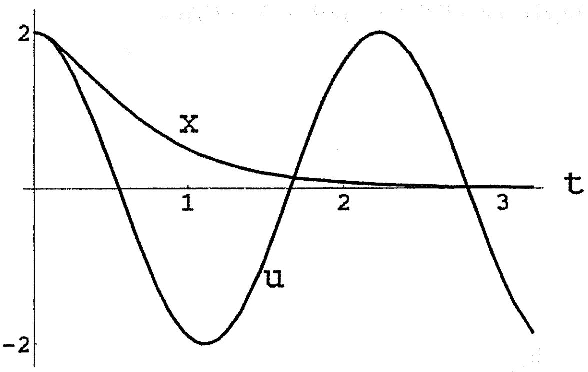
(b) The derivative $x'(t) = 25e^{-t/2} - 20e^{-2t/5} = 5e^{-2t/5}\left(5e^{-t/10} - 4\right) = 0$ when $t = 10\ln(5/4) \approx 2.23144$. Hence the mass's farthest distance to the right is given by $x(10\ln(5/4)) = 512/125 = 4.096$.

14. **(a)** The characteristic equation $25r^2 + 10r + 226 = (5r+1)^2 + 15^2 = 0$ has roots $r = (-1 \pm 15i)/5 = -1/5 \pm 3i$. When we impose the initial conditions $x(0) = 20,\ x'(0) = 41$ on the general solution $x(t) = e^{-t/5}\left(A\cos 3t + B\sin 3t\right)$ we get $A = 20,\ B = 15$. The corresponding particular solution is given by

$x(t) = e^{-t/5}(20\cos 3t + 15\sin 3t) = 25e^{-t/5}\cos(3t - \alpha)$ where $\alpha = \tan^{-1}(3/4) \approx 0.6435$.

(b) Thus the oscillations are "bounded" by the curves $x = \pm 25e^{-t/5}$ and the pseudoperiod of oscillation is $T = 2\pi/3$ (because $\omega = 3$).

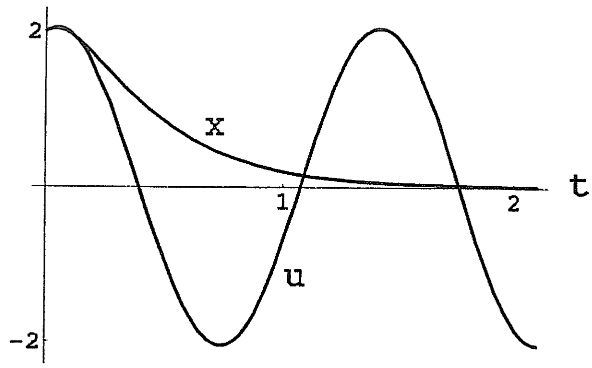
15. **With damping** The characteristic equation $(1/2)r^2 + 3r + 4 = 0$ has roots $r = -2, -4$. When we impose the initial conditions $x(0) = 2$, $x'(0) = 0$ on the general solution $x(t) = c_1e^{-2t} + c_2e^{-4t}$ we get the particular solution $x(t) = 4e^{-2t} - 2e^{-4t}$ that describes overdamped motion.
Without damping The characteristic equation $(1/2)r^2 + 4 = 0$ has roots $r = \pm 2i\sqrt{2}$. When we impose the initial conditions $x(0) = 2$, $x'(0) = 0$ on the general solution $u(t) = A\cos(2\sqrt{2}\,t) + B\sin(2\sqrt{2}\,t)$ we get the particular solution $u(t) = 2\cos(2\sqrt{2}\,t)$. The graphs of $x(t)$ and $u(t)$ are shown in the following figure.



16. **With damping** The characteristic equation $3r^2 + 30r + 63 = 0$ has roots $r = -3, -7$. When we impose the initial conditions $x(0) = 2$, $x'(0) = 2$ on the general solution $x(t) = c_1e^{-3t} + c_2e^{-7t}$ we get the particular solution $x(t) = 4e^{-3t} - 2e^{-7t}$ that describes overdamped motion.
Without damping The characteristic equation $3r^2 + 63 = 0$ has roots $r = \pm i\sqrt{21}$. When we impose the initial conditions $x(0) = 2$, $x'(0) = 2$ on the general solution $u(t) = A\cos(\sqrt{21}\,t) + B\sin(\sqrt{21}\,t)$ we get the particular solution

$$u(t) = 2\cos(\sqrt{21}\,t) + \frac{2}{\sqrt{21}}\sin(\sqrt{21}\,t) \approx 2\sqrt{\frac{22}{21}}\cos(\sqrt{21}\,t - 0.2149).$$

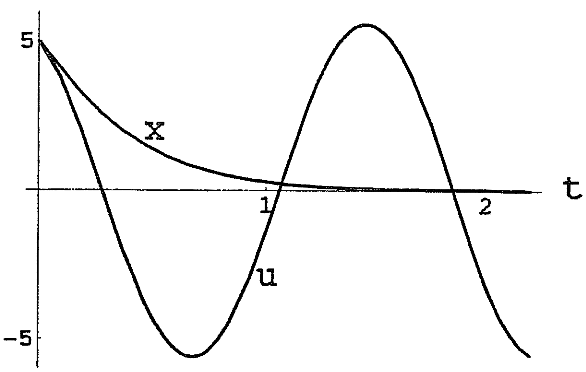
The graphs of $x(t)$ and $u(t)$ are shown in the figure at the top of the next page.



17. **With damping** The characteristic equation $r^2 + 8r + 16 = 0$ has roots $r = -4, -4$. When we impose the initial conditions $x(0) = 5,\ x'(0) = -10$ on the general solution $x(t) = (c_1 + c_2 t)e^{-4t}$ we get the particular solution $x(t) = 5e^{-4t}(2t+1)$ that describes critically damped motion.
Without damping The characteristic equation $r^2 + 16 = 0$ has roots $r = \pm 4i$. When we impose the initial conditions $x(0) = 5,\ x'(0) = -10$ on the general solution $u(t) = A\cos(4t) + B\sin(4t)$ we get the particular solution

$$u(t) = 5\cos(4t) + \frac{5}{2}\sin(4t) \approx \frac{5}{2}\sqrt{5}\cos(4t - 5.8195).$$

The graphs of $x(t)$ and $u(t)$ are shown in the following figure.

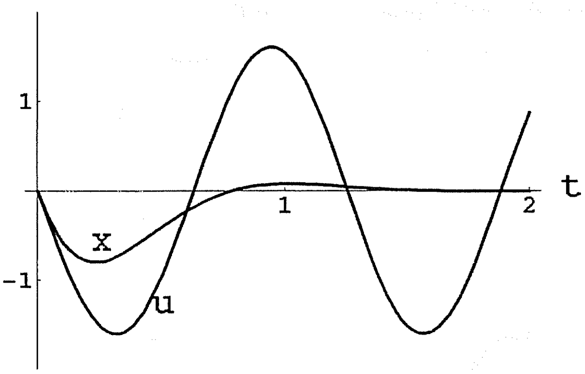


18. **With damping** The characteristic equation $2r^2 + 12r + 50 = 0$ has roots $r = -3 \pm 4i$. When we impose the initial conditions $x(0) = 0,\ x'(0) = -8$ on the general solution $x(t) = e^{-3t}(A\cos 4t + B\sin 4t)$ we get the particular solution $x(t) = -2e^{-3t}\sin 4t \ = \ 2e^{-3t}\cos(4t - 3\pi/2)$ that describes underdamped motion.

Without damping The characteristic equation $2r^2+50=0$ has roots $r=\pm 5i$. When we impose the initial conditions $x(0)=0,\ x'(0)=-8$ on the general solution $u(t)=A\cos(5t)+B\sin(5t)$ we get the particular solution

$$u(t)=-\frac{8}{5}\sin(5t)=\frac{8}{5}\cos\left(5t-\frac{3\pi}{2}\right).$$

The graphs of $x(t)$ and $u(t)$ are shown in the following figure.



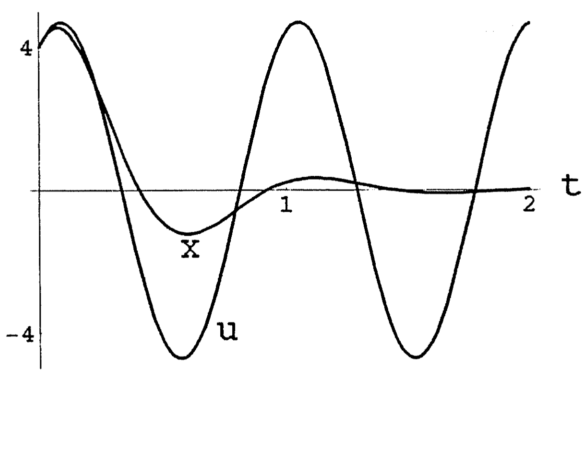
19. The characteristic equation $4r^2+20r+169=0$ has roots $r=-5/2\pm 6i$. When we impose the initial conditions $x(0)=4,\ x'(0)=16$ on the general solution $x(t)=e^{-5t/2}\left(A\cos 6t+B\sin 6t\right)$ we get the particular solution

$$x(t)\ =\ e^{-5t/2}\left[\,4\cos 6t+\frac{13}{3}\sin 6t\,\right]\ \approx\ \frac{1}{3}\sqrt{313}\ e^{-5t/2}\cos(6t-0.8254)$$

that describes underdamped motion.

Without damping The characteristic equation $4r^2+169=0$ has roots $r=\pm 13i/2$. When we impose the initial conditions $x(0)=4,\ x'(0)=16$ on the general solution $u(t)=A\cos(13t/2)+B\sin(13t/2)$ we get the particular solution

$$u(t)\ =\ 4\cos\left(\frac{13t}{2}\right)+\frac{32}{13}\sin\left(\frac{13t}{2}\right)\ \approx\ \frac{4}{13}\sqrt{233}\cos\left(\frac{13}{2}t-0.5517\right).$$



20. **With damping** The characteristic equation $2r^2 + 16r + 40 = 0$ has roots $r = -4 \pm 2i$. When we impose the initial conditions $x(0) = 5$, $x'(0) = 4$ on the general solution $x(t) = e^{-4t}(A\cos 2t + B\sin 2t)$ we get the particular solution

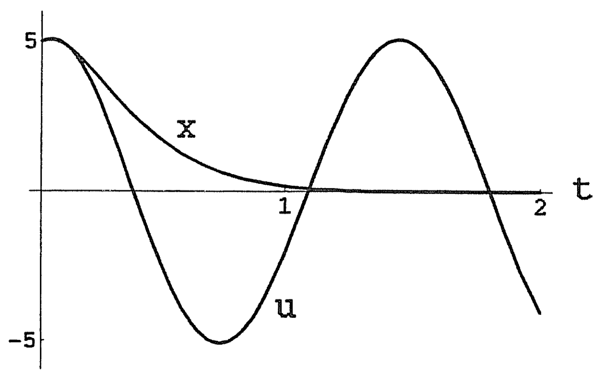
$$x(t) = e^{-4t}(5\cos 2t + 12\sin 2t) \approx 13\, e^{-4t}\cos(2t - 1.1760)$$

that describes underdamped motion.

Without damping The characteristic equation $2r^2 + 40 = 0$ has roots $r = \pm 2\sqrt{5}\, i$. When we impose the initial conditions $x(0) = 5$, $x'(0) = 4$ on the general solution $u(t) = A\cos(2\sqrt{5}\,t) + B\sin(2\sqrt{5}\,t)$ we get the particular solution

$$u(t) = 5\cos\left(2\sqrt{5}\,t\right) + \frac{2}{\sqrt{5}}\sin\left(2\sqrt{5}\,t\right) \approx \sqrt{\frac{129}{5}}\cos\left(2\sqrt{5}\,t - 0.1770\right).$$

The graphs of $x(t)$ and $u(t)$ are shown in the following figure.



21. **With damping** The characteristic equation $r^2 + 10r + 125 = 0$ has roots $r = -5 \pm 10i$. When we impose the initial conditions $x(0) = 6$, $x'(0) = 50$ on the general solution $x(t) = e^{-5t}(A\cos 10t + B\sin 10t)$ we get the particular solution

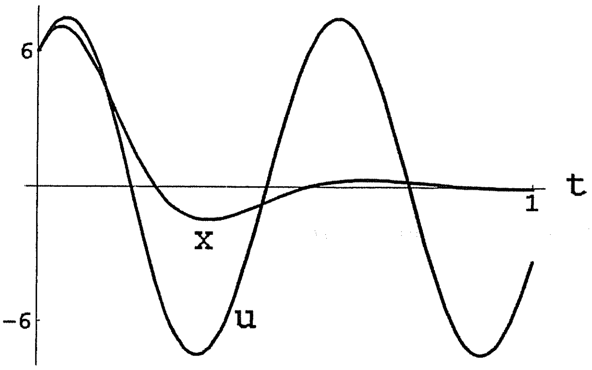
$$x(t) = e^{-5t}(6\cos 10t + 8\sin 10t) \approx 10\, e^{-5t}\cos(10t - 0.9273)$$

that describes underdamped motion.

Without damping The characteristic equation $r^2 + 125 = 0$ has roots $r = \pm 5\sqrt{5}\, i$. When we impose the initial conditions $x(0) = 6$, $x'(0) = 50$ on the general solution $u(t) = A\cos(5\sqrt{5}\,t) + B\sin(5\sqrt{5}\,t)$ we get the particular solution

$$u(t) = 6\cos\left(5\sqrt{5}\,t\right) + 2\sqrt{5}\sin\left(5\sqrt{5}\,t\right) \approx 2\sqrt{14}\cos\left(5\sqrt{5}\,t - 0.6405\right).$$

The graphs of $x(t)$ and $u(t)$ are shown in the figure at the top of the next page.



22. **(a)** With $m = 12/32 = 3/8$ slug, $c = 3$ lb-sec/ft, and $k = 24$ lb/ft, the differential equation is equivalent to $3x'' + 24x' + 192x = 0$. The characteristic equation $3r^2 + 24r + 192 = 0$ has roots $r = -4 \pm 4\sqrt{3}\,i$. When we impose the initial conditions $x(0) = 1,\ x'(0) = 0$ on the general solution $x(t) = e^{-4t}\left(A\cos 4t\sqrt{3} + B\sin 4t\sqrt{3}\right)$ we get the particular solution

$$\begin{aligned} x(t) &= e^{-4t}[\cos 4t\sqrt{3} + (1/\sqrt{3})\sin 4t\sqrt{3}\,] \\ &= (2/\sqrt{3})e^{-4t}[(\sqrt{3}/2)\cos 4t\sqrt{3} + (1/2)\sin 4t\sqrt{3}\,] \\ x(t) &= (2/\sqrt{3})e^{-4t}\cos(4t\sqrt{3} - \pi/6). \end{aligned}$$

(b) The time-varying amplitude is $2/\sqrt{3} \approx 1.15$ ft; the frequency is $4\sqrt{3} \approx 6.93$ rad/sec; and the phase angle is $\pi/6$.

23. **(a)** With $m = 100$ slugs we get $\omega = \sqrt{k/100}$. But we are given that

$$\omega = (80 \text{ cycles/min})(2\pi)(1 \text{ min}/60 \text{ sec}) = 8\pi/3,$$

and equating the two values yields $k \approx 7018$ lb/ft.

(b) With $\omega_1 = 2\pi(78/60)\text{ sec}^{-1}$, Equation (21) in the text yields $c \approx 372.31$ lb/(ft/sec). Hence $p = c/2m \approx 1.8615$. Finally $e^{-pt} = 0.01$ gives $t \approx 2.47$ sec.

30. In the underdamped case we have

$$\begin{aligned} x(t) &= e^{-pt}[A\cos\omega_1 t + B\sin\omega_1 t], \\ x'(t) &= -pe^{pt}[A\cos\omega_1 t + B\sin\omega_1 t] + e^{-pt}[-A\omega_1\sin\omega_1 t + B\omega_1\cos\omega_1 t]. \end{aligned}$$

The conditions $x(0) = x_0,\ x'(0) = v_0$ yield the equations $A = x_0$ and $-pA + B\omega_1 = v_0$, whence $B = (v_0 + px_0)/\omega_1$.

31. The binomial series

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots$$

converges if $|x|<1$. (See, for instance, Section 11.8 of Edwards and Penney, *Calculus*, 6th edition, Prentice Hall, 2002.) With $\alpha = 1/2$ and $x = -c^2/4mk$ in Eq. (21) of Section 2.4 in this text, the binomial series gives

$$\omega_1 = \sqrt{\omega_0^2 - p^2} = \sqrt{\frac{k}{m}-\frac{c^2}{4m^2}} = \sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{4mk}}$$
$$= \sqrt{\frac{k}{m}}\left(1-\frac{c^2}{8mk}-\frac{c^4}{128m^2k^2}-\cdots\right) \approx \omega_0\left(1-\frac{c^2}{8mk}\right).$$

32. If $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$ then

$$x'(t) = -pCe^{-pt}\cos(\omega_1 t - \alpha) + C\omega_1 e^{-pt}\sin(\omega_1 t - \alpha) = 0$$

yields $\tan(\omega_1 t - \alpha) = -p/\omega_1$.

33. If $x_1 = x(t_1)$ and $x_2 = x(t_2)$ are two successive local maxima, then $\omega_1 t_2 = \omega_1 t_1 + 2\pi$ so

$$x_1 = C\exp(-pt_1)\cos(\omega_1 t_1 - \alpha),$$
$$x_2 = C\exp(-pt_2)\cos(\omega_1 t_2 - \alpha) = C\exp(-pt_2)\cos(\omega_1 t_1 - \alpha).$$

Hence $x_1/x_2 = \exp[-p(t_1 - t_2)]$, and therefore

$$\ln(x_1/x_2) = -p(t_1 - t_2) = 2\pi p/\omega_1.$$

34. With $t_1 = 0.34$ and $t_2 = 1.17$ we first use the equation $\omega_1 t_2 = \omega_1 t_1 + 2\pi$ from Problem 32 to calculate $\omega_1 = 2\pi/(0.83) \approx 7.57$ rad/sec. Next, with $x_1 = 6.73$ and $x_2 = 1.46$, the result of Problem 33 yields

$$p = (1/0.83)\ln(6.73/1.46) \approx 1.84.$$

Then Equation (16) in this section gives

$$c = 2mp = 2(100/32)(1.84) \approx 11.51 \text{ lb-sec/ft},$$

and finally Equation (21) yields

$$k = (4m^2\omega_1^2 + c^2)/4m \approx 189.68 \text{ lb/ft}.$$

35. The characteristic equation $r^2+2r+1=0$ has roots $r=-1,-1$. When we impose the initial conditions $x(0)=0,\ x'(1)=0$ on the general solution $x(t)=(c_1+c_2t)e^{-t}$ we get the particular solution $x_1(t) \;=\; t\,e^{-t}$.

36. The characteristic equation $r^2+2r+(1-10^{-2n})=0$ has roots $r=-1\pm10^{-n}$. When we impose the initial conditions $x(0)=0,\ x'(1)=0$ on the general solution

$$x(t) \;=\; c_1\exp\left[\left(-1+10^{-n}\right)t\right]+c_1\exp\left[\left(-1-10^{-n}\right)t\right]$$

we get the equations

$$c_1+c_2 = 0, \qquad \left(-1+10^{-n}\right)c_1+\left(-1-10^{-n}\right)c_2 \;=\; 1$$

with solution $c_1=2^{n-1}5^n,\ c_2=2^{n-1}5^n$. This gives the particular solution

$$x_2(t) \;=\; 10^n e^{-t}\left(\frac{\exp(10^{-n}t)-\exp(-10^{-n}t)}{2}\right) \;=\; 10^n e^{-t}\sinh(10^{-n}t).$$

37. The characteristic equation $r^2+2r+(1+10^{-2n})=0$ has roots $r=-1\pm10^{-n}i$. When we impose the initial conditions $x(0)=0,\ x'(1)=0$ on the general solution

$$x(t) \;=\; e^{-t}\left[A\cos\left(10^{-n}t\right)+B\sin\left(10^{-n}t\right)\right]$$

we get the equations $c_1 = 0, \quad -c_1+10^{-n}c_2 \;=\; 1$ with solution $c_1=0,\ c_2=10^n$. This gives the particular solution $x_3(t) \;=\; 10^n e^{-t}\sin(10^{-n}t)$.

38. $$\lim_{n\to\infty} x_2(t) \;=\; \lim_{n\to\infty} 10^n e^{-t}\sinh(10^{-n}t) \;=\; t\,e^{-t}\cdot\lim_{n\to\infty}\frac{\sinh(10^{-n}t)}{10^{-n}t} \;=\; t\,e^{-t} \quad\text{and}$$

$$\lim_{n\to\infty} x_3(t) \;=\; \lim_{n\to\infty} 10^n e^{-t}\sin(10^{-n}t) \;=\; t\,e^{-t}\cdot\lim_{n\to\infty}\frac{\sin(10^{-n}t)}{10^{-n}t} \;=\; t\,e^{-t},$$

using the fact that $\lim_{\theta\to0}(\sin\theta)/\theta \;=\; \lim_{\theta\to0}(\sinh\theta)/\theta \;=\; 0$ (by L'Hôpital's rule, for instance).

SECTION 2.5

NONHOMOGENEOUS EQUATIONS AND THE METHOD OF UNDETERMINED COEFFICIENTS

The method of undetermined coefficients is based on "educated guessing". If we can guess correctly the **form** of a particular solution of a nonhomogeneous linear equation with constant coefficients, then we can determine the particular solution explicitly by substitution in the given differential equation. It is pointed out at the end of Section 2.5 that this simple approach is not always successful — in which case the method of variation of parameters is available if a complementary function is known. However, undetermined coefficients *does* turn out to work well with a surprisingly large number of the nonhomogeneous linear differential equations that arise in elementary scientific applications.

In each of Problems 1–20 we give first the form of the trial solution y_{trial}, then the equations in the coefficients we get when we substitute y_{trial} into the differential equation and collect like terms, and finally the resulting particular solution y_p.

1. $y_{\text{trial}} = Ae^{3x}$; $25A = 1$; $y_p = (1/25)e^{3x}$

2. $y_{\text{trial}} = A + Bx$; $-2A - B = 4$, $-2B = 3$; $y_p = -(5 + 6x)/4$

3. $y_{\text{trial}} = A\cos 3x + B\sin 3x$; $-15A - 3B = 0$, $3A - 15B = 2$;
$y_p = (\cos 3x - 5\sin 3x)/39$

4. $y_{\text{trial}} = Ae^x + Bxe^x$; $9A + 12B = 0$, $9B = 3$; $y_p = (-4e^x + 3xe^x)/9$

5. First we substitute $\sin^2 x = (1 - \cos 2x)/2$ on the right-hand side of the differential equation. Then:

$y_{\text{trial}} = A + B\cos 2x + C\sin 2x$; $A = 1/2$, $-3B + 2C = -1/2$, $-2B - 3C = 0$;
$y_p = (13 + 3\cos 2x - 2\sin 2x)/26$

6. $y_{\text{trial}} = A + Bx + Cx^2$; $7A + 4B + 4C = 0$, $7B + 8C = 0$, $7C = 1$;
$y_p = (4 - 56x + 49x^2)/343$

7. First we substitute $\sinh x = (e^x - e^{-x})/2$ on the right-hand side of the differential equation. Then:

$y_{\text{trial}} = Ae^x + Be^{-x}$; $-3A = 1/2$, $-3B = -1/2$; $y_p = (e^{-x} - e^x)/6 = -(1/3)\sinh x$

8. First we note that $\cosh 2x$ is part of the complementary function $y_c = c_1 \cosh 2x + c_2 \sinh 2x$.. Then:

$y_{\text{trial}} = x(A\cosh 2x + B\sinh 2x)$; $4A = 0,\ 4B = 1$; $y_p = (1/4)x \sinh 2x$

9. First we note that e^x is part of the complementary function $y_c = c_1e^x + c_2e^{-3x}$. Then:

$y_{\text{trial}} = A + x(B + Cx)e^x$; $-3A = 1,\ 4B + 2C = 0,\ 8C = 1$;
$y_p = -(1/3) + (2x^2 - x)e^x/16$.

10. First we note the duplication with the complementary function $y_c = c_1 \cos 3x + c_2 \sin 3x$. Then:

$y_{\text{trial}} = x(A\cos 3x + B\sin 3x)$; $6B = 2,\ -6A = 3$; $y_p = (2x \sin 3x - 3x \cos 3x)/6$

11. First we note the duplication with the complementary function $y_c = c_1x + c_2 \cos 2x + c_3 \sin 2x$. Then:

$y_{\text{trial}} = x(A + Bx)$; $4A = -1,\ 8B = 3$; $y_p = (3x^2 - 2x)/8$

12. First we note the duplication with the complementary function $y_c = c_1x + c_2 \cos x + c_3 \sin x$. Then:

$y_{\text{trial}} = Ax + x(B\cos x + C\sin x)$; $A = 2,\ -2B = 0,\ -2C = -1$;
$y_p = 2x + (1/2)x \sin x$

13. $y_{\text{trial}} = e^x(A\cos x + B\sin x)$; $7A + 4B = 0,\ -4A + 7B = 1$;
$y_p = e^x(7 \sin x - 4 \cos x)/65$

14. First we note the duplication with the complementary function $y_c = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^x$. Then:

$y_{\text{trial}} = x^2(A + Bx)e^x$; $8A + 24B = 0,\ 24B = 1$; $y_p = (-3x^2e^x + x^3e^x)/24$

15. This is something of a trick problem. We cannot solve the characteristic equation $r^5 + 5r^4 - 1 = 0$ to find the complementary function, but we can see that it contains no constant term (why?). Hence the trial solution $y_{\text{trial}} = A$ leads immediately to the particular solution $y_p = -17$.

16. $y_{\text{trial}} = A + (B + Cx + Dx^2)e^{3x}$;
$9A = 5,\ 18B + 6C + 2D = 0,\ 18C + 12D = 0,\ 18D = 2$;

$$y_p = (45 + e^{3x} - 6xe^{3x} + 9x^2e^{3x})/81$$

17. First we note the duplication with the complementary function $y_c = c_1 \cos x + c_2 \sin x$. Then:

$$y_{\text{trial}} = x[(A + Bx)\cos x + (C + Dx)\sin x];$$
$$2B + 2C = 0, \; 4D = 1, -2A + 2D = 1, -4B = 0;$$
$$y_p = (x^2 \sin x - x \cos x)/4$$

18. First we note the duplication with the complementary function $y_c = c_1e^{-x} + c_2e^{x} + c_3e^{-2x} + c_4e^{2x}$. Then:

$$y_{\text{trial}} = x(Ae^x) + x(B + Cx)\, e^{2x}; \quad -6A = 1, \; 12B + 38C = 0, \; 24C = -1;$$
$$y_p = -(24xe^x - 19xe^{2x} + 6x^2e^{2x})/144$$

19. First we note the duplication with the part $c_1 + c_2x$ of the complementary function (which corresponds to the factor r^2 of the characteristic polynomial). Then:

$$y_{\text{trial}} = x^2(A + Bx + Cx^2); \qquad 4A + 12B = -1, \; 12B + 48C = 0, \; 24C = 3;$$
$$y_p = (10x^2 - 4x^3 + x^4)/8$$

20. First we note that the characteristic polynomial $r^3 - r$ has the zero $r = 1$ corresponding to the duplicating part e^x of the complementary function. Then:

$$y_{\text{trial}} = A + x\left(Be^x\right); \quad -A = 7, \; 3B = 1; \quad y_p = -7 + (1/3)xe^x$$

In Problems 21–30 we list first the complementary function y_c, then the initially proposed trial function y_i, and finally the actual trial function y_p in which duplication with the complementary function has been eliminated.

21.
$$y_c = e^x\left(c_1 \cos x + c_2 \sin x\right);$$
$$y_i = e^x\left(A\cos x + B\sin x\right)$$
$$y_p = x \cdot e^x\left(A\cos x + B\sin x\right)$$

22.
$$y_c = \left(c_1 + c_2x + c_3x^2\right) + \left(c_4e^x\right) + \left(c_5e^{-x}\right);$$
$$y_i = \left(A + Bx + Cx^2\right) + \left(De^x\right)$$
$$y_p = x^3 \cdot \left(A + Bx + Cx^2\right) + x \cdot \left(De^x\right)$$

23.
$$y_c = c_1 \cos x + c_2 \sin x;$$
$$y_i = (A + Bx)\cos 2x + (C + Dx)\sin 2x$$
$$y_p = x \cdot [(A + Bx)\cos 2x + (C + Dx)\sin 2x]$$

24. $$\begin{aligned} y_c &= c_1 + c_2 e^{-3x} + c_3 e^{4x}; \\ y_i &= (A+Bx) + (C+Dx)e^{-3x} \\ y_p &= x\cdot(A+Bx) + x\cdot(C+Dx)e^{-3x} \end{aligned}$$

25. $$\begin{aligned} y_c &= c_1 e^{-x} + c_2 e^{-2x}; \\ y_i &= (A+Bx)e^{-x} + (C+Dx)e^{-2x} \\ y_p &= x\cdot(A+Bx)e^{-x} + x\cdot(C+Dx)e^{-2x} \end{aligned}$$

26. $$\begin{aligned} y_c &= e^{3x}\left(c_1\cos 2x + c_2 \sin 2x\right); \\ y_i &= (A+Bx)e^{3x}\cos 2x + (C+Dx)e^{3x}\sin 2x \\ y_p &= x\cdot\left[(A+Bx)e^{3x}\cos 2x + (C+Dx)e^{3x}\sin 2x\right] \end{aligned}$$

27. $$\begin{aligned} y_c &= \left(c_1\cos x + c_2\sin x\right) + \left(c_3\cos 2x + c_3\sin 2x\right) \\ y_i &= \left(A\cos x + B\sin x\right) + \left(C\cos 2x + D\sin 2x\right) \\ y_p &= x\cdot\left[\left(A\cos x + B\sin x\right) + \left(C\cos 2x + D\sin 2x\right)\right] \end{aligned}$$

28. $$\begin{aligned} y_c &= \left(c_1 + c_2 x\right) + \left(c_3\cos 3x + c_3\sin 3x\right) \\ y_i &= \left(A + Bx + Cx^2\right)\cos 3x + \left(D + Ex + Fx^2\right)\sin 3x \\ y_p &= x\cdot\left[\left(A + Bx + Cx^2\right)\cos 3x + \left(D + Ex + Fx^2\right)\sin 3x\right] \end{aligned}$$

29. $$\begin{aligned} y_c &= \left(c_1 + c_2 x + c_3 x^2\right)e^x + c_4 e^{2x} + c_5 e^{-2x}; \\ y_i &= \left(A+Bx\right)e^x + Ce^{2x} + De^{-2x} \\ y_p &= x^3\cdot\left(A+Bx\right)e^x + x\cdot\left(Ce^{2x}\right) + x\cdot\left(De^{-2x}\right) \end{aligned}$$

30. $$\begin{aligned} y_c &= \left(c_1 + c_2 x\right)e^{-x} + \left(c_3 + c_4 x\right)e^x \\ y_i &= y_p = \left(A + Bx + Cx^2\right)\cos x + \left(D + Ex + Fx^2\right)\sin x \end{aligned}$$

In Problems 31–40 we list first the complementary function y_c, the trial solution y_{tr} for the method of undetermined coefficients, and the corresponding general solution $y_g = y_c + y_p$ where y_p results from determining the coefficients in y_{tr} so as to satisfy the given nonhomogeneous differential equation. Then we list the linear equations obtained by imposing the given initial conditions, and finally the resulting particular solution $y(x)$.

31. $y_c = c_1\cos 2x + c_2\sin 2x; \qquad y_{tr} = A + Bx$

$$y_g = c_1\cos 2x + c_2\sin 2x + x/2$$

$$c_1 = 1, \quad 2c_2 + 1/2 = 2$$
$$y(x) = \cos 2x + (3/4)\sin 2x + x/2$$

32. $y_c = c_1 e^{-x} + c_2 e^{-2x}; \qquad y_{tr} = Ae^x$

$$y_g = c_1 e^{-x} + c_2 e^{-2x} + e^x/6$$

$$c_1 + c_2 + 1/6 = 0, \quad -c_1 - 2c_2 + 1/6 = 3$$
$$y(x) = \left(15e^{-x} - 16e^{-2x} + e^x\right)/6$$

33. $y_c = c_1 \cos 3x + c_2 \sin 3x; \quad y_{tr} = A\cos 2x + B\sin 2x$

$$y_g = c_1 \cos 3x + c_2 \sin 3x + (1/5)\sin 2x$$

$$c_1 = 1, \quad 3c_2 + 2/5 = 0$$
$$y(x) = \left(15\cos 3x - 2\sin 3x + 3\sin 2x\right)/15$$

34. $y_c = c_1 \cos x + c_2 \sin x; \quad y_{tr} = x \cdot \left(A\cos x + B\sin x\right)$

$$y_g = c_1 \cos x + c_2 \sin x + \tfrac{1}{2} x \sin x$$

$$c_1 = 1, \quad c_2 = -1; \quad y(x) = \cos x - \sin x + \tfrac{1}{2} x \sin x$$

35. $y_c = e^x\left(c_1 \cos x + c_2 \sin x\right); \quad y_{tr} = A + Bx$

$$y_g = e^x\left(c_1 \cos x + c_2 \sin x\right) + 1 + x/2$$

$$c_1 + 1 = 3, \quad c_1 + c_2 + 1/2 = 0$$
$$y(x) = e^x\left(4\cos x - 5\sin x\right)/2 + 1 + x/2$$

36. $y_c = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^{2x}; \quad y_{tr} = x^2 \cdot \left(A + Bx + Cx^2\right)$

$$y_g = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^{2x} - x^2/16 - x^4/48$$

$$c_1 + c_3 + c_4 = 1, \quad c_2 - 2c_3 + 2c_4 = 1, \quad 4c_3 + 4c_4 - 1/8 = -1, \quad -8c_3 + 8c_4 = -1$$

$$y(x) = \left(234 + 240x - 9e^{-2x} - 33e^{2x} - 12x^2 - 4x^4\right)/192$$

37. $y_c = c_1 + c_2 e^x + c_3 x e^x; \quad y_{tr} = x \cdot \left(A\right) + x^2 \cdot \left(B + Cx\right)e^x$

$$y_g = c_1 + c_2 e^x + c_3 x e^x + x - x^2 e^x/2 + x^3 e^x/6$$

$$c_1 + c_2 = 0, \quad c_2 + c_3 + 1 = 0, \quad c_2 + 2c_3 - 1 = 1$$

$$y(x) = 4 + x + e^x\left(-24 + 18x - 3x^2 + x^3\right)/6$$

38. $$y_c = e^{-x}\left(c_1 \cos x + c_2 \sin x\right); \qquad y_{tr} = A\cos 3x + B\sin 3x$$

$$y_g = e^{-x}\left(c_1 \cos x + c_2 \sin x\right) - \left(6\cos 3x + 7\sin 3x\right)/85$$

$$c_1 - 6/185 = 2, \quad -c_1 + c_2 - 21/85 = 0$$

$$y(x) = \left[e^{-x}\left(176\cos x + 197\sin x\right) - \left(6\cos 3x + 7\sin 3x\right)\right]/85$$

39. $$y_c = c_1 + c_2 x + c_3 e^{-x}; \qquad y_{tr} = x^2 \cdot \left(A + Bx\right) + x \cdot \left(Ce^{-x}\right)$$

$$y_g = c_1 + c_2 x + c_3 e^{-x} - x^2/2 + x^3/6 + x\,e^{-x}$$

$$c_1 + c_3 = 1, \quad c_2 - c_3 + 1 = 0, \quad c_3 - 3 = 1$$

$$y(x) = \left(-18 + 18x - 3x^2 + x^3\right)/6 + (4 + x)e^{-x}$$

40. $$y_c = c_1 e^{-x} + c_2 e^x + c_3 \cos x + c_4 \sin x; \qquad y_{tr} = A$$

$$y_g = c_1 e^{-x} + c_2 e^x + c_3 \cos x + c_4 \sin x - 5$$

$$c_1 + c_2 + c_3 - 5 = 0, \quad -c_1 + c_2 + c_4 = 0, \quad c_1 + c_2 - c_3 = 0, \quad -c_1 + c_2 - c_4 = 0$$

$$y(x) = \left(5e^{-x} + 5e^x + 10\cos x - 20\right)/4$$

41. The trial solution $y_{tr} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$ leads to the equations

$$\begin{aligned}
2A - B - 2C - 6D + 24E &= 0 \\
-2B - 2C - 6D - 24E + 120F &= 0 \\
-2C - 3D - 12E - 60F &= 0 \\
-2D - 4E - 20F &= 0 \\
-2E - 5F &= 0 \\
-2F &= 8
\end{aligned}$$

that are readily solve by back-substitution. The resulting particular solution is

$$y(x) = -255 - 450x + 30x^2 + 20x^3 + 10x^4 - 4x^5.$$

42. The characteristic equation $r^4 - r^3 - r^2 - r - 2 = 0$ has roots $r = -1, 2, \pm i$ so the complementary function is $y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x$. We find that the coefficients satisfy the equations

$$\begin{aligned} c_1 + c_2 + c_3 - 255 &= 0 \\ -c_1 + 2c_2 + c_4 - 450 &= 0 \\ c_1 + 4c_2 - c_3 + 60 &= 0 \\ -c_1 + 8c_2 - c_4 + 120 &= 0 \end{aligned}$$

Solution of this system gives finally the particular solution $y = y_c + y_p$ where y_p is the particular solution of Problem 41 and

$$y_c = 10e^{-x} + 35e^{2x} + 210\cos x + 390\sin x.$$

43. **(a)** $$\begin{aligned} \cos 3x + i\sin 3x &= (\cos x + i\sin x)^3 \\ &= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x \end{aligned}$$

When we equate real parts we get the equation

$$\cos^3 x - 3(\cos x)(1 - \cos^2 x) = 4\cos^3 x - 3\cos x$$

and readily solve for $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$. The formula for $\sin^3 x$ is derived similarly by equating imaginary parts in the first equation above.

(b) Upon substituting the trial solution $y_p = A\cos x + B\sin x + C\cos 3x + D\sin 3x$ in the differential equation $y'' + 4y = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$, we find that $A = 1/4$, $B = 0$, $C = -1/20$, $D = 0$. The resulting general solution is

$$y(x) = c_1\cos 2x + c_2\sin 2x + (1/4)\cos x - (1/20)\cos 3x.$$

44. We use the identity $\sin x\sin 3x = \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x$, and hence substitute the trial solution $y_p = A\cos 2x + B\sin 2x + C\cos 4x + D\sin 4x$ in the differential equation $y'' + y' + y = \frac{1}{2}\cos 2x - \frac{1}{2}\cos 4x$. We find that $A = -3/26$, $B = 1/13$, $C = -14/482$, $D = 2/141$. The resulting general solution is

$$\begin{aligned} y(x) &= e^{-x/2}(c_1\cos x\sqrt{3}/2 + c_2\sin x\sqrt{3}/2) \\ &\quad + (-3\cos 2x + 2\sin 2x)/26 + (-15\cos 4x + 4\sin 4x)/482. \end{aligned}$$

45. We substitute

$$\sin^4 x = (1 - \cos 2x)^2/4$$

$$= (1 - 2\cos 2x + \cos^2 2x)/4 = (3 - 4\cos 2x + \cos 4x)/8$$

on the right-hand side of the differential equation, and then substitute the trial solution $y_p = A\cos 2x + B\sin 2x + C\cos 4x + D\sin 4x + E$. We find that $A = -1/10$, $B = 0$, $C = -1/56$, $D = 0$, $E = 1/24$. The resulting general solution is

$$y = c_1\cos 3x + c_2\sin 3x + 1/24 - (1/10)\cos 2x - (1/56)\cos 4x.$$

46. By the formula for $\cos^3 x$ in Problem 43, the differential equation can be written as

$$y'' + y = \tfrac{3}{4}x\cos x + \tfrac{1}{4}x\cos 3x.$$

The complementary solution is $y_c = c_1\cos x + c_2\sin x$, so we substitute the trial solution

$$y_p = x\cdot\left[(A+Bx)\cos x + (C+Dx)\sin x\right] + \left[(E+Fx)\cos 3x + (G+Hx)\sin 3x\right].$$

We find that $A = 3/16, B = C = 0, D = 3/16, E = 0, F = -1/32, G = 3/128, H = 0$. Hence the general solution is given by $y = y_c + y_1 + y_2$ where

$$y_1 = (3x\cos x + 3x^2\sin x)/16 \quad\text{and}\quad y_2 = (3\sin 3x - 4x\cos 3x)/128.$$

In Problems 47–49 we list the independent solutions y_1 and y_2 of the associated homogeneous equation, their Wronskian $W = W(y_1, y_2)$, the coefficient functions

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\,dx \quad\text{and}\quad u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\,dx$$

in the particular solution $y_p = u_1y_1 + u_2y_2$ of Eq. (32) in the text, and finally y_p itself.

47. $y_1 = e^{-2x}$, $\quad y_2 = e^{-x}$, $\quad W = e^{-3x}$

$u_1 = -(4/3)e^{3x}$, $\quad u_2 = 2e^{2x}$,

$y_p = (2/3)e^x$

48. $y_1 = e^{-2x}$, $\quad y_2 = e^{4x}$, $\quad W = 6e^{2x}$

$u_1 = -x/2$, $\quad u_2 = -e^{-6x}/12$,

$y_p = -(6x+1)e^{-2x}/12$

49. $y_1 = e^{2x}$, $\quad y_2 = xe^{2x}$, $\quad W = e^{4x}$

$u_1 = -x^2$, $\quad u_2 = 2x$,

$y_p = x^2e^{2x}$

50. The complementary function is $y_1 = c_1\cosh 2x + c_2\sinh 2x$, so the Wronskian is

$$W = 2\cosh^2 2x - 2\sinh^2 2x = 2,$$

so when we solve Equations (31) simultaneously for u_1' and u_2', integrate each and substitute in $y_p = y_1u_1 + y_2u_2$, the result is

$$y_p = -(\cosh 2x)\int \tfrac{1}{2}(\sinh 2x)(\sinh 2x)\,dx + (\sinh 2x)\int \tfrac{1}{2}(\cosh 2x)(\sinh 2x)\,dx\,.$$

Using the identities $2\sinh^2 x = \cosh 2x - 1$ and $2\sinh x\cosh x = \sinh 2x$, we evaluate the integrals and find that

$$y_p = (4x\cosh 2x - \sinh 4x\cosh 2x + \cosh 4x\sinh 2x)/16,$$
$$y_p = (4x\cosh 2x - \sinh 2x)/16.$$

51. $y_1 = \cos 2x, \qquad y_2 = \sin 2x, \qquad W = 2$

Liberal use of trigonometric sum and product identities yields

$u_1 = (\cos 5x - 5\cos x)/20, \qquad u_1 = (\sin 5x - 5\sin x)/20$

$y_p = -(1/4)(\cos 2x\cos x - \sin 2x\sin x) + (1/20)(\cos 5x\cos 2x + \sin 5x\sin 2x)$

$\quad = -(1/5)\cos 3x$ (!)

52. $y_1 = \cos 3x, \qquad y_2 = \sin 3x, \qquad W = 3$

$u_1 = -(6x - \sin 6x)/36, \qquad u_1 = -(1 + \cos 6x)/36$

$y_p = -(x\cos 3x)/6$

53. $y_1 = \cos 3x, \qquad y_2 = \sin 3x, \qquad W = 3$

$u_1' = -(2/3)\tan 3x, \qquad u_2' = 2/3$

$y_p = (2/9)[3x\sin 3x + (\cos 3x)\ln|\cos 3x|]$

54. $y_1 = \cos x, \qquad y_2 = \sin x, \qquad W = 1$

$u_1' = -\csc x, \qquad u_2' = \cos x\csc^2 x$

$y_p = -1 - (\cos x)\ln|\csc x - \cot x|$

55. $y_1 = \cos 2x, \qquad y_2 = \sin 2x, \qquad W = 2$

$u_1' = -(1/2)\sin^2 x\sin 2x = -(1/4)(1 - \cos 2x)\sin 2x$

$$u_2' = (1/2)\sin^2 x \cos 2x = (1/4)(1 - \cos 2x)\cos 2x$$

$$y_p = (1 - x \sin 2x)/8$$

56. $y_1 = e^{-2x}$, $\quad y_2 = e^{2x}$, $\quad W = 4$

$u_1 = -(3x - 1)e^{3x}/36$, $\quad u_2 = -(x + 1)e^{-x}/4$

$y_p = -e^x(3x + 2)/9$

57. With $y_1 = x$, $y_2 = x^{-1}$, and $f(x) = 72x^3$, Equations (31) in the text take the form

$$x u_1' + x^{-1} u_2' = 0,$$
$$u_1' - x^{-2} u_2' = 72x^3.$$

Upon multiplying the second equation by x and then adding, we readily solve first for

$$u_1' = 36x^3, \quad \text{so} \quad u_1 = 9x^4$$

and then

$$u_2' = -x^2 u_1' = -36x^5, \quad \text{so} \quad u_2 = -6x^6.$$

Then it follows that

$$y_p = y_1 u_1 + y_2 u_2 = (x)(9x^4) + (x^{-1})(-6x^6) = 3x^5.$$

58. Here it is important to remember that — for variation of parameters — the differential equation must be written in standard form with leading coefficient 1. We therefore rewrite the given equation with complementary function $y_c = c_1 x^2 + c_2 x^3$ as

$$y'' - (4/x)y' + (6/x^2)y = x.$$

Thus $f(x) = x$, and $W = x^4$, so simultaneous solution of Equations (31) as in Problem 50 (followed by integration of u_1' and u_2') yields

$$\begin{aligned} y_p &= -x^2 \int x^3 \cdot x \cdot x^{-4}\, dx + x^3 \int x^2 \cdot x \cdot x^{-4}\, dx \\ &= -x^2 \int dx + x^3 \int (1/x)\, dx = x^3(\ln x - 1). \end{aligned}$$

59. $y_1 = x^2$, $\quad y_2 = x^2 \ln x$,

$W = x^3$, $\quad f(x) = x^2$

$u_1' = -x \ln x$, $\quad u_2' = x$

$y_p = x^4/4$

60. $y_1 = x^{1/2}$, $\quad y_2 = x^{3/2}$

$f(x) = 2x^{-2/3}$; $\quad W = x$

$u_1 = -12x^{5/6}/5$, $\quad u_2 = -12x^{-1/6}$

$y_p = -72x^{4/3}/5$

61. $y_1 = \cos(\ln x)$, $\quad y_2 = \sin(\ln x)$, $\quad W = 1/x$,

$f(x) = (\ln x)/x^2$

$u_1 = (\ln x)\cos(\ln x) - \sin(\ln x)$

$u_2 = (\ln x)\sin(\ln x) + \cos(\ln x)$

$y_p = \ln x$ (!)

62. $y_1 = x$, $\quad y_2 = 1 + x^2$,

$W = x^2 - 1$, $\quad f(x) = 1$

$u_1' = (1 + x^2)/(1 - x^2)$, $\quad u_2' = x/(x^2 - 1)$

$y_p = -x^2 + x\ln|(1 + x)/(1 - x)| + (1/2)(1 + x^2)\ln|1 - x^2|$

63. This is simply a matter of solving the equations in (31) for the derivatives

$$u_1' = -\frac{y_2(x)f(x)}{W(x)} \quad \text{and} \quad u_2' = \frac{y_1(x)f(x)}{W(x)},$$

integrating each, and then substituting the results in (32).

64. Here we have $y_1(x) = \cos x,\ y_2(x) = \sin x,\ W(x) = 1,\ f(x) = 2\sin x$, so (33) gives

$$\begin{aligned} y_p(x) &= -(\cos x)\int \sin x \cdot 2\sin x\, dx + (\sin x)\int \cos x \cdot 2\sin x\, dx \\ &= -(\cos x)\int (1 - \cos 2x)\, dx + (\sin x)\int 2(\sin x)\cdot \cos x\, dx \\ &= -(\cos x)(x - \sin x\cos x) + (\sin x)(\sin^2 x) \\ &= -x\cos x + (\sin x)(\cos^2 x + \sin^2 x) \\ y_p(x) &= -x\cos x + \sin x \end{aligned}$$

But we can drop the term $\sin x$ because it satisfies the associated homogeneous equation $y'' + y = 0$.

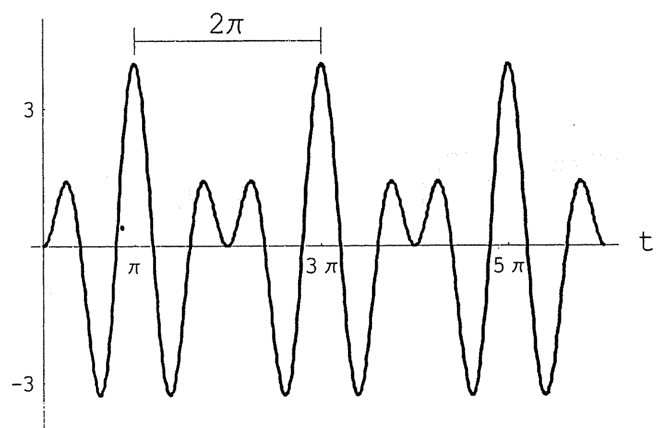
SECTION 2.6

FORCED OSCILLATIONS AND RESONANCE

1. Trial of $x = A\cos 2t$ yields the particular solution $x_p = 2\cos 2t$. (Can you see that — because the differential equation contains no first-derivative term — there is no need to include a $\sin 2t$ term in the trial solution?) Hence the general solution is

$$x(t) = c_1\cos 3t + c_2\sin 3t + 2\cos 2t.$$

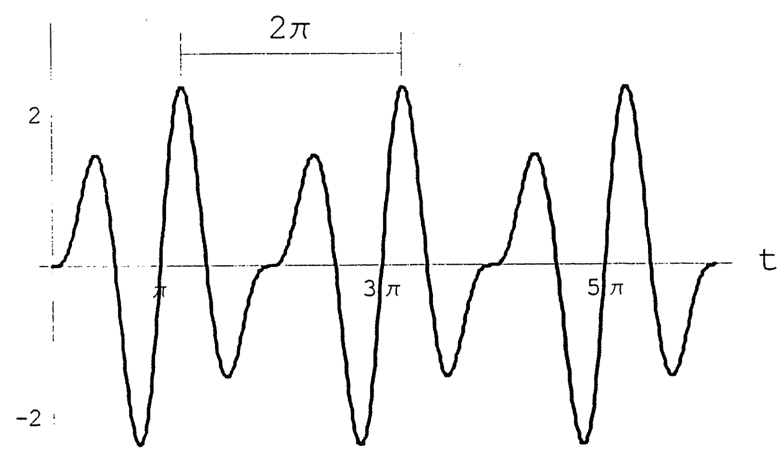
The initial conditions imply that $c_1 = -2$ and $c_2 = 0$, so $x(t) = 2\cos 2t - 2\cos 3t$. The following figure shows the graph of $x(t)$.



2. Trial of $x = A\sin 3t$ yields the particular solution $x_p = -\sin 3t$. Then we impose the initial conditions $x(0) = x'(0) = 0$ on the general solution

$$x(t) = c_1\cos 2t + c_2\sin 2t - \sin 3t,$$

and find that $x(t) = \frac{3}{2}\sin 2t - \sin 3t$. The following figure shows the graph of $x(t)$.



3. First we apply the method of undetermined coefficients — with trial solution $x = A\cos 5t + B\sin 5t$ — to find the particular solution

$$\begin{aligned} x_p &= 3\cos 5t + 4\sin 5t \\ &= 5\left[\frac{3}{5}\cos 5t + \frac{4}{5}\sin 5t\right] = 5\cos(5t-\beta) \end{aligned}$$

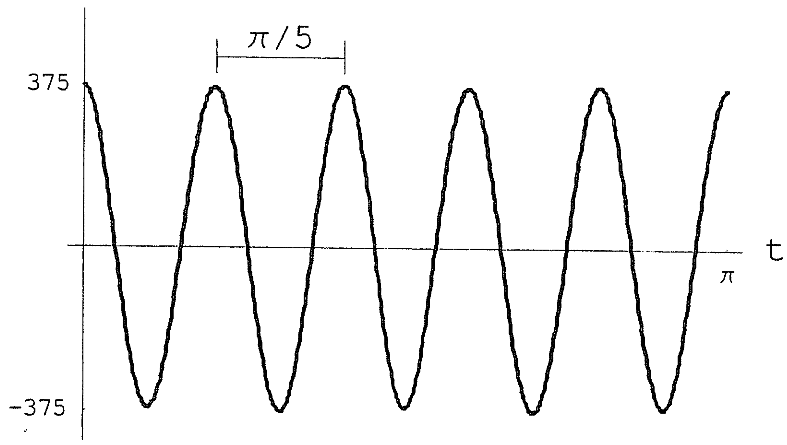
where $\beta = \tan^{-1}(4/3) \approx 0.9273$. Hence the general solution is

$$x(t) = c_1\cos 10t + c_2\sin 10t + 3\cos 5t + 4\sin 5t.$$

The initial conditions $x(0) = 375$, $x'(0) = 0$ now yield $c_1 = 372$ and $c_2 = -2$, so the part of the solution with frequency $\omega = 10$ is

$$\begin{aligned} x_c &= 372\cos 10t - 2\sin 10t \\ &= \sqrt{138388}\left[\frac{372}{\sqrt{138388}}\cos 10t - \frac{2}{\sqrt{138388}}\sin 10t\right] \\ &= \sqrt{138388}\cos(10t-\alpha) \end{aligned}$$

where $\alpha = 2\pi - \tan^{-1}(1/186) \approx 6.2778$ is a fourth-quadrant angle. The following figure shows the graph of $x(t)$.



4. Noting that there is no first-derivative term, we try $x = A\cos 4t$ and find the particular solution $x_p = 10\cos 4t$. Then imposition of the initial conditions on the general solution $x(t) = c_1\cos 5t + c_2\sin 5t + 10\cos 4t$ yields

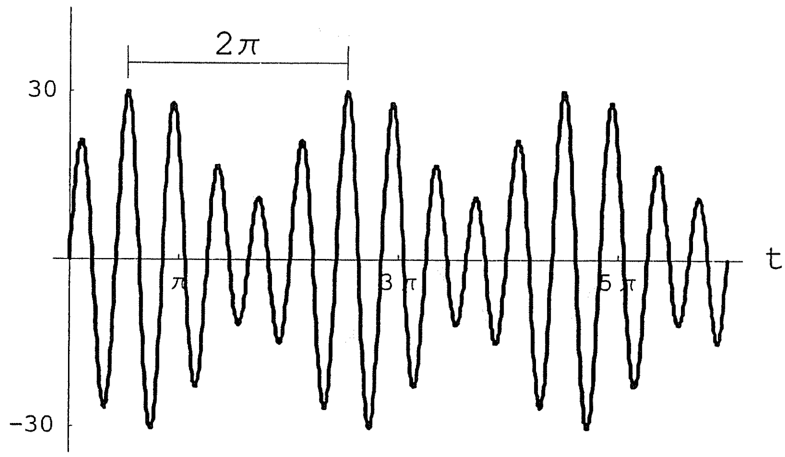
$$x(t) = (-10\cos 5t + 18\sin 5t) + 10\cos 4t$$

$$= 2(-5\cos 5t + 9\sin 5t) + 10\cos 4t$$

$$= 2\sqrt{106}\left(-\frac{5}{\sqrt{106}}\cos 5t + \frac{9}{\sqrt{106}}\sin 5t\right) + 10\cos 4t$$

$$= 2\sqrt{106}\cos(5t - \alpha)$$

where $\alpha = \pi - \tan^{-1}(9/5) \approx 2.0779$ is a second-quadrant angle. The following figure shows the graph of $x(t)$.



5. Substitution of the trial solution $x = C\cos\omega t$ gives $C = F_0/(k - m\omega^2)$. Then imposition of the initial conditions $x(0) = x_0,\ x'(0) = 0$ on the general solution

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + C\cos\omega t \qquad (\text{where } \omega_0 = \sqrt{k/m})$$

gives the particular solution $x(t) = (x_0 - C)\cos\omega_0 t + C\cos\omega t$.

6. First, let's write the differential equation in the form $x'' + \omega_0^2 x = (F_0/m)\cos\omega_0 t$, which is the same as Eq. (13) in the text, and therefore has the particular solution $x_p = (F_0/2m\omega_0)t\sin\omega_0 t$ given in Eq. (14). When we impose the initial conditions $x(0) = 0,\ x'(0) = v_0$ on the general solution

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + (F_0/2m\omega_0)\,t\sin\omega_0 t$$

we find that $c_1 = 0,\ c_2 = v_0/\omega_0$. The resulting resonance solution of our initial value problem is

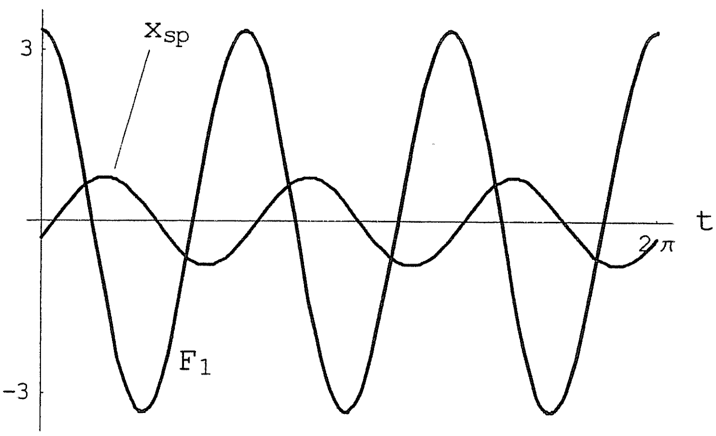
$$x(t) = \frac{2mv_0 + F_0 t}{2m\omega_0}\sin\omega_0 t.$$

In Problems 7–10 we give first the trial solution x_p involving undetermined coefficients A and B, then the equations that determine these coefficients, and finally the resulting steady periodic solution x_{sp}. In each case the figure shows the graphs of $x_{sp}(t)$ and the adjusted forcing function $F_1(t) = F(t)/m\omega$.

7. $x_p = A\cos 3t + B\sin 3t$; $\quad -5A + 12B = 10, \quad 12A + 5B = 0$

$$x_{sp}(t) = -\frac{50}{169}\cos 3t + \frac{120}{169}\sin 3t = \frac{10}{13}\left(-\frac{5}{13}\cos 3t + \frac{12}{13}\sin 3t\right) = \frac{10}{13}\cos(3t - \alpha)$$

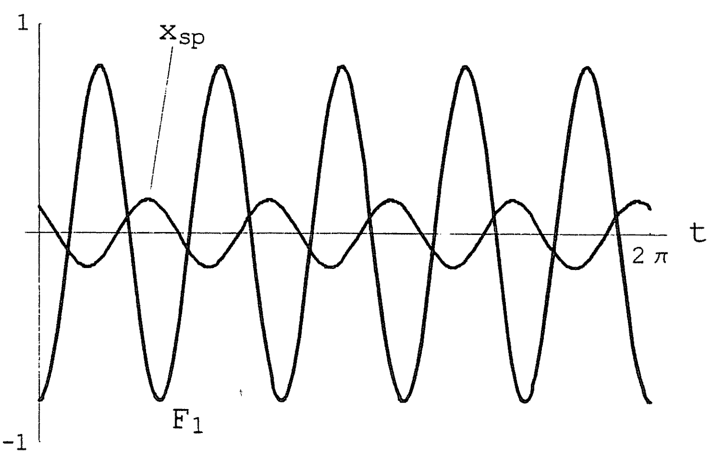
$\alpha = \pi - \tan^{-1}(12/5) \approx 1.9656$ (2nd quadrant angle)



8. $x_p = A\cos 5t + B\sin 5t$; $\quad -20A + 15B = -4, \quad 15A + 20B = 0$

$$x_{sp}(t) = \frac{16}{125}\cos 5t - \frac{12}{125}\sin 5t = \frac{4}{25}\left(\frac{4}{5}\cos 5t - \frac{3}{5}\sin 5t\right) = \frac{4}{25}\cos(5t - \alpha)$$

$\alpha = 2\pi - \tan^{-1}(3/4) \approx 5.6397$ (4th quadrant angle)

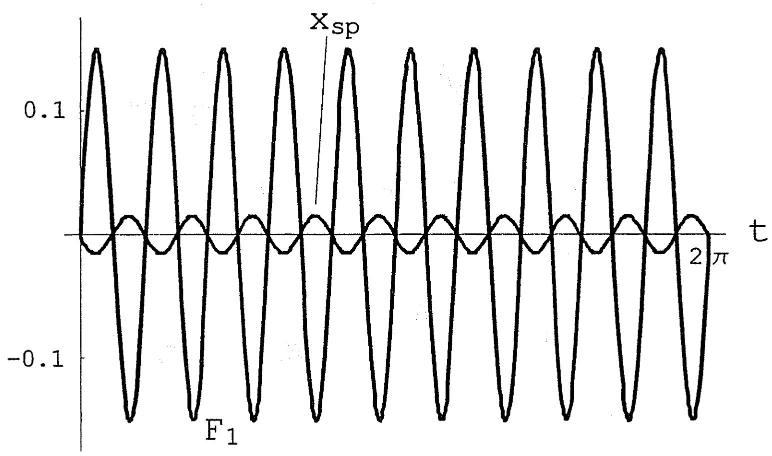


9. $x_p = A\cos 10t + 10\sin 5t; \qquad -199A + 20B = 0, \quad 20A + 199B = -3$

$$x_{sp}(t) = -\frac{60}{40001}\cos 10t - \frac{597}{40001}\sin 10t$$

$$= \frac{3}{\sqrt{40001}}\left(-\frac{20}{\sqrt{40001}}\cos 10t - \frac{199}{\sqrt{40001}}\sin 10t\right) = \frac{3}{\sqrt{40001}}\cos(10t - \alpha)$$

$\alpha = \pi + \tan^{-1}(199/20) \approx 4.6122$ (3rd quadrant angle)

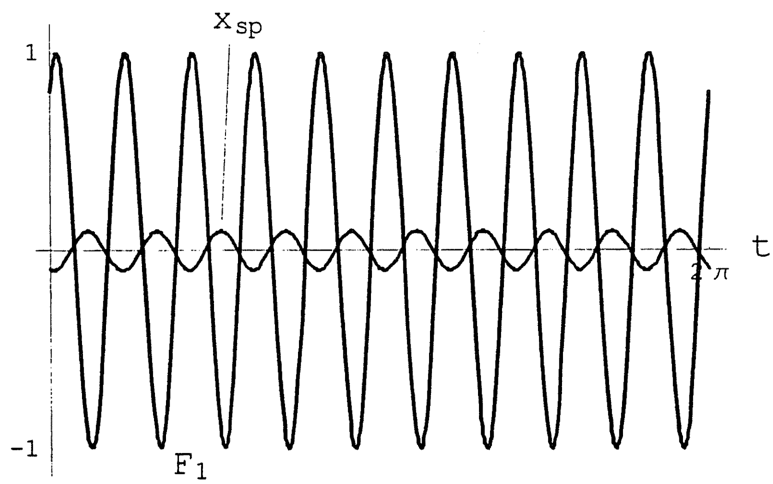


10. $x_p = A\cos 10t + 10\sin 5t; \qquad -97A + 30B = 8, \quad 30A + 97B = -6$

$$x_{sp}(t) = -\frac{956}{10309}\cos 10t - \frac{342}{10309}\sin 10t$$

$$= \frac{2\sqrt{257725}}{10309}\left(-\frac{478}{\sqrt{257725}}\cos 10t - \frac{171}{\sqrt{257725}}\sin 10t\right) = \frac{10}{793}\sqrt{61}\cos(10t - \alpha)$$

$\alpha = \pi + \tan^{-1}(171/478) \approx 3.4851$ (3rd quadrant angle)



Each solution in Problems 11–14 has two parts. For the first part, we give first the trial solution x_p involving undetermined coefficients A and B, then the equations that determine these coefficients, and finally the resulting steady periodic solution x_{sp}. For the second part, we give first the general solution $x(t)$ involving the coefficients c_1 and c_2 in the transient solution, then the equations that determine these coefficients, and finally the resulting transient solution x_{tr} so that $x(t) = x_{tr}(t) + x_{sp}(t)$ satisfies the given initial conditions. For each problem, the graph shows the graphs of both $x(t)$ and $x_{sp}(t)$.

11. $x_p = A\cos 3t + B\sin 3t; \qquad -4A + 12B = 10, \quad 12A + 4B = 0$

$$x_{sp}(t) = -\frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t = \frac{\sqrt{10}}{4}\left(-\frac{1}{\sqrt{10}}\cos 3t + \frac{3}{\sqrt{10}}\sin 3t\right) = \frac{\sqrt{10}}{4}\cos(3t - \alpha)$$

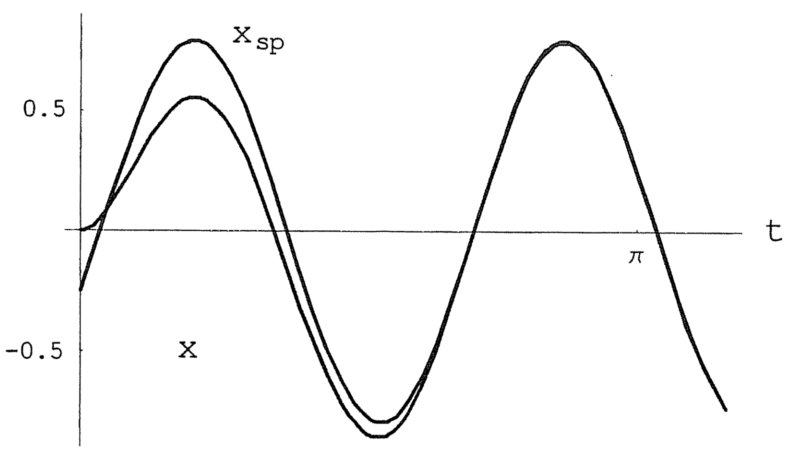
$\alpha = \pi - \tan^{-1}(3) \approx 1.8925$ (2nd quadrant angle)

$x(t) = e^{-2t}(c_1\cos t + c_2\sin t) + x_{sp}(t); \qquad c_1 - \frac{1}{4} = 0, \quad -2c_1 + c_2 + \frac{9}{4} = 0$

$$x_{tr}(t) = e^{-2t}\left(\frac{1}{4}\cos t - \frac{7}{4}\sin t\right) = \frac{\sqrt{50}}{4}e^{-2t}\left(\frac{1}{\sqrt{50}}\cos t - \frac{7}{\sqrt{50}}\sin t\right)$$

$$= \frac{5}{4}\sqrt{2}\,e^{-2t}\cos(t - \beta)$$

$\beta = 2\pi - \tan^{-1}(7) \approx 4.8543$ (4th quadrant angle)



12. $x_p = A\cos 5t + B\sin 5t; \qquad 12A - 30B = 0, \quad 30A + 12B = -10$

$$x_{sp}(t) = -\frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t$$

$$= \frac{5\sqrt{29}}{87}\left(-\frac{5}{\sqrt{29}}\cos 3t - \frac{2}{\sqrt{29}}\sin 3t\right) = \frac{5}{3\sqrt{29}}\cos(3t-\alpha)$$

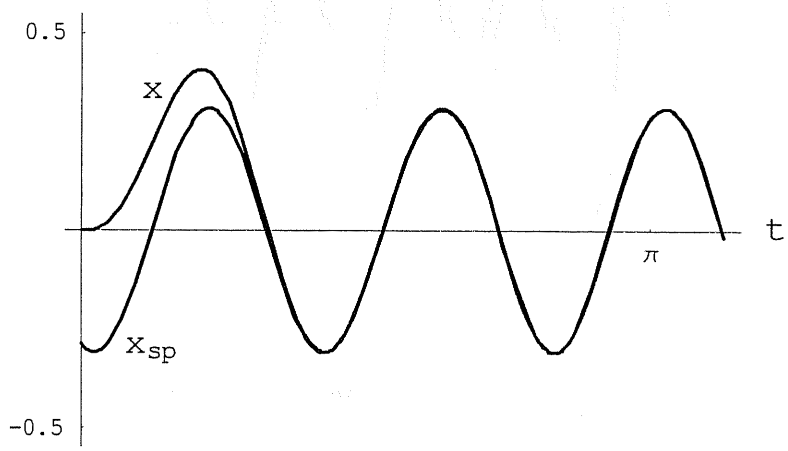
$\alpha = \pi + \tan^{-1}(2/5) \approx 3.5221$ (3rd quadrant angle)

$x(t) = e^{-3t}(c_1\cos 2t + c_2\sin 2t) + x_{sp}(t);\qquad c_1 - 25/87 = 0,\quad -3c_1 + 2c_2 - 50/87 = 0$

$$x_{tr}(t) = e^{-3t}\left(\frac{50}{174}\cos 2t + \frac{125}{174}\sin 2t\right)$$

$$= \frac{25\sqrt{29}}{174}e^{-3t}\left(\frac{2}{\sqrt{29}}\cos 2t + \frac{5}{\sqrt{29}}\sin 2t\right) = \frac{25}{6\sqrt{29}}e^{-3t}\cos(2t-\beta)$$

$\beta = \tan^{-1}(5/2) \approx 1.1903$ (1st quadrant angle)



13. $x_p = A\cos 10t + B\sin 10t;\qquad -74A + 20B = 600,\quad 20A + 74B = 0$

$$x_{sp}(t) = -\frac{11100}{1469}\cos 10t + \frac{3000}{1469}\sin 10t$$

$$= \frac{300}{\sqrt{1469}}\left(-\frac{37}{\sqrt{1469}}\cos 10t + \frac{10}{\sqrt{1469}}\sin 10t\right) = \frac{300}{\sqrt{1469}}\cos(10t-\alpha)$$

$\alpha = \pi - \tan^{-1}(10/37) \approx 2.9320$ (2nd quadrant angle)

$x(t) = e^{-t}(c_1\cos 5t + c_2\sin 5t) + x_{sp}(t);$

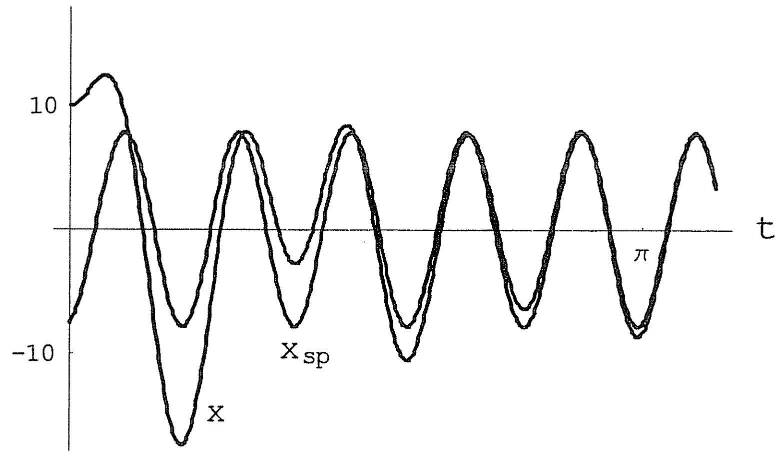
$c_1 - 11100/1469 = 10,\qquad -c_1 + 5c_2 = -30000/1469$

$$x_{tr}(t) = \frac{e^{-t}}{1469}(25790\cos 5t - 842\sin 5t)$$

$$= \frac{2\sqrt{166458266}}{1469} e^{-t} \left(\frac{12895}{\sqrt{166458266}} \cos 5t - \frac{421}{\sqrt{166458266}} \sin 5t \right)$$

$$= 2\sqrt{\frac{113314}{1469}} \, e^{-t} \cos(5t - \beta)$$

$\beta = 2\pi - \tan^{-1}(421/12895) \approx 6.2505$ (4th quadrant angle)



14. $x_p = A\cos t + B\sin t$; $\quad 24A + 8B = 200, \quad -8A + 24B = 520$

$$x_{sp}(t) = \cos t + 22\sin t = \sqrt{485}\left(\frac{1}{\sqrt{485}} \cos t + \frac{22}{\sqrt{485}} \sin t \right) = \sqrt{485} \cos(t - \alpha)$$

$\alpha = \tan^{-1}(22) \approx 1.5254$ (1st quadrant angle)

$x(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t) + x_{sp}(t)$;

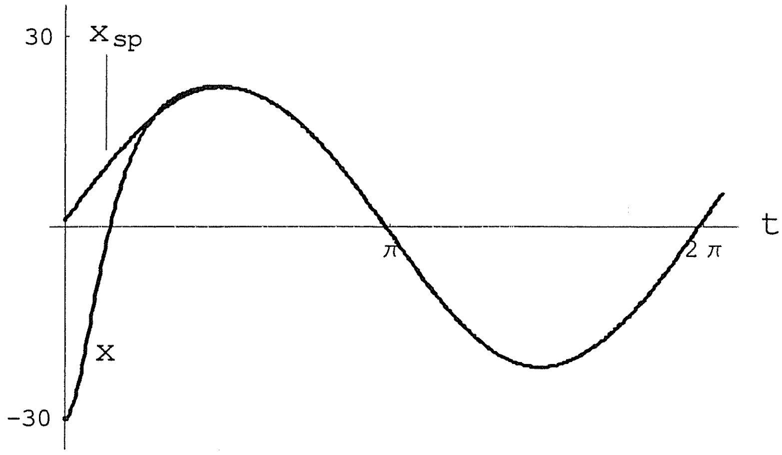
$c_1 + 1 = -30, \quad -4c_1 + 3c_2 + 22 = -10$

$x_{tr}(t) = e^{-4t}(-31\cos 3t - 52\sin 3t)$

$$= \sqrt{3665}\, e^{-4t} \left(-\frac{31}{\sqrt{3665}} \cos 3t - \frac{52}{\sqrt{3665}} \sin 3t \right) = \sqrt{3665}\, e^{-4t} \cos(3t - \beta)$$

$\beta = \pi + \tan^{-1}(52/31) \approx 4.1748$ (3rd quadrant angle)

The figure at the top of the next page shows the graphs of $x(t)$ and $x_{sp}(t)$.

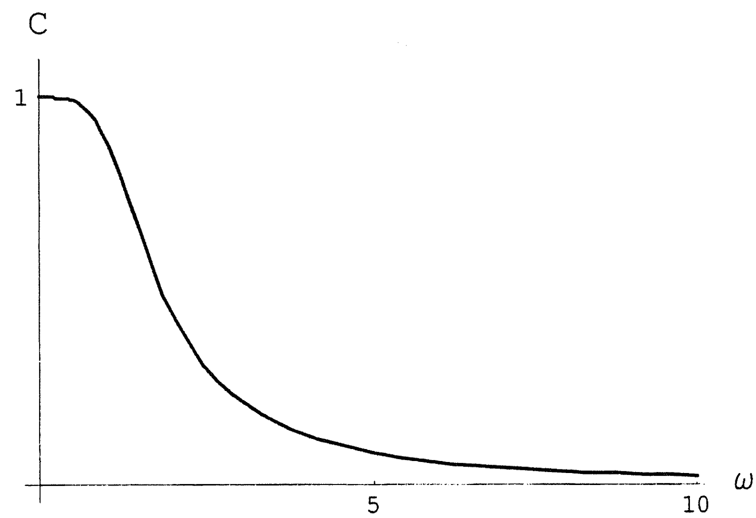


In Problems 15–18 we substitute $x(t) = A(\omega)\cos\omega t + B(\omega)\sin\omega t$ into the differential equation $mx'' + cx' + kx = F_0 \cos\omega t$ with the given numerical values of m, c, k, and F_0. We give first the equations in A and B that result upon collection of coefficients of $\cos\omega t$ and $\sin\omega t$, and then the values of $A(\omega)$ and $B(\omega)$ that we get by solving these equations. Finally, $C = \sqrt{A^2 + B^2}$ gives the amplitude of the resulting forced oscillations as a function of the forcing frequency ω, and we show the graph of the function $C(\omega)$.

15. $(2-\omega^2)A + 2\omega B = 2, \quad -2\omega A + (2-\omega^2)B = 0$

$$A = \frac{2(2-\omega^2)}{4+\omega^4}, \qquad B = \frac{4\omega}{4+\omega^4}$$

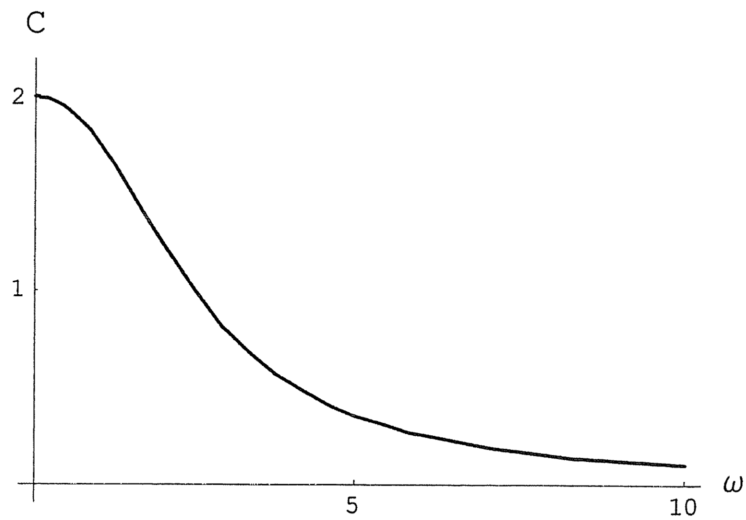
$C(\omega) = 2/\sqrt{4+\omega^4}$ begins with $C(0) = 1$ and steadily decreases as ω increases. Hence there is no practical resonance frequency.



16. $(5-\omega^2)A + 4\omega B = 10, \quad -4\omega A + (5-\omega^2)B = 0$

$$A = \frac{10(5-\omega^2)}{25+6\omega^2+\omega^4}, \quad B = \frac{40\omega}{25+6\omega^2+\omega^4}$$

$C(\omega) = 10/\sqrt{25+6\omega^2+\omega^4}$ begins with $C(0) = 2$ and steadily decreases as ω increases. Hence there is no practical resonance frequency.



17. $(45-\omega^2)A + 6\omega B = 50, \quad -6\omega A + (45-\omega^2)B = 0$

$$A = \frac{50(45-\omega^2)}{2025-54\omega^2+\omega^4}, \quad B = \frac{300\omega}{2025-54\omega^2+\omega^4}$$

$C(\omega) = 50/\sqrt{2025-54\omega^2+\omega^4}$ so, to find its maximum value, we calculate the derivative

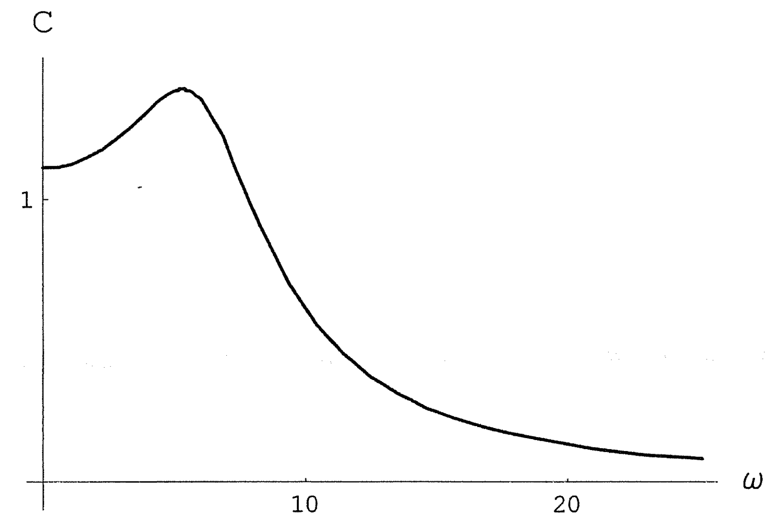
$$C'(\omega) = \frac{-100\omega(-27+\omega^2)}{(2025-54\omega^2+\omega^4)^{3/2}}.$$

Hence the practical resonance frequency (where the derivative vanishes) is $\omega = \sqrt{27} = 3\sqrt{3}$. The graph of $C(\omega)$ is shown at the top of the next page.

18. $(650-\omega^2)A + 10\omega B = 100, \quad -10\omega A + (650-\omega^2)B = 0$

$$A = \frac{100(650-\omega^2)}{422500-1200\omega^2+\omega^4}, \quad B = \frac{1000\omega}{422500-1200\omega^2+\omega^4}$$

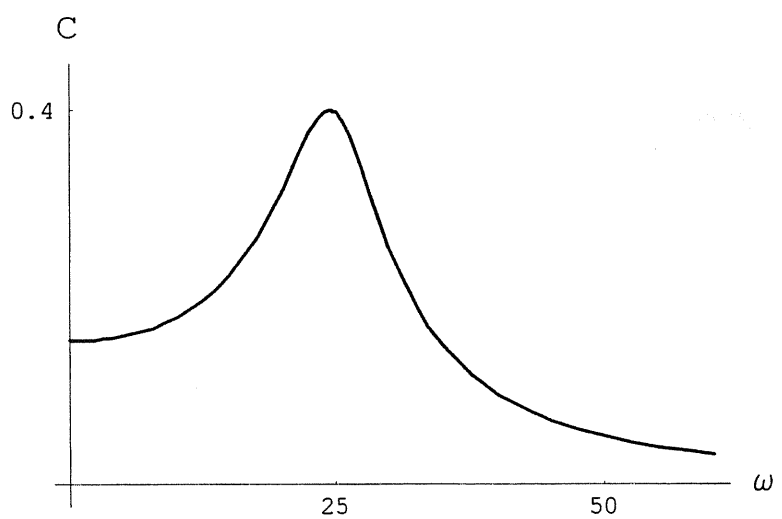
$C(\omega) = 100/\sqrt{422500-1200\omega^2+\omega^4}$ so, to find its maximum value,



we calculate the derivative

$$C'(\omega) = \frac{-200\,\omega(-600+\omega^2)}{(422500-1200\omega^2+\omega^4)^{3/2}}.$$

Hence the practical resonance frequency (where the derivative vanishes) is $\omega = \sqrt{600} = 10\sqrt{6}$.



19. m = 100/32 slugs and k = 1200 lb/ft, so the critical frequency is $\omega_0 = \sqrt{k/m}$ $= \sqrt{384}$ rad/sec $= \sqrt{384}/2\pi \approx 3.12$ Hz.

20. Let the machine have mass m. Then the force $F = mg = 9.8m$ (the machine's weight) causes a displacement of x = 0.5 cm = 1/200 meters, so Hooke's law

$$F = kx, \quad \text{that is,} \quad mg = k(1/200)$$

gives the spring constant is $k = 200mg$ (N/m). Hence the resonance frequency is

$$\omega = \sqrt{k/m} = \sqrt{200g} \approx \sqrt{200 \times 9.8} \approx 44.27 \text{ rad/sec} \approx 7.05 \text{ Hz},$$

which is about 423 rpm (revolutions per minute).

21. If θ is the angular displacement from the vertical, then the (essentially horizontal) displacement of the mass is $x = L\theta$, so twice its total energy (KE + PE) is

$$m(x')^2 + kx^2 + 2mgh = mL^2(\theta')^2 + kL^2\theta^2 + 2mgL(1 - \cos\theta) = C.$$

Differentiation, substitution of $\theta \approx \sin\theta$, and simplification yields

$$\theta'' + (k/m + g/L)\theta = 0$$

so

$$\omega_0 = \sqrt{k/m + g/L}.$$

22. Let x denote the displacement of the mass from its equilibrium position, $v = x'$ its velocity, and $\omega = v/a$ the angular velocity of the pulley. Then conservation of energy yields

$$mv^2/2 + I\omega^2/2 + kx^2/2 - mgx = C.$$

When we differentiate both sides with respect to t and simplify the result, we get the differential equation

$$(m + I/a^2)x'' + kx = mg.$$

Hence $\omega = \sqrt{k/(m + I/a^2)}$.

23. **(a)** In ft-lb-sec units we have $m = 1000$ and $k = 10000$, so $\omega_0 = \sqrt{10}$ rad/sec ≈ 0.50 Hz.

(b) We are given that $\omega = 2\pi/2.25 \approx 2.79$ rad/sec, and the equation $mx'' + kx = F(t)$ simplifies to

$$x'' + 10x = (1/4)\omega^2 \sin\omega t.$$

When we substitute $x(t) = A \sin \omega t$ we find that the amplitude is

$$A = \omega^2/4(10 - \omega^2) \approx 0.8854 \text{ ft} \approx 10.63 \text{ in.}$$

24. By the identity of Problem 43 in Section 2.5, the differential equation is

$$mx'' + kx = F_0(3\cos\omega t + \cos 3\omega t)/4.$$

Hence resonance occurs when either ω or 3ω equals $\omega_0 = \sqrt{k/m}$, that is, when either $\omega = \omega_0$ or $\omega = \omega_0/3$.

25. Substitution of the trial solution $x = A\cos\omega t + B\sin\omega t$ in the differential equation, and then collection of coefficients as usual yields the equations

$$\left(k - m\omega^2\right)A + (c\omega)B = 0, \quad -(c\omega)A + \left(k - m\omega^2\right)B = F_0$$

with coefficient determinant $\Delta = \left(k - m\omega^2\right)^2 + (c\omega)^2$ and solution $A = -(c\omega)F_0/\Delta$, $B = \left(k - m\omega^2\right)F_0/\Delta$. Hence

$$x(t) = \frac{F_0}{\sqrt{\Delta}}\left[\frac{k - m\omega^2}{\sqrt{\Delta}}\sin\omega t - \frac{c\omega}{\sqrt{\Delta}}\cos\omega t\right] = C\sin(\omega t - \alpha),$$

where $C = F_0/\sqrt{\Delta}$ and $\sin\alpha = c\omega/\sqrt{\Delta}$, $\cos\alpha = \left(k - m\omega^2\right)/\sqrt{\Delta}$.

26. Let $G_0 = \sqrt{E_0^2 + F_0^2}$ and $\rho = 1/\sqrt{(k - m\omega^2) + (c\omega)^2}$. Then

$$\begin{aligned}
x_{sp}(t) &= \rho E_0\cos(\omega t - \alpha) + \rho F_0\sin(\omega t - \alpha) \\
&= \rho G_0\left[\frac{E_0}{G_0}\cos(\omega t - \alpha) + \frac{F_0}{G_0}\sin(\omega t - \alpha)\right] \\
&= \rho G_0\left[\cos\beta\,\cos(\omega t - \alpha) + \sin\beta\sin(\omega t - \alpha)\right] \\
x_{sp}(t) &= \rho G_0\cos(\omega t - \alpha - \beta)
\end{aligned}$$

where $\tan\beta = F_0/E_0$. The desired formula now results when we substitute the value of ρ defined above.

27. The derivative of $C(\omega) = F_0/\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}$ is given by

$$C'(\omega) = -\frac{\omega F_0}{2}\frac{(c^2 - 2km) + 2(m\omega)^2}{\left[\left(k - m\omega^2\right)^2 + (c\omega)^2\right]^{3/2}}.$$

(a) Therefore, if $c^2 \geq 2km$, it is clear from the numerator that $C'(\omega) < 0$ for all ω, so $C(\omega)$ steadily decreases as ω increases.

(b) But if $c^2 < 2km$, then the numerator (and hence $C'(\omega)$) vanishes when $\omega = \omega_m = \sqrt{k/m - c^2/2m^2} < \sqrt{k/m} = \omega_0$. Calculation then shows that

$$C''(\omega_m) = \frac{16F_0 m^3 (c^2 - 2km)}{c^3 \left(4km - c^2\right)^{3/2}} < 0,$$

so it follows from the second-derivative test that $C(\omega_m)$ is a local maximum value.

28. **(a)** The given differential equation corresponds to Equation (17) with $F_0 = mA\omega^2$. It therefore follows from Equation (21) that the amplitude of the steady periodic vibrations at frequency ω is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{mA\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

(b) Now we calculate

$$C'(\omega) = \frac{mA\omega\left[2k^2 - (2mk - c^2)\omega^2\right]}{\left[\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2\right]^{3/2}},$$

and we see that the numerator vanishes when

$$\omega = \sqrt{\frac{2k^2}{2mk - c^2}} = \sqrt{\frac{k}{m}\left(\frac{2mk}{2mk - c^2}\right)} > \sqrt{\frac{k}{m}} = \omega_0.$$

29. We need only substitute $E_0 = ac\omega$ and $F_0 = ak$ in the result of Problem 26.

30. When we substitute the values $\omega = 2\pi v/L$, $m = 800$, $k = 7 \times 10^4$, $c = 3000$ and $L = 10$, $a = 0.05$ in the formula of Problem 29, simplify, and square, we get the function

$$Csq(v) = \frac{25\left(9\pi^2 v^2 + 122500\right)}{16\left(16\pi^4 v^4 - 64375\pi^2 v^2 + 76562500\right)^2}$$

giving the *square* of the amplitude C (in meters) as a function of the velocity v (in meters per second). Differentiation gives

$$Csq'(v) = -\frac{50\pi^2 v(9\pi^4 v^4 + 245000\pi^2 v^2 - 535937500)}{(16\pi^4 v^4 - 64375\pi^2 v^2 - 76562500)^2}.$$

Because the principal factor in the numerator is a quadratic in v^2, it is easy to solve the equation $Csq'(v) = 0$ to find where the maximum amplitude occurs; we find that the only positive solution is $v \approx 14.36$ m/sec ≈ 32.12 mi/hr. The corresponding amplitude of the car's vibrations is $\sqrt{Csq(14.36)} \approx 0.1364$ m $= 13.64$ cm.

SECTION 2.7

ELECTRICAL CIRCUITS

1. With $E(t) \equiv 0$ we have the simple exponential equation $5I' + 25I = 0$ whose solution with $I(0) = 4$ is $I(t) = 4e^{-5t}$.

2. With $E(t) \equiv 100$ we have the simple linear equation $5I' + 25I = 100$ whose solution with $I(0) = 0$ is $I(t) = 4(1 - e^{-5t})$.

3. Now the differential equation is $5I' + 25I = 100 \cos 60t$. Substitution of the trial solution

$$I_p = A \cos 60t + B \sin 60t$$

yields

$$I_p = 4(\cos 60t + 12 \sin 60t)/145.$$

The complementary function is $I_c = ce^{-5t}$; the solution with $I(0) = 0$ is

$$I(t) = 4(\cos 60t + 12 \sin 60t - e^{-5t})/145.$$

4. The solution of the initial value problem $2I' + 40I = 100e^{-10t}$, $I(0) = 0$ is $I(t) = 5(e^{-10t} - e^{-20t})$. To find the maximum current we solve the equation $I'(t) = -50e^{-10t} + 100e^{-20t} = -50e^{-20t}\left(e^{20t} - 2\right) = 0$ for $t = (\ln 2)/10$. Then $I_{\max} = I((\ln 2)/10) = 5/4$.

5. The linear equation $I' + 10I = 50\, e^{-10t} \cos 60t$ has integrating factor $\rho = e^{10t}$. The resulting general solution is $I(t) = e^{-10t}\left[(5/6)\sin 60t + C\right].$ To satisfy the initial condition $I(0) = 0$, we take $C = 0$ and get $I(t) = (5/6)e^{-10t} \sin 60t$.

6. Substitution of the trial solution $I = A\cos 60t + B\sin 60t$ in the differential equation $I' + 10I = 30\cos 60t + 40\sin 60t$ gives the equations $10A + 60B = 30,\ -60A + 10B = 40$ with solution $A = -21/37,\ B = 22/37$. The resulting steady periodic solution is $I_{sp}(t) = (1/37)(-21\cos 60t + 22\sin 60t) = (5/\sqrt{37})\cos(60t - \alpha)$, where $\alpha = \pi - \tan^{-1}(22/21) \approx 2.3329$ (2nd quadrant angle).

7. **(a)** The linear differential equation $RQ' + (1/C)Q = E_0$ has integrating factor $\rho = e^{t/RC}$. The resulting solution with $Q(0) = 0$ is $Q(t) = E_0C(1 - e^{-t/RC})$. Then $I(t) = Q'(t) = (E_0/R)e^{-t/RC}$.

(b) These solutions make it obvious that $\lim_{t\to\infty} Q(t) = E_0C$ and $\lim_{t\to\infty} I(t) = 0$.

8. **(a)** The linear equation $Q' + 5Q = 10e^{-5t}$ has integrating factor $\rho = e^{5t}$. The resulting solution with $Q(0) = 0$ is $Q(t) = 10te^{-5t}$, so $I(t) = Q'(t) = 10(1 - 5t)e^{-5t}$.
(b) $I(t) = 0$ when $t = 1/5$, so $Q_{max} = Q(1/5) = 2e^{-1}$.

9. Substitution of the trial solution $Q = A\cos 120t + B\sin 120t$ into the differential equation $200Q' + 4000Q = 100\cos 120t$ yields the equations

$$4000A + 24000B = 100, \qquad -24000A + 4000B = 0$$

with solution $A = 1/1480,\ B = 3/740$. The complementary function is $Q_c = ce^{-20t}$, and imposition of the initial condition $Q(0) = 0$ yields the solution
$Q(t) = (\cos 120t + 6\sin 120t - e^{-20t})/1480$. The current function is then
$I(t) = Q'(t) = (36\cos 120t - 6\sin 120t + e^{-20t})/74$. Thus the steady-periodic current is

$$\begin{aligned} I_{sp} &= \frac{6}{74}(6\cos 120t - \sin 120t) \\ &= \frac{6\sqrt{37}}{74}\left(\frac{6}{\sqrt{37}}\cos 120t - \frac{1}{\sqrt{37}}\sin 120t\right) = \frac{3}{\sqrt{37}}\cos(120t - \alpha) \end{aligned}$$

(with $\alpha = 2\pi - \tan^{-1}\frac{1}{6}$), so the steady-state amplitude is $3/\sqrt{37}$.

10. Substitution of the trial solution $Q = A\cos\omega t + B\sin\omega t$ into the differential equation $RQ' + (1/C)Q = E_0\cos\omega t$ yields the equations

$$(1/C)A + \omega r B = E_0, \qquad -r\omega A + (1/C)B = 0$$

with solution $A = E_0C/\left(1 + \omega^2R^2C^2\right),\ B = E_0\omega RC^2/\left(1 + \omega^2R^2C^2\right)$, so

$$
\begin{aligned}
Q_{\mathrm{sp}}(t) &= \frac{E_0 C}{1+\omega^2 R^2 C^2}\left(\cos\omega t + \omega RC \sin\omega t\right) \\
&= \frac{E_0 C}{\sqrt{1+\omega^2 R^2 C^2}}\left(\frac{1}{\sqrt{1+\omega^2 R^2 C^2}}\cos\omega t + \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}\sin\omega t\right) \\
&= \frac{E_0 C}{\sqrt{1+\omega^2 R^2 C^2}}\cos\left(\omega t - \beta\right)
\end{aligned}
$$

where $\beta = \tan^{-1}\omega RC$ (1st quadrant angle).

In Problems 11–16, we give first the trial solution $I_{\mathrm{p}} = A\cos\omega t + B\sin\omega t$, then the equations in A and B that we get upon substituting this trial solution into the RLC equation $LI'' + RI' + (1/C)I = E'(t)$, and finally the resulting steady periodic solution.

11. $I_{\mathrm{p}} = A\cos 2t + B\sin 2t$; $A + 6B = 10$, $-6A + B = 0$

$$I_{\mathrm{sp}}(t) = \frac{10}{37}\cos 2t + \frac{60}{37}\sin 2t = \frac{10}{\sqrt{37}}\left(\frac{1}{\sqrt{37}}\cos 2t + \frac{6}{\sqrt{37}}\sin 2t\right) = \frac{10}{\sqrt{37}}\sin(2t - \delta)$$

$\delta = 2\pi - \tan^{-1}(1/6) \approx 6.1180$ (4th quadrant angle)

12. $I_{\mathrm{p}} = A\cos 10t + B\sin 10t$; $A + 4B = 2$, $-4A + B = 0$

$$I_{\mathrm{sp}}(t) = \frac{2}{17}\cos 10t + \frac{8}{17}\sin 10t = \frac{2}{\sqrt{17}}\left(\frac{1}{\sqrt{17}}\cos 10t + \frac{4}{\sqrt{17}}\sin 10t\right) = \frac{2}{\sqrt{17}}\sin(10t - \delta)$$

$\delta = 2\pi - \tan^{-1}(1/4) \approx 6.0382$ (4th quadrant angle)

13. $I_{\mathrm{p}} = A\cos 5t + B\sin 5t$; $3A - 2B = 0$, $2A + 3B = 20$

$$I_{\mathrm{sp}}(t) = \frac{40}{13}\cos 5t + \frac{60}{13}\sin 5t = \frac{20}{\sqrt{13}}\left(\frac{2}{\sqrt{13}}\cos 5t + \frac{3}{\sqrt{13}}\sin 5t\right) = \frac{20}{\sqrt{13}}\sin(5t - \delta)$$

$\delta = 2\pi - \tan^{-1}(2/3) \approx 5.6952$ (4th quadrant angle)

14. $I_{\mathrm{p}} = A\cos 100t + B\sin 100t$; $-249A + 25B = 200$, $-25A - 249B = -150$

$$
\begin{aligned}
I_{\mathrm{sp}}(t) &= \frac{25}{31313}\left(-921\cos 100t + 847\sin 100t\right) \\
&= \frac{25\sqrt{1565650}}{31313}\left(-\frac{921}{\sqrt{1565650}}\cos 100t + \frac{847}{\sqrt{1565650}}\sin 100t\right) \approx 0.9990\sin(100t - \delta)
\end{aligned}
$$

$\delta = \tan^{-1}(921/847) \approx 0.8272$ (1st quadrant angle)

15. $I_p = A\cos 60\pi t + B\sin 60\pi t;$

$(1000 - 36\pi^2)A + 30\pi B = 33\pi, \quad 15\pi A - (500 - 18\pi^2)B = 0$

$$A = \frac{33\pi(250 - 9\pi^2)}{250000 - 17775\pi^2 + 324\pi^4}, \quad B = \frac{495\pi^2}{2(250000 - 17775\pi^2 + 324\pi^4)}$$

$$I_{sp}(t) \approx I_0 \sin(60\pi t - \delta); \quad I_0 = \frac{33\pi}{2\sqrt{250000 - 17775\pi^2 + 324\pi^4}} \approx 0.1591$$

$$\delta = 2\pi - \tan^{-1}\left(\frac{500 - 18\pi^2}{15\pi}\right) \approx 4.8576$$

16. $I_p = A\cos 377t + B\sin 377t;$

$-132129A + 47125B = 226200, \quad 47125A + 132129B = 0$

$$A = -\frac{14943789900}{9839419133}, \quad B = \frac{5329837500}{9839419133}$$

$$I_{sp}(t) \approx I_0 \sin(377t - \delta); \quad I_0 = \sqrt{\frac{25583220000}{9839419133}} \approx 1.6125$$

$$\delta = \tan^{-1}\left(\frac{132129}{47215}\right) \approx 1.2282$$

In each of Problems 17–22, the first step is to substitute the given RLC parameters, the initial values $I(0)$ and $Q(0)$, and the voltage $E(t)$ into Eq. (16) and solve for the remaining initial value

$$I'(0) = \frac{1}{L}\left[E(0) - R\,I(0) - (1/C)Q(0)\right]. \qquad (*)$$

17. With $I(0) = 0$ and $Q(0) = 5$, Equation (*) gives $I'(0) = -75$. The solution of the RLC equation $2I'' + 16I' + 50I = 0$ with these initial conditions is $I(t) = -25e^{-4t}\sin 3t$.

18. Our differential equation to solve is

$$2I'' + 60I' + 400I = -100e^{-t}.$$

We find the particular solution $I_p = (-50/171)e^{-t}$ by substituting the trial solution $A\,e^{-t}$; the general solution is

$$I(t) = c_1 e^{-10t} + c_2 e^{-20t} - (50/171)e^{-t}.$$

The initial conditions are $I(0) = 0$ and $I'(0) = 50$, the latter found by substituting $L = 2, R = 60, 1/C = 400, I(0) = Q(0) = 0$, and $E(0) = 100$ into Equation (*). Imposition of these initial values on the general solution above yields the equations $c_1 + c_2 - 50/171 = 0, \; -10c_1 - 20c_2 + 50/171 = 50$ with solution $c_1 = 50/9$, $c_2 = -100/19$. This gives the solution

$$I(t) = (50/171)(19e^{-10t} - 18e^{-20t} - e^{-t}).$$

19. Now our differential equation to solve is

$$2I'' + 60I' + 400I = -1000e^{-10t}.$$

We find the particular solution $I_p = -50t\,e^{-10t}$ by substituting the trial solution $At\,e^{-10t}$; the general solution is

$$I(t) = c_1e^{-10t} + c_2e^{-20t} - 50t\,e^{-10t}.$$

The initial conditions are $I(0) = 0$ and $I'(0) = -150$, the latter found by substituting $L = 2, R = 60, 1/C = 400, I(0) = 0, Q(0) = 1$, and $E(0) = 100$ into Equation (*). Imposition of these initial values on the general solution above yields the equations $c_1 + c_2 = 0, \; -10c_1 - 20c_2 - 50 = -150$ with solution $c_1 = -10, \; c_2 = 10$. Thus we get the solution

$$I(t) = 10e^{-20t} - 10e^{-10t} - 50te^{-10t}.$$

20. The differential equation $10I'' + 30I' + 50I = 100\cos 2t$ has transient solution

$$I_{tr}(t) = e^{-3t/2}\left(c_1\cos t\sqrt{11}/2 + c_2\sin t\sqrt{11}/2\right),$$

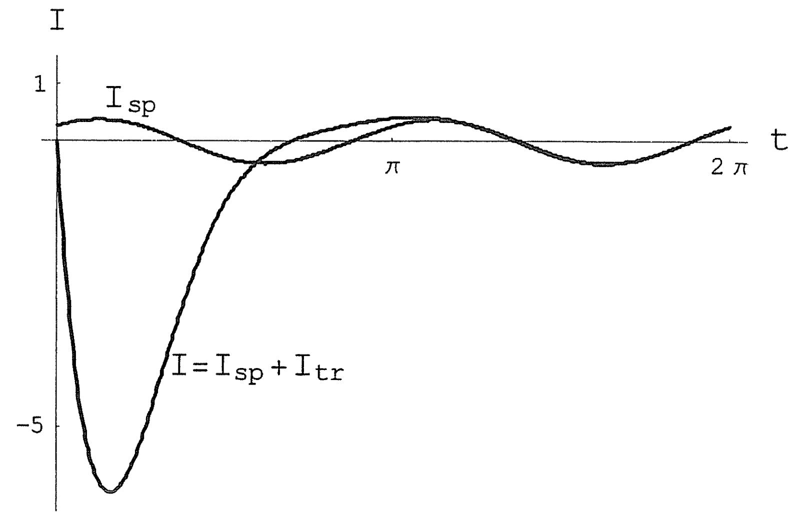
and in Problem 11 we found the steady periodic solution

$$I_{sp}(t) = \frac{10}{37}(\cos 2t + 6\sin 2t).$$

When we impose the initial conditions $I(0) = I'(0) = 0$ on the general solution $I(t) = I_{tr}(t) + I_{sp}(t)$, we get the equations

$$c_1 + 10/37 = 0, \quad -3c_1/2 + \sqrt{11}\,c_2/2 + 120/37 = 0$$

with solution $c_1 = -10/37, \; c_2 = -270/37\sqrt{11}$. The following figure shows the graphs of $I(t)$ and $I_{sp}(t)$.



21. The differential equation $10I'' + 20I' + 100I = -1000 \sin 5t$ has transient solution

$$I_{\text{tr}}(t) = e^{-t}\left(c_1 \cos 3t + c_2 \sin 3t\right),$$

and in Problem 13 we found the steady periodic solution

$$I_{sp}(t) = \frac{20}{13}\left(2\cos 5t + 3\sin 5t\right).$$

When we impose the initial conditions $I(0) = 0$, $I'(0) = -10$ on the general solution $I(t) = I_{\text{tr}}(t) + I_{\text{sp}}(t)$, we get the equations

$$c_1 + 40/13 = 0, \qquad -c_1 + 3c_2 + 300/13 = -10$$

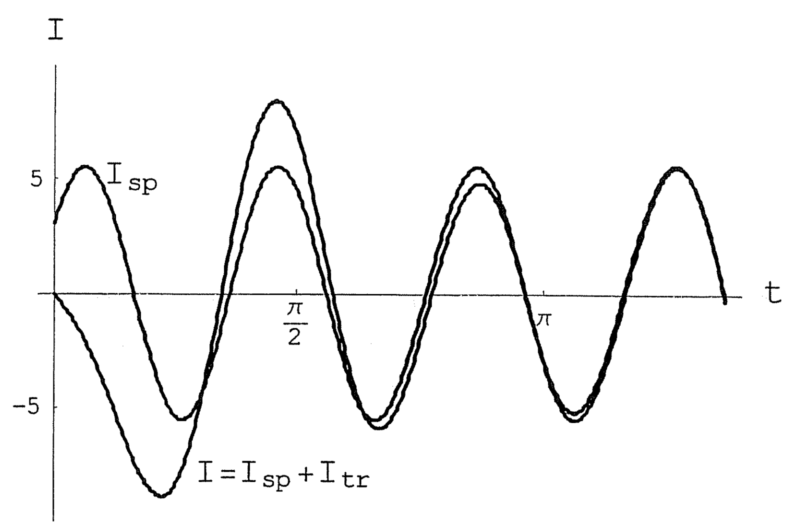
with solution $c_1 = -40/13$, $c_2 = -470/39$. The figure at the top of the next page shows the graphs of $I(t)$ and $I_{\text{sp}}(t)$.

22. The differential equation $2I'' + 100I' + 200000I = 6600\pi \cos 60\pi t$ has transient solution

$$I_{\text{tr}}(t) = e^{-25t}\left(c_1 \cos 25t\sqrt{159} + c_2 \sin 25t\sqrt{159}\right),$$

and in Problem 15 we found the steady periodic solution

$$I_{sp}(t) = A\cos 60\pi t + B\sin 60\pi t \approx 0.157444\cos 60\pi t + 0.023017\sin 60\pi t$$

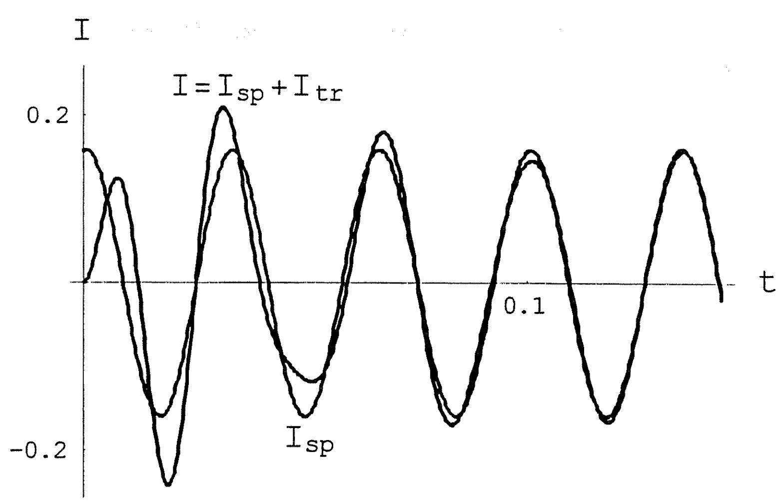


(with the exact values of A and B given there). When we impose the initial conditions $I(0) = I'(0) = 0$ on the general solution $I(t) = I_{tr}(t) + I_{sp}(t)$, we find (with the aid of a computer algebra system) that

$$c_1 = -\frac{33\pi(250-9\pi^2)}{250000-17775\pi^2+324\pi^4} \approx -0.157444,$$

$$c_2 = -\frac{11\pi\sqrt{159}\,(250+9\pi^2)}{53(250000-17775\pi^2+324\pi^4)} \approx -0.026249$$

The following figure shows the graphs of $I(t)$ and $I_{sp}(t)$.



23. The LC equation $LI'' + (1/C)I = 0$ has general solution $I(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$ with critical frequency $\omega_0 = 1/\sqrt{LC}$.

24. We need only observe that the roots

$$r = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

both necessarily have *negative* real parts.

25. According to Eq. (8) in the text, the amplitude of the steady periodic current is $E_0 / \sqrt{R^2 + (\omega L - 1/\omega C)^2}$. Because the radicand in the denominator is a sum of squares, it is obvious that the denominator is least when $\omega L - 1/\omega C = 0$, that is, when $\omega = 1/\sqrt{LC}$.

SECTION 2.8

ENDPOINT PROBLEMS AND EIGENVALUES

The material on eigenvalues and endpoint problems in Section 2.8 can be considered optional at this point in a first course. It will not be needed until we discuss boundary value problems in the last three sections of Chapter 8 and in Chapter 9. However, after the concentration thus far on initial value problems, the inclusion of this section can give students a view of a new class of problems that have diverse and important applications (as illustrated by the subsection on the whirling string). If Section 2.8 is not covered at this point in the course, then it can be inserted just prior to Section 8.5.

1. If $\lambda = 0$ then $y'' = 0$ implies that $y(x) = A + Bx$. The endpoint conditions $y'(0) = 0$ and $y(1) = 0$ yield $B = 0$ and $A = 0$, respectively. Hence $\lambda = 0$ is *not* an eigenvalue.

   If $\lambda = \alpha^2 > 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

   so

$$y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

   Then $y'(0) = 0$ yields $B = 0$, so $y(x) = A\cos\alpha x$. Next $y(1) = 0$ implies that $\cos\alpha = 0$, so α is an odd multiple of $\pi/2$. Hence the positive eigenvalues are $\{(2n-1)^2\pi^2/4\}$ with associated eigenfunctions $\{\cos(2n-1)\pi x/2\}$ for $n = 1, 2, 3, \cdots$.

2. If $\lambda = 0$ then $y'' = 0$ implies that $y(x) = A + Bx$. The endpoint conditions $y'(0) = y'(\pi) = 0$ imply only that $B = 0$, so $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = 1$.

If $\lambda = \alpha^2 > 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A\cos \alpha x + B\sin \alpha x.$$

Then

$$y'(x) = -A\alpha \sin \alpha x + B\alpha \cos \alpha x,$$

so $y'(0) = 0$ implies that $B = 0$. Next, $y'(\pi) = 0$ implies that $\alpha\pi$ is an integral multiple of π. Hence the positive eigenvalues are $\{n^2\}$ with associated eigenfunctions $\{\cos nx\}$, $n = 1, 2, 3, \cdots$.

3. Much as in Problem 1 we see that $\lambda = 0$ is not an eigenvalue. Suppose that $\lambda = \alpha^2 > 0$, so

$$y(x) = A\cos \alpha x + B\sin \alpha x.$$

Then the conditions $y(-\pi) = y(\pi) = 0$ yield

$$A\cos \alpha\pi + B\sin \alpha\pi = 0,$$

$$A\cos \alpha\pi - B\sin \alpha\pi = 0.$$

It follows that

$$A\cos \alpha\pi = 0 = B\sin \alpha\pi.$$

Hence either $A = 0$ and $B \neq 0$ with $\alpha\pi$ an even multiple of $\pi/2$, or $A \neq 0$ and $B = 0$ with $\alpha\pi$ an odd multiple of $\pi/2$. Thus the eigenvalues are $\{n^2/4\}$ for n a positive integer, and the nth eigenfunction is $y_n(x) = \cos(nx/2)$ if n is odd, $y_n(x) = \sin(nx/2)$ if n is even.

4. Just as in Problem 2, $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = 1$. If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos \alpha x + B\sin \alpha x,$$

then the equations

$$y'(-\pi) = \alpha(A\sin \alpha\pi + B\cos \alpha\pi) = 0,$$

$$y'(\pi) = \alpha(-A\sin \alpha\pi + B\cos \alpha\pi) = 0$$

yield $A\sin \alpha\pi = B\cos \alpha\pi = 0$. If $A = 0$ and $B \neq 0$, then $\cos \alpha\pi = 0$ so $\alpha\pi$ must be an odd multiple of $\pi/2$. If $A \neq 0$ and $B = 0$, then $\sin \alpha\pi = 0$ so $\alpha\pi$ must be an even multiple of $\pi/2$. Therefore the positive eigenvalues are $\{n^2/4\}$ with associated eigenfunctions $y_n(x) = \cos(nx/2)$ if the integer n is even, $y_n(x) = \sin(nx/2)$ if n is odd.

5. If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos \alpha x + B\sin \alpha x,$$

$$y'(x) = -A\alpha \sin \alpha x + B\alpha \cos \alpha x$$

then the conditions $y(-2) = y'(2) = 0$ yield

$$A \cos 2\alpha - B \sin 2\alpha = 0,$$

$$-A \sin 2\alpha + B \cos 2\alpha = 0.$$

It follows either that $A = B$ and $\cos 2\alpha = \sin 2\alpha$, or that $A = -B$ and $\cos 2\alpha = -\sin 2\alpha$. The former occurs if

$$2\alpha = \pi/4, 5\pi/4, 9\pi/4, \cdots,$$

the latter if

$$2\alpha = 3\pi/4, 7\pi/4, 11\pi/4, \cdots.$$

Hence the nth eigenvalue is

$$\lambda_n = \alpha_n^2 = (2n-1)^2\pi^2/64$$

for $n = 1, 2, 3, \cdots$, and the associated eigenfunction is

$$y_n(x) = \cos \alpha_n x + \sin \alpha_n x \quad (n \text{ odd})$$

or

$$y_n(x) = \cos \alpha_n x - \sin \alpha_n x \quad (n \text{ even}).$$

6. **(a)** If $\lambda = 0$ and $y(x) = A + Bx$, then $y'(0) = B = 0$, so $y(x) = A$. But then $y(1) + y'(1) = A = 0$ also, so $\lambda = 0$ is not an eigenvalue.

(b) If $\lambda = \alpha^2 > 0$ and

$$y(x) = A \cos \alpha x + B \sin \alpha x,$$

then

$$y'(x) = \alpha(-A \sin \alpha x + B \cos \alpha x),$$

so $y'(0) = B\alpha = 0$. Hence $B = 0$ so $y(x) = A \cos \alpha x$. Then

$$y(1) + y'(1) = A(\cos \alpha - \alpha \sin \alpha) = 0,$$

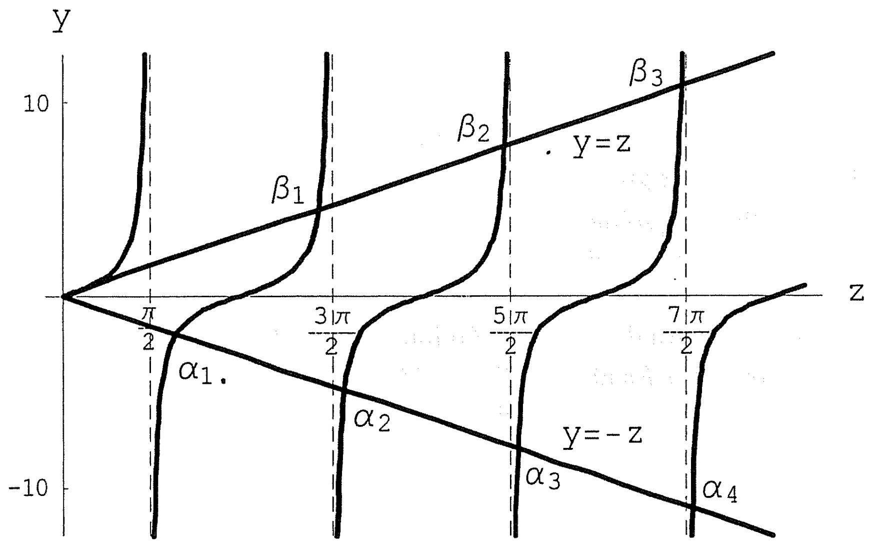
so α must be a positive root of the equation $\tan \alpha = 1/\alpha$.

7. **(a)** If $\lambda = 0$ and $y(x) = A + Bx$, then $y(0) = A = 0$, so $y(x) = Bx$. But then $y(1) + y'(1) = 2B = 0$, so $A = B = 0$ and $\lambda = 0$ is not an eigenvalue.

(b) If $\lambda = \alpha^2 > 0$ and $y(x) = A \cos \alpha x + B \sin \alpha x$, then $y(0) = A = 0$ so $y(x) = B \sin \alpha x$. Hence

$$y(1) + y'(1) = B(\sin\alpha + \alpha\cos\alpha) = 0.$$

so α must be a positive root of the equation $\tan\alpha = -\alpha$, and hence the abscissa of a point of intersection of the lines $y = \tan z$ and $y = -z$. We see from the figure below that α_n lies just to the right of the vertical line $z = (2n-1)\pi/2$, and lies closer and closer to this line as n gets larger and larger.



8. **(a)** If $\lambda = 0$ and $y(x) = A + Bx$, then $y(0) = A = 0$, so $y(x) = Bx$. But then $y(1) = y'(1)$ says only that $B = B$. Hence $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = x$.

(b) If $\lambda = \beta^2 > 0$ and $y(x) = A\cos\beta x + B\sin\beta x$, then $y(0) = A = 0$ so $y(x) = B\sin\beta x$. Then $y(1) = y'(1)$ says that $B\sin\beta = B\beta\cos\beta$, so β must be a positive root of the equation $\tan\beta = \beta$, and hence the abscissa of a point of intersection of the lines $y = \tan z$ and $y = z$. We see from the figure above that β_n lies just to the left of the vertical line $z = (2n+1)\pi/2$, and lies closer and closer to this line as n gets larger and larger.

9. If $y'' + \lambda y = 0$ and $\lambda = -\alpha^2 < 0$, then

$$y(x) = Ae^{\alpha x} + Be^{-\alpha x}.$$

Then $y(0) = A + B = 0$, so $B = -A$ and therefore

$$y(x) = A(e^{\alpha x} - e^{-\alpha x}).$$

Hence

$$y'(L) = A\alpha(e^{\alpha L} + e^{-\alpha L}) = 0.$$

But $\alpha \neq 0$ and $e^{\alpha L} + e^{-\alpha L} > 0$, so $A = 0$. Thus $\lambda = -\alpha^2$ is not an eigenvalue.

10. If $\lambda = -\alpha^2 < 0$, then the general solution of $y'' + \lambda y = 0$ is $y(x) = A\cosh\alpha x + B\sinh\alpha x$. Then $y(0) = 0$ implies that $A = 0$, so $y(x) = \sinh\alpha x$ (or a nonzero multiple thereof). Next,

$$y(1) + y'(1) = \sinh\alpha + \alpha\cosh\alpha = 0$$

implies that $\tanh\alpha = -\alpha$. But the graph of $y = \tanh\alpha$ lies in the first and third quadrants, while the graph of $y = -\alpha$ lies in the second and fourth quadrants. It follows that the only solution of $\tanh\alpha = -\alpha$ is $\alpha = 0$, and hence that our eigenvalue problem has no negative eigenvalues.

11. If $\lambda = -\alpha^2 < 0$, then the general solution of $y'' + \lambda y = 0$ is $y(x) = A\cosh\alpha x + B\sinh\alpha x$. Then $y'(0) = 0$ implies that $B = 0$, so $y(x) = \cosh\alpha x$ (or a nonzero multiple thereof). Next,

$$y(1) + y'(1) = \cosh\alpha + \alpha\sinh\alpha = 0$$

implies that $\tanh\alpha = -1/\alpha$. But the graph of $y = \tanh\alpha$ lies in the first and third quadrants, while the graph of $y = -1/\alpha$ lies in the second and fourth quadrants. It follows that the only solution of $\tanh\alpha = -1/\alpha$ is $\alpha = 0$, and hence that our eigenvalue problem has no negative eigenvalues.

12. **(a)** If $\lambda = 0$ and $y(x) = A + Bx$, then $y(-\pi) = y(\pi)$ means that $A + B\pi = A - B\pi$, so $B = 0$ and $y(x) = A$. But then $y'(-\pi) = y'(\pi)$ implies nothing about A. Hence $\lambda_0 = 0$ is an eigenvalue with $y_0(x) = 1$.

(b) If $\lambda = -\alpha^2 < 0$ and

$$y(x) = Ae^{\alpha x} + Be^{-\alpha x},$$

then the conditions $y(-\pi) = y(\pi)$ and $y'(-\pi) = y'(\pi)$ yield the equations

$$Ae^{\alpha\pi} + Be^{-\alpha\pi} = Ae^{-\alpha\pi} + Be^{\alpha\pi},$$

$$Ae^{\alpha\pi} - Be^{-\alpha\pi} = Ae^{-\alpha\pi} - Be^{\alpha\pi}.$$

Addition of these equations yields $2Ae^{\alpha\pi} = 2Ae^{-\alpha\pi}$. Since $e^{\alpha\pi} \neq e^{-\alpha\pi}$ because $\alpha \neq 0$, it follows that $A = 0$. Similarly $B = 0$. Thus there are no negative eigenvalues.

(c) If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

then the endpoint conditions yield the equations

$$A\cos\alpha\pi + B\sin\alpha\pi = A\cos\alpha\pi - B\sin\alpha\pi,$$

$$-A\sin\alpha\pi + B\cos\alpha\pi = A\sin\alpha\pi + B\cos\alpha\pi.$$

The first equation implies that $B\sin\alpha\pi = 0$, the second that $A\sin\alpha\pi = 0$. If A and B are not both zero, then it follows that $\sin\alpha\pi = 0$, so $\alpha = n$, an integer. In this case A and B are both arbitrary. Thus $\cos nx$ and $\sin nx$ are two different eigenfunctions associated with the single eigenvalue n^2.

13. **(a)** With $\lambda = 1$, the general solution of $y'' + 2y' + y = 0$ is

$$y(x) = Ae^{-x} + Bxe^{-x}.$$

But then $y(0) = A = 0$ and $y(1) = e^{-1}(A + B) = 0$. Hence $\lambda = 1$ is not an eigenvalue.

(b) If $\lambda < 1$, then the equation $y'' + 2y' + \lambda y = 0$ has characteristic equation $r^2 + 2r + \lambda = 0$. This equation has the two distinct real roots $\left(-2 \pm \sqrt{4 - 4\lambda}\right)/2$; call them r and s. Then the general solution is

$$y(x) = Ae^{rx} + Be^{sx},$$

and the conditions $y(0) = y(1) = 0$ yield the equations

$$A + B = 0, \quad Ae^r + Be^s = 0.$$

If $A, B \neq 0$, then it follows that $e^r = e^s$. But $r \neq s$, so there is no eigenvalue $\lambda < 1$.

(c) If $\lambda > 1$ let $\lambda - 1 = \alpha^2$, so $\lambda = 1 + \alpha^2$. Then the characteristic equation

$$r^2 + 2r + \lambda = (r + 1)^2 + \alpha^2 = 0$$

has roots $-1 \pm \alpha i$, so

$$y(x) = e^{-x}(A\cos\alpha x + B\sin\alpha x).$$

Now $y(0) = A = 0$, so $y(x) = Ae^{-x}\sin\alpha x$. Next, $y(1) = Ae^{-1}\sin\alpha = 0$, so $\alpha = n\pi$ with n an integer. Thus the nth positive eigenvalue is $\lambda_n = n^2\pi^2 + 1$. Because $\lambda = \alpha^2 + 1$, the eigenfunction associated with λ_n is

$$y_n(x) = e^{-x}\sin n\pi x.$$

14. If $\lambda = 1 + \alpha^2$ then we first impose the condition $y(0) = 0$ on the solution

$$y(x) = e^{-x}(A \cos \alpha x + B \sin \alpha x)$$

found in Problem 13, and find that $A = 0$. Hence

$$y(x) = Be^{-x}\sin \alpha x,$$
$$y'(x) = B(-e^{-x}\sin \alpha x + e^{-x}\alpha \cos \alpha x),$$

so the condition $y'(1) = 0$ yields $-\sin \alpha + \alpha \cos \alpha = 0$, that is, $\tan \alpha = \alpha$.

15. **(a)** The endpoint conditions are

$$y(0) = y'(0) = y''(L) = y^{(3)}(L) = 0.$$

With these conditions, four successive integrations as in Example 5 yield the indicated shape function $y(x)$.

(b) The maximum value y_{max} of $y(x)$ on the closed interval $[0, L]$ must occur either at an interior point where $y'(x) = 0$ or at one of the endpoints $x = 0$ and $x = L$. Now

$$y'(x) = k(4x^3 - 12Lx^2 + 12L^2x) = 4kx(x^2 - 3Lx + 3L^2)$$

where $k = w/24EI$, and the quadratic factor has no real zero. Hence $x = 0$ is the only zero of $y'(x)$. But $y(0) = 0$, so it follows that $y_{max} = y(L)$.

16. **(a)** The endpoint conditions are

$$y(0) = y'(0) = 0 \text{ and } y(L) = y'(L) = 0.$$

(b) The derivative

$$y'(x) = k(4x^3 - 6Lx^2 + 2L^2x) = 2kx(2x - L)(x - L)$$

vanishes at $x = 0, L/2, L$. Because $y(0) = y(L) = 0$, the argument of Problem 15(b) implies that $y_{max} = y(L/2)$.

17. If $y(x) = k(x^4 - 2Lx^3 + L^3x)$ with $k = w/24EI$, then

$$y'(x) = k(4x^3 - 6Lx^2 + L^3) = 0$$

has the solution $x = L/2$ that we can verify by inspection. Now long division of the cubic $4x^3 - 6Lx^2 + L^3$ by $2x - L$ yields the quadratic factor $2x^2 - 2Lx - L^2$ whose zeros $\left(2L \pm \sqrt{12L^2}\right)/4 = \left(1 \pm \sqrt{3}\right)L/2$ both lie outside the interval $[0, L]$. Thus $x = L/2$ is, indeed, the only zero of $y'(x) = 0$ in this interval.

18. **(a)** The endpoint conditions are

$$y(0) = y'(0) = 0 \text{ and } y(L) = y''(L) = 0.$$

(b) The only zero of the derivative

$$y'(x) = 2kx(8x^2 - 15Lx + 6L^2)$$

interior to the interval $[0, L]$ is

$$x_m = (15 - \sqrt{33})L/16,$$

and $y(0) = y(L) = 0$, so it follows by the argument of Problem 15(b) that $y_{\max} = y(x_m)$.

CHAPTER 3

POWER SERIES METHODS

SECTION 3.1

INTRODUCTION AND REVIEW OF POWER SERIES

The power series method consists of substituting a series $y = \Sigma c_n x^n$ into a given differential equation in order to determine what the coefficients $\{c_n\}$ must be in order that the power series will satisfy the equation. It might be pointed out that, if we find a recurrence relation in the form $c_{n+1} = \phi(n) c_n$, then we can determine the radius of convergence ρ of the series solution directly from the recurrence relation,

$$\rho = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{\phi(n)}\right|.$$

In Problems 1–10 we give first that recurrence relation that can be used to find the radius of convergence and to calculate the succeeding coefficients $c_1, c_2, c_3, \cdots$ in terms of the arbitrary constant c_0. Then we give the series itself

1. $c_{n+1} = \dfrac{c_n}{n+1}$; it follows that $c_n = \dfrac{c_0}{n!}$ and $\rho = \lim\limits_{n\to\infty}(n+1) = \infty$.

$$y(x) = c_0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) = c_0\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = c_0 e^x$$

2. $c_{n+1} = \dfrac{4c_n}{n+1}$; it follows that $c_n = \dfrac{4^n c_0}{n!}$ and $\rho = \lim\limits_{n\to\infty}\dfrac{n+1}{4} = \infty$.

$$y(x) = c_0\left(1 + 4x + 8x^2 + \frac{32x^3}{3} + \frac{32x^4}{4} + \cdots\right)$$

$$= c_0\left(1 + \frac{4x}{1!} + \frac{4^2x^2}{2!} + \frac{4^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots\right) = c_0 e^{4x}$$

3. $c_{n+1} = -\dfrac{3c_n}{2(n+1)}$; it follows that $c_n = \dfrac{(-1)^n 3^n c_0}{2^n n!}$ and $\rho = \lim\limits_{n\to\infty}\dfrac{2(n+1)}{3} = \infty$.

$$y(x) = c_0\left(1 - \frac{3x}{2} + \frac{9x^2}{8} - \frac{9x^3}{16} + \frac{27x^4}{128} - \cdots\right)$$

$$= c_0\left(1 - \frac{3x}{1!2} + \frac{3^2x^2}{2!2^2} - \frac{3^3x^3}{3!2^3} + \frac{3^4x^4}{4!2^4} - \cdots\right) = c_0 e^{-3x/2}$$

4. When we substitute $y = \Sigma c_n x^n$ into the equation $y' + 2xy = 0$, we find that

$$c_1 + \sum_{n=0}^{\infty}\left[(n+2)c_{n+2} + 2c_n\right]x^{n+1} = 0.$$

Hence $c_1 = 0$ — which we see by equating constant terms on the two sides of this equation — and $c_{n+2} = -\dfrac{2c_n}{n+2}$. It follows that

$$c_1 = c_3 = c_5 = \cdots = c_{\text{odd}} = 0 \quad \text{and} \quad c_{2k} = \frac{(-1)^k c_0}{k!}.$$

Hence

$$y(x) = c_0\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3} + \cdots\right) = c_0\left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) = c_0 e^{-x^2}$$

and $\rho = \infty$.

5. When we substitute $y = \Sigma c_n x^n$ into the equation $y' = x^2 y$, we find that

$$c_1 + 2c_2 x + \sum_{n=0}^{\infty}\left[(n+3)c_{n+3} - c_n\right]x^{n+1} = 0.$$

Hence $c_1 = c_2 = 0$ — which we see by equating constant terms and x-terms on the two sides of this equation — and $c_3 = \dfrac{c_n}{n+3}$. It follows that

$$c_{3k+1} = c_{3k+2} = 0 \quad \text{and} \quad c_{3k} = \frac{c_0}{3\cdot 6\cdot\cdots\cdot(3k)} = \frac{c_0}{k!3^k}.$$

Hence

$$y(x) = c_0\left(1 + \frac{x^3}{3} + \frac{x^6}{18} + \frac{x^9}{162} + \cdots\right) = c_0\left(1 + \frac{x^3}{1!3} + \frac{x^6}{2!3^2} + \frac{x^9}{3!3^3} + \cdots\right) = c_0 e^{(x^3/3)}.$$

and $\rho = \infty$.

6. $c_{n+1} = \dfrac{c_n}{2}$; it follows that $c_n = \dfrac{c_0}{2^n}$ and $\rho = \lim\limits_{n\to\infty} 2 = 2.$

$$y(x) = c_0\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16} + \cdots\right)$$

$$= c_0\left[1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+\cdots\right] = \frac{c_0}{1-\frac{x}{2}} = \frac{2c_0}{2-x}$$

7. $c_{n+1} = 2c_n$; it follows that $c_n = 2^n c_0$ and $\rho = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}$.

$$y(x) = c_0\left(1+2x+4x^2+8x^3+16x^4+\cdots\right)$$

$$= c_0\left[1+(2x)+(2x)^2+(2x)^3+(2x)^4+\cdots\right] = \frac{c_0}{1-2x}$$

8. $c_{n+1} = -\frac{(2n-1)c_n}{2n+2}$; it follows that $\rho = \lim_{n\to\infty}\frac{2n+2}{2n-1} = 1$.

$$y(x) = c_0\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots\right)$$

Separation of variables gives $y(x) = c_0\sqrt{1+x}$.

9. $c_{n+1} = \frac{(n+2)c_n}{n+1}$; it follows that $c_n = (n+1)c_0$ and $\rho = \lim_{n\to\infty}\frac{n+1}{n+2} = 1$.

$$y(x) = c_0\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)$$

Separation of variables gives $y(x) = \frac{c_0}{(1-x)^2}$.

10. $c_{n+1} = \frac{(2n-3)c_n}{2n+2}$; it follows that $\rho = \lim_{n\to\infty}\frac{2n+2}{2n-3} = 1$.

$$y(x) = c_0\left(1-\frac{3x}{2}+\frac{3x^2}{8}+\frac{x^3}{16}+\frac{3x^4}{128}+\cdots\right)$$

Separation of variables gives $y(x) = c_0(1-x)^{3/2}$.

In Problems 11–14 the differential equations are second-order, and we find that the two initial coefficients c_0 and c_1 are both arbitrary. In each case we find the even-degree coefficients in terms of c_0 and the odd-degree coefficients in terms of c_1. The solution series in these problems are all recognizable power series that have infinite radii of convergence.

11. $c_{n+1} = \frac{c_n}{(n+1)(n+2)}$; it follows that $c_{2k} = \frac{c_0}{(2k)!}$ and $c_{2k+1} = \frac{c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)+c_1\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots\right) = c_0\cosh x + c_1\sinh x$$

12. $$c_{n+1} = \frac{4c_n}{(n+1)(n+2)};\text{ it follows that } c_{2k} = \frac{2^{2k}c_0}{(2k)!} \text{ and } c_{2k+1} = \frac{2^{2k}c_1}{(2k+1)!}.$$

$$\begin{aligned}
y(x) &= c_0\left(1+2x^2+\frac{2x^4}{3}+\frac{4x^6}{45}+\cdots\right)+c_1\left(x+\frac{2x^3}{3}+\frac{2x^5}{15}+\frac{4x^7}{315}+\cdots\right) \\
&= c_0\left(1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\frac{(2x)^6}{6!}+\cdots\right)+\frac{c_1}{2}\left((2x)+\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\frac{(2x)^7}{7!}+\cdots\right) \\
&= c_0\cosh 2x+\frac{c_1}{2}\sinh 2x
\end{aligned}$$

13. $$c_{n+1} = -\frac{9c_n}{(n+1)(n+2)};\text{ it follows that } c_{2k} = \frac{(-1)^k 3^{2k}c_0}{(2k)!} \text{ and } c_{2k+1} = \frac{(-1)^k 3^{2k}c_1}{(2k+1)!}.$$

$$\begin{aligned}
y(x) &= c_0\left(1-\frac{9x^2}{2}+\frac{27x^4}{8}-\frac{81x^6}{80}+\cdots\right)+c_1\left(x-\frac{3x^3}{2}+\frac{27x^5}{40}-\frac{81x^7}{560}+\cdots\right) \\
&= c_0\left(1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\frac{(3x)^6}{6!}+\cdots\right)+\frac{c_1}{3}\left((3x)-\frac{(3x)^3}{3!}+\frac{(3x)^5}{5!}-\frac{(3x)^7}{7!}+\cdots\right) \\
&= c_0\cos 3x+\frac{c_1}{3}\sin x
\end{aligned}$$

14. When we substitute $y = \Sigma c_n x^n$ into $y''+y-x = 0$ and split off the terms of degrees 0 and 1, we get

$$(2c_2+c_0)+(6c_3+c_1-1)x+\sum_{n=2}^{\infty}[(n+1)(n+2)c_{n+2}+c_n]x^n = 0.$$

Hence $c_2 = -\frac{c_0}{2}$, $c_3 = -\frac{c_1-1}{6}$, and $c_{n+2} = -\frac{c_n}{(n+1)(n+2)}$ for $n \geq 2$. It follows that

$$\begin{aligned}
y(x) &= c_0+c_0\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+c_1x+(c_1-1)\left(-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\
&= x+c_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+(c_1-1)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\
&= x+c_0\cos x+(c_1-1)\sin x.
\end{aligned}$$

15. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $xy'+y=0$ and find that $(n+1)c_n = 0$ for all $n \geq 0$. This implies that $c_n = 0$ for all $n \geq 0$, which means that the only power series solution of our differential equation is the trivial solution $y(x) \equiv 0$. Therefore the equation has no *non-trivial* power series solution.

16. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $2xy' = y$ and find that $2nc_n = c_n$ for all $n \geq 0$. This implies that $0c_0 = c_0$, $2c_1 = c_1$, $4c_2 = c_2, \cdots$, and hence that $c_n = 0$ for all $n \geq 0$, which means that the only power series solution of our differential equation is the trivial solution $y(x) \equiv 0$. Therefore the equation has no *non-trivial* power series solution.

17. Assuming a power series solution of the form $y = \Sigma c_n x^n$, we substitute it into the differential equation $x^2 y' + y = 0$. We find that $c_0 = c_1 = 0$ and that $c_{n+1} = -nc_n$ for $n \geq 1$, so it follows that $c_n = 0$ for all $n \geq 0$. Just as in Problems 15 and 16, this means that the equation has no *non-trivial* power series solution.

18. When we substitute and assumed power series solution $y = \Sigma c_n x^n$ into $x^3 y' = 2y$, we find that $c_0 = c_1 = c_2 = 0$ and that $c_{n+2} = nc_n/2$ for $n \geq 1$. Hence $c_n = 0$ for all $n \geq 0$, just as in Problems 15–17.

In Problems 19–22 we first give the recurrence relation that results upon substitution of an assumed power series solution $y = \Sigma c_n x^n$ into the given second-order differential equation. Then we give the resulting general solution, and finally apply the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ to determine the desired particular solution.

19. $c_{n+2} = -\dfrac{2^2 c_n}{(n+1)(n+2)}$ for $n \geq 0$, so $c_{2k} = \dfrac{(-1)^k 2^{2k} c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{(-1)^k 2^{2k} c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots\right) + c_1\left(x - \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} - \frac{2^6 x^7}{7!} + \cdots\right)$$

$c_0 = y(0) = 0$ and $c_1 = y'(0) = 3$, so

$$\begin{aligned} y(x) &= 3\left(x - \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} - \frac{2^6 x^7}{7!} + \cdots\right) \\ &= \frac{3}{2}\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\right] = \frac{3}{2}\sin 2x. \end{aligned}$$

20. $c_{n+2} = \dfrac{2^2 c_n}{(n+1)(n+2)}$ for $n \geq 0$, so $c_{2k} = \dfrac{2^{2k} c_0}{(2k)!}$ and $c_{2k+1} = \dfrac{2^{2k} c_1}{(2k+1)!}$.

$$y(x) = c_0\left(1 + \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} + \cdots\right) + c_1\left(x + \frac{2^2 x^3}{3!} + \frac{2^4 x^5}{5!} + \frac{2^6 x^7}{7!} + \cdots\right)$$

$c_0 = y(0) = 2$ and $c_1 = y'(0) = 0$, so

$$y(x) = 2\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \cdots\right) = 2\cosh 2x.$$

21. $c_{n+1} = \dfrac{2nc_n - c_{n-1}}{n(n+1)}$ for $n \geq 1$; with $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$, we obtain

$c_2 = 1,\ c_3 = \dfrac{1}{2},\ c_4 = \dfrac{1}{6} = \dfrac{1}{3!},\ c_5 = \dfrac{1}{24} = \dfrac{1}{4!},\ c_6 = \dfrac{1}{120} = \dfrac{1}{5!}$. Evidently $c_n = \dfrac{1}{(n-1)!}$, so

$$y(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = xe^x.$$

22. $c_{n+1} = -\dfrac{nc_n - 2c_{n-1}}{n(n+1)}$ for $n \geq 1$; with $c_0 = y(0) = 1$ and $c_1 = y'(0) = -2$, we obtain

$c_2 = 2,\ c_3 = -\dfrac{4}{3} = -\dfrac{2^3}{3!},\ c_4 = \dfrac{2}{3} = \dfrac{2^4}{4!},\ c_5 = -\dfrac{4}{15} = -\dfrac{2^5}{5!}$. Apparently $c_n = \pm\dfrac{2^n}{n!}$, so

$$y(x) = 1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \frac{(2x)^5}{5!} + \cdots = e^{-2x}.$$

23. $c_0 = c_1 = 0$ and the recursion relation

$$(n^2 - n + 1)c_n + (n-1)c_{n-1} = 0$$

for $n \geq 2$ imply that $c_n = 0$ for $n \geq 0$. Thus any assumed power series solution $y = \Sigma c_n x^n$ must reduce to the trivial solution $y(x) \equiv 0$.

24. **(a)** The fact that $y(x) = (1+x)^\alpha$ satisfies the differential equation $(1+x)y' = \alpha y$ follows immediately from the fact that $y'(x) = \alpha(1+x)^{\alpha-1}$.

(b) When we substitute $y = \Sigma c_n x^n$ into the differential equation $(1+x)y' = \alpha y$ we get the recurrence formula

$$c_{n+1} = \frac{(\alpha - n)c_n}{n+1}. c_{n+1} = (\alpha - n)c_n/(n+1).$$

Since $c_0 = 1$ because of the initial condition $y(0) = 1$, the binomial series (Equation (12) in the text) follows.

(c) The function $(1+x)^\alpha$ and the binomial series must agree on $(-1, 1)$ because of the uniqueness of solutions of linear initial value problems.

25. Substitution of $\sum_{n=0}^{\infty} c_n x^n$ into the differential equation $y'' = y' + y$ leads routinely — via shifts of summation to exhibit x^n-terms throughout — to the recurrence formula

$$(n+2)(n+1)c_{n+2} = (n+1)c_{n+1} + c_n,$$

and the given initial conditions yield $c_0 = 0 = F_0$ and $c_1 = 1 = F_1$. But instead of proceeding immediately to calculate explicit values of further coefficients, let us first multiply the recurrence relation by $n!$. This trick provides the relation

$$(n+2)!c_{n+2} = (n+1)!c_{n+1} + n!c_n,$$

that is, the Fibonacci-defining relation $F_{n+2} = F_{n+1} + F_n$ where $F_n = n!c_n$, so we see that $c_n = F_n / n!$ as desired.

26. This problem is pretty fully outlined in the textbook. The only hard part is squaring the power series:

$$\begin{aligned}&\left(1 + c_3x^3 + c_5x^5 + c_7x^7 + c_9x^9 + c_{11}x^{11} + \cdots\right)^2 \\ &\quad = x^2 + 2c_3x^4 + \left(c_3^2 + 2c_5\right)x^6 + \left(2c_3c_5 + 2c_7\right)x^8 + \\ &\qquad \left(c_5^2 + 2c_3c_7 + 2c_9\right)x^{10} + \left(2c_5c_7 + 2c_3c_9 + 2c_{11}\right)x^{12} + \cdots\end{aligned}$$

27. **(b)** The roots of the characteristic equation $r^3 = 1$ are $r_1 = 1$, $r_2 = \alpha = (-1 + i\sqrt{3})/2$, and $r_3 = \beta = (-1 - i\sqrt{3})/2$. Then the general solution is

$$y(x) = Ae^x + Be^{\alpha x} + Ce^{\beta x}. \qquad (*)$$

Imposing the initial conditions, we get the equations

$$\begin{aligned} A + B + C &= 1 \\ A + \alpha B + \beta C &= 1 \\ A + \alpha^2 B + \beta^2 C &= -1. \end{aligned}$$

The solution of this system is $A = 1/3$, $B = (1 - i\sqrt{3})/3$, $C = (1 + i\sqrt{3})/3$. Substitution of these coefficients in (*) and use of Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$ finally yields the desired result.

SECTION 3.2

SERIES SOLUTIONS NEAR ORDINARY POINTS

Instead of deriving in detail the recurrence relations and solution series for Problems 1 through 15, we indicate where some of these problems and answers originally came from. Each of the differential equations in Problems 1–10 is of the form

$$(Ax^2 + B)y'' + Cxy' + Dy = 0$$

with selected values of the constants A, B, C, D. When we substitute $y = \Sigma c_n x^n$, shift indices where appropriate, and collect coefficients, we get

$$\sum_{n=0}^{\infty}\left[An(n-1)c_n + B(n+1)(n+2)c_{n+2} + Cnc_n + Dc_n\right]x^n = 0.$$

Thus the recurrence relation is

$$c_{n+2} = -\frac{An^2 + (C-A)n + D}{B(n+1)(n+2)}c_n \qquad \text{for } n \geq 0.$$

It yields a solution of the form

$$y = c_0\, y_{\text{even}} + c_1\, y_{\text{odd}}$$

where y_{even} and y_{odd} denote series with terms of even and odd degrees, respectively. The even-degree series $c_0 + c_2x^2 + c_4x^4 + \cdots$ converges (by the ratio test) provided that

$$\lim_{n\to\infty}\left|\frac{c_{n+2}x^{n+2}}{c_n x^n}\right| = \left|\frac{Ax^2}{B}\right| < 1.$$

Hence its radius of convergence is at least $\rho = \sqrt{|B/A|}$, as is that of the odd-degree series $c_1x + c_3x^3 + c_5x^4 + \cdots$. (See Problem 6 for an example in which the radius of convergence is, surprisingly, greater than $\sqrt{|B/A|}$.)

In Problems 1–15 we give first the recurrence relation and the radius of convergence, then the resulting power series solution.

1. $c_{n+2} = c_n$; $\quad \rho = 1$; $\quad c_0 = c_2 = c_4 = \cdots$; $\quad c_1 = c_3 = c_4 = \cdots$

$$y(x) = c_0\sum_{n=0}^{\infty} x^{2n} + c_1\sum_{n=0}^{\infty} x^{2n+1} = \frac{c_0 + c_1x}{1-x^2}$$

2. $c_{n+2} = -\dfrac{1}{2}c_n$; $\quad \rho = 2$; $\quad c_{2n} = \dfrac{(-1)^n c_0}{2^n}$; $\quad c_{2n+1} = \dfrac{(-1)^n c_1}{2^n}$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n} + c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n}$$

3. $c_{n+2} = -\dfrac{c_n}{(n+2)}$; $\quad \rho = \infty$;

$$c_{2n} = \frac{(-1)^n c_0}{(2n)(2n-2)\cdots\cdots 4\cdot 2} = \frac{(-1)^n c_0}{n!2^n};$$

$$c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)(2n-1)\cdots\cdots 5\cdot 3} = \frac{(-1)^n c_1}{(2n+1)!!}$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!2^n}+c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!!}$$

4. $$c_{n+2} = -\frac{n+4}{n+2}c_n; \qquad \rho=1$$

$$c_{2n} = \left(-\frac{2n+2}{2n}\right)\left(-\frac{2n}{2n-2}\right)\cdots\cdots\left(-\frac{6}{4}\right)\left(-\frac{4}{2}\right)c_0 = (-1)^n\frac{2n+2}{2}c_0 = (-1)^n(n+1)c_0$$

$$c_{2n} = \left(-\frac{2n+3}{2n+1}\right)\left(-\frac{2n+1}{2n-1}\right)\cdots\cdots\left(-\frac{7}{5}\right)\left(-\frac{5}{3}\right)c_0 = (-1)^n\frac{2n+3}{3}c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n}+\frac{1}{3}c_1\sum_{n=0}^{\infty}(-1)^n(2n+3)x^{2n+1}$$

5. $$c_{n+2} = \frac{nc_n}{3(n+2)}; \qquad \rho=3; \qquad c_2=c_4=c_6=\cdots=0$$

$$c_{2n+1} = \frac{2n-1}{3(2n+1)}\cdot\frac{2n-3}{3(2n-1)}\cdots\cdots\frac{3}{3(5)}\cdot\frac{1}{3(3)}c_1 = \frac{c_1}{(2n+1)3^n}$$

$$y(x) = c_0+c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)3^n}$$

6. $$c_{n+2} = \frac{(n-3)(n-4)}{(n+1)(n+2)}c_n$$

The factor $(n-3)$ in the numerator yields $c_5=c_7=c_9=\cdots=0$, and the factor $(n-4)$ yields $c_6=c_8=c_{10}=\cdots=0$. Hence y_{even} and y_{odd} are both polynomials with radius of convergence $\rho=\infty$.

$$y(x) = c_0(1+6x^2+x^4)+c_1(x+x^3)$$

7. $$c_{n+2} = -\frac{(n-4)^2}{3(n+1)(n+2)}c_n; \qquad \rho\ge\sqrt{3}$$

The factor $(n-4)$ yields $c_6=c_8=c_{10}=\cdots=0$, so y_{even} is a 4th-degree polynomial.

We find first that $c_3=-c_1/2$ and $c_5=c_1/120$, and then for $n\ge 3$ that

$$c_{2n+1} = \left(-\frac{(2n-5)^2}{3(2n)(2n+1)}\right)\left(-\frac{(2n-7)^2}{3(2n-2)(2n-1)}\right)\cdots\cdots\left(-\frac{1^2}{3(6)(7)}\right)c_5 =$$

$$= (-1)^{n-2}\frac{[(2n-5)!!]^2}{3^{n-2}(2n+1)(2n-1)\cdots\cdots 7\cdot 6}\cdot\frac{c_1}{120} = 9\cdot(-1)^n\frac{[(2n-5)!!]^2}{3^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-\frac{8}{3}x^2+\frac{8}{27}x^4\right)+c_1\left[x-\frac{1}{2}x^3+\frac{1}{120}x^5+9\sum_{n=3}^{\infty}\frac{[(2n-5)!!]^2(-1)^n}{(2n+1)!\,3^n}x^{2n+1}\right]$$

8. $$c_{n+2} = \frac{(n-4)(n+4)}{2(n+1)(n+2)}c_n; \qquad \rho \geq \sqrt{2}$$

We find first that $c_3 = -5c_1/4$ and $c_5 = 7c_1/32$, and then for $n \geq 3$ that

$$c_{2n+1} = \left(\frac{(2n-5)(2n+3)}{2(2n)(2n+1)}\right)\left(\frac{(2n-7)(2n+1)}{2(2n-2)(2n-1)}\right)\cdots\cdots\left(\frac{1\cdot 9}{2(6)(7)}\right)c_5 =$$

$$= \frac{(2n-5)!!(2n+3)(2n+1)\cdots\cdots 9}{2^{n-2}(2n+1)(2n)\cdots\cdots 7\cdot 6}\cdot\frac{7c_1}{32} = 4\cdot\frac{5!}{7\cdot 5\cdot 3}\cdot\frac{7}{32}\frac{(2n-5)!!(2n+3)!!}{2^n(2n+1)!}c_1$$

$$c_{2n+1} = \frac{(2n-5)!!(2n+3)!!}{2^n(2n+1)!}c_1$$

$$y(x) = c_0\left(1-4x^2+2x^4\right)+c_1\left[x-\frac{5}{4}x^3+\frac{7}{32}x^5+\sum_{n=3}^{\infty}\frac{(2n-5)!!(2n+3)!!}{(2n+1)!\,2^n}x^{2n+1}\right]$$

9. $$c_{n+2} = \frac{(n+3)(n+4)}{(n+1)(n+2)}c_n; \qquad \rho = 1$$

$$c_{2n} = \frac{(2n+1)(2n+2)}{(2n-1)(2n)}\cdot\frac{(2n-1)(2n)}{(2n-3)(2n-2)}\cdots\cdots\frac{3\cdot 4}{1\cdot 2}c_0 = \frac{1}{2}(n+1)(2n+1)c_0$$

$$c_{2n+1} = \frac{(2n+2)(2n+3)}{(2n)(2n+1)}\cdot\frac{(2n)(2n+1)}{(2n-2)(2n-1)}\cdots\cdots\frac{4\cdot 5}{2\cdot 3}c_1 = \frac{1}{3}(n+1)(2n+3)c_1$$

$$y(x) = c_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}c_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}$$

10. $$c_{n+2} = -\frac{(n-4)}{3(n+1)(n+2)}c_n; \qquad \rho = \infty$$

The factor $(n-4)$ yields $c_6 = c_8 = c_{10} = \cdots = 0$, so y_{even} is a 4th-degree polynomial. We find first that $c_3 = c_1/6$ and $c_5 = c_1/360$, and then for $n \geq 3$ that

$$c_{2n+1} = \frac{-(2n-5)}{3(2n+1)(2n)} \cdot \frac{-(2n-3)}{3(2n-1)(2n-2)} \cdot \cdots \cdot \frac{-1}{3(7)(6)} c_5$$

$$= \frac{(2n-5)!!(-1)^{n-2}}{3^{n-2}(2n+1)(2n) \cdot \cdots \cdot (7)(6)} \cdot \frac{c_1}{360} =$$

$$= \frac{3^2 \cdot 5!}{360} \cdot \frac{(2n-5)!!(-1)^n}{3^n(2n+1)(2n) \cdot \cdots \cdot (7)(6) \cdot 5!} \cdot c_1 = 3 \cdot \frac{(2n-5)!!(-1)^n}{3^n(2n+1)!} c_1$$

$$y(x) = c_0\left(1 + \frac{2}{3}x^2 + \frac{1}{27}x^4\right) + c_1\left[x + \frac{1}{6}x^3 + \frac{1}{360}x^5 + 3\sum_{n=3}^{\infty} \frac{(2n-5)!!(-1)^n}{(2n+1)!\,3^n} x^{2n+1}\right]$$

11. $c_{n+2} = \dfrac{2(n-5)}{5(n+1)(n+2)} c_n; \qquad \rho = \infty$

The factor $(n-5)$ yields $c_7 = c_9 = c_{11} = \cdots = 0,$ so y_{odd} is a 5th-degree polynomial. We find first that $c_2 = -c_1$, $c_4 = c_0/10$ and $c_6 = c_0/750$, and then for $n \ge 4$ that

$$c_{2n} = \frac{2(2n-7)}{5(2n)(2n-1)} \cdot \frac{2(2n-5)}{5(2n-2)(2n-3)} \cdot \cdots \cdot \frac{2(1)}{5(8)(7)} c_6$$

$$= \frac{2^{n-3}(2n-7)!!}{5^{n-3}(2n)(2n-1) \cdot \cdots \cdot (8)(7)} \cdot \frac{c_0}{750} =$$

$$= \frac{5^3 \cdot 6!}{2^3 \cdot 750} \cdot \frac{2^n(2n-7)!!}{5^n(2n)(2n) \cdot \cdots \cdot (8)(7) \cdot 6!} \cdot c_1 = 15 \cdot \frac{2^n(2n-7)!!}{5^n(2n)!} c_0$$

$$y(x) = c_1\left(x - \frac{4x^3}{15} + \frac{4x^5}{375}\right) + c_0\left[1 - x^2 + \frac{x^4}{10} + \frac{x^6}{750} + 15\sum_{n=4}^{\infty} \frac{(2n-7)!!\,2^n}{(2n)!\,5^n} x^{2n}\right]$$

12. $c_{n+3} = \dfrac{c_n}{n+2}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0,$ so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0\left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 5 \cdot \cdots \cdot (3n-1)}\right] + c_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!\,3^n}$$

13. $c_{n+3} = -\dfrac{c_n}{n+3}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0,$ so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also.

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!\,3^n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{1 \cdot 4 \cdot \cdots \cdot (3n+1)}$$

14. $c_{n+3} = -\dfrac{c_n}{(n+2)(n+3)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = 0,$ so the recurrence relation yields $c_5 = c_8 = c_{11} = \cdots = 0$ also. Then

$$c_{3n} = \frac{-1}{(3n)(3n-1)}\cdot\frac{-1}{(3n-3)(3n-4)}\cdots\cdots\frac{-1}{3\cdot 2}c_0 = \frac{(-1)^n c_0}{3^n n!\cdot(3n-1)(3n-4)\cdots\cdots 5\cdot 2},$$

$$c_{3n+1} = \frac{-1}{(3n+1)(3n)}\cdot\frac{-1}{(3n-2)(3n-3)}\cdots\cdots\frac{-1}{4\cdot 3}c_1 = \frac{(-1)^n c_1}{3^n n!\cdot(3n+1)(3n-2)\cdots\cdots 4\cdot 1}.$$

$$y(x) = c_0\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{3n}}{3^n n!\cdot 2\cdot 5\cdots\cdots(3n-1)}\right]+c_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{3^n n!\cdot 1\cdot 4\cdots\cdots(3n+1)}$$

15. $c_{n+4} = -\dfrac{c_n}{(n+3)(n+4)}; \qquad \rho = \infty$

When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_3 = 0,$ so the recurrence relation yields $c_6 = c_{10} = \cdots = 0$ and $c_7 = c_{11} = \cdots = 0$ also. Then

$$c_{4n} = \frac{-1}{(4n)(4n-1)}\cdot\frac{-1}{(4n-4)(4n-5)}\cdots\cdots\frac{-1}{4\cdot 3}c_0 = \frac{(-1)^n c_0}{4^n n!\cdot(4n-1)(4n-5)\cdots\cdots 5\cdot 3},$$

$$c_{3n+1} = \frac{-1}{(4n+1)(4n)}\cdot\frac{-1}{(4n-3)(4n-4)}\cdots\cdots\frac{-1}{5\cdot 4}c_1 = \frac{(-1)^n c_1}{4^n n!\cdot(4n+1)(4n-3)\cdots\cdots 9\cdot 5}.$$

$$y(x) = c_0\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{4n}}{4^n n!\cdot 3\cdot 7\cdots\cdots(4n-1)}\right]+c_1\left[x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{4n+1}}{4^n n!\cdot 5\cdot 9\cdots\cdots(4n+1)}\right]$$

16. The recurrence relation is $c_{n+2} = -\dfrac{n-1}{n+1}c_n$ for $n \ge 1$. The factor $(n-1)$ in the numerator yields $c_3 = c_5 = c_7 = \cdots = 0$. When we substitute $y = \Sigma c_n x^n$ into the given differential equation, we find first that $c_2 = c_0$, and then the recurrence relation gives

$$c_{2n} = -\frac{2n-3}{2n-1}\cdot -\frac{2n-5}{2n-3}\cdots\cdots -\frac{3}{5}\cdot -\frac{1}{3}c_2 = \frac{(-1)^{n-1}}{2n-1}c_0.$$

Hence

$$y(x) = c_1 x + c_0\left(1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \cdots\right)$$

$$= c_1 x + c_0 + c_0 x\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = c_1 x + c_0\left(1 + x\tan^{-1} x\right).$$

With $c_0 = y(0) = 0$ and $c_1 = y'(0) = 1$ we obtain the particular solution $y(x) = x$.

17. The recurrence relation

$$c_{n+2} = -\frac{(n-2)c_n}{(n+1)(n+2)}$$

yields $c_2 = c_0 = y(0) = 1$ and $c_4 = c_6 = \cdots = 0$. Because $c_1 = y'(0) = 0$, it follows also that $c_1 = c_3 = c_5 = \cdots = 0$. Thus the desired particular solution is $y(x) = 1 + x^2$.

18. The substitution $t = x - 1$ yields $y'' + ty' + y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{c_n}{n+2}.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = \infty$.. The initial conditions give $c_0 = 2$ and $c_1 = 0$, so $c_{\text{odd}} = 0$ and it follows that

$$y = 2\left(1 - \frac{t^2}{2} + \frac{t^4}{2\cdot 4} - \frac{t^6}{2\cdot 4\cdot 6} + \cdots\right),$$

$$y(x) = 2\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^4}{2\cdot 4} - \frac{(x-1)^6}{2\cdot 4\cdot 6} + \cdots\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{2n}}{n!2^n}.$$

19. The substitution $t = x - 1$ yields $(1 - t^2)y'' - 6ty' - 4y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{n+4}{n+2}c_n.$$

for $n \geq 0$, so the solution series has radius of convergence $\rho = 1$, and therefore converges if $-1 < t < 1$. The initial conditions give $c_0 = 0$ and $c_1 = 1$, so $c_{\text{even}} = 0$ and

$$c_{2n+1} = \frac{2n+3}{2n+1}\cdot\frac{2n+1}{2n-1}\cdot\cdots\cdot\frac{7}{5}\cdot\frac{5}{3}c_1 = \frac{2n+3}{3}.$$

Thus

$$y = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)t^{2n+1} = \frac{1}{3}\sum_{n=0}^{\infty}(2n+3)(x-1)^{2n+1},$$

and the x-series converges if $0 < x < 2$.

20. The substitution $t = x - 3$ yields $(t^2 + 1)y'' - 4ty' + 6y = 0$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 2$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = -6$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2 - 6t^2 = 2 - 6(x-3)^2.$$

21. The substitution $t = x+2$ yields $(4t^2+1)y'' = 8y$, where primes now denote differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = -\frac{4(n-2)}{(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 1$ and $c_1 = 0$. It follows that $c_{\text{odd}} = 0$, $c_2 = 4$ and $c_4 = c_6 = \cdots = 0$, so the solution reduces to

$$y = 2 + 4t^2 = 1 + 4(x+2)^2.$$

22. The substitution $t = x+3$ yields $(t^2-9)y'' + 3ty' - 3y = 0$, with primes now denoting differentiation with respect to t. When we substitute $y = \Sigma c_n t^n$ we get the recurrence relation

$$c_{n+2} = \frac{(n+3)(n-1)}{9(n+1)(n+2)}c_n$$

for $n \geq 0$. The initial conditions give $c_0 = 0$ and $c_1 = 2$. It follows that $c_{\text{even}} = 0$ and $c_3 = c_5 = \cdots = 0$, so

$$y = 2t = 2x + 6.$$

In Problems 23–26 we first derive the recurrence relation, and then calculate the solution series $y_1(x)$ with $c_0 = 1$ and $c_1 = 0$, the solution series $y_2(x)$ with $c_0 = 0$ and $c_1 = 1$.

23. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + \sum_{n=1}^{\infty}\left[c_{n-1} + c_n + (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = -\frac{1}{2}c_0, \qquad c_{n+2} = -\frac{c_{n-1}+c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 1.$$

$$y_1(x) = 1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots; \qquad y_2(x) = x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{120} + \cdots$$

24. Substitution of $y = \Sigma c_n x^n$ yields

$$-2c_2 + \sum_{n=1}^{\infty}\left[2c_{n-1} + n(n+1)c_n - (n+1)(n+2)c_{n+2}\right]x^n = 0,$$

so

$$c_2 = 0, \qquad c_{n+2} = \frac{c_{n-1} + n(n+1)c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 1.$$

$$y_1(x) = 1 + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^6}{45} + \cdots; \qquad y_2(x) = x + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{5} + \cdots$$

25. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + \sum_{n=2}^{\infty} \left[c_{n-2} + (n-1)c_{n-1} + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = c_3 = 0, \qquad c_{n+2} = -\frac{c_{n-2} + (n-1)c_{n-1}}{(n+1)(n+2)} \quad \text{for} \quad n \geq 2.$$

$$y_1(x) = 1 - \frac{x^4}{12} + \frac{x^7}{126} + \frac{x^8}{672} + \cdots; \qquad y_2(x) = x - \frac{x^4}{12} - \frac{x^5}{20} + \frac{x^7}{126} + \cdots$$

26. Substitution of $y = \Sigma c_n x^n$ yields

$$2c_2 + 6c_3 x + 12c_4 x^2 + (2c_2 + 20c_5)x^3 +$$
$$\sum_{n=4}^{\infty} \left[c_{n-4} + (n-1)(n-2)c_{n-1} + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = c_3 = c_4 = c_5 = 0, \qquad c_{n+2} = -\frac{c_{n-4} + (n-1)(n-2)c_{n-1}}{(n+1)(n+2)} \quad \text{for} \quad n \geq 4.$$

$$y_1(x) = 1 - \frac{x^6}{30} + \frac{x^9}{72} - \frac{29x^{12}}{3960} + \cdots; \qquad y_2(x) = x - \frac{x^7}{42} + \frac{x^{10}}{90} - \frac{41x^{13}}{6552} + \cdots$$

27. Substitution of $y = \Sigma c_n x^n$ yields

$$c_0 + 2c_2 + (2c_1 + 6c_3)x + \sum_{n=2}^{\infty} \left[2c_{n-2} + (n+1)c_n + (n+1)(n+2)c_{n+2} \right] x^n = 0,$$

so

$$c_2 = -\frac{c_0}{2}, \quad c_3 = -\frac{c_1}{3}, \qquad c_{n+2} = -\frac{2c_{n-2} + (n+1)c_n}{(n+1)(n+2)} \quad \text{for} \quad n \geq 2.$$

With $c_0 = y(0) = 1$ and $c_1 = y'(0) = -1$, we obtain

$$y(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{24} + \frac{x^5}{30} + \frac{29x^6}{720} - \frac{13x^7}{630} - \frac{143x^8}{40320} + \frac{31x^9}{22680} + \cdots.$$

Finally, $x = 0.5$ gives

$$\begin{aligned} y(0.5) &= 1 - 0.5 - 0.125 + 0.041667 - 0.002604 + 0.001042 \\ &\qquad + 0.000629 - 0.000161 - 0.000014 + 0.000003 + \cdots \\ y(0.5) &\approx 0.415562 \approx 0.4156. \end{aligned}$$

28. When we substitute $y = \Sigma c_n x^n$ and $e^{-x} = \sum (-1)^n x^n / n!$ and then collect coefficients of the terms involving 1, x, x^2, and x^3, we find that

$$c_2 = -\frac{c_0}{2}, \quad c_3 = \frac{c_0 - c_1}{6}, \quad c_4 = \frac{c_1}{12}, \quad c_5 = -\frac{3c_0 + 2c_1}{120}.$$

With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{40} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{60} + \cdots.$$

29. When we substitute $y = \Sigma c_n x^n$ and $\cos x = \sum (-1)^n x^{2n} / (2n)!$ and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^6$, we obtain the equations

$$c_0 + 2c_2 = 0, \quad c_1 + 6c_3 = 0, \quad 12c_4 = 0, \quad -2c_3 + 20c_5 = 0,$$

$$\frac{1}{12}c_2 - 5c_4 + 30c_6 = 0, \quad \frac{1}{4}c_3 - 9c_5 + 42c_6 = 0,$$

$$-\frac{1}{360}c_2 + \frac{1}{2}c_4 - 14c_6 + 56c_8 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_8$ in turn. With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^6}{720} + \frac{13x^8}{40320} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{6} - \frac{x^5}{60} - \frac{13x^7}{5040} + \cdots.$$

30. When we substitute $y = \Sigma c_n x^n$ and $\sin x = \Sigma (-1)^n x^{2n+1}/(2n+1)!$, and then collect coefficients of the terms involving $1, x, x^2, \cdots, x^5$, we obtain the equations

$$c_0 + c_1 + 2c_2 = 0, \quad c_1 + 2c_2 + 6c_3 = 0, \quad -\frac{c_1}{6} + c_2 + 3c_3 + 12c_4 = 0,$$

$$-\frac{c_2}{3} + c_3 + 4c_4 + 20c_5 = 0, \quad \frac{c_1}{120} - \frac{c_3}{2} + c_4 + 5c_5 + 30c_6 = 0.$$

Given c_0 and c_1, we can solve easily for $c_2, c_3, \cdots, c_6$ in turn. With the choices $c_0 = 1,\ c_1 = 0$ and $c_0 = 0,\ c_1 = 1$ we obtain the two series solutions

$$y_1(x) = 1-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^5}{60}+\frac{x^6}{180}+\cdots \text{ and } y_2(x) = x-\frac{x^2}{2}+\frac{x^4}{18}-\frac{7x^5}{360}+\frac{x^6}{900}+\cdots.$$

33. Substitution of $y = \Sigma c_n x^n$ in Hermite's equation leads in the usual way to the recurrence formula

$$c_{n+2} = -\frac{2(\alpha-n)c_n}{(n+1)(n+2)}.$$

Starting with $c_0 = 1$, this formula yields

$$c_2 = -\frac{2\alpha}{2!}, \quad c_4 = +\frac{2^2\alpha(\alpha-2)}{4!}, \quad c_6 = -\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}, \quad \ldots.$$

Starting with $c_1 = 1$, it yields

$$c_3 = -\frac{2(\alpha-1)}{3!}, \quad c_5 = +\frac{2^2(\alpha-1)(\alpha-3)}{5!}, \quad c_7 = -\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}, \quad \ldots.$$

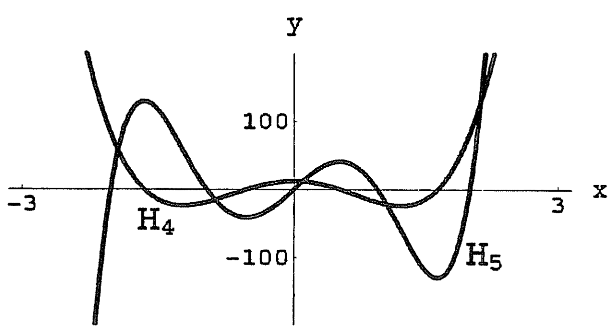
This gives the desired even-term and odd-term series y_1 and y_2. If α is an integer, then obviously one series or the other has only finitely many non-zero terms. For instance, with $\alpha = 4$ we get

$$y_1(x) = 1-\frac{2\cdot 4}{2}x^2+\frac{2^2\cdot 4\cdot 2}{24}x^4 = 1-4x^2+\frac{4}{3}x^4 = \frac{1}{12}\left(16x^4-48x^2+12\right),$$

and with $\alpha = 5$ we get

$$y_2(x) = x-\frac{2\cdot 4}{6}x^3+\frac{2^2\cdot 4\cdot 2}{120}x^5 = x-\frac{4}{3}x^3+\frac{4}{15}x^5 = \frac{1}{120}\left(32x^5-160x^3+120\right).$$

The figure below shows the interlaced zeros of the 4th and 5th Hermite polynomials.



34. Substitution of $y = \Sigma c_n x^n$ in the Airy equation leads upon shift of index and collection of terms to

$$2c_2 + \sum_{n=1}^{\infty}[(n+1)(n+2)c_{n+2} - c_{n-1}]x^n = 0.$$

The identity principle then gives $c_2 = 0$ and the recurrence formula

$$c_{n+3} = \frac{c_n}{(n+2)(n+3)}.$$

Because of the "3-step" in indices, it follows that $c_2 = c_5 = c_8 = c_{11} = \cdots = 0$. Starting with $c_0 = 1$, we calculate

$$c_3 = \frac{1}{2\cdot 3} = \frac{1}{3!} =, \quad c_6 = \frac{1}{3!\cdot 5\cdot 6} = \frac{1\cdot 4}{6!}, \quad c_9 = \frac{1\cdot 4}{6!\cdot 8\cdot 9} = \frac{1\cdot 4\cdot 7}{9!}, \quad \ldots.$$

Starting with $c_1 = 1$, we calculate

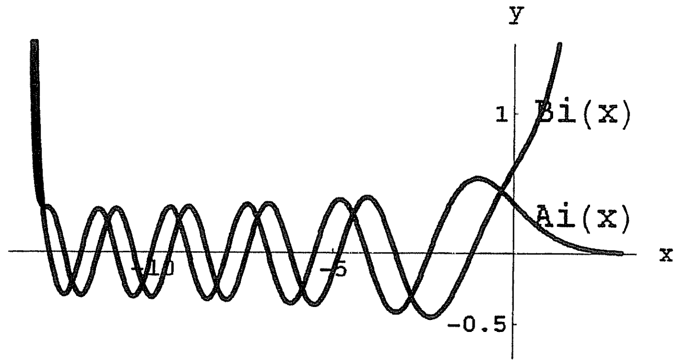
$$c_4 = \frac{1}{3\cdot 4} = \frac{2}{4!}, \quad c_7 = \frac{2}{4!\cdot 6\cdot 7} = \frac{2\cdot 5}{7!}, \quad c_{10} = \frac{2\cdot 5}{7!\cdot 9\cdot 10} = \frac{2\cdot 5\cdot 8}{10!}, \quad \ldots.$$

Evidently we are building up the coefficients

$$c_{3k} = \frac{1\cdot 4\cdot \cdots \cdot (3k-2)}{(3k)!} \quad \text{and} \quad c_{3k+1} = \frac{2\cdot 5\cdot \cdots \cdot (3k-1)}{(3k+1)!}$$

that appear in the desired series for $y_1(x)$ and $y_2(x)$. Finally, the Mathematica commands

```
A[1] = 1/6; A[k_] := A[k - 1]/(3k (3k - 1))
B[1] = 1/12; B[k_] := B[k - 1]/(3k (3k + 1))
n = 40;
y1 = 1 + Sum[A[k] x^(3k), {k, 1, n}];
y2 = x + Sum[B[k] x^(3k+1), {k, 1, n}];
yA = y1/(3^(2/3) Gamma[2/3]) - y2/(3^(1/3) Gamma[1/3]);
yB = y1/(3^(1/6) Gamma[2/3]) + y2/(3^(-1/6) Gamma[1/3]);
Plot[ {yA, yB}, {x, -13.5, 3}, PlotRange -> {-0.75, 1.5}];
```



produce the figure above. But with $n = 50$ (instead of $n = 40$) terms we get a figure that is visually indistinguishable from Figure 3.2.3 in the textbook.

SECTION 3.3

REGULAR SINGULAR POINTS

1. Upon division of the given differential equation by x we see that $P(x) = 1 - x^2$ and $Q(x) = (\sin x)/x$. Because both are analytic at $x = 0$ — in particular, $(\sin x)/x \to 1$ as $x \to 0$ because

$$\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

— it follows that $x = 0$ is an ordinary point.

2. Division of the differential equation by x yields

$$y'' + xy' + \frac{e^x - 1}{x}y = 0.$$

Because the function

$$\frac{e^x - 1}{x} = \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - 1\right) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

is analytic at the origin, we see that $x = 0$ is an ordinary point.

3. When we rewrite the given equation in the standard form of Equation (3) in this section, we see that $p(x) = (\cos x)/x$ and $q(x) = x$. Because $(\cos x)/x \to \infty$ as $x \to 0$ it follows that $p(x)$ is not analytic, so $x = 0$ is an irregular singular point.

4. When we rewrite the given equation in the standard form of Equation (3), we have $p(x) = 2/3$ and $q(x) = (1-x^2)/3x$. Since $q(x)$ is not analytic at the origin, $x = 0$ is an irregular singular point.

5. In the standard form of Equation (3) we have $p(x) = 2/(1+x)$ and $q(x) = 3x^2/(1+x)$. Both are analytic, so $x = 0$ is a regular singular point. The indicial equation is

$$r(r-1)+2r = r^2+r = r(r+1) = 0,$$

so the exponents are $r_1 = 0$ and $r_2 = -1$.

6. In the standard form of Equation (3) we have $p(x) = 2/(1-x^2)$ and $q(x) = -2/(1-x^2)$, so $x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = -2$. The indicial equation is $r^2+r-2 = 0$, so the exponents are $r = -2, 1$.

7. In the standard form of Equation (3) we have $p(x) = (6\sin x)/x$ and $q(x) = 6$, so $x = 0$ is a regular singular point with $p_0 = q_0 = 6$. The indicial equation is $r^2+5r+6 = 0$, so the exponents are $r_1 = -2$ and $r_2 = -3$.

8. In the standard form of Equation (3) we have $p(x) = 21/(6+2x)$ and $q(x) = 9(x^2-1)/(6+2x)$, so $x = 0$ is a regular singular point with $p_0 = 7/2$ and $q_0 = -3/2$. The indicial equation simplifies to $2r^2+5r-3 = 0$, so the exponents are $r = -3, 1/2$.

9. The only singular point of the differential equation $y''+\dfrac{x}{1-x}y'+\dfrac{x^2}{1-x}y = 0$ is $x = 1$. Upon substituting $t = x-1$, $x = t+1$ we get the transformed equation $y''-\dfrac{t+1}{t}y'-\dfrac{(t+1)^2}{t}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = -(1+t)$ and $q(t) = -t(1+t)^2$. Both these functions are analytic, so it follows that $x = 1$ is a regular singular point of the original equation.

10. The only singular point of the differential equation $y''+\dfrac{2}{x-1}y'+\dfrac{1}{(x-1)^2}y = 0$ is $x = 1$. Upon substituting $t = x-1$, $x = t+1$ we get the transformed equation $y''+\dfrac{2}{t}y'+\dfrac{1}{t^2}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 2$ and $q(t) \equiv 1$. Both these functions are analytic, so it follows that $x = 1$ is a regular singular point of the original equation.

11. The only singular points of the differential equation $y''-\dfrac{2x}{1-x^2}y'+\dfrac{12}{1-x^2}y = 0$ are $x = +1$ and $x = -1$.

$x = +1$: Upon substituting $t = x-1,\ x = t+1$ we get the transformed equation $y'' + \dfrac{2(t+1)}{t(t+2)}y' - \dfrac{12}{t(t+2)}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{2(t+1)}{t+2}$ and $q(t) = -\dfrac{12t}{t+2}$. Both these functions are analytic at $t = 0$, so it follows that $x = +1$ is a regular singular point of the original equation.

$x = -1$: Upon substituting $t = x+1,\ x = t-1$ we get the transformed equation $y'' + \dfrac{2(t-1)}{t(t-2)}y' - \dfrac{12}{t(t-2)}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{2(t-1)}{t-2}$ and $q(t) = -\dfrac{12t}{t-2}$. Both these functions are analytic at $t = 0$, so it follows that $x = -1$ is a regular singular point of the original equation.

12. The only singular point of the differential equation $y'' + \dfrac{3}{x-2}y' + \dfrac{x^3}{(x-2)^3}y = 0$ is $x = 2$. Upon substituting $t = x-2,\ x = t+2$ we get the transformed equation $y'' + \dfrac{3}{t}y' + \dfrac{(t+2)^3}{t^3}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 3$ and $q(t) = \dfrac{(t+2)^3}{t}$. Because q is *not* analytic at $t = 0$, it follows that $x = 2$ is an irregular singular point of the original equation.

13. The only singular points of the differential equation $y'' + \dfrac{1}{x-2}y' + \dfrac{1}{x+2}y = 0$ are $x = +2$ and $x = -2$.

$x = +2$: Upon substituting $t = x-2,\ x = t+2$ we get the transformed equation $y'' + \dfrac{1}{t+4}y' + \dfrac{1}{t}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = \dfrac{t}{t+4}$ and $q(t) = t$. Both these functions are analytic at $t = 0$, so it follows that $x = +2$ is a regular singular point of the original equation.

$x = -2$: Upon substituting $t = x+2,\ x = t-2$ we get the transformed equation $y'' + \dfrac{1}{t}y' + \dfrac{1}{t-4}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) \equiv 1$ and $q(t) = \dfrac{t^2}{t-4}$. Both these functions are analytic at $t = 0$, so it follows that $x = -2$ is a regular singular point of the original equation.

14. The only singular points of the differential equation $y'' + \dfrac{x^2+9}{(x^2-9)^2}y' + \dfrac{x^2+4}{(x^2-9)^2}y = 0$ are $x = +3$ and $x = -3$.

$x = +3$: Upon substituting $t = x-3$, $x = t+3$ we get the transformed equation $y'' + \dfrac{t^2+6t+13}{t^2(t^2+6)^2}y' + \dfrac{t^2+6t+18}{t^2(t^2+6)^2}y = 0$, where primes now denote differentiation with respect to t. Because $p(t) = \dfrac{t^2+6t+13}{t(t^2+6)^2}$ is *not* analytic at $t = 0$, it follows that $x = 3$ is an irregular singular point of the original equation.

$x = -3$: Upon substituting $t = x+3$, $x = t-3$ we get the transformed equation $y'' + \dfrac{t^2-6t+13}{t^2(t^2-6)^2}y' + \dfrac{t^2-6t+18}{t^2(t^2-6)^2}y = 0$, where primes now denote differentiation with respect to t. Because $p(t) = \dfrac{t^2-6t+13}{t(t^2-6)^2}$ is *not* analytic at $t = 0$, it follows that $x = -3$ is an irregular singular point of the original equation.

15. The only singular point of the differential equation $y'' - \dfrac{x^2-4}{(x-2)^2}y' + \dfrac{x+2}{(x-2)^2}y = 0$ is $x = 2$. Upon substituting $t = x-2$, $x = t+2$ we get the transformed equation $y'' - \dfrac{t+4}{t}y' + \dfrac{t+4}{t^2}y = 0$, where primes now denote differentiation with respect to t. In the standard form of Equation (3) we have $p(t) = -(t+4)$ and $q(t) = t+4$. Both these functions are analytic, so it follows that $x = 2$ is a regular singular point of the original equation.

16. The only singular points of the differential equation $y'' + \dfrac{3x+2}{x^3(1-x)}y' + \dfrac{1}{x^2(1-x)}y = 0$ are $x = 0$ and $x = 1$.

$x = 0$: In the standard form of Equation (3) we have $p(x) = \dfrac{3x+2}{x^2(1-x)}$ and $q(x) = \dfrac{1}{1-x}$. Since p is not analytic at $x = 0$, it follows that $x = 0$ is an irregular singular point.

$x = 1$: Upon substituting $t = x-1$, $x = t+1$ we get the transformed equation $y'' - \dfrac{3t+5}{(t+1)^3}y' - \dfrac{t}{(t+1)^2}y = 0$, where primes now denote differentiation with respect to t. Both $p(t) \equiv -\dfrac{t(3t+5)}{(t+1)^3}$ and $q(t) = -\dfrac{t^3}{(t+1)^2}$ are analytic at $t = 0$, so it follows that $x = 1$ is a regular singular point of the original equation.

Each of the differential equations in Problems 17–20 is of the form

$$Axy'' + By' + Cy = 0$$

with indicial equation $Ar^2 + (B-A)r = 0$. Substitution of $y = \Sigma c_n x^{n+r}$ into the differential equation yields the recurrence relation

$$c_n = -\frac{C c_{n-1}}{A(n+r)^2 + (B-A)(n+r)}$$

for $n \geq 1$. In these problems the exponents $r_1 = 0$ and $r_2 = (A-B)/A$ do *not* differ by an integer, so this recurrence relation yields two linearly independent Frobenius series solutions when we apply it separately with $r = r_1$ and with $r = r_2$.

17. With exponent $r_1 = 0$: $c_n = -\dfrac{c_{n-1}}{4n^2 - 2n}$

$$y_1(x) = x^0\left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\sqrt{x}\right)^{2n}}{(2n)!} = \cos\sqrt{x}$$

With exponent $r_2 = \dfrac{1}{2}$: $c_n = -\dfrac{c_{n-1}}{4n^2 + 2n}$

$$y_2(x) = x^{1/2}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \cdots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\sqrt{x}\right)^{2n+1}}{(2n+1)!} = \sin\sqrt{x}$$

18. With exponent $r_1 = 0$: $c_n = \dfrac{c_{n-1}}{2n^2 + n}$

$$y_1(x) = x^0\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \cdots\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!(2n+1)!!}$$

With exponent $r_2 = -\dfrac{1}{2}$: $c_n = \dfrac{c_{n-1}}{2n^2 - n}$

$$y_2(x) = x^{-1/2}\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \cdots\right) = \frac{1}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!(2n-1)!!}\right]$$

19. With exponent $r_1 = 0$: $\quad c_n = \dfrac{c_{n-1}}{2n^2 - 3n}$

$$y_1(x) = x^0\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \frac{x^4}{360} - \cdots\right) = 1 - x - \sum_{n=2}^{\infty} \frac{x^n}{n!(2n-3)!!}$$

With exponent $r_2 = \dfrac{3}{2}$: $\quad c_n = \dfrac{c_{n-1}}{2n^2 + 3n}$

$$y_2(x) = x^{3/2}\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \frac{x^4}{83160} + \cdots\right) = x^{3/2}\left[1 + 3\sum_{n=1}^{\infty} \frac{x^n}{n!(2n+3)!!}\right]$$

20. With exponent $r_1 = 0$: $\quad c_n = -\dfrac{2c_{n-1}}{3n^2 - n}$

$$y_1(x) = x^0\left(1 - x + \frac{x^2}{5} - \frac{x^3}{60} + \frac{x^4}{1320} - \cdots\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{n! \cdot 2 \cdot 5 \cdot \cdots \cdot (3n-1)}$$

With exponent $r_2 = \dfrac{1}{3}$: $\quad c_n = -\dfrac{2c_{n-1}}{3n^2 + n}$

$$y_2(x) = x^{1/3}\left(1 - \frac{x}{2} + \frac{x^2}{14} - \frac{x^3}{210} + \frac{x^4}{5460} - \cdots\right) = x^{1/3}\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n! \cdot 1 \cdot 4 \cdot \cdots \cdot (3n+1)}$$

The differential equations in Problems 21–24 are all of the form

$$Ax^2y'' + Bxy' + (C + Dx^2)y = 0 \tag{1}$$

with indical equation

$$\phi(r) = Ar^2 + (B - A)r + C = 0. \tag{2}$$

Substitution of $y = \Sigma c_n x^{n+r}$ into the differential equation yields

$$\phi(r)c_0x^r + \phi(r+1)c_1x^{r+1} + \sum_{n=2}^{\infty}\left[\phi(r+n)c_n + Dc_{n-2}\right]x^{n+r} = 0. \tag{3}$$

In each of Problems 21–24 the exponents r_1 and r_2 do *not* differ by an integer. Hence when we substitute either $r = r_1$ or $r = r_2$ into Equation (*) above, we find that c_0 is arbitrary because $\phi(r)$ is then zero, that $c_1 = 0$ — because its coefficient $\phi(r+1)$ is then nonzero — and that

$$c_n = -\frac{Dc_{n-2}}{\phi(r+n)} = -\frac{Dc_{n-2}}{A(n+r)^2 + (B-A)(n+r) + C} \tag{4}$$

for $n \geq 2$. Thus this recurrence formula yields two linearly independent Frobenius series solutions when we apply it separately with $r = r_1$ and with $r = r_2$.

21. With exponent $r_1 = 1$: $c_1 = 0, \quad c_n = \dfrac{2c_{n-2}}{n(2n+3)}$

$$y_1(x) = x^1\left(1 + \frac{x^2}{7} + \frac{x^4}{154} + \frac{x^6}{6930} + \cdots\right) = x\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\cdot 7\cdot 11\cdot\cdots\cdot(4n+3)}\right]$$

With exponent $r_2 = -\dfrac{1}{2}$: $c_1 = 0, \quad c_n = \dfrac{2c_{n-2}}{n(2n-3)}$

$$y_2(x) = x^{-1/2}\left(1 + x^2 + \frac{x^4}{10} + \frac{x^6}{270} + \cdots\right) = \frac{1}{\sqrt{x}}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\cdot 1\cdot 5\cdot\cdots\cdot(4n-3)}\right]$$

22. With exponent $r_1 = \dfrac{3}{2}$: $c_1 = 0, \quad c_n = -\dfrac{2c_{n-2}}{n(2n+5)}$

$$y_1(x) = x^{3/2}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} - \frac{x^6}{11934} + \cdots\right) = x^{3/2}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n!\cdot 9\cdot 13\cdot\cdots\cdot(4n+5)}\right]$$

With exponent $r_2 = -1$: $c_1 = 0, \quad c_n = -\dfrac{2c_{n-2}}{n(2n-5)}$

$$y_2(x) = x^{-1}\left(1 + x^2 - \frac{x^4}{6} + \frac{x^6}{126} - \frac{x^8}{5544} + \cdots\right) = \frac{1}{x}\left[1 + x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}x^{2n}}{n!\cdot 3\cdot 7\cdot\cdots\cdot(4n-5)}\right]$$

23. With exponent $r_1 = \dfrac{1}{2}$: $c_1 = 0, \quad c_n = \dfrac{c_{n-2}}{n(6n+7)}$

$$y_1(x) = x^{1/2}\left(1 + \frac{x^2}{38} + \frac{x^4}{4712} + \frac{x^6}{1215696} + \cdots\right) = \sqrt{x}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n!\cdot 19\cdot 31\cdot\cdots\cdot(12n+7)}\right]$$

With exponent $r_2 = -\dfrac{2}{3}$: $c_1 = 0, \quad c_n = \dfrac{c_{n-2}}{n(6n-7)}$

$$y_2(x) = x^{-2/3}\left(1 + \frac{x^2}{10} + \frac{x^4}{680} + \frac{x^6}{118320} + \cdots\right) = x^{-2/3}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n!\cdot 5\cdot 17\cdot\cdots\cdot(12n-7)}\right]$$

24. With exponent $r_1 = \dfrac{1}{3}$: $c_1 = 0, \quad c_n = -\dfrac{c_{n-2}}{n(3n+1)}$

$$y_1(x) = x^{1/3}\left(1 - \frac{x^2}{14} + \frac{x^4}{728} - \frac{x^6}{82992} + \cdots\right) = \sqrt[3]{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\cdot 7\cdot 13\cdot\cdots\cdot(6n+1)}\right]$$

With exponent $r_2 = 0$: $c_1 = 0, \quad c_n = -\dfrac{c_{n-2}}{n(3n-1)}$

$$y_2(x) = x^0\left(1 - \frac{x^2}{10} + \frac{x^4}{440} - \frac{x^6}{44880} + \cdots\right) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\cdot 5\cdot 11\cdot\cdots\cdot(6n-1)}$$

25. With exponent $r_1 = \frac{1}{2}$: $c_n = -\frac{c_{n-1}}{2n}$

$$y_1(x) = x^{1/2}\left(1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \cdots\right) = \sqrt{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!2^n} = \sqrt{x}\,e^{-x/2}$$

With exponent $r_2 = 0$: $c_n = -\frac{c_{n-1}}{2n-1}$

$$y_2(x) = x^0\left(1 - x + \frac{x^2}{3} - \frac{x^3}{15} + \frac{x^4}{105} - \cdots\right) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{(2n-1)!!}$$

26. With exponent $r_1 = \frac{1}{2}$: $c_1 = 0$, $c_n = \frac{c_{n-2}}{n}$

$$y_1(x) = x^{1/2}\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \frac{x^8}{384} + \cdots\right) = \sqrt{x}\sum_{n=0}^{\infty}\frac{x^{2n}}{n!2^n} = \sqrt{x}\,e^{x^2/2}$$

With exponent $r_2 = 0$: $c_1 = 0$, $c_n = \frac{2c_{n-2}}{2n-1}$

$$y_2(x) = x^0\left(1 + \frac{2x^2}{3} + \frac{4x^4}{21} + \frac{8x^6}{231} + \frac{16x^8}{3465} + \cdots\right) = 1 + \sum_{n=1}^{\infty}\frac{2^n x^{2n}}{3\cdot 7\cdot\cdots\cdot(4n-1)}$$

The differential equations in Problems 27–29 (after multiplication by x) and the one in Problem 31 are of the same form (1) above as those in Problems 21–24. However, now the exponents r_1 and $r_2 = r_1 - 1$ *do* differ by an integer. Hence when we substitute the smaller exponent $r = r_2$ into Equation (3), we find that c_0 and c_1 are *both* arbitrary, and that c_n is given (for $n \geq 2$) by the recurrence relation in (4). Thus the *smaller* exponent r_2 yields the general solution $y(x) = c_0 y_1(x) + c_1 y_2(x)$ in terms of the two linearly independent Frobenius series solutions $y_1(x)$ and $y_2(x)$.

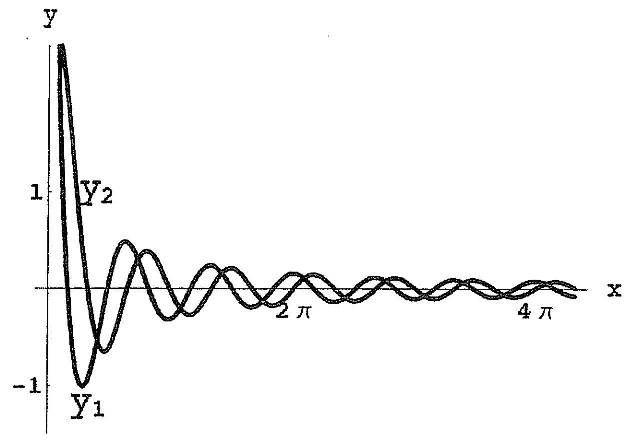
27. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $c_n = -\frac{9c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \cdots\right) + \frac{c_1}{x}\left(x - \frac{3x^3}{2} + \frac{27x^5}{40} - \frac{81x^7}{560} + \cdots\right)$$

$$= \frac{c_0}{x}\left(1 - \frac{9x^2}{2} + \frac{81x^4}{24} - \frac{729x^6}{720} + \cdots\right) + \frac{c_1}{3x}\left(3x - \frac{27x^3}{6} + \frac{243x^5}{120} - \frac{2187x^7}{5040} + \cdots\right)$$

$$y(x) = c_0\frac{\cos 3x}{x} + \frac{1}{3}c_1\frac{\sin 3x}{x}$$

The figure at the top of the next page shows the graphs of the independent solutions $y_1(x) = \frac{\cos 3x}{x}$ and $y_2(x) = \frac{\sin 3x}{x}$.



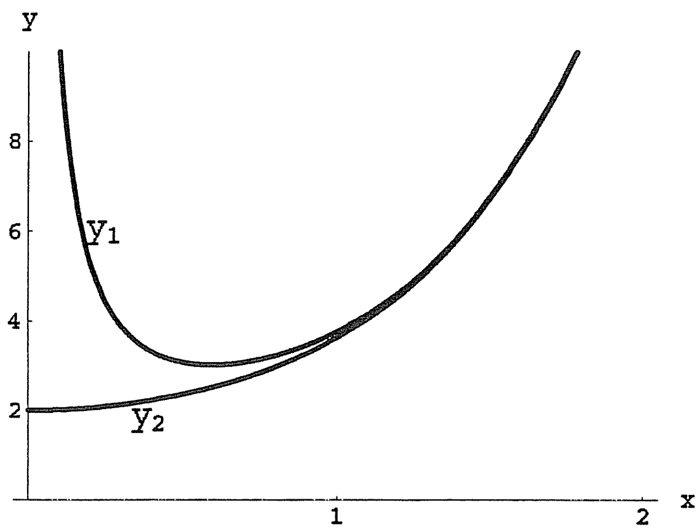
28. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $c_n = \dfrac{4c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1 + 2x^2 + \frac{2x^4}{3} + \frac{4x^6}{45} + \cdots\right) + \frac{c_1}{x}\left(x + \frac{2x^3}{3} + \frac{2x^5}{15} + \frac{4x^7}{315} + \cdots\right)$$

$$= \frac{c_0}{x}\left(1 + \frac{4x^2}{2} + \frac{16x^4}{24} + \frac{96x^6}{720} + \cdots\right) + \frac{c_1}{2x}\left(2x + \frac{8x^3}{6} + \frac{32x^5}{120} + \frac{128x^7}{5040} + \cdots\right)$$

$$y(x) = c_0\frac{\cosh 2x}{x} + \frac{1}{2}c_1\frac{\sinh 2x}{x}$$

The figure below shows the graphs of the independent solutions $y_1(x) = \dfrac{\cosh 2x}{x}$ and $y_2(x) = \dfrac{\sinh 2x}{x}$.



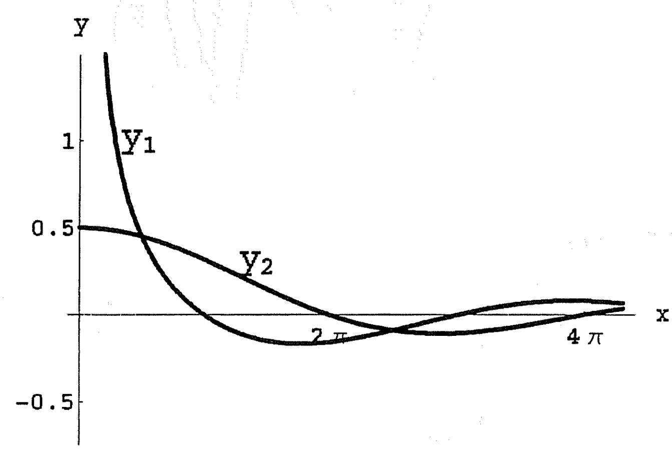
29. Exponents $r_1 = 0$ and $r_2 = -1$; with $r = -1$: $c_n = -\dfrac{c_{n-2}}{4n(n-1)}$

$$y(x) = \frac{c_0}{x}\left(1 - \frac{x^2}{8} + \frac{x^4}{384} - \frac{x^6}{46080} + \cdots\right) + \frac{c_1}{x}\left(x - \frac{x^3}{24} + \frac{x^5}{1920} - \frac{x^7}{322560} + \cdots\right)$$

$$= \frac{c_0}{x}\left(1 - \frac{x^2}{2^2 \cdot 2} + \frac{x^4}{2^4 \cdot 24} - \frac{x^6}{2^6 \cdot 720} + \cdots\right) + \frac{2c_1}{x}\left(\frac{x}{2} - \frac{x^3}{2^3 \cdot 6} + \frac{x^5}{2^5 \cdot 120} - \frac{x^7}{2^7 \cdot 5040} + \cdots\right)$$

$$y(x) = \frac{c_0}{x}\cos\frac{x}{2} + \frac{2c_1}{x}\sin\frac{x}{2}$$

The figure below shows the graphs of the independent solutions $y_1(x) = \dfrac{\cos x/2}{x}$ and $y_2(x) = \dfrac{\sin x/2}{x}$.



30. The given differential equation $xy'' - y' + 4x^3 y = 0$ has indicial equation $r^2 - 2r = r(r-2) = 0$, so its exponents are $r_1 = 2$ and $r_2 = 0$. Taking $r = 0$, substitution of the power series $y = \sum_{n=0}^{\infty} c_n x^n$ gives

$$-c_1 + 2c_3x^2 + (4c_0 + 8c_4)x^3 + (4c_1 + 15c_5)x^4 + (4c_2 + 24c_6)x^5 +$$
$$(4c_3 + 35c_7)x^6 + (4c_4 + 48c_8)x^7 + (4c_5 + 63c_9)x^8 + \cdots = 0.$$

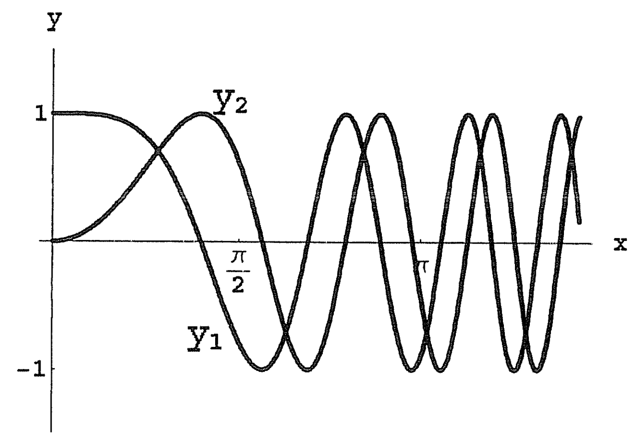
We see that $c_1 = c_3 = 0$ and

$$c_n = -\frac{4c_{n-4}}{n(n-2)} \quad \text{for } n \geq 4.$$

Hence the odd subscripts all vanish, and we obtain

$$y(x) = c_0 x\left(1-\frac{x^4}{2}+\frac{x^8}{24}-\frac{x^{12}}{720}+\cdots\right)+c_2\left(x^2-\frac{x^6}{6}+\frac{x^{10}}{120}-\frac{x^{14}}{5040}+\cdots\right)$$
$$y(x) = c_0 \cos x^2 + c_2 \sin x^2.$$

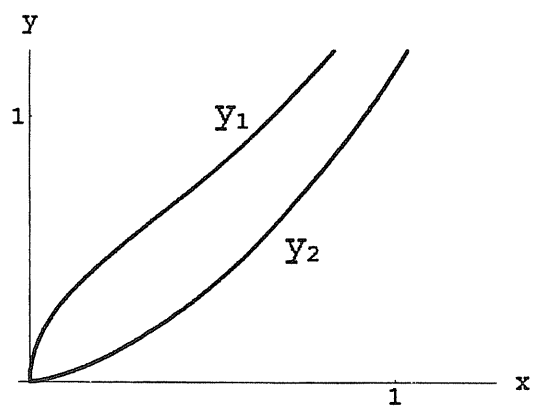
The figure below shows the graphs of the independent solutions $y_1(x) = \cos x^2$ and $y_2(x) = \sin x^2$.



31. The given differential equation $4x^2y'' - 4xy' + (3-4x^2)y = 0$ has indicial equation $4r^2 - 8r + 3 = (2r-3)(2r-1) = 0$, so its exponents are $r_1 = 3/2$ and $r_2 = 1/2$. With $r = 3/2$, the recurrence relation $c_n = c_{n-2}/n(n-1)$ yields the general solution

$$y(x) = c_0 x^{1/2}\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+\cdots\right)+c_1 x^{1/2}\left(x+\frac{x^3}{6}+\frac{x^5}{120}+\frac{x^7}{5040}+\cdots\right)$$
$$y(x) = c_0\sqrt{x}\cosh x + c_1\sqrt{x}\sinh x.$$

The figure below shows the graphs of the independent solutions $y_1(x) = \sqrt{x}\cosh x$ and $y_2(x) = \sqrt{x}\sinh x$.



32. The two indicial exponents are $r_1 = 1$ and $r_2 = -1/2$.

With $r_1 = 1$: Substitution of $y = x\sum c_n x^n$ in the differential equation yields

$$(5c_1 - c_0)x^2 + 14c_2x^3 + (c_2 + 27c_3)x^4 + (2c_3 + 44c_4)x^4 + (3c_4 + 65c_6)x^6 + \cdots = 0.$$

Hence we see that $c_1 = c_0/5$ and $c_2 = c_3 = c_4 = c_5 = \cdots 0$. Thus the series terminates and we obtain the polynomial solution

$$y_1(x) = x\left(1+\frac{x}{5}\right) = x+\frac{x^2}{5}.$$

With $r_2 = -1/2$: We substitute $y = x^{-1/2}\sum c_n x^n$ and obtain the Frobenius solution

$$y_2(x) = \frac{1}{\sqrt{x}}\left(1-\frac{5x}{2}-\frac{15x^2}{8}-\frac{5x^3}{48}+\frac{x^4}{384}+\cdots\right).$$

Remark: The Mathematica **DSolve** function yields the two closed form solutions $y_1(x)$ and

$$y_3(x) = x^{-1/2}e^{-x/2}\left(x^2+4x-2\right)+\sqrt{\frac{\pi}{2}}\,x(x+1)\operatorname{erf}\sqrt{\frac{x}{2}}.$$

Inquiring minds naturally want to know! The Mathematica **Series** command reveals the answer that $y_2(x) = -\frac{1}{2}y_3(x)$.

33. Exponents $r_1 = 1/2$ and $r_2 = -1$. With each exponent we find that c_0 is arbitrary and we can solve recursively for c_n in terms of c_{n-1}.

$$y_1(x) = \sqrt{x}\left(1+\frac{11x}{20}-\frac{11x^2}{224}+\frac{671x^3}{24192}-\frac{9577x^4}{387072}+\cdots\right)$$

$$y_2(x) = \frac{1}{x}\left(1+10x+5x^2+\frac{10x^3}{9}-\frac{7x^4}{18}+\cdots\right)$$

34. Exponents $r_1 = 1$ and $r_2 = -1/2$. With each exponent we find that $c_1 = 0$ and we can solve recursively for c_n in terms of c_{n-2}.

$$y_1(x) = x\left(1-\frac{x^2}{42}+\frac{x^4}{1320}-\frac{37x^6}{2494800}-\cdots\right)$$

$$y_2(x) = \frac{1}{\sqrt{x}}\left(1-\frac{7x^2}{24}+\frac{19x^4}{3200}-\frac{7661x^6}{43545600}+\cdots\right)$$

35. Substitution of $y = x^r \sum c_n x^n$ into the differential equation yields a result of the form

$$-rc_0 x^{r-1} + (\cdots)x^r + (\cdots)x^{r+1} + \cdots = 0,$$

so we see immediately that $c_0 \neq 0$ implies that $r = 0$. Then substitution of the power series $y = \sum c_n x^n$ yields

$$(c_0 - c_1) + (4c_1 - 2c_2)x + (9c_2 - 3c_3)x^2 + (16c_3 - 4c_4)x^4 + \cdots = 0$$

Evidently $c_n = nc_{n-1}$, so if $c_0 = 1$ it follows that $c_n = n!$ for $n \geq 1$. But the series $\sum n!x^n$ has zero radius of convergence, and hence converges only if $x = 0$. We therefore conclude that the given differential equation has *no* nontrivial Frobenius series solution.

36. **(a)** Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^2 y'' + Ay' + By = 0$ yields a result of the form

$$Arc_0 x^{r-1} + (\cdots)x^r + (\cdots)x^{r+1} + \cdots = 0,$$

so we see immediately that $A \neq 0$ and $c_0 \neq 0$ imply that $r = 0$.

(b) Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' + Axy' + By = 0$ yields a result of the form

$$(Ar + B)c_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

so we see immediately that $c_0 \neq 0$ implies that $r = -B/A$.

(c) Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' + Ax^2 y' + By = 0$ yields a result of the form

$$Bc_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

which is impossible because both $c_0 \neq 0$ and $B \neq 0$. It follows that *no* Frobenius series can satisfy this equation.

37. Substitution of $y = x^r \sum c_n x^n$ into the differential equation $x^3 y'' - y' + y = 0$ yields a result of the form

$$(r-1)^2 c_0 x^r + (\cdots)x^{r+1} + (\cdots)x^{r+2} + \cdots = 0,$$

so it follows that $r = 1$. But then substitution of $y = x\sum c_n x^n$ into the differential equation yields

$$c_1 x^2 + 4c_2 x^3 + 9c_3 x^4 + 16c_4 x^5 + 25c_5 x^6 + \cdots = 0,$$

so it follows that $c_1 = c_2 = c_3 = c_4 = \cdots = 0$. Hence $y(x) = c_0 x$,

38. Exponents $r_1 = 1/2$ and $r_2 = -1/2$; with $r = -1/2$: $c_n = -\dfrac{c_{n-2}}{n(n-1)}$

$$y(x) = \frac{c_0}{\sqrt{x}}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) + \frac{c_1}{\sqrt{x}}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)$$

$$= \frac{c_0}{\sqrt{x}}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \frac{c_1}{\sqrt{x}}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$y(x) = c_0 \frac{\cos x}{\sqrt{x}} + c_1 \frac{\sin 3x}{\sqrt{x}}$$

39. Exponents $r_1 = 1$ and $r_2 = -1$; with $r = +1$: $c_1 = 0$, $c_n = -\dfrac{c_{n-2}}{n(n+2)}$

$$y(x) = c_0 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + \frac{x^8}{737280} - \cdots\right)$$

$$= c_0 x\left(1 - \frac{x^2}{2^2 1!2!} + \frac{x^4}{2^4 2!3!} - \frac{x^6}{2^6 3!4!} + \frac{x^8}{2^8 4!5!} - \cdots\right)$$

If $c_0 = 1/2$, then

$$y(x) = J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)}\left(\frac{x}{2}\right)^{2n}.$$

Now, consider the smaller exponent $r_2 = -1$. A Frobenius series with $r = -1$ is of the form $y = x^{-1}\sum_{n=0}^{\infty} c_n x^n$ with $c_0 \neq 0$. However, substitution of this series into Bessel's equation of order 1 gives

$$-c_1 + c_0 x + (c_1 + 3c_3)x^2 + (c_2 + 8c_4)x^3 + (c_3 + 15c_5)x^5 + \cdots = 0,$$

so it follows that $c_0 = 0$, after all. Thus Bessel's equation of order 1 does not have a Frobenius series solution with leading term $c_0 x^{-1}$. However, there is a little more here that meets the eye. We see further that c_2 is arbitrary and that $c_1 = 0$ and $c_n = c_{n-2}/n(n-2)$ for $n > 2$. It follows that our assumed Frobenius series $y = x^{-1}\sum_{n=0}^{\infty} c_n x^n$ actually reduces to

$$y(x) = c_2 x\left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9216} + \frac{x^8}{737280} - \cdots\right).$$

But this is the same as our series solution obtained above using the larger exponent $r = +1$ (calling the arbitrary constant c_2 rather than c_0).

SECTION 3.4

METHOD OF FROBENIUS — THE EXCEPTIONAL CASES

Each of the differential equations in Problems 1–6 is of (or can be written in) the form

$$xy'' + (A + Bx)y' + Cy = 0.$$

The origin is a regular singular point with exponents $r = 0$ and $r = 1 - A$, so if A is an integer then we have an exceptional case of the method of Frobenius. When we substitute $y = \Sigma c_n x^{n+r}$ in the differential equation we find that the coefficient of x^{n+r} is

$$[(n + r)^2 + (A - 1)(n + r)]c_n + [B(n + r) + C - B]c_{n-1} = 0. \qquad (*)$$

Case 1: In each of Problems 1–4 we have $A \geq 2$ and $B = C$, so the larger exponent $r_1 = 0$ and the smaller exponent $r_2 = 1 - A = -N$ differ by a positive integer. When we substitute the smaller exponent $r = -N$ in Equation (*) above, it simplifies to

$$n(n - N)c_n + B(n - N)c_{n-1} = 0. \qquad (1)$$

This equation determines $c_1, c_2, \cdots, c_{N-1}$ in terms of c_0, thereby yielding the solution

$$y_1(x) = x^{-N}(c_0 + c_1 x + \cdots + c_{N-1}x^{n-1}), \qquad (2)$$

provided it is possible to choose $c_N = 0$. But when $n = N$, Equation (1) reduces to

$$0 \cdot c_N + 0 \cdot c_{N-1} = 0,$$

so c_N may be chosen arbitrarily. With $C_N = 0$ we get the terminating Frobenius series solution in (2). For $n > N$, Equation (1) yields the recurrence formula $c_n = -Bc_{n-1}/n$, which if $C_N \neq 0$ gives a second (non-terminating) Frobenius series solution of the form

$$y_2(x) = c_N + c_{N+1}x + c_{N+2}x^2 + \cdots. \qquad (3)$$

Case 2: If $A \leq 0$ then the larger exponent $r_1 = 1 - A = N$ and the smaller exponent $r_2 = 0$ again differ by a positive integer. In Problems 5 and 6 we have this case with $B = -1$. When we substitute the smaller exponent $r = 0$ in Equation (*), it simplifies to

$$n(n - N)c_n - (n - C - 1)c_{n-1} = 0. \qquad (4)$$

This equation determines $c_1, c_2, \cdots, c_{N-1}$ in terms of c_0. When $n = N$ it reduces to

$$0 \cdot c_N - (N - C - 1)c_{N-1} = 0. \qquad (5)$$

If either $N - C - 1 = 0$ or $c_{N-1} = 0$ (the latter happens in Problem 5) then c_N can be chosen arbitrarily, and finally $c_{N+1}, c_{N+2}, \cdots$ are determined in terms of c_N. Thus we get *two* Frobenius series solutions

$$y_1 = c_0 + c_1 x + \cdots + c_{N-1}x^{N-1}, \qquad \text{(terminating)}$$

$$y_2 = c_N x^N + c_{N+1}x^{N+1} + \cdots. \qquad \text{(not terminating)}$$

On the other hand, if (as in Problem 6) neither $N - C - 1 = 0$ nor $c_{N-1} = 0$, then c_N cannot be chosen so as to satisfy Equation (5), and hence there is no Frobenius series solution corresponding to the smaller exponent $r_2 = 0$. We therefore find the *single* Frobenius series solution by substituting the larger exponent $r_1 = N$ in Equation (*) and using the resulting recurrence relation to determine $c_1, c_2, c_3, \cdots$ in terms of c_0.

Problems 1–4 correspond to case 1 above. We give first the indicial roots and the critical index N, then the recurrence relation that defines c_n in terms of c_{n-1}, for both the N-term solution $y_1(x)$ in (2) and the non-terminating series solution $y_2(x)$ in (3)..

1. $r_1 = 0,\ r_2 = -2,\ N = 2;\ c_n = \dfrac{c_{n-1}}{n};\qquad y_1(x) = x^{-2}(1+x);$

$$y_2(x) = 1 + \frac{x}{3} + \frac{x^2}{3\cdot 4} + \frac{x^3}{3\cdot 4\cdot 5} + \cdots = 1 + 2\sum_{n=1}^{\infty}\frac{x^n}{(n+2)!}$$

2. $r_1 = 0,\ r_2 = -4,\ N = 4,\ c_n = \dfrac{c_{n-1}}{n};\qquad y_1(x) = x^{-4}\left(1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3\right)$

$$y_2(x) = 1 + \frac{x}{5} + \frac{x^2}{5\cdot 6} + \frac{x^3}{5\cdot 6\cdot 7} + \cdots = 1 + 24\sum_{n=1}^{\infty}\frac{x^n}{(n+4)!}$$

3. $r_1 = 0,\ r_2 = -4,\ N = 4,\ c_n = -\dfrac{3c_{n-1}}{n};\qquad y_1(x) = x^{-4}\left(1 - 3x + \dfrac{9}{2}x^2 - \dfrac{9}{2}x^3\right)$

$$y_2(x) = 1 - \frac{3x}{5} + \frac{3^2x^2}{5\cdot 6} - \frac{3^3x^3}{5\cdot 6\cdot 7} + \cdots = 1 + 24\sum_{n=1}^{\infty}\frac{(-1)^n 3^n x^n}{(n+4)!}$$

4. $r_1 = 0,\ r_2 = -5,\ N = 5,\ c_n = -\dfrac{3c_{n-1}}{5n}$

$$y_1(x) = x^{-5}\left(1 - \frac{3}{5}x + \frac{9}{50}x^2 - \frac{9}{250}x^3 + \frac{27}{5000}x^4\right)$$

$$y_2(x) = 1 - \frac{3x}{5\cdot 6} + \frac{3^2x^2}{5^2\cdot 6\cdot 7} - \frac{3^3x^3}{5^3\cdot 6\cdot 7\cdot 8} + \cdots = 1 + 120\sum_{n=1}^{\infty}\frac{(-1)^n 3^n x^n}{(n+5)!\ 5^n}$$

Problems 5 and 6 correspond to case 2 described above.

5. $r_1 = 5,\ r_2 = 0,\ N = 5,\ c_n = \dfrac{(n-4)c_{n-1}}{n(n-5)}$ for $n \neq 5$

$$y_1(x) = 1 + \frac{3}{4}x + \frac{1}{4}x^2 + \frac{1}{24}x^3$$

With $n = 5$ the recurrence relation is $0 \cdot c_5 - c_4 = 0$. Because $c_4 = 0$ we can choose $c_5 = 1$ arbitrarily and proceed:

$$y_2(x) = x^5 + \frac{2x^6}{6} + \frac{3x^7}{6 \cdot 7} + \frac{4x^8}{6 \cdot 7 \cdot 8} + \cdots = x^5\left[1 + 120\sum_{n=1}^{\infty}\frac{(n+1)x^n}{(n+5)!}\right]$$

6. Here $A = -3$, $B = -1$, $C = 1/2$, $r_1 = N = 4$, and $r_2 = 0$, so Equation (4) above is

$$n(n-4)c_n - (n - \tfrac{3}{2})c_{n-1} = 0.$$

Starting with $c_0 = 1$, this equation gives $c_1 = 1/6$, $c_2 = -1/48$, $c_3 = 1/96$. With $n = 4$ it reduces to

$$0 \cdot c_4 - \frac{7}{2} \cdot \frac{1}{96} = 0,$$

so c_4 cannot be chosen. We therefore start over by substituting $r_1 = 4$ in Equation (*) above and get the recurrence relation

$$c_n = \frac{2n+5}{2n(n+4)}c_{n-1}$$

for the coefficients in $y = x^4\sum_{n=0}^{\infty} c_n x^n$. This yields the single Frobenius series solution

$$y_1(x) = x^4\left(1 + \frac{7x}{2 \cdot 1 \cdot 5} + \frac{7 \cdot 9x^2}{2^2 \cdot 1 \cdot 2 \cdot 5.6} + \frac{7 \cdot 9 \cdot 11x^2}{2^3 \cdot 1 \cdot 2 \cdot 3 \cdot 5 \cdot 6 \cdot 7} + \cdots\right)$$

$$= x^4\left(1 + \frac{8}{5}\sum_{n=1}^{\infty}\frac{(2n+5)!!\,x^n}{2^n n!(n+4)!}\right).$$

7. The indicial exponents are $r = -2,\ 1$. Substitution of $y = x^{-2}\sum_{n=0}^{\infty} c_n x^n$ in the differential equation leads to the recurrence relation

$$n(n-3)c_n + 3(n-3)c_{n-1} = 0$$

that reduces to $0 \cdot c_3 + 0 \cdot c_2 = 0$ when $n = 3$ so — having found c_1 and c_2 — c_3 can be chosen arbitrarily. With $c_0 = 2$ and $c_3 = 0$ we get the terminating Frobenius series

$$y_1(x) = x^{-2}(2 - 6x + 9x^2).$$

Starting afresh with $c_3 = 3/3! = 1/2$, the recurrence relation $c_n = -3c_{n-1}/n$ for $n > 3$ yields the second Frobenius series solution

$$y_2(x) = x^{-2}\left(\frac{3x^3}{3!} - \frac{3^2x^4}{4!} + \frac{3^3x^5}{5!} - \cdots\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^n x^n}{(n+2)!}.$$

8. The exponents are $r = 0, 4$. When we substitute $y = \sum_{n=0}^{\infty} c_n x^n$ $\Sigma c_n x^n$ (corresponding to $r = 0$) in the differential equation we get the recurrence relation.

$$(n-4)c_n - (n-3)c_{n-1} = 0$$

for $n \geq 1$. Starting with $c_0 =$ e, we compute $c_1 = 2$, $c_2 = 1$, and $c_3 = 0$. Because of the latter, we can select $c_4 = 0$ and get the terminating Frobenius series solution

$$y_1(x) = 3 + 2x + x^2.$$

But we also can choose $c_4 = 1$. Then our recurrence formula above yields $c_5 = 2$, $c_6 = 3$, $c_7 = 4, \cdots$. Hence the second Frobenius series solution is

$$y_2(x) = x^4(1 + 2x + 3x^2 + 4x^3 + \cdots) = x^4/(1-x)^2,$$

with the closed form coming from the derivative of the geometric series $1/(1-x) = \Sigma x^n$.

In Problems 11–15, we give first the Frobenius series solution $y_1(x)$ corresponding to the larger indicial exponent r_1 of the given differential equation. Then, writing the equation in the form $y'' + P(x)y' + Q(x)y = 0$, we apply the reduction of order formula

$$y_2(x) = \int \frac{\exp\left(-\int P(x)\,dx\right)}{y_1(x)^2}\,dx$$

to derive a second independent solution $y_2(x)$.

9. $r_1 = r_2 = 0$

$$y_1 = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \frac{x^8}{147456} + \cdots$$

$$P(x) = 1/x; \qquad \exp\left(-\int P(x)\,dx\right) = 1/x$$

$$y_2 = y_1 \int x^{-1} \cdot \left(1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \frac{x^8}{147456} + \cdots\right)^{-2} dx$$

$$= y_1 \int x^{-1}\left(1+\frac{x^2}{2}+\frac{3x^4}{32}+\frac{5x^6}{576}+\frac{35x^8}{73728}+\cdots\right)^{-1} dx$$

$$= y_1 \int x^{-1}\left(1-\frac{x^2}{2}+\frac{5x^4}{32}-\frac{23x^6}{576}+\frac{677x^8}{73728}+\cdots\right) dx$$

$$y_2 = y_1\left(\ln x-\frac{x^2}{4}+\frac{5x^4}{128}-\frac{23x^6}{3456}+\frac{677x^6}{589824}-\cdots\right)$$

10. $r_1 = r_2 = 1$

$$y_1 = x\left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147456}-\cdots\right)$$

$$P(x) = -1/x; \qquad \exp\left(-\int P(x)\,dx\right) = x$$

$$y_2 = y_1 \int x^{-1}\cdot\left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147456}-\cdots\right)^{-2} dx$$

$$= y_1 \int x^{-1}\left(1-\frac{x^2}{2}+\frac{3x^4}{32}-\frac{5x^6}{576}+\frac{35x^8}{73728}-\cdots\right)^{-1} dx$$

$$= y_1 \int x^{-1}\left(1+\frac{x^2}{2}+\frac{5x^4}{32}+\frac{23x^6}{576}+\frac{677x^8}{73728}+\cdots\right) dx$$

$$y_2 = y_1\left(\ln x+\frac{x^2}{4}+\frac{5x^4}{128}+\frac{23x^6}{3456}+\frac{677x^6}{589824}+\cdots\right)$$

11. $r_1 = r_2 = 2$

$$y_1 = x^2\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\cdots\right)$$

$$P(x) = 1-3/x; \qquad \exp\left(-\int P(x)\,dx\right) = x^3e^{-x}$$

$$y_2 = y_1 \int x^3e^{-x}\cdot x^{-4}\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\cdots\right)^{-2} dx$$

$$= y_1 \int x^{-1}e^{-x}\left(1-4x+7x^2-\frac{22x^3}{3}+\frac{16x^4}{3}-\cdots\right)^{-1} dx$$

$$= y_1 \int x^{-1}\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\cdots\right)\left(1+4x+9x^2+\frac{46x^3}{3}+\frac{67x^4}{3}+\cdots\right) dx$$

$$= y_1 \int x^{-1}\left(1+3x+\frac{11x^2}{2}+\frac{49x^3}{6}+\frac{87x^4}{8}+\cdots\right) dx$$

$$y_2 = y_1\left(\ln x + 3x + \frac{11x^2}{4} + \frac{49x^3}{18} + \frac{87x^4}{32} + \cdots\right)$$

12. $r_1 = 2, \quad r_2 = -1$

$$y_1 = x^2\left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \cdots\right)$$

$$P(x) = 1; \qquad \exp\left(-\int P(x)\,dx\right) = e^{-x}$$

$$y_2 = y_1 \int e^{-x}\cdot x^{-4}\left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \cdots\right)^{-2} dx$$

$$= y_1 \int x^{-4}e^{-x}\left(1 - x + \frac{11x^2}{20} - \frac{13x^3}{60} + \frac{569x^4}{8400} - \cdots\right)^{-1} dx$$

$$= y_1 \int x^{-4}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right)\left(1 + x + \frac{9x^2}{20} + \frac{7x^3}{60} + \frac{19x^4}{1050} + \cdots\right) dx$$

$$= y_1 \int x^{-4}\left(1 - \frac{x^2}{20} + \frac{x^4}{700} - \frac{47x^6}{1512000} + \cdots\right) dx$$

$$y_2 = y_1\left(-\frac{1}{3x^3} + \frac{1}{20x} + \frac{x}{700} - \frac{47x^3}{4536000} + \cdots\right) \qquad \text{[no logarithmic term]}$$

13. $r_1 = 3, \quad r_2 = 1$

$$y_1 = x^3\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \cdots\right)$$

$$P(x) = 2 - 3/x; \qquad \exp\left(-\int P(x)\,dx\right) = x^3e^{-2x}$$

$$y_2 = y_1 \int x^3e^{-2x}\cdot x^{-6}\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \cdots\right)^{-2} dx$$

$$= y_1 \int x^{-3}e^{-2x}\left(1 - 4x + 8x^2 - \frac{32x^3}{3} + \frac{32x^4}{3} - \cdots\right)^{-1} dx$$

$$= y_1 \int x^{-3}\left(1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \cdots\right)\left(1 + 4x + 8x^2 + \frac{32x^3}{3} + \frac{32x^4}{3} + \cdots\right) dx$$

$$= y_1 \int x^{-3}\left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots\right) dx$$

$$y_2 = y_1\left(2\ln x - \frac{1}{2x^2} - \frac{2}{x} + \frac{4x}{3} + \frac{x^2}{3} + \cdots\right)$$

14. $r_1 = 2, \quad r_2 = -2$

$$y_1 = x^2\left(1-\frac{2x}{5}+\frac{x^2}{10}-\frac{2x^3}{105}+\frac{x^4}{336}-\frac{x^5}{2520}+\cdots\right)$$

$$P(x) = 1+1/x; \qquad \exp\left(-\int P(x)\,dx\right) = x^{-1}e^{-x}$$

$$y_2 = y_1\int x^{-1}e^{-x}\cdot x^{-4}\left(1-\frac{2x}{5}+\frac{x^2}{10}-\frac{2x^3}{105}+\frac{x^4}{336}-\frac{x^5}{2520}+\cdots\right)^{-2}dx$$

$$= y_1\int x^{-5}e^{-x}\left(1-\frac{4x}{4}+\frac{9x^2}{25}-\frac{62x^3}{525}+\frac{131x^4}{4200}-\frac{11x^5}{1575}+\cdots\right)^{-1}dx$$

$$= y_1\int x^{-5}\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+\cdots\right)\cdot$$

$$\left(1+\frac{4x}{5}+\frac{7x^2}{25}+\frac{142x^3}{2625}+\frac{121x^4}{21000}+\frac{46x^5}{196875}-\cdots\right)dx$$

$$= y_1\int x^{-5}\left(1-\frac{x}{5}-\frac{x^2}{50}+\frac{13x^3}{1750}-\frac{29x^5}{196875}+\cdots\right)dx$$

$$y_2 = y_1\left(-\frac{1}{4x^4}+\frac{15}{x^3}+\frac{1}{100x^2}-\frac{13}{1750x}+0\cdot\ln x-\frac{29x}{196875}+\cdots\right)$$

Thus the second solution $y_2(x)$ contains no logarithmic term.

15. $r_1 = r_2 = 0$

$$J_0(x) = 1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147456}-\cdots$$

$$P(x) = 1/x; \qquad \exp\left(-\int P(x)\,dx\right) = 1/x$$

$$y_2(x) = J_0(x)\int x^{-1}\cdot\left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147456}-\cdots\right)^{-2}dx$$

$$= J_0(x)\int x^{-1}\left(1-\frac{x^2}{2}+\frac{3x^4}{32}-\frac{5x^6}{576}+\frac{35x^8}{73728}-\cdots\right)^{-1}dx$$

$$= J_0(x)\int x^{-1}\left(1+\frac{x^2}{2}+\frac{5x^4}{32}+\frac{23x^6}{576}+\frac{677x^8}{73728}+\cdots\right)dx$$

$$= J_0(x)\left(\ln x+\frac{x^2}{4}+\frac{5x^4}{128}+\frac{23x^6}{3456}+\frac{677x^6}{589824}-\cdots\right)$$

$$= J_0(x)\ln x+$$

$$\left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147456}-\cdots\right)\left(\frac{x^2}{4}+\frac{5x^4}{128}+\frac{23x^6}{3456}+\frac{677x^6}{589824}-\cdots\right)$$

$$y_2(x) = J_0(x)\ln x + \frac{x^2}{4} - \frac{3x^4}{128} + \frac{11x^6}{13284} - \cdots$$

16. The indicial exponents are $r = \pm\frac{3}{2}$. We start with the larger exponent $r_1 = +\frac{3}{2}$ and substitute $y = x^{3/2}\sum_{n=0}^{\infty} a_n x^n$ into Bessel's equation of order $\frac{3}{2}$. We find that $c_1 = 0$, and then the recurrence relation

$$a_n = -\frac{a_{n-2}}{n(n+3)},$$

implies that all odd subscripts vanish. Starting with a0 = 1, this recurrence relation yields in the usual manner the first solution

$$\begin{aligned} y_1(x) &= x^{3/2}\left[1 - \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 7} - \frac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 7\cdot 8} + \cdots\right] \\ &= x^{3/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\cdot 5\cdot 7\cdot\cdots\cdot(2n+3)}\right]. \end{aligned}$$

Now we start afresh with the smaller exponent $r_1 = -\frac{3}{2}$ and substitute $y = x^{-3/2}\sum_{n=0}^{\infty} b_n x^n$ into Bessel's equation of order $\frac{3}{2}$. This time, we find that $n(n-3)b_n + b_{n-2} = 0$ for $n \geq 2$. We can satisfy the critical case $0\cdot b_3 + b_1 = 0$ by choosing $b_1 = 0$, which then implies that all odd coefficients vanish. Then the recurrence relation

$$b_n = -\frac{b_{n-2}}{n(n-3)},$$

yields routinely the second solution

$$\begin{aligned} y_2(x) &= x^{-3/2}\left[1 - \frac{x^2}{2\cdot(-1)} + \frac{x^4}{2\cdot 4\cdot(-1)\cdot 1} - \frac{x^6}{2\cdot 4\cdot 6\cdot(-1)\cdot 1\cdot 3} + \cdots\right] \\ &= x^{-3/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\cdot(-1)\cdot 1\cdot 3\cdot\cdots\cdot(2n-3)}\right]. \end{aligned}$$

17. The given first solution

$$y_1(x) = xe^x = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right)$$

can be derived by starting with the single exponent $r = 1$, substituting $y = x\sum_{n=0}^{\infty} c_n x^n$ into the differential equation, and calculating successive coefficient recursively as usual. We can verify the alleged second solution by applying the method of reduction of order as in Problems 9–14:

$$P(x) = -1-1/x; \qquad \exp\left(-\int P(x)\,dx\right) = x e^x$$

$$y_2 = y_1 \int x e^x \cdot \left(x e^x\right)^{-2} dx = y_1 \int x^{-1} e^{-x}\,dx$$

$$= y_1 \int x^{-1}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right) dx$$

$$= y_1\left(\ln x - x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + \cdots\right)$$

$$= y_1 \ln x + \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right)\left(-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{96} - \frac{x^5}{600} + \cdots\right)$$

$$= y_1 \ln x - \left(x^2 + \frac{3x^3}{4} + \frac{11x^4}{36} + \frac{25x^5}{288} + \frac{137x^6}{7200} + \cdots\right)$$

$$y_2(x) = x e^x \ln x - \sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n!}$$

18. When we substitute

$$y(x) = C\, y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n$$

in the differential equation $xy'' - x = 0$ we find that $b_1 = -b_0 = -C$ and that

$$n(n+1)b_{n+1} - b_n = -\frac{(2n+1)C}{n!(n+1)!}$$

for $n \geq 1$. To solve this recurrence relation we take $C = 1$ and substitute $b_n = c_n / (n-1)!n!$. The result is

$$c_{n+1} - c_n = -\frac{2n+1}{n(n+1)} = -\frac{1}{n} - \frac{1}{n+1}.$$

Starting with $c_1 = b_1 = -1$, it follows readily by induction on n that $c_n = -(H_n + H_{n+1})$.

SECTION 3.5

BESSEL'S EQUATION

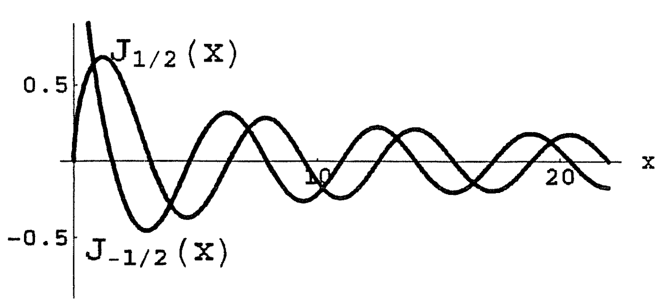
Of course Bessel's equation is the most important special ordinary differential equation in mathematics, and every student should be exposed at least to Bessel functions of the first kind. Though Bessel functions of integral order can be treated without the gamma function, the subsection on the gamma function is also needed for Chapter 4 on Laplace transforms. The final subsections on Bessel function identities and the parametric Bessel equation will not be needed until Section 9.4, and therefore may be considered optional at this point in the course.

1.
$$J_0'(x) = D_x\left(1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right) = \sum_{m=1}^{\infty}\frac{(-1)^m 2m\,x^{2m-1}}{2^{2m}(m!)^2}$$
$$= \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-1}(m-1)!(m!)} = \sum_{m=0}^{\infty}\frac{(-1)^{m+1} x^{2m+1}}{2^{2m+1}(m)!(m+1!)}$$
$$= -\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2^{2m+1}(m)!(m+1!)} = -J_1(x)$$

2. **(a)**
$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{2n-1}{2}\cdot\Gamma\left(\frac{2n-1}{2}\right)$$
$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\Gamma\left(\frac{2n-3}{2}\right)$$
$$\vdots$$
$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\cdot\left(\frac{1}{2}\right)$$
$$= \frac{(2n-1)(2n-3)\cdot\cdots\cdot 3\cdot 1}{2^n}\cdot\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

(b)
$$J_{1/2}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+\frac{1}{2}}}{m!\,\Gamma(m+\frac{3}{2})2^{2m+\frac{1}{2}}} = \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{m!\,2^{-m-1}(2m+1)!!\,\sqrt{\pi}\,2^{2m+1}}$$
$$= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2\cdot 4\cdot\cdots\cdot 2m)(1\cdot 3\cdot\cdots\cdot(2m+1))}$$
$$= \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}}\sin x$$

$J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$ similarly (See the figure below for the graphs.)



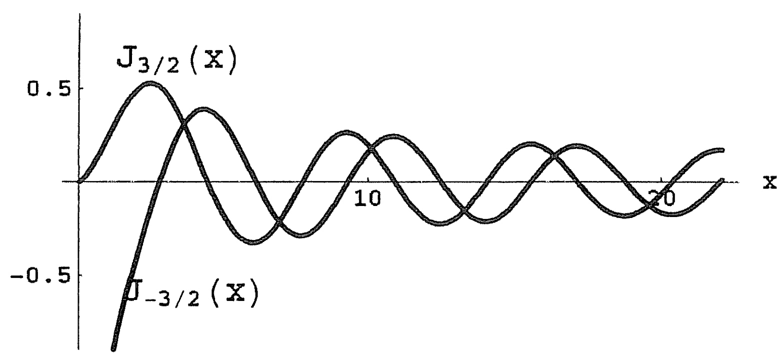
3. (a) $$\Gamma\left(m+\frac{2}{3}\right) = \Gamma\left(\frac{3m+2}{3}\right) = \frac{3m-1}{3}\cdot\frac{3m-4}{3}\cdot\Gamma\left(\frac{3m-4}{3}\right)$$
$$= \frac{3m-1}{3}\cdot\frac{3m-4}{3}\cdot\cdots\cdot\frac{5}{3}\cdot\frac{2}{3}\cdot\Gamma\left(\frac{2}{3}\right) = \frac{2\cdot5\cdot8\ \cdots\cdot(3m-1)}{3^m}\Gamma\left(\frac{2}{3}\right)$$

(b) $$J_{-1/3}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+2/3)}\left(\frac{x}{2}\right)^{2m-1/3} = \frac{(x/2)^{-1/3}}{\Gamma(2/3)}\sum_{m=0}^{\infty}\frac{(-1)^m 3^m x^{2m}}{m!\cdot2\cdot3\cdot8\cdots\cdot(3m-1)}$$

4. With $p = 1/2$ in Equation (26) in the text we have

$$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$
$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\left(\sin x - x\cos x\right) = \sqrt{\frac{2}{\pi x^3}}\left(\sin x - x\cos x\right)$$

The figure below shows the graphs of $J_{3/2}(x)$ and $J_{-3/2}(x)$.



5. Starting with $p = 3$ in Equation (26) we get

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) = \frac{6}{x}\left[\frac{4}{x}J_2(x) - J_1(x)\right] - J_2(x)$$
$$= \left(\frac{24}{x^2}-1\right)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \frac{6}{x}J_1(x)$$
$$= \frac{x^2-24}{x^2}J_0(x) + \frac{8(6-x^2)}{x^3}J_1(x)$$

8. When we carry out the differentiations indicated in Equations (22) and (23) in the text, we get

$$p x^{p-1}J_p(x) + x^p J_p'(x) = x^p J_{p-1}(x),$$
$$-p x^{-p-1}J_p(x) + x^{-p}J_p'(x) = -x^p J_{p+1}(x).$$

When we solve these two equations for $J'_p(x)$ we get Equations (24) and (25) in the text.

10. When we add equations (24) and (25) we get

$$J'_p(x) = \frac{1}{2}\left[J_{p-1}(x) - J_{p+1}(x)\right],$$

so

$$J''_p(x) = \frac{1}{2}\left[J'_{p-1}(x) - J'_{p+1}(x)\right].$$

Replacing p with $p-1$ and with $p+1$ in the first equation, we get

$$J'_{p-1}(x) = \frac{1}{2}\left[J_{p-2}(x) - J_p(x)\right]$$

and

$$J'_{p+1}(x) = \frac{1}{2}\left[J_p(x) - J_{p+2}(x)\right].$$

When we use these equations to substitute for $J'_{p-1}(x)$ and $J_{p+1}(x)$ in the equation for $J''_p(x)$ above, we find that

$$J''_p(x) = \frac{1}{4}\left[J_{p-2}(x) - 2J_p(x) + J_{p+2}(x)\right].$$

11. $\Gamma(p+m+1) = (p+m)(p+m-1)\cdots\cdots(p+2)(p+1)\Gamma(p+1)$, so

$$\begin{aligned} J_p(x) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(p+m+1)}\left(\frac{x}{2}\right)^{2m+p} \\ &= \frac{(x/2)^p}{\Gamma(p+1)}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(p+1)(p+2)\cdots\cdots(p+m)}\left(\frac{x}{2}\right)^{2m}. \end{aligned}$$

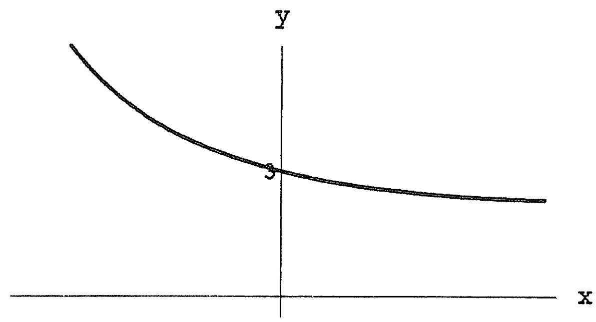
12. Substitution of the power series of Problem 11 yields

$$y(x) = x^2\cdot\frac{x^{5/2}(A+\cdots)+x^{-5/2}(B+\cdots)}{x^{1/2}(C+\cdots)+x^{-1/2}(D+\cdots)} = \frac{x^5(A+\cdots)+(B+\cdots)}{x(C+\cdots)+(D+\cdots)}$$

where $A = 1/(2^{5/2}\Gamma(7/2))$, $B = 1/(2^{-5/2}\Gamma(-3/2),)$ $C = 1/(2^{1/2}\Gamma(3/2))$, and $D = (1/2^{-1/2})\Gamma(1/2)$. Hence

$$y(0) = \frac{0\cdot(A+\cdots)+(B+\cdots)}{0\cdot(C+\cdots)+(D+\cdots)} = \frac{B}{D} = \frac{2^{-1/2}\Gamma(1/2)}{2^{-5/2}\Gamma(-3/2)} = \frac{2^2\Gamma(1/2)}{(4/3)\Gamma(1/2} = 3.$$

The graph of $y(x)$ shown at the top of the next page corroborates this value.



In Problems 13–21 we use a conspicuous dot • to indicate our choice of u and dv in the integration by parts formula $\int u \bullet dv = uv - \int v\,du$. We use repeatedly the facts (from Example 1) that $\int xJ_0(x)\,dx = xJ_1(x) + C$ and $\int J_1(x)\,dx = -J_0(x) + C$.

13.
$$\begin{aligned}\int x^2 J_0(x)\,dx &= \int x \bullet xJ_0(x)\,dx \\ &= x^2 J_1(x) - \int x \bullet J_1(x)\,dx \\ &= x^2 J_1(x) - \left(-xJ_0(x) + \int J_0(x)\,dx\right) \\ &= x^2 J_1(x) + xJ_0(x) - \int J_0(x)\,dx + C\end{aligned}$$

14.
$$\begin{aligned}\int x^3 J_0(x)\,dx &= \int x^2 \bullet xJ_0(x)\,dx \\ &= x^3 J_1(x) - 2\int x^2 \bullet J_1(x)\,dx \\ &= x^3 J_1(x) - 2\left(-x^2 J_0(x) + 2\int xJ_0(x)\,dx\right) \\ &= x^3 J_1(x) + 2x^2 J_0(x) - 4xJ_1(x) + C = (x^3 - 4x)J_1(x) + 2x^2 J_0(x) + C\end{aligned}$$

15.
$$\begin{aligned}\int x^4 J_0(x)\,dx &= \int x^3 \bullet xJ_0(x)\,dx \\ &= x^4 J_1(x) - 3\int x^3 \bullet J_1(x)\,dx \\ &= x^4 J_1(x) - 3\left(-x^3 J_0(x) + 3\int x \bullet xJ_0(x)\,dx\right) \\ &= x^4 J_1(x) + 3x^3 J_0(x) - 9\left(x^2 J_1(x) - \int x \bullet J_1(x)\,dx\right) \\ &= x^4 J_1(x) + 3x^3 J_0(x) - 9x^2 J_1(x) + 9\left(-xJ_0(x) + \int J_0(x)\,dx\right) \\ &= (x^4 - 9x^2)J_1(x) + (3x^3 - 9x)J_0(x) + 9\int J_0(x)\,dx + C\end{aligned}$$

16. $\int xJ_1(x)\,dx = \int x \bullet J_1(x)\,dx = -xJ_0(x) + \int J_0(x)\,dx + C$

17. $\int x^2 J_1(x)\,dx = \int x^2 \bullet J_1(x)\,dx$

$$= -x^2 J_0(x) + 2\int x J_0(x)\,dx = -x^2 J_0(x) + 2x J_1(x) + C$$

18. $\int x^3 J_1(x)\,dx = \int x^3 \bullet J_1(x)\,dx$

$$= -x^3 J_0(x) + 3\int x \bullet x J_0(x)\,dx$$

$$= -x^3 J_0(x) + 3\left(x^2 J_1(x) - \int x \bullet J_1(x)\,dx\right)$$

$$= -x^3 J_0(x) + 3x^2 J_1(x) - 3\left(-x J_0(x) + \int J_0(x)\,dx\right)$$

$$= (-x^3 + 3x) J_0(x) + 3x^2 J_1(x) - 3\int J_0(x)\,dx + C$$

19. $\int x^4 J_1(x)\,dx = \int x^4 \bullet J_1(x)\,dx$

$$= -x^4 J_0(x) + 4\int x^2 \bullet x J_0(x)\,dx$$

$$= -x^4 J_0(x) + 4\left(x^3 J_1(x) - 2\int x^2 \bullet J_1(x)\,dx\right)$$

$$= -x^4 J_0(x) + 4x^3 J_1(x) - 8\left(-x^2 J_0(x) + 2\int x J_0(x)\,dx\right)$$

$$= (-x^4 + 8x^2) J_0(x) + (4x^3 - 16x) J_1(x) + C$$

20. With $p = 1$, Eq. (23) in the text gives $\int x^{-1} J_2(x)\,dx = -x^{-1} J_1(x) + C$. Hence

$$\int J_2(x)\,dx = \int x \bullet x^{-1} J_2(x)\,dx$$

$$= x\left(-x^{-1} J_1(x)\right) + \int x^{-1} J_1(x) = -J_1(x) + \int x^{-1} J_1(x)\,dx.$$

But Eq. (26) with $p = 1$ gives $x^{-1} J_1(x) = \tfrac{1}{2}\left[J_0(x) + J_2(x)\right]$, so

$$\int J_2(x)\,dx = -J_1(x) + \tfrac{1}{2}\int J_0(x)\,dx + \tfrac{1}{2}\int J_2(x)\,dx.$$

Finally, we can solve this last equation for

$$\int J_2(x)\,dx = -2J_1(x) + \int J_0(x)\,dx + C.$$

21. With $p = 2$, Eq. (23) in the text gives $\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x) + C$. Hence

$$\int J_3(x)\,dx = \int x^2 \bullet x^{-2} J_3(x)\,dx$$

$$= x^2\left(-x^{-2} J_2(x)\right) + 2\int x^{-1} J_2(x)\,dx$$

$$= -J_2(x) - \frac{2}{x} J_1(x) + C \qquad \text{(by Example 3)}$$

$$= -\left(\frac{2}{x}J_1(x) - J_0(x)\right) - \frac{2}{x}J_1(x) + C \qquad \text{(By Eq. (26) with } p = 1\text{)}$$

$$= J_0(x) - \frac{4}{x}J_1(x) + C.$$

22. Let us define

$$g(x) = \int_0^\pi \cos(x\sin\theta)\,d\theta$$

and note first that

$$g(0) = \int_0^\pi \cos(0)\,d\theta = \pi = \pi J_0(0).$$

Differentiation under the integral sign yields

$$g'(x) = -\int_0^\pi \sin(x\sin\theta)\sin\theta\,d\theta.$$

When we integrate by parts with

$$\begin{aligned} u &= \sin(x\sin\theta) & dv &= \sin\theta\,d\theta \\ du &= (x\cos\theta)\cos(x\sin\theta)\,d\theta & v &= -\cos\theta \end{aligned}$$

we get

$$g'(x) = -x\int_0^\pi \cos^2\theta\,\cos(x\sin\theta)\,d\theta.$$

But differentiation of the first equation for $g'(x)$ yields

$$g''(x) = -\int_0^\pi \sin^2\theta\cos(x\sin\theta)\,d\theta.$$

Finally, because $\cos^2\theta + \sin^2\theta = 1$, it follows that

$$g''(x) + \frac{1}{x}g'(x) = -\int_0^\pi \cos(x\sin\theta)\,d\theta = -g(x).$$

Thus $y = g(x)$ satisfies Bessel's equation of order zero in the form $y'' + (1/x)y' + y = 0$. Therefore the function g takes the form

$$g(x) = a\,J_0(x) + b\,Y_0(x).$$

Since $g(0) = \pi$ is finite and $J_0(0) = 1$, we must have $a = \pi$ and $b = 0$, so $g(x) = \pi J_0(x)$, as desired.

23. This is a special case of the discussion below in Problem 24.

24. Given an integer $n \geq 1$, let us define

$$g_n(x) = \int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta.$$

Differentiation yields

$$g_n'(x) = \int_0^{\pi} \sin(n\theta - x\sin\theta)\sin\theta\,d\theta.$$

Integration by parts with $u = \sin(n\theta - x \sin \theta)$ and $dv = \sin\theta\,d\theta$ yields

$$g_n'(x) = n\int_0^{\pi} \cos\theta\,\cos(n\theta - x\sin\theta)\,d\theta - x\int_0^{\pi} \cos^2\theta\,\cos(n\theta - x\sin\theta)\,d\theta.$$

But differentiation of the first equation for $g_n'(x)$ yields

$$g_n''(x) = -\int_0^{\pi} \sin^2\theta\,\cos(n\theta - x\sin\theta)\,d\theta.$$

It follows that

$$\begin{aligned}
g_n''(x)+\frac{1}{x}g_n'(x) &= -g_n(x)+\frac{n}{x}\int_0^{\pi} \cos\theta\,\cos(n\theta - x\sin\theta)\,d\theta \\
&= -g_n(x)-\frac{n}{x^2}\int_0^{\pi} \big[(n - x\cos\theta)-n\big]\cos(n\theta - x\sin\theta)\,d\theta \\
&= -g_n(x)-\frac{n}{x^2}\big[\sin(n\theta - x\sin\theta\big]_0^{\pi}+\frac{n^2}{x^2}g_n(x) = -\left(1-\frac{n^2}{x^2}\right)g_n(x).
\end{aligned}$$

Upon equating the first and last members of this continued inequality and multiplying by x^2, we see that $y = g_n(x)$ satisfies Bessel's equation of order $n \geq 1$. The initial values of $g_n(x)$ are

$$g_n(0) = \int_0^{\pi} \cos(n\theta)\,d\theta = 0 \quad\text{and}\quad g_n'(0) = \int_0^{\pi} \sin(\theta)\sin(n\theta)\,d\theta = 0.$$

If $n = 1$ then $g_1'(0) = \pi/2$, whereas $g_n'(0) = 0$ if $n \geq 1$. In either case the values of $g_n(0)$ and $g_n'(0)$ are π times those of $J_n(0)$ and $J_n'(0)$, respectively. Now we know from the general solution of Bessel's equation that $g_n(x) = c\,J_n(x)$ for some constant c. If $n = 1$ than the fact that

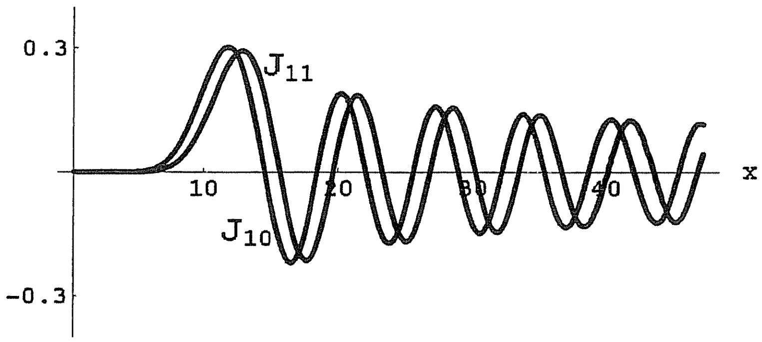
$$\pi/2 = g_n'(0) = c\,J_1'(0) = c/2$$

implies that $c = \pi$, as desired. But if $n > 1$ the fact that

$$0 = g_n'(0) = c\,J_n'(0) = c\cdot 0$$

does not suffice to determine c.

26. The graph shown at the top of the next page illustrates the interlaced zeros of the Bessel functions $J_{10}(x)$ and $J_{11}(x)$.



SECTION 3.6

APPLICATIONS OF BESSEL FUNCTIONS

Problems 1–12 are routine applications of the theorem in this section. In each case it is necessary only to identify the coefficients A, B, C and the exponent q in the differential equation

$$x^2y'' + Axy' + (B + Cx^q)y = 0. \tag{1}$$

Then we can calculate the values

$$\alpha = \frac{1-A}{2}, \quad \beta = \frac{q}{2}, \quad k = \frac{2\sqrt{C}}{q}, \quad p = \frac{\sqrt{(1-A)^2 - 4B}}{q} \tag{2}$$

and finally write the general solution

$$y(x) = x^\alpha\left[c_1J_p(kx^\beta) + c_2J_{-p}(kx^\beta)\right] \tag{3}$$

specified in Theorem 1 of this section. This is a "template procedure" that we illustrate only in a couple of problems.

1. We have $A = -1, B = 1, C = 1, q = 2$ so

$$\alpha = \frac{1-(-1)}{2} = 1, \quad \beta = \frac{2}{2} = 1, \quad k = \frac{2\sqrt{1}}{2} = 1, \quad p = \frac{\sqrt{(1-(-1))^2 - 4(1)}}{2} = 0,$$

so our general solution is $y(x) = x[c_1J_0(x) + c_2Y_0(x)]$, using $Y_0(x)$ because $p = 0$ is an integer.

2. $y(x) = x^{-1}[c_1J_1(x) + c_2Y_1(x)]$

3. $y(x) = x[c_1J_{1/2}(3x^2) + c_2J_{-1/2}(3x^2)]$

4. $y(x) = x^3[c_1J_2(2x^{1/2}) + c_2Y_2(2x^{1/2})]$

5. To match the given equation with Eq. (1) above, we first divide through by the leading coefficient 16 to obtain the equation

$$x^2y'' + \frac{5}{3}xy' + \left(-\frac{5}{36} + \frac{1}{4}x^3\right)y = 0$$

with $A = 5/3, B = -5/36, C = 1/4,$ and $q = 3$. Then

$$\alpha = \frac{1-5/3}{3} = -\frac{1}{3}, \quad \beta = \frac{3}{2}, \quad k = \frac{2\sqrt{1/4}}{3} = \frac{1}{3}, \quad p = \frac{\sqrt{(1-5/3)^2 - 4(-5/36)}}{3} = \frac{1}{3},$$

so our general solution is $y(x) = x^{-1/3}[c_1J_{1/3}(x^{3/2}/3) + c_2J_{-1/3}(x^{3/2}/3)]$.

6. $y(x) = x^{-1/4}[c_1J_0(2x^{3/2}) + c_2Y_0(2x^{3/2})]$

7. $y(x) = x^{-1}[c_1J_0(x) + c_2Y_0(x)]$

8. $y(x) = x^2[c_1J_1(4x^{1/2}) + c_2Y_1(4x^{1/2})]$

9. $y(x) = x^{1/2}[c_1J_{1/2}(2x^{3/2}) + c_2J_{-1/2}(2x^{3/2})]$

10. $y(x) = x^{-1/4}[c_1J_{3/2}(2x^{5/2}/5) + c_2J_{-3/2}(2x^{5/2}/5)]$

11. $y(x) = x^{1/2}[c_1J_{1/6}(x^3/3) + c_2J_{-1/6}(x^3/3)]$

12. $y(x) = x^{1/2}[c_1J_{1/5}(4x^{5/2}/5) + c_2J_{-1/5}(4x^{5/2}/5)]$

13. We want to solve the equation $xy'' + 2y' + xy = 0$. If we rewrite it as

$$x^2y'' + 2xy' + x^2y = 0$$

then we have the form in Equation (1) with $A = 2,\ B = 0,\ C = 1,$ and $q = 2$. Then Equation (2) gives $\alpha = -1/2,\ \beta = 1,\ k = 1,$ and $p = 1/2,$ so by Equation (3) the general solution is

$$\begin{aligned} y(x) &= x^{-1/2}\left[c_1J_{1/2}(x) + c_1J_{-1/2}(x)\right] \\ &= x^{-1/2}\left[c_1\sqrt{\frac{2}{\pi x}}\cos x + c_2\sqrt{\frac{2}{\pi x}}\sin x\right] = \frac{1}{x}(a_1\cos x + a_2\sin x), \end{aligned}$$

(with $a_i = c_i\sqrt{2/\pi}$) using Equations (19) in Section 3.5.

15. The substitution

$$y = -\frac{u'}{u}, \quad y' = \frac{(u')^2}{u^2} - \frac{u''}{u}$$

immediately transforms $y' = x^2 + y^2$ to $u'' + x^2u = 0$. The equivalent equation

$$x^2u'' + x^4u = 0$$

is of the form in (1) with $A = B = 0$, $C = 1$, and $q = 4$. Equations (2) give $\alpha = 1/2$, $\beta = 2$, $k = 1/2$, and $p = 1/4$, so the general solution is

$$u(x) = x^{1/2}[c_1J_{1/4}(x^2/2) + c_2J_{-1/4}(x^2/2)].$$

To compute $u'(x)$, let $z = x^2/2$ so $x = 2^{1/2}z^{1/2}$. Then Equation (22) in Section 3.5 with $p = 1/4$ yields

$$\begin{aligned}\frac{d}{dx}\left(x^{1/2}J_{1/4}(x^2/2)\right) &= \frac{d}{dz}\left(2^{1/4}z^{1/4}J_{1/4}(z)\right)\cdot\frac{dz}{dx}\\ &= 2^{1/4}z^{1/4}J_{-3/4}(z)\cdot\frac{dz}{dx}\\ &= 2^{1/4}\cdot\frac{x^{1/2}}{2^{1/4}}J_{-3/4}(x^2/2)\cdot x = x^{3/2}J_{-3/4}(x^2/2).\end{aligned}$$

Similarly, Equation (23) in Section 3.5 with $p = -1/4$ yields

$$\frac{d}{dx}\left(x^{1/2}J_{-1/4}(x^2/2)\right) = \frac{d}{dz}\left(2^{1/4}z^{1/4}J_{-1/4}(z)\right)\cdot\frac{dz}{dx} = -x^{3/2}J_{3/4}(x^2/2).$$

Therefore

$$u'(x) = x^{3/2}[c_1J_{-3/4}(x^2/2) - c_2J_{3/4}(x^2/2)].$$

It follows finally that the general solution of the Riccati equation $y' = x^2 + y^2$ is

$$y(x) = -\frac{u'}{u} = x\cdot\frac{J_{3/4}(\frac{1}{2}x^2) - c\,J_{-3/4}(\frac{1}{2}x^2)}{c\,J_{1/4}(\frac{1}{2}x^2) + J_{-1/4}(\frac{1}{2}x^2)}$$

where the arbitrary constant is $c = c_1/c_2$.

16. Substitution of the series expressions for the Bessel functions in the formula for $y(x)$ in Problem 15 yields

$$y(x) = x\cdot\frac{A\left(\frac{1}{2}x^2\right)^{3/4}(1+\cdots) - c\,B\left(\frac{1}{2}x^2\right)^{-3/4}(1+\cdots)}{c\,C\left(\frac{1}{2}x^2\right)^{1/4}(1+\cdots) + D\left(\frac{1}{2}x^2\right)^{-1/4}(1+\cdots)}$$

where each pair of parentheses encloses a power series in x with constant term 1, and

$$A = 2^{-3/4}/\Gamma(7/4) \qquad B = 2^{3/4}/\Gamma(1/4)$$

$$C = 2^{-1/4}/\Gamma(5/4) \qquad D = 2^{1/4}/\Gamma(3/4).$$

Multiplication of numerator and denominator by $x^{1/2}$ and a bit of simplification gives

$$y(x) = \frac{2^{-3/4}Ax^3(1+\cdots)-2^{3/4}cB(1+\cdots)}{2^{-1/4}cCx(1+\cdots)+2^{1/4}D(1+\cdots)}.$$

It now follows that

$$y(0) = \frac{-2^{3/4}cB}{2^{1/4}D} = \frac{-2^{1/2}\left(2^{3/4}/\Gamma(1/4)\right)}{2^{1/4}/\Gamma(3/4)} = -2c\cdot\frac{\Gamma(3/4)}{\Gamma(1/4)}. \qquad (*)$$

(a) If $y(0) = 0$ then (*) gives $c = 0$ in the general solution formula of Problem 15.

(b) If $y(0) = 1$ then (*) gives $c = -\Gamma(1/4)/2\Gamma(3/4)$. More generally, (*) yields the formula

$$y(x) = x\cdot\frac{2\Gamma\left(\frac{3}{4}\right)J_{3/4}\left(\frac{1}{2}x^2\right)+y_0\Gamma\left(\frac{1}{4}\right)J_{-3/4}\left(\frac{1}{2}x^2\right)}{2\Gamma\left(\frac{3}{4}\right)J_{-1/4}\left(\frac{1}{2}x^2\right)-y_0\Gamma\left(\frac{1}{4}\right)J_{1/4}\left(\frac{1}{2}x^2\right)}$$

for the solution of the initial value problem

$$y' = x^2 + y^2, \qquad y(0) = y_0.$$

17. If we write the equation $x^4y'' + \gamma^2 y = 0$ in the form

$$x^2y'' + \gamma^2x^{-2}y = 0,$$

then we see that it is of the form in Equation (3) of this section with $A = B = 0$, $C = \gamma^2$, and $q = -2$. Then Equations (5) give $\alpha = 1/2$, $\beta = -1$, $k = \gamma$, and $p = -1/2$, so the theorem yields the general solution

$$y(x) = x^{1/2}[c_1J_{1/2}(\gamma/x) + c_2J_{-1/2}(\gamma/x)] = x[A\cos(\gamma/x) + B\sin(\gamma/x)],$$

using Equations (19) in Section 3.5 for $J_{1/2}(x)$ and $J_{-1/2}(x)$. With a and b both nonzero, the initial conditions $y(a) = y(b) = 0$ yield the equations

$$A\cos(\gamma/a) + B\sin(\gamma/a) = 0$$

$$A\cos(\gamma/b) + B\sin(\gamma/b) = 0.$$

These equations have a nontrivial solution for A and B only if the coefficient determinant

$$\begin{aligned}\Delta &= \sin(\gamma/b)\cos(\gamma/a) - \sin(\gamma/a)\cos(\gamma/b) \\ &= \sin(\gamma/b - \gamma/a) = \sin(\gamma L/ab)\end{aligned}$$

is nonzero. Hence $\gamma L/ab$ must be an integral multiple $n\pi$ of π, and then the nth buckling force is

$$P_n = \frac{EI_0\gamma_n^2}{b^4} = \frac{EI_0}{b^4}\left(\frac{n\pi ab}{L}\right)^2 = EI_0\left(\frac{n\pi}{L}\right)^2\left(\frac{a}{b}\right)^2.$$

18. The substitution $L = a + bt$ in $L\theta'' + 2L'\theta' + g\theta = 0$ yields the transformed equation

$$L^2\theta''(L) + 2L\theta'(L) + (g/b^2)L\theta = 0$$

with independent variable L that is of the form in (1) with $A = 2,\ B = 0,\ q = 1,$ and $C = g/b^2$. Hence

$$\alpha = -1/2,\ \ \beta = 1/2,\ \ k = 2g^{1/2}/b,\ \ \text{and}\ \ p = 1,$$

so

$$\theta(L) = \frac{1}{\sqrt{L}}\left[AJ_1\left(\frac{2}{b}\sqrt{gL}\right) + BY_1\left(\frac{2}{b}\sqrt{gL}\right)\right].$$

CHAPTER 4

LAPLACE TRANSFORM METHODS

SECTION 4.1

LAPLACE TRANSFORMS AND INVERSE TRANSFORMS

The objectives of this section are especially clearcut. They include familiarity with the definition of the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ that is given in Equation (1) in the textbook, the direct application of this definition to calculate Laplace transforms of simple functions (as in Examples 1–3), and the use of known transforms (those listed in Figure 4.1.2) to find Laplace transforms and inverse transforms (as in Examples 4–6). Perhaps students need to be told explicitly to memorize the transforms that are listed in the short table that appears in Figure 4.1.2.

1. $\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\,dt \qquad (u = -st, \quad du = -s\,dt)$

$$= \int_0^{-\infty} \left[\frac{1}{s^2}\right] ue^u\,du = \frac{1}{s^2}\Big[(u-1)e^u\Big]_0^{-\infty} = \frac{1}{s^2}$$

2. We substitute $u = -st$ in the tabulated integral

$$\int u^2 e^u\,du = e^u\left(u^2 - 2u + 2\right) + C$$

(or, alternatively, integrate by parts) and get

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2\,dt = \left[-e^{-st}\left(\frac{t^2}{s} + \frac{2t}{s^2} + \frac{2}{s^3}\right)\right]_{t=0}^{\infty} = \frac{2}{s^3}.$$

3. $\mathcal{L}\{e^{3t+1}\} = \int_0^\infty e^{-st} e^{3t+1}\,dt = e\int_0^\infty e^{-(s-3)t}\,dt = \dfrac{e}{s-3}$

4. With $a = -s$ and $b = 1$ the tabulated integral

$$\int e^{au}\cos bu\,du = e^{au}\left[\frac{a\cos bu + b\sin bu}{a^2 + b^2}\right] + C$$

yields

$$\mathcal{L}\{\cos t\} = \int_0^\infty e^{-st}\cos t\,dt = \left[\frac{e^{-st}(-s\cos t + \sin t)}{s^2 + 1}\right]_{t=0}^{\infty} = \frac{s}{s^2+1}.$$

5. $\mathcal{L}\{\sinh t\} = \frac{1}{2}\mathcal{L}\{e^t - e^{-t}\} = \frac{1}{2}\int_0^\infty e^{-st}\left(e^t - e^{-t}\right)dt = \frac{1}{2}\int_0^\infty \left(e^{-(s-1)t} - e^{-(s+1)t}\right)dt$

$$= \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = \frac{1}{s^2-1}$$

6. $\mathcal{L}\{\sin^2 t\} = \int_0^\infty e^{-st}\sin^2 t\,dt = \frac{1}{2}\int_0^\infty e^{-st}\left(1-\cos 2t\right)dt$

$$= \frac{1}{2}\left[e^{-st}\left(-\frac{1}{s}\right) - e^{-st}\cdot\frac{-s\cos 2t + 2\sin 2t}{s^2+4}\right]_{t=0}^{\infty} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

7. $\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^1 = \frac{1-e^{-s}}{s}$

8. $\mathcal{L}\{f(t)\} = \int_1^2 e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_1^2 = \frac{e^{-s}-e^{-2s}}{s}$

9. $\mathcal{L}\{f(t)\} = \int_0^1 e^{-st}t\,dt = \frac{1-e^{-s}-se^{-s}}{s^2}$

10. $\mathcal{L}\{f(t)\} = \int_0^1 (1-t)e^{-st}\,dt = \left[-e^{-st}\left(\frac{1}{s} - \frac{t}{s} - \frac{1}{s^2}\right)\right]_0^1 = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}$

11. $\mathcal{L}\{\sqrt{t} + 3t\} = \frac{\Gamma(3/2)}{s^{3/2}} + 3\cdot\frac{1}{s^2} = \frac{\sqrt{\pi}}{2s^{3/2}} + \frac{3}{s^2}$

12. $\mathcal{L}\{3t^{5/2} - 4t^3\} = 3\cdot\frac{\Gamma(7/2)}{s^{7/2}} - 4\cdot\frac{3!}{s^4} = \frac{45\sqrt{\pi}}{8s^{7/2}} - \frac{24}{s^2}$

13. $\mathcal{L}\{t - 2e^{3t}\} = \frac{1}{s^2} - \frac{2}{s-3}$

14. $\mathcal{L}\{t^{3/2} + e^{-10t}\} = \frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{s+10} = \frac{3\sqrt{\pi}}{4s^{5/2}} + \frac{1}{s+10}$

15. $\mathcal{L}\{1 + \cosh 5t\} = \frac{1}{s} + \frac{s}{s^2-25}$

16. $\mathcal{L}\{\sin 2t + \cos 2t\} = \dfrac{2}{s^2+4} + \dfrac{s}{s^2+4} = \dfrac{s+2}{s^2+4}$

17. $\mathcal{L}\{\cos^2 2t\} = \dfrac{1}{2}\mathcal{L}\{1+\cos 4t\} = \dfrac{1}{2}\left(\dfrac{1}{s} + \dfrac{s}{s^2+16}\right)$

18. $\mathcal{L}\{\sin 3t \cos 3t\} = \dfrac{1}{2}\mathcal{L}\{\sin 6t\} = \dfrac{1}{2}\cdot\dfrac{6}{s^2+36} = \dfrac{3}{s^2+36}$

19. $\mathcal{L}\{(1+t)^3\} = \mathcal{L}\{1+3t+3t^2+t^3\} = \dfrac{1}{s} + 3\cdot\dfrac{1!}{s^2} + 3\cdot\dfrac{2!}{s^3} + \dfrac{3!}{s^4} = \dfrac{1}{s} + \dfrac{3}{s^2} + \dfrac{6}{s^3} + \dfrac{6}{s^4}$

20. Integrating by parts with $u = t,\ dv = e^{-(s-1)t}dt,$ we get

$$\mathcal{L}\{te^t\} = \int_0^\infty e^{-st}\, te^t\, dt = \int_0^\infty te^{-(s-1)t}\, dt$$

$$= \left[\frac{-te^{-(s-1)t}}{s-1}\right]_0^\infty + \frac{1}{s-1}\int_0^\infty e^{-st}e^t\, dt = \frac{1}{s-1}\mathcal{L}\{t\} = \frac{1}{(s-1)^2}.$$

21. Integration by parts with $u = t$ and $dv = e^{-st}\cos 2t\, dt$ yields

$$\mathcal{L}\{t\cos 2t\} = \int_0^\infty te^{-st}\cos 2t\ dt = -\frac{1}{s^2+4}\int_0^\infty e^{-st}\left(-s\cos 2t + 2\sin 2t\right)\, dt$$

$$= -\frac{1}{s^2+4}\left[-s\,\mathcal{L}\{\cos 2t\} + 2\,\mathcal{L}\{\sin 2t\}\right]$$

$$= -\frac{1}{s^2+4}\left[\frac{-s^2}{s^2+4} + \frac{4}{s^2+4}\right] = \frac{s^2-4}{\left(s^2+4\right)^2}.$$

22. $\mathcal{L}\{\sinh^2 3t\} = \dfrac{1}{2}\mathcal{L}\{\cosh 6t - 1\} = \dfrac{1}{2}\left(\dfrac{s}{s^2-36} - \dfrac{1}{s}\right)$

23. $\mathcal{L}^{-1}\left\{\dfrac{3}{s^4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{6}{s^4}\right\} = \dfrac{1}{2}t^3$

24. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{3/2}}\right\} = \mathcal{L}^{-1}\left\{\dfrac{2}{\sqrt{\pi}}\cdot\dfrac{\sqrt{\pi}}{2s^{3/2}}\right\} = \dfrac{2}{\sqrt{\pi}}\cdot t^{1/2} = 2\sqrt{\dfrac{t}{\pi}}$

25. $\mathcal{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{2}{s^{5/2}}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{2}{\Gamma(5/2)}\cdot\dfrac{\Gamma(5/2)}{s^{5/2}}\right\} = 1 - \dfrac{2}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\cdot t^{3/2} = 1 - \dfrac{8t^{3/2}}{3\sqrt{\pi}}$

26. $\mathcal{L}^{-1}\left\{\dfrac{1}{s+5}\right\} = = e^{-5t}$

27. $\mathcal{L}^{-1}\left\{\dfrac{3}{s-4}\right\} = 3\cdot\mathcal{L}^{-1}\left\{\dfrac{1}{s-4}\right\} = 3e^{4t}$

28. $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{s^2+4}\right\} = 3\cdot\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\}+\dfrac{1}{2}\cdot\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = 3\cos 2t+\dfrac{1}{2}\sin 2t$

29. $\mathcal{L}^{-1}\left\{\dfrac{5-3s}{s^2+9}\right\} = \dfrac{5}{3}\cdot\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\}-3\cdot\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}\right\} = \dfrac{5}{3}\sin 3t-3\cos 3t$

30. $\mathcal{L}^{-1}\left\{\dfrac{9+s}{4-s^2}\right\} = -\dfrac{9}{2}\cdot\mathcal{L}^{-1}\left\{\dfrac{2}{s^2-4}\right\}-\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-4}\right\} = -\dfrac{9}{2}\sinh 2t-\cosh 2t$

31. $\mathcal{L}^{-1}\left\{\dfrac{10s-3}{25-s^2}\right\} = -10\cdot\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-25}\right\}+\dfrac{3}{5}\cdot\mathcal{L}^{-1}\left\{\dfrac{5}{s^2-25}\right\} = -10\cosh 5t+\dfrac{3}{5}\sinh 5t$

32. $\mathcal{L}^{-1}\left\{2\cdot\dfrac{e^{-3s}}{s}\right\} = 2u(t-3) = 2u_3(t)$ [See Example 8 in the textbook.]

33. $$\mathcal{L}\{\sin kt\} = \mathcal{L}\left\{\frac{e^{ikt}-e^{-ikt}}{2i}\right\} = \frac{1}{2i}\left(\frac{1}{s-ik}-\frac{1}{s+ik}\right)$$
$$= \frac{1}{2i}\cdot\frac{2ik}{(s-ik)(s-ik)} = \frac{k}{s^2+k^2} \qquad \text{(because } i^2=-1\text{)}$$

34. $$\mathcal{L}\{\sinh kt\} = \mathcal{L}\left\{\frac{e^{kt}-e^{-kt}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s-k}-\frac{1}{s+k}\right) = \frac{1}{2}\cdot\frac{2k}{s^2-k^2} = \frac{k}{s^2-k^2}$$

35. Using the given tabulated integral with $a=-s$ and $b=k$, we find that

$$\mathcal{L}\{\cos kt\} = \int_0^\infty e^{-st}\cos kt\,dt = \left[\frac{e^{-st}}{s^2+k^2}\left(-s\cos kt+k\sin kt\right)\right]_{t=0}^{\infty}$$
$$= \lim_{t\to\infty}\left(\frac{e^{-st}}{s^2+k^2}\left(-s\cos kt+k\sin kt\right)\right)-\frac{e^0}{s^2+k^2}(-s\cdot 1+k\cdot 0) = \frac{s}{s^2+k^2}.$$

36. Evidently the function $f(t) = \sin(e^{t^2})$ is of exponential order because it is bounded; we can simply take $c = 0$ and $M = 1$ in Eq. (23) of this section in the text. However, its derivative $f'(t) = 2te^{t^2}\cos(e^{t^2})$ is *not* bounded by any exponential function e^{ct}, because $e^{t^2}/e^{ct} = e^{t^2-ct} \to \infty$ as $t \to \infty$.

37. $f(t) = 1 - u_a(t) = 1 - u(t-a)$ so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{u_a(t)\} = \frac{1}{s} - \frac{e^{-as}}{s} = s^{-1}(1 - e^{-as}).$$

For the graph of f, note that $f(a) = 1 - u(a) = 1 - 1 = 0$.

38. $f(t) = u(t-a) - u(t-b)$, so

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_a(t)\} - \mathcal{L}\{u_b(t)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = s^{-1}\left(e^{-as} - e^{-bs}\right).$$

For the graph of f, note that $f(a) = u(0) - u(a-b) = 1 - 0 = 1$ because $a < b$, but $f(b) = u(b-a) - u(0) = 1 - 1 = 0$.

39. Use of the geometric series gives

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} \mathcal{L}\{u(t-n)\} = \sum_{n=0}^{\infty} \frac{e^{-ns}}{s} = \frac{1}{s}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots\right)$$

$$= \frac{1}{s}\left(1 + (e^{-s}) + (e^{-s})^2 + (e^{-s})^3 + \cdots\right) = \frac{1}{s} \cdot \frac{1}{1 - e^{-s}} = \frac{1}{s\left(1 - e^{-s}\right)}.$$

40. Use of the geometric series gives

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} (-1)^n \mathcal{L}\{u(t-n)\} = \sum_{n=0}^{\infty} \frac{(-1)^n e^{-ns}}{s} = \frac{1}{s}\left(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots\right)$$

$$= \frac{1}{s}\left(1 + (-e^{-s}) + (-e^{-s})^2 + (-e^{-s})^3 + \cdots\right) = \frac{1}{s} \cdot \frac{1}{1 - (-e^{-s})} = \frac{1}{s\left(1 + e^{-s}\right)}.$$

41. By checking values at sample points, you can verify that $g(t) = 2f(t) - 1$ in terms of the square wave function $f(t)$ of Problem 40. Hence

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{2f(t) - 1\} = \frac{2}{s\left(1 + e^{-s}\right)} - \frac{1}{s} = \frac{1}{s}\left(\frac{2}{1 + e^{-s}} - 1\right) = \frac{1}{s} \cdot \frac{1 - e^{-s}}{1 + e^{-s}}$$

$$= \frac{1}{s} \cdot \frac{1 - e^{-s}}{1 + e^{-s}} \cdot \frac{e^{s/2}}{e^{s/2}} = \frac{1}{s} \cdot \frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}} = \frac{1}{s} \cdot \frac{\frac{1}{2}\left(e^{s/2} - e^{-s/2}\right)}{\frac{1}{2}\left(e^{s/2} + e^{-s/2}\right)}$$

$$= \frac{1}{s} \cdot \frac{\sinh(s/2)}{\cosh(s/2)} = \frac{1}{s} \tanh \frac{s}{2}.$$

42. Let's refer to $(n-1, n]$ as an odd interval if the integer n is odd, and even interval if n is even. Then our function $h(t)$ has the value a on odd intervals, the value b on even intervals. Now the unit step function $f(t)$ of Problem 40 has the value 1 on odd intervals, the value 0 on even intervals. Hence the function $(a-b)f(t)$ has the value $(a-b)$ on odd intervals, the value 0 on even intervals. Finally, the function $(a-b)f(t)+b$ has the value $(a-b)+b=a$ on odd intervals, the value b on even intervals, and hence $(a-b)f(t)+b=h(t)$. Therefore

$$L\{h(t)\} = L\{(a-b)f(t)\} + L\{b\} = \frac{a-b}{s(1+e^{-s})} + \frac{b}{s} = \frac{a+be^{-s}}{s(1+e^{-s})}.$$

SECTION 4.2

TRANSFORMATION OF INITIAL VALUE PROBLEMS

The focus of this section is on the use of transforms of derivatives (Theorem 1) to solve initial value problems (as in Examples 1 and 2). Transforms of integrals (Theorem 2) appear less frequently in practice, and the extension of Theorem 1 at the end of Section 4.2 may be considered entirely optional (except perhaps for electrical engineering students).

In Problems 1–10 we give first the transformed differential equation, then the transform $X(s)$ of the solution, and finally the inverse transform $x(t)$ of $X(s)$.

1. $[s^2X(s) - 5s] + 4\{X(s)\} = 0$

$$X(s) = \frac{5s}{s^2+4} = 5 \cdot \frac{s}{s^2+4}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 5\cos 2t$$

2. $[s^2X(s) - 3s - 4] + 9[X(s)] = 0$

$$X(s) = \frac{3s+4}{s^2+9} = 3 \cdot \frac{s}{s^2+9} + \frac{4}{3} \cdot \frac{3}{s^2+9}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 3\cos 3t + (4/3)\sin 3t$$

3. $[s^2X(s) - 2] - [sX(s)] - 2[X(s)] = 0$

$$X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s-2)(s+1)} = \frac{2}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

$x(t) = (2/3)(e^{2t} - e^{-t})$

4. $[s^2X(s) - 2s + 3] + 8[s\,X(s) - 2] + 15[X(s)] = 0$

$$X(s) = \frac{2s+13}{s^2+8s+15} = \frac{7}{2}\cdot\frac{1}{s+3} - \frac{3}{2}\cdot\frac{1}{s+5}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (7/2)e^{-3t} - (3/2)e^{-5t}$

5. $[s^2X(s)] + [X(s)] = 2/(s^2 + 4)$

$$X(s) = \frac{2}{(s^2+1)(s^2+4)} = \frac{2}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\cdot\frac{2}{s^2+4}$$

$x(t) = (2\sin t - \sin 2t)/3$

6. $[s^2X(s)] + 4[X(s)] = \mathcal{L}\{\cos t\} = s/(s^2 + 1)$

$$X(s) = \frac{2}{(s^2+1)(s^2+4)} = \frac{1}{3}\cdot\frac{s}{s^2+1} - \frac{1}{3}\cdot\frac{s}{s^2+4}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (\cos t - \cos 2t)/3$

7. $[s^2X(s) - s] + [X(s)] = s/s^2 + 9)$

$(s^2 + 1)X(s) = s + s/(s^2 + 9) = (s^3 + 10s)/(s^2 + 9)$

$$X(s) = \frac{s^2+10s}{(s^2+1)(s^2+9)} = \frac{9}{8}\cdot\frac{s}{s^2+1} - \frac{1}{8}\cdot\frac{s}{s^2+9}$$

$x(t) = (9\cos t - \cos 3t)/8$

8. $[s^2X(s)] + 9[X(s)] = \mathcal{L}\{1\} = 1/s$

$$X(s) = \frac{1}{s(s^2+9)} = \frac{1}{9}\cdot\frac{1}{s} - \frac{1}{9}\cdot\frac{s}{s^2+9}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (1 - \cos 3t)/9$

9. $s^2X(s) + 4sX(s) + 3X(s) = 1/s$

$$X(s) = \frac{1}{s(s^2+4s+3)} = \frac{1}{s(s+1)(s+3)} = \frac{1}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{6}\cdot\frac{1}{s+3}$$

$x(t) = (2 - 3e^{-t} + e^{-3t})/6$

10. $[s^2X(s) - 2] + 3[sX(s)] + 2[X(s)] = \mathcal{L}\{t\} = 1/s^2$

$(s^2 + 3s + 2)X(s) = 2 + 1/s^2 = (2s^2 + 1)/s^2$

$$X(s) = \frac{2s^2+1}{s^2(s^2+3s+2)} = \frac{2s^2+1}{s^2(s+1)(s+2)} = -\frac{3}{4}\cdot\frac{1}{s}+\frac{1}{2}\cdot\frac{1}{s^2}+3\cdot\frac{1}{s+1}-\frac{9}{4}\cdot\frac{1}{s+2}$$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = (-3 + 2t + 12e^{-t} - 9e^{-2t})/4$

11. The transformed equations are

$$sX(s) - 1 = 2X(s) + Y(s)$$
$$sY(s) + 2 = 6X(s) + 3Y(s).$$

We solve for the Laplace transforms

$$X(s) = \frac{s-5}{s(s-5)} = \frac{1}{s}$$

$$Y(s) = X(s) = \frac{-2s+10}{s(s-5)} = -\frac{2}{s}.$$

Hence the solution is given by

$$x(t) = 1, \qquad y(t) = -2.$$

12. The transformed equations are

$$s\,X(s) = X(s) + 2Y(s)$$
$$s\,Y(s) = X(s) + 1/(s+1),$$

which we solve for

$$X(s) = \frac{2}{(s-2)(s+1)^2} = \frac{2}{9}\left(\frac{1}{s-2}-\frac{1}{s+1}-3\cdot\frac{1}{(s+1)^2}\right)$$

$$Y(s) = \frac{s-1}{(s-2)(s+1)^2} = \frac{1}{9}\left(\frac{1}{s-2}-\frac{1}{s+1}-6\cdot\frac{1}{(s+1)^2}\right).$$

Hence the solution is

$$x(t) = (2/9)(e^{2t} - e^{-t} - 3t\,e^{-t})$$
$$y(t) = (1/9)(e^{2t} - e^{-t} + 6t\,e^{-t}).$$

13. The transformed equations are

$$\begin{aligned} sX(s) + 2[sY(s) - 1] + X(s) &= 0 \\ sX(s) - [sY(s) - 1] + Y(s) &= 0, \end{aligned}$$

which we solve for the transforms

$$X(s) = -\frac{2}{3s^2-1} = -\frac{2}{3}\cdot\frac{1}{s^2-1/3} = -\frac{2}{\sqrt{3}}\cdot\frac{1/\sqrt{3}}{s^2-\left(1/\sqrt{3}\right)^2}$$

$$X(s) = \frac{3s+1}{3s^2-1} = \frac{s+1/3}{s^2-1/3} = \frac{s}{s^2-\left(1/\sqrt{3}\right)^2} + \frac{1}{\sqrt{3}}\cdot\frac{1/\sqrt{3}}{s^2-\left(1/\sqrt{3}\right)^2}.$$

Hence the solution is

$$\begin{aligned} x(t) &= -\left(2/\sqrt{3}\right)\sinh\left(t/\sqrt{3}\right) \\ y(t) &= \cosh\left(t/\sqrt{3}\right) + \left(1/\sqrt{3}\right)\sinh\left(t/\sqrt{3}\right). \end{aligned}$$

14. The transformed equations are

$$\begin{aligned} s^2X(s) + 1 + 2X(s) + 4Y(s) &= 0 \\ s^2Y(s) + 1 + X(s) + 2Y(s) &= 0, \end{aligned}$$

which we solve for

$$X(s) = \frac{-s^2+2}{s^2(s^2+4)} = \frac{1}{4}\left(2\cdot\frac{1}{s^2} - 3\cdot\frac{2}{s^2+4}\right)$$

$$Y(s) = \frac{-s^2-1}{s^2(s^2+4)} = -\frac{1}{8}\left(2\cdot\frac{1}{s^2} + 3\cdot\frac{2}{s^2+4}\right).$$

Hence the solution is

$$\begin{aligned} x(t) &= (1/4)(2t - 3\sin 2t) \\ y(t) &= (-1/8)(2t + 3\sin 2t). \end{aligned}$$

15. The transformed equations are

$$\begin{aligned} [s^2X - s] + [sX - 1] + [sY - 1] + 2X - Y &= 0 \\ [s^2Y - s] + [sX - 1] + [sY - 1] + 4X - 2Y &= 0, \end{aligned}$$

which we solve for

$$X(s) = \frac{s^2+3s+2}{s^3+3s^2+3s} = \frac{1}{3}\left(\frac{2}{s}+\frac{s+3}{s^2+3s+3}\right) = \frac{1}{3}\left(\frac{2}{s}+\frac{s+3}{(s+3/2)^2+(3/4)}\right)$$

$$= \frac{1}{3}\left(\frac{2}{s}+\frac{s+3/2}{(s+3/2)^2+(\sqrt{3}/2)^2}+\sqrt{3}\cdot\frac{\sqrt{3}/2}{(s+3/2)^2+(\sqrt{3}/2)^2}\right)$$

$$Y(s) = \frac{-s^3-2s^2+2s+4}{s^3+3s^2+3s} = \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+\frac{2s+15}{s^2+3s+3}\right)$$

$$= \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+\frac{2s+15}{(s+3/2)^2+3/4}\right)$$

$$= \frac{1}{21}\left(\frac{28}{s}-\frac{9}{s-1}+2\cdot\frac{s+3/2}{(s+3/2)^2+(\sqrt{3}/2)^2}+8\sqrt{3}\cdot\frac{\sqrt{3}/2}{(s+3/2)^2+(\sqrt{3}/2)^2}\right).$$

Here we've used some fairly heavy-duty partial fractions (Section 4.3). The transforms

$$\mathcal{L}\left\{e^{at}\cos kt\right\} = \frac{s-a}{(s-a)^2+k^2}, \quad \mathcal{L}\left\{e^{at}\sin kt\right\} = \frac{k}{(s-a)^2+k^2}$$

from the inside-front-cover table (with $a=-3/2,\ k=\sqrt{3}/2$) finally yield

$$x(t) = \frac{1}{3}\left\{2+e^{-3t/2}\left[\cos\left(\sqrt{3}t/2\right)+\sqrt{3}\sin\left(\sqrt{3}t/2\right)\right]\right\}$$

$$y(t) = \frac{1}{21}\left\{28-9e^t+e^{-3t/2}\left[2\cos\left(\sqrt{3}t/2\right)+8\sqrt{3}\sin\left(\sqrt{3}t/2\right)\right]\right\}.$$

16. The transformed equations are

$$\begin{aligned} s\,X(s) - 1 &= X(s) + Z(s) \\ s\,Y(s) &= X(s) + Y(s) \\ s\,Z(s) &= -2X(s) - Z(s), \end{aligned}$$

which we solve for

$$X(s) = \frac{s^2-1}{(s-1)(s^2+1)} = \frac{s+1}{s^2+1}$$

$$Y(s) = \frac{s+1}{(s-1)(s^2+1)} = \frac{1}{s-1}-\frac{s}{s^2+1}$$

$$Z(s) = \frac{-2s+2}{(s-1)(s^2+1)} = -\frac{2}{s^2+1}.$$

Hence the solution is

$$\begin{aligned} x(t) &= \cos t + \sin t \\ y(t) &= e^t - \cos t \\ z(t) &= -2 \sin t. \end{aligned}$$

17. $f(t) = \int_0^t e^{3\tau}\,d\tau = \left[\frac{1}{3}e^{3\tau}\right]_{\tau=0}^{t} = \frac{1}{3}\left(e^{3t}-1\right)$

18. $f(t) = \int_0^t 3e^{-5\tau}\,d\tau = \left[-\frac{3}{5}e^{-5\tau}\right]_{\tau=0}^{t} = \frac{3}{5}\left(1-e^{-5t}\right)$

19. $f(t) = \int_0^t \tfrac{1}{2}\sin 2\tau\,d\tau = \left[-\frac{1}{4}\cos 2\tau\right]_{\tau=0}^{t} = \frac{1}{4}(1-\cos 2t)$

20. $f(t) = \int_0^t \left(2\cos 3\tau + \tfrac{1}{3}\sin 3\tau\right)d\tau = \left[\frac{2}{3}\sin 3\tau - \frac{1}{9}\cos 3\tau\right]_{\tau=0}^{t} = \frac{1}{9}(6\sin 3t - \cos 3t + 1)$

21. $f(t) = \int_0^t \left[\int_0^\tau \sin t\,dt\right]d\tau = \int_0^t (1-\cos\tau)\,d\tau = \left[\tau - \sin\tau\right]_{\tau=0}^{t} = t - \sin t$

22. $f(t) = \int_0^t \tfrac{1}{3}\sinh 3\tau\,d\tau = \left[\frac{1}{9}\cosh 3\tau\right]_{\tau=0}^{t} = \frac{1}{9}(\cosh 3t - 1)$

23. $f(t) = \int_0^t \left[\int_0^\tau \sinh t\,dt\right]d\tau = \int_0^t (\cosh\tau - 1)\,d\tau = \left[\sinh\tau - \tau\right]_{\tau=0}^{t} = \sinh t - t$

24. $f(t) = \int_0^t \left(e^{-\tau} - e^{-2\tau}\right)d\tau = \left[-e^{-\tau} + \frac{1}{2}e^{-2\tau}\right]_{\tau=0}^{t} = \frac{1}{2}\left(e^{-2t} - 2e^{-t} + 1\right)$

25. With $f(t) = \cos kt$ and $F(s) = s/(s^2+k^2)$, Theorem 1 in this section yields

$$\mathcal{L}\{-k\sin kt\} = \mathcal{L}\{f'(t)\} = sF(s) - 1 = s\cdot\frac{s}{s^2+k^2} - 1 = -\frac{k^2}{s^2+k^2},$$

so division by $-k$ yields $\mathcal{L}\{\sin kt\} = k/(s^2+k^2)$.

26. With $f(t) = \sinh kt$ and $F(s) = k/(s^2 - k^2)$, Theorem 1 yields

$$\mathcal{L}\{f'(t)\} = \mathcal{L}\{k \cosh kt\} = ks/(s^2 - k^2) = sF(s),$$

so it follows upon division by k that $\mathcal{L}\{\cosh kt\} = s/(s^2 - k^2)$.

27. **(a)** With $f(t) = t^n e^{at}$ and $f'(t) = nt^{n-1}e^{at} + at^n e^{at}$, Theorem 1 yields

$$\mathcal{L}\{nt^{n-1}e^{at} + at^n e^{at}\} = s\,\mathcal{L}\{t^n e^{at}\}$$

so

$$n\,\mathcal{L}\{t^{n-1}e^{at}\} = (s - a)\mathcal{L}\{t^n e^{at}\}$$

and hence

$$\mathcal{L}\{t^n e^{at}\} = \frac{n}{s-a}\mathcal{L}\{t^{n-1}e^{at}\}.$$

(b) $n=1$: $\mathcal{L}\{t e^{at}\} = \frac{1}{s-a}\mathcal{L}\{e^{at}\} = \frac{1}{s-a}\cdot\frac{1}{s-a} = \frac{1}{(s-a)^2}$

$n=2$: $\mathcal{L}\{t^2 e^{at}\} = \frac{2}{s-a}\mathcal{L}\{t e^{at}\} = \frac{2}{s-a}\cdot\frac{1}{(s-a)^2} = \frac{2!}{(s-a)^3}$

$n=3$: $\mathcal{L}\{t^3 e^{at}\} = \frac{3}{s-a}\mathcal{L}\{t^2 e^{at}\} = \frac{3}{s-a}\cdot\frac{2!}{(s-a)^3} = \frac{3!}{(s-a)^4}$

And so forth.

28. Problems 28 and 30 are the trigonometric and hyperbolic versions of essentially the same computation. For Problem 30 we let $f(t) = t\cosh kt$, so $f(0) = 0$. Then

$$f'(t) = \cosh kt + kt\sinh kt$$
$$f''(t) = 2k\sinh kt + k^2 t\cosh kt,$$

and thus $f'(0) = 1$, so Formula (5) in this section yields

$$\mathcal{L}\{2k\sinh kt + k^2 t\cosh kt\} = s^2\mathcal{L}\{t\cosh kt\} - 1,$$

$$2k\cdot\frac{k}{s^2-k^2} + k^2F(s) = s^2F(s) - 1.$$

We readily solve this last equation for

$$\mathcal{L}\{t\cosh kt\} = F(s) = \frac{s^2+k^2}{\left(s^2-k^2\right)^2}.$$

29. Let $f(t) = t\sinh kt$, so $f(0) = 0$. Then

$$f'(t) = \sinh kt + kt \cosh kt$$

$$f''(t) = 2k \cosh kt + k^2 t \sinh kt,$$

and thus $f'(0) = 0$, so Formula (5) in this section yields

$$\mathcal{L}\{2k \cosh kt + k^2 t \sinh kt\} = s^2 \mathcal{L}\{\sinh kt\},$$

$$2k \cdot \frac{s}{s^2 - k^2} + k^2 F(s) = s^2 F(s).$$

We readily solve this last equation for

$$\mathcal{L}\{t \cosh kt\} = F(s) = \frac{2ks}{\left(s^2 - k^2\right)^2}.$$

30. See Problem 28.

31. Using the known transform of $\sin kt$ and the Problem 28 transform of $t \cos kt$, we obtain

$$\mathcal{L}\left\{\frac{1}{2k^3}(\sin kt - kt \cos kt)\right\} = \frac{1}{2k^3} \cdot \frac{k}{s^2 + k^2} - \frac{k}{2k^3} \cdot \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$$

$$= \frac{1}{2k^2}\left(\frac{1}{s^2 + k^2} - \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}\right) = \frac{1}{2k^2} \cdot \frac{2k^2}{\left(s^2 + k^2\right)^2} = \frac{1}{\left(s^2 + k^2\right)^2}$$

32. If $f(t) = u(t - a)$, then the only jump in $f(t)$ is $j_1 = 1$ at $t_1 = a$. Since $f(0) = 0$ and $f'(t) = 0$, Formula (21) in this section yields

$$0 = s\,F(s) - 0 - e^{as}(1).$$

Hence $\mathcal{L}\{u(t - a)\} = F(s) = s^{-1} e^{-as}$.

33. $f(t) = u_a(t) - u_b(t) = u(t - a) - u(t - b)$, so the result of Problem 32 gives

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t - a)\} - \mathcal{L}\{u(t - b)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{e^{-as} - e^{-bs}}{s}.$$

34. The square wave function of Figure 4.2.9 has a sequence $\{t_n\}$ of jumps with $t_n = n$ and $j_n = 2(-1)^n$ for $n = 1, 2, 3, \ldots$. Hence Formula (21) yields

$$0 = s\,F(s) - 1 - \sum_{n=1}^{\infty} e^{-ns} \cdot 2(-1)^n.$$

It follows that

$$s\,F(s) = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-ns}$$

$$= -1 + 2(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots)$$

$$= -1 + 2/(1 + e^{-s})$$

$$= (1 - e^{-s})/(1 + e^{-s})$$

$$= (e^{s/2} - e^{-s/2})/(e^{s/2} + e^{-s/2})$$

$$s\,F(s) = \tanh(s/2),$$

because $2\cosh(s/2) = e^{s/2} + e^{-s/2}$ and $2\sinh(s/2) = e^{s/2} - e^{-s/2}$.

35. Let's write $g(t)$ for the on-off function of this problem to distinguish it from the square wave function of Problem 34. Then comparison of Figures 4.2.9 and 4.2.10 makes it clear that $g(t) = \frac{1}{2}(1 + f(t))$, so (using the result of Problem 34) we obtain

$$G(s) = \frac{1}{2s} + \frac{1}{2}F(s) = \frac{1}{2s} + \frac{1}{2s}\tanh\frac{s}{2} = \frac{1}{2s}\left(1 + \frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}} \cdot \frac{e^{-s/2}}{e^{-s/2}}\right)$$

$$= \frac{1}{2s}\left(1 + \frac{1 - e^{-s}}{1 + e^{-s}}\right) = \frac{1}{2s} \cdot \frac{2}{1 + e^{-s}} = \frac{1}{s\left(1 + e^{-s}\right)}.$$

36. If $g(t)$ is the triangular wave function of Figure 4.2.11 and $f(t)$ is the square wave function of Problem 34, then $g'(t) = f(t)$. Hence Theorem 1 and the result of Problem 34 yield

$$\mathcal{L}\{g'(t)\} = s\,\mathcal{L}\{g(t)\} - g(0),$$

$$F(s) = s\,G(s), \qquad \text{(because } g(0) = 0\text{)}$$

$$\mathcal{L}\{g(t)\} = s^{-1}F(s) = s^{-2}\tanh(s/2).$$

37. We observe that $f(0) = 0$ and that the sawtooth function has jump -1 at each of the points $t_n = n = 1, 2, 3, \cdots$. Also, $f'(t) \equiv 1$ wherever the derivative is defined. Hence Eq. (21) in this section gives

$$\frac{1}{s} = sF(s) + \sum_{n=1}^{\infty} e^{-ns} = sF(s) - 1 + \sum_{n=0}^{\infty} e^{-ns} = sF(s) - 1 + \frac{1}{1 - e^{-ns}},$$

using the geometric series $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$ with $x = e^{-s}$. Solution for $F(s)$ gives

$$F(s) = \frac{1}{s^2}+\frac{1}{s}-\frac{1}{s\left(1-e^{-s}\right)} = \frac{1}{s^2}-\frac{e^{-s}}{s\left(1-e^{-s}\right)}.$$

SECTION 4.3

TRANSLATION AND PARTIAL FRACTIONS

This section is devoted to the computational nuts and bolts of the staple technique for the inversion of Laplace transforms — partial fraction decompositions. If time does not permit going further in this chapter, Sections 4.1–4.3 provide a self-contained introduction to Laplace transforms that suffices for the most common elementary applications.

1. $\mathcal{L}\{t^4\} = \dfrac{24}{s^5}$, so $\mathcal{L}\{t^4 e^{\pi t}\} = \dfrac{24}{(s-\pi)^5}$

2. $\mathcal{L}\{t^{3/2}\} = \dfrac{3\sqrt{\pi}}{4s^{5/2}}$, so $\mathcal{L}\{t^{3/2}e^{-4t}\} = \dfrac{3\sqrt{\pi}}{4(s+4)^{5/2}}$.

3. $\mathcal{L}\{\sin 3\pi t\} = \dfrac{3\pi}{s^2+9\pi^2}$, so $\mathcal{L}\{e^{-2t}\sin 3\pi t\} = \dfrac{3\pi}{(s+2)^2+9\pi^2}$.

4.
$$\cos 2\left(t-\frac{\pi}{8}\right) = \cos\left(2t-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\cos 2t+\sin 2t)$$
$$\mathcal{L}\left\{\cos 2\left(t-\frac{\pi}{8}\right)\right\} = \frac{1}{\sqrt{2}}\frac{s+2}{s^2+4}$$
$$\mathcal{L}\left\{e^{-t/2}\cos 2\left(t-\frac{\pi}{8}\right)\right\} = \frac{1}{\sqrt{2}}\frac{(s+1/2)+2}{(s+1/2)^2+4} = \frac{1}{\sqrt{2}}\frac{2s+5}{4s^2+4s+17}$$

5. $F(s) = \dfrac{3}{2s-4} = \dfrac{3}{2}\cdot\dfrac{1}{s-2}$, so $f(t) = \dfrac{3}{2}e^{2t}$

6. $F(s) = \dfrac{(s+1)-2}{(s+1)^3} = \dfrac{1}{(s+1)^2}-\dfrac{2}{(s+1)^3}$, so $f(t) = te^{-t}-t^2e^{-t} = e^{-t}\left(t-t^2\right)$

7. $F(s) = \dfrac{1}{(s+2)^2}$, so $f(t) = t\,e^{-2t}$

8. $F(s) = \dfrac{s+2}{(s+2)^2+1}$, so $f(t) = e^{-2t}\cos t$

9. $F(s) = 3\cdot\dfrac{s-3}{(s-3)^2+16}+\dfrac{7}{2}\cdot\dfrac{4}{(s-3)^2+16}$, so $f(t) = e^{3t}[3\cos 4t + (7/2)\sin 4t]$

10. $$F(s) = \frac{2s-3}{(3s-2)^2+16} = \frac{1}{9}\cdot\frac{2s-3}{(s-2/3)^2+16/9}$$
$$= \frac{2}{9}\cdot\frac{s-2/3}{(s-2/3)^2+(4/3)^2}-\frac{5}{36}\cdot\frac{4/3}{(s-2/3)^2+(4/3)^2}$$
$$f(t) = \frac{1}{36}e^{2t/3}\left(8\cos\frac{4t}{3}-5\sin\frac{4t}{3}\right)$$

11. $F(s) = \dfrac{1}{4}\cdot\dfrac{1}{s-2}-\dfrac{1}{4}\cdot\dfrac{1}{s+2}$, so $f(t) = \dfrac{1}{4}\left(e^{2t}-e^{-2t}\right) = \dfrac{1}{2}\sinh 2t$

12. $F(s) = 2\cdot\dfrac{1}{s}+3\cdot\dfrac{1}{s-3}$, so $f(t) = 2+3e^{3t}$

13. $F(s) = 3\cdot\dfrac{1}{s+2}-5\cdot\dfrac{1}{s+5}$, so $f(t) = 3e^{-2t}-5e^{-5t}$

14. $F(s) = 2\cdot\dfrac{1}{s}-3\cdot\dfrac{1}{s+1}+\dfrac{1}{s-2}$, so $f(t) = 2-3e^{-t}+e^{2t}$

15. $F(s) = \dfrac{1}{25}\left(-1\cdot\dfrac{1}{s}-5\cdot\dfrac{1}{s^2}+\dfrac{1}{s-5}\right)$, so $f(t) = \dfrac{1}{25}\left(-1-5t+e^{5t}\right)$

16. $$F(s) = \frac{1}{(s+3)^2(s-2)^2} = \frac{1}{125}\left(\frac{2}{s+3}+\frac{5}{(s+3)^2}-\frac{2}{s-2}+\frac{5}{(s-2)^2}\right)$$
$$f(t) = \frac{1}{125}\left[e^{-3t}(2+5t)+e^{2t}(-2+5t)\right]$$

17. $$F(s) = \frac{1}{8}\left(\frac{1}{s^2-4}-\frac{1}{s^2+4}\right) = \frac{1}{16}\left(\frac{2}{s^2-4}-\frac{2}{s^2+4}\right)$$
$$f(t) = \frac{1}{16}(\sinh 2t-\sin 2t)$$

18. $$F(s) = \frac{1}{s-4}+\frac{1}{(s-4)^2}+\frac{48}{(s-4)^3}+\frac{64}{(s-4)^4}$$

$$f(t) = e^{4t}\left(1+12t+24t^2+\frac{32}{3}t^3\right)$$

19. $$F(s) = \frac{s^2-2s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{-2s-1}{s^2+1}+\frac{2s+4}{s^2+4}\right)$$

$$f(t) = \frac{1}{3}(-2\cos t-\sin t+2\cos 2t+2\sin 2t)$$

20. $$F(s) = \frac{1}{(s^2-4)^2} = \frac{1}{(s-2)^2(s+2)^2} = \frac{1}{32}\left(\frac{1}{s+2}+\frac{2}{(s+2)^2}-\frac{1}{s-2}+\frac{2}{(s-2)^2}\right)$$

$$f(t) = \frac{1}{32}\left[e^{-2t}(1+2t)+e^{2t}(-1+2t)\right]$$

21. First we need to find A, B, C, D so that

$$\frac{s^2+3}{(s^2+2s+2)^2} = \frac{As+B}{s^2+2s+2}+\frac{Cs+D}{(s^2+2s+2)^2}.$$

When we multiply both sides by the quadratic factor s^2+2s+2 and collect coefficients, we get the linear equations

$$\begin{aligned} -2B-D+3 &= 0 \\ -2A-2B-C &= 0 \\ -2A-B+1 &= 0 \\ -A &= 0 \end{aligned}$$

which we solve for $A=0,\ B=1,\ C=-2,\ D=1$. Thus

$$F(s) = \frac{1}{(s+1)^2+1}+\frac{-2s+1}{\left[(s+1)^2+1\right]^2} = \frac{1}{(s+1)^2+1}-2\cdot\frac{s+1}{\left[(s+1)^2+1\right]^2}+3\cdot\frac{1}{\left[(s+1)^2+1\right]^2}.$$

We now use the inverse Laplace transforms given in Eq. (16) and (17) of Section 4.3 — supplying the factor e^{-t} corresponding to the translation $s \to s+1$ — and get

$$f(t) = e^{-t}\left[\sin t-2\cdot\frac{1}{2}t\sin t+3\cdot\frac{1}{2}(\sin t-t\cos t)\right] = \frac{1}{2}e^{-t}(5\sin t-2t\sin t-3t\cos t).$$

22. First we need to find A, B, C, D so that

$$\frac{2s^3-s^2}{\left(4s^2-4s+5\right)^2} = \frac{As+B}{4s^2-4s+5}+\frac{Cs+D}{\left(4s^2-4s+5\right)^2}.$$

When we multiply each side by the quadratic factor (squared) 5 we get the identity

$$2s^3 - s^2 = (As + B)(4s^2 - 4s + 5) + Cs + D.$$

When we substitute the root $s = 1/2 + i$ of the quadratic into this identity, we find that $C = -3/2$ and $D = -5/4$. When we first differentiate each side of the identity and then substitute the root, we find that $A = 1/2$ and $B = 1/4$. Writing

$$4s^2 - 4s + 5 = 4[(s - 1/2)^2 + 1],$$

it follows that

$$F(s) = \frac{1}{8}\cdot\frac{\left(s-\frac{1}{2}\right)+1}{\left(s-\frac{1}{2}\right)^2+1}-\frac{1}{32}\cdot\frac{3\left(s-\frac{1}{2}\right)+4}{\left[\left(s-\frac{1}{2}\right)^2+1\right]^2}.$$

Finally the results

$$\mathcal{L}^{-1}\{2s/(s^2 + 1)^2\} = t\sin t$$

$$\mathcal{L}^{-1}\{2/(s^2 + 1)^2\} = \sin t - t\cos t$$

of Eqs. (16) and (17) in Section 4.3, together with the translation theorem, yield

$$f(t) = e^{t/2}\left[\frac{1}{8}\cdot\left(\cos t+\sin t\right)-\frac{3}{32}\cdot\frac{1}{2}t\sin t-\frac{4}{32}\cdot\frac{1}{2}\left(\sin t-t\cos t\right)\right]$$

$$= \frac{1}{64}e^{t/2}\left[\left(8+4t\right)\cos t+\left(4-3t\right)\sin t\right].$$

23. $$\frac{s^3}{s^4+4a^4} = \frac{1}{2}\left(\frac{s-a}{s^2-2as+2a^2}+\frac{s+a}{s^2+2as+2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \frac{1}{2}\left(e^{at}+e^{-at}\right)\cos at = \cosh at\cos at.$$

24. $$\frac{s}{s^4+4a^4} = \frac{1}{4a^2}\left(\frac{a}{s^2-2as+2a^2}-\frac{a}{s^2+2as+2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s^3}{s^4+4a^4}\right\} = \frac{1}{4a^2}\left(e^{at}-e^{-at}\right)\sin at = \frac{1}{2a^2}\sinh at \sin at.$$

25. $$\frac{s}{s^4+4a^4} = \frac{1}{4a}\left(\frac{s}{s^2-2as+2a^2}-\frac{s}{s^2+2as+2a^2}\right)$$
$$= \frac{1}{4a}\left(\frac{s-a}{s^2-2as+2a^2}+\frac{a}{s^2-2as+2a^2}-\frac{s+a}{s^2+2as+2a^2}+\frac{a}{s^2+2as+2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{4a}\left[e^{at}(\cos at+\sin at)-e^{-at}\left(\cos at-\sin at\right)\right]$$
$$= \frac{1}{2a}\left[\frac{1}{2}\left(e^{at}+e^{-at}\right)\sin at+\frac{1}{2}\left(e^{at}-e^{-at}\right)\cos at\right]$$
$$= \frac{1}{2a}\left(\cosh at \sin at+\sinh at \cos at\right).$$

26. $$\frac{1}{s^4+4a^4} = \frac{1}{8a^3}\left(\frac{-s+2a}{s^2-2as+2a^2}+\frac{s+2a}{s^2+2as+2a^2}\right)$$
$$= \frac{1}{8a^3}\left(-\frac{s-a}{s^2-2as+2a^2}+\frac{a}{s^2-2as+2a^2}+\frac{s+a}{s^2+2as+2a^2}+\frac{a}{s^2+2as+2a^2}\right),$$

and $s^2 \pm 2as + 2a^2 = (s \pm a)^2 + a^2$, so it follows that

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{8a^3}\left[e^{at}(-\cos at+\sin at)+e^{-at}\left(\cos at+\sin at\right)\right]$$
$$= \frac{1}{4a^3}\left[\frac{1}{2}\left(e^{at}+e^{-at}\right)\sin at-\frac{1}{2}\left(e^{at}-e^{-at}\right)\cos at\right]$$
$$= \frac{1}{4a^3}\left(\cosh at \sin at-\sinh at \cos at\right).$$

In Problems 27–40 we give first the transformed equation, then the Laplace transform $X(s)$ of the solution, and finally the desired solution $x(t)$.

27. $[s^2X(s) - 2s - 3] + 6[sX(s) - 2] + 25X(s) = 0$

$$X(s) = \frac{2s+15}{s^2+6s+25} = 2\cdot\frac{s+3}{(s+3)^2+16}+\frac{9}{4}\cdot\frac{4}{(s+3)^2+16}$$

$x(t) = e^{-3t}[2\cos 4t + (9/4)\sin 4t]$

28. $$s^2X(s) - 6sX(s) + 8X(s) = \frac{2}{s}$$

$$X(s) = \frac{2}{s(s^2 - 6s + 8)} = \frac{1}{4}\left(\frac{1}{s} + \frac{1}{s-4} - \frac{2}{s-2}\right)$$

$$x(t) = \frac{1}{4}\left(1 + e^{4t} - 2e^{2t}\right)$$

29. $$s^2X(s) - 4X(s) = \frac{3}{s^2}$$

$$X(s) = \frac{3}{s^2\left(s^2 - 4\right)} = \frac{3}{4}\left(\frac{1}{s^2 - 4} - \frac{1}{s^2}\right)$$

$$x(t) = \frac{3}{8}\sinh 2t - \frac{3}{4}t = \frac{3}{8}(\sinh 2t - 2t)$$

30. $$s^2X(s) + 4sX(s) + 8X(s) = \frac{1}{s+1}$$

$$X(s) = \frac{1}{(s+1)\left(s^2 + 4s + 8\right)} = \frac{1}{5}\left(\frac{1}{s+1} - \frac{s+3}{s^2 + 4s + 8}\right)$$

$$= \frac{1}{5}\left(\frac{1}{s+1} - \frac{s+2}{(s+2)^2 + 4} - \frac{1}{2}\cdot\frac{2}{(s+2)^2 + 4}\right)$$

$$x(t) = \frac{1}{10}\left[2e^{-t} - e^{-2t}(2\cos 2t + \sin 2t)\right]$$

31. $$[s^3X(s) - s - 1] + [s^2X(s) - 1] - 6[sX(s)] = 0$$

$$X(s) = \frac{s+2}{s^3 + s^2 - 6s} = \frac{1}{15}\left(-\frac{5}{s} - \frac{1}{s+3} + \frac{6}{s-2}\right)$$

$$x(t) = \frac{1}{15}\left(-5 - e^{-3t} + 6e^{2t}\right)$$

32. $$\left[s^4X(s) - s^3\right] - X(s) = 0$$

$$X(s) = \frac{s^3}{s^4 - 1} = \frac{1}{2}\left(\frac{s}{s^2 + 1} + \frac{s}{s^2 - 1}\right)$$

$$x(t) = \frac{1}{2}(\cos t + \cosh t)$$

33. $[s^4X(s) - 1] + X(s) = 0$

$$X(s) = \frac{1}{s^4+1}$$

It therefore follows from Problem 26 with $a = \sqrt[4]{1/4} = 1/\sqrt{2}$ that

$$x(t) = \frac{1}{\sqrt{2}}\left(\cosh\frac{t}{\sqrt{2}}\sin\frac{t}{\sqrt{2}} - \sinh\frac{t}{\sqrt{2}}\cos\frac{t}{\sqrt{2}}\right).$$

34. $[s^4X(s) - 2s^2 + 13] + 13[s^2X(s) - 2] + 36\,X(s) = 0$

$$X(s) = \frac{2s^2+13}{s^4+13s^2+36} = \frac{1}{s^2+4} + \frac{1}{s^2+9}$$

$$x(t) = \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t$$

35. $\left[s^4X(s) - 1\right] + 8s^2X(s) + 16X(s) = 0$

$$X(s) = \frac{1}{s^4+8s^2+16} = \frac{1}{\left(s^2+4\right)^2}$$

$$x(t) = \frac{1}{16}\left(\sin 2t - 2t\cos 2t\right) \qquad \text{(by Eq. (17) in Section 4.3)}$$

36. $s^4X(s) + 2s^2X(s) + X(s) = \dfrac{1}{s-2}$

$$X(s) = \frac{1}{(s-2)\left(s^4+2s^2+1\right)} = \frac{1}{25}\left(\frac{1}{s-2} - \frac{s+2}{s^2+1} - \frac{5(s+2)}{\left(s^2+1\right)^2}\right)$$

$$x(t) = \frac{1}{25}\left(e^{-2t} - \cos t - 2\sin t - 5\cdot\frac{1}{2}t\sin t - 10\cdot\frac{1}{2}\left(\sin t - t\cos t\right)\right)$$

$$= \frac{1}{50}\left[2e^{2t} + (10t-2)\cos t - (5t+14)\sin t\right]$$

37. $\left[s^2X(s) - 2\right] + 4sX(s) + 13X(s) = \dfrac{1}{(s+1)^2}$

$$X(s) = \frac{2+1/(s+1)^2}{s^2+4s+13} = \frac{2s^2+4s+13}{(s+1)^2\left(s^2+4s+13\right)}$$

$$= \frac{1}{50}\left[-\frac{1}{s+1}+\frac{5}{(s+1)^2}+\frac{s+98}{(s+2)^2+9}\right]$$

$$= \frac{1}{50}\left[-\frac{1}{s+1}+\frac{5}{(s+1)^2}+\frac{s+2}{(s+2)^2+9}+32\cdot\frac{3}{(s+2)^2+9}\right]$$

$$x(t) = \frac{1}{50}\left[(-1+5t)e^{-t}+e^{-2t}(\cos 3t+32\sin 3t)\right]$$

38. $$\left[s^2X(s)-s+1\right]+6\left[sX(s)-1\right]+18X(s) = \frac{s}{s^2+4}$$

$$X(s) = \frac{s+5}{s^2+6s+18}+\frac{s}{\left(s^2+4\right)\left(s^2+6s+18\right)}$$

$$= \frac{s+5}{s^2+6s+18}+\frac{1}{170}\left(\frac{7s+12}{s^2+4}-\frac{7s+54}{s^2+6s+18}\right)$$

$$= \frac{1}{170}\left(\frac{7s+12}{s^2+4}+\frac{163s+796}{s^2+6s+18}\right)$$

$$X(s) = \frac{1}{170}\left(\frac{7s+12}{s^2+4}+\frac{163(s+3)}{(s+3)^2+9}+\frac{307}{(s+3)^2+9}\right)$$

$$x(t) = \frac{1}{170}(7\cos 2t+6\sin 2t)+\frac{1}{510}e^{-3t}(489\cos 3t+307\sin 3t)$$

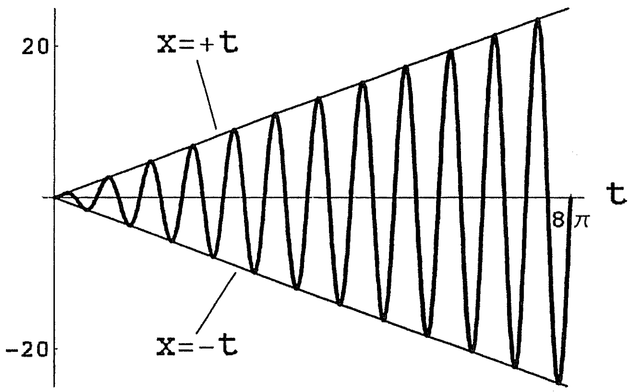
39. $$x''+9x = 6\cos 3t, \qquad x(0)=x'(0)=0$$

$$s^2X(s)+9X(s) = \frac{6s}{s^2+9}$$

$$X(s) = \frac{6s}{\left(s^2+9\right)^2}$$

$$x(t) = 6\cdot\frac{1}{2\cdot 3}t\sin 3t = t\sin 3t \qquad \text{(by Eq. (16) in Section 4.3)}$$

The graph of this resonance is shown in the figure at the top of the next page.



40. $x'' + 0.4x' + 9.04x = x'' + \frac{2}{5}x' + \frac{226}{25} = 6e^{-t/5}\cos 3t$

$$\left(s^2 + \frac{2}{5}s + \frac{226}{25}\right)X(s) = \frac{6(s+1/5)}{(s+1/5)^2+9}$$

$$X(s) = \frac{6(s+1/5)}{\left[(s+1/5)^2+9\right]^2}$$

$$x(t) = te^{-t/5}\sin 3t \qquad \text{(by Eq. (16) in Section 4.3)}$$

SECTION 4.4

DERIVATIVES, INTEGRALS, AND PRODUCTS OF TRANSFORMS

This section completes the presentation of the standard "operational properties" of Laplace transforms, the most important one here being the convolution property $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$, where the **convolution** $f*g$ is defined by

$$f*g(t) = \int_0^t f(x)g(t-x)\,dx.$$

Here we use x rather than τ as the variable of integration; compare with Eq. (3) in Section 4.4 of the textbook.

1. With $f(t)=t$ and $g(t)=1$ we calculate

$$t*1 = \int_0^t x\cdot 1\,dx = \left[\frac{1}{2}x^2\right]_{x=0}^{x=t} = \frac{1}{2}t^2.$$

2. With $f(t)=t$ and $g(t)=e^{at}$ we calculate

$$\begin{aligned}
t*e^{at} &= \int_0^t x\cdot e^{a(t-x)}\,dx = e^{at}\int_0^t x\cdot e^{-ax}\,dx \\
&= e^{at}\int_{-}^{-}\left(-\frac{u}{a}\right)e^u\left(-\frac{du}{a}\right) = \frac{e^{at}}{a^2}\int ue^u\,du \qquad \text{(with } u=-ax\text{)} \\
&= \frac{e^{at}}{a^2}\left[(u-1)e^u\right]_{-}^{-} \qquad \text{(integral formula \#46 inside back cover)} \\
&= \frac{e^{at}}{a^2}\left[(-ax-1)e^{-ax}\right]_{x=0}^{x=t} = \frac{e^{at}}{a^2}\left[(-at-1)e^{-at}+1\right] \\
t*e^{at} &= \frac{1}{a^2}\left(e^{at}-at-1\right).
\end{aligned}$$

3. To compute $(\sin t)*(\sin t) = \int_0^t \sin x\sin(t-x)\,dx,$ we first apply the identity $\sin A\sin B = [\cos(A-B)-\cos(A+B)]/2$. This gives

$$\begin{aligned}
(\sin t)*(\sin t) &= \int_0^t \sin x\sin(t-x)\,dx \\
&= \frac{1}{2}\int_0^t\left[\cos(2x-t)-\cos t\right]dx \\
&= \frac{1}{2}\left[\frac{1}{2}\sin(2x-t)-x\cos t\right]_{x=0}^{x=t} \\
(\sin t)*(\sin t) &= \frac{1}{2}\left(\sin t - t\cos t\right).
\end{aligned}$$

4. To compute $t^2*\cos t = \int_0^t x^2\cos(t-x)\,dx,$ we first substitute

$$\cos(t-x) = \cos t\cos x + \sin t\sin x,$$

and then use the integral formulas

$$\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C$$

$$\int x^2\sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

from #40 and #41 inside the back cover of the textbook. This gives

$$\begin{aligned}
t^2 * \cos t &= \int_0^t x^2(\cos t\cos x + \sin t\sin x)\,dx \\
&= (\cos t)\int_0^t x^2\cos x\,dx + (\sin t)\int_0^t x^2\sin x\,dx \\
&= (\cos t)\left[x^2\sin x + 2x\cos x - 2\sin x\right]_{x=0}^{x=t} \\
&\qquad + (\sin t)\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_{x=0}^{x=t} \\
t^2 * \cos t &= 2(t - \sin t).
\end{aligned}$$

5. $$e^{at} * e^{at} = \int_0^t e^{ax}e^{a(t-x)}dx = \int_0^t e^{at}dx = e^{at}\left[x\right]_{x=0}^{x=t} = t\,e^{at}$$

6. $$\begin{aligned}
e^{at} * e^{bt} &= \int_0^t e^{ax}e^{b(t-x)}dx = e^{bt}\int_0^t e^{(a-b)x}dx \\
&= e^{bt}\left[\frac{e^{(a-b)x}}{a-b}\right]_{x=0}^{x=t} = \frac{e^{bt}\left(e^{(a-b)t}-1\right)}{a-b} = \frac{e^{at}-e^{bt}}{a-b}
\end{aligned}$$

7. $$f(t) = 1 * e^{3t} = e^{3t} * 1 = \int_0^t e^{3x}\cdot 1\,dx = \frac{1}{3}\left(e^{3t}-1\right)$$

8. $$f(t) = 1 * \frac{1}{2}\sin 2t = \int_0^t \frac{1}{2}\sin 2x\,dx = \frac{1}{4}\left(1-\cos 2t\right)$$

9. $$\begin{aligned}
f(t) &= \frac{1}{9}\sin 3t * \sin 3t = \frac{1}{9}\int_0^t \sin 3x\sin 3(t-x)\,dx \\
&= \frac{1}{9}\int_0^t \sin 3x\left[\sin 3t\cos 3x - \cos 3t\sin 3x\right]dx \\
&= \frac{1}{9}\sin 3t\int_0^t \sin 3x\cos 3x\,dx - \frac{1}{9}\cos 3t\int_0^t \sin^2 3x\,dx \\
&= \frac{1}{9}\sin 3t\left[\frac{1}{6}\sin^2 3x\right]_{x=0}^{x=t} - \frac{1}{9}\cos 3t\left[\frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right)\right]_{x=0}^{x=t} \\
f(t) &= \frac{1}{54}\left(\sin 3t - 3t\cos 3t\right)
\end{aligned}$$

10. $$f(t) = t * (\sin kt)/k = \frac{1}{k}\int_0^t \sin kx\cdot(t-x)\,dx$$

$$= \frac{t}{k}\int_0^t \sin kx\,dx - \frac{1}{k}\int_0^t x\sin kx\,dx = \frac{kt-\sin kt}{k^3}$$

11. $$f(t) = \cos 2t * \cos 2t = \int_0^t \cos 2x\cos 2(t-x)\,dx$$

$$= \int_0^t \cos 2x\left(\cos 2t\cos 2x + \sin 2t\sin 2x\right)dx$$

$$= (\cos 2t)\int_0^t \cos^2 2x\,dx + (\sin 2t)\int_0^t \cos 2x\sin 2x\,dx$$

$$= (\cos 2t)\left[\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]_{x=0}^{x=t} + (\sin 2t)\left[\frac{1}{4}\sin^2 2x\right]_{x=0}^{x=t}$$

$$f(t) = \frac{1}{4}\left(\sin 2t + 2t\cos 2t\right)$$

12. $$f(t) = (e^{-2t}\sin t)*(1) = \int_0^t e^{-2x}\sin x\,dx = \frac{1}{5}\left[1-e^{-2t}(\cos t + 2\sin t)\right]$$

13. $$f(t) = e^{3t} * \cos t = \int_0^t (\cos x)e^{3(t-x)}dx$$

$$= e^{3t}\int_0^t e^{-3x}\cos x\,dx$$

$$= e^{3t}\left[\frac{e^{-3x}}{10}(-3\cos x + \sin x)\right]_{x=0}^{x=t} \qquad \text{(by integral formula \#50)}$$

$$f(t) = \frac{1}{10}\left(3e^{3t} - 3\cos t + \sin t\right)$$

14. $$f(t) = \cos 2t * \sin t = \int_0^t \cos 2x\sin(t-x)\,dx$$

$$= \int_0^t \cos 2x(\sin t\cos x - \cos t\sin x)\,dx$$

$$= (\sin t)\int_0^t \cos 2x\cos x\,dx - (\cos t)\int_0^t \cos 2x\sin x\,dx$$

$$= \frac{1}{2}(\sin t)\int_0^t (\cos 3x + \cos x)\,dx - \frac{1}{2}(\cos t)\int_0^t (\sin 3x - \sin x)\,dx$$

$$f(t) = \frac{1}{3}(\cos t - \cos 2t)$$

15. $$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\left(\mathcal{L}\{\sin t\}\right) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{\left(s^2+9\right)^2}$$

16. $\mathcal{L}\{t^2\cos 2t\} = \dfrac{d^2}{ds^2}\left(\mathcal{L}\{\cos 2t\}\right) = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+4}\right) = \dfrac{2s(s^2-12)}{\left(s^2+4\right)^3}$

17. $\mathcal{L}\{e^{2t}\cos 3t\} = (s-2)/(s^2-4s+13)$

$\mathcal{L}\{te^{2t}\cos 3t\} = -(d/ds)[(s-2)/(s^2-4s+13)] = (s^2-4s-5)/(s^2-4s+13)^2$

18. $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\{(1-\cos 2t)/2\} = 2/s(s^2+4)$

$\mathcal{L}\{e^{-t}\sin^2 t\} = 2/[(s+1)(s^2+2s+5)]$

$\mathcal{L}\{te^{-t}\sin^2 t\} = -(d/ds)[2/((s+1)(s^2+2s+5))]$

$= 2(3s^2+6s+7)/[(s+1)^2(s^2+2s+5)^2]$

19. $\mathcal{L}\left\{\dfrac{\sin t}{t}\right\} = \displaystyle\int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2}-\tan^{-1}s = \tan^{-1}\left(\frac{1}{s}\right)$

20. $\mathcal{L}\{1-\cos 2t\} = \dfrac{1}{s}-\dfrac{s}{s^2+4}$, so

$$\mathcal{L}\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+4}\right)ds = \left[\ln\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty = \ln\left(\frac{\sqrt{s^2+4}}{s}\right)$$

21. $\mathcal{L}\{e^{3t}-1\} = \dfrac{1}{s-3}-\dfrac{1}{s}$, so

$$\mathcal{L}\left\{\frac{e^{3t}-1}{t}\right\} = \int_s^\infty\left(\frac{1}{s-3}-\frac{1}{s}\right)ds = \left[\ln\left(\frac{s-3}{s}\right)\right]_s^\infty = \ln\left(\frac{s}{s-3}\right)$$

22. $\mathcal{L}\{e^t-e^{-t}\} = \dfrac{1}{s-1}-\dfrac{1}{s+1} = \dfrac{2}{s^2-1}$, so

$$\mathcal{L}\left\{\frac{e^t-e^{-t}}{t}\right\} = \int_s^\infty\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds = \left[\ln\left(\frac{s-1}{s+1}\right)\right]_s^\infty = \ln\left(\frac{s+1}{s-1}\right)$$

23. $f(t) = -\dfrac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\dfrac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}-\dfrac{1}{s+2}\right\} = -\dfrac{1}{t}\left(e^{2t}-e^{-2t}\right) = -\dfrac{2\sinh 2t}{t}$

24. $f(t) = -\dfrac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\dfrac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2+1}-\dfrac{2s}{s^2+4}\right\} = \dfrac{2}{t}(\cos 2t-\cos t)$

25. $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s+2} - \frac{1}{s-3}\right\} = \frac{1}{t}\left(e^{-2t} + e^{3t} - 2\cos t\right)$

26. $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{3}{(s+2)^2+9}\right\} = \frac{e^{-2t}\sin 3t}{t}$

27. $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{-2/s^3}{1+1/s^2}\right\}$

$$= \frac{2}{t}\mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\} = \frac{2}{t}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = \frac{2}{t}(1-\cos t)$$

28. An empirical approach works best with this one. We can construct transforms with powers of (s^2+1) in their denominators by differentiating the transforms of $\sin t$ and $\cos t$. Thus,

$$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$\mathcal{L}\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}$$

$$\mathcal{L}\{t^2\cos t\} = -\frac{d}{ds}\left(\frac{s^2-1}{(s^2+1)^2}\right) = \frac{2s^3-6s}{(s^2+1)^3}.$$

From the first and last of these formulas it follows readily that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^3}\right\} = \frac{1}{8}\left(t\sin t - t^2\cos t\right).$$

Alternatively, one could work out the repeated convolution

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^3}\right\} = (\cos t) * (\sin t * \sin t).$$

29. $-[s^2X(s) - x'(0)]' - [s\,X(s)]' - 2[s\,X(s)] + X(s) = 0$

$s(s+1)X'(s) + 4s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s+1)^4} \quad \text{with } A \neq 0$$

$$x(t) = Ct^3 e^{-t} \text{ with } C \neq 0$$

30. $-[s^2X(s) - x'(0)]' - 3[s\,X(s)]' - [s\,X(s)] + 3X(s) = 0$

$-(s^2 + 3s)X'(s) - 3s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s+3)^3} \quad \text{with } A \neq 0$$

$$x(t) = Ct^2 e^{-3t} \text{ with } C \neq 0$$

31. $-[s^2X(s) - x'(0)]' + 4[s\,X(s)]' - [s\,X(s)] - 4[X(s)]' + 2X(s) = 0$

$(s^2 - 4s + 4)X'(s) + (3s - 6)X(s) = 0$ (separable)

$(s - 2)X'(s) + 3X(s) = 0$

$$X(s) = \frac{A}{(s-2)^3} \quad \text{with } A \neq 0$$

$$x(t) = Ct^2 e^{2t} \text{ with } C \neq 0$$

32. $-[s^2X(s) - x'(0)]' - 2[s\,X(s)]' - 2[s\,X(s)] - 2X(s) = 0$

$-(s^2 + 2s)X'(s) - (4s + 4)X(s) = 0$ (separable)

$$X(s) = \frac{A}{s^2(s+2)^2} = C\left[\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+2} - \frac{1}{(s+2)^2}\right]$$

$$x(t) = C(1 - t - e^{-2t} - te^{-2t}) \text{ with } C = -A/4 \neq 0$$

33. $-[s^2X(s) - x(0)]' - 2[s\,X(s)] - [X(s)]' = 0$

$(s^2 + 1)X'(s) + 4s\,X(s) = 0$ (separable)

$$X(s) = \frac{A}{(s^2+1)^2} \quad \text{with } A \neq 0$$

$$x(t) = C(\sin t - t\cos t) \text{ with } C \neq 0$$

34. $-(s^2 + 4s + 13)X'(s) - (4s + 8)X(s) = 0$

$$X(s) = \frac{C}{(s^2+4s+13)^2} = \frac{C}{\left[(s+2)^2+9\right]^2}$$

It now follows from Problem 31 in Section 4.2 that

$$x(t) = Ae^{-2t}(\sin 3t - 3t\cos 3t) \text{ with } A \neq 0.$$

35. $$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\} = e^t * \frac{1}{\sqrt{\pi t}} = \int_0^t \frac{1}{\sqrt{\pi x}}\cdot e^{t-x}\,dx$$

$$= \frac{e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} \frac{1}{u}\cdot e^{-u^2}\cdot 2u\,du = \frac{2e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}du = e^t\operatorname{erf}\left(\sqrt{t}\right)$$

36. $$s^2X(s) + 4X(s) = F(s)$$

$$X(s) = \frac{1}{2}F(s)\cdot\frac{2}{s^2+4}$$

$$x(t) = \frac{1}{2}f(t) * \sin 2t = \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\, d\tau\,(1/2)$$

37. $$s^2X(s) + 2sX(s) + X(s) = F(s)$$

$$X(s) = F(s)\cdot\frac{1}{(s+1)^2}$$

$$x(t) = te^{-t} * f(t) = \int_0^t \tau e^{-\tau} f(t-\tau)\,d\tau$$

38. $$s^2X(s) + 4sX(s) + 13X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2+4s+13} = \frac{1}{3}F(s)\cdot\frac{3}{(s+2)^2+9}$$

$$x(t) = \frac{1}{3}f(t) * e^{-2t}\sin 3t = \frac{1}{3}\int_0^t e^{-2\tau} f(t-\tau)\sin 3\tau\, d\tau$$

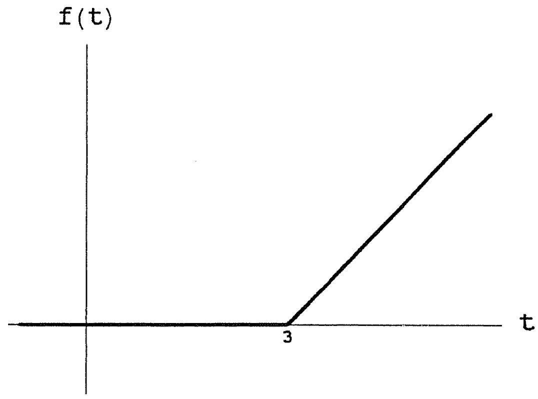
SECTION 4.5

PERIODIC AND PIECEWISE CONTINUOUS FORCING FUNCTIONS

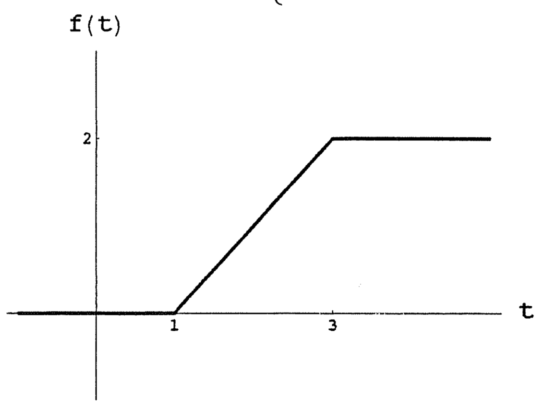
In Problems 1 through 10, we first derive the inverse Laplace transform $f(t)$ of $F(s)$ and then show the graph of $f(t)$.

1. $F(s) = e^{-3s}\mathcal{L}\{t\}$ so Eq. (3b) in Theorem 1 gives

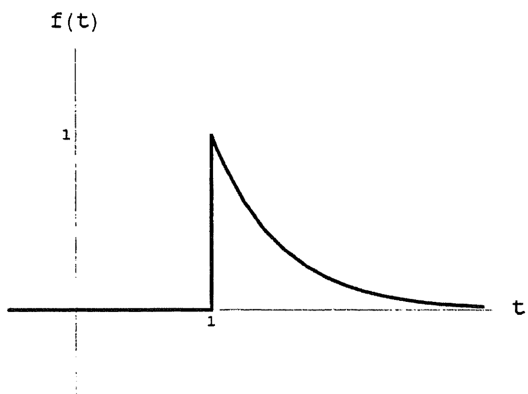
$$f(t) = u(t-3)\cdot(t-3) = \begin{cases} 0 & \text{if } t<3, \\ t-3 & \text{if } t \geq 3. \end{cases}$$



2. $f(t) = (t-1)u(t-1)-(t-3)u(t-3) = \begin{cases} 0 & \text{if } t<1, \\ t-1 & \text{if } 1 \leq t<3, \\ 2 & \text{if } t \geq 3. \end{cases}$

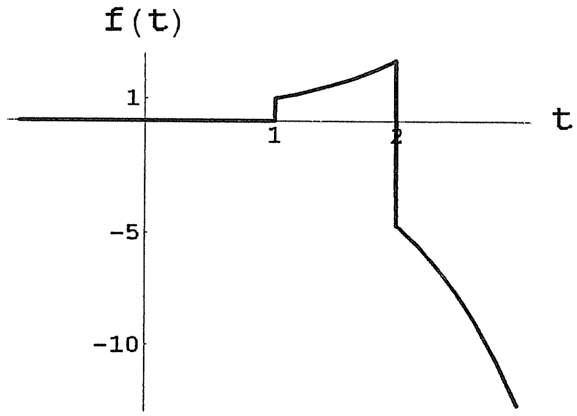


3. $F(s) = e^{-s}\mathcal{L}\{e^{-2t}\}$ so $f(t) = u(t-1)\cdot e^{-2(t-1)} = \begin{cases} 0 & \text{if } t<1, \\ e^{-2(t-1)} & \text{if } t \geq 1. \end{cases}$



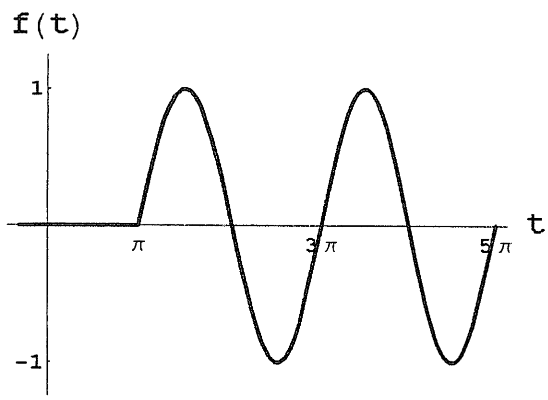
4. $F(s) = e^{-s}\mathcal{L}\{t\} - e^2 e^{-2s}\mathcal{L}\{t\}$ so

$$f(t) = e^{t-1}u(t-1) - e^2 e^{t-2}u(t-2) = \begin{cases} 0 & \text{if } t<1, \\ e^{t-1} & \text{if } 1 \le t < 2, \\ e^{t-1} - e^t & \text{if } t \ge 2. \end{cases}$$



5. $F(s) = e^{-\pi s}\mathcal{L}\{\sin t\}$ so

$$f(t) = u(t-\pi)\cdot\sin(t-\pi) = -u(t-\pi)\sin t = \begin{cases} 0 & \text{if } t<\pi, \\ -\sin t & \text{if } t \ge \pi. \end{cases}$$



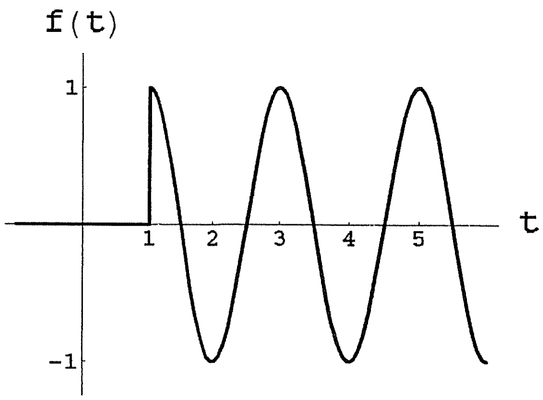
6. $F(s) = e^{-s}\mathcal{L}\{\cos \pi t\}$ so

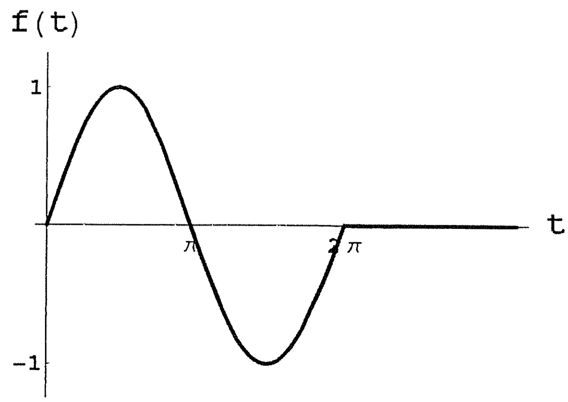
$$f(t) = u(t-1)\cdot\cos\pi(t-1) = -u(t-1)\cos\pi t = \begin{cases} 0 & \text{if } t<1, \\ -\cos\pi t & \text{if } t \ge 1. \end{cases}$$

7. $F(s) = \mathcal{L}\{\sin t\} - e^{-2\pi s}\mathcal{L}\{\sin t\}$ so

$$f(t) = \sin t - u(t-2\pi)\sin(t-2\pi) = [1-u(t-2\pi)]\sin t = \begin{cases} \sin t & \text{if } t<2\pi, \\ 0 & \text{if } t \ge 2\pi. \end{cases}$$

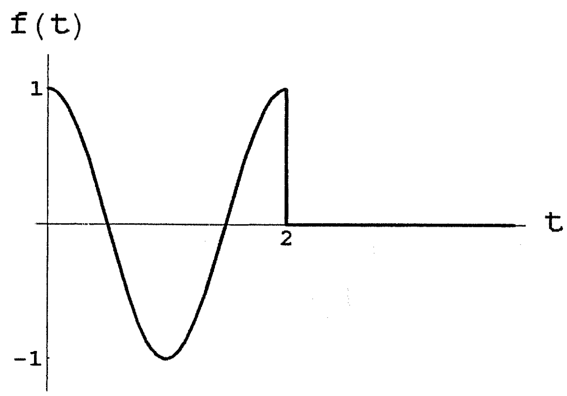
The left-hand figure below is the graph for Problem 6 on the preceding page, and the right-hand figure is the graph for Problem 7.





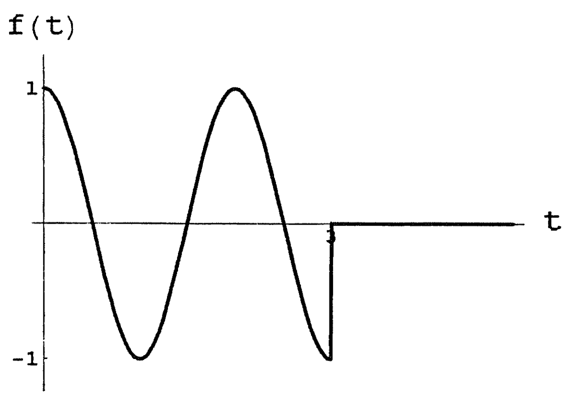
8. $F(s) = \mathcal{L}\{\cos \pi t\} - e^{-2s}\mathcal{L}\{\cos \pi t\}$ so

$$f(t) = \cos \pi t - u(t-2)\cos \pi(t-2) = [1-u(t-2)]\cos \pi t = \begin{cases} \cos \pi t & \text{if } t < 2, \\ 0 & \text{if } t \geq 2. \end{cases}$$



9. $F(s) = \mathcal{L}\{\cos \pi t\} + e^{-3s}\mathcal{L}\{\cos \pi t\}$ so

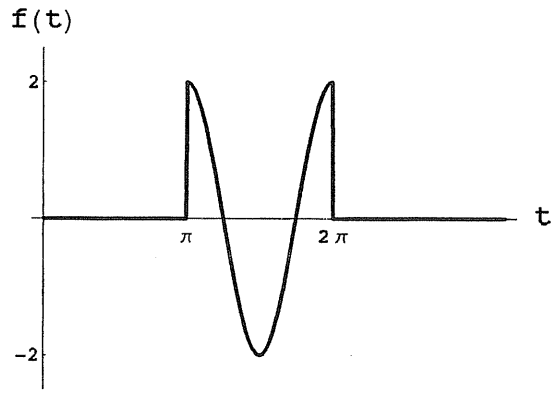
$$f(t) = \cos \pi t + u(t-3)\cos \pi(t-3) = [1-u(t-3)]\cos \pi t = \begin{cases} \cos \pi t & \text{if } t < 3, \\ 0 & \text{if } t \geq 3. \end{cases}$$



10. $F(s) = e^{-\pi s}\mathcal{L}\{2\cos 2t\} + e^{-2\pi s}\mathcal{L}\{2\cos 2t\}$ so

$$f(t) = 2u(t-\pi)\cos 2(t-\pi) - 2u(t-2\pi)\cos 2(t-2\pi)$$

$$= 2\big[u(t-\pi) - u(t-2\pi)\big]\cos 2t = \begin{cases} 0 \text{ if } t<\pi \text{ or } t \ge 2\pi, \\ 2\cos 2t \text{ if } \pi \le t < 2\pi. \end{cases}$$



11. $f(t) = 2 - u(t-3)\cdot 2$ so $F(s) = \dfrac{2}{s} - e^{-3s}\dfrac{2}{s} = \dfrac{2}{s}\left(1-e^{-3s}\right).$

12. $f(t) = u(t-1) - u(t-4)$ so $F(s) = \dfrac{e^{-s}}{s} - \dfrac{e^{-3s}}{s} = \dfrac{1}{s}\left(e^{-s} - e^{-3s}\right).$

13. $f(t) = [1-u(t-2\pi)]\sin t = \sin t - u(t-2\pi)\sin(t-2\pi)$ so

$$F(s) = \frac{1}{s^2+1} - e^{-2\pi s}\cdot\frac{1}{s^2+1} = \frac{1-e^{-2\pi s}}{s^2+1}.$$

14. $f(t) = [1-u(t-2)]\cos \pi t = \cos \pi t - u(t-2)\cos \pi(t-2)$ so

$$F(s) = \frac{s}{s^2+\pi^2} - e^{-2s}\cdot\frac{s}{s^2+\pi^2} = \frac{s\left(1-e^{-2s}\right)}{s^2+\pi^2}.$$

15. $f(t) = [1-u(t-3\pi)]\sin t = \sin t + u(t-3\pi)]\sin(t-3\pi)$ so

$$F(s) = \frac{1}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1} = \frac{1+e^{-3\pi s}}{s^2+1}.$$

16. $f(t) = [u(t-\pi) - u(t-2\pi)]\sin 2t = u(t-\pi)\sin 2(t-\pi) - u(t-2\pi)\sin 2(t-2\pi)$ so

$$F(s) = \left(e^{-\pi s} - e^{-2\pi s}\right)\cdot\frac{2}{s^2+4} = \frac{2\left(e^{-\pi s} - e^{-2\pi s}\right)}{s^2+4}.$$

17. $f(t) = [u(t-2)-u(t-3)]\sin \pi t = u(t-2)\sin \pi(t-2) + u(t-3)\sin \pi(t-3)$ so

$$F(s) = \left(e^{-2s} + e^{-3s}\right)\cdot\frac{\pi}{s^2+\pi^2} = \frac{\pi\left(e^{-2s}+e^{-3s}\right)}{s^2+\pi^2}.$$

18. $f(t) = [u(t-3)-u(t-5)]\cos\frac{\pi t}{2} = u(t-3)\sin\frac{\pi}{2}(t-3) + u(t-5)\sin\frac{\pi}{2}(t-5)$ so

$$F(s) = \left(e^{-3s} + e^{-5s}\right)\cdot\frac{\pi/2}{s^2+\pi^2/4} = \frac{2\pi\left(e^{-3s}+e^{-5s}\right)}{4s^2+\pi^2}.$$

19. If $g(t) = t+1$ then $f(t) = u(t-1)\cdot t = u(t-1)\cdot g(t-1)$ so

$$F(s) = e^{-s}G(s) = e^{-s}L\{t+1\} = e^{-s}\cdot\left(\frac{1}{s^2}+\frac{1}{s}\right) = \frac{e^{-s}(s+1)}{s^2}.$$

20. If $g(t) = t+1$ then $f(t) = [1-u(t-1)]t + u(t-1) = t - u(t-1)g(t-1) + u(t-1)$ so

$$F(s) = \frac{1}{s^2} - e^{-s}\cdot G(s) + \frac{e^{-s}}{s} = \frac{1}{s^2} - e^{-s}\cdot\left(\frac{1}{s^2}+\frac{1}{s}\right) + \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s^2}.$$

21. If $g(t) = t+1$ and $h(t) = t+2$ then

$$\begin{aligned} f(t) &= t[1-u(t-1)] + (2-t)[u(t-1)-u(t-2)] \\ &= t - 2t\,u(t-1) + 2u(t-1) - 2u(t-2) + t\,u(t-2) \\ &= t - 2u(t-1)g(t-1) + 2u(t-1) - 2u(t-2) + u(t-2)h(t-2) \end{aligned}$$

so

$$F(s) = \frac{1}{s^2} - 2e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) + \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s} + e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) = \frac{\left(1-e^{-s}\right)^2}{s^2}.$$

22. $f(t) = [u_1(t) - u_2(t)]\, t^3 = u_1(t)g(t-1) - u_2(t)h(t-2)$ where

$$\begin{aligned} g(t) &= (t+1)^3 = t^3 + 3t^2 + 3t + 1, \\ h(t) &= (t+2)^3 = t^3 + 6t^2 + 12t + 8. \end{aligned}$$

It follows that

$$\begin{aligned} F(s) &= e^{-s}G(s) - e^{-2s}H(s) \\ &= [(s^3 + 3s^2 + 6s + 6)e^{-s} - (8s^3 + 12s^2 + 12s + 6)e^{-2s}]/s^4. \end{aligned}$$

23. With $f(t) = 1$ and $p = 1$, Formula (12) in the text gives

$$\mathcal{L}\{1\} = \frac{1}{1-e^{-s}}\int_0^1 e^{-st}\cdot 1\,dt = \frac{1}{1-e^{-s}}\left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=1} = \frac{1}{s}.$$

24. With $f(t) = \cos kt$ and $p = 2\pi/k$, Formula (12) the integral formula

$$\int e^{at}\cos bt\,dt = e^{at}\left[\frac{a\cos bt + b\sin bt}{a^2+b^2}\right]+C$$

give

$$\begin{aligned}
\mathcal{L}\{\cos kt\} &= \frac{1}{1-e^{-2\pi s/k}}\int_0^{2\pi/k} e^{-st}\cdot\cos kt\,dt \\
&= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-st}\left(\frac{-s\cos kt + k\sin kt}{s^2+k^2}\right)\right]_{t=0}^{t=2\pi/k} \\
&= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-2\pi s/k}\left(\frac{-s}{s^2+k^2}\right) - e^{-0}(-s)\right] = \frac{s}{s^2+k^2}.
\end{aligned}$$

25. With $p = 2a$ and $f(t) = 1$ if $0 \le t \le a$, $f(t) = 0$ if $a < t \le 2a$, Formula (12) gives

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2as}}\int_0^a e^{-st}\cdot 1\,dt = \frac{1}{1-e^{-2as}}\left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=a} \\
&= \frac{1-e^{-as}}{s\left(1-e^{-as}\right)\left(1+e^{-as}\right)} = \frac{1}{s\left(1+e^{-as}\right)}.
\end{aligned}$$

26. With $p = a$ and $f(t) = t/a$, Formula (12) and the integral formula $\int ue^u\,du = (u-1)e^u$ (with $u = -st$) give

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{a\left(1-e^{-as}\right)}\int_0^a e^{-st}\cdot t\,dt = \frac{1}{a\left(1-e^{-as}\right)}\int_0^{-as} e^u\cdot\left(-\frac{u}{s}\right)\left(-\frac{du}{s}\right) \\
&= \frac{1}{as^2\left(1-e^{-as}\right)}\int_0^{-as} e^u u\,du = \frac{1}{as^2\left(1-e^{-as}\right)}\left[(u-1)e^u\right]_0^{-as} \\
&= \frac{1}{as^2\left(1-e^{-as}\right)}\left[(-as-1)e^{-as}+1\right] = \frac{1}{as^2} - \frac{e^{-as}}{s\left(1-e^{-as}\right)}.
\end{aligned}$$

27. $G(s) = \mathcal{L}\{t/a - f(t)\} = (1/as^2) - F(s)$. Now substitution of the result of Problem 26 in place of $F(s)$ immediately gives the desired transform.

28. This computation is very similar to the one in Problem 26, except that $p = 2a$:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2as}}\int_0^a e^{-st}\cdot t\,dt = \frac{1}{1-e^{-2as}}\int_0^{-as} e^u\cdot\left(-\frac{u}{s}\right)\left(-\frac{du}{s}\right) \\
&= \frac{1}{s^2\left(1-e^{-2as}\right)}\int_0^{-as} e^u u\,du = \frac{1}{s^2\left(1-e^{-2as}\right)}\left[(u-1)e^u\right]_0^{-as}
\end{aligned}$$

$$= \frac{1}{s^2\left(1-e^{-2as}\right)}\left[(-as-1)e^{-as}+1\right] = \frac{1-e^{-as}(1+as)}{s^2\left(1-e^{-2as}\right)}.$$

29. With $p = 2\pi/k$ and $f(t) = \sin kt$ for $0 \le t \le \pi/k$ while $f(t) = 0$ for $\pi/k \le t \le 2\pi/k$, Formula (12) the integral formula

$$\int e^{at}\sin bt\,dt = e^{at}\left[\frac{a\sin bt - b\cos bt}{a^2+b^2}\right]+C$$

give

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s/k}}\int_0^{\pi/k} e^{-st}\cdot \sin kt\,dt$$

$$= \frac{1}{1-e^{-2\pi s/k}}\left[e^{-st}\left(\frac{-s\sin kt - k\cos kt}{s^2+k^2}\right)\right]_{t=0}^{t=\pi/k}$$

$$= \frac{1}{1-e^{-2\pi s/k}}\left[\frac{e^{-\pi s/k}(k)-(-k)}{s^2+k^2}\right]$$

$$= \frac{k\left(1+e^{-\pi s/k}\right)}{\left(1-e^{-\pi s/k}\right)\left(1+e^{-\pi s/k}\right)\left(s^2+k^2\right)} = \frac{k}{\left(s^2+k^2\right)\left(1-e^{-\pi s/k}\right)}.$$

30. $h(t) = f(t)+g(t) = f(t)+u(t-\pi/k)f(t-\pi/k)$, so Problem 29 gives

$$H(s) = F(s)+e^{-\pi s/k}F(s) = \left(1+e^{-\pi s/k}\right)F(s)$$

$$= \left(1+e^{-\pi s/k}\right)\cdot\frac{k}{\left(s^2+k^2\right)\left(1-e^{-\pi s/k}\right)} = \frac{k}{s^2+k^2}\cdot\frac{1+e^{-\pi s/k}}{1-e^{-\pi s/k}}\cdot\frac{e^{\pi s/2k}}{e^{\pi s/2k}}$$

$$= \frac{k}{s^2+k^2}\cdot\frac{e^{\pi s/2k}+e^{-\pi s/2k}}{e^{\pi s/2k}-e^{-\pi s/2k}} = \frac{k}{s^2+k^2}\,\frac{\cosh(\pi s/2k)}{\sinh(\pi s/2k)} = \frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}.$$

In Problems 31-42, we first write and transform the appropriate differential equation. Then we solve for the transform of the solution, and finally inverse transform to find the desired solution.

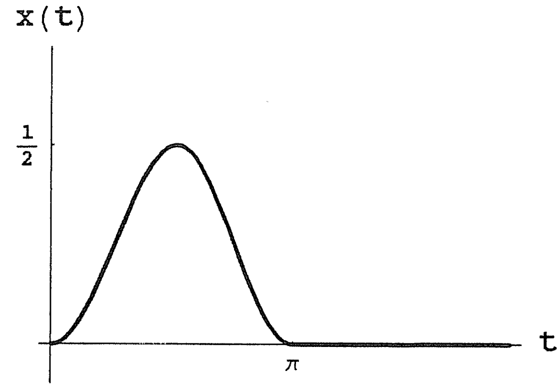
31. $x'' + 4x = 1 - u(t-\pi)$

$$s^2X(s) + 4X(s) = \frac{1-e^{-\pi s}}{s}$$

$$X(s) = \frac{1-e^{-\pi s}}{s\left(s^2+4\right)} = \frac{1}{4}\left(1-e^{-\pi s}\right)\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$x(t) = (1/4)[1 - u(t-\pi)][1 - \cos 2(t-\pi)] = (1/2)[1 - u(t-\pi)]\sin^2 t$

The graph of the position function $x(t)$ is shown at the top of the next page.



32. $x'' + 5x' + 4x = 1 - u(t-2)$

$$s^2X(s) + 5s\,X(s) + 4X(s) = \frac{1-e^{-2s}}{s}$$

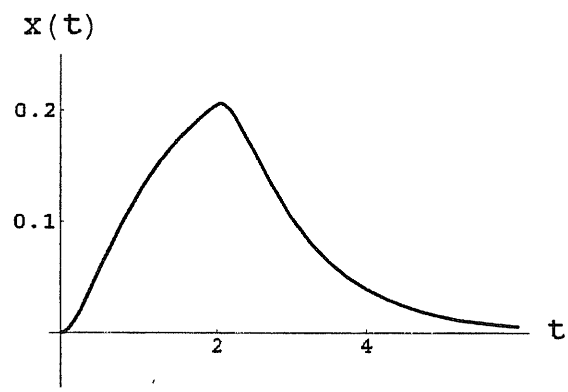
$$X(s) = \frac{1-e^{-2s}}{s\left(s^2+5s+4\right)} = (1 - e^{-2s})G(s)$$

where

$$G(s) = \frac{1}{12}\left(\frac{3}{s}-\frac{4}{s+1}+\frac{1}{s+4}\right), \text{ so } g(t) = \frac{1}{12}\left(3-4e^{-t}+e^{-4t}\right).$$

It follows that

$$x(t) = g(t)-u(t-2)g(t-2) = \begin{cases} g(t) \text{ if } t<2, \\ g(t)-g(t-2) \text{ if } t\geq 2. \end{cases}$$

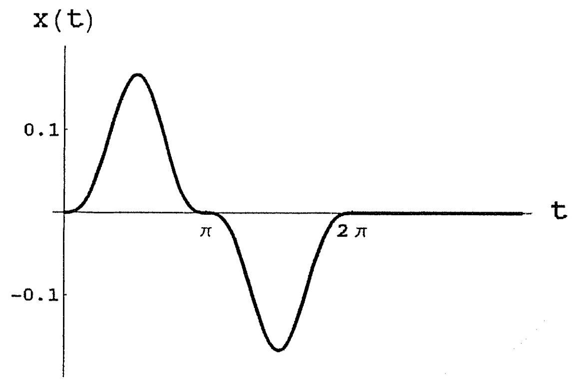


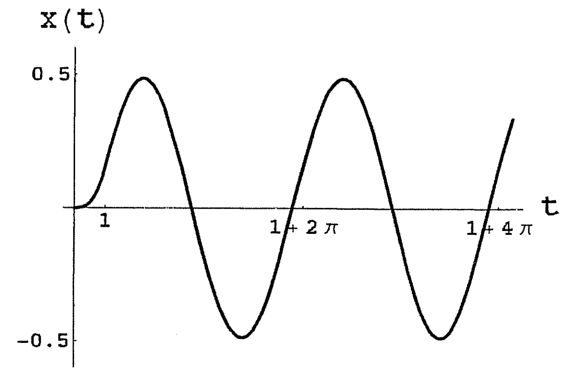
33. $x'' + 9x = [1 - u(t-2\pi)]\sin t$

$$X(s) = \frac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)} = \frac{1}{8}(1-e^{-2\pi s})\left(\frac{1}{s^2+1}-\frac{1}{s^2+9}\right)$$

$$x(t) = \frac{1}{8}[1-u(t-2\pi)]\left(\sin t-\frac{1}{3}\sin 3t\right)$$

The left-hand figure below show the graph of this position function.





34. $x''+x = [1 - u(t-1)]\,t = 1-u(t-1)f(t-1)$, where $f(t) = t+1$

$$s^2X(s)+X(s) = \frac{1}{s^2}-e^{-s}G(s) = \frac{1}{s^2}-e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

It follows that

$$X(s) = \frac{1}{s^2(s^2+1)}-\frac{e^{-s}(s+1)}{s^2(s^2+1)}$$
$$= (1-e^{-s})\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)-e^{-s}\left(\frac{1}{s}-\frac{s}{s^2+1}\right) = (1-e^{-s})G(s)-e^{-s}H(s)$$

where $g(t) = t - \sin t$, $h(t) = 1 - \cos t$. Hence

$$x(t) = g(t) - u(t-1)g(t-1) - u(t-1)h(t-1)$$

and so

$$x(t) = t - \sin t \text{ if } t < 1,$$
$$x(t) = -\sin t + \sin(t-1) + \cos(t-1) \text{ if } t > 1.$$

The right-hand figure above shows the graph of this position function.

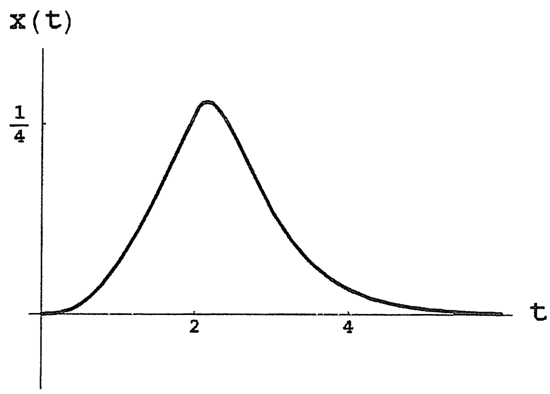
35. $x'' + 4x' + 4x = [1 - u(t-2)]t = t - u(t-2)g(t-2)$ where $g(t) = t+2$

$$(s+2)^2X(s) = \frac{1}{s^2}-e^{-2s}\left(\frac{2}{s}+\frac{1}{s^2}\right)$$

$$X(s) = \frac{1}{s^2(s+2)^2} - e^{-2s}\frac{2s+1}{s^2(s+2)^2}$$

$$= \frac{1}{4}\left(-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+2}+\frac{1}{(s+2)^2}\right)-\frac{1}{4}e^{-2s}\left(\frac{1}{s}+\frac{1}{s^2}-\frac{1}{s+2}-\frac{3}{(s+2)^2}\right)$$

$$x(t) = (1/4)\{-1 + t + (1 + t)e^{-2t} + u(t-2)[1 - t + (3t - 5)e^{-2(t-2)}]\}$$



36. $$100I(s) + 1000\frac{I(s)}{s} = 100\left(\frac{1}{s}-\frac{e^{-s}}{s}\right)$$

$$I(s) = \frac{1-e^{-s}}{s+10} = \left(1-e^{-s}\right)\mathcal{L}\{e^{-10t}\}$$

$$i(t) = e^{-10t} - u(t-1)e^{-10(t-1)}$$

37. $$i'(t) + 10^4\int i(t)\,dt = 100[1 - u(t-2\pi)]$$

$$sI(s) + 10^4\frac{I(s)}{s} = 100\frac{1-e^{-2\pi s}}{s}$$

$$I(s) = \frac{100\left(1-e^{-2\pi s}\right)}{s^2+10^4} = \left(1-e^{-2\pi s}\right)\mathcal{L}\{\sin 100t\}$$

$$i(t) = \sin 100t - u(t-2\pi)\sin 100(t-2\pi) = [1 - u(t-2\pi)]\sin 100t$$

38. $$i'(t) + 10000\int i(t)\,dt = [1 - u(t-\pi)](100\sin 10t)$$

$$sI(s) + 10000\,I(s)/s = 1000(1 - e^{-\pi s})/(s^2 + 100)$$

$$I(s) = \left(1-e^{-\pi s}\right)\cdot\frac{1000s}{\left(s^2+100\right)\left(s^2+10000\right)} = \left(1-e^{-\pi s}\right)\cdot\frac{10}{99}\left(\frac{s}{s^2+10^2}-\frac{s}{s^2+100^2}\right)$$

$$= \frac{10}{99}\left(1-e^{-\pi s}\right)\mathcal{L}\{\cos 10t - \cos 100t\}$$

$$i(t) = \frac{10}{99}\left(\cos 10t - \cos 100t\right) - \frac{10}{99}u(t-\pi)\left[\cos 10(t-\pi) - \cos 100(t-\pi)\right]$$

$$= \frac{10}{99}\left[1-u(t-\pi)\right]\left(\cos 10t - \cos 100t\right)$$

$$i(t) = \begin{cases} \frac{10}{99}\left(\cos 10t - \cos 100t\right) & \text{if } t \le \pi, \\ 0 & \text{if } t > \pi \end{cases}$$

39. $i'(t) + 150\, i(t) + 5000 \int i(t)\, dt = 100t[1 - u(t-1)]$

$$sI(s) + 150I(s) + 5000\frac{I(s)}{s} = \frac{100}{s^2} - 100e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

$$I(s) = \frac{100}{s(s+50)(s+100)} - e^{-s}\cdot\frac{100(s+1)}{s(s+50)(s+100)}$$

$$= \frac{1}{50}\left(\frac{1}{s}-\frac{2}{s+50}+\frac{1}{s+100}\right) - \frac{1}{50}e^{-s}\left(\frac{1}{s}+\frac{98}{s+50}-\frac{99}{s+100}\right)$$

$$i(t) = (1/50)[1 - 2e^{-50t} + e^{-100t}] - (1/50)u(t-1)[1 + 98e^{-50(t-1)} - 99e^{-100(t-1)}]$$

40. $i'(t) + 100\, i(t) + 2500 \int i(t)\, dt = 50t[1 - u(t-1)]$

$$sI(s) + 100I(s) + 2500\frac{I(s)}{s} = \frac{50}{s^2} - 50e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$$

$$I(s) = \frac{50}{s(s+50)^2} - e^{-s}\cdot\frac{50(s+1)}{s(s+50)^2}$$

$$= \frac{1}{50}\left(\frac{1}{s}-\frac{1}{s+50}-\frac{50}{(s+50)^2}\right) - \frac{1}{50}e^{-s}\left(\frac{1}{s}-\frac{50}{s+50}+\frac{2450}{(s+50)^2}\right)$$

$$i(t) = \frac{1}{50}\left(1-e^{-50t}-50t\,e^{-50t}\right) - \frac{1}{50}u(t-1)\left(1-e^{-50(t-1)}+2450t\,e^{-50(t-1)}\right)$$

41. $x'' + 4x = f(t), \qquad x(0) = x'(0) = 0$

$$(s^2+4)X(s) = \frac{4\left(1-e^{-\pi s}\right)}{s\left(1+e^{-\pi s}\right)} \qquad \text{(by Example 6 of Section 4.5)}$$

$$(s^2+4)X(s) = \frac{4}{s}+\frac{8}{s}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s} \qquad \text{(as in Eq. (16) of Section 4.5)}$$

Now let

$$g(t) = \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)}\right\} = 1-\cos 2t = 2\sin^2 t.$$

Then it follows that

$$x(t) = g(t)+2\sum_{n=1}^{\infty}(-1)^n u_{n\pi}(t)g(t-n\pi) = 2\sin^2 t+4\sum_{n=1}^{\infty}(-1)^n u_{n\pi}(t)\sin^2 t.$$

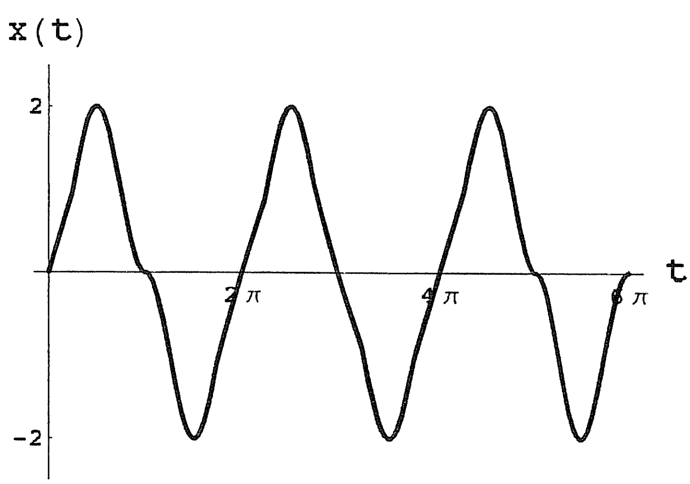
Hence

$$x(t) = \begin{cases} 2\sin^2 t & \text{if } 2n\pi \le t < (2n+1)\pi, \\ -2\sin^2 t & \text{if } (2n-1)\pi \le t < 2n\pi. \end{cases}$$

Consequently the complete solution

$$x(t) = 2|\sin t|\sin t$$

is periodic, so the transient solution is zero. The graph of $x(t)$:



42. $x'' + 2x' + 10x = f(t), \quad x(0) = x'(0) = 0$

As in the solution of Example 8 we find first that

$$\left(s^2+2s+10\right)X(s) = \frac{10}{s}+\frac{20}{s}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s},$$

so

$$X(s) = \frac{10}{s(s^2+2s+10)}+2\sum_{n=1}^{\infty}\frac{10(-1)^n e^{-n\pi s}}{s(s^2+2s+10)}.$$

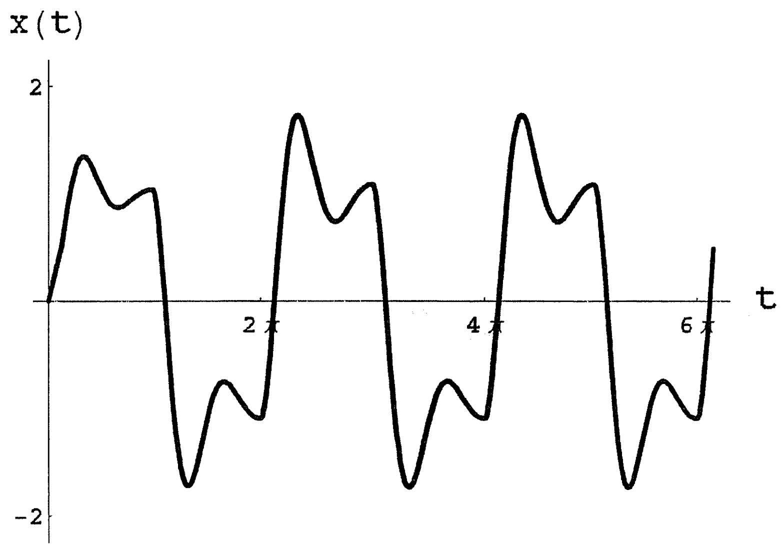
If

$$g(t) = \mathcal{L}^{-1}\left\{\frac{10}{s\left[(s+1)^2+9\right]}\right\} = 1-\frac{1}{3}e^{-t}\left(3\cos 3t+\sin 3t\right),$$

then it follows that

$$x(t) = g(t)+2\sum_{n=1}^{\infty}(-1)^n u_{n\pi}(t)g(t-n\pi).$$

The graph of $x(t)$:



SECTION 4.6

IMPULSES AND DELTA FUNCTIONS

Among the several ways of introducing delta functions, we consider the physical approach of the first two pages of this section to be the most tangible one for elementary students. Whatever the approach, however, the practical consequences are the same — as described in the discussion associated with equations (11)–(19) in the text. That is, in order to solve a differential equation of the form

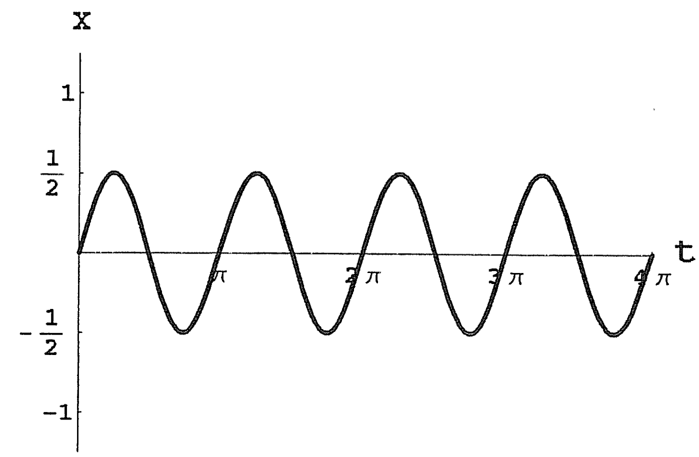
$$a\,x''(t) + b\,x'(t) + c\,x(t) = f(t)$$

where $f(t)$ involves delta functions, we transform the equation using the operational principle $\mathcal{L}\{\delta_a(t)\} = e^{-as}$, then solve for $X(s)$, and finally invert as usual to find the formal solution $x(t)$. Then we show the graph of the inverse transform $x(t)$.

1. $s^2X(s) + 4X(s) = 1$

$$X(s) = \frac{1}{s^2+4}$$

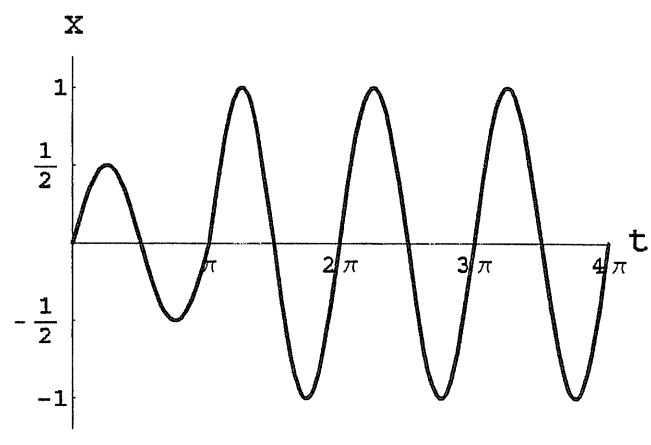
$$x(t) = \frac{1}{2}\sin 2t$$



2. $s^2X(s) + 4X(s) = 1 + e^{-\pi s}$

$$X(s) = \frac{1+e^{-\pi s}}{s^2+4}$$

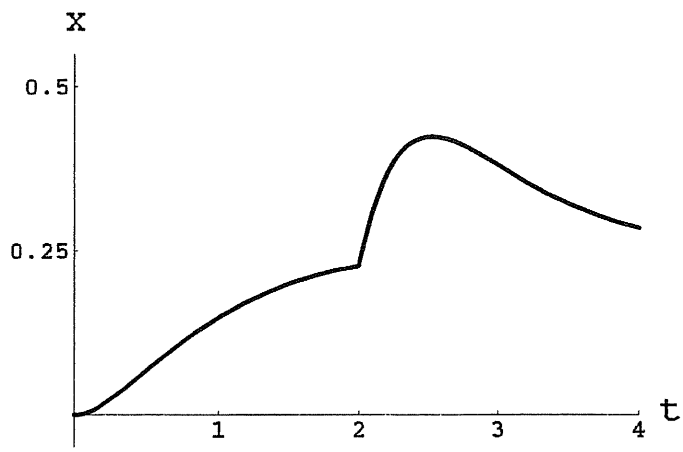
$$x(t) = \frac{1}{2}\left[1+u(t-\pi)\right]\sin 2t = \begin{cases} \frac{1}{2}\sin 2t & \text{if } t \le \pi, \\ \sin 2t & \text{if } t > \pi. \end{cases}$$



3. $s^2X(s) + 4sX(s) + 4X(s) = \dfrac{1}{s} + e^{-2s}$

$$X(s) = \frac{1}{s(s+2)^2} + \frac{e^{-2s}}{(s+2)^2} = \frac{1}{4}\left(\frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}\right) + \frac{e^{-2s}}{(s+2)^2}$$

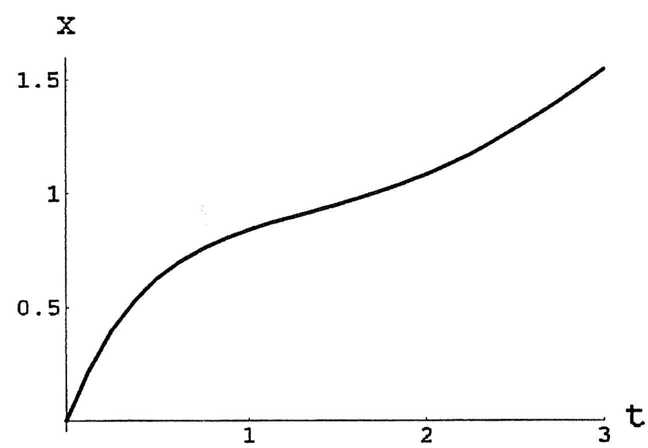
$$x(t) = \frac{1}{4}\left[1 - e^{-2t} - 2te^{-2t}\right] + u(t-2)(t-2)e^{-2(t-2)}$$



4. $[s^2X(s) - 1] + 2sX(s) + X(s) = 1 + \dfrac{1}{s^2}$

$$X(s) = \frac{2s^2+1}{s^2(s+1)^2} = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{3}{(s+1)^2}$$

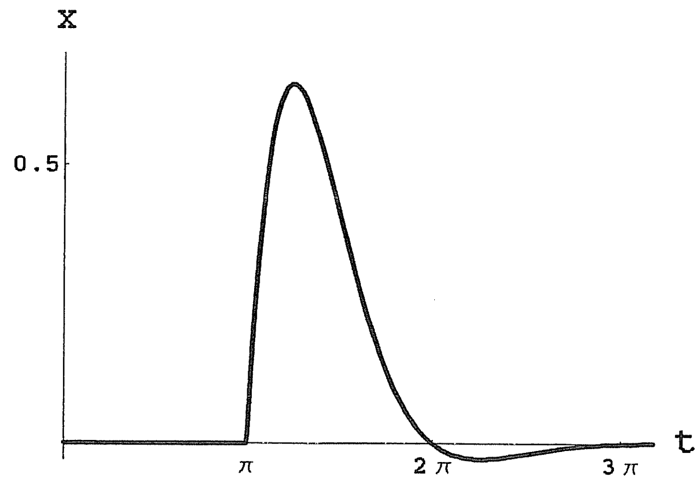
$$x(t) = -2 + t + 2e^{-t} + 3te^{-t}$$



5. $(s^2 + 2s + 2)X(s) = 2e^{-\pi s}$

$$X(s) = \frac{2e^{-\pi s}}{(s+1)^2+1}$$

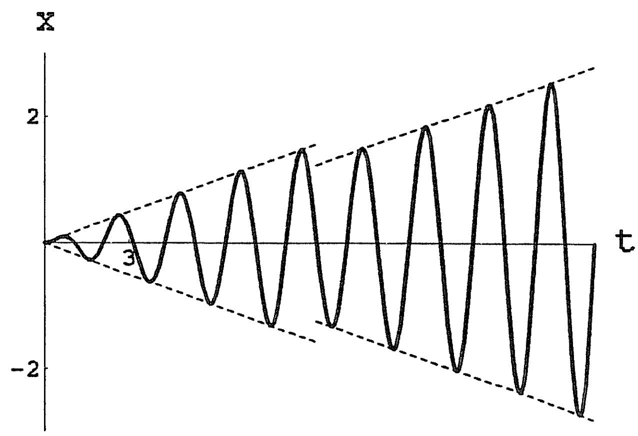
$$x(t) = 2u(t-\pi)e^{-(t-\pi)}\sin(t-\pi) = \begin{cases} 0 & \text{if} \quad 0 \le t \le \pi, \\ -2e^{-(t-\pi)}\sin t & \text{if} \quad t \ge \pi. \end{cases}$$



6. $s^2X(s) + 9X(s) = e^{-3\pi s} + \dfrac{s}{s^2+9}$

$$X(s) = \frac{s}{\left(s^2+9\right)^2} + \frac{e^{-3\pi s}}{s^2+9}$$

$$x(t) = \frac{1}{6}t\sin 3t + \frac{1}{3}u(t-3\pi)\sin 3(t-3\pi) = \frac{1}{6}t\sin 3t - \frac{1}{3}u(t-3\pi)\sin 3t$$

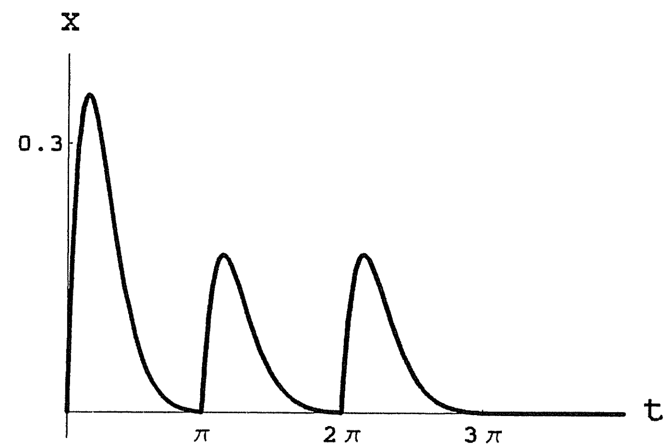


7. $[s^2X(s) - 2] + 4sX(s) + 5X(s) = e^{-\pi s} + e^{-2\pi s}$

$$X(s) = \frac{2 + e^{-\pi s} + e^{-2\pi s}}{(s+2)^2 + 1}$$

$$x(t) = 2e^{-2t}\sin t + u_\pi(t)e^{-2(t-\pi)}\sin(t - \pi) + u_{2\pi}(t)e^{-2(t-2\pi)}\sin(t - 2\pi)$$

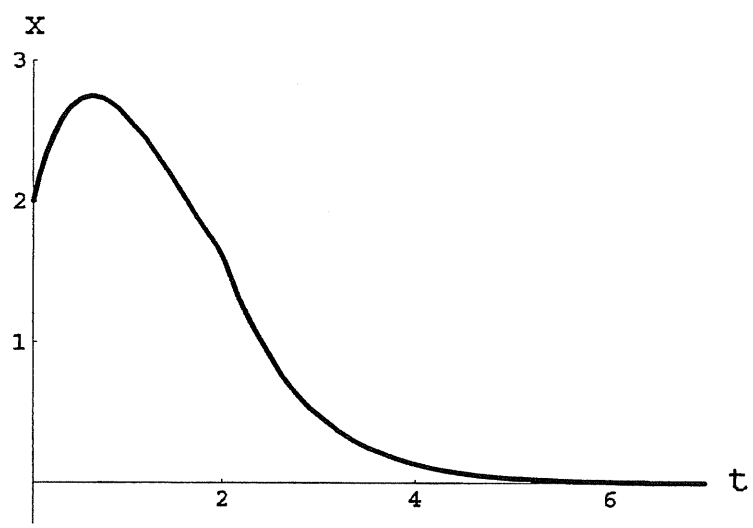
$$= [2 - e^{2\pi}u(t-\pi) + e^{4\pi}u(t-2\pi)]\, e^{-2t}\sin t$$



8. $[s^2X(s) - 2s - 2] + 2[sX(s) - 2] + X(s) = 1 - e^{-2s}$

$$X(s) = \frac{2s + 7 - e^{-2s}}{(s+1)^2} = \frac{2}{s+1} + \frac{5}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}$$

$$x(t) = (2 + 5t)e^{-t} - u(t-2)(t-2)e^{-(t-2)}$$



9. $s^2X(s) + 4X(s) = F(s)$

$$X(s) = \frac{1}{s^2+4}\cdot F(s)$$

$$x(t) = \tfrac{1}{2}\int_0^t (\sin 2u)\, f(t-u)\, du$$

10. $s^2X(s) + 6s\, X(s) + 9X(s) = F(s)$

$$X(s) = \frac{1}{(s+3)^2}\cdot F(s)$$

$$x(t) = \int_0^t ue^{-3u} f(t-u)\, du$$

11. $(s^2 + 6s + 8)X(s) = F(s)$

$$X(s) = \frac{1}{(s+3)^2-1}\cdot F(s)$$

$$x(t) = \int_0^t e^{-3u}(\sinh u)\, f(t-u)\, du$$

12. $s^2X(s) + 4sX(s) + 8X(s) = F(s)$

$$X(s) = \frac{1}{(s+2)^2+4}\cdot F(s)$$

$$x(t) = \tfrac{1}{2}\int_0^t e^{-2u}(\sin 2u)\, f(t-u)\, du$$

13. **(a)** $mx\varepsilon''(t) = (p/\varepsilon)[u_0(t) - u_\varepsilon(t)]$

$$ms^2X_\varepsilon(s) = (p/\varepsilon)[1/s - e^{-\varepsilon s}/s]$$

$$mX_\varepsilon(s) = (p/\varepsilon)[(1 - e^{-\varepsilon s})/s^3]$$

$$mx_\varepsilon(t) = (p/2\varepsilon)[t^2 - u_\varepsilon(t)(t-\varepsilon)^2]$$

(b) If $t \geq \varepsilon$ then

$$mx_\varepsilon(t) = (p/2\varepsilon)[t^2 - (t^2 - 2\varepsilon t + \varepsilon^2)] = (p/2\varepsilon)(2\varepsilon t - \varepsilon^2).$$

Hence $mx_\varepsilon(t) \to pt$ as $\varepsilon \to 0$.

(c) $mv = (mx)' = (pt)' = p.$

14. $sX(s) = e^{-as}$; $X(s) = e^{-as}/s$; $x(t) = u(t-a)$

15. Each of the two given initial value problems transforms to

$$(ms^2 + k)X(s) = mv_0 = p_0.$$

16. Each of the two given initial value problems transforms to

$$(as^2 + bs + c)X(s) = F(s) + av_0$$

17. **(b)** $i' + 100i = \delta_1(t) - \delta_2(t),\ \ i(0) = 0$

$$I(s) = \frac{e^{-s} - e^{-2s}}{s+100} I(s)$$

$$i(t) = u_1(t)e^{-100(t-1)} - u_2(t)e^{-100(t-2)}$$

18. **(b)** $i''(t) + 100\, i(t) = 10\,\delta(t) - 10\,\delta(t-\pi)$

$$(s^2 + 100)\, I(s) = 10 - 10\, e^{-\pi s}$$

$$I(s) = \frac{10}{s^2+100} - \frac{10e^{-\pi s}}{s^2+100}$$

$$i(t) = \sin 10t - u_\pi(t)\sin 10(t-\pi)$$

$$= [1 - u(t-\pi)]\sin 10t = \begin{cases} \sin 10t & \text{if } t \le \pi, \\ 0 & \text{if } t \ge \pi \end{cases}$$

19. $\left(s^2+100\right)I(s) = 10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/10}$

$$I(s) = \frac{10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/10}}{s^2+100} = \sum_{n=0}^{\infty}\left((-1)^n e^{-n\pi s/10}\cdot\frac{10}{s^2+100}\right)$$

$$i(t) = \sum_{n=0}^{\infty}(-1)^n u_{n\pi/10}(t)\sin 10(t - n\pi/10) = \sum_{n=0}^{\infty} u(t - n\pi/10)\sin 10t$$

because $\sin(10t - n\pi) = (-1)^n \sin 10t$. Hence

$$i(t) = (n+1)\sin 10t$$

if $n\pi/10 < t < (n+1)\pi/10$.

20. $\left(s^2+100\right)I(s) \;=\; 10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/5}$

$$I(s) \;=\; \frac{10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/5}}{s^2+100} \;=\; \sum_{n=0}^{\infty}\left((-1)^n e^{-n\pi s/5}\cdot\frac{10}{s^2+100}\right)$$

$$i(t) \;=\; \sum_{n=0}^{\infty}(-1)^n u_{n\pi/5}(t)\sin 10(t-n\pi/5) \;=\; \sum_{n=0}^{\infty}(-1)^n u(t-n\pi/5)\sin 10t$$

Hence

$$i(t) \;=\; \sin 10t + (-1)^1\sin 10t + \cdots + (-1)^n\sin 10t$$

if $n\pi/5 \;<\; t \;<\; (n+1)\pi/5,\; n\geq 0.$ Thus $i(t) = \sin 10t$ in this interval if n is even, but is zero in this interval if n is odd.

21. $\left(s^2+60s+1000\right)I(s) \;=\; 10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/10}$

$$I(s) \;=\; \frac{10\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s/10}}{s^2+60s+1000} \;=\; \sum_{n=0}^{\infty}\left((-1)^n e^{-n\pi s/10}\cdot\frac{10}{(s+30)^2+100}\right)$$

$i(t) \;=\; \Sigma\,(-1)^n u_{n\pi/10}(t)\, g(t - n\pi/10)$

where $g(t) \;=\; e^{-30t}\sin 10t,$ and so

$$\begin{aligned} g(t - n\pi/10) \;&=\; \exp[-30(t - n\pi/10)]\,\sin 10(t - n\pi/10) \\ &=\; e^{3n\pi}e^{-30t}\cdot(-1)^n\sin 10t \end{aligned}$$

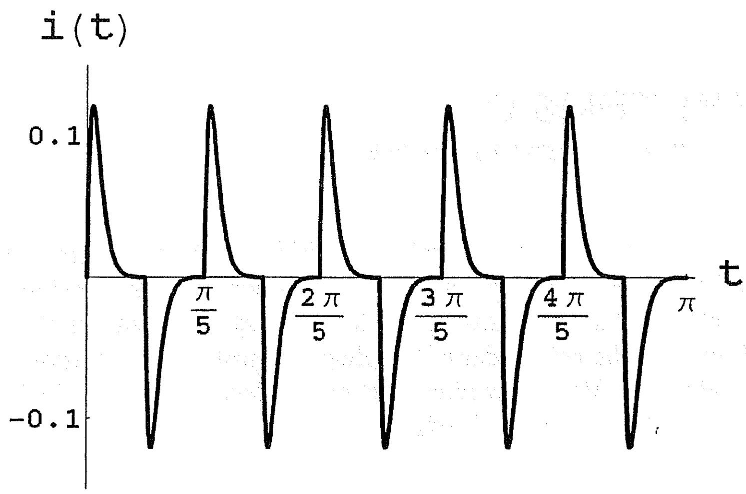
Therefore

$$i(t) \;=\; \sum_{n=0}^{\infty} u(t-n\pi/10)\,e^{3n\pi}e^{-30t}\sin t.$$

If $n\pi/10 < t < (n+1)\pi/10$ then it follows that

$$i(t) \;=\; (1 + e^{3\pi} + \cdots + e^{3n\pi})e^{-30t}\sin 10t \;=\; \frac{e^{(3n+1)\pi}-1}{e^{3\pi}-1}e^{-30t}\sin 10t.$$

The graph of $i(t)$ is shown at the top of the next page.

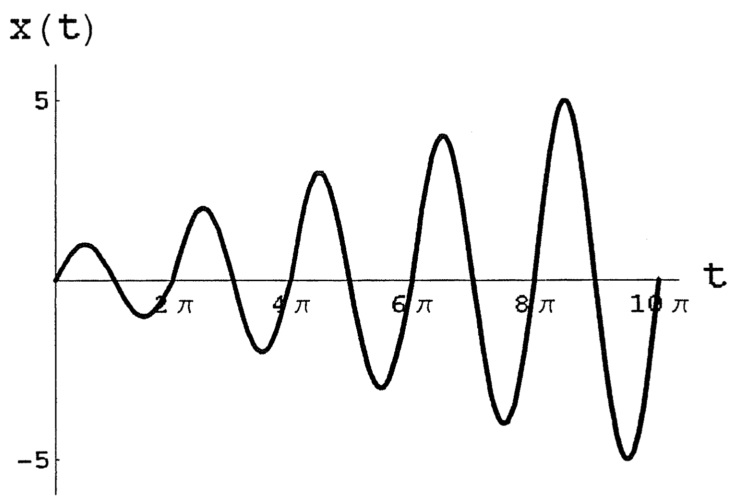


22. $(s^2+1)X(s) = \sum_{n=0}^{\infty} e^{-2n\pi s}$

$$X(s) = \sum_{n=0}^{\infty} \frac{e^{-2n\pi s}}{s^2+1}$$

$$x(t) = \sum_{n=0}^{\infty} u_{2n\pi}(t)\sin(t-2n\pi) = \sum_{n=0}^{\infty} u(t-2n\pi)\sin t$$

Hence $x(t) = (n+1)\sin t$ if $2n\pi < t < 2(n+1)\pi$. The graph of $x(t)$.



CHAPTER 5

LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

This chapter is designed to offer considerable flexibility in the treatment of linear systems, depending on the background in linear algebra that students are assumed to have. Sections 5.1 and 5.2 can stand alone as a brief introduction to linear systems without the use of linear algebra and matrices. However, the remainder of the chapter employs the notation and terminology of elementary linear algebra. For ready reference and review, Section 5.3 includes a complete and self-contained account of the needed background of determinants, matrices, and vectors. The additional linear algebra that is needed in subsequent sections is introduced along the way.

SECTION 5.1

FIRST-ORDER SYSTEMS AND APPLICATIONS

1. Let $x_1 = x$ and $x_2 = x_1' = x'$, so $x_2' = x'' = -7x - 3x' + t^2$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = -7x_1 - 3x_2 + t^2$

2. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, $x_4 = x_3' = x'''$, so $x_4' = x^{(4)} = x + 3x' - 6x'' + \cos 3t$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_4, \qquad x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t$

3. Let $x_1 = x$ and $x_2 = x_1' = x'$, so $x_2' = x'' = \left[\left(1 - t^2\right)x - tx'\right]/t^2$.

Equivalent system:

$x_1' = x_2, \qquad t^2 x_2' = (1 - t^2)x_1 - tx_2$

4. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, so $x_3' = x''' = \left(-5x - 3tx' - 2t^2x'' + \ln t\right)/t^3$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = x_3, \qquad t^3 x_3' = -5x_1 - 3tx_2 + 2t^2x_3 + \ln t$

5. Let $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2' = x''$, so $x_3' = x''' = \left(x'\right)^2 + \cos x$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_2^2 + \cos x_1$

6. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = 5x - 4y,\ y_2' = y'' = -4x + 5y$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = 5x_1 - 4y_1$

$y_1' = y_2, \qquad y_2' = -4x_1 + 5y_1$

7. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = -kx/(x^2+y^2)^{3/2}$,
$y_2' = y'' = -ky/(x^2+y^2)^{3/2}$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = -kx_1/\left(x_1^2+y_1^2\right)^{3/2}$

$y_1' = y_2, \qquad y_2' = -ky_1/\left(x_1^2+y_1^2\right)^{3/2}$

8. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = -4x + 2y - 3x'$,
$y_2' = y'' = 3x - y - 2y' + \cos t$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = -4x_1 + 2y_1 - 3x_2$

$y_1' = y_2, \qquad y_2' = 3x_1 - y_1 - 2y_2 + \cos t$

9. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y',\ z_1 = z,\ z_2 = z_1' = z'$, so
$x_2' = x'' = 3x - y + 2z,\ y_2' = y'' = x + y - 4z,\ z_2' = z'' = 5x - y - z$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = 3x_1 - y_1 + 2z_1$

$y_1' = y_2, \qquad y_2' = x_1 + y_1 - 4z_1$

$z_1' = z_2, \qquad z_2' = 5x_1 - y_1 - z_1$

10. Let $x_1 = x,\ x_2 = x_1' = x',\ y_1 = y,\ y_2 = y_1' = y'$ so $x_2' = x'' = x(1-y),\ y_2' = y'' = y(1-x)$.

Equivalent system:

$x_1' = x_2, \qquad x_2' = x_1(1 - y_1)$

$y_1' = y_2 \qquad y_2' = y_1(1 - x_1)$

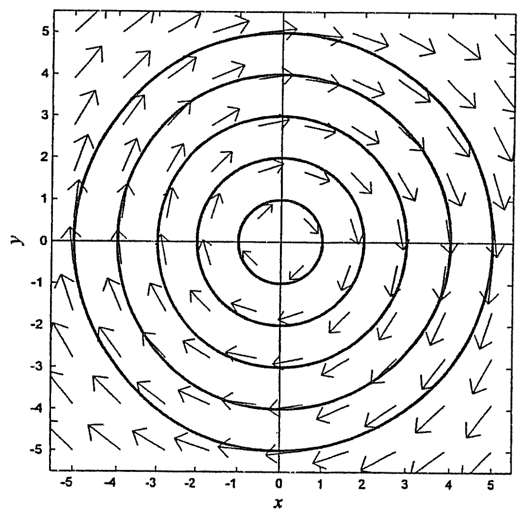
11. The computation $x'' = y' = -x$ yields the single linear second-order equation $x''+x = 0$ with characteristic equation $r^2+1 = 0$ and general solution

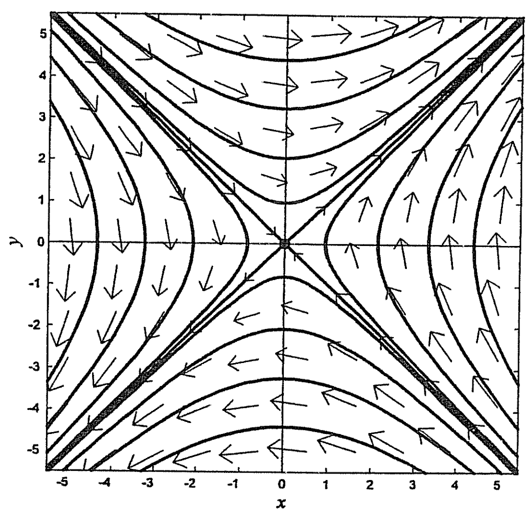
$$x(t) = A\cos t + B\sin t.$$

Then the original first equation $y = x'$ gives

$$y(t) = B\cos t - A\sin t.$$

The figure on the left below shows a direction field and typical solution curves (obviously circles?) for the given system.





12. The computation $x'' = y' = x$ yields the single linear second-order equation $x''-x = 0$ with characteristic equation $r^2-1 = 0$ and general solution

$$x(t) = A\,e^{t} + B\,e^{-t}.$$

Then the original first equation $y = x'$ gives

$$y(t) = A\,e^{t} - B\,e^{-t}.$$

The figure on the right above shows a direction field and some typical solution curves of this system. It appears that the typical solution curve is a branch of a hyperbola.

13. The computation $x'' = -2y' = -4x$ yields the single linear second-order equation $x''+4x = 0$ with characteristic equation $r^2+4 = 0$ and general solution

$$x(t) = A\cos 2t + B\sin 2t.$$

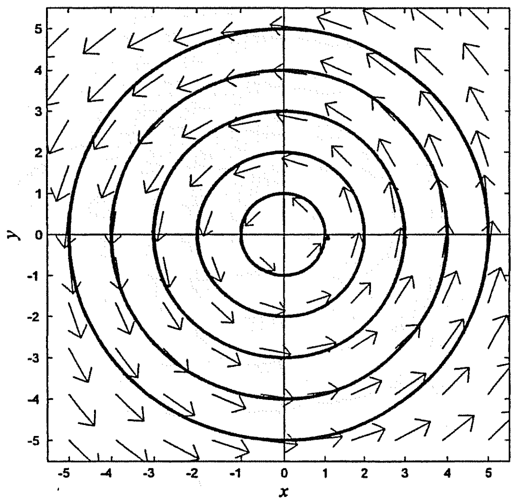
Then the original first equation $y = -x'/2$ gives

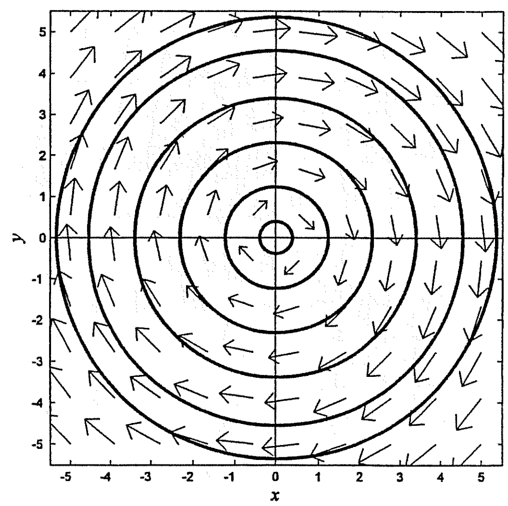
$$y(t) = -B\cos 2t + A\sin 2t.$$

Finally, the condition $x(0) = 1$ implies that $A = 1$, and then the condition $y(0) = 0$ gives $B = 0$. Hence the desired particular solution is given by

$$x(t) = \cos 2t, \qquad y(t) = \sin 2t.$$

The figure on the left below shows a direction field and some typical circular solution curves for the given system.





14. The computation $x'' = 10y' = -100x$ yields the single linear second-order equation $x'' + 100x = 0$ with characteristic equation $r^2 + 100 = 0$ and general solution

$$x(t) = A\cos 10t + B\sin 10t.$$

Then the original first equation $y = x'/10$ gives

$$y(t) = B\cos 10t - A\sin 10t.$$

Finally, the condition $x(0) = 3$ implies that $A = 3$, and then the condition $y(0) = 4$ gives $B = 4$. Hence the desired particular solution is given by

$$x(t) = 3\cos 10t + 4\sin 10t,$$
$$y(t) = 4\cos 10t - 3\sin 10t.$$

The typical solution curve is a circle. See the figure on the right above.

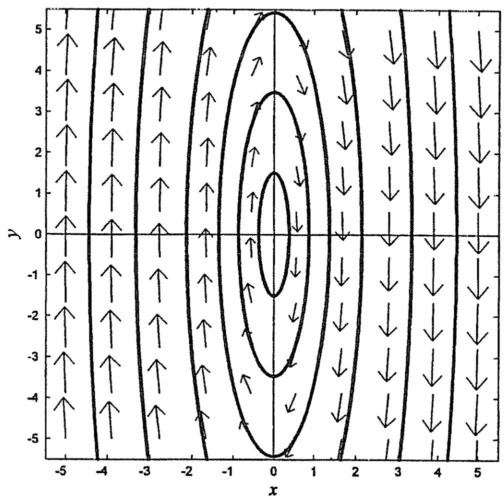
15. The computation $x'' = y'/2 = -4x$ yields the single linear second-order equation $x'' + 4x = 0$ with characteristic equation $r^2 + 4 = 0$ and general solution

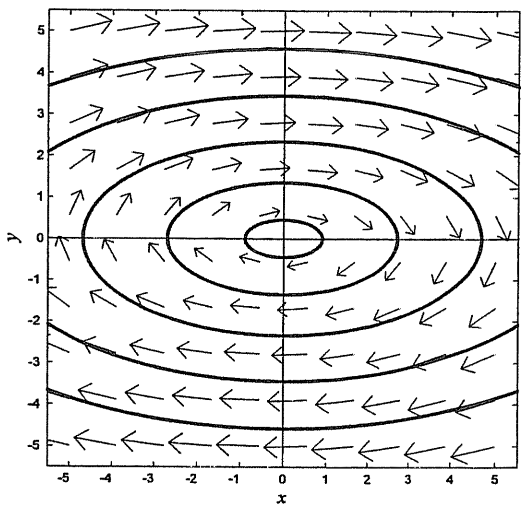
$$x(t) = A\cos 2t + B\sin 2t.$$

Then the original first equation $y = 2x'$ gives

$$y(t) = 4B\cos 2t - 4A\sin 2t.$$

The figure on the left below shows a direction field and some typical elliptical solution curves.





16. The computation $x'' = 8y' = -16x$ yields the single linear second-order equation $x'' + 16x = 0$ with characteristic equation $r^2 + 16 = 0$ and general solution

$$x(t) = A\cos 4t + B\sin 4t.$$

Then the original first equation $y = x'/8$ gives

$$y(t) = (B/2)\cos 4t - (A/2)\sin 4t.$$

The typical solution curve is an ellipse. The figure on the right above shows a direction field and some typical solution curves.

17. The computation $x'' = y' = 6x - y = 6x - x'$ yields the single linear second-order equation $x'' + x' - 6x = 0$ with characteristic equation $r^2 + r - 6 = 0$ and characteristic roots $r = -3$ and 2, so the general solution

$$x(t) = A\,e^{-3t} + B\,e^{2t}.$$

Then the original first equation $y = x'$ gives

$$y(t) = -3A\,e^{-3t} + 2B\,e^{2t}.$$

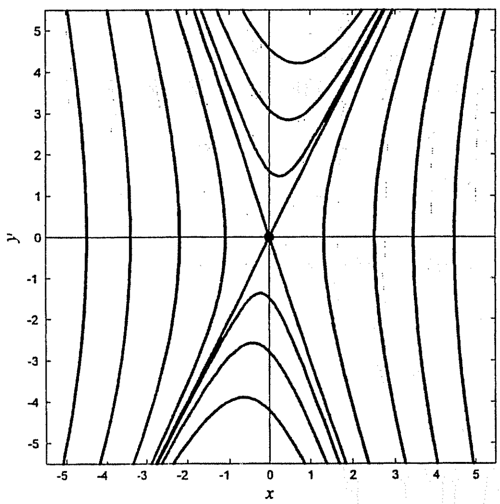
Finally, the initial conditions

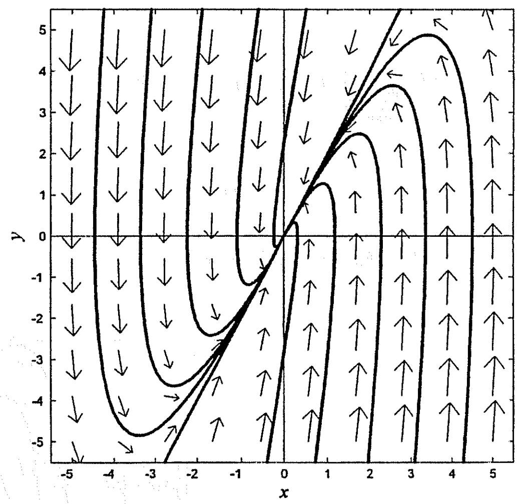
$$x(0) = A + B = 1, \quad y(0) = -3A + 2B = 2$$

imply that $A = 0$ and $B = 1$, so the desired particular solution is given by

$$x(t) = e^{-3t}, \qquad y(t) = 2\,e^{2t}.$$

The figure on the left below shows a direction field and some typical solution curves.





18. The computation $x'' = -y' = -10x + 7y = -10x - 7x'$ yields the single linear second-order equation $x'' + 7x' + 10x = 0$ with characteristic equation $r^2 + 7r + 10 = 0$, characteristic roots $r = -2$ and -5, and general solution

$$x(t) = A\,e^{-2t} + B\,e^{-5t}.$$

Then the original first equation $y = -x'$ gives

$$y(t) = 2A\,e^{-2t} + 5B\,e^{-5t}.$$

Finally, the initial conditions

$$x(0) = A + B = 2, \quad y(0) = 2A + 5B = -7$$

imply that $A = 17/3$, $B = -11/3$, so the desired particular solution is given by

$$x(t) = (17\,e^{-2t} - 11\,e^{-5t})/3, \qquad y(t) = (34e^{-2t} - 55e^{-5t})/3.$$

It appears that the typical solution curve is tangent to the straight line $y = 2x$. See the right-hand figure on the preceding page for a direction field and typical solution curves.

19. The computation $x'' = -y' = -13x - 4y = -13x + 4x'$ yields the single linear second-order equation $x'' - 4x' + 13x = 0$ with characteristic equation $r^2 - 4r + 13 = 0$ and characteristic roots $r = 2 \pm 3i$, hence the general solution is

$$x(t) = e^{2t}(A\cos 3t + B\sin 3t).$$

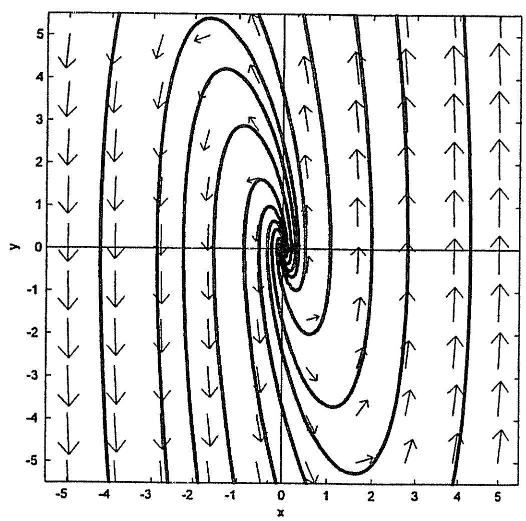
The initial condition $x(0) = 0$ then gives $A = 0$, so $x(t) = B\,e^{2t}\sin 3t$. Then the original first equation $y = -x'$ gives

$$y(t) = -e^{2t}(3B\cos 3t + 2B\sin 3t).$$

Finally, the initial condition $y(0) = 3$ gives $B = -1$, so the desired particular solution is given by

$$x(t) = -e^{2t}\sin 3t, \qquad y(t) = e^{2t}(3\cos 3t + 2\sin 3t).$$

The figure below shows a direction field and some typical solution curves.



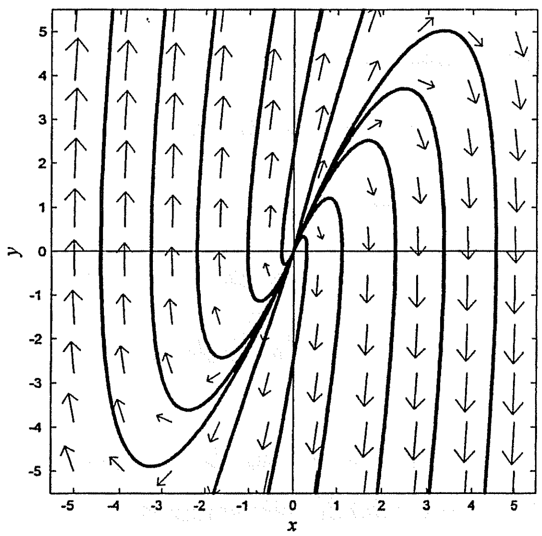
20. The computation $x'' = y' = -9x + 6y = -9x + 6x'$ yields the single linear second-order equation $x'' - 6x' + 9x = 0$ with characteristic equation $r^2 - 6r + 9 = 0$ and repeated characteristic root $r = 3, 3$, so its general solution is given by

$$x(t) = (A + Bt)e^{3t}.$$

Then the original first equation $y = x'$ gives

$$y(t) = (3A + B + 3Bt)e^{3t}.$$

It appears that the typical solution curve is tangent to the straight line $y = 3x$. The figure below shows a direction field and some typical solution curves.



21. **(a)** Substituting the general solution found in Problem 11 we get

$$\begin{aligned} x^2 + y^2 &= (A\cos t + B\sin t)^2 + (B\cos t - A\sin t)^2 \\ &= (A^2 + B^2)(\cos^2 t + \sin^2 t) = A^2 + B^2 \\ x^2 + y^2 &= C^2, \end{aligned}$$

the equation of a circle of radius $C = (A^2 + B^2)^{1/2}$.

(b) Substituting the general solution found in Problem 12 we get

$$x^2 - y^2 = (Ae^t + Be^{-t})^2 - (Ae^t - Be^{-t})^2 = 4AB,$$

the equation of a hyperbola.

22. **(a)** Substituting the general solution found in Problem 13 we get

$$\begin{aligned} x^2 + y^2 &= (A\cos 2t + B\sin 2t)^2 + (-B\cos 2t + A\sin 2t)^2 \\ &= (A^2 + B^2)(\cos^2 2t + \sin^2 2t) = A^2 + B^2 \\ x^2 + y^2 &= C^2, \end{aligned}$$

the equation of a circle of radius $C = (A^2 + B^2)^{1/2}$.

(b) Substituting the general solution found in Problem 15 we get

$$\begin{aligned} 16x^2 + y^2 &= 16(A\cos 2t + B\sin 2t)^2 + (4B\cos 2t - 4A\sin 2t)^2 \\ &= 16(A^2 + B^2)(\cos^2 2t + \sin^2 2t) = 16(A^2 + B^2) \\ 16x^2 + y^2 &= C^2, \end{aligned}$$

the equation of an ellipse with semi-axes 1 and 4. .

23. When we solve Equations (20) and (21) in the text for e^{-t} and e^{2t} we get

$$2x - y = 3Ae^{-t} \text{and} \qquad x + y = 3Be^{2t}.$$

Hence

$$(2x - y)^2(x + y) = (3Ae^{-t})^2(3Be^{2t}) = 27A^2B = C.$$

Clearly $y = 2x$ or $y = -x$ if $C = 0$, and expansion gives the equation $4x^3 - 3xy^2 + y^3 = C$.

24. Looking at Fig. 5.1.11 in the text, we see that the first spring is stretched by x_1, the second spring is stretched by $x_2 - x_1$, and the third spring is compressed by x_2. Hence Newton's second law gives $m_1x_1'' = -k_1(x_1) + k_2(x_2 - x_1)$ and $m_2x_2'' = -k_2(x_2 - x_1) - k_3(x_2)$.

25. Looking at Fig. 5.1.12 in the text, we see that

$$my_1'' = -T\sin\theta_1 + T\sin\theta_2 \approx -T\tan\theta_1 + T\tan\theta_2 = -Ty_1/L + T(y_2 - y_1)/L,$$

$$my_2'' = -T\sin\theta_2 - T\sin\theta_3 \approx -T\tan\theta_2 - T\tan\theta_3 = -T(y_2 - y_1)/L - Ty_2/L.$$

We get the desired equations when we multiply each of these equations by L/T and set $k = mL/T$.

26. The concentration of salt in tank i is $c_i = x_i/100$ for $i = 1, 2, 3$ and each inflow-outflow rate is $r = 10$. Hence

$$x_1' = -rc_1 + rc_3 = \tfrac{1}{10}(-x_1 + x_3),$$

$$x_2' = +rc_1 - rc_2 = \tfrac{1}{10}(x_1 - x_2),$$

$$x_3' = +rc_2 - rc_3 = \tfrac{1}{10}(x_2 - x_3).$$

27. We apply Kirchhoff's law to each loop in Figure 5.1.14 in the text, and immediately get the equations

$$2(I_1' - I_2') + 50I_1 = 100\sin 60t, \qquad 2(I_2' - I_1') + 25I_2 = 0.$$

28. First we apply Kirchhoff's law to each loop in Figure 5.1.14 in the text, denoting by Q the charge on the capacitor, and get the equations

$$50I_1 + 1000Q = 100, \qquad 25I_2 - 1000Q = 0.$$

Then we differentiate each equation and substitute $Q' = I_1 - I_2$ to get the system

$$I_1' = -20(I_1 - I_2), \qquad I_2' = 40(I_1 - I_2).$$

29. If θ is the polar angular coordinate of the point (x, y) and we write $F = k/\left(x^2+y^2\right) = k/r^2$, then Newton's second law gives

$$\begin{aligned} mx'' &= -F\cos\theta = -(k/r^2)(x/r) = -kx/r^3, \\ my'' &= -F\sin\theta = -(k/r^2)(y/r) = -ky/r^3, \end{aligned}$$

30. If we write (x', y') for the velocity vector and $v = \sqrt{(x')^2 + (y')^2}$ for the speed, then $(x'/v, y'/v)$ is a unit vector pointing in the direction of the velocity vector, and so the components of the air resistance force F_r are given by

$$F_r = -kv^2(x'/v, y'/v) = (-kvx', -kvy').$$

31. If $\mathbf{r} = (x, y, z)$ is the particle's position vector, then Newton's law $m\mathbf{r}'' = \mathbf{F}$ gives

$$m\mathbf{r}'' = q\mathbf{v}\times\mathbf{B} = q\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & z' \\ 0 & 0 & B \end{vmatrix} = +qBy'\mathbf{i} - qBx'\mathbf{j} = qB(-y', x', 0).$$

SECTION 5.2

THE METHOD OF ELIMINATION

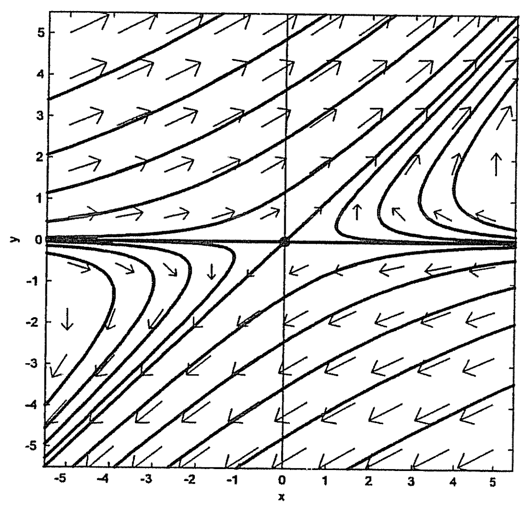
1. The second differential equation $y' = 2y$ has the exponential solution

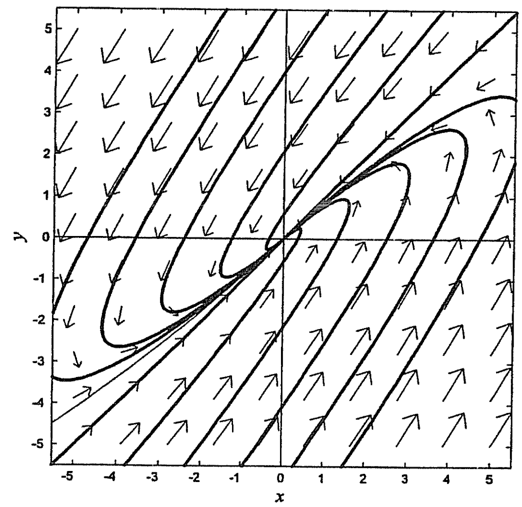
$$y(t) = c_2 e^{2t}.$$

Substitution in the first differential equation gives the linear first-order equation $x' + x = 3c_2e^{2t}$ with integrating factor $\rho = e^t$. Solution of this equation in the usual way gives

$$x(t) = e^{-t}\left(c_1 + c_2e^{3t}\right) = c_1e^{-t} + c_2e^{2t}.$$

The left-hand figure below shows a direction field and some typical solution curves.





2. From the first differential equation we get $y = (x - x')/2$, so $y' = (x' - x'')/2$. Substitution of these expressions for y and y' into the second differential equation yields the second-order equation

$$x'' + 2x' + x = 0$$

with general solution

$$x(t) = (c_1 + c_2t)e^{-t}.$$

Substitution in $y = (x - x')/2$ now yields

$$y(t) = (c_1 - c_2/2 + c_2t)e^{-t}.$$

The right-hand figure above shows a direction field and some typical solution curves.

3. From the first differential equation we get $y = (3x + x')/2$, so $y' = (3x' + x'')/2$. Substitution of these expressions for y and y' into the second differential equation yields the second-order equation

$$x'' - x' + 6x = 0$$

with general solution

$$x(t) = c_1e^{-2t} + c_2e^{3t}.$$

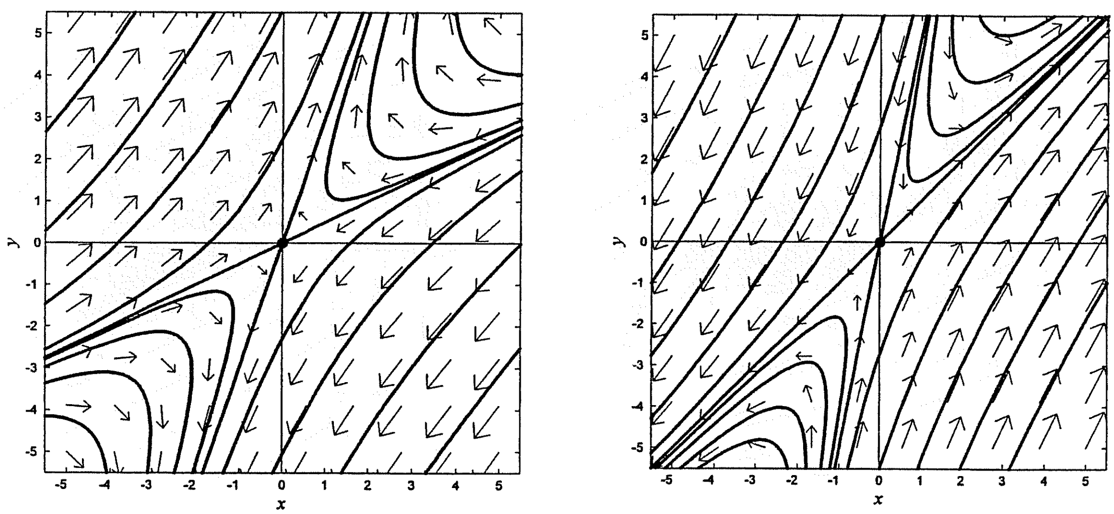
Substitution in $y = (3x + x')/2$ now yields

$$y(t) = \tfrac{1}{2}c_1 e^{-2t} + 3c_2 e^{3t}.$$

Imposition of the initial conditions $x(0) = 0,\ y(0) = 2$ now gives the equations $c_1 + c_2 = 0,\ c_1/2 + 3c_2 = 0$ with solution $c_1 = -4/5,\ c_2 = 4/5$. These coefficients give the desired particular solution

$$x(t) = 4(e^{3t} - e^{-2t})/5, \qquad y(t) = 2(6e^{3t} - e^{-2t})/5.$$

The left-hand figure below shows a direction field and some typical solution curves.



4. Substitution of $y = 3x - x'$ and $y' = 3x' - x''$ — from the first equation — into the second equation yields the second-order equation $x'' - 4x = 0$ with general solution

$$x = c_1 e^{2t} + c_2 e^{-2t}.$$

Substitution of this solution in $y = 3x - x'$ gives

$$y = c_1 e^{2t} + 5c_2 e^{-2t}.$$

The initial conditions yield the equations $c_1 + c_2 = 1,\ c_1 + 5c_2 = -1$ with solution $c_1 = 3/2,\ c_2 = -1/2$. Hence the desired particular solution is

$$x(t) = (3e^{2t} - e^{-2t})/2, \qquad y(t) = (3e^{2t} - 5e^{-2t})/2.$$

The right-hand figure above shows a direction field and some typical solution curves.

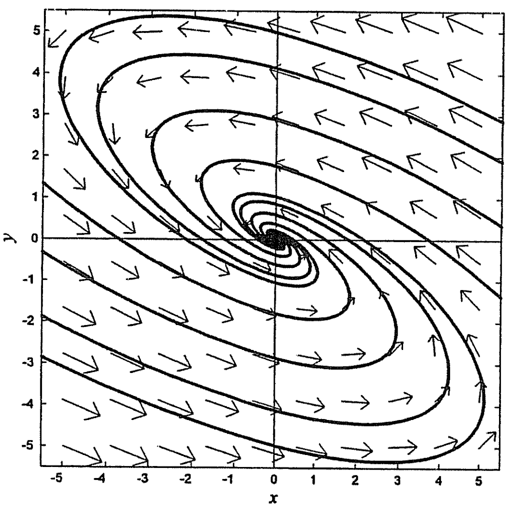
5. Substitution of $y = -(x'+3x)/4$ and $y' = -(x''+3x')/4$ — from the first equation — into the second equation yields the second-order equation $x''+2x'+5x = 0$ with general solution

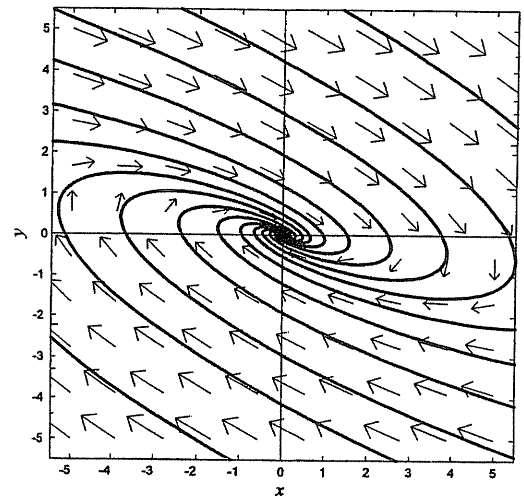
$$x(t) = e^{-t}\left(c_1 \cos 2t + c_2 \sin 2t\right).$$

Substitution of this solution in $y = -(x'+3x)/4$ gives

$$y(t) = \frac{1}{2}e^{-t}\left[-(c_1+c_2)\cos 2t + (c_1-c_2)\sin 2t\right].$$

The left-hand figure below shows a direction field and some typical solution curves.





6. Substitution of $y = (x'-x)/9$ and $y' = (x''-x')/9$ — from the first equation — into the second equation yields the second-order equation $x''+4x'+13x = 0$ with general solution

$$x(t) = e^{-2t}\left(c_1 \cos 3t + c_2 \sin 3t\right).$$

Substitution in $y = (x'-x)/9$ now yields

$$y(t) = \frac{1}{3}e^{-2t}\left[(-c_1+c_2)\cos 3t + (-c_1-c_2)\sin 3t\right].$$

Imposition of the initial conditions $x(0)=3,\ y(0)=2$ now gives the equations $c_1 = 3,\ -c_1/3 + c_2/3 = 2$ with solution $c_1 = 3,\ c_2 = 9$. These coefficients give the desired particular solution

$$x(t) = e^{-2t}(3\cos 3t + 9\sin 3t), \qquad y(t) = e^{-2t}(2\cos 3t - 4\sin 3t).$$

The right-hand figure on the preceding page shows a direction field and some typical solution curves.

7. Substitution of $y = x' - 4x - 2t$ and $y' = x'' - 4x' - 2$ — from the first equation — into the second equation yields the nonhomogeneous second-order equation $x'' - 5x' + 6x = 2 - 2t$. Substitution of the trial solution $x_p = A + Bt$ yields $A = 1/18$, $B = -1/3$ so $x_p = 1/18 - t/3$. Hence the general solution for x is

$$x(t) = c_1 e^{2t} + c_2 e^{3t} - t/3 + 1/18.$$

Substitution in $y = x' - 4x - 2t$ now yields

$$y(t) = -2c_1 e^{2t} - c_2 e^{3t} - 2t/3 - 5/9.$$

8. Substitution of $y = x' - 2x$ and $y' = x'' - 2x'$ — from the first equation — into the second equation yields the nonhomogeneous second-order equation $x'' - 4x' + 3x = -e^{2t}$. Substitution of the trial solution $x_p = Ae^{2t}$ yields $A = 1$, so $x_p = e^{2t}$. Hence the general solution for x is

$$x(t) = c_1 e^{t} + c_2 e^{3t} + e^{2t}.$$

Substitution in $y = x' - 2x$ now yields

$$y(t) = -c_1 e^{t} + c_2 e^{3t}.$$

9. Substitution of $y = (-x' + 2x + 2\sin 2t)/3$ and $y' = (-x'' + 2x' + 4\cos 2t)/3$ — from the first equation — into the second equation yields the second-order equation $x'' - x = 7\cos 2t + 4\sin 2t$ with general solution

$$x(t) = c_1 e^{-t} + c_2 e^{t} - \frac{1}{5}(7\cos 2t + 4\sin 2t).$$

Substitution in $y = (-x' + 2x + 2\sin 2t)/3$ now yields

$$y(t) = c_1 e^{-t} + \frac{1}{3}c_2 e^{t} - \frac{1}{5}(2\cos 2t + 4\sin 2t).$$

Imposition of the initial conditions $x(0) = 3$, $y(0) = 2$ now gives the equations $c_1 = 3$, $-c_1/3 + c_2/3 = 2$ with solution $c_1 = 3$, $c_2 = 9$. These coefficients give the desired particular solution

$$x(t) = e^{-2t}\left(3\cos 3t + 9\sin 3t\right), \qquad y(t) = e^{-2t}\left(2\cos 3t - 4\sin 3t\right).$$

10. First we solve the given equations for the normal-form first-order equations

$$x' = 2x + y, \qquad y' = x + 2y.$$

Substitution of $y = x' - 2x$ and $y' = x'' - 2x'$ — from the first equation — into the second equation yields the second-order equation $x'' - 4x' + 3x = 0$ with general solution

$$x(t) = c_1 e^t + c_2 e^{3t}.$$

Substitution in $y = x' - 2x$ now yields

$$y(t) = -c_1 e^t + c_2 e^{3t}.$$

Imposition of the initial conditions $x(0) = 1$, $y(0) = -1$ now gives the equations $c_1 + c_2 = 1$, $-c_1 + c_2 = -1$ with solution $c_1 = 1$, $c_2 = 0$. These coefficients give the desired particular solution

$$x(t) = e^t, \qquad y(t) = -e^t.$$

11. First we solve the given equations for the normal-form first-order equations

$$x' = 3x - 9y + e^{-t} + 2e^t,$$
$$y' = 2x - 3y + e^{-t}/2 + 3e^t/2.$$

Substitution of $y = (-x' + 3x + e^{-t} + 2e^t)/9$ and $y' = (-x'' + 3x' - e^{-t} + 2e^t)/9$ — from the first equation — into the second equation yields the nonhomogeneous second-order equation $x'' + 9x = \left(5e^{-t} + 11e^t\right)/2$. Substitution of the trial solution $x_p = Ae^{-t} + Be^t$ yields $A = -1/4$, $B = -11/20$ so $x_p = -e^{-t}/4 - 11e^t/20$. Hence the general solution for x is

$$x(t) = c_1\cos 3t + c_2\sin 3t - \frac{1}{4}e^{-t} - \frac{11}{20}e^t.$$

Substitution in $y = (-x' + 3x + e^{-t} + 2e^t)/9$ now yields

$$y(t) = \frac{1}{3}\left(c_1 - c_2\right)\cos 3t + \frac{1}{3}\left(c_1 + c_2\right)\sin 3t + \frac{1}{2}e^{-t} + \frac{8}{5}e^t.$$

12. The first equation yields $y = (x'' - 6x)/2$, so $y'' = (x^{(4)} - 6x'')/2$. Substitution in the second equation yields

$$x^{(4)} - 13x'' + 36x = 0.$$

The characteristic equation is $r^4 - 13r^2 + 36 = (r^2 - 4)(r^2 - 9) = 0$, so the general solution for x is

$$x(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 e^{3t} + c_4 e^{-3t}.$$

Substitution in $y = (x'' - 6x)/2$ now gives

$$y(t) = -c_1 e^{2t} - c_2 e^{-2t} + \frac{3}{2} c_3 e^{3t} + \frac{3}{2} c_4 e^{-3t}.$$

13. The first equation yields $y = (x'' + 5x)/2$, so $y'' = (x^{(4)} + 5x'')/2$. Substitution in the second equation yields

$$x^{(4)} + 13x'' + 36x = 0.$$

The characteristic equation is $r^4 + 13r^2 + 36 = (r^2 + 4)(r^2 + 9) = 0$, so the general solution for x is

$$x(t) = a_1 \cos 2t + a_2 \cos 2t + b_1 \cos 3t + b_2 \sin 3t.$$

Substitution in $y = (x'' + 5x)/2$ now gives

$$y(t) = \frac{1}{2} a_1 \cos 2t + \frac{1}{2} a_2 \cos 2t - 2b_1 \cos 3t - 2b_2 \sin 3t.$$

14. The first equation $x'' + 4x = \sin t$ has complementary function $x_c = c_1 \cos 2t + c_2 \sin 2t$, and substitution of the trial solution $x = A \sin t$ yields the particular solution $x_p = (1/3) \sin t$. Hence the general solution for x is

$$x(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \sin t.$$

Substitution in the second differential equation gives the equation

$$y'' + 8y = 4c_1 \cos 2t + 4c_2 \sin 2t + (4/3) \sin t$$

with complementary function $y_c = c_3 \cos 2t\sqrt{2} + c_4 \sin 2t\sqrt{2}$. Substitution of the trial solution $y_p = A \cos 2t + B \sin 2t + C \sin t$ now yields $A = c_1$, $B = c_2$, $C = 4/21$, so the general solution for y is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 \cos 2t\sqrt{2} + c_4 \sin 2t\sqrt{2} + \frac{4}{21}\sin t.$$

15. If we write the given differential equations in operator notation as

$$\begin{aligned}(D^2-2)x - 3Dy &= 0\\ 3Dx + (D^2-2)y &= 0,\end{aligned}$$

we see that the system has operational determinant

$$(D^2-2)^2 + 9D^2 = D^4 + 5D^2 + 4 = (D^2+1)(D^2+4).$$

Therefore (as in Example 3) we see that x satisfies the fourth-order differential equation $(D^2+1)(D^2+4)x = 0$ with characteristic equation $(r^2+1)(r^2+4) = 0$ and general solution

$$x(t) = a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t.$$

Similarly, the general solution for y is of the form

$$y(t) = c_1 \cos t + c_2 \sin t + d_1 \cos 2t + d_2 \sin 2t.$$

Now, substitution of these two general solutions in the first equation $x'' - 3y' - 2x = 0$ and collection of coefficients gives

$$(-3a_1 - 3c_2)\cos t + (3c_1 - 3a_2)\sin t + (-6b_1 - 6d_2)\cos 2t + (3d_1 - 3b_2)\sin 2t = 0.$$

Thus we see finally that $c_1 = a_2$, $c_2 = -a_1$, $d_1 = b_2$, $d_2 = -b_1$. Hence

$$y(t) = a_2 \cos t - a_1 \sin t + b_2 \cos 2t - b_1 \sin 2t.$$

16. In operational form our system is

$$\begin{aligned}(D^2-4)x + \quad 13Dy &= 6\sin t\\ -2Dx + (D^2-9)y &= 0\end{aligned}$$

When we operate on the first equation with $D^2 - 9$, on the second with $13D$, and subtract, the result is

$$(D^4 + 13D^2 + 36)x = -60 \sin t.$$

The associated homogeneous equation has characteristic equation $(r^2+4)(r^2+9) = 0$ and complementary function $x_c = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 3t + b_2 \sin 3t$. Upon substituting the trial solution $x_p = A \sin t$ we find that $A = 5/2$. Thus the general solution for x is

$$x(t) = a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 3t + b_2 \sin 3t + \frac{5}{2}\sin t.$$

We find similarly that

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + d_1 \cos 3t + d_2 \sin 3t + \frac{1}{2}\cos t.$$

When we substitute these expressions into either of the original differential equations we find that $c_1 = -4a_2/13,\ c_2 = 4a_1/13,\ d_1 = -b_2/3,\ d_1 = b_1/3$. Therefore

$$y(t) = \frac{4}{13}\left(-a_2 \cos 2t + a_1 \sin 2t\right) + \frac{1}{3}\left(-b_2 \cos 3t + b_1 \sin 3t\right) + \frac{1}{2}\cos t.$$

17. If we write the given equations in the operational form

$$\begin{aligned} \left(D^2 - 3D - 2\right)x + \left(D^2 - D + 2\right)y &= 0, \\ \left(2D^2 - 9D - 4\right)x + \left(3D^2 - 2D + 6\right)y &= 0 \end{aligned}$$

we see (thinking of the operational determinant) that x satisfies a homogeneous fourth-order equation with characteristic equation

$$\begin{aligned} &\left(r^2 - 3r - 2\right)\left(3r^2 - 2r + 6\right) - \left(r^2 - r + 2\right)\left(2r^2 - 9r - 4\right) \\ &\quad = r^4 - 3r^2 - 4 = \left(r^2 + 1\right)\left(r^2 - 4\right) = \left(r^2 + 1\right)(r+2)(r-2) = 0. \end{aligned}$$

Hence the general solution for x is

$$x(t) = a_1 \cos t + a_2 \sin t + b_1 e^{-2t} + b_2 e^{2t},$$

and, similarly, the general solution for y is

$$y(t) = c_1 \cos t + c_2 \sin t + d_1 e^{-2t} + d_2 e^{2t}.$$

To determine the relations between the arbitrary constants in these two general solutions, we substitute them in the first of the original differential equations and get

$$\begin{aligned}
&\left(-a_1\cos t - a_2\sin t + 4b_1e^{-2t} + 4b_2e^{2t}\right) + \left(-c_1\cos t - c_2\sin t + 4d_1e^{-2t} + 4d_2e^{2t}\right) + \\
&\left(3a_1\sin t - 3a_2\cos t + 6b_1e^{-2t} - 6b_2e^{2t}\right) + \left(c_1\sin t - c_2\cos t + 2d_1e^{-2t} - 2d_2e^{2t}\right) + \\
&\left(-2a_1\cos t - 2a_2\sin t - 2b_1e^{-2t} - 2b_2e^{2t}\right) + \left(+2c_1\cos t + 2c_2\sin t + 2d_1e^{-2t} + 2d_2e^{2t}\right) \;=\; 0
\end{aligned}$$

If we collect coefficients of the trigonometric and exponential terms we get the equations

$$\begin{aligned} -3a_1 - 3a_2 + c_1 - c_2 &= 0, \\ 3a_1 - 3a_2 + c_1 + c_2 &= .0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 8b_1 + 8d_1 &= 0, \\ -4b_2 + 4d_2 &= 0. \end{aligned}$$

The first two of these equations imply that $c_1 = -3a_2$ and $c_2 = 3a_1$, while the latter two give $d_1 = -b_1,\ d_2 = b_{2.}$ We therefore see finally that

$$y(t) \;=\; -3a_2\cos t + 3a_1\sin t - b_1e^{-2t} + b_2e^{2t}.$$

18. From the first and third equations we see that $x' + z' = 0$, so $z = -x$. Hence the first equation reduces to $x' = 2y$, and substitution in the second equation yields $y'' + y' - 12y$, so it follows that

$$y(t) \;=\; c_1e^{3t} + c_2e^{-4t}.$$

Since $x = (y + y')/6$, it follows that

$$\begin{aligned} x(t) &= (\ 4c_1e^{3t} - 3c_2e^{-4t})/6, \\ z(t) &= (-4c_1e^{3t} + 3c_2e^{-4t})/6. \end{aligned}$$

19. The operational determinant of the given system is

$$L \;=\; \begin{vmatrix} D-4 & 2 & 0 \\ 4 & D-4 & 2 \\ 0 & 4 & D-4 \end{vmatrix} \;=\; D^3 - 12D^2 + 32D,$$

so x, y, and z all satisfy a third-order homogeneous linear differential equation with characteristic equation $r^3 - 12r^2 + 32r = r(r-4)(r-8) = 0$. The corresponding general solutions are

$$x(t) = a_1 + a_2e^{4t} + a_3e^{8t}, \qquad y(t) = b_1 + b_2e^{4t} + b_3e^{8t}, \qquad z(t) = c_1 + c_2e^{4t} + c_3e^{8t}.$$

If we substitute $x(t)$ and $y(t)$ in the first differential equation $x' = 4x - 2y$ and collect coefficients of like terms, we find quickly that $b_1 = 2a_1$, $b_2 = 0$, and $b_3 = -2a_3$. Similarly, we find by substitution in the other two equations that

$c_1 = 2a_1$, $c_2 = -2a_2$, and $c_3 = 2a_3$. Thus y and z are given by

$$y(t) = 2a_1 - 2a_3e^{8t} \quad \text{and} \quad z(t) = 2a_1 - 2a_2e^{4t} + 2a_3e^{8t}.$$

20. The operational determinant of the given system is

$$L = D^3 - 3D - 2 = (D+1)^2(D-2),$$

and we find that

$$Lx = Ly = Lz = 0.$$

Hence

$$x = a_1e^{2t} + a_2e^{-t} + a_3te^{-t},$$
$$y = b_1e^{2t} + b_2e^{-t} + b_3te^{-t},$$
$$z = c_1e^{2t} + c_2e^{-t} + c_3te^{-t}.$$

When we substitute these expressions in the three differential equations and compare coefficients of e^{2t}, we find that $a_1 = b_1 = c_1$. When we compare coefficients of te^{-t} we find that $a_3 + b_3 + c_3 = 0$. Comparison of coefficients of e^{-t} yields

$$a_2 + b_2 + c_2 = a_3 - 1 = b_3 = c_3.$$

It follows that $a_3 = 2/3$ and $b_3 = c_3 = -1/3$. If a_2 and b_2 are chosen arbitrarily, then $c_2 = -a_2 - b_2 - 1/3$. Hence the general solution is

$$x = a_1e^{2t} + a_2e^{-t} + (2/3)te^{-t},$$
$$y = a_1e^{2t} + b_2e^{-t} - (1/3)te^{-t},$$
$$z = a_1e^{2t} - (a_2 + b_2 + 1/3)e^{-t} - (1/3)te^{-t}.$$

21. $L_1L_2 = L_2L_1$ because both sides simplify to the same thing upon multiplying out and collecting terms in the usual fashion of polynomial algebra. This "works" because different powers of D commute — that is, $D^iD^j = D^jD^i$ because $D^i(D^jx) = D^{i+j}x = D^j(D^ix)$.

22.
$$L_1(L_2x) = (tD+1)(Dx+tx) = tD(Dx+tx)+(Dx+tx) =$$
$$= t(D^2x+tDx+x)+(Dx+tx) = tD^2x+t^2Dx+Dx+2tx$$

$$L_2(L_1x) = (D+t)(tDx+x) = D(tDx+x)+t(tDx+x) =$$
$$= (tD^2x+2Dx)+(t^2Dx+tx) = tD^2x+t^2Dx+2Dx+tx$$

23. Subtraction of the two equations yields $x + y = e^{-2t} - e^{-3t}$. We then verify readily that any two differentiable functions $x(t)$ and $y(t)$ satisfying this condition will constitute a solution of the given system, which thus has infinitely many solutions.

24. Subtraction of one equation from the other yields $x+y = t^2 - t$. But then

$$\begin{aligned}(D+2)x+(D+2)y &= D(x+y)+2(x+y)\\ &= (2t-1)+2(t^2-t) = 2t^2-1 \neq t.\end{aligned}$$

Thus the given system has no solution.

25. Infinitely many solutions, because any solution of the second equation also satisfies the first equation (because it is $D+2$ times the second one).

26. Subtraction of the second equation from the first one gives $x = e^{-t}$. Then substitution in the second equation yields $D^2y = 0$, so $y = b_1t+b_2$. Thus there are *two* arbitrary constants.

27. Subtraction of the second equation from the first one gives $x+y = e^{-t}$. Then substitution in the second equation yields

$$x(t) = D^2(x+y) = e^{-t}.$$

It follows that $y(t) \equiv 0$, so there are *no* arbitrary constants.

28. Differentiation of the difference of the two given equations yields

$$(D^2+D)x+D^2y = -2e^{-t},$$

which contradicts the first equation. Hence the system has *no* solution.

29. Addition of the two given equations yields $D^2x = e^{-t}$, so $x(t) = e^{-t}+a_1t+a_2$. Then the second equation gives $D^2y = a_1t+a_2$, so

$$y(t) = (1/6)a_1t^3+(1/2)a_2t^2+a_3t+a_4.$$

Thus there are *four* arbitrary constants.

30. Substitution of $y = 20x'+6x$ and $y' = 20x''+6x'$ — from the first equation — into the second equation yields the second-order equation $100x''+45x'+3x = 0$ with general solution

$$x(t) = c_1e^{r_1t}+c_2e^{r_2t}, \quad \text{where} \quad r_1 = \frac{-9+\sqrt{33}}{40}, \; r_2 = \frac{-9-\sqrt{33}}{40}.$$

Substitution in $y = 20x'+6x$ now yields

$$y(t) = \frac{1}{2}\left(3+\sqrt{33}\right)c_1e^{r_1t}+\frac{1}{2}\left(3-\sqrt{33}\right)c_2e^{r_2t}.$$

Imposition of the initial conditions $x(0) = 50,\ y(0) = 100$ now gives the equations $c_1 + c_2 = 50,\ \left(3+\sqrt{33}\right)c_1 + \left(3-\sqrt{33}\right)c_2 = 200$ with solution $c_1 = 25\left(1+\sqrt{33}\right)/\sqrt{33}$, $c_2 = 25\left(-1+\sqrt{33}\right)/\sqrt{33}$. These coefficients give the desired particular solution

$$x(t) = \frac{25}{\sqrt{33}}\left[\left(1+\sqrt{33}\right)e^{r_1 t} + \left(-1+\sqrt{33}\right)e^{r_2 t}\right],$$
$$y(t) = \frac{50}{11}\left[\left(11+3\sqrt{33}\right)e^{r_1 t} + \left(11-3\sqrt{33}\right)e^{r_2 t}\right].$$

31. Substitution of $I_2 = (I_1' + 25I_1 - 50)/25$ and $I_2' = (I_1'' + 25I_1')/25$ — from the first equation — into the second equation yields the second-order equation $3I_1'' + 30I_1' + 125I_1 = 250$ with general solution

$$I_1(t) = 2 + e^{-5t}\left[c_1 \cos\left(5t\sqrt{6}/3\right) + c_2 \sin\left(5t\sqrt{6}/3\right)\right].$$

Substitution in $I_2 = (I_1' + 25I_1 - 50)/25$ now yields

$$I_2(t) = \frac{1}{15}e^{-5t}\left[\left(12c_1 + \sqrt{6}\,c_2\right)\cos\left(5t\sqrt{6}/3\right) + \left(12c_2 - \sqrt{6}\,c_1\right)\sin\left(5t\sqrt{6}/3\right)\right].$$

Imposition of the initial conditions $I_1(0) = 0,\ I_2(0) = 0$ now gives the equations $c_1 + 2 = 0,\ 4c_1/5 + \sqrt{6}\,c_2/15 = 0$ with solution $c_1 = -2,\ c_2 = 4\sqrt{6}$. These coefficients give the desired particular solution

$$I_1(t) = 2 + e^{-5t}\left[-2\cos\left(5t\sqrt{6}/3\right) + 4\sqrt{6}\sin\left(5t\sqrt{6}/3\right)\right],$$
$$I_2(t) = \frac{20}{\sqrt{6}}e^{-5t}\sin\left(5t\sqrt{6}/3\right).$$

32. To solve the system

$$2(I_1' - I_2') + 50\,I_1 = 100 \sin 60t$$
$$2(I_2' - I_1') + 25\,I_2 = 0$$

we first note from the second equation that $2(I_1' - I_2') = 25I_2$, so the first equation then tells us that $25I_2 + 50I_1 = 100\sin 60t$, hence $I_2 = -2I_1 + 4\sin 60t,\ I_2' = -2I_1' + 240\cos 60t$. Substitution of these into the second equation gives the first-order equation

$$6\,I_1' + 50\,I_1 = 480\cos 60t + 100\sin 60t$$

with complementary function $I_{1c} = Ce^{-25t/3}$. Substitution of the trial solution $I_{1p} = A\cos 60t + B\sin 60t$ yields $A = -120/1321$, $B = 1778/1321$. Finally the initial condition $I_1(0) = 0$ gives $C = A$. Consequently we find that

$$I_1(t) = (\ 120e^{-25t/3} - 120\cos 60t + 1778\sin 60t)/1321,$$

$$I_2(t) = (-240e^{-25t/3} + 240\cos 60t + 1728\sin 60t)/1321.$$

(Yes, one coefficient of $\sin 60t$ is 1778 and the other is 1728.)

33. To solve the system

$$I_1' = -20(I_1 - I_2), \qquad I_2' = 40(I_1 - I_2)$$

we first note that $I_2' = -2I_1'$, so $I_2 = -2I_1 + K$. Then $K = 2I_1(0) + I_2(0) = 2(2) + 0 = 4$, so $I_2' = -2I_1'$, so $I_2 = -2I_1 + 4$. Substitution of this into the first equation gives the simple first-order linear equation $I_1' + 60I_1 = 80$ with general solution $I_1(t) = 4/3 + ce^{-60t}$. The initial condition $I_1(0) = 2$ gives $c = 2/3$, so

$$I_1(t) = \frac{2}{3}\left(2 + e^{-60t}\right), \qquad I_2(t) = \frac{4}{3}\left(1 - e^{-60t}\right).$$

34. The operational determinant of the system

$$10x_1' = -x_1 + x_3, \quad 10x_2' = x_1 - x_2, \quad 10x_3' = x_2 - x_3$$

is

$$L = \begin{vmatrix} 10D+1 & 0 & -1 \\ -1 & 10D+1 & 0 \\ 0 & -1 & 10D+1 \end{vmatrix} = 1000D^3 + 300D^2 + 30D,$$

so x_1, x_2, and x_3 all satisfy a third-order homogeneous linear differential equation with characteristic equation $1000r^3 + 300r^2 + 30r = 10r(100r^2 + 30r + 3) = 0$ and characteristic roots $r = 0, \left(-3 \pm i\sqrt{3}\right)/20$. The corresponding general solutions are

$$x_1(t) = a_1 + e^{-3t/20}\left[b_1\cos\left(t\sqrt{3}/20\right) + c_1\sin\left(t\sqrt{3}/20\right)\right],$$

$$x_2(t) = a_2 + e^{-3t/20}\left[b_2\cos\left(t\sqrt{3}/20\right) + c_2\sin\left(t\sqrt{3}/20\right)\right],$$

$$x_3(t) = a_3 + e^{-3t/20}\left[b_3\cos\left(t\sqrt{3}/20\right) + c_3\sin\left(t\sqrt{3}/20\right)\right].$$

Substituting in the original differential equations, we see that the constant terms are all equal: $a_1 = a_2 = a_3 = a$ (say). Then the initial conditions $x_1(0) = 100$, $x_2(0) = x_3(0) = 0$

imply that $b_1 = 100 - a,\ b_2 = b_3 = -a$. After these substitutions, collection of coefficients gives the equations

$$\frac{3}{2}a + \frac{\sqrt{3}}{2}c_1 - 50 = 0, \qquad \frac{\sqrt{3}}{2}a - \frac{1}{2}c_1 - c_3 - 50\sqrt{3} = 0,$$
$$\frac{3}{2}a + \frac{\sqrt{3}}{2}c_2 - 100 = 0, \qquad \frac{\sqrt{3}}{2}a - c_1 - \frac{1}{2}c_2 = 0$$

that we solve for $a = 100/3,\ c_1 = 0,\ c_2 = 100/\sqrt{3},\ c_3 = -100/\sqrt{3}$. The resulting solution of the original system is given by

$$\begin{aligned} x_1(t) &= \frac{100}{3}\left[1 + 2e^{-3t/20}\cos\left(t\sqrt{3}/20\right)\right], \\ x_2(t) &= \frac{100}{3}\left[1 + e^{-3t/20}\left(-\cos\left(t\sqrt{3}/20\right) + \sqrt{3}\sin\left(t\sqrt{3}/20\right)\right)\right], \\ x_3(t) &= \frac{100}{3}\left[1 + e^{-3t/20}\left(-\cos\left(t\sqrt{3}/20\right) - \sqrt{3}\sin\left(t\sqrt{3}/20\right)\right)\right]. \end{aligned}$$

35. The two given equations yield

$$mx^{(3)} = qBy'' = -q^2B^2x'/m,$$

so $x^{(3)} + \omega^2 x' = 0$. The general solution is

$$x(t) = A\cos\omega t + B\sin\omega t + C.$$

Now $x'(0) = 0$ implies $B = 0$, and then $x(0) = r_0$ gives $A + C = r_0$. Next,

$$\omega y' = x'' = -A\omega^2\cos\omega t,$$

so $y'(0) = -\omega r_0$ implies $A = r_0$, hence $C = 0$. It now follows readily that the trajectory is the circle

$$x(t) = r_0\cos\omega t, \qquad y(t) = -r_0\sin\omega t.$$

36. With $\omega = qB/m$ our differential equations are

$$x'' = \omega y' + qE/m, \qquad y'' = -\omega x'.$$

Elimination gives

$$x^{(4)} + \omega^2 x'' = y^{(4)} + \omega^2 y'' = 0,$$

so

$$x = a_1 + b_1 t + c_1\cos\omega t + d_1\sin\omega t,$$

$$y = a_2 + b_2 t + c_2 \cos \omega t + d_2 \sin \omega t.$$

The initial conditions $x(0) = x'(0) = 0 = y(0) = y'(0)$ yield

$$c_1 = -a_1, \;\; b_1 = -\omega d_1, \;\; c_2 = -a_2, \;\; b_2 = -\omega d_2.$$

Then substitution in $y'' = -\omega x'$ yields

$$d_1 = b_1 = a_2 = c_2 = 0 \;\; \text{and} \;\; d_2 = a_1.$$

Finally, substitution in $x'' = \omega y' + qE/m$ yields $a = a_1 = E/\omega B$, so the solution is

$$x(t) = a(1 - \cos \omega t), \quad y(t) = -a(\omega t - \sin \omega t).$$

37. **(a)** If we set $m_1 = 2,\; m_2 = 1/2,\; k_1 = 75, k_2 = 25$ in Eqs. (3) of Section 5.1, we get the system $2x'' = -100x + 25y,\; \frac{1}{2}y'' = 25x - 25y$ with operational determinant $D^4 + 100D^2 + 1875 = (D^2 + 25)(D^2 + 75)$. Hence the general form of the solution is

$$\begin{aligned} x(t) &= a_1 \cos 5t + a_2 \sin 5t + b_1 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3}, \\ y(t) &= c_1 \cos 5t + c_2 \sin 5t + d_1 \cos 5t\sqrt{3} + d_2 \sin 5t\sqrt{3}. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = 2a_1,\; c_2 = 2a_2$ and $d_1 = -2b_1,\; d_2 = -2b_2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos 5t + a_2 \sin 5t + b_1 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3}, \\ y(t) &= 2a_1 \cos 5t + 2a_2 \sin 5t - 2b_1 \cos 5t\sqrt{3} - 2b_2 \sin 5t\sqrt{3}. \end{aligned}$$

(b) In the natural mode with frequency $\omega_1 = 5$ the masses move in the same direction, while in the natural mode with frequency $\omega_2 = 5\sqrt{3}$ they move in opposite directions. In each case the amplitude of the motion of m_2 is twice that of m_1.

38. Looking at Fig. 5.2.6 in the text, we see that the first spring is stretched by x, the second spring is stretched by $y - x$, and the third spring is compressed by y. Hence Newton's second law gives $m_1 x'' = -k_1(x) + k_2(y - x)$ and $m_2 y'' = -k_2(y - x) - k_3(y)$.

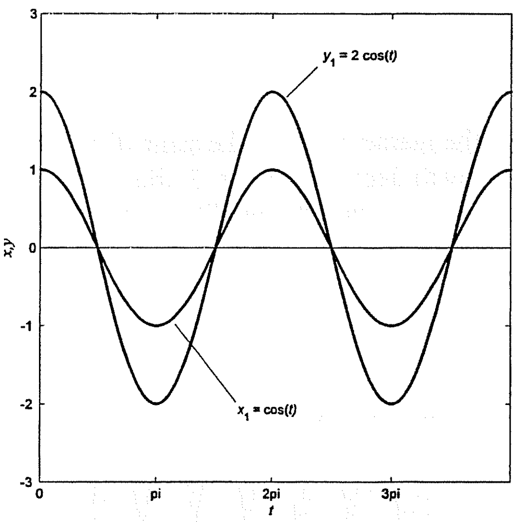
39. The system has operational determinant $8D^4 + 40D^2 + 32 = 8(D^2 + 1)(D^2 + 4)$. Hence the general form of the solution is

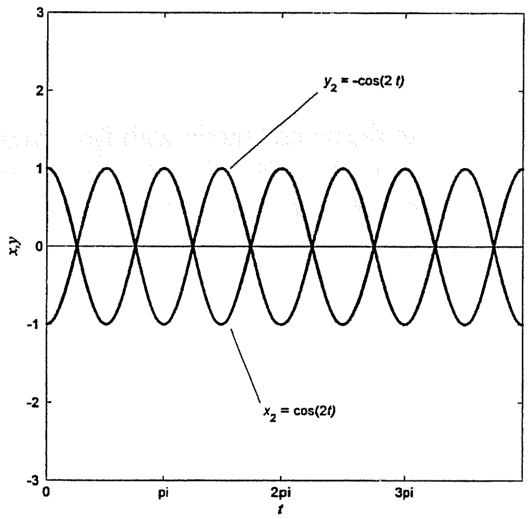
$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t, \\ y(t) &= c_1 \cos t + c_2 \sin t + d_1 \cos 2t + d_2 \sin 2t. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = 2a_1,\ c_2 = 2a_2$ and $d_1 = -b_1,\ d_2 = -b_2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t, \\ y(t) &= 2a_1 \cos t + 2a_2 \sin t - b_1 \cos 2t - b_2 \sin 2t. \end{aligned}$$

In the natural mode with frequency $\omega_1 = 1$ the masses move in the same direction, with the amplitude of motion of the second mass twice that of the first mass (figure on the left below). In the natural mode with frequency $\omega_2 = 2$ they move in opposite directions with the same amplitude of motion (figure on the right below).





40. The system has operational determinant $2D^4 + 250D^2 + 5000 = 2(D^2+25)(D^2+100)$. Hence the general form of the solution is

$$\begin{aligned} x(t) &= a_1 \cos 5t + a_2 \sin 5t + b_1 \cos 10t + b_2 \sin 10t, \\ y(t) &= c_1 \cos 5t + c_2 \sin 5t + d_1 \cos 10t + d_2 \sin 10t. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = 2a_1,\ c_2 = 2a_2$ and $d_1 = -b_1,\ d_2 = -b_2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos 5t + a_2 \sin 5t + b_1 \cos 10t + b_2 \sin 10t, \\ y(t) &= 2a_1 \cos 5t + 2a_2 \sin 5t - b_1 \cos 10t - b_2 \sin 10t. \end{aligned}$$

In the natural mode with frequency $\omega_1 = 5$ the masses move in the same direction, with the amplitude of motion of the second mass twice that of the first mass. In the natural mode with frequency $\omega_2 = 10$ they move in opposite directions with the same amplitude of motion.

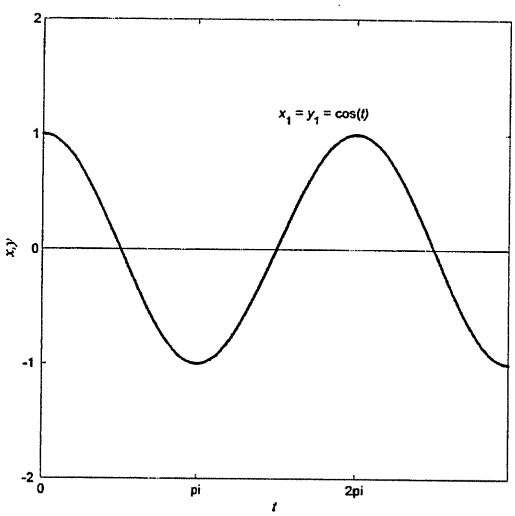
41. The system has operational determinant $D^4 + 10D^2 + 9 = (D^2+1)(D^2+9)$. Hence the general form of the solution is

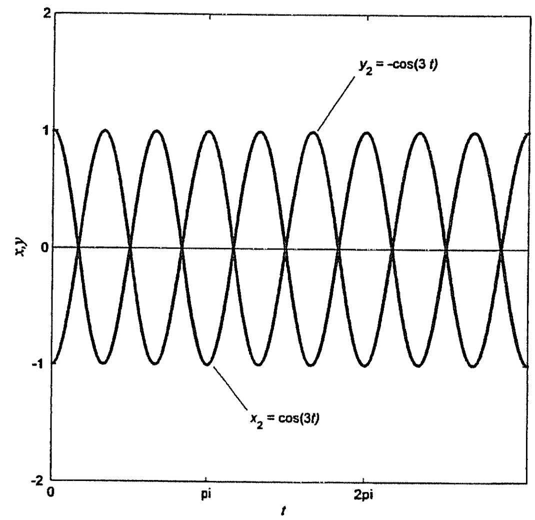
$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t, \\ y(t) &= c_1 \cos t + c_2 \sin t + d_1 \cos 3t + d_2 \sin 3t. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = a_1,\ c_2 = a_2$ and $d_1 = -b_1,\ d_2 = -b_2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t, \\ y(t) &= a_1 \cos t + a_2 \sin t - b_1 \cos 3t - b_2 \sin 3t. \end{aligned}$$

In the natural mode with frequency $\omega_1 = 1$ the masses move in the same direction (left-hand figure below), while in the natural mode with frequency $\omega_2 = 3$ they move in opposite directions (right-hand figure below). In each case the amplitudes of motion of the two masses are equal.





42. The system has operational determinant $2D^4 + 10D^2 + 8 = 2(D^2+1)(D^2+4)$. Hence the general form of the solution is

$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 2t + b_2 \sin 2t, \\ y(t) &= c_1 \cos t + c_2 \sin t + d_1 \cos 2t + d_2 \sin 2t. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = a_1,\ c_2 = a_2$ and $d_1 = -b_1/2,\ d_2 = -b_2/2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + \ b_1 \cos 2t + \ b_2 \sin 2t, \\ y(t) &= a_1 \cos t + a_2 \sin t - \tfrac{1}{2} b_1 \cos 2t - \tfrac{1}{2} b_2 \sin 2t. \end{aligned}$$

In the natural mode with frequency $\omega_1 = 1$ the two masses move in the same direction with equal amplitudes of oscillation. In the natural mode with frequency $\omega_2 = 2$ the two masses move in opposite directions with the amplitude of m_2 being half that of m_1.

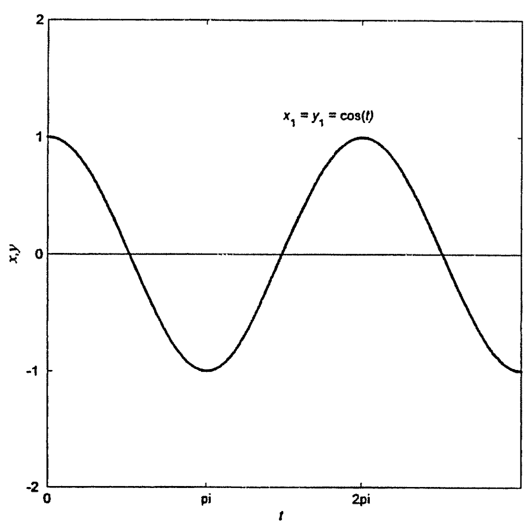
43. The system has operational determinant $D^4 + 6D^2 + 5 = (D^2+1)(D^2+5)$. Hence the general form of the solution is

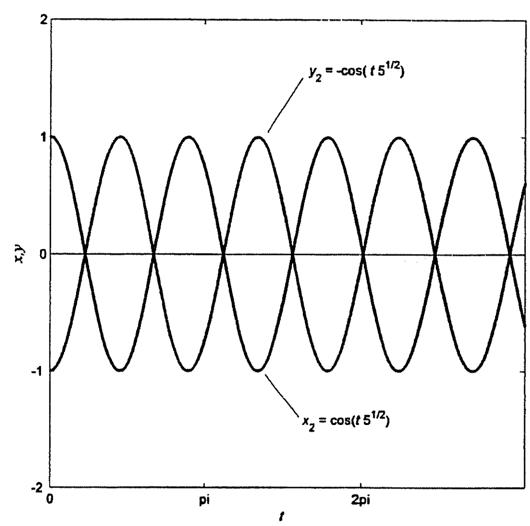
$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos t\sqrt{5} + b_2 \sin t\sqrt{5}, \\ y(t) &= c_1 \cos t + c_2 \sin t + d_1 \cos t\sqrt{5} + d_2 \sin t\sqrt{5}. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = a_1,\ c_2 = a_2$ and $d_1 = -b_1,\ d_2 = -b_2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + \ b_1 \cos t\sqrt{5} + b_2 \sin t\sqrt{5}, \\ y(t) &= a_1 \cos t + a_2 \sin t - b_1 \cos t\sqrt{5} - b_2 \sin t\sqrt{5}. \end{aligned}$$

In the natural mode with frequency $\omega_1 = 1$ the masses move in the same direction (left-hand figure below), while in the natural mode with frequency $\omega_2 = \sqrt{5}$ they move in opposite directions (right-hand figure below). In each case the amplitudes of motion of the two masses are equal.





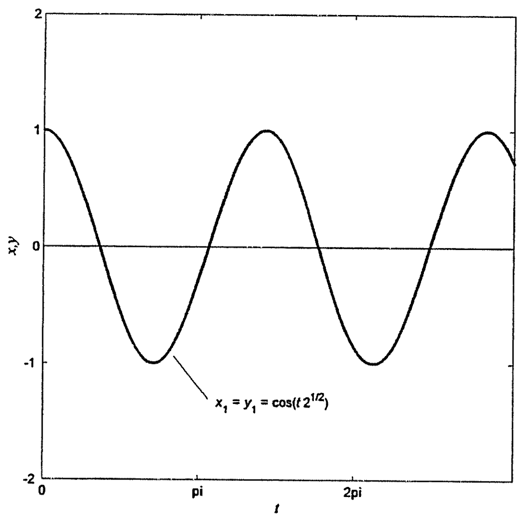
44. The system has operational determinant $D^4 + 6D^2 + 8 = (D^2+2)(D^2+4)$. Hence the general form of the solution is

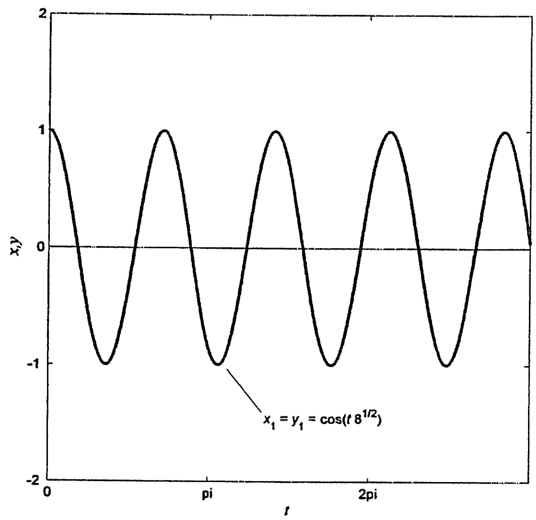
$$\begin{aligned} x(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + b_1 \cos 2t + b_2 \sin 2t, \\ y(t) &= c_1 \cos t\sqrt{2} + c_2 \sin t\sqrt{2} + d_1 \cos 2t + d_2 \sin 2t. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = a_1,\ c_2 = a_2$ and $d_1 = -b_1,\ d_2 = -b_2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + b_1 \cos 2t + b_2 \sin 2t, \\ y(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} - b_1 \cos 2t - b_2 \sin 2t. \end{aligned}$$

In the natural mode with frequency $\omega_1 = \sqrt{2}$ the two masses move in the same direction; in the natural mode with frequency $\omega_2 = 2$ they move in opposite directions. In each natural mode their amplitudes of oscillation are equal.





45. The system has operational determinant $2D^4 + 20D^2 + 32 = 2(D^2+2)(D^2+8)$. Hence the general form of the solution is

$$\begin{aligned} x(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + b_1 \cos t\sqrt{8} + b_2 \sin t\sqrt{8}, \\ y(t) &= c_1 \cos t\sqrt{2} + c_2 \sin t\sqrt{2} + d_1 \cos t\sqrt{8} + d_2 \sin t\sqrt{8}. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = a_1,\ c_2 = a_2$ and $d_1 = -b_1/2,\ d_2 = -b_2/2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + \ \ b_1 \cos t\sqrt{8} + \ b_2 \sin t\sqrt{8}, \\ y(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} - \tfrac{1}{2} b_1 \cos t\sqrt{8} - \tfrac{1}{2} b_2 \sin t\sqrt{8}. \end{aligned}$$

In the natural mode with frequency $\omega_1 = \sqrt{2}$ the two masses move in the same direction with equal amplitudes of oscillation (see the left-hand figure on the preceding page). In the natural mode with frequency $\omega_2 = \sqrt{8} = 2\sqrt{2}$ the two masses move in opposite directions with the amplitude of m_2 being half that of m_1 (see the right-hand figure on the preceding page).

46. The system has operational determinant $D^4 + 20D^2 + 64 = \left(D^2+4\right)\left(D^2+16\right)$. Hence the general form of the solution is

$$\begin{aligned} x(t) &= a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t, \\ y(t) &= c_1 \cos 2t + c_2 \sin 2t + d_1 \cos 4t + d_2 \sin 4t. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = a_1,\ c_2 = a_2$ and $d_1 = -b_1,\ d_2 = -b_2$. This gives

$$\begin{aligned} x(t) &= a_1 \cos 2t + a_2 \sin 2t + \ b_1 \cos 4t + b_2 \sin 4t, \\ y(t) &= a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t. \end{aligned}$$

In the natural mode with frequency $\omega_1 = 2$ the masses move in the same direction with equal amplitudes of motion. In the natural mode with frequency $\omega_2 = 4$ they move in opposite directions with the same amplitude of motion.

47. **(a)** Looking at Fig. 5.2.7 in the text, we see that the first spring is stretched by x, the second spring is stretched by $y - x$, the third spring is stretched by $z - y$, and the fourth spring is compressed by z. Hence Newton's second law gives $mx'' = -k(x) + k(y-x)$, $my'' = -k(y-x) + k(z-y)$, and $mz'' = -k(z-y) - k(z)$.

(b) The operational determinant is

$$(D^2+2)[(D^2+2)^2 - 1] + [\,-(D^2+2)] = (D^2+2)[(D^2+2)^2 - 2],$$

and the characteristic equation $(r^2+2)[(r^2+2)^2 - 2] = 0$ has roots $\pm i\sqrt{2}$ and $\pm i\sqrt{2 \pm \sqrt{2}}$.

48. The given system has operational determinant $D^4 + 10D^2 + 9 = (D^2+1)(D^2+9)$. Hence the general form of the solution is

$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t, \\ y(t) &= c_1 \cos t + c_2 \sin t + d_1 \cos 3t + d_2 \sin 3t. \end{aligned}$$

Upon substitution in either differential equation we see that $c_1 = -a_2$, $c_2 = a_1$ and $d_1 = b_2$, $d_2 = -b_1$. This gives

$$\begin{aligned} x(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t, \\ y(t) &= -a_2 \cos t + a_1 \sin t + b_2 \cos 3t - b_1 \sin 3t. \end{aligned}$$

When we impose the initial conditions $x(0) = 4$, $y(0) = x'(0) = y'(0) = 0$ we find that $a_1 = 3$, $b_1 = 1$, and $a_2 = b_2 = 0$.

SECTION 5.3

MATRICES AND LINEAR SYSTEMS

The first half-dozen pages of this section are devoted to a review of matrix notation and terminology. With students who've had some prior exposure to matrices and determinants, this review material can be skimmed rapidly. In this event serious study of the section can begin with the subsections on matrix-valued functions and first-order linear systems. About all that's actually needed for this purpose is some acquaintance with determinants, with matrix multiplication and inverse matrices, and with the fact that a square matrix is invertible if and only if its determinant is nonzero.

1. **(a)** $2\mathbf{A} + 3\mathbf{B} = \begin{bmatrix} 4 & -6 \\ 8 & 14 \end{bmatrix} + \begin{bmatrix} 9 & -12 \\ 15 & 3 \end{bmatrix} = \begin{bmatrix} 13 & -18 \\ 23 & 17 \end{bmatrix}$

(b) $3\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} 6 & -9 \\ 12 & 21 \end{bmatrix} - \begin{bmatrix} 6 & -8 \\ 10 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 19 \end{bmatrix}$

(c) $\mathbf{AB} = \begin{bmatrix} -9 & -11 \\ 47 & -9 \end{bmatrix}$ **(d)** $\mathbf{BA} = \begin{bmatrix} -10 & -37 \\ 14 & -8 \end{bmatrix}$

2. $$(\mathbf{AB})\mathbf{C} = \begin{bmatrix} -9 & -11 \\ 47 & -9 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -33 & -7 \\ -27 & 103 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} -12 & 10 \\ 3 & 9 \end{bmatrix} = \mathbf{A}(\mathbf{BC})$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{bmatrix} 2 & -3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} -18 & -4 \\ 68 & -8 \end{bmatrix} = \begin{bmatrix} -9 & -11 \\ 47 & -9 \end{bmatrix} + \begin{bmatrix} -9 & 7 \\ 21 & 1 \end{bmatrix} = \mathbf{AB}+\mathbf{AC}$$

3. $$\mathbf{AB} = \begin{bmatrix} -1 & 8 \\ 46 & -1 \end{bmatrix}; \qquad \mathbf{BA} = \begin{bmatrix} 11 & -12 & 14 \\ -14 & 0 & 7 \\ 0 & 8 & -13 \end{bmatrix}$$

4. $$\mathbf{Ay} = \begin{bmatrix} 2t^2 - \cos t \\ 3t^2 - 4\sin t + 5\cos t \end{bmatrix}; \qquad \mathbf{Bx} = \begin{bmatrix} 2t + 3e^{-t} \\ -14t \\ 6t - 2e^{-t} \end{bmatrix}$$

The products $\mathbf{Ax}$ and $\mathbf{By}$ are not defined, because in neither case is the number of columns of the first factor equal to the number of rows of the second factor.

5. (a) $$7\mathbf{A}+4\mathbf{B} = \begin{bmatrix} 21 & 14 & -7 \\ 0 & 28 & 21 \\ -35 & 14 & 49 \end{bmatrix} + \begin{bmatrix} 0 & -12 & 8 \\ 4 & 16 & -12 \\ 8 & 20 & -4 \end{bmatrix} = \begin{bmatrix} 21 & 2 & 1 \\ 4 & 44 & 9 \\ -27 & 34 & 45 \end{bmatrix}$$

(b) $$3\mathbf{A}-5\mathbf{B} = \begin{bmatrix} 9 & 6 & -3 \\ 0 & 12 & 9 \\ -15 & 6 & 21 \end{bmatrix} - \begin{bmatrix} 0 & -15 & 10 \\ 5 & 20 & -15 \\ 10 & 25 & -5 \end{bmatrix} = \begin{bmatrix} 9 & 21 & -13 \\ -5 & -8 & 24 \\ -25 & -19 & 26 \end{bmatrix}$$

(c) $$\mathbf{AB} = \begin{bmatrix} 0 & -6 & 1 \\ 10 & 31 & -15 \\ 16 & 58 & -23 \end{bmatrix}$$

(d) $$\mathbf{BA} = \begin{bmatrix} -10 & -8 & 5 \\ 18 & 12 & -10 \\ 11 & 22 & 6 \end{bmatrix}$$

(e) $$\mathbf{A}-t\mathbf{I} = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 3 \\ -5 & 2 & 7 \end{bmatrix} - \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} = \begin{bmatrix} 3-t & 2 & -1 \\ 0 & 4-t & 3 \\ -5 & 2 & 7-t \end{bmatrix}$$

6. (a) $\mathbf{A}_1\mathbf{B} = \mathbf{A}_2\mathbf{B} = \begin{bmatrix} 5 & 10 \\ -4 & -8 \end{bmatrix}$

(b) $\mathbf{AB} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$

7. $\det(\mathbf{AB}) = 0 = 0\cdot 0 = \det(\mathbf{A})\cdot\det(\mathbf{B})$

8. $\det(\mathbf{AB}) = \det(\mathbf{BA}) = 144$ (with $\mathbf{AB}$ and $\mathbf{BA}$ as in Problem 5)

9.
$$(\mathbf{AB})' = \begin{bmatrix} t-4t^2+6t^3 & t+t^2-4t^3+8t^4 \\ 3t+t^3-t^4 & 4t^2+t^3+t^4 \end{bmatrix}' = \begin{bmatrix} 1-8t+18t^2 & 1+2t-12t^2+32t^3 \\ 3+3t^2-4t^3 & 8t+3t^2+4t^3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A'B}+\mathbf{AB'} &= \begin{bmatrix} 1 & 2 \\ 3t^2 & -\dfrac{1}{t^2} \end{bmatrix}\begin{bmatrix} 1-t & 1+t \\ 3t^2 & 4t^3 \end{bmatrix} + \begin{bmatrix} t & 2t-1 \\ t^3 & \dfrac{1}{t} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 6t & 12t^2 \end{bmatrix} \\ &= \begin{bmatrix} 1-t+6t^2 & 1+t+8t^3 \\ -3+3t^2-3t^3 & -4t+3t^2+3t^3 \end{bmatrix} + \begin{bmatrix} -7t+12t^2 & t-12t^2+24t^3 \\ 6-t^3 & 12t+t^3 \end{bmatrix} \\ &= \begin{bmatrix} 1-8t+18t^2 & 1+2t-12t^2+32t^3 \\ 3+3t^2-4t^3 & 8t+3t^2+4t^3 \end{bmatrix} \end{aligned}$$

10.
$$(\mathbf{AB})' = \begin{bmatrix} 3t^3+3e^t+2t\,e^{-t} \\ 3t \\ 3t^4+24t-2e^{-t} \end{bmatrix}' = \begin{bmatrix} 9t^2+3e^t+2e^{-t}-2t\,e^{-t} \\ 3 \\ 12t^3+24+2e^{-t} \end{bmatrix}$$

$$\begin{aligned} \mathbf{A'B}+\mathbf{AB'} &= \begin{bmatrix} e^t & 1 & 2t \\ -1 & 0 & 0 \\ 8 & 0 & 3t^2 \end{bmatrix}\begin{bmatrix} 3 \\ 2e^{-t} \\ 3t \end{bmatrix} + \begin{bmatrix} e^t & t & t^2 \\ -t & 0 & 2 \\ 8t & -1 & t^3 \end{bmatrix}\begin{bmatrix} 0 \\ -2e^{-t} \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 6t^2+3e^t+2e^{-t} \\ -3 \\ 24+9t^3 \end{bmatrix} + \begin{bmatrix} 3t^2-2t\,e^{-t} \\ 6 \\ 3t^3+2e^{-t} \end{bmatrix} = \begin{bmatrix} 9t^2+3e^t+2e^{-t}-2t\,e^{-t} \\ 3 \\ 12t^3+24+2e^{-t} \end{bmatrix} \end{aligned}$$

11. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

12. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

13. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$

14. $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} t & -e^t \\ e^{-t} & t^2 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

15. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

16. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 0 & 5 & -7 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

17. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$

18. $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} t & -1 & e^t \\ 2 & t^2 & -1 \\ e^{-t} & 3t & t^3 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

19. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

20. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\quad \mathbf{P}(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $\quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ t \\ t^2 \\ t^3 \end{bmatrix}$

21. $$W(t) = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = e^{3t} \neq 0$$

$$\mathbf{x}_1' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \mathbf{A}\mathbf{x}_1$$

$$\mathbf{x}_2' = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \mathbf{A}\mathbf{x}_2$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} + c_2\begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 2c_1e^t + c_2e^{2t} \\ -3c_1e^t - c_2e^{2t} \end{bmatrix}$$

In most of Problems 22–30, we omit the verifications of the given solutions. In each case, this is simply a matter of calculating both the derivative $\mathbf{x}_i'$ of the given solution vector and the product $\mathbf{A}\mathbf{x}_i$ (where $\mathbf{A}$ is the coefficient matrix in the given differential equation) to verify that $\mathbf{x}_i' = \mathbf{A}\mathbf{x}_i$ (just as in the verification of the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ in Problem 21 above).

22. $$W(t) = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = -5e^{3t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} + c_2\begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} c_1e^{3t} + 2c_2e^{-2t} \\ 3c_1e^{3t} + c_2e^{-2t} \end{bmatrix}$$

23. $$W(t) = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 5 \end{bmatrix}e^{-2t} = \begin{bmatrix} c_1e^{2t} + c_2e^{-2t} \\ c_1e^{2t} + 5c_2e^{-2t} \end{bmatrix}$$

24. $$W(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{vmatrix} = e^{5t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{2t} = \begin{bmatrix} c_1e^{3t} + c_2e^{2t} \\ -c_1e^{3t} - 2c_2e^{2t} \end{bmatrix}$$

25. $$W(t) = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 7e^{-3t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} 3c_1e^{2t} + c_2e^{-5t} \\ 2c_1e^{2t} + 3c_2e^{-5t} \end{bmatrix}$$

26. $$W(t) = \begin{vmatrix} 2e^{t} & -2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & e^{3t} & e^{5t} \end{vmatrix} = 16e^{9t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}e^{3t} + c_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}e^{5t} = \begin{bmatrix} 2c_1e^{t} - 2c_2e^{3t} + 2c_3e^{5t} \\ 2c_1e^{t} - 2c_3e^{5t} \\ c_1e^{t} + c_2e^{3t} + c_3e^{5t} \end{bmatrix}$$

27. $$W(t) = \begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 3 \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-t} = \begin{bmatrix} c_1e^{2t} + c_2e^{-t} \\ c_1e^{2t} + c_3e^{-t} \\ c_1e^{2t} - c_2e^{-t} - c_3e^{-t} \end{bmatrix}$$

$$\mathbf{x}_1' = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}e^{t} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{t} = \mathbf{A}\mathbf{x}_1$$

$$\mathbf{x}_2' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{-t} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-t} = \mathbf{A}\mathbf{x}_2$$

$$\mathbf{x}_3' = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}e^{t} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-t} = \mathbf{A}\mathbf{x}_3$$

28. $$W(t) = \begin{vmatrix} 1 & 2e^{3t} & -e^{4t} \\ 6 & 3e^{3t} & 2e^{4t} \\ -13 & -2e^{3t} & e^{4t} \end{vmatrix} = -84e^{7t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1\begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}e^{3t} + c_3\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}e^{4t} = \begin{bmatrix} c_1 + 2c_2e^{3t} - c_3e^{4t} \\ 6c_1 + 3c_2e^{3t} + 2c_3e^{4t} \\ -13c_1 - 2c_2e^{3t} + c_3e^{4t} \end{bmatrix}$$

29. $$W(t) = \begin{vmatrix} 3e^{-2t} & e^{t} & e^{3t} \\ -2e^{-2t} & -e^{t} & -e^{3t} \\ 2e^{-2t} & e^{t} & 0 \end{vmatrix} = e^{2t} \neq 0$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}e^{t} + c_3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}e^{3t} = \begin{bmatrix} 3c_1e^{-2t} + c_2e^{t} + c_3e^{3t} \\ -2c_1e^{-2t} - c_2e^{t} - c_3e^{3t} \\ 2c_1e^{-2t} + c_2e^{t} \end{bmatrix}$$

30. $$W(t) = \begin{vmatrix} e^{-t} & 0 & 0 & e^{t} \\ 0 & 0 & e^{t} & 0 \\ 0 & e^{-t} & 0 & 3e^{t} \\ e^{-t} & 0 & -2e^{t} & 0 \end{vmatrix} = -e^{-t}\begin{vmatrix} e^{-t} & 0 & e^{t} \\ 0 & e^{t} & 0 \\ e^{-t} & -2e^{t} & 0 \end{vmatrix} = -\begin{vmatrix} 0 & e^{t} \\ e^{1t} & -2e^{t} \end{vmatrix} = 1 \neq 0$$

$$\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}e^{t} + c_4\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}e^{t} = \begin{bmatrix} c_1e^{-t} + c_4e^{t} \\ c_3e^{t} \\ c_2e^{-t} + 3c_4e^{t} \\ c_1e^{-t} - 2c_3e^{t} \end{bmatrix}$$

In Problems 31–34 (and similarly in Problems 35–40) we give first the scalar components $x_1(t)$ and $x_2(t)$ of a general solution, then the equations in the coefficients c_1 and c_2 that are obtained when the given initial conditions are imposed, and finally the resulting particular solution of the given system.

31. $x_1(t) = c_1e^{3t} + 2c_2e^{-2t}, \quad x_2(t) = 3c_1e^{3t} + c_2e^{-2t}$

$c_1 + 2c_2 = 0, \qquad 3c_1 + c_2 = 5$

$x_1(t) = 2e^{3t} - 2e^{-2t}, \quad x_2(t) = 6e^{3t} - e^{-2t}$

32. $x_1(t) = c_1e^{2t} + c_2e^{-2t}, \quad x_2(t) = c_1e^{2t} + 5c_2e^{-2t}$

$c_1 + c_2 = 5, \quad c_1 + 5c_2 = -3$

$x_1(t) = 7e^{2t} - 2e^{-2t}, \quad x_2(t) = 7e^{2t} - 10e^{-2t}$

33. $x_1(t) = c_1e^{3t} + c_2e^{2t}, \quad x_2(t) = -c_1e^{3t} - 2c_2e^{2t}$

$c_1 + c_2 = 11, \quad -c_1 - 2c_2 = -7$

$x_1(t) = 15e^{3t} - 4e^{2t}, \quad x_2(t) = -15e^{3t} + 8e^{2t}$

34. $x_1(t) = 3c_1e^{2t} + c_2e^{-5t}, \quad x_2(t) = 2c_1e^{2t} + 3c_2e^{-5t}$

$3c_1 + c_2 = 8, \quad 2c_1 + 3c_2 = 0$

$x_1(t) = \frac{8}{7}\left(9e^{2t} - 2e^{-5t}\right), \quad x_2(t) = \frac{48}{7}\left(e^{2t} - e^{-5t}\right)$

35. $x_1(t) = 2c_1e^{t} - 2c_2e^{3t} + 2c_3e^{5t}, \quad x_2(t) = 2c_1e^{t} - 2c_3e^{5t}, \quad x_3(t) = c_1e^{t} + c_2e^{3t} + c_3e^{5t}$

$2c_1 - 2c_2 + 2c_3 = 0, \quad 2c_1 - 2c_3 = 0, \quad c_1 + c_2 + c_3 = 4$

$x_1(t) = 2e^{t} - 4e^{3t} + 2e^{5t}, \quad x_2(t) = 2e^{t} - 2e^{5t}, \quad x_3(t) = e^{t} + 2e^{3t} + e^{5t}$

36. $x_1(t) = c_1e^{2t} + c_2e^{-t}, \quad x_2(t) = c_1e^{2t} + c_3e^{-t}, \quad x_3(t) = c_1e^{2t} - c_2e^{-t} - c_3e^{-t}$

$c_1 + c_2 = 10, \quad c_1 + c_3 = 12, \quad c_1 - c_2 - c_3 = -1$

$x_1(t) = 7e^{2t} + 3e^{-t}, \quad x_2(t) = 7e^{2t} + 5e^{-t}, \quad x_3(t) = 7e^{2t} - 8e^{-t}$

37. $x_1(t) = 3c_1e^{-2t} + c_2e^{t} + c_3e^{3t}, \quad x_2(t) = -2c_1e^{-2t} - c_2e^{t} - c_3e^{3t}, \quad x_3(t) = 2c_1e^{-2t} + c_2e^{t}$

$3c_1 + c_2 + c_3 = 1, \quad -2c_1 - c_2 - c_3 = 2, \quad 2c_1 + c_2 = 3$

$x_1(t) = 9e^{-2t} - 3e^{t} - 5e^{3t}, \quad x_2(t) = -6e^{-2t} + 3e^{t} + 5e^{3t}, \quad x_3(t) = 6e^{-2t} - 3e^{t}$

38. $x_1(t) = 3c_1e^{-2t} + c_2e^{t} + c_3e^{3t}, \quad x_2(t) = -2c_1e^{-2t} - c_2e^{t} - c_3e^{3t}, \quad x_3(t) = 2c_1e^{-2t} + c_2e^{t}$

$3c_1 + c_2 + c_3 = 5, \quad -2c_1 - c_2 - c_3 = -7, \quad 2c_1 + c_2 = 11$

$x_1(t) = -6e^{-2t} + 15e^{t} - 4e^{3t}, \quad x_2(t) = 4e^{-2t} - 15e^{t} + 4e^{3t}, \quad x_3(t) = -4e^{-2t} + 15e^{t}$

39. $x_1(t) = c_1e^{-t} + c_4e^t, \quad x_2(t) = c_3e^t, \quad x_3(t) = c_2e^{-t} + 3c_4e^t, \quad x_4(t) = c_1e^{-t} - 2c_3e^t$

$c_1 + c_4 = 1, \quad c_3 = 1, \quad c_2 + 3c_4 = 1, \quad c_1 - 2c_3 = 1$

$x_1(t) = 3e^{-t} - 2e^t, \quad x_2(t) = e^t, \quad x_3(t) = 7e^{-t} - 6e^t, \quad x_4(t) = 3e^{-t} - 2e^t$

40. $x_1(t) = c_1e^{-t} + c_4e^t, \quad x_2(t) = c_3e^t, \quad x_3(t) = c_2e^{-t} + 3c_4e^t, \quad x_4(t) = c_1e^{-t} - 2c_3e^t$

$c_1 + c_4 = 1, \quad c_3 = 3, \quad c_2 + 3c_4 = 4, \quad c_1 - 2c_3 = 7$

$x_1(t) = 13e^{-t} - 12e^t, \quad x_2(t) = 3e^t, \quad x_3(t) = 40e^{-t} - 36e^t, \quad x_4(t) = 13e^{-t} - 6e^t$

41. **(a)** $\mathbf{x}_2 = t\mathbf{x}_1$, so neither is a constant multiple of the other.

(b) $W(\mathbf{x}_1, \mathbf{x}_2) = 0$, whereas Theorem 2 would imply that $W \neq 0$ if $\mathbf{x}_1$ and $\mathbf{x}_2$ were independent solutions of a system of the indicated form.

42. If $x_{12}(t) = c\,x_{11}(t)$ and $x_{22}(t) = c\,x_{21}(t)$ then

$$W(t) = x_{11}(t)x_{22}(t) - x_{12}(t)x_{21}(t) = c\,x_{11}(t)x_{21}(t) - c x_{11}(t)x_{21}(t) = 0.$$

43. Suppose $W(a) = x_{11}(a)x_{22}(a) - x_{12}(a)x_{21}(a) = 0$. Then the coefficient determinant of the homogeneous linear system $c_1x_{11}(a) + c_2x_{12}(a) = 0, \ c_1x_{21}(a) + c_2x_{22}(a) = 0$ vanishes. The system therefore has a non-trivial solution $\{c_1, c_2\}$ such that $c_1\mathbf{x}_1(a) + c_2\mathbf{x}_2(a) = \mathbf{0}$. Then $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is a solution of $\mathbf{x}' = \mathbf{P}\mathbf{x}$ such that $\mathbf{x}(a) = \mathbf{0}$. It therefore follows (by uniqueness of solutions) that $\mathbf{x}(t) \equiv \mathbf{0}$, that is, $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) \equiv 0$ with c_1 and c_2 not both zero. Thus the solution vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent.

44. The argument is precisely the same, except with n solution vectors each having n component functions (rather than 2 solution vectors each having 2 component functions).

45. Suppose that $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \equiv \mathbf{0}$. Then the ith scalar component of this vector equation is $c_1x_{i1}(t) + c_2x_{i2}(t) + \cdots + c_nx_{in}(t) \equiv 0$. Hence the fact that the scalar functions $x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)$ are linear linearly independent implies that $c_1 = c_2 = \cdots c_n = 0$. Consequently the vector functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_n(t)$ are linearly independent.

SECTION 5.4

THE EIGENVALUE METHOD FOR HOMOGENEOUS LINEAR SYSTEMS

In each of Problems 1–16 we give the characteristic equation, the eigenvalues λ_1 and λ_2 of the coefficient matrix of the given system, the corresponding equations determining the associated eigenvectors $\mathbf{v}_1 = [a_1 \quad b_1]^T$ and $\mathbf{v}_2 = [a_2 \quad b_2]^T$, these eigenvectors, and the resulting scalar components $x_1(t)$ and $x_2(t)$ of a general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$ of the system.

1. Characteristic equation $\lambda^2 - 2\lambda - 3 = 0$

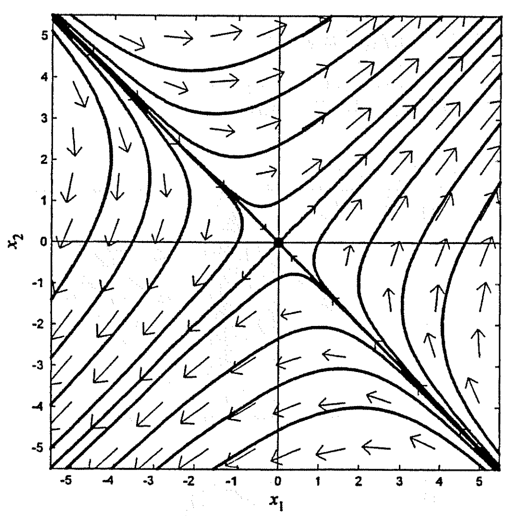
Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$

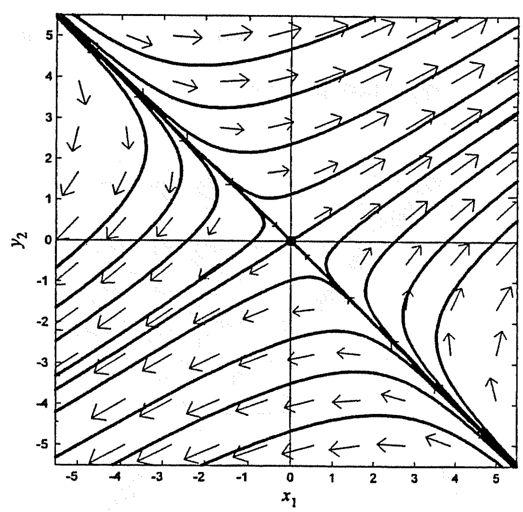
Eigenvector equations $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad -1]^T$ and $\mathbf{v}_2 = [1 \quad 1]^T$

$x_1(t) = c_1 e^{-t} + c_2 e^{3t}, \quad x_2(t) = -c_1 e^{-t} + c_2 e^{3t}$

The left-hand figure below shows a direction field and some typical solution curves for the system in Problem 1.





2. Characteristic equation $\lambda^2 - 3\lambda - 4 = 0$

Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$

Eigenvector equations $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad -1]^T$ and $\mathbf{v}_2 = [3 \quad 2]^T$

$x_1(t) = c_1e^{-t} + 3c_2e^{4t}, \quad x_2(t) = -c_1e^{-t} + 2c_2e^{4t}$

The right-hand figure at the bottom of the preceding page shows a direction field and some typical solution curves.

3. Characteristic equation $\lambda^2 - 5\lambda - 6 = 0$

Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 6$

Eigenvector equations $\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad -1]^T$ and $\mathbf{v}_2 = [4 \quad 3]^T$

$x_1(t) = c_1e^{-t} + 4c_2e^{6t}, \quad x_2(t) = -c_1e^{-t} + 3c_2e^{6t}$

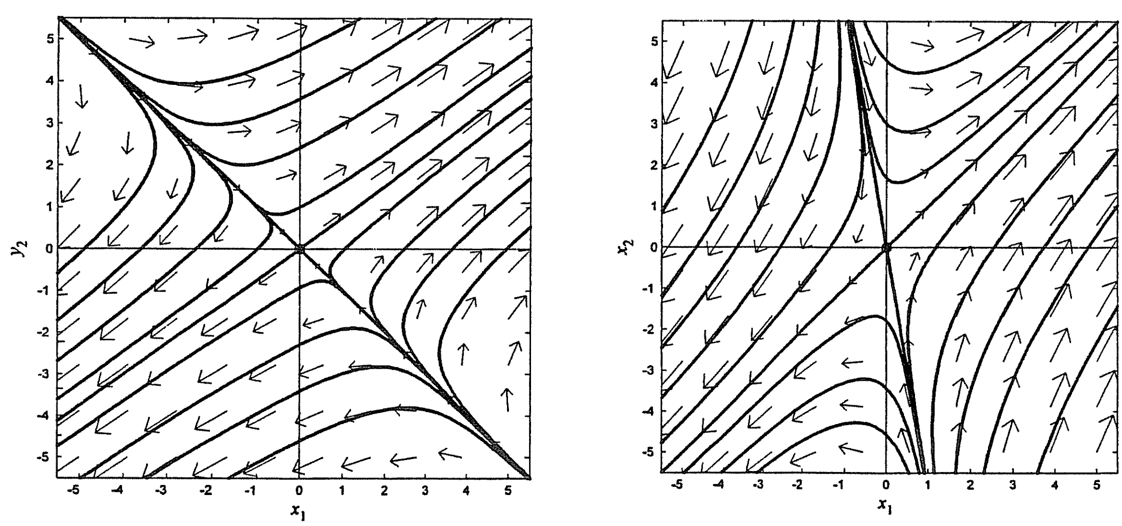
The equations

$$x_1(0) = c_1 + 4c_2 = 1$$
$$x_2(0) = -c_1 + 3c_2 = 1$$

yield $c_1 = -1/7$ and $c_2 = 2/7$, so the desired particular solution is given by

$$x_1(t) = \tfrac{1}{7}(-e^{-t} + 8e^{6t}), \quad x_2(t) = \tfrac{1}{7}(e^{-t} + 6e^{6t}).$$

The left-hand figure below shows a direction field and some typical solution curves.



4. Characteristic equation $\lambda^2 - 3\lambda - 10 = 0$

Eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$

Eigenvector equations $\begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \ \ -6]^T$ and $\mathbf{v}_2 = [1 \ \ 1]^T$

$x_1(t) = c_1e^{-2t} + c_2e^{5t}, \quad x_2(t) = -6c_1e^{-2t} + c_2e^{5t}$

The right-hand figure at the bottom of the preceding page shows a direction field and some typical solution curves.

5. Characteristic equation $\lambda^2 - 4\lambda - 5 = 0$

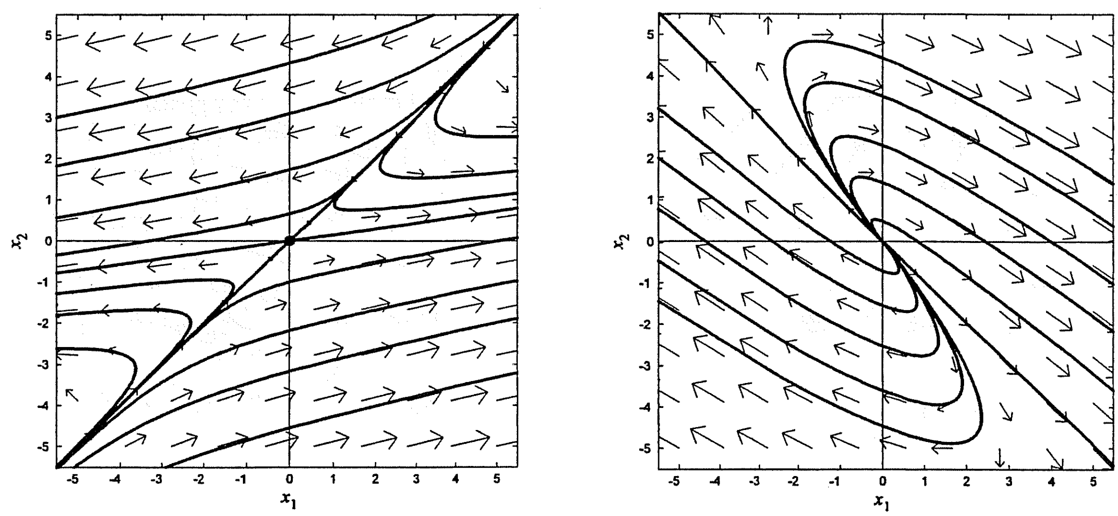
Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$

Eigenvector equations $\begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \ \ 1]^T$ and $\mathbf{v}_2 = [7 \ \ 1]^T$

$x_1(t) = c_1e^{-t} + 7c_2e^{5t}, \quad x_2(t) = c_1e^{-t} + c_2e^{5t}$

The left-hand figure below shows a direction field and some typical solution curves.



6. Characteristic equation $\lambda^2 - 7\lambda + 12 = 0$

Eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 4$

Eigenvector equations $\begin{bmatrix} 6 & 5 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 5 & 5 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [5 \quad -6]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$

$x_1(t) = 5c_1e^{3t} + c_2e^{4t}, \quad x_2(t) = -6c_1e^{3t} - c_2e^{4t}$

The initial conditions yield $c_1 = -1$ and $c_2 = 6$, so

$$x_1(t) = -5e^{3t} + 6e^{4t}, \quad x_2(t) = 6e^{3t} - 6e^{4t}.$$

The right-hand figure at the bottom of the preceding page shows a direction field and some typical solution curves.

7. Characteristic equation $\lambda^2 + 8\lambda - 9 = 0$

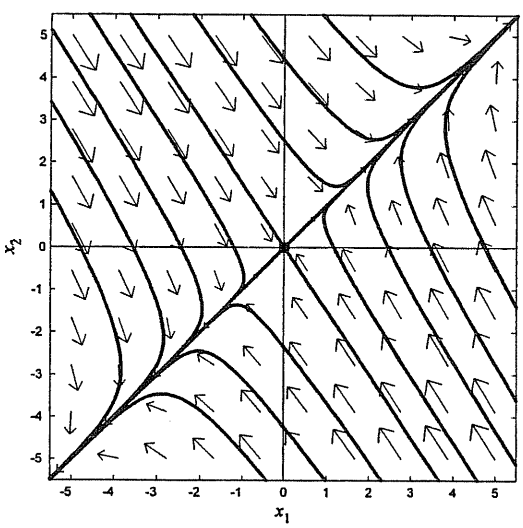
Eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -9$

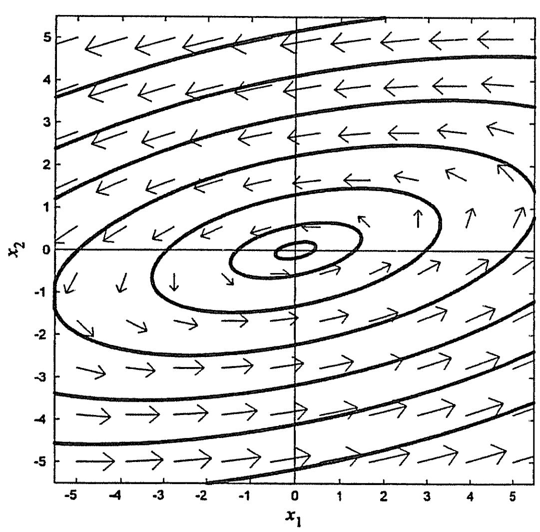
Eigenvector equations $\begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 6 & 4 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [2 \quad -3]^T$

$x_1(t) = c_1e^{t} + 2c_2e^{-9t}, \quad x_2(t) = c_1e^{t} - 3c_2e^{-9t}$

The left-hand figure below shows a direction field and some typical solution curves.





8. Characteristic equation $\lambda^2 + 4 = 0$

Eigenvalue $\lambda = 2i$

Eigenvector equation $\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [5 \quad 1-2i]^T$

$$\mathbf{x}(t) = \mathbf{v}e^{2it} = \begin{bmatrix} 5\cos 2t + 5i\sin 2t \\ (\cos 2t + 2\sin 2t) + i(\sin 2t - 2\cos 2t) \end{bmatrix}$$

$$x_1(t) = 5c_1\cos 2t + 5c_2\sin 2t$$

$$\begin{aligned} x_2(t) &= c_1(\cos 2t + 2\sin 2t) + c_2(\sin 2t - 2\cos 2t) \\ &= (c_1 - 2c_2)\cos 2t + (2c_1 + c_2)\sin 2t \end{aligned}$$

The right-hand figure at the bottom of the preceding page shows a direction field and some typical solution curves.

9. Characteristic equation $\lambda^2 + 16 = 0$

Eigenvalue $\lambda = 4i$

Eigenvector equation $\begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [5 \quad 2-4i]^T$

The real and imaginary parts of

$$\mathbf{x}(t) = \mathbf{v}e^{4it} = \begin{bmatrix} 5\cos 4t + 5i\sin 4t \\ (2\cos 4t + 4\sin 4t) + i(2\sin 4t - 4\cos 4t) \end{bmatrix}$$

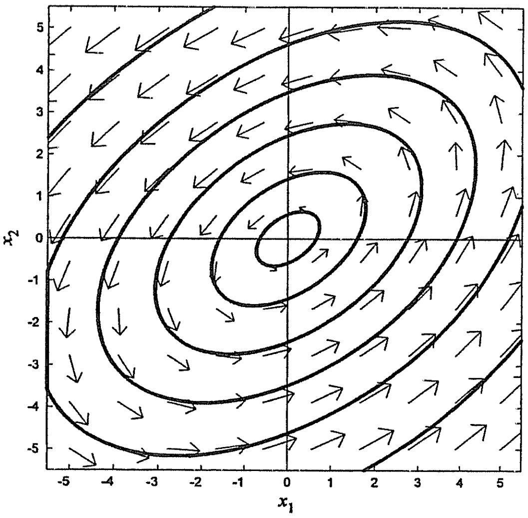
yield the general solution

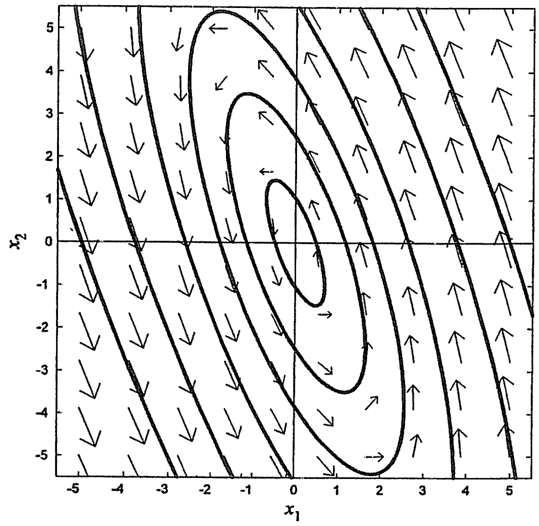
$$\begin{aligned} x_1(t) &= 5c_1\cos 4t + 5c_2\sin 4t \\ x_2(t) &= c_1(2\cos 4t + 4\sin 4t) + c_2(2\sin 4t - 4\cos 4t). \end{aligned}$$

The initial conditions $x_1(0) = 2$ and $x_2(0) = 3$ give $c_1 = 2/5$ and $c_2 = -11/20$, so the desired particular solution is

$$\begin{aligned} x_1(t) &= 2\cos 4t - \tfrac{11}{4}\sin 4t \\ x_2(t) &= 3\cos 4t + \tfrac{1}{2}\sin 4t. \end{aligned}$$

The left-hand figure below at the top of the next page shows a direction field and some typical solution curves.





10. Characteristic equation $\lambda^2 + 9 = 0$

Eigenvalue $\lambda = 3i$

Eigenvector equation $\begin{bmatrix} -3-3i & -2 \\ 9 & 3-3i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [-2 \quad 3+3i]^T$

$$\mathbf{x}(t) = \mathbf{v}e^{3it} = \begin{bmatrix} -2\cos 3t - 2i\sin 3t \\ (3\cos 3t - 3\sin 3t) + i(3\sin 3t + 3\cos 3t) \end{bmatrix}$$

$x_1(t) = -2c_1\cos 3t - 2c_2\sin 3t$

$x_2(t) = c_1(3\cos 3t - 3\sin 3t) + c_2(3\cos 3t + 3\sin 3t)$

$= (3c_1 + 3c_2)\cos 3t + (3c_2 - 3c_1)\sin 3t$

The right-hand figure above shows a direction field and some typical solution curves for the system in Problem 10.

11. Characteristic equation $\lambda^2 - 2\lambda + 5 = 0$

Eigenvalue $\lambda = 1 - 2i$

Eigenvector equation $\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [1 \quad i]^T$

The real and imaginary parts of

$$\mathbf{x}(t) = [1 \quad i]^T e^t(\cos 2t - i \sin 2t)$$
$$= e^t [\cos 2t \quad \sin 2t]^T + i e^t [-\sin 2t \quad \cos 2t]^T$$

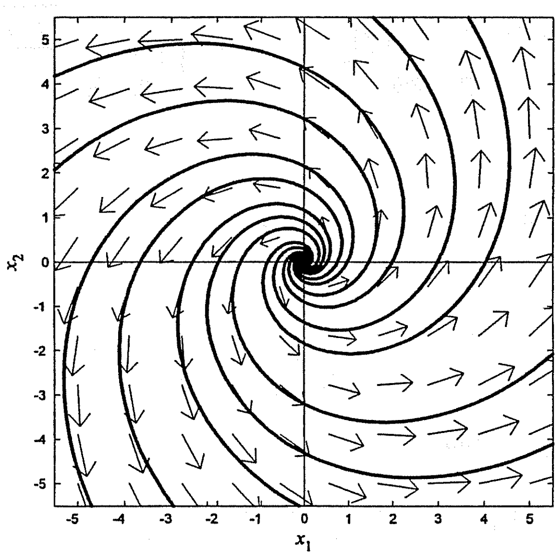
yield the general solution

$$x_1(t) = e^t(c_1 \cos 2t - c_2 \sin 2t)$$
$$x_2(t) = e^t(c_1 \sin 2t + c_2 \cos 2t).$$

The particular solution with $x_1(0) = 0$ and $x_2(0) = 4$ is obtained with $c_1 = 0$ and $c_2 = 4$, so

$$x_1(t) = -4e^t \sin 2t, \qquad x_2(t) = 4e^t \cos 2t.$$

The figure below shows a direction field and some typical solution curves.



12. Characteristic equation $\lambda^2 - 4\lambda + 8 = 0$

Eigenvalue $\lambda = 2 + 2i$

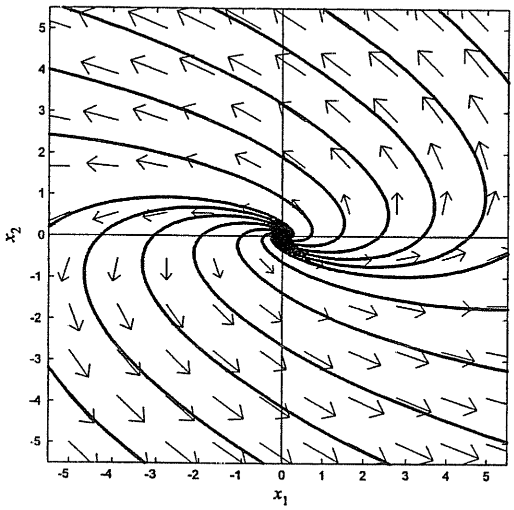
Eigenvector equation $\begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

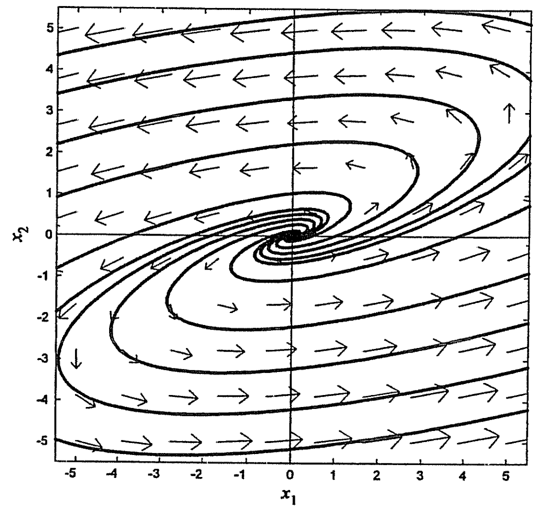
Eigenvector $\mathbf{v} = [-5 \quad 1+2i]^T$

$$\mathbf{x}(t) = \mathbf{v} e^{(2+2i)t} = e^{2t} \begin{bmatrix} -5\cos 2t - 5i \sin 2t \\ (\cos 2t - 2\sin 2t) + i(\sin 2t + 2\cos 2t) \end{bmatrix}$$

$$x_1(t) = e^{2t}(-5c_1 \cos 2t - 5c_2 \sin 2t)$$
$$x_2(t) = e^{2t} [c_1(\cos 2t - 2 \sin 2t) + c_2(2 \cos 2t + \sin 2t)]$$
$$= e^{2t} [(c_1 + 2c_2)\cos 2t + (-2c_1 + c_2)\sin 2t]$$

The left-hand figure below shows a direction field and some typical solution curves.





13. Characteristic equation $\lambda^2 - 4\lambda + 13 = 0$

Eigenvalue $\lambda = 2 - 3i$

Eigenvector equation $\begin{bmatrix} 3+3i & -9 \\ 2 & -3+3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvector $\mathbf{v} = [3 \quad 1+i]^T$

$$\mathbf{x}(t) = \mathbf{v}e^{(2-3i)t} = e^{2t}\begin{bmatrix} 3\cos 3t - 3i\sin 3t \\ (\cos 3t + \sin 3t) + i(\cos 3t - \sin 3t) \end{bmatrix}$$

$x_1(t) = 3e^{2t}(c_1\cos 3t - c_2\sin 3t)$

$x_2(t) = e^{2t}\,[(c_1 + c_2)\cos 3t + (c_1 - c_2)\sin 3t)]$.

The right-hand figure above page shows a direction field and some typical solution curves for the system in Problem 13.

14. Characteristic equation $\lambda^2 - 2\lambda + 5 = 0$

Eigenvalue $\lambda = 3 + 4i$

Eigenvector equation $\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

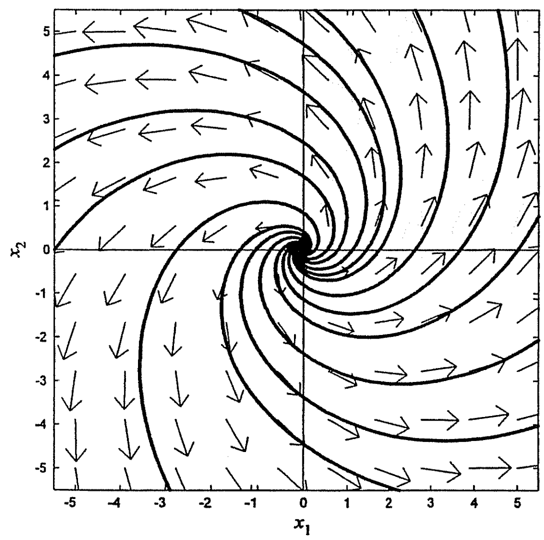
Eigenvector $\mathbf{v} = [1 \quad -i]^T$

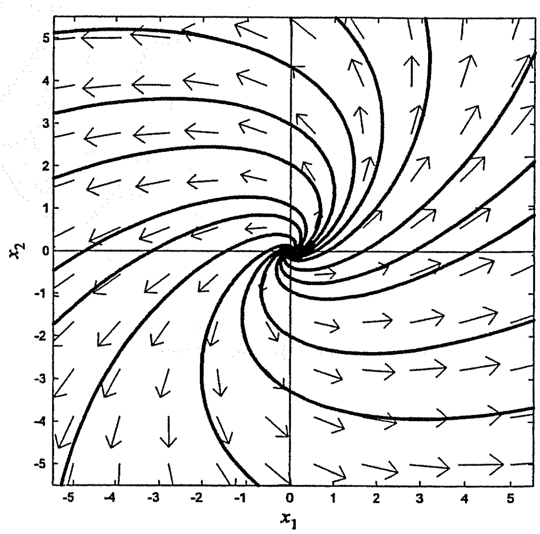
$$\mathbf{x}(t) = \mathbf{v}\,e^{(3+4i)\,t} = e^{3t}\begin{bmatrix}\cos 4t + i\sin 4t\\ \sin 4t - i\cos 4t\end{bmatrix}$$

$$x_1(t) = e^{3t}(c_1\cos 4t + c_2\sin 4t)$$

$$x_2(t) = e^{3t}(c_1\sin 4t - c_2\cos 4t)$$

The left hand figure below shows a direction field and some typical solution curves.





15. Characteristic equation $\lambda^2 - 10\lambda + 41 = 0$

Eigenvalue $\lambda = 5 - 4i$

Eigenvector equation $\begin{bmatrix}2+4i & -5\\ 4 & -2+4i\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

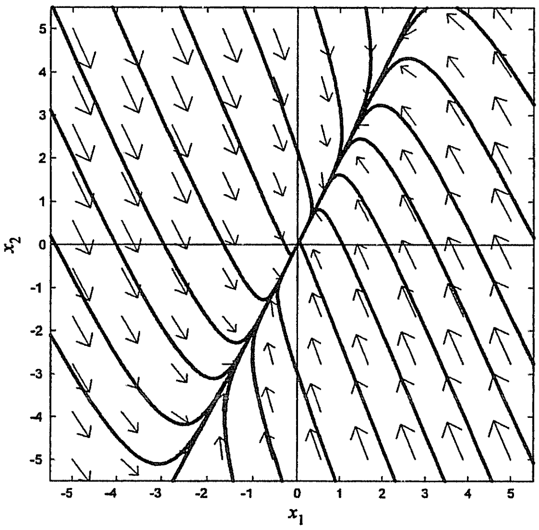
Eigenvector $\mathbf{v} = [5 \quad 2+4i]^T$

$$\mathbf{x}(t) = \mathbf{v}\,e^{(5-4i)\,t} = e^{5t}\begin{bmatrix}5\cos 4t - 5i\sin 4t\\ (2\cos 4t + 4\sin 4t) + i(4\cos 4t - 2\sin 4t)\end{bmatrix}$$

$$x_1(t) = 5e^{5t}(c_1\cos 4t - c_2\sin 4t)$$

$$x_2(t) = e^{5t}[(2c_1 + 4c_2)\cos 4t + (4c_1 - 2c_2)\sin 4t)]$$

The right-hand figure above shows a direction field and some typical solution curves.



16. Characteristic equation $\lambda^2 + 110\lambda + 1000 = 0$

Eigenvalues $\lambda_1 = -10$ and $\lambda_2 = -100$

Eigenvector equations $\begin{bmatrix} -40 & 20 \\ 100 & -50 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 50 & 20 \\ 100 & 40 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Eigenvectors $\mathbf{v}_1 = [1 \quad 2]^T$ and $\mathbf{v}_2 = [2 \ \ -5]^T$

$x_1(t) = c_1 e^{-10t} + 2c_2 e^{-100t}$

$x_2(t) = 2c_1 e^{-10t} - 5c_2 e^{-100t}$

The right-hand figure above shows a direction field and some typical solution curves.

17. Characteristic equation $-\lambda^3 + 15\lambda^2 - 54\lambda = 0$

Eigenvalues $\lambda_1 = 9$, $\lambda_2 = 6$, $\lambda_3 = 0$

Eigenvector equations

$$\begin{bmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \ \ 1 \ \ 1]^T$, $\mathbf{v}_2 = [1 \ \ -2 \ \ 1]^T$, $\mathbf{v}_3 = [1 \ \ 0 \ \ -1]^T$

$$x_1(t) = c_1e^{9t} + c_2e^{6t} + c_3$$
$$x_2(t) = c_1e^{9t} - 2c_2e^{6t}$$
$$x_3(t) = c_1e^{9t} + c_2e^{6t} - c_3$$

18. Characteristic equation $-\lambda^3 + 15\lambda^2 - 54\lambda = 0$

Eigenvalues $\lambda_1 = 9, \ \lambda_2 = 6, \ \lambda_3 = 0$

Eigenvector equations

$$\begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \ \ 2 \ \ 2]^T, \quad \mathbf{v}_2 = [0 \ \ 1 \ \ -1]^T, \quad \mathbf{v}_3 = [4 \ \ -1 \ \ -1]^T$

$$x_1(t) = c_1e^{9t} + 4c_3$$
$$x_2(t) = 2c_1e^{9t} + c_2e^{6t} - c_3$$
$$x_3(t) = 2c_1e^{9t} - c_2e^{6t} - c_3$$

19. Characteristic equation $-\lambda^3 + 12\lambda^2 - 45\lambda + 54 = 0$

Eigenvalues $\lambda_1 = 6, \ \lambda_2 = 3, \ \lambda_3 = 3$

Eigenvector equations

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \ \ 1 \ \ 1]^T, \quad \mathbf{v}_2 = [1 \ \ -2 \ \ 1]^T, \quad \mathbf{v}_3 = [1 \ \ 0 \ \ -1]^T$

$$x_1(t) = c_1e^{6t} + c_2e^{3t} + c_3e^{3t}$$
$$x_2(t) = c_1e^{6t} - 2c_2e^{3t}$$
$$x_3(t) = c_1e^{6t} + c_2e^{3t} - c_3e^{3t}$$

20. Characteristic equation $-\lambda^3 + 17\lambda^2 - 84\lambda + 108 = 0$

Eigenvalues $\lambda_1 = 9, \ \lambda_2 = 6, \ \lambda_3 = 2$

Eigenvector equations

$$\begin{bmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1 \;\; 1 \;\; 1]^T$, $\quad \mathbf{v}_2 = [1 \;\; -2 \;\; 1]^T$, $\quad \mathbf{v}_3 = [1 \;\; 0 \;\; -1]^T$

$x_1(t) = c_1e^{9t} + c_2e^{6t} + c_3e^{2t}$

$x_2(t) = c_1e^{9t} - 2c_2e^{6t}$

$x_3(t) = c_1e^{9t} + c_2e^{6t} - c_3e^{2t}$

21. Characteristic equation $-\lambda^3 + \lambda = 0$

Eigenvalues $\lambda_1 = 0, \; \lambda_2 = 1, \; \lambda_3 = -1$

Eigenvector equations

$$\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [6 \;\; 2 \;\; 5]^T$, $\quad \mathbf{v}_2 = [3 \;\; 1 \;\; 2]^T$, $\quad \mathbf{v}_3 = [2 \;\; 1 \;\; 2]^T$

$x_1(t) = 6c_1 + 3c_2e^{t} + 2c_3e^{-t}$

$x_2(t) = 2c_1 + c_2e^{t} + c_3e^{-t}$

$x_3(t) = 5c_1 + 2c_2e^{t} + 2c_3e^{-t}$

22. Characteristic equation $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$

Distinct eigenvalues $\lambda_1 = -2, \; \lambda_2 = 1, \; \lambda_3 = 3$

Eigenvector equations

$$\begin{bmatrix} 5 & 2 & 2 \\ -5 & -2 & -2 \\ 5 & 5 & 5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 2 \\ -5 & -5 & -2 \\ 5 & 5 & 2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 2 \\ -5 & -7 & -2 \\ 5 & 5 & 0 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [0 \;\; 1 \;\; -1]^T$, $\quad \mathbf{v}_2 = [1 \;\; -1 \;\; 0]^T$, $\quad \mathbf{v}_3 = [1 \;\; -1 \;\; 1]^T$

$x_1(t) = \qquad\quad c_2e^{t} + c_3e^{3t}$

$x_2(t) = c_1e^{-2t} - c_2e^{t} - c_3e^{3t}$

$x_3(t) = -c_1e^{-2t} \qquad\quad + c_3e^{3t}$

23. Characteristic equation $-\lambda^3+3\lambda^2+4\lambda-12 = 0$

Eigenvalues $\lambda_1 = 2,\ \lambda_2 = -2,\ \lambda_3 = 3$

Eigenvector equations

$$\begin{bmatrix} 1 & 1 & 1 \\ -5 & -5 & -1 \\ 5 & 5 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 & 1 & 1 \\ -5 & -1 & -1 \\ 5 & 5 & 5 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ -5 & -6 & -1 \\ 5 & 5 & 0 \end{bmatrix}\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors $\mathbf{v}_1 = [1\ \ -1\ \ \ 0]^T,\quad \mathbf{v}_2 = [0\ \ 1\ \ -1]^T,\qquad \mathbf{v}_3 = [1\ \ -1\ \ \ 1]^T$

$$\begin{aligned} x_1(t) &= c_1e^{2t} \qquad\qquad + c_3e^{3t} \\ x_2(t) &= -c_1e^{2t} + c_2e^{-2t} - c_3e^{3t} \\ x_3(t) &= \qquad\quad\ - c_2e^{-2t} + c_3e^{3t} \end{aligned}$$

24. Characteristic equation $-\lambda^3+\lambda^2-4\lambda+4 = 0$

Eigenvalues $\lambda = 1$ and $\lambda = \pm 2i$

With $\lambda = 1$ the eigenvector equation

$$\begin{bmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives eigenvector } \mathbf{v}_1 = [1\ \ -1\ \ \ 0]^T.$$

To find an eigenvector $\mathbf{v} = [a\ \ b\ \ c]^T$ associated with $\lambda = 2i$ we must find a nontrivial solution of the equations

$$\begin{aligned} (2-2i)a + \qquad\quad b \ - \qquad c &= 0 \\ -4a + (-3-2i)b \ - \qquad c &= 0 \\ 4a + \qquad 4b + (2-2i)c &= 0. \end{aligned}$$

Subtraction of the first two equations yields

$$(6-2i)a + (4+2i)b = 0,$$

so we take $a = 2+i$ and $b = -3+i$. Then the first equation gives $c = 3-i$. Thus $\mathbf{v} = [2+i\ \ \ -3+i\ \ \ \ 3-i]^T$. Finally

$$\begin{aligned} (2+i)e^{2it} &= (2\cos 2t - \sin 2t) + i\,(\cos 2t + 2\sin 2t) \\ (3-i)e^{2it} &= (3\cos 2t + \sin 2t) + i\,(3\sin 2t - \cos 2t), \end{aligned}$$

so the solution is

$$\begin{aligned} x_1(t) &= c_1e^t + c_2(2\cos 2t - \sin 2t) + c_3(\cos 2t + 2\sin 2t) \\ x_2(t) &= -c_1e^t - c_2(3\cos 2t + \sin 2t) + c_3(\cos 2t - 3\sin 2t) \\ x_3(t) &= \qquad\quad c_2(3\cos 2t + \sin 2t) + c_3(3\sin 2t - \cos 2t). \end{aligned}$$

25. Characteristic equation $-\lambda^3 + 4\lambda^2 - 13\lambda = 0$

Eigenvalues $\lambda = 0$ and $2 \pm 3i$

With $\lambda = 1$ the eigenvector equation

$$\begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ gives eigenvector $\mathbf{v}_1 = [1 \ \ -1 \ \ 0]^T$.

With $\lambda = 2 + 3i$ we solve the eigenvector equation

$$\begin{bmatrix} 3-3i & 5 & 2 \\ -6 & -8-3i & -5 \\ 6 & 6 & 3-3i \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to find the complex-valued eigenvector $\mathbf{v} = [1+i \ \ -2 \ \ 2]^T$. The corresponding complex-valued solution is

$$\mathbf{x}(t) = \mathbf{v}e^{(2+3i)t} = e^{2t}\begin{bmatrix} (\cos 3t - \sin 3t) + i(\cos 3t + \sin 3t \\ -2\cos 3t - 2i\sin 3t \\ 2\cos 3t + 2i\sin 3t \end{bmatrix}.$$

The scalar components of the resulting general solution are

$$\begin{aligned} x_1(t) &= c_1 + e^{2t}[(c_2 + c_3)\cos 3t + (-c_2 + c_3)\sin 3t] \\ x_2(t) &= -c_1 + 2e^{2t}(-c_2\cos 3t - c_3\sin 3t) \\ x_3(t) &= \qquad 2e^{2t}(c_2\cos 3t + c_3\sin 3t) \end{aligned}$$

26. Characteristic equation $-\lambda^3 + \lambda^2 + 4\lambda + 6 = 0$

Eigenvalues $\lambda = 3$ and $\lambda = -1 \pm i$

With $\lambda = 3$ the eigenvector equation

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & -4 & 2 \\ -9 & 4 & -4 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{gives eigenvector } \mathbf{v}_1 = [4 \quad 9 \quad 0]^T.$$

With $\lambda = -1 + i$ we solve the eigenvector equation

$$\begin{bmatrix} 4-i & 0 & 1 \\ 9 & i & 2 \\ -9 & 4 & i \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

to find the complex-valued eigenvector $\mathbf{v} = [1 \quad 2-i \quad -4+i\,]^T$. The corresponding complex-valued solution is

$$\mathbf{x}(t) = \mathbf{v}\,e^{(-1+i)t} = e^{-t}\begin{bmatrix} \cos t + i\sin t \\ (2\cos t + \sin t) + i(-\cos t + 2\sin t) \\ (-4\cos t - \sin t) + i(\cos t - 4\sin t) \end{bmatrix}$$

with real and imaginary parts $\mathbf{x}_2(t)$ and $\mathbf{x}_3(t)$. Assembling the general solution $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$, we get the scalar equations

$$x_1(t) = 4c_1e^{3t} + e^{-t}\,[c_2\cos t + c_3\sin t]$$

$$x_2(t) = 9c_1e^{3t} + e^{-t}\,[(2c_2 - c_3)\cos t + (c_2 + 2c_3)\sin t]$$

$$x_3(t) = e^{-t}\,[(-4c_2 + c_3)\cos t + (-c_2 - 4c_3)\sin t].$$

Finally, the given initial conditions yield the values $c_1 = 1$, $c_2 = -4$, $c_3 = 1$, so the desired particular solution is

$$x_1(t) = 4e^{3t} - e^{-t}(4\cos t - \sin t)$$

$$x_2(t) = 9e^{3t} - e^{-t}(9\cos t + 2\sin t)$$

$$x_3(t) = 17e^{-t}\cos t.$$

27. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.4 \end{bmatrix}$$

has characteristic equation $\lambda^2 + 0.6\lambda + 0.08 = 0$ with eigenvalues $\lambda_1 = -0.2$ and $\lambda_2 = -0.4$. We find easily that the associated eigenvectors are $\mathbf{v}_1 = [1 \quad 1]^T$ and

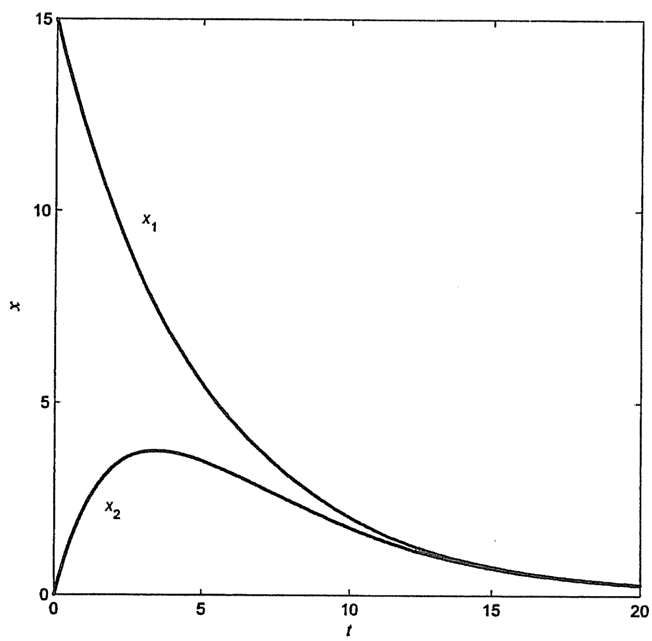
$\mathbf{v}_2 = [0 \quad 1]^T$, so we get the general solution

$$x_1(t) = c_1 e^{-0.2t}, \qquad x_2(t) = c_1 e^{-0.2t} + c_2 e^{-0.4t}.$$

The initial conditions $x_1(0) = 15$, $x_2(0) = 0$ give $c_1 = 15$ and $c_2 = -15$, so we get

$$x_1(t) = 15e^{-0.2t}, \qquad x_2(t) = 15e^{-0.2t} - 15e^{-0.4t}.$$

To find the maximum value of $x_2(t)$, we solve the equation $x_2'(t) = 0$ for $t = 5 \ln 2$, which gives the maximum value $x_2(5 \ln 2) = 3.75$ lb. The following figure shows the graphs of $x_1(t)$ and $x_2(t)$.



28. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -0.4 & 0 \\ 0.4 & -0.25 \end{bmatrix}$$

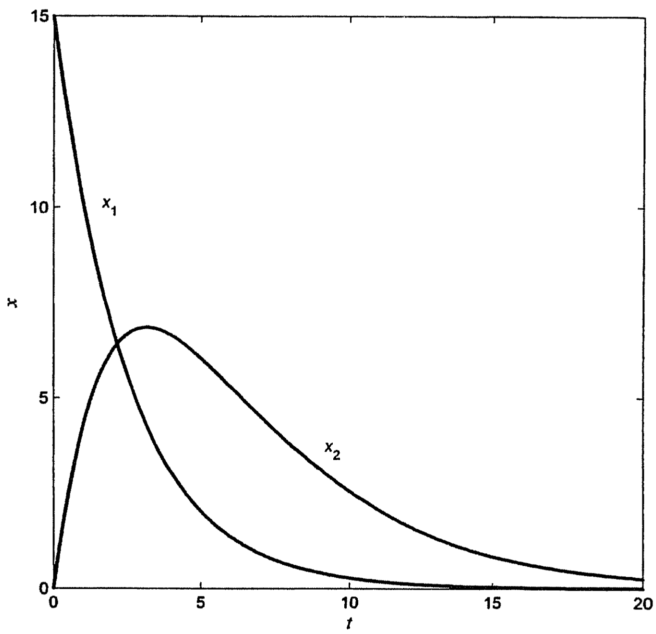
has characteristic equation $\lambda^2 + 0.65\lambda + 0.10 = 0$ with eigenvalues $\lambda_1 = -0.4$ and $\lambda_2 = -0.25$. We find easily that the associated eigenvectors are $\mathbf{v}_1 = [3 \quad -8]^T$ and $\mathbf{v}_2 = [0 \quad 1]^T$, so we get the general solution

$$x_1(t) = 3c_1 e^{-0.2t}, \qquad x_2(t) = -8c_1 e^{-0.2t} + c_2 e^{-0.4t}.$$

The initial conditions $x_1(0)=15,\ x_2(0)=0$ give $c_1=5$ and $c_2=40$, so we get

$$x_1(t) = 15e^{-0.4t}, \qquad x_2(t) = -40e^{-0.4t}+40e^{-0.25t}.$$

To find the maximum value of $x_2(t)$, we solve the equation $x_2'(t)=0$ for $t_m = \frac{20}{3}\ln\frac{8}{5}$, which gives the maximum value $x_2(t_m)\approx 6.85$ lb. The following figure shows the graphs of $x_1(t)$ and $x_2(t)$.



29. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}$$

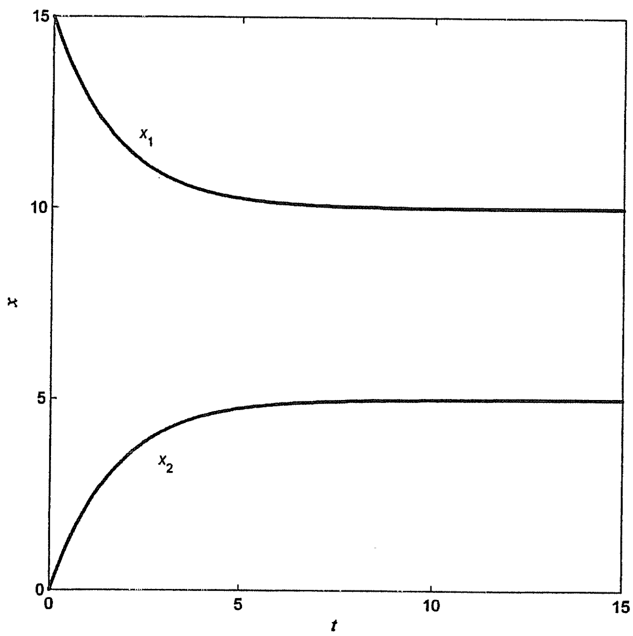
has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.6$, with eigenvectors $\mathbf{v}_1 = [2 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$ that yield the general solution

$$x_1(t) = 2c_1 + c_2e^{-0.6t}, \qquad x_2(t) = c_1 - c_2e^{-0.6t}.$$

The initial conditions $x_1(0)=15,\ x_2(0)=0$ give $c_1=c_2=5$, so we get

$$x_1(t) = 10+5e^{-0.6t}, \qquad x_2(t) = 5-5e^{-0.6t}.$$

The figure at the top of the next page shows the graphs of $x_1(t)$ and $x_2(t)$.



30. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -0.4 & 0.25 \\ 0.4 & -0.25 \end{bmatrix}$$

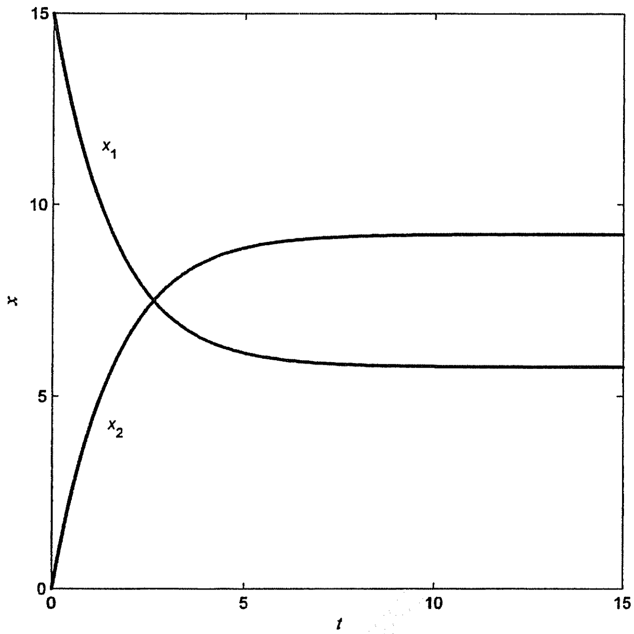
has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.65$, with eigenvectors $\mathbf{v}_1 = [5 \quad 8]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$ that yield the general solution

$$x_1(t) = 5c_1 + c_2e^{-0.65t}, \quad x_2(t) = 8c_1 - c_2e^{-0.65t}.$$

The initial conditions $x_1(0) = 15$, $x_2(0) = 0$ give $c_1 = 15/13$, $c_2 = 120/13$, so we get

$$x_1(t) = (\ 75 + 120e^{-0.65t})/13$$
$$x_2(t) = (120 - 120e^{-0.65t})/13.$$

The figure at the top of the next page shows the graphs of $x_1(t)$ and $x_2(t)$.



31. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -3$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = [1 \ \ 1 \ \ 1]^T$, $\mathbf{v}_2 = [0 \ \ 1 \ \ 2]^T$, and $\mathbf{v}_3 = [0 \ \ 0 \ \ 1]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_1 e^{-t} + c_2 e^{-2t} \\ x_3(t) &= c_1 e^{-t} + 2c_2 e^{-2t} + c_3 e^{-3t}. \end{aligned}$$

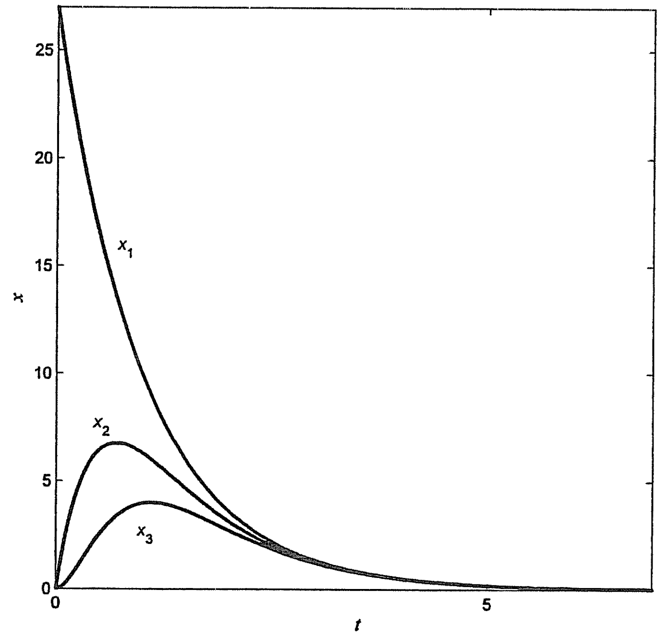
The initial conditions $x_1(0) = 27,\ x_2(0) = x_2(0) = 0$ give $c_1 = c_3 = 27,\ c_2 = -27,$ so we get

$$\begin{aligned} x_1(t) &= 27e^{-t} \\ x_2(t) &= 27e^{-t} - 27e^{-2t} \\ x_3(t) &= 27e^{-t} - 54e^{-2t} + 27e^{-3t}. \end{aligned}$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$3e^{-2t} - 4e^{-t} + 1 = \left(3e^{-t} - 1\right)\left(e^{-t} - 1\right) = 0$$

with positive solution $t_m = \ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3(\ln 3) = 4$ pounds. The figure below shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.



32. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -3$, $\lambda_2 = -2$, and $\lambda_3 = -1$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = [1 \;\; -3 \;\; 3]^T$, $\mathbf{v}_2 = [0 \;\; -1 \;\; 2]^T$, and $\mathbf{v}_3 = [0 \;\; 0 \;\; 1]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= c_1 e^{-3t} \\ x_2(t) &= -3c_1 e^{-3t} - c_2 e^{-2t} \\ x_3(t) &= 3c_1 e^{-3t} + 2c_2 e^{-2t} + c_3 e^{-t}. \end{aligned}$$

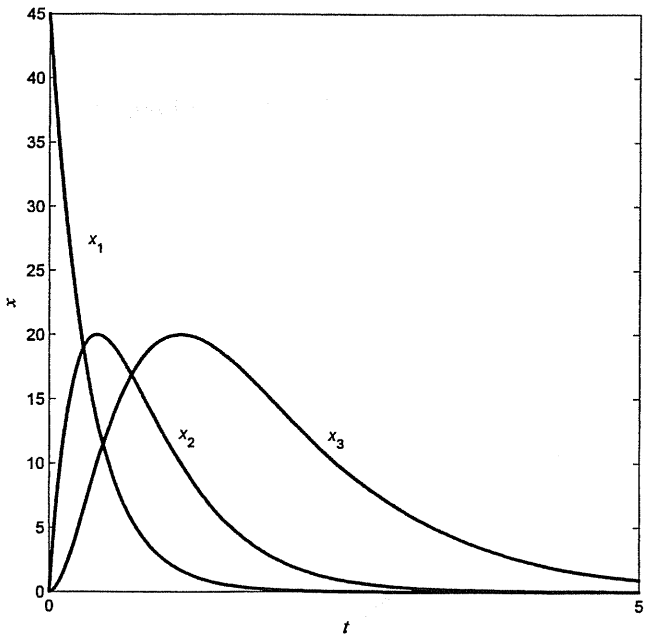
The initial conditions $x_1(0) = 45,\ x_2(0) = x_2(0) = 0$ give $c_1 = 45,\ c_2 = -135,\ c_3 = 135$, so we get

$$\begin{aligned} x_1(t) &= 45e^{-3t} \\ x_2(t) &= -135e^{-3t} + 135e^{-2t} \\ x_3(t) &= 135e^{-3t} - 270e^{-2t} + 135e^{-t}. \end{aligned}$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$3e^{-2t} - 4e^{-t} + 1 = \left(3e^{-t} - 1\right)\left(e^{-t} - 1\right) = 0$$

with positive solution $t_m = \ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3(\ln 3) = 20$ pounds. The figure below shows the graphs of $x_1(t), x_2(t)$, and $x_3(t)$.



33. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ 4 & -6 & 0 \\ 0 & 6 & -2 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -4$, $\lambda_2 = -6$, and $\lambda_3 = -2$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = [-1 \quad -2 \quad 6]^T$, $\mathbf{v}_2 = [0 \quad -2 \quad 3]^T$, and $\mathbf{v}_3 = [0 \quad 0 \quad 1]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= -c_1 e^{-4t} \\ x_2(t) &= -2c_1 e^{-4t} - 2c_2 e^{-6t} \\ x_3(t) &= 6c_1 e^{-4t} + 3c_2 e^{-6t} + c_3 e^{-2t}. \end{aligned}$$

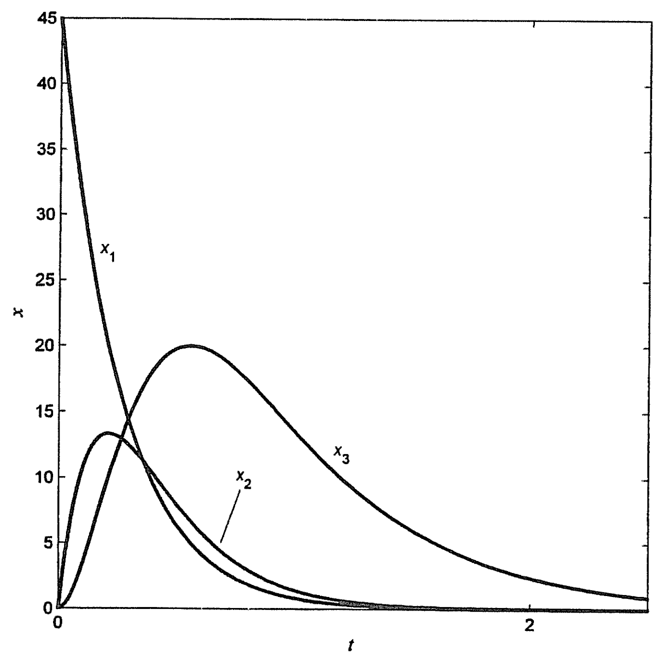
The initial conditions $x_1(0) = 45$, $x_2(0) = x_2(0) = 0$ give $c_1 = -45$, $c_2 = 45$, $c_3 = 135$, so we get

$$x_1(t) = 45e^{-4t}$$
$$x_2(t) = 90e^{-4t} - 90e^{-6t}$$
$$x_3(t) = -270e^{-4t} + 135e^{-6t} + 135e^{-2t}.$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$3e^{-4t} - 4e^{-2t} + 1 = \left(3e^{-2t} - 1\right)\left(e^{-2t} - 1\right) = 0$$

with positive solution $t_m = \frac{1}{2}\ln 3$. Thus the maximum amount of salt ever in tank 3 is $x_3(\frac{1}{2}\ln 3) = 20$ pounds. The figure below shows the graphs of $x_1(t), x_2(t)$, and $x_3(t)$.



34. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -5 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$

has as eigenvalues its diagonal elements $\lambda_1 = -3$, $\lambda_2 = -5$, and $\lambda_3 = -1$. We find readily that the associated eigenvectors are $\mathbf{v}_1 = [-4 \quad -6 \quad 15]^T$, $\mathbf{v}_2 = [0 \quad -4 \quad 5]^T$, and $\mathbf{v}_3 = [0 \quad 0 \quad 1]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= -4c_1 e^{-3t} \\ x_2(t) &= -6c_1 e^{-3t} - 4c_2 e^{-5t} \\ x_3(t) &= 15c_1 e^{-3t} + 5c_2 e^{-5t} + c_3 e^{-t}. \end{aligned}$$

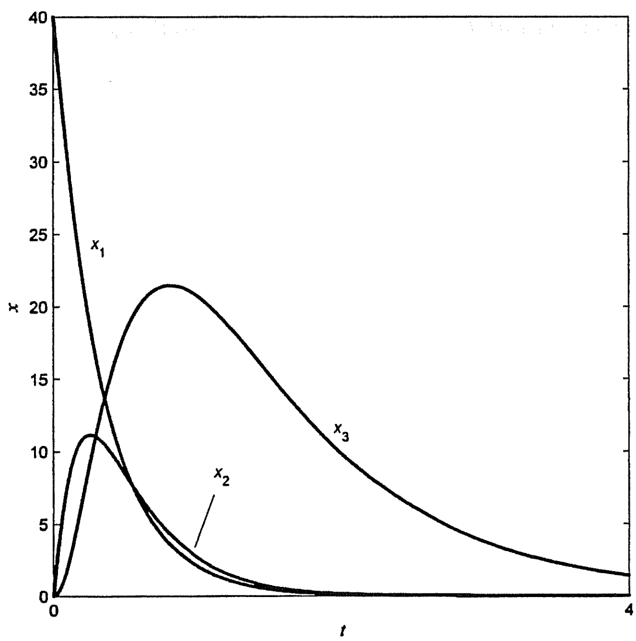
The initial conditions $x_1(0) = 40,\ x_2(0) = x_2(0) = 0$ give $c_1 = -10,\ c_2 = 15,\ c_3 = 75$, so we get

$$\begin{aligned} x_1(t) &= 40e^{-3t} \\ x_2(t) &= 60e^{-3t} - 60e^{-5t} \\ x_3(t) &= -150e^{-3t} + 75e^{-5t} + 75e^{-t}. \end{aligned}$$

The equation $x_3'(t) = 0$ simplifies to the equation

$$5e^{-4t} - 6e^{-2t} + 1 = \left(5e^{-2t} - 1\right)\left(e^{-2t} - 1\right) = 0$$

with positive solution $t_m = \frac{1}{2}\ln 5$. Thus the maximum amount of salt ever in tank 3 is $x_3(\frac{1}{2}\ln 5) \approx 21.4663$ pounds. The figure below shows the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.



35. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -6 & 0 & 3 \\ 6 & -20 & 0 \\ 0 & 20 & -3 \end{bmatrix}$$

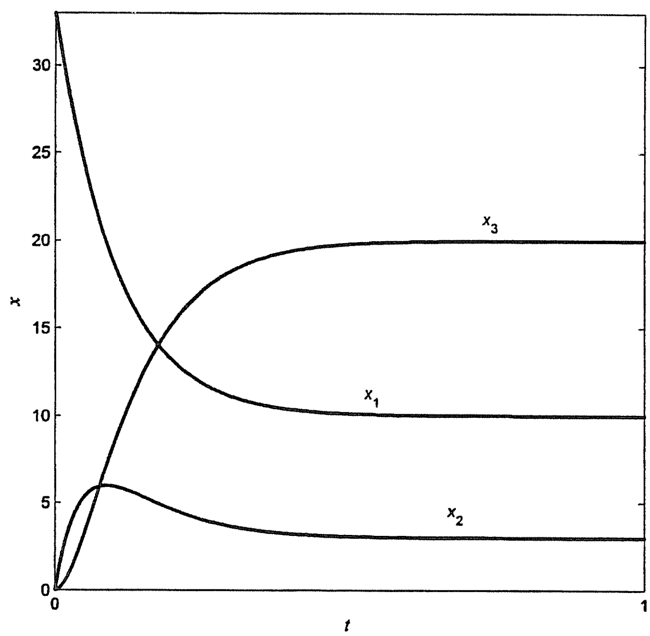
has characteristic equation $-\lambda^3 - 29\lambda^2 - 198\lambda = -\lambda(\lambda-18)(\lambda-11) = 0$ with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -18$, and $\lambda_2 = -11$. We find that associated eigenvectors are $\mathbf{v}_0 = [10 \quad 3 \quad 20]^T$, $\mathbf{v}_1 = [-1 \quad -3 \quad 4]^T$, and $\mathbf{v}_2 = [-3 \quad -2 \quad 5]^T$. The resulting general solution is solution is given by

$$\begin{aligned} x_1(t) &= 10c_0 - c_1 e^{-18t} - 3c_2 e^{-11t} \\ x_2(t) &= 3c_0 - 3c_1 e^{-18t} - 2c_2 e^{-11t} \\ x_3(t) &= 20c_0 + 4c_1 e^{-18t} + 5c_2 e^{-11t}. \end{aligned}$$

The initial conditions $x_1(0) = 33,\ x_2(0) = x_2(0) = 0$ give $c_1 = 1,\ c_2 = 55/7,\ c_3 = -72/7$, so we get

$$\begin{aligned} x_1(t) &= 10 - \tfrac{1}{7}\left(55e^{-18t} - 216e^{-11t}\right) \\ x_2(t) &= 3 - \tfrac{1}{7}\left(165e^{-18t} - 144e^{-11t}\right) \\ x_3(t) &= 20 + \tfrac{1}{7}\left(220e^{-18t} - 360e^{-11t}\right). \end{aligned}$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 10 lb, 3 lb, and 20 lb. The figure below shows the graphs of $x_1(t),\ x_2(t),$ and $x_3(t)$.



36. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix}$$

has characteristic equation $-\lambda^3 - (6/5)\lambda^2 - (9/20)\lambda = 0$ with eigenvalues $\lambda_0 = 0$, $\lambda_1 = -3(2+i)/10$, and $\lambda_2 = -3(2-i)/10$. The eigenvector equation

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with the eigenvalue $\lambda_0 = 0$ yields the associated eigenvector $\mathbf{v}_0 = [1 \quad 5/2 \quad 1]^T$ and consequently the constant solution $\mathbf{x}_0(t) \equiv \mathbf{v}_0$. Then the eigenvector equation

$$\begin{bmatrix} \frac{1}{10}(1+3i) & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{10}(4+3i) & 0 \\ 0 & \frac{1}{5} & \frac{1}{10}(1+3i) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with $\lambda_1 = -3(2+i)/10$ yields the complex-valued eigenvector $\mathbf{v}_1 = [-(1-3\,i)/2 \quad -(1+3\,i)/2 \quad 1]^T$. The corresponding complex-valued solution is

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{v}_1\, e^{(-6-3i)\,t/10} \\ &= \frac{1}{2} e^{-3t/5} \begin{bmatrix} \left(-\cos(3t/10) + 3\sin(3t/10)\right) + i\left(3\cos(3t/10) + \sin(3t/10)\right) \\ \left(-\cos(3t/10) - 3\sin(3t/10)\right) + i\left(-3\cos(3t/10) + \sin(3t/10)\right) \\ 2\cos(3t/10) - 2i\sin(3t/10) \end{bmatrix}. \end{aligned}$$

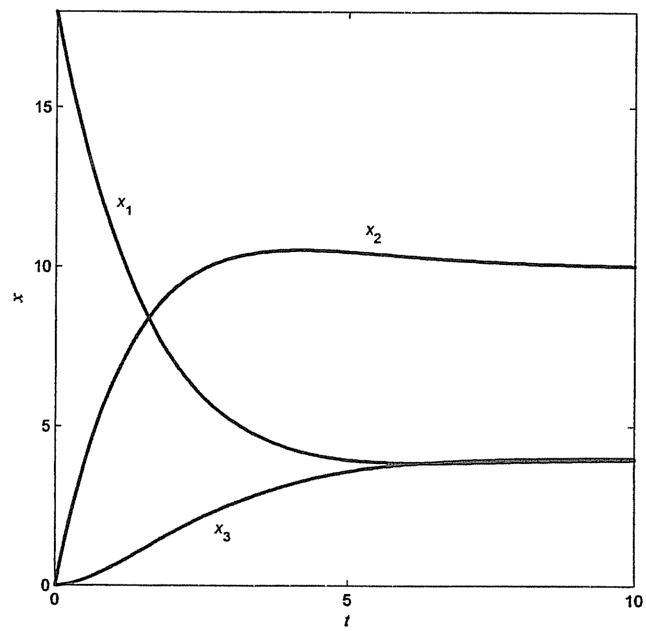
The scalar components of resulting general solution $\mathbf{x} = c_0\mathbf{x}_0 + c_1 \operatorname{Re}(\mathbf{x}_1) + c_2 \operatorname{Im}(\mathbf{x}_1)$ are given by

$$\begin{aligned} x_1(t) &= c_0 + \tfrac{1}{2} e^{-3t/5}\left[\left(-c_1 + 3c_2\right)\cos(3t/10) + \left(3c_1 + c_2\right)\sin(3t/10)\right] \\ x_2(t) &= \tfrac{5}{2} c_0 + \tfrac{1}{2} e^{-3t/5}\left[\left(-c_1 - 3c_2\right)\cos(3t/10) + \left(-3c_1 + c_2\right)\sin(3t/10)\right] \\ x_3(t) &= c_0 + e^{-3t/5}\left[c_1\cos(3t/10) - c_2\sin(3t/10)\right]. \end{aligned}$$

When we impose the initial conditions $x_1(0) = 18$, $x_2(0) = x_2(0) = 0$ we find that $c_0 = 4$, $c_1 = -4$, and $c_2 = 8$. This finally gives the particular solution

$$
\begin{aligned}
x_1(t) &= 4 + e^{-3t/5}\left[14\cos(3t/10) - 2\sin(3t/10)\right] \\
x_2(t) &= 10 - e^{-3t/5}\left[10\cos(3t/10) - 10\sin(3t/10)\right] \\
x_3(t) &= 4 - e^{-3t/5}\left[\ 4\cos(3t/10) + 8\sin(3t/10)\right].
\end{aligned}
$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 4 lb, 10 lb, and 4 lb. The figure below shows the graphs of $x_1(t),\ x_2(t),$ and $x_3(t)$.



37. The coefficient matrix

$$
\mathbf{A} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix}
$$

has characteristic equation $-\lambda^3 - 6\lambda^2 - 11\lambda = 0$ with eigenvalues $\lambda_0 = 0,\ \lambda_1 = -3 - i\sqrt{2}$, and $\lambda_2 = -3 + i\sqrt{2}$. The eigenvector equation

$$
\begin{bmatrix} -1 & 0 & 2 \\ 1 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

associated with the eigenvalue $\lambda_0 = 0$ yields the associated eigenvector

$\mathbf{v}_0 = [6 \quad 2 \quad 3]^T$ and consequently the constant solution $\mathbf{x}_0(t) \equiv \mathbf{v}_0$. Then the eigenvector equation

$$\begin{bmatrix} 2+i\sqrt{2} & 0 & 2 \\ 1 & i\sqrt{2} & 0 \\ 0 & 3 & 1+i\sqrt{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

associated with $\lambda_1 = -3 - i\sqrt{2}$ yields the complex-valued eigenvector $\mathbf{v}_1 = \left[(-2+i\sqrt{2})/3 \quad (-1-i\sqrt{2})/3 \quad 1\right]^T$. The corresponding complex-valued solution is

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{v}_1 e^{(-3-i\sqrt{2})t} \\ &= \frac{1}{3}e^{-3t} \begin{bmatrix} \left(-2\cos(t\sqrt{2}) + \sqrt{2}\sin(t\sqrt{2})\right) + i\left(\sqrt{2}\cos(t\sqrt{2}) + 2\sin(t\sqrt{2})\right) \\ \left(-\cos(t\sqrt{2}) - \sqrt{2}\sin(t\sqrt{2})\right) + i\left(-\sqrt{2}\cos(t\sqrt{2}) + \sin(t\sqrt{2})\right) \\ 3\cos(t\sqrt{2}) - 3i\sin(t\sqrt{2}) \end{bmatrix}. \end{aligned}$$

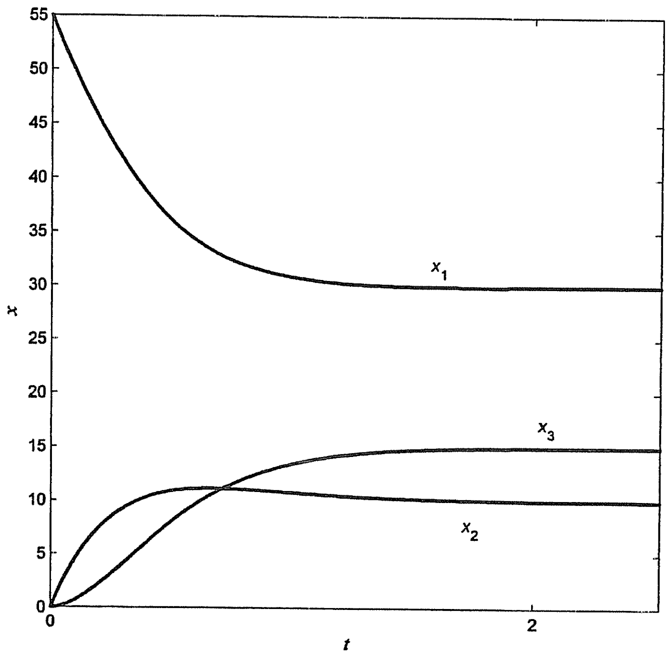
The scalar components of resulting general solution $\mathbf{x} = c_0\mathbf{x}_0 + c_1 \operatorname{Re}(\mathbf{x}_1) + c_2 \operatorname{Im}(\mathbf{x}_1)$ are given by

$$\begin{aligned} x_1(t) &= 6c_0 + \tfrac{1}{3}e^{-3t}\left[\left(-2c_1 + \sqrt{2}\,c_2\right)\cos(t\sqrt{2}) + \left(\sqrt{2}\,c_1 + 2c_2\right)\sin(t\sqrt{2})\right] \\ x_2(t) &= 2c_0 + \tfrac{1}{3}e^{-3t}\left[\left(-c_1 - \sqrt{2}\,c_2\right)\cos(t\sqrt{2}) + \left(-\sqrt{2}\,c_1 + c_2\right)\sin(t\sqrt{2})\right] \\ x_3(t) &= 3c_0 + e^{-3t}\left[c_1\cos(t\sqrt{2}) - c_2\sin(t\sqrt{2})\right]. \end{aligned}$$

When we impose the initial conditions $x_1(0) = 55,\ x_2(0) = x_2(0) = 0$ we find that $c_0 = 5,\ c_1 = -15,$ and $c_2 = 45/\sqrt{2}$. This finally gives the particular solution

$$\begin{aligned} x_1(t) &= 30 + e^{-3t}\left[25\cos(t\sqrt{2}) + 10\sqrt{2}\sin(t\sqrt{2})\right] \\ x_2(t) &= 10 - e^{-3t}\left[10\cos(t\sqrt{2}) - \tfrac{25}{2}\sqrt{2}\sin(t\sqrt{2})\right] \\ x_3(t) &= 15 - e^{-3t}\left[15\cos(t\sqrt{2}) + \tfrac{45}{2}\sqrt{2}\sin(t\sqrt{2})\right]. \end{aligned}$$

Thus the limiting amounts of salt in tanks 1, 2, and 3 are 30 lb, 10 lb, and 15 lb. The figure at the top of the next page shows the graphs of $x_1(t),\ x_2(t),$ and $x_3(t)$.



In Problems 38–41 the Maple command `with(linalg):eigenvects(A)`, the Mathematica command `Eigensystem[A]`, or the MATLAB command `[V,D] = eig(A)` can be used to find the eigenvalues and associated eigenvectors of the given coefficient matrix **A**.

38. Characteristic equation: $(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) = 0$

Eigenvalues and associated eigenvectors:

$$\begin{array}{ll} \lambda = 1, & \mathbf{v} = [1 \quad -2 \quad 3 \quad -4]^T \\ \lambda = 2, & \mathbf{v} = [0 \quad 1 \quad -3 \quad 6]^T \\ \lambda = 3, & \mathbf{v} = [0 \quad 0 \quad 1 \quad -4]^T \\ \lambda = 4, & \mathbf{v} = [0 \quad 0 \quad 0 \quad 1]^T \end{array}$$

Scalar solution equations:

$$\begin{aligned} x_1(t) &= c_1e^t \\ x_2(t) &= -2c_1e^t + c_2e^{2t} \\ x_3(t) &= 3c_1e^t - 3c_2e^{2t} + c_3e^{3t} \\ x_4(t) &= -4c_1e^t + 6c_2e^{2t} - 4c_3e^{3t} + c_4e^{4t} \end{aligned}$$

39. Characteristic equation: $(\lambda^2 - 1)(\lambda^2 - 4) = 0$

Eigenvalues and associated eigenvectors:

$$\begin{array}{ll} \lambda = 1, & \mathbf{v} = [3 \quad -2 \quad 4 \quad 1]^T \\ \lambda = -1, & \mathbf{v} = [0 \quad 0 \quad 1 \quad 0]^T \\ \lambda = 2, & \mathbf{v} = [0 \quad 1 \quad 0 \quad 0]^T \\ \lambda = -2, & \mathbf{v} = [1 \quad -1 \quad 0 \quad 0]^T \end{array}$$

Scalar solution equations:

$$\begin{aligned} x_1(t) &= 3c_1e^t + c_4e^{-2t} \\ x_2(t) &= -2c_1e^t + c_3e^{2t} - c_4e^{-2t} \\ x_3(t) &= 4c_1e^t + c_2e^{-t} \\ x_4(t) &= c_1e^t \end{aligned}$$

40. Characteristic equation: $(\lambda^2 - 4)(\lambda^2 - 25) = 0$

Eigenvalues and associated eigenvectors:

$$\begin{array}{ll} \lambda = 2, & \mathbf{v} = [1 \quad -3 \quad 0 \quad 0]^T \\ \lambda = -2, & \mathbf{v} = [0 \quad 3 \quad 0 \quad -1]^T \\ \lambda = 5, & \mathbf{v} = [0 \quad 0 \quad 1 \quad -3]^T \\ \lambda = -5, & \mathbf{v} = [0 \quad 1 \quad 0 \quad 0]^T \end{array}$$

Scalar solution equations:

$$\begin{aligned} x_1(t) &= c_1e^{2t} \\ x_2(t) &= -3c_1e^{2t} + 3c_2e^{-2t} - c_4e^{-5t} \\ x_3(t) &= c_3e^{5t} \\ x_4(t) &= -c_2e^{-2t} - 3c_3e^{5t} \end{aligned}$$

41. The eigenvectors associated with the respective eigenvalues $\lambda_1 = -3$, $\lambda_2 = -6$, $\lambda_3 = 10$, and $\lambda_4 = 15$ are

$$\begin{aligned} \mathbf{v}_1 &= [\ 1 \quad 0 \quad 0 \quad -1]^T \\ \mathbf{v}_2 &= [\ 0 \quad 1 \quad -1 \quad 0]^T \\ \mathbf{v}_3 &= [-2 \quad 1 \quad 1 \quad -2]^T \\ \mathbf{v}_4 &= [\ 1 \quad 2 \quad 2 \quad 1]^T. \end{aligned}$$

Hence the general solution has scalar component functions

$$
\begin{aligned}
x_1(t) &= c_1e^{-3t} \qquad\qquad - 2c_3e^{10t} + c_4e^{15t} \\
x_2(t) &= \qquad\qquad c_2e^{-6t} + c_3e^{10t} + 2c_4e^{15t} \\
x_3(t) &= \qquad\quad - c_2e^{-6t} + c_3e^{10t} + 2c_4e^{15t} \\
x_4(t) &= -c_1e^{-3t} \qquad\quad - 2c_3e^{10t} + c_4e^{15t}.
\end{aligned}
$$

The given initial conditions are satisfied by choosing $c_1 = c_2 = 0$, $c_3 = -1$, and $c_4 = 1$, so the desired particular solution is given by

$$
\begin{aligned}
x_1(t) &= 2e^{10t} + e^{15t} = x_4(t) \\
x_2(t) &= -e^{10t} + 2e^{15t} = x_3(t)\ .
\end{aligned}
$$

In Problems 42–50 we give a general solution in the form $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + \cdots$ that exhibits explicitly the eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ of the given coefficient matrix $\mathbf{A}$.

42. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}e^{5t}$$

43. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{4t} + c_3\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}e^{8t}$$

44. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}e^{6t} + c_3\begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix}e^{12t}$$

45. $$\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}e^{3t} + c_4\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}e^{6t}$$

46. $$\mathbf{x}(t) = c_1\begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}e^{-4t} + c_2\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}e^{4t} + c_4\begin{bmatrix} 3 \\ -2 \\ 3 \\ -3 \end{bmatrix}e^{8t}$$

47. $\mathbf{x}(t) = c_1\begin{bmatrix}2\\2\\1\\-1\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}1\\2\\-1\\1\end{bmatrix}e^{3t} + c_3\begin{bmatrix}2\\1\\1\\1\end{bmatrix}e^{6t} + c_4\begin{bmatrix}1\\-1\\2\\-1\end{bmatrix}e^{9t}$

48. $\mathbf{x}(t) = c_1\begin{bmatrix}1\\2\\-1\\2\end{bmatrix}e^{16t} + c_2\begin{bmatrix}2\\5\\1\\-1\end{bmatrix}e^{32t} + c_3\begin{bmatrix}3\\-1\\1\\2\end{bmatrix}e^{48t} + c_4\begin{bmatrix}1\\1\\2\\-3\end{bmatrix}e^{64t}$

49. $\mathbf{x}(t) = c_1\begin{bmatrix}1\\0\\3\\1\\1\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}0\\3\\0\\-1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\7\\1\\1\\1\end{bmatrix}e^{3t} + c_4\begin{bmatrix}0\\1\\0\\1\\1\end{bmatrix}e^{6t} + c_5\begin{bmatrix}2\\0\\5\\2\\1\end{bmatrix}e^{9t}$

50. $\mathbf{x}(t) = c_1\begin{bmatrix}0\\1\\1\\1\\0\\1\end{bmatrix}e^{-7t} + c_2\begin{bmatrix}1\\0\\0\\0\\1\\1\end{bmatrix}e^{-4t} + c_3\begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix}e^{3t} + c_4\begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix}e^{5t} + c_5\begin{bmatrix}1\\1\\0\\0\\0\\1\end{bmatrix}e^{9t} + c_6\begin{bmatrix}0\\0\\1\\-1\\-1\\0\end{bmatrix}e^{11t}$

SECTION 5.5

SECOND-ORDER SYSTEMS AND MECHANICAL APPLICATIONS

This section uses the eigenvalue method to exhibit realistic applications of linear systems. If a computer system like Maple, Mathematica, MATLAB, or even a TI-85/86/89/92 calculator is available, then a system of more than three railway cars, or a multistory building with four or more floors (as in the project), can be investigated. However, the problems in the text are intended for manual solution.

Problems 1–7 involve the system

$$m_1x_1'' = -(k_1 + k_2)x_1 + \quad k_2x_2$$
$$m_2x_2'' = \quad k_2x_1 - (k_2 + k_3)x_2$$

with various values of m_1, m_2 and k_1, k_2, k_3. In each problem we divide the first equation by m_1 and the second one by m_2 to obtain a second-order linear system $\mathbf{x}'' = \mathbf{A}\mathbf{x}$ in the standard form of Theorem 1 in this section. If the eigenvalues λ_1 and λ_2 are both negative, then the natural (circular) frequencies of the system are $\omega_1 = \sqrt{-\lambda_1}$ and $\omega_2 = \sqrt{-\lambda_2}$, and — according to Eq. (11) in Theorem 1 of this section — the eigenvalues $\mathbf{v}_1$ and $\mathbf{v}_2$ associated with λ_1 and λ_2 determine the natural modes of oscillations at these frequencies.

1. The matrix $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ has eigenvalues $\lambda_0 = 0$ and $\lambda_1 = -4$ with associated eigenvalues $\mathbf{v}_0 = [1 \quad 1]^T$ and $\mathbf{v}_1 = [1 \quad -1]^T$. Thus we have the special case described in Eq. (12) of Theorem 1, and a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 + a_2 t + b_1\cos 2t + b_2\sin 2t, \\ x_2(t) &= a_1 + a_2 t - b_1\cos 2t - b_2\sin 2t. \end{aligned}$$

The natural frequencies are $\omega_1 = 0$ and $\omega_2 = 2$. In the degenerate natural mode with "frequency" $\omega_1 = 0$ the two masses move by translation without oscillating. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

2. The matrix $\mathbf{A} = \begin{bmatrix} -5 & 4 \\ 5 & -5 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -9$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1\cos t + a_2\sin t + b_1\cos 3t + b_2\sin 3t, \\ x_2(t) &= a_1\cos t + a_2\sin t - b_1\cos 3t - b_2\sin 3t. \end{aligned}$$

3. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -4$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [2 \quad -1]^T$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1\cos t + a_2\sin t + 2b_1\cos 2t + 2b_2\sin 2t, \\ x_2(t) &= a_1\cos t + a_2\sin t - b_1\cos 2t - b_2\sin 2t. \end{aligned}$$

The natural frequencies are $\omega_1 = 1$ and $\omega_2 = 2$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. In the natural mode with frequency ω_2 they move in opposite directions with the amplitude of oscillation of m_1 twice that of m_2.

4. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -5$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos t\sqrt{5} + b_2 \sin t\sqrt{5}, \\ x_2(t) &= a_1 \cos t + a_2 \sin t - b_1 \cos t\sqrt{5} - b_2 \sin t\sqrt{5}. \end{aligned}$$

The natural frequencies are $\omega_1 = 1$ and $\omega_2 = \sqrt{5}$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

5. The matrix $\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + b_1 \cos 2t + b_2 \sin 2t, \\ x_2(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} - b_1 \cos 2t - b_2 \sin 2t. \end{aligned}$$

The natural frequencies are $\omega_1 = \sqrt{2}$ and $\omega_2 = 2$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

6. The matrix $\mathbf{A} = \begin{bmatrix} -6 & 4 \\ 2 & -4 \end{bmatrix}$ has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -8$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [2 \quad -1]^T$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} + 2b_1 \cos t\sqrt{8} + 2b_2 \sin t\sqrt{8}, \\ x_2(t) &= a_1 \cos t\sqrt{2} + a_2 \sin t\sqrt{2} - b_1 \cos t\sqrt{8} - b_2 \sin t\sqrt{8}. \end{aligned}$$

The natural frequencies are $\omega_1 = \sqrt{2}$ and $\omega_2 = \sqrt{8}$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. In the natural mode with frequency ω_2 they move in opposite directions with the amplitude of oscillation of m_1 twice that of m_2.

7. The matrix $\mathbf{A} = \begin{bmatrix} -10 & 6 \\ 6 & -10 \end{bmatrix}$ has eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -16$ with associated eigenvalues $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -1]^T$. Hence a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t, \\ x_2(t) &= a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t. \end{aligned}$$

The natural frequencies are $\omega_1 = 2$ and $\omega_2 = 4$. In the natural mode with frequency ω_1, the two masses m_1 and m_2 move in the same direction with equal amplitudes of oscillation. At frequency ω_2 they move in opposite directions with equal amplitudes.

8. Substitution of the trial solution $x_1 = c_1 \cos 5t,\ x_2 = c_2 \cos 5t$ in the system

$$x_1'' = -5x_1 + 4x_2 + 96\cos 5t, \qquad x_2'' = 4x_1 - 5x_2$$

yields $c_1 = -5,\ c_2 = 1$, so a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t + a_2 \sin t + b_1 \cos 3t + b_2 \sin 3t - 5\cos 5t, \\ x_2(t) &= a_1 \cos t + a_2 \sin t - b_1 \cos 3t - b_2 \sin 3t + \cos 5t. \end{aligned}$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 2,\ a_2 = 0,\ b_1 = 3,\ b_2 = 0$. The resulting particular solution is

$$\begin{aligned} x_1(t) &= 2\cos t + 3\cos 3t - 5\cos 5t, \\ x_2(t) &= 2\cos t - 3\cos 3t + \cos 5t. \end{aligned}$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and equal amplitudes;
- in opposite directions with frequency $\omega_2 = 3$ and equal amplitudes;
- in opposite directions with frequency $\omega_3 = 5$ and with the amplitude of motion of m_1 being 5 times that of m_2.

9. Substitution of the trial solution $x_1 = c_1 \cos 3t,\ x_2 = c_2 \cos 3t$ in the system

$$x_1'' = -3x_1 + 2x_2, \qquad 2x_2'' = 2x_1 - 4x_2 + 120\cos 3t$$

yields $c_1 = 3,\ c_2 = -9,$ so a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos t + a_2 \sin t + 2b_1 \cos 2t + 2b_2 \sin 2t + 3\cos 3t, \\ x_2(t) &= a_1 \cos t + a_2 \sin t - b_1 \cos 2t - b_2 \sin 2t - 9\cos 3t. \end{aligned}$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 5,\ a_2 = 0,\ b_1 = -4,\ b_2 = 0.$ The resulting particular solution is

$$\begin{aligned} x_1(t) &= 5\cos t - 8\cos 2t + 3\cos 3t, \\ x_2(t) &= 5\cos t + 4\cos 2t - 9\cos 3t. \end{aligned}$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and equal amplitudes;
- in opposite directions with frequency $\omega_2 = 2$ and with the amplitude of motion of m_1 being twice that of m_2;
- in opposite directions with frequency $\omega_3 = 3$ and with the amplitude of motion of m_2 being 3 times that of m_1.

10. Substitution of the trial solution $x_1 = c_1 \cos t,\ x_2 = c_2 \cos t$ in the system

$$x_1'' = -10x_1 + 6x_2 + 30\cos t, \qquad x_2'' = 6x_1 - 10x_2 + 60\cos t$$

yields $c_1 = 14,\ c_2 = 16,$ so a general solution is given by

$$\begin{aligned} x_1(t) &= a_1 \cos 2t + a_2 \sin 2t + b_1 \cos 4t + b_2 \sin 4t + 14\cos t, \\ x_2(t) &= a_1 \cos 2t + a_2 \sin 2t - b_1 \cos 4t - b_2 \sin 4t + 16\cos t. \end{aligned}$$

Imposition of the initial conditions $x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0$ now yields $a_1 = 1,\ a_2 = 0,\ b_1 = -15,\ b_2 = 0.$ The resulting particular solution is

$$\begin{aligned} x_1(t) &= \cos 2t - 15\cos 4t + 14\cos t, \\ x_2(t) &= \cos 2t + 15\cos 4t + 16\cos t. \end{aligned}$$

We have a superposition of three oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 1$ and with the amplitude of motion of m_2 being 8/7 times that of m_1;
- in the same direction with frequency $\omega_2 = 2$ and equal amplitudes;
- in opposite directions with frequency $\omega_3 = 4$ and equal amplitudes.

11. (a) The matrix $A = \begin{bmatrix} -40 & 8 \\ 12 & -60 \end{bmatrix}$ has eigenvalues $\lambda_1 = -36$ and $\lambda_2 = -64$ with associated eigenvalues $\mathbf{v}_2 = [2 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -3]^T$. Hence a general solution is given by

$$x(t) = 2a_1 \cos 6t + 2a_2 \sin 6t + b_1 \cos 8t + b_2 \sin 8t,$$
$$y(t) = a_1 \cos 6t + a_2 \sin 6t - 3b_1 \cos 8t - 3b_2 \sin 8t.$$

The natural frequencies are $\omega_1 = 6$ and $\omega_2 = 8$. In mode 1 the two masses oscillate in the same direction with frequency $\omega_1 = 6$ and with the amplitude of motion of m_1 being twice that of m_2. In mode 2 the two masses oscillate in opposite directions with frequency $\omega_2 = 8$ and with the amplitude of motion of m_2 being 3 times that of m_1.

(b) Substitution of the trial solution $x = c_1 \cos 7t,\ y = c_2 \cos 7t$ in the system

$$x'' = -40x + 8y - 195\cos 7t, \qquad y'' = 12x - 60y - 195\cos 7t$$

yields $c_1 = 19,\ c_2 = 3$, so a general solution is given by

$$x(t) = 2a_1 \cos 6t + 2a_2 \sin 6t + b_1 \cos 8t + b_2 \sin 8t + 19\cos 7t,$$
$$y(t) = a_1 \cos 6t + a_2 \sin 6t - 3b_1 \cos 8t - 3b_2 \sin 8t + 3\cos 7t.$$

Imposition of the initial conditions $x(0) = 19,\ x'(0) = 12,\ y(0) = 3,\ y'(0) = 6$ now yields $a_1 = 0,\ a_2 = 1,\ b_1 = 0,\ b_2 = 0$. The resulting particular solution is

$$x(t) = 2\sin 6t + 19\cos 7t,$$
$$y(t) = \sin 6t + 3\cos 7t.$$

Thus the expected oscillation with frequency $\omega_2 = 8$ is missing, and we have a superposition of (only two) oscillations, in which the two masses move

- in the same direction with frequency $\omega_1 = 6$ and with the amplitude of motion of m_1 being twice that of m_2;
- in the same direction with frequency $\omega_3 = 7$ and with the amplitude of motion of m_1 being 19/3 times that of m_2.

12. The coefficient matrix $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ has characteristic polynomial

$$\lambda^3 + 6\lambda^2 + 10\lambda + 4 = (\lambda + 2)(\lambda^2 + 4\lambda + 2).$$

Its eigenvalues $\lambda_1 = -2,\ \lambda_2 = -2 - \sqrt{2},\ \lambda_3 = -2 + \sqrt{2}$ have associated eigenvectors

$\mathbf{v}_1 = [1 \quad 0 \quad -1]^T$, $\mathbf{v}_2 = [1 \quad -\sqrt{2} \quad 1]^T$, $\mathbf{v}_3 = [1 \quad \sqrt{2} \quad 1]^T$. Hence the system's three natural modes of oscillation have

- Natural frequency $\omega_1 = \sqrt{2}$ with amplitude ratios $1 : 0 : -1$.
- Natural frequency $\omega_2 = \sqrt{2+\sqrt{2}}$ with amplitude ratios $1 : -\sqrt{2} : 1$.
- Natural frequency $\omega_2 = \sqrt{2-\sqrt{2}}$ with amplitude ratios $1 : \sqrt{2} : 1$.

13. The coefficient matrix $\mathbf{A} = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -4 \end{bmatrix}$ has characteristic polynomial

$$-\lambda^3 - 12\lambda^2 - 40\lambda - 32 = -(\lambda+4)(\lambda^2 + 8\lambda + 8).$$

Its eigenvalues $\lambda_1 = -4$, $\lambda_2 = -4 - 2\sqrt{2}$, $\lambda_3 = -4 + 2\sqrt{2}$ have associated eigenvectors $\mathbf{v}_1 = [1 \quad 0 \quad -1]^T$, $\mathbf{v}_2 = [1 \quad -\sqrt{2} \quad 1]^T$, $\mathbf{v}_3 = [1 \quad \sqrt{2} \quad 1]^T$. Hence the system's three natural modes of oscillation have

- Natural frequency $\omega_1 = 2$ with amplitude ratios $1 : 0 : -1$.
- Natural frequency $\omega_2 = \sqrt{4+2\sqrt{2}}$ with amplitude ratios $1 : -\sqrt{2} : 1$.
- Natural frequency $\omega_2 = \sqrt{4-2\sqrt{2}}$ with amplitude ratios $1 : \sqrt{2} : 1$.

14. The equations of motion of the given system are

$$\begin{aligned} x_1'' &= -50x_1 + 10(x_2 - x_1) + 5\cos 10t \\ m_2 x_2'' &= -10(x_2 - x_1). \end{aligned}$$

When we substitute $x_1 = A\cos 10t$, $x_2 = B\cos 10t$ and cancel $\cos 10t$ throughout we get the equations

$$\begin{aligned} -40A - \qquad\qquad 10B &= 5 \\ -10A + (10 - 100m_2)B &= 0. \end{aligned}$$

If $m_2 = 0.1$ (slug) then it follows that $A = 0$, so the mass m_1 remains at rest.

15. First we need the general solution of the homogeneous system $\mathbf{x}'' = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{bmatrix} -50 & 25/2 \\ 50 & -50 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = -25$ and $\lambda_2 = -75$, so the natural frequencies of the system are $\omega_1 = 5$ and $\omega_2 = 5\sqrt{3}$. The associated eigenvectors are $\mathbf{v}_1 = [1 \quad 2]^T$ and $\mathbf{v}_2 = [1 \quad -2]^T$, so the complementary solution $\mathbf{x}_c(t)$ is given by

$$\begin{aligned} x_1(t) &= a_1\cos 5t + a_2\sin 5t + b_1\cos 5\sqrt{3}t + b_2\sin 5\sqrt{3}t, \\ x_2(t) &= 2a_1\cos 5t + 2a_2\sin 5t - 2b_1\cos 5\sqrt{3}t - 2b_2\sin 5\sqrt{3}t. \end{aligned}$$

When we substitute the trial solution $\mathbf{x}_p(t) = [c_1 \quad c_2]^T\cos 10t$ in the nonhomogeneous system, we find that $c_1 = 4/3$ and $c_2 = -16/3$, so a particular solution $\mathbf{x}_p(t)$ is described by

$$x_1(t) = (4/3)\cos 10t, \qquad x_2(t) = -(16/3)\cos 10t.$$

Finally, when we impose the zero initial conditions on the solution $\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$ we find that $a_1 = 2/3$, $a_2 = 0$, $b_1 = -2$, and $b_2 = 0$. Thus the solution we seek is described by

$$x_1(t) = \tfrac{2}{3}\cos 5t - 2\cos 5\sqrt{3}t + \tfrac{4}{3}\cos 10t$$

$$x_2(t) = \tfrac{4}{3}\cos 5t + 4\cos 5\sqrt{3}t + \tfrac{16}{3}\cos 10t.$$

We have a superposition of two oscillations with the natural frequencies $\omega_1 = 5$ and $\omega_2 = 5\sqrt{3}$ and a forced oscillation with frequency $\omega = 10$. In each of the two natural oscillations the amplitude of motion of m_2 is twice that of m_1, while in the forced oscillation the amplitude of motion of m_2 is four times that of m_1.

16. The characteristic equation of A is

$$(-c_1 - \lambda)(-c_2 - \lambda) - c_1c_2 = \lambda^2 + (c_1 + c_2)\lambda = 0,$$

whence the given eigenvalues and eigenvectors follow readily.

17. With $c_1 = c_2 = 2$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 2$, $\mathbf{v}_2 = [1 \quad -1]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1t + a_2\cos 2t + b_2\sin 2t$$

$$x_2(t) = a_1 + b_1t - a_2\cos 2t - b_2\sin 2t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'0) = 0$ yield $a_1 = a_2 = 0$ and $b_1 = v_0/2$, $b_2 = v_0/4$, so

$$x_1(t) = (v_0/4)(2t + \sin 2t)$$

$$x_2(t) = (v_0/4)(2t - \sin 2t)$$

while $x_2 - x_1 = (v_0/4)(-2\sin 2t) < 0$, that is, until $t = \pi/2$. Finally, $x_1'(\pi/2) = 0$ and $x_2'(\pi/2) = v_0$.

18. With $c_1 = 6$ and $c_2 = 3$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 3$, $\mathbf{v}_2 = [2 \quad -1]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1 t + 2a_2\cos 3t + 2b_2\sin 3t$$

$$x_2(t) = a_1 + b_1 t - a_2\cos 3t - b_2\sin 3t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'(0) = 0$ yield $a_1 = a_2 = 0$ and $b_1 = v_0/3$, $b_2 = v_0/9$, so

$$x_1(t) = (v_0/9)(3t + 2\sin 3t)$$

$$x_2(t) = (v_0/9)(3t - \sin 3t)$$

while $x_2 - x_1 = (v_0/9)(-3\sin 3t) < 0$; that is, until $t = \pi/3$. Finally, $x_1'(\pi/3) = -v_0/3$ and $x_2'(\pi/3) = 2v_0/3$.

19. With $c_1 = 1$ and $c_2 = 3$, it follows from Problem 16 that the natural frequencies and associated eigenvectors are $\omega_1 = 0$, $\mathbf{v}_1 = [1 \quad 1]^T$ and $\omega_2 = 2$, $\mathbf{v}_2 = [1 \quad -3]^T$. Hence Theorem 1 gives the general solution

$$x_1(t) = a_1 + b_1 t + a_2\cos 2t + b_2\sin 2t$$

$$x_2(t) = a_1 + b_1 t - 3a_2\cos 2t - 3b_2\sin 2t.$$

The initial conditions $x_1'(0) = v_0$, $x_1(0) = x_2(0) = x_2'(0) = 0$ yield $a_1 = a_2 = 0$ and $b_1 = 3v_0/4$, $b_2 = v_0/8$, so

$$x_1(t) = (v_0/8)(6t + \sin 2t)$$

$$x_2(t) = (v_0/8)(6t - 3\sin 2t)$$

while $x_2 - x_1 = (v_0/8)(-4\sin 2t) < 0$; that is, until $t = \pi/2$. Finally, $x_1'(\pi/2) = v_0/2$ and $x_2'(\pi/2) = 3v_0/2$.

20. With $c_1 = c_3 = 4$ and $c_2 = 16$ the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 4 & 0 \\ 16 & -32 & 16 \\ 0 & 4 & -4 \end{bmatrix}$$

is

$$\lambda^3 + 40\lambda^2 + 144\lambda = \lambda(\lambda+4)(\lambda+36) = 0.$$

The resulting eigenvalues, natural frequencies, and associated eigenvectors are

$$\begin{array}{lll} \lambda_1 = 0, & \omega_1 = 0, & \mathbf{v}_1 = [1 \quad 1 \quad 1]^T \\ \lambda_2 = -4, & \omega_2 = 2, & \mathbf{v}_2 = [1 \quad 0 \quad -1]^T \\ \lambda_3 = -36, & \omega_3 = 6, & \mathbf{v}_3 = [1 \quad -8 \quad 1]^T. \end{array}$$

Theorem 1 then gives the general solution

$$\begin{aligned} x_1(t) &= a_1 + b_1 t + a_2\cos 2t + b_2\sin 2t + a_3\cos 6t + b_3\sin 6t \\ x_2(t) &= a_1 + b_1 t \qquad\qquad\qquad\qquad\qquad - 8a_3\cos 6t - 8b_3\sin 6t \\ x_3(t) &= a_1 + b_1 t - a_2\cos 2t - b_2\sin 2t + a_3\cos 6\,t + b_3\sin 6t. \end{aligned}$$

The initial conditons yield $a_1 = a_2 = a_3 = 0$ and $b_1 = 4v_0/9$, $b_2 = v_0/4$, $b_3 = v_0/108$, so

$$\begin{aligned} x_1(t) &= (v_0/108)(48t + 27\sin 2t + \sin 6t) \\ x_2(t) &= (v_0/108)(48t \qquad\qquad - 8\sin 6t) \\ x_3(t) &= (v_o/108)(48t - 27\sin 2t + \sin 6t) \end{aligned}$$

while

$$\begin{aligned} x_2 - x_1 &= -18(\sin 2t)(3 - 2\sin^3 2t) < 0, \\ x_3 - x_2 &= -9(4\sin^3 2t) < 0; \end{aligned}$$

that is, until $t = \pi/2$. Finally

$$x_1'(\pi/2) = -v_0/9, \quad x_2'(\pi/2) = 8v_0/9, \quad x_3'(\pi/2) = 8v_0/9.$$

21. **(a)** The matrix

$$\mathbf{A} = \begin{bmatrix} -160/3 & 320/3 \\ 8 & -116 \end{bmatrix}$$

has eigenvalues $\lambda_1 \approx -41.8285$ and $\lambda_2 \approx -127.5049$, so the natural frequencies are

$$\omega_1 \approx 6.4675 \text{ rad/sec} \approx 1.0293 \text{ Hz}$$
$$\omega_2 \approx 11.2918 \text{ rad/sec} \approx 1.7971 \text{ Hz.}$$

(b) Resonance occurs at the two critical speeds

$$v_1 = 20\omega_1/\pi \approx 41 \text{ ft/sec} \approx 28 \text{ mi/h}$$
$$v_2 = 20\omega_2/\pi \approx 72 \text{ ft/sec} \approx 49 \text{ mi/h.}$$

22. With $k_1 = k_2 = k$ and $L_1 = L_2 = L/2$ the equations in (42) reduce to

$$mx'' = -2kx \quad \text{and} \quad I\theta'' = -kL^2\theta/2.$$

The first equation yields $\omega_1 = \sqrt{2k/m}$ and the second one yields $\omega_2 = \sqrt{kL^2/2I}$.

In Problems 23–25 we substitute the given physical parameters into the equations in (42):

$$mx'' = -(k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta$$

$$I\theta'' = (k_1L_1 - k_2L_2)x - (k_1L_1^2 + k_2L_2^2)\theta$$

As in Problem 21, a critical frequency of ω rad/sec yields a critical velocity of $v = 20\omega/\pi$ ft/sec.

23. $100x'' = -4000x, \quad 800\theta'' = 100000\theta$

Obviously the matrix $\mathbf{A} = \begin{bmatrix} -40 & 0 \\ 0 & -125 \end{bmatrix}$ has eigenvalues $\lambda_1 = -40$ and $\lambda_2 = -125$.

Up-and-down:	$\omega_1 = \sqrt{40}$,	$v_1 \approx 40.26$ ft/sec ≈ 27 mph
Angular:	$\omega_2 = \sqrt{125}$,	$v_2 \approx 71.18$ ft/sec ≈ 49 mph

24.
$$100x'' = -4000x + 4000\theta$$
$$1000\theta'' = 4000x - 104000\theta$$

The matrix $\mathbf{A} = \begin{bmatrix} -40 & 40 \\ 4 & -104 \end{bmatrix}$ has eigenvalues $\lambda_1, \lambda_2 = 4\left(-18 \pm \sqrt{74}\right)$.

$\omega_1 \approx 6.1311, \quad v_1 \approx 39.03$ ft/sec ≈ 27 mph
$\omega_2 \approx 10.3155, \quad v_2 \approx 65.67$ ft/sec ≈ 45 mph

25.
$$100x'' = -3000x - 5000\theta$$
$$800\theta'' = -5000x - 75000\theta$$

The matrix $\mathbf{A} = \begin{bmatrix} -30 & -50 \\ -25/4 & -375/4 \end{bmatrix}$ has eigenvalues $\lambda_1, \lambda_2 = \frac{5}{8}\left(-99 \pm \sqrt{3401}\right)$.

$\omega_1 \approx 5.0424,\qquad v_1 \approx 32.10$ ft/sec ≈ 22 mph

$\omega_2 \approx 9.9158,\qquad v_2 \approx 63.13$ ft/sec ≈ 43 mph

SECTION 5.6

MULTIPLE EIGENVALUE SOLUTIONS

In each of Problems 1–6 we give first the characteristic equation with repeated (multiplicity 2) eigenvalue λ. In each case we find that $(\mathbf{A} - \lambda\mathbf{I})^2 = \mathbf{0}$. Then $\mathbf{w} = [1 \quad 0]^T$ is a generalized eigenvector and $\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} \neq \mathbf{0}$ is an ordinary eigenvector associated with λ. We give finally the scalar component functions $x_1(t)$, $x_2(t)$ of the general solution

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2(\mathbf{v}t + \mathbf{w})e^{\lambda t}$$

of the given system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

1. Characteristic equation $\qquad \lambda^2 + 6\lambda + 9 = 0$

Repeated eigenvalue $\qquad \lambda = -3$

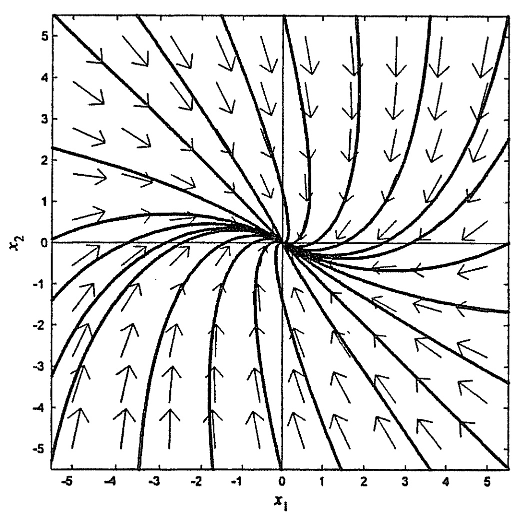
Generalized eigenvector $\qquad \mathbf{w} = [1 \quad 0]^T$

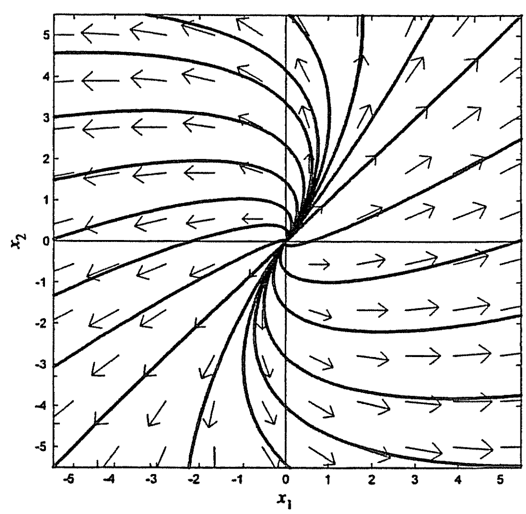
$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x_1(t) = (\ c_1 + c_2 + c_2 t)e^{-3t}$$

$$x_2(t) = (-c_1 \quad - c_2 t)e^{-3t}.$$

The left-hand figure below shows a direction field and typical solution curves.





2. Characteristic equation $\lambda^2 - 4\lambda + 4 = 0$
Repeated eigenvalue $\lambda = 2$
Generalized eigenvector $\mathbf{w} = [1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_1(t) = (c_1 + c_2 + c_2 t)e^{2t}$$
$$x_2(t) = (c_1 + \qquad c_2 t)e^{2t}.$$

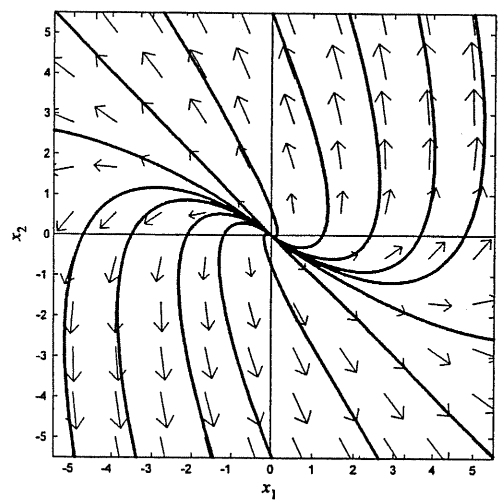
The right-hand figure at the bottom of the preceding page shows a direction field and typical solution curves.

3. Characteristic equation $\lambda^2 - 6\lambda + 9 = 0$
Repeated eigenvalue $\lambda = 3$
Generalized eigenvector $\mathbf{w} = [\,1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$x_1(t) = (-2c_1 + c_2 - 2c_2 t)e^{3t}$$
$$x_2(t) = (\,2c_1 + \qquad 2c_2 t)e^{3t}.$$

The figure below shows a direction field and typical solution curves.



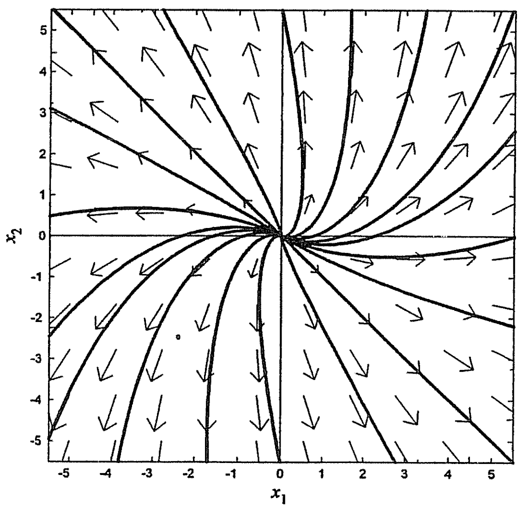
4. Characteristic equation $\lambda^2 - 8\lambda + 16 = 0$
Repeated eigenvalue $\lambda = 4$
Generalized eigenvector $\mathbf{w} = [\,1 \quad 0]^T$

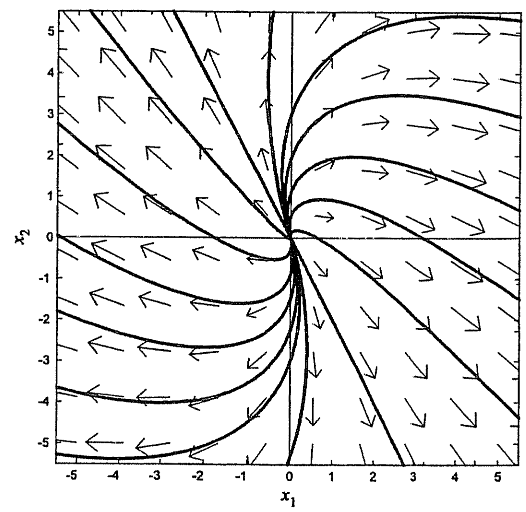
$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_1(t) = (-c_1 + c_2 - c_2 t)e^{4t}$$

$$x_2(t) = (\ c_1 + \qquad c_2 t)e^{4t}.$$

The left-hand figure below shows a direction field and typical solution curves.





5. Characteristic equation $\lambda^2 - 10\lambda + 25 = 0$
Repeated eigenvalue $\lambda = 5$
Generalized eigenvector $\mathbf{w} = [1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$x_1(t) = (\ 2c_1 + c_2 + 2c_2 t)e^{5t}$$

$$x_2(t) = (-4c_1 \qquad - 4c_2 t)e^{5t}.$$

The right-hand figure above shows a direction field and typical solution curves.

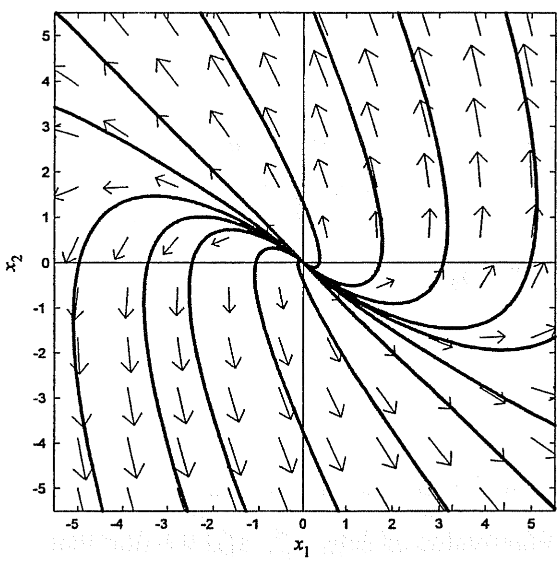
6. Characteristic equation $\lambda^2 - 10\lambda + 25 = 0$
Repeated eigenvalue $\lambda = 5$
Generalized eigenvector $\mathbf{w} = [\,1 \quad 0]^T$

$$\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \begin{bmatrix} -4 & -4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

$x_1(t) = (-4c_1 + c_2 - 4c_2 t)e^{5t}$

$x_2(t) = (4c_1 \qquad + 4c_2 t)e^{5t}.$

The figure below shows a direction field and typical solution curves.



In each of Problems 7–10 the characteristic polynomial is easily calculated by expansion along the row or column of **A** that contains two zeros. The matrix **A** has only two distinct eigenvalues, so we write $\lambda_1, \lambda_2, \lambda_3$ with either $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$. Nevertheless, we find that it has 3 linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. We list also the scalar components $x_1(t), x_2(t), x_3(t)$ of the general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$ of the system.

7. Characteristic equation $\quad -\lambda^3 + 13\lambda^2 - 40\lambda + 36 = -(\lambda-2)^2(\lambda-9)$

Eigenvalues $\quad \lambda = 2,\ 2,\ 9$

Eigenvectors $\quad [1 \ \ 1 \ \ 0]^T,\ [1 \ \ 0 \ \ 1]^T,\ [0 \ \ 1 \ \ 0]^T$

$x_1(t) = c_1 e^{2t} + c_2 e^{2t}$

$x_2(t) = c_1 e^{2t} \qquad + c_3 e^{9t}$

$x_3(t) = \qquad c_2 e^{2t}$

8. Characteristic equation $\quad -\lambda^3 + 33\lambda^2 - 351\lambda + 1183 = -(\lambda-13)^2(\lambda-7)$

Eigenvalues $\quad \lambda = 7,\ 13,\ 13$

Eigenvectors $\quad [2 \ \ -3 \ \ 1]^T,\ [0 \ \ 0 \ \ 1]^T,\ [-1 \ \ 1 \ \ 0]^T,$

$x_1(t) = 2c_1 e^{7t} \qquad - c_3 e^{13t}$

$x_2(t) = -3c_1 e^{7t} \qquad + c_3 e^{13t}$

$x_3(t) = c_1 e^{7t} + c_2 e^{13t}$

9. Characteristic equation $-\lambda^3+19\lambda^2-115\lambda+225 = -(\lambda-5)^2(\lambda-9)$

Eigenvalues $\lambda = 5,\ 5,\ 9$

Eigenvectors $[1 \quad 2 \quad 0]^T,\ [7 \quad 0 \quad 2]^T,\ [3 \quad 0 \quad 1]^T$

$$x_1(t) = c_1e^{5t} + 7c_2e^{5t} + 3c_3e^{9t}$$
$$x_2(t) = 2c_1e^{5t}$$
$$x_3(t) = 2c_2e^{5t} + c_3e^{9t}$$

10. Characteristic equation $-\lambda^3+13\lambda^2-51\lambda+63 = -(\lambda-3)^2(\lambda-7)$

Eigenvalues $\lambda = 3,\ 3,\ 7$

Eigenvectors $[5 \quad 2 \quad 0]^T,\ [-3 \quad 0 \quad 1]^T,\ [2 \quad 1 \quad 0]^T$

$$x_1(t) = 5c_1e^{3t} - 3c_2e^{3t} + 2c_3e^{7t}$$
$$x_2(t) = 2c_1e^{3t} + c_3e^{7t}$$
$$x_3(t) = c_2e^{3t}$$

In each of Problems 11–14, the characteristic equation is $-\lambda^3-3\lambda^2-3\lambda-1 = -(\lambda+1)^3$. Hence $\lambda = -1$ is a triple eigenvalue of defect 2, and we find that $(\mathbf{A}-\lambda\mathbf{I})^3 = \mathbf{0}$. In each problem we start with $\mathbf{v}_3 = [1 \quad 0 \quad 0]^T$ and then calculate $\mathbf{v}_2 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 \neq 0$. It follows that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2 = (\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v}_3 = \mathbf{0}$, so the vector $\mathbf{v}_1$ (if nonzero) is an ordinary eigenvector associated with the triple eigenvalue λ. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain of generalized eigenvectors, and the corresponding general solution is described by

$$\mathbf{x}(t) = e^{-t}[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3)].$$

We give the scalar components $x_1(t)$, $x_2(t)$, $x_3(t)$ of $\mathbf{x}(t)$.

11. $\mathbf{v}_1 = [0 \quad 1 \quad 0]^T, \quad \mathbf{v}_2 = [-2 \quad -1 \quad 1]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$

$$x_1(t) = e^{-t}(-2c_2 + c_3 - 2c_3 t)$$
$$x_2(t) = e^{-t}(c_1 - c_2 + c_2 t - c_3 t + c_3 t^2/2)$$
$$x_3(t) = e^{-t}(c_2 + c_3 t)$$

12. $\mathbf{v}_1 = [1 \quad 1 \quad 0]^T, \quad \mathbf{v}_2 = [0 \quad 0 \quad 1]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$

$$x_1(t) = e^{-t}(c_1 + c_3 + c_2 t + c_3 t^2/2)$$
$$x_2(t) = e^{-t}(c_1 + c_2 t + c_3 t^2/2)$$
$$x_3(t) = e^{-t}(c_2 + c_3 t)$$

13. Here we are stymied initially, because if $\mathbf{v}_3 = [1 \quad 0 \quad 0]^T$ then $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3 = \mathbf{0}$ does not qualify as a (nonzero) generalized eigenvector. We there make a fresh start with $\mathbf{v}_3 = [0 \quad 1 \quad 0]^T$, and now we get the desired nonzero generalized eigenvectors upon successive multiplication by $\mathbf{A}-\lambda\mathbf{I}$.

$$\mathbf{v}_1 = [1 \quad 0 \quad 0]^T, \qquad \mathbf{v}_2 = [0 \quad 2 \quad 1]^T, \qquad \mathbf{v}_3 = [0 \quad 1 \quad 0]^T$$

$$x_1(t) = e^{-t}(c_1 + c_2 t + c_3 t^2/2)$$

$$x_2(t) = e^{-t}(2c_2 + c_3 + 2c_3 t)$$

$$x_3(t) = e^{-t}(c_2 + c_3 t)$$

14. $\mathbf{v}_1 = [5 \quad -25 \quad -5]^T, \quad \mathbf{v}_2 = [1 \quad -5 \quad 4]^T, \quad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$

$$x_1(t) = e^{-t}(5c_1 + c_2 + c_3 + 5c_2 t + c_3 t + 5c_3 t^2/2)$$

$$x_2(t) = e^{-t}(-25c_1 - 5c_2 - 25c_2 t - 5c_3 t - 25c_3 t^2/2)$$

$$x_3(t) = e^{-t}(-5c_1 + 4c_2 - 5c_2 t + 4c_3 t - 5c_3 t^2/2)$$

In each of Problems 15–18, the characteristic equation is $-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda-1)^3$. Hence $\lambda = 1$ is a triple eigenvalue of defect 1, and we find that $(\mathbf{A}-\lambda\mathbf{I})^2 = \mathbf{0}$. First we find the two linearly independent (ordinary) eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ associated with λ. Then we start with $\mathbf{v}_2 = [1 \quad 0 \quad 0]^T$ and calculate $\mathbf{v}_1 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 \neq 0$. It follows that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_1$ is an ordinary eigenvector associated with λ. However, $\mathbf{v}_1$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, so $\mathbf{v}_1 e^t$ is a linear combination of the independent solutions $\mathbf{u}_1 e^t$ and $\mathbf{u}_2 e^t$. But $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a length 2 chain of generalized eigenvectors associated with λ, so $(\mathbf{v}_1 t + \mathbf{v}_2)e^t$ is the desired third independent solution. The corresponding general solution is described by

$$\mathbf{x}(t) = e^t[c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3(\mathbf{v}_1 t + \mathbf{v}_2)]$$

We give the scalar components $x_1(t), x_2(t), x_3(t)$ of $\mathbf{x}(t)$.

15. $\mathbf{u}_1 = [3 \quad -1 \quad 0]^T \qquad \mathbf{u}_2 = [0 \quad 0 \quad 1]^T$

$$\mathbf{v}_1 = [-3 \quad 1 \quad 1]^T \qquad \mathbf{v}_2 = [1 \quad 0 \quad 0]^T$$

$$x_1(t) = e^t(3c_1 + c_3 - 3c_3 t)$$

$$x_2(t) = e^t(-c_1 + c_3 t)$$

$$x_3(t) = e^t(c_2 + c_3 t)$$

16. $\mathbf{u}_1 = [3 \;\; -2 \;\; 0]^T \qquad \mathbf{u}_2 = [3 \;\; 0 \;\; -2]^T$

$\mathbf{v}_1 = [0 \;\; -2 \;\; 2]^T \qquad \mathbf{v}_2 = [1 \;\; 0 \;\; 0]^T$

$$x_1(t) = e^t(3c_1 + 3c_2 + c_3)$$
$$x_2(t) = e^t(-2c_1 - 2c_3 t)$$
$$x_3(t) = e^t(-2c_2 + 2c_3 t)$$

17. $\mathbf{u}_1 = [2 \;\; 0 \;\; -9]^T \qquad \mathbf{u}_2 = [1 \;\; -3 \;\; 0]^T$

$\mathbf{v}_1 = [0 \;\; 6 \;\; -9]^T \qquad \mathbf{v}_2 = [0 \;\; 1 \;\; 0]^T$

(Either $\mathbf{v}_2 = [1 \;\; 0 \;\; 0]^T$ or $\mathbf{v}_2 = [0 \;\; 0 \;\; 1]^T$ can be used also, but they yield different forms of the solution than given in the book's answer section.)

$$x_1(t) = e^t(2c_1 + c_2)$$
$$x_2(t) = e^t(-3c_2 + c_3 + 6c_3 t)$$
$$x_3(t) = e^t(-9c_1 - 9c_3 t)$$

18. $\mathbf{u}_1 = [-1 \;\; 0 \;\; 1]^T \qquad \mathbf{u}_2 = [-2 \;\; 1 \;\; 0]^T$

$\mathbf{v}_1 = [0 \;\; 1 \;\; -2]^T \qquad \mathbf{v}_2 = [1 \;\; 0 \;\; 0]^T$

$$x_1(t) = e^t(-c_1 - 2c_2 + c_3)$$
$$x_2(t) = e^t(c_2 + c_3 t)$$
$$x_3(t) = e^t(c_1 - 2c_3 t)$$

19. Characteristic equation $\qquad \lambda^4 - 2\lambda^2 + 1 = 0$

Double eigenvalue $\lambda = -1$ with eigenvectors

$$\mathbf{v}_1 = [1 \;\; 0 \;\; 0 \;\; 1]^T \text{ and } \mathbf{v}_2 = [0 \;\; 0 \;\; 1 \;\; 0]^T.$$

Double eigenvalue $\lambda = +1$ with eigenvectors

$$\mathbf{v}_3 = [0 \;\; 1 \;\; 0 \;\; -2]^T \text{ and } \mathbf{v}_4 = [1 \;\; 0 \;\; 3 \;\; 0]^T.$$

General solution

$$\mathbf{x}(t) = e^{-t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + e^t(c_3\mathbf{v}_3 + c_4\mathbf{v}_4)$$

Scalar components

$$x_1(t) = c_1 e^{-t} + c_4 e^t$$
$$x_2(t) = c_3 e^t$$
$$x_3(t) = c_2 e^{-t} + 3c_4 e^t$$
$$x_4(t) = c_1 e^{-t} - 2c_3 e^t$$

20. Characteristic equation $\lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 16 = (\lambda - 2)^4 = 0$
Eigenvalue $\lambda = 2$ with multiplicity 4 and defect 3.

We find that $(\mathbf{A} - \lambda\mathbf{I})^3 \neq 0$ but $(\mathbf{A} - \lambda\mathbf{I})^4 = 0$. We therefore start with $\mathbf{v}_4 = [0 \quad 0 \quad 0 \quad 1]^T$ and define $\mathbf{v}_3 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_4$, $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$, $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$, and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq 0$. This gives the length 4 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ with

$$\mathbf{v}_1 = [1 \quad 0 \quad 0 \quad 0]^T \qquad \mathbf{v}_2 = [0 \quad 1 \quad 0 \quad 0]^T$$
$$\mathbf{v}_3 = [1 \quad 0 \quad 1 \quad 0]^T \qquad \mathbf{v}_4 = [0 \quad 0 \quad 0 \quad 1]^T.$$

The corresponding general solution is given by

$$\begin{aligned}\mathbf{x}(t) &= e^{-t}\,[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3) \\ &\quad + c_4(\mathbf{v}_1 t^3/6 + \mathbf{v}_2 t^2/2 + \mathbf{v}_3 t + \mathbf{v}_4)]\end{aligned}$$

with scalar components

$$x_1(t) = e^{2t}(c_1 + c_3 + c_2 t + c_4 t + c_3 t^2/2 + c_4 t^3/6)$$
$$x_2(t) = e^{2t}(c_2 + c_3 t + c_4 t^2/2)$$
$$x_3(t) = e^{2t}(c_3 + c_4 t)$$
$$x_4(t) = e^{2t}(c_4).$$

21. Characteristic equation $\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = (\lambda - 1)^4 = 0$
Eigenvalue $\lambda = 1$ with multiplicity 4 and defect 2.

We find that $(\mathbf{A} - \lambda\mathbf{I})^2 \neq 0$ but $(\mathbf{A} - \lambda\mathbf{I})^3 = 0$. We therefore start with $\mathbf{v}_3 = [1 \quad 0 \quad 0 \quad 0]^T$ and define $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq 0$, thereby obtaining the length 3 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [0 \;\; 0 \;\; 0 \;\; 1]^T, \quad \mathbf{v}_2 = [-2 \;\; 1 \;\; 1 \;\; 0]^T, \quad v3 = [1 \;\; 0 \;\; 0 \;\; 0]^T.$$

Then we find the second ordinary eigenvector $\mathbf{v}_4 = [0 \quad 0 \;\; 1 \;\; 0]^T$. The corresponding general solution

$$\mathbf{x}(t) = e^{t}\,[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3) + c_4\mathbf{v}_4]$$

has scalar components

$$x_1(t) = e^{t}(-2c_2 + c_3 - 2c_3 t)$$
$$x_2(t) = e^{t}(c_2 + c_3 t)$$
$$x_3(t) = e^{t}(c_2 + c_4 + c_3 t).$$
$$x_4(t) = e^{t}(c_1 + c_2 t + c_3 t^2/2.)$$

22. Same eigenvalue and chain structure as in Problem 21, but with generalized eigenvectors

$$\mathbf{v}_1 = [1 \quad 0 \quad 0 \quad -2]^T \qquad \mathbf{v}_2 = [3 \quad -2 \quad 1 \quad -6]^T$$
$$\mathbf{v}_3 = [0 \quad 1 \quad 0 \quad 0]^T \qquad \mathbf{v}_4 = [1 \quad 0 \quad 0 \quad 0]^T$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain and $\mathbf{v}_4$ is an ordinary eigenvector. The general solution $\mathbf{x}(t)$ defined as in Problem 21 has scalar components

$$\begin{aligned}
x_1(t) &= e^t(c_1 + 3c_2 + c_4 + c_2 t + 3c_3 t + c_3 t^2/2) \\
x_2(t) &= e^t(-2c_2 + c_3 - 2c_3 t) \\
x_3(t) &= e^t(c_2 + c_3 t) \\
x_4(t) &= e^t(-2c_1 - 6c_2 - 2c_2 t - 6c_3 t - c_3 t^2)
\end{aligned}$$

In Problems 23 and 24 there are only two distinct eigenvalues λ_1 and λ_2. However, the eigenvector equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$ yields the three linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ that are given. We list the scalar components of the corresponding general solution $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_2 t}$.

23. $\lambda_1 = -1$: $\{\mathbf{v}_1\}$ with $\mathbf{v}_1 = [1 \quad -1 \quad 2]^T$
$\lambda_2 = 3$: $\{\mathbf{v}_2\}$ with $\mathbf{v}_2 = [4 \quad 0 \quad 9]^T$ and
$\{\mathbf{v}_3\}$ with $\mathbf{v}_3 = [0 \quad 2 \quad 1]^T$

Scalar components

$$\begin{aligned}
x_1(t) &= c_1 e^{-t} + 4c_2 e^{3t} \\
x_2(t) &= -c_1 e^{-t} + 2c_3 e^{3t} \\
x_3(t) &= 2c_1 e^{-t} + 9c_2 e^{3t} + c_3 e^{3t}
\end{aligned}$$

24. $\lambda_1 = -2$: $\{\mathbf{v}_1\}$ with $\mathbf{v}_1 = [5 \quad 3 \quad -3]^T$

$\lambda_2 = 3$: $\{\mathbf{v}_2\}$ with $\mathbf{v}_2 = [4 \quad 0 \quad -1]^T$ and
$\{\mathbf{v}_3\}$ with $\mathbf{v}_3 = [2 \quad -1 \quad 0]^T$

Scalar components

$$\begin{aligned}
x_1(t) &= 5c_1 e^{-2t} + 4c_2 e^{3t} + 2c_3 e^{3t} \\
x_2(t) &= 3c_1 e^{-2t} - c_3 e^{3t} \\
x_3(t) &= -3c_1 e^{-2t} - c_2 e^{3t}
\end{aligned}$$

In Problems 25, 26, and 28 there is given a single eigenvalue λ of multiplicity 3. We find that $(\mathbf{A} - \lambda\mathbf{I})^2 \neq 0$ but $(\mathbf{A} - \lambda\mathbf{I})^3 = 0$. We therefore start with $\mathbf{v}_3 = [1 \quad 0 \quad 0]^T$ and define

$\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq 0$, thereby obtaining the length 3 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of generalized eigenvectors based on the ordinary eigenvector $\mathbf{v}_1$. We list the scalar components of the corresponding general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3)e^{\lambda t}.$$

25. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [-1 \quad 0 \quad -1]^T, \qquad \mathbf{v}_2 = [-4 \quad -1 \quad 0]^T, \qquad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$$

Scalar components

$$\begin{aligned} x_1(t) &= e^{2t}(-c_1 - 4c_2 + c_3 - c_2 t - 4c_3 t - c_3 t^2/2) \\ x_2(t) &= e^{2t}(-c_2 - c_3 t) \\ x_3(t) &= e^{2t}(-c_1 - c_2 t - c_3 t^2/2) \end{aligned}$$

26. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [0 \quad 2 \quad 2]^T, \qquad \mathbf{v}_2 = [2 \quad 1 \quad -3]^T, \qquad \mathbf{v}_3 = [1 \quad 0 \quad 0]^T$$

General solution

$$\begin{aligned} x_1(t) &= e^{3t}(2c_2 + c_3 + 2c_3 t) \\ x_2(t) &= e^{3t}(2c_1 + c_2 + 2c_2 t + c_3 t + c_3 t^2) \\ x_3(t) &= e^{3t}(2c_1 - 3c_2 + 2c_2 t - 3c_3 t + c_3 t^2) \end{aligned}$$

27. We find that the triple eigenvalue $\lambda = 2$ has the two linearly independent eigenvectors $[1 \quad 1 \quad 0]^T$ and $[-1 \quad 0 \quad 1]^T$. Next we find that $(\mathbf{A} - \lambda\mathbf{I}) \neq 0$ but $(\mathbf{A} - \lambda\mathbf{I})^2 = 0$. We therefore start with $\mathbf{v}_2 = [1 \quad 0 \quad 0]^T$ and define

$$\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = [-5 \quad 3 \quad 8]^T \neq \mathbf{0},$$

thereby obtaining the length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ of generalized eigenvectors based on the ordinary eigenvector $\mathbf{v}_1$. If we take $\mathbf{v}_3 = [1 \quad 1 \quad 0]^T$, then the general solution $\mathbf{x}(t) = e^{2t}[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3\mathbf{v}_3]$ has scalar components

$$\begin{aligned} x_1(t) &= e^{2t}(-5c_1 + c_2 + c_3 - 5c_2 t) \\ x_2(t) &= e^{2t}(3c_1 + 3c_2 t) \\ x_3(t) &= e^{2t}(8c_1 + 8c_2 t). \end{aligned}$$

$c_1\begin{bmatrix}-5\\3\\8\end{bmatrix} + tc_2\begin{bmatrix}1\\3\\8\end{bmatrix} + c_3$

$c_1\begin{bmatrix}-5\\3\\8\end{bmatrix} + c_2\left[t\begin{bmatrix}-5\\3\\8\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}\right) + c_3\begin{bmatrix}1\\1\\0\end{bmatrix}$

28. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with

$$\mathbf{v}_1 = [119\ \ -289\ \ 0]^T,\quad \mathbf{v}_2 = [-17\ \ 34\ \ 17]^T,\quad \mathbf{v}_3 = [1\ \ 0\ \ 0]^T$$

General solution

$$\begin{aligned}
x_1(t) &= e^{2t}(119c_1 - 17c_2 + c_3 + 119c_2t - 17c_3t + 119c_3t^2/2)\\
x_2(t) &= e^{2t}(-289c_1 + 34c_2 - 289c_2t + 34c_3t - 289c_3t^2/2)\\
x_3(t) &= e^{2t}(17c_2 + 17c_3t)
\end{aligned}$$

In Problems 29 and 30 the matrix $\mathbf{A}$ has two distinct eigenvalues λ_1 and λ_2 each having multiplicity 2 and defect 1. First, we select $\mathbf{v}_2$ so that $\mathbf{v}_1 = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$ but $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a length 2 chain based on $\mathbf{v}_1$. Next, we select $\mathbf{u}_2$ so that $\mathbf{u}_1 = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u}_2 \neq \mathbf{0}$ but $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{u}_1 = \mathbf{0}$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on $\mathbf{u}_1$. We give the scalar components of the corresponding general solution

$$\mathbf{x}(t) = e^{\lambda_1 t}[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1t + \mathbf{v}_2)] + e^{\lambda_2 t}[c_3\mathbf{u}_1 + c_4(\mathbf{u}_1t + \mathbf{u}_2)].$$

29. $\lambda = -1$: $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = [1\ \ -3\ \ -1\ \ -2]^T$ and $\mathbf{v}_2 = [0\ \ 1\ \ 0\ \ 0]^T$,
$\lambda = 2$: $\{\mathbf{u}_1, \mathbf{u}_2\}$ with $\mathbf{u}_1 = [0\ \ -1\ \ 1\ \ 0]^T$ and $\mathbf{u}_2 = [0\ \ 0\ \ 2\ \ 1]^T$

Scalar components

$$\begin{aligned}
x_1(t) &= e^{-t}(c_1 + c_2t)\\
x_2(t) &= e^{-t}(-3c_1 + c_2 - 3c_2t) + e^{2t}(-c_3 - c_4t)\\
x_3(t) &= e^{-t}(-c_1 - c_2t) + e^{2t}(c_3 + 2c_4 + c_4t)\\
x_4(t) &= e^{-t}(-2c_1 - 2c_2t) + e^{2t}(c_4)
\end{aligned}$$

30. $\lambda = -1$: $\{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = [0\ \ 1\ \ -1\ \ -3]^T$ and $\mathbf{v}_2 = [0\ \ 0\ \ 1\ \ 2]^T$,
$\lambda = 2$: $\{\mathbf{u}_1, \mathbf{u}_2\}$ with $\mathbf{u}_1 = [-1\ \ 0\ \ 0\ \ 0]^T$ and $\mathbf{u}_2 = [0\ \ 0\ \ 3\ \ 5]^T$

Scalar components

$$\begin{aligned}
x_1(t) &= e^{2t}(-c_3 - c_4t)\\
x_2(t) &= e^{-t}(c_1 + c_2t)\\
x_3(t) &= e^{-t}(-c_1 + c_2 - c_2t) + e^{2t}(3c_4)\\
x_4(t) &= e^{-t}(-3c_1 + 2c_2 - 3c_2t) + e^{2t}(5c_4)
\end{aligned}$$

31. We have the single eigenvalue $\lambda = 1$ of multiplicity 4. Starting with $\mathbf{v}_3 = [1 \quad 0 \quad 0 \quad 0]^T$, we calculate $\mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3$ and $\mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 \neq \mathbf{0}$, and find that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0}$. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a length 3 chain based on the ordinary eigenvector $\mathbf{v}_1$. Next, the eigenvector equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ yields the second linearly independent eigenvector $\mathbf{v}_4 = [0 \quad 1 \quad 3 \quad 0]^T$. With

$$\mathbf{v}_1 = [42 \quad 7 \quad -21 \quad -42]^T, \qquad \mathbf{v}_2 = [34 \quad 22 \quad -10 \quad -27]^T,$$

$$\mathbf{v}_3 = [1 \quad 0 \quad 0 \quad 0]^T \quad \text{and} \quad \mathbf{v}_4 = [0 \quad 1 \quad 3 \quad 0]$$

the general solution

$$\mathbf{x}(t) = e^t\,[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3) + c_4\mathbf{v}_4]$$

has scalar components

$$\begin{aligned}
x_1(t) &= e^t(42c_1 + 34c_2 + c_3 + 42c_2 t + 34c_3 t + 21c_3 t^2) \\
x_2(t) &= e^t(7c_1 + 22c_2 + c_4 + 7c_2 t + 22c_3 t + 7c_3 t^2/2) \\
x_3(t) &= e^t(-21c_1 - 10c_2 + 3c_4 - 21c_2 t - 10c_3 t - 21c_3 t^2/2) \\
x_4(t) &= e^t(-42c_1 - 27c_2 - 42c_2 t - 27c_3 t - 21c_3 t^2).
\end{aligned}$$

32. Here we find that the matrix $\mathbf{A}$ has five linearly independent eigenvectors:

$\lambda = 2$: eigenvectors $\mathbf{v}_1 = [8 \quad 0 \quad -3 \quad 1 \quad 0]^T$ and $\mathbf{v}_2 = [1 \quad 0 \quad 0 \quad 0 \quad 3]^T$

$\lambda = 3$: eigenvectors $\mathbf{v}_3 = [3 \quad -2 \quad -1 \quad 0 \quad 0]^T$, $\mathbf{v}_4 = [2 \quad -2 \quad 0 \quad -3 \quad 0]^T$, $\mathbf{v}_5 = [1 \quad -1 \quad 0 \quad 0 \quad 3]^T$

The general solution

$$\mathbf{x}(t) = e^{2t}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + e^{3t}(c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5)$$

has scalar components

$$\begin{aligned}
x_1(t) &= e^{2t}(8c_1 + c_2) + e^{3t}(3c_3 + 2c_4 + c_5) \\
x_2(t) &= e^{3t}(-2c_3 - 2c_4 - c_5) \\
x_3(t) &= e^{2t}(-3c_1) + e^{3t}(-c_3) \\
x_4(t) &= e^{2t}(c_1) + e^{3t}(-3c_4) \\
x_3(t) &= e^{2t}(3c_2) + e^{3t}(3c_5)
\end{aligned}$$

33. The chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ was found using the matrices

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{bmatrix} \to \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} -32 & -32i & 8i & -8 \\ 32i & -32 & 8 & 8i \\ 0 & 0 & -32 & -32i \\ 0 & 0 & 32i & -32 \end{bmatrix} \to \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\to$ signifies reduction to row-echelon form. The resulting real-valued solution vectors are

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{3t}\,[\ \cos 4t \quad \sin 4t \quad 0 \quad 0\ ]^T \\
\mathbf{x}_2(t) &= e^{3t}\,[-\sin 4t \quad \cos 4t \quad 0 \quad 0\ ]^T \\
\mathbf{x}_3(t) &= e^{3t}\,[\ t\cos 4t \quad t\sin 4t \quad \cos 4t \quad \sin 4t\ ]^T \\
\mathbf{x}_4(t) &= e^{3t}\,[-t\sin 4t \quad t\cos 4t \quad -\sin 4t \quad \cos 4t\ ]^T.
\end{aligned}$$

34. The chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ was found using the matrices

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 3i & 0 & -8 & -3 \\ -18 & -3-3i & 0 & 0 \\ -9 & -3 & -27-3i & -9 \\ 33 & 10 & 90 & 30-3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3+3i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{bmatrix} -36 & -6 & -54+48i & -18+18i \\ 54+108i & 18i & 144 & 54 \\ 54i & 18i & -18+162i & 54i \\ -198i & -60i & 6-540i & -18-180i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3i & -i \\ 0 & 1 & 9+10i & 3+3i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\to$ signifies reduction to row-echelon form. The resulting real-valued solution vectors are

$$\mathbf{x}_1(t) = e^{2t}[\ \sin 3t \quad 3\cos 3t - 3\sin 3t \quad 0 \quad \sin 3t\ ]^T$$
$$\mathbf{x}_2(t) = e^{2t}[\ -\cos 3t \quad 3\sin 3t + 3\cos 3t \quad 0 \quad -\cos 3t\ ]^T$$
$$\mathbf{x}_3(t) = e^{2t}[3\cos 3t + t\sin 3t \quad (3t-10)\cos 3t - (3t+9)\sin 3t \quad \sin 3t \quad t\sin 3t\ ]^T$$
$$\mathbf{x}_4(t) = e^{2t}[-t\cos 3t + 3\sin 3t \quad (3t+9)\cos 3t + (3t-10)\sin 3t \quad -\cos 3t \quad -t\cos 3t]^T.$$

35. The coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix}$$

has eigenvalues

$\lambda = 0$ with eigenvector $\mathbf{v}_1 = [1 \quad 1 \quad 0 \quad 0]^T$
$\lambda = -1$ with eigenvectors $\mathbf{v}_2 = [1 \quad 0 \quad -1 \quad 0]^T$ and $\mathbf{v}_3 = [0 \quad 1 \quad 0 \quad -1]^T$,
$\lambda = -2$ with eigenvector $\mathbf{v}_4 = [1 \quad -1 \quad -2 \quad 2]^T$.

When we impose the given initial conditions on the general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 e^{-t} + c_3\mathbf{v}_3 e^{-t} + c_4\mathbf{v}_4 e^{-2t}$$

we find that $c_1 = v_0$, $c_2 = c_3 = -v_0$, $c_4 = 0$. Hence the position functions of the two masses are given by

$$x_1(t) = x_2(t) = v_0(1 - e^{-t}).$$

Each mass travels a distance v_0 before stopping.

36. The coefficient matrix is the same as in Problem 35 except that $a_{44} = -1$. Now the matrix $\mathbf{A}$ has the eigenvalue $\lambda = 0$ with eigenvector $\mathbf{v}_0 = [1 \quad 1 \quad 0 \quad 0]^T$, and the triple eigenvalue $\lambda = -1$ with associated length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ consisting of the generalized eigenvectors

$$\mathbf{v}_1 = [0 \quad 1 \quad 0 \quad -1]^T$$
$$\mathbf{v}_2 = [1 \quad 0 \quad -1 \quad 1]^T$$
$$\mathbf{v}_3 = [1 \quad 0 \quad 0 \quad 0]^T.$$

When we impose the given initial conditions on the general solution

$$\mathbf{x}(t) = c_0\mathbf{v}_0 + e^{-t}\,[c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 t + \mathbf{v}_2) + c_3(\mathbf{v}_1 t^2/2 + \mathbf{v}_2 t + \mathbf{v}_3)]$$

we find that $c_0 = 2v_0,\ c_1 = -2v_0,\ c_2 = c_3 = -v_0$. Hence the position functions of the two masses are given by

$$x_1(t) = v_0(2 - 2e^{-t} - te^{-t}),$$
$$x_2(t) = v_0(2 - 2e^{-t} - te^{-t} - t^2e^{-t}/2).$$

Each travels a distance $2v_0$ before stopping.

SECTION 5.7

MATRIX EXPONENTIALS AND LINEAR SYSTEMS

In Problems 1–8 we first use the eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$ to find first a fundamental matrix $\mathbf{\Phi}(t)$ for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then we apply the formula

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0,$$

to find the solution vector $\mathbf{x}(t)$ that satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Formulas (11) and (12) in the text provide inverses of 2-by-2 and 3-by-3 matrices.

1. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [1 \ \ -1]^T; \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = [1 \ \ 1]^T$

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5e^t + e^{3t} \\ -5e^t + e^{3t} \end{bmatrix}$$

2. Eigensystem: $\lambda_1 = 0, \quad \mathbf{v}_1 = [1 \ \ 2]^T; \qquad \lambda_2 = 4, \quad \mathbf{v}_2 = [1 \ \ -2]^T$

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 + 5e^{4t} \\ 6 - 10e^{4t} \end{bmatrix}$$

3. Eigensystem: $\lambda = 4i, \quad \mathbf{v} = [1+2i \quad 2]^T$

$$\Phi(t) = \left[\operatorname{Re}(\mathbf{v}e^{\lambda t}) \quad \operatorname{Im}(\mathbf{v}e^{\lambda t})\right] = \begin{bmatrix} \cos 4t - 2\sin 4t & 2\cos 4t + \sin 4t \\ 2\cos 4t & 2\sin 4t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \cos 4t - 2\sin 4t & 2\cos 4t + \sin 4t \\ 2\cos 4t & 2\sin 4t \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{bmatrix}$$

4. Eigensystem: $\lambda = 2, 2; \quad \{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = [1 \quad 1]^T$, $\mathbf{v}_2 = [1 \quad 0]^T$

$$\Phi(t) = \left[\mathbf{v}_1 e^{\lambda t} \quad (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t}\right] = e^{2t}\begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix}$$

$$\mathbf{x}(t) = e^{2t}\begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 1+t \\ t \end{bmatrix}$$

5. Eigensystem: $\lambda = 3i, \quad \mathbf{v} = [-1+i \quad 3]^T$

$$\Phi(t) = \left[\operatorname{Re}(\mathbf{v}e^{\lambda t}) \quad \operatorname{Im}(\mathbf{v}e^{\lambda t})\right] = \begin{bmatrix} -\cos 3t - \sin 3t & \cos 3t - \sin 3t \\ 3\cos 3t & 3\sin 3t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} -\cos 3t - \sin 3t & \cos 3t - \sin 3t \\ 3\cos 3t & 3\sin 3t \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3\cos 3t - \sin 3t \\ -3\cos 3t + 6\sin 3t \end{bmatrix}$$

6. Eigensystem: $\lambda = 5+4i, \quad \mathbf{v} = [1+2i \quad 2]^T$

$$\Phi(t) = \left[\operatorname{Re}(\mathbf{v}e^{\lambda t}) \quad \operatorname{Im}(\mathbf{v}e^{\lambda t})\right] = e^{5t}\begin{bmatrix} \cos 4t - 2\sin 4t & 2\cos 4t + 2\sin 4t \\ 2\cos 4t & 2\sin 4t \end{bmatrix}$$

$$\mathbf{x}(t) = e^{5t}\begin{bmatrix} \cos 4t - 2\sin 4t & 2\cos 4t + 2\sin 4t \\ 2\cos 4t & 2\sin 4t \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 0 & 2 \\ 2 & -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2e^{5t}\begin{bmatrix} \cos 4t + \sin 4t \\ \sin 4t \end{bmatrix}$$

7. Eigensystem:
$\lambda_1 = 0, \ \mathbf{v}_1 = [6 \ 2 \ 5]^T; \quad \lambda_2 = 1, \ \mathbf{v}_2 = [3 \ 1 \ 2]^T; \quad \lambda_3 = -1, \ \mathbf{v}_3 = [2 \ 1 \ 2]^T$

$$\Phi(t) = \left[e^{\lambda_1 t}\mathbf{v}_1 \quad e^{\lambda_2 t}\mathbf{v}_2 \quad e^{\lambda_3 t}\mathbf{v}_3\right] = \begin{bmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{bmatrix} \cdot \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -12+12e^t+2e^{-t} \\ -4+4e^t+e^{-t} \\ -10+8e^t+2e^{-t} \end{bmatrix}$$

8. Eigensystem:

$\lambda_1 = -2,\ \mathbf{v}_1 = [0 \quad 1 \quad -1]^T;\quad \lambda_2 = 1,\ \mathbf{v}_2 = [1 \quad -1 \quad 0]^T;\quad \lambda_3 = 3,\ \mathbf{v}_3 = [1 \quad -1 \quad 1]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 & e^{\lambda_3 t}\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & e^t & e^{3t} \\ e^{-2t} & -e^t & -e^{3t} \\ -e^{-2t} & 0 & e^{3t} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0 & e^t & e^{3t} \\ e^{-2t} & -e^t & -e^{3t} \\ -e^{-2t} & 0 & e^{3t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t+e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

In each of Problems 9–20 we first solve the given linear system to find two linearly independent solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, then set up the fundamental matrix $\Phi(t) = \begin{bmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t)\end{bmatrix}$, and finally calculate the matrix exponential $e^{\mathbf{A}t} = \Phi(t)\,\Phi(0)^{-1}$.

9. Eigensystem: $\lambda_1 = 1,\quad \mathbf{v}_1 = [1 \quad 1]^T;\qquad \lambda_2 = 3,\quad \mathbf{v}_2 = [2 \quad 1]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -e^t+2e^{3t} & 2e^t-2e^{3t} \\ -e^t+e^{3t} & 2e^t-e^{3t} \end{bmatrix}$$

10. Eigensystem: $\lambda_1 = 0,\quad \mathbf{v}_1 = [1 \quad 1]^T;\qquad \lambda_2 = 2,\quad \mathbf{v}_2 = [3 \quad 2]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3e^{2t} \\ 1 & 2e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 3e^{2t} \\ 1 & 2e^{2t} \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2+3e^{2t} & 3-3e^{2t} \\ -2+2e^{2t} & 3-2e^{2t} \end{bmatrix}$$

11. Eigensystem: $\lambda_1 = 2,\quad \mathbf{v}_1 = [1 \quad 1]^T;\qquad \lambda_2 = 3,\quad \mathbf{v}_2 = [3 \quad 2]^T$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2e^{2t}+3e^{3t} & 3e^{2t}-3e^{3t} \\ -2e^{2t}+2e^{3t} & 3e^{2t}-2e^{3t} \end{bmatrix}$$

12. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}; \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = [4 \quad 3]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{t} & 4e^{2t} \\ e^{t} & 3e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{t} & 4e^{2t} \\ e^{t} & 3e^{2t} \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3e^{t}+4e^{2t} & 4e^{t}-4e^{2t} \\ -3e^{t}+3e^{2t} & 4e^{t}-3e^{2t} \end{bmatrix}$$

13. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [1 \quad 1]^{\mathrm{T}}; \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = [4 \quad 3]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{t} & 4e^{3t} \\ e^{t} & 3e^{3t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{t} & 4e^{3t} \\ e^{t} & 3e^{3t} \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3e^{t}+4e^{3t} & 4e^{t}-4e^{3t} \\ -3e^{t}+3e^{3t} & 4e^{t}-3e^{3t} \end{bmatrix}$$

14. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [2 \quad 3]^{\mathrm{T}}; \qquad \lambda_2 = 3, \quad \mathbf{v}_2 = [3 \quad 4]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2e^{t} & 3e^{2t} \\ 3e^{t} & 4e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{t} & 3e^{2t} \\ 3e^{t} & 4e^{2t} \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -8e^{t}+9e^{2t} & 6e^{t}-6e^{2t} \\ -12e^{t}+12e^{2t} & 9e^{t}-8e^{2t} \end{bmatrix}$$

15. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [2 \quad 1]^{\mathrm{T}}; \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = [5 \quad 2]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2e^{t} & 5e^{2t} \\ e^{t} & 2e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{t} & 5e^{2t} \\ e^{t} & 2e^{2t} \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -4e^{t}+5e^{2t} & 10e^{t}-10e^{2t} \\ -2e^{t}+2e^{2t} & 5e^{t}-4e^{2t} \end{bmatrix}$$

16. Eigensystem: $\lambda_1 = 1, \quad \mathbf{v}_1 = [3 \quad 2]^{\mathrm{T}}; \qquad \lambda_2 = 2, \quad \mathbf{v}_2 = [5 \quad 3]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 3e^{t} & 5e^{2t} \\ 2e^{t} & 3e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^t & 5e^{2t} \\ 2e^t & 3e^{2t} \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -9e^t+10e^{2t} & 15e^t-15e^{2t} \\ -6e^t+6e^{2t} & 10e^t-9e^{2t} \end{bmatrix}$$

17. Eigensystem: $\lambda_1 = 2, \quad \mathbf{v}_1 = [1 \quad -1]^{\mathrm{T}}; \qquad \lambda_2 = 4, \quad \mathbf{v}_2 = [1 \quad 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t}+e^{4t} & -e^{2t}+e^{4t} \\ -e^{2t}+e^{4t} & e^{2t}+e^{4t} \end{bmatrix}$$

18. Eigensystem: $\lambda_1 = 2, \quad \mathbf{v}_1 = [1 \quad -1]^{\mathrm{T}}; \qquad \lambda_2 = 6, \quad \mathbf{v}_2 = [1 \quad 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t}+e^{6t} & -e^{2t}+e^{6t} \\ -e^{2t}+e^{6t} & e^{2t}+e^{6t} \end{bmatrix}$$

19. Eigensystem: $\lambda_1 = 5, \quad \mathbf{v}_1 = [1 \quad -2]^{\mathrm{T}}; \qquad \lambda_2 = 10, \quad \mathbf{v}_2 = [2 \quad 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{5t} & 2e^{10t} \\ -2e^{5t} & e^{10t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 2e^{10t} \\ -2e^{5t} & e^{10t} \end{bmatrix}\cdot\frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^{5t}+4e^{10t} & -2e^{5t}+2e^{10t} \\ -2e^{5t}+2e^{10t} & 4e^{5t}+e^{10t} \end{bmatrix}$$

20. Eigensystem: $\lambda_1 = 5, \quad \mathbf{v}_1 = [1 \quad -2]^{\mathrm{T}}; \qquad \lambda_2 = 15, \quad \mathbf{v}_2 = [2 \quad 1]^{\mathrm{T}}$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} e^{5t} & 2e^{15t} \\ -2e^{5t} & e^{15t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 2e^{15t} \\ -2e^{5t} & e^{15t} \end{bmatrix}\cdot\frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^{5t}+4e^{15t} & -2e^{5t}+2e^{15t} \\ -2e^{5t}+2e^{15t} & 4e^{5t}+e^{15t} \end{bmatrix}$$

21. $\mathbf{A}^2 = \mathbf{0}$ so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$

22. $\mathbf{A}^2 = \mathbf{0}$ so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1+6t & 4t \\ -9t & 1-6t \end{bmatrix}$

23. $\mathbf{A}^3 = \mathbf{0}$ so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 = \begin{bmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{bmatrix}$

24. $\mathbf{A}^3 = \mathbf{0}$ so $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 = \begin{bmatrix} 1+3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{bmatrix}$

25. $\mathbf{A} = 2\mathbf{I}+\mathbf{B}$ where $\mathbf{B}^2 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{2\mathbf{I}t}e^{\mathbf{B}t} = (e^{2t}\mathbf{I})(\mathbf{I}+\mathbf{B}t)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}, \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = e^{2t}\begin{bmatrix} 4+35t \\ 7 \end{bmatrix}$$

26. $\mathbf{A} = 7\mathbf{I}+\mathbf{B}$ where $\mathbf{B}^2 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{7\mathbf{I}t}e^{\mathbf{B}t} = (e^{7t}\mathbf{I})(\mathbf{I}+\mathbf{B}t)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{7t} & 0 \\ 11te^{7t} & e^{7t} \end{bmatrix}, \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 5 \\ -10 \end{bmatrix} = e^{7t}\begin{bmatrix} 5 \\ -10+55t \end{bmatrix}$$

27. $\mathbf{A} = \mathbf{I}+\mathbf{B}$ where $\mathbf{B}^3 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{\mathbf{I}t}e^{\mathbf{B}t} = (e^{t}\mathbf{I})(\mathbf{I}+t+\frac{1}{2}\mathbf{B}^2t^2)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{t} & 2te^{t} & (3t+2t^2)e^{t} \\ 0 & e^{t} & 2te^{t} \\ 0 & 0 & e^{t} \end{bmatrix}, \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = e^{t}\begin{bmatrix} 4+28t+12t^2 \\ 5+12t \\ 6 \end{bmatrix}$$

28. $\mathbf{A} = 5\mathbf{I}+\mathbf{B}$ where $\mathbf{B}^3 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{5\mathbf{I}t}e^{\mathbf{B}t} = (e^{5t}\mathbf{I})(\mathbf{I}+\mathbf{B}t+\frac{1}{2}\mathbf{B}^2t^2)$. Hence

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{5t} & 0 & 0 \\ 10te^{5t} & e^{5t} & 0 \\ (20t+150t^2)e^{5t} & 30te^{5t} & e^{5t} \end{bmatrix}, \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix} = e^{5t}\begin{bmatrix} 40 \\ 50+400t \\ 60+2300t+6000t^2 \end{bmatrix}$$

29. $\mathbf{A} = \mathbf{I}+\mathbf{B}$ where $\mathbf{B}^4 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{\mathbf{I}t}e^{\mathbf{B}t} = (e^{t}\mathbf{I})(\mathbf{I}+\mathbf{B}t+\frac{1}{2}\mathbf{B}^2t^2+\frac{1}{6}\mathbf{B}^3t^3)$. Hence

$$e^{\mathbf{A}t} = e^{t}\begin{bmatrix} 1 & 2t & 3t+6t^2 & 4t+6t^2+4t^3 \\ 0 & 1 & 6t & 3t+6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e^{t}\begin{bmatrix} 1+9t+12t^2+4t^3 \\ 1+9t+6t^2 \\ 1+2t \\ 1 \end{bmatrix}$$

30. $\mathbf{A} = 3\mathbf{I}+\mathbf{B}$ where $\mathbf{B}^4 = \mathbf{0}$, so $e^{\mathbf{A}t} = e^{3\mathbf{I}t}e^{\mathbf{B}t} = (e^{3t}\,\mathbf{I})(\mathbf{I}+\mathbf{B}t+\tfrac{1}{2}\mathbf{B}^2t^2+\tfrac{1}{6}\mathbf{B}^3t^3)$.
Hence

$$e^{\mathbf{A}t} = e^{3t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 6t & 1 & 0 & 0 \\ 9t+18t^2 & 6t & 1 & 0 \\ 12t+54t^2+36t^3 & 9t+18t^2 & 6t & 1 \end{bmatrix}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 \\ 1+6t \\ 1+15t+18t^2 \\ 1+27t+72t^2+36t^3 \end{bmatrix}$$

33. $e^{\mathbf{A}t} = \mathbf{I}\cosh t + \mathbf{A}\sinh t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$, so the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{c} = \begin{bmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{bmatrix}.$$

34. Direct calculation gives $\mathbf{A}^2 = -4\mathbf{I}$, and it follows that $\mathbf{A}^3 = -4\mathbf{A}$ and $\mathbf{A}^4 = 16\mathbf{I}$.
Therefore

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I}+\mathbf{A}t - \frac{4\mathbf{I}t^2}{2!} - \frac{4\mathbf{A}t^3}{3!} + \frac{16\mathbf{I}t^4}{4!} + \frac{16\mathbf{A}t^5}{5!} + \cdots \\ &= \mathbf{I}\left[1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} + \cdots\right] + \frac{1}{2}\mathbf{A}\left[(2t) - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} + \cdots\right] \\ e^{\mathbf{A}t} &= \mathbf{I}\cos 2t + \tfrac{1}{2}\mathbf{A}\sin 2t \end{aligned}$$

In Problems 35–40 we give first the linearly independent generalized eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ of the matrix $\mathbf{A}$ and the corresponding solution vectors $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_n(t)$ defined by Eq. (34) in the text, then the fundamental matrix $\Phi(t) = \begin{bmatrix}\mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t)\end{bmatrix}$. Finally we calculate the exponential matrix $e^{\mathbf{A}t} = \Phi(t)\Phi(0)^{-1}$.

35. $\lambda = 3: \quad \mathbf{u}_1 = [4 \quad 0]^T, \quad \mathbf{u}_2 = [0 \quad 1]^T$

$\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\left(\mathbf{u}_2 + (\mathbf{A}-\lambda\mathbf{I})\mathbf{u}_2 t\right)$$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t)] = e^{3t}\begin{bmatrix} 4 & 4t \\ 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^{3t}\begin{bmatrix} 4 & 4t \\ 0 & 1 \end{bmatrix}\cdot\frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 & 4t \\ 0 & 1 \end{bmatrix}$$

36. $\lambda = 1: \quad \mathbf{u}_1 = [8 \quad 0 \quad 0]^T, \quad \mathbf{u}_2 = [5 \quad 4 \quad 0]^T, \quad \mathbf{u}_3 = [0 \quad 1 \quad 1]^T$

$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a length 3 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ and $\mathbf{u}_3$ are generalized eigenvectors of ranks 2 and 3 (respectively).

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\left(\mathbf{u}_2 + (\mathbf{A}-\lambda\mathbf{I})\mathbf{u}_2 t\right),$$

$$\mathbf{x}_3(t) = e^{\lambda t}\left(\mathbf{u}_3 + (\mathbf{A}-\lambda\mathbf{I})\mathbf{u}_3 t + (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{u}_3 t^2/2\right)$$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)] = e^t\begin{bmatrix} 8 & 5+8t & 5t+4t^2 \\ 0 & 4 & 1+4t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = e^t\begin{bmatrix} 8 & 5+8t & 5t+4t^2 \\ 0 & 4 & 1+4t \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{32}\begin{bmatrix} 4 & -5 & 5 \\ 0 & 8 & -8 \\ 0 & 0 & 32 \end{bmatrix} = e^t\begin{bmatrix} 1 & 2t & 3t+4t^2 \\ 0 & 1 & 4t \\ 0 & 0 & 1 \end{bmatrix}$$

37. $\lambda_1 = 2: \quad \mathbf{u}_1 = [1 \quad 0 \quad 0]^T, \quad \mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$

$\lambda_2 = 1: \quad \mathbf{u}_2 = [9 \quad -3 \quad 0]^T, \quad \mathbf{u}_3 = [10 \quad 1 \quad -1]^T$

$\{\mathbf{u}_2, \mathbf{u}_3\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left(\mathbf{u}_3 + (\mathbf{A}-\lambda_2\mathbf{I})\mathbf{u}_3 t\right)$$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)] = \begin{bmatrix} e^{2t} & 9e^t & (10+9t)e^t \\ 0 & -3e^t & (1-3t)e^t \\ 0 & 0 & -e^t \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{2t} & 9e^t & (10+9t)e^t \\ 0 & -3e^t & (1-3t)e^t \\ 0 & 0 & -e^t \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 3 & 9 & 13 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & -3e^t + 3e^{2t} & (-13-9t)e^t + 13e^{2t} \\ 0 & e^t & 3t\,e^t \\ 0 & 0 & e^t \end{bmatrix}$$

38. $\lambda_1 = 10: \quad \mathbf{u}_1 = [4 \quad 1 \quad 0]^T, \quad \mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$

$\lambda_2 = 5: \quad \mathbf{u}_2 = [50 \quad 0 \quad 0]^T, \quad \mathbf{u}_3 = [0 \quad 4 \quad -1]^T$

$\{\mathbf{u}_2, \mathbf{u}_3\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left(\mathbf{u}_3 + (\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{u}_3 t\right)$$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t)] = \begin{bmatrix} 4e^{10t} & 50e^{5t} & 50te^{5t} \\ e^{10t} & 0 & 4e^{5t} \\ 0 & 0 & -e^{5t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 4e^{10t} & 50e^{5t} & 50te^{5t} \\ e^{10t} & 0 & 4e^{5t} \\ 0 & 0 & -e^{5t} \end{bmatrix} \cdot \frac{1}{50}\begin{bmatrix} 0 & 50 & 200 \\ 1 & -4 & -16 \\ 0 & 0 & -50 \end{bmatrix}$$

$$= \begin{bmatrix} e^{5t} & 4e^{10t} - 4e^{5t} & 16e^{10t} - (16+50t)e^{5t} \\ 0 & e^{10t} & 4e^{10t} - 4e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$$

39. $\lambda_2 = 1: \quad \mathbf{u}_1 = [3 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_2 = [0 \quad 1 \quad 0 \quad 0]^{\mathrm{T}}$

$\{\mathbf{u}_1, \mathbf{u}_2\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_1$, so $\mathbf{u}_2$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1, \quad \mathbf{x}_2(t) = e^{\lambda_1 t}\left(\mathbf{u}_2 + (\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{u}_2 t\right)$$

$\lambda_2 = 2: \quad \mathbf{u}_3 = [144 \quad 36 \quad 12 \quad 0]^{\mathrm{T}}, \quad \mathbf{u}_4 = [0 \quad 27 \quad 17 \quad 4]^{\mathrm{T}}$

$\{\mathbf{u}_3, \mathbf{u}_4\}$ is a length 2 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_3$, so $\mathbf{u}_4$ is a generalized eigenvector of rank 2.

$$\mathbf{x}_3(t) = e^{\lambda_2 t}\mathbf{u}_3, \quad \mathbf{x}_4(t) = e^{\lambda_2 t}\left(\mathbf{u}_4 + (\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{u}_4 t\right)$$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t) \quad \mathbf{x}_4(t)] = \begin{bmatrix} 3e^{t} & 3te^{t} & 144e^{2t} & 144te^{2t} \\ 0 & e^{t} & 36e^{2t} & (27+36t)e^{2t} \\ 0 & 0 & 12e^{2t} & (17+12t)e^{2t} \\ 0 & 0 & 0 & 4e^{2t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^{t} & 3te^{t} & 144e^{2t} & 144te^{2t} \\ 0 & e^{t} & 36e^{2t} & (27+36t)e^{2t} \\ 0 & 0 & 12e^{2t} & (17+12t)e^{2t} \\ 0 & 0 & 0 & 4e^{2t} \end{bmatrix} \cdot \frac{1}{48}\begin{bmatrix} 16 & 0 & -192 & 816 \\ 0 & 48 & -144 & 288 \\ 0 & 0 & 4 & -17 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 3te^t & (-12-9t)e^t+12te^{2t} & (51+18t)e^t+(-51+36t)e^{2t} \\ 0 & e^t & -3e^t+3e^{2t} & 6e^t+(-6+9t)e^{2t} \\ 0 & 0 & e^{2t} & 3te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}$$

40. $\lambda_1 = 3: \quad \mathbf{u}_1 = [100 \quad 20 \quad 4 \quad 1]^T, \quad \mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{u}_1$

$\lambda = 2: \quad \mathbf{u}_2 = [16 \quad 0 \quad 0 \quad 0]^T, \quad \mathbf{u}_3 = [0 \quad 4 \quad 0 \quad 0]^T, \quad \mathbf{u}_4 = [0 \quad -1 \quad 1 \quad 0]^T$

$\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a length 3 chain based on the ordinary (rank 1) eigenvector $\mathbf{u}_2$, so $\mathbf{u}_3$ and $\mathbf{u}_4$ are generalized eigenvectors of ranks 2 and 3 (respectively).

$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{u}_2, \quad \mathbf{x}_3(t) = e^{\lambda_2 t}\left(\mathbf{u}_3 + (\mathbf{A}-\lambda_2\mathbf{I})\mathbf{u}_3 t\right),$

$\mathbf{x}_4(t) = e^{\lambda_2 t}\left(\mathbf{u}_4 + (\mathbf{A}-\lambda_2\mathbf{I})\mathbf{u}_4 t + (\mathbf{A}-\lambda_2\mathbf{I})^2\mathbf{u}_4 t^2/2\right)$

$$\Phi(t) = [\mathbf{x}_1(t) \quad \mathbf{x}_2(t) \quad \mathbf{x}_3(t) \quad \mathbf{x}_4(t)] = \begin{bmatrix} 100e^{3t} & 16e^{2t} & 16te^{2t} & 8t^2e^{2t} \\ 20e^{3t} & 0 & 4e^{2t} & (-1+4t)e^{2t} \\ 4e^{3t} & 0 & 0 & e^{2t} \\ e^{3t} & 0 & 0 & 0 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 100e^{3t} & 16e^{2t} & 16te^{2t} & 8t^2e^{2t} \\ 20e^{3t} & 0 & 4e^{2t} & (-1+4t)e^{2t} \\ 4e^{3t} & 0 & 0 & e^{2t} \\ e^{3t} & 0 & 0 & 0 \end{bmatrix} \cdot \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 16 \\ 1 & 0 & 0 & -100 \\ 0 & 4 & 4 & -96 \\ 0 & 0 & 16 & -64 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 4te^{2t} & (4t+8t^2)e^{2t} & 100e^{3t}-(100+96t+32t^2)e^{2t} \\ 0 & e^{2t} & 4te^{2t} & 20e^{3t}-(20+16t)e^{2t} \\ 0 & 0 & e^{2t} & 4e^{3t}-4e^{2t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

SECTION 5.8

NONHOMOGENEOUS LINEAR SYSTEMS

1. Substitution of the trial solution $x_p(t) = a,\ y_p(t) = b$ yields the equations $a + 2b + 3 = 0,\ 2a + b - 2 = 0$ with solution $a = 7/3,\ b = -8/3$. Thus we obtain the particular solution $x(t) = 7/3,\ y(t) = -8/3$.

2. When we substitute the trial solution $x_p(t) = a_1 + b_1 t,\ y_p(t) = a_2 + b_2 t$ and collect coefficients, we get the equations

$$\begin{aligned} 2a_1 + 3a_2 + 5 &= b_1 \qquad & 2b_1 + 3b_2 &= 0 \\ 2a_1 + a_2 &= b_2 \qquad & 2b_1 + b_2 &= 2. \end{aligned}$$

We first solve the second pair for $b_1 = 3/2,\ b_2 = -1$. Then we can solve the first pair for $a_1 = 1/8,\ a_2 = -5/4$. This gives the particular solution

$$x(t) = \frac{1}{8}(1 + 12t), \qquad y(t) = -\frac{1}{4}(5 + 4t).$$

3. When we substitute the trial solution

$$x_p = a_1 + b_1 t + c_1 t^2, \qquad y_p = a_2 + b_2 t + c_2 t^2$$

and collect coefficients, we get the equations

$$\begin{aligned} 3a_1 + 4a_2 &= b_1 \qquad & 3b_1 + 4b_2 &= 2c_1 \qquad & 3c_1 + 4c_2 &= 0 \\ 3a_1 + 2a_2 &= b_2 \qquad & 3b_1 + 2b_2 &= 2c_2 \qquad & 3c_1 + 2c_2 + 1 &= 0. \end{aligned}$$

Working backwards, we solve first for $c_1 = -2/3,\ c_2 = 1/2$, then for $b_1 = 10/9,\ b_2 = -7/6$, and finally for $a_1 = -31/27,\ a_2 = 41/36$. This determines the particular solution $x_p(t),\ y_p(t)$. Next, the coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 6$ with eigenvectors $\mathbf{v}_1 = [1 \quad -1]^T$ and $\mathbf{v}_2 = [4 \quad 3]^T$, respectively, so the complementary solution is given by

$$x_c(t) = c_1 e^{-t} + 4c_2 e^{6t}, \qquad x_c(t) = -c_1 e^{-t} + 3c_2 e^{6t}.$$

When we impose the initial conditions $x(0) = 0,\ y(0) = 0$ on the general solution $x(t) = x_c(t) + x_p(t),\ y(t) = y_c(t) + y_p(t)$ we find that $c_1 = 8/7,\ c_2 = 1/756$. This finally gives the desired particular solution

$$x(t) = \frac{1}{756}(864e^{-t} + 4e^{6t} - 868 + 840t - 504t^2)$$

$$y(t) = \frac{1}{756}(-864e^{-t} + 3e^{6t} + 861 - 882t + 378t^2).$$

4. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = -5$ and $\lambda_2 = -2$ with eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [1 \quad -6]^T$, respectively, so the complementary solution is given by

$$x_c(t) = c_1e^{5t} + c_2e^{-2t}, \quad y_c(t) = c_1e^{5t} - 6c_2e^{-2t}.$$

Then we try $x_p(t) = a\,e^t$, $y_p(t) = b\,e^t$ and find readily the particular solution $x_p(t) = -\frac{1}{12}e^t$, $y_p(t) = -\frac{3}{4}e^t$. Thus the general solution is

$$x(t) = c_1e^{5t} + c_2e^{-2t} - \tfrac{1}{12}e^t, \quad y(t) = c_1e^{5t} - 6c_2e^{-2t} - \tfrac{3}{4}e^t.$$

Finally we apply the initial conditions $x(0) = y(0) = 1$ to determine $c_1 = 33/28$ and $c_2 = -2/21$. The resulting particular solution is given by

$$x(t) = \frac{1}{84}(99e^{5t} - 8e^{-2t} - 7e^t), \quad y(t) = \frac{1}{84}(99e^{5t} + 48e^{-2t} - 63e^t).$$

5. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$, so the nonhomogeneous term e^{-t} duplicates part of the complementary solution. We therefore try the particular solution

$$x_p(t) = a_1 + b_1e^{-t} + c_1t\,e^{-t}, \quad y_p(t) = a_2 + b_2e^{-t} + c_2t\,e^{-t}.$$

Upon solving the six linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we obtain the particular solution

$$x(t) = \frac{1}{3}(-12 - e^{-t} - 7t\,e^{-t}), \quad y(t) = \frac{1}{3}(-6 - 7t\,e^{-t}).$$

6. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \frac{1}{2}\left(7 \pm \sqrt{89}\right)$ so there is no duplication. We therefore try the particular solution

$$x_p(t) = b_1e^t + c_1t\,e^t, \quad y_p(t) = b_2e^t + c_2t\,e^t.$$

Upon solving the four linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we obtain the particular solution

$$x(t) = -\frac{1}{256}(91+16t)e^t, \quad y(t) = \frac{1}{32}(25+16t)e^t.$$

7. First we try the particular solution

$$x_p(t) = a_1 \sin t + b_1 \cos t, \qquad y_p(t) = a_2 \sin t + b_2 \cos t.$$

Upon solving the four linear equations we get by collecting coefficients after substitution of this trial solution into the given nonhomogeneous system, we find that $a_1 = -21/82$, $b_1 = -25/82$, $a_2 = -15/41$, $b_2 = -12/41$. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -9$ with eigenvectors $\mathbf{v}_1 = [1 \quad 1]^T$ and $\mathbf{v}_2 = [2 \quad -3]^T$, respectively, so the complementary solution is given by

$$x_c(t) = c_1 e^t + 2c_2 e^{-9t}, \qquad y_c(t) = c_1 e^t - 3c_2 e^{-9t}.$$

When we impose the initial conditions $x(0) = 1$, $y(0) = 0$, we find that $c_1 = 9/10$ and $c_2 = 83/410$. It follows that the desired particular solution $x = x_c + x_p$, $y = y_c + y_p$ is given by

$$x(t) = \frac{1}{410}(369e^t + 166e^{-9t} - 125\cos t - 105\sin t)$$

$$y(t) = \frac{1}{410}(369e^t - 249e^{-9t} - 120\cos t - 150\sin t).$$

8. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \pm 2i$, so the complementary function involves $\cos 2t$ and $\sin 2t$. There being therefore no duplication, we substitute the trial solution

$$x_p(t) = a_1 \sin t + b_1 \cos t, \qquad y_p(t) = a_2 \sin t + b_2 \cos t$$

into the given nonhomogeneous system. Upon solving the four linear equations that result upon collection of coefficients, we obtain the particular solution

$$x(t) = \frac{1}{3}(17\cos t + 2\sin t), \qquad y(t) = \frac{1}{3}(3\cos t + 5\sin t).$$

9. Here the associated homogeneous system is the same as in Problem 8, so the nonhomogeneous term $\cos 2t$ term duplicates the complementary function. We therefore substitute the trial solution

$$\begin{aligned} x_p(t) &= a_1 \sin 2t + b_1 \cos 2t + c_1 t \sin 2t + d_1 t \cos 2t \\ y_p(t) &= a_2 \sin 2t + b_2 \cos 2t + c_2 t \sin 2t + d_2 t \cos 2t \end{aligned}$$

and use a computer algebra system to solve the system of 8 linear equations that results when we collect coefficients in the usual way. This gives the particular solution

$$x(t) = \frac{1}{4}(\sin 2t + 2t\cos 2t + t\sin 2t), \qquad y(t) = \frac{1}{4}t\sin 2t.$$

10. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda = \pm i\sqrt{3}$, so there is no duplication. Substitution of the trial solution

$$x_p(t) = a_1 e^t \cos t + b_1 e^t \sin t, \qquad y_p(t) = a_2 e^t \cos t + b_2 e^t \sin t$$

yields the equations

$$\begin{aligned} 2a_2 + b_1 &= 0 & \qquad -2a_1 + 2a_2 + b_2 &= 0 \\ 2b_2 - a_1 &= 0 & \qquad -a_2 - 2b_1 + 2b_2 &= 1. \end{aligned}$$

The first two equations enable us to eliminate two of the variables immediately, and we readily solve for the values $a_1 = 4/13,\ a_2 = 3/13,\ b_1 = -6/13,\ b_2 = 2/13$ that give the particular solution

$$x(t) = \frac{1}{13}e^t(4\cos t - 6\sin t), \qquad y(t) = \frac{1}{13}e^t(3\cos t + 2\sin t).$$

11. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$, so there is duplication of constant terms. We therefore substitute the particular solution

$$x_p(t) = a_1 + b_1 t, \qquad y_p(t) = a_2 + b_2 t$$

and solve the resulting equations for $a_1 = -2,\ a_2 = 0,\ b_1 = -2,\ b_2 = 1$. The eigenvectors of the coefficient matrix associated with the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$ are $\mathbf{v}_1 = [2 \quad -1]^T$ and $\mathbf{v}_2 = [2 \quad 1]^T$, respectively, so the general solution of the given nonhomogeneous system is given by

$$x(t) = 2c_1 + 2c_2 e^{4t} - 2 - 2t, \qquad y(t) = -c_1 + c_2 e^{4t} + t.$$

When we impose the initial conditions $x(0) = 1,\ y(0) = -1$ we find readily that $c_1 = 5/4,\ c_2 = 1/4$. This gives the desired particular solution

$$x(t) = \tfrac{1}{2}(1-4t+e^{4t}), \qquad y(t) = \tfrac{1}{4}(-5+4t+e^{4t}).$$

12. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$, so there is duplication of constant terms in the first natural attempt. We must multiply the t-terms by t and include all lower degree terms in the trial solution. Thus we substitute the the trial solution

$$x_p(t) = a_1 + b_1 t + c_1 t^2, \qquad y_p(t) = a_2 + b_2 t + c_2 t^2.$$

The resulting six equations in the coefficients are satisfied by $a_1 = b_1 = a_2 = b_2 = 0$, $c_1 = 1$, $c_2 = -1$. This gives the particular solution $x(t) = t^2$, $y(t) = -t^2$.

13. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, so there is duplication of e^t terms. We therefore substitute the trial solution

$$x_p(t) = (a_1 + b_1 t)e^t, \quad y_p(t) = (a_2 + b_2 t)e^t.$$

This leads readily to the particular solution

$$x(t) = \frac{1}{2}(1+5t)e^t, \qquad y(t) = -\frac{5}{2}te^t.$$

14. The coefficient matrix of the associated homogeneous system has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$, so there is duplication of both constant terms and e^{4t} terms. We therefore substitute the particular solution

$$x_p(t) = a_1 + b_1 t + c_1 e^{4t} + d_1 t e^{4t}, \quad y_p(t) = a_2 + b_2 t + c_2 e^{4t} + d_2 t e^{4t}.$$

When we use a computer algebra system to solve the resulting system of 8 equations in 8 unknowns, we find that a_2 and c_2 can be chosen arbitrarily. With both zero we get the particular solution

$$x(t) = \frac{1}{8}\left(-2+4t-e^{4t}+2te^{4t}\right), \qquad y(t) = \frac{t}{2}\left(-2+e^{4t}\right).$$

In Problems 15 and 16 the amounts $x_1(t)$ and $x_2(t)$ in the two tanks satisfy the equations

$$x_1' = rc_0 - k_1 x_1, \qquad x_2' = k_1 x_1 - k_2 x_2$$

where $k_i = r/V_i$ in terms of the flow rate r, the inflowing concentration c_0, and the volumes V_1 and V_2 of the two tanks.

15. **(a)** We solve the initial value problem

$$\begin{aligned} x_1' &= 20 - x_1/10, & x_1(0) &= 0 \\ x_2' &= x_1/10 - x_2/20, & x_2(0) &= 0 \end{aligned}$$

for $x_1(t) = 200(1 - e^{-t/10})$, $x_2(t) = 400(1 + e^{-t/10} - 2e^{-t/20})$.

(b) Evidently $x_1(t) \to 200$ gal and $x_2(t) \to 400$ gal as $t \to \infty$.

(c) It takes about 6 min 56 sec for tank 1 to reach a salt concentration of 1 lb/gal, and about 24 min 34 sec for tank 2 to reach this concentration.

16. **(a)** We solve the initial value problem

$$\begin{aligned} x_1' &= 30 - x_1/20, & x_1(0) &= 0 \\ x_2' &= x_1/20 - x_2/10, & x_2(0) &= 0 \end{aligned}$$

for $x_1(t) = 600(1 - e^{-t/20})$, $x_2(t) = 300(1 + e^{-t/10} - 2e^{-t/20})$.

(b) Evidently $x_1(t) \to 600$ gal and $x_2(t) \to 300$ gal as $t \to \infty$.

(c) It takes about 8 min 7 sec for tank 1 to reach a salt concentration of 1 lb/gal, and about 17 min 13 sec for tank 2 to reach this concentration.

In Problems 17–34 we apply the variation of parameters formula in Eq. (28) of Section 5.8. The answers shown below were actually calculated using the Mathematica. For instance, for Problem 17 we first enter the coefficient matrix

```
A = {{6,  -7},
     {1,  -2}};
```

the initial vector

```
x0 = {{0},
      {0}};
```

and the vector

```
f[t_] := {{60},
          {90}};
```

of nonhomogeneous terms. It simplifies the notation to rename Mathematica's exponential matrix function by defining

```
exp[A_] := MatrixExp[A]
```

Then the integral in the variation of parameters formula is given by

```
integral =
Integrate[exp[-A*s] . f[s], {s, 0, t}] // Simplify
```

$$\begin{bmatrix} -102+7e^{-5t}+95e^{t} \\ -96+e^{-5t}+95e^{t} \end{bmatrix}.$$

Finally the desired particular solution is given by

```
solution =
exp[A*t] . (x0 + integral) // Simplify
```

$$\begin{bmatrix} 102-7e^{-5t}-95e^{t} \\ 96-e^{-5t}-95e^{t} \end{bmatrix}.$$

In each succeeding problem, we need only substitute the given coefficient matrix **A**, initial vector **x0**, and the vector **f** of nonhomogeneous terms in the above commands, and then re-execute them in turn. We give below only the component functions of the final results.

17. $x_1(t) = 102-95e^{-t}-7e^{5t}, \quad x_2(t) = 96-95e^{-t}-e^{5t}$

18. $x_1(t) = 68-110t-75e^{-t}+7e^{5t}, \quad x_2(t) = 74-80t-75e^{-t}+e^{5t}$

19. $x_1(t) = -70-60t+16e^{-3t}+54e^{2t}, \quad x_2(t) = 5-60t-32e^{-3t}+27e^{2t}$

20. $x_1(t) = 3e^{2t}+60te^{2t}-3e^{-3t}, \quad x_2(t) = -6e^{2t}+30te^{2t}+6e^{-3t}$

21. $x_1(t) = -e^{-t}-14e^{2t}+15e^{3t}, \quad x_2(t) = -5e^{-t}-10e^{2t}+15e^{3t}$

22. $x_1(t) = -10e^{-t}-7te^{-t}+10e^{3t}-5te^{3t}, \quad x_2(t) = -15e^{-t}-35te^{-t}+15e^{3t}-5te^{3t}$

23. $x_1(t) = 3+11t+8t^2, \quad x_2(t) = 5+17t+24t^2$

24. $x_1(t) = 2+t+\ln t, \quad x_2(t) = 5+3t-\frac{1}{t}+3\ln t$

25. $x_1(t) = -1+8t+\cos t-8\sin t, \quad x_2(t) = -2+4t+2\cos t-3\sin t$

26. $x_1(t) = 3\cos t - 32\sin t + 17t\cos t + 4t\sin t$
$x_2(t) = 5\cos t - 13\sin t + 6t\cos t + 5t\sin t$

27. $x_1(t) = 8t^3 + 6t^4, \quad x_2(t) = 3t^2 - 2t^3 + 3t^4$

28. $x_1(t) = -7 + 14t - 6t^2 + 4t^2\ln t, \quad x_2(t) = -7 + 9t - 3t^2 + \ln t - 2t\ln t + 2t^2\ln t$

29. $x_1(t) = t\cos t - \ln(\cos t)\sin t, \quad x_2(t) = t\sin t - \ln(\cos t)\cos t$

30. $x_1(t) = \frac{1}{2}t^2\cos 2t, \quad x_2(t) = \frac{1}{2}t^2\sin 2t$

31. $x_1(t) = (9t^2 + 4t^3)e^t, \quad x_2(t) = 6t^2e^t, \quad x_3(t) = 6te^t$

32. $x_1(t) = (44 + 18t)e^t + (-44 + 26t)e^{2t}, \quad x_2(t) = 6e^t + (-6 + 6t)e^{2t}, \quad x_3(t) = 2te^{2t}$

33. $x_1(t) = 15t^2 + 60t^3 + 95t^4 + 12t^5, \quad x_2(t) = 15t^2 + 55t^3 + 15t^4,$
$x_3(t) = 15t^2 + 20t^3, \quad x_4(t) = 15t^2$

34. $x_1(t) = 4t^3 + (4 + 16t + 8t^2)e^{2t}, \quad x_2(t) = 3t^2 + (2 + 4t)e^{2t},$
$x_3(t) = (2 + 4t + 2t^2)e^{2t}, \quad x_4(t) = (1 + t)e^{2t}$

CHAPTER 6

NUMERICAL METHODS

SECTION 6.1

NUMERICAL APPROXIMATION: EULER'S METHOD

In each of Problems 1–10 we also give first the explicit form of Euler's iterative formula for the given differential equation $y' = f(x, y)$. As we illustrate in Problem 1, the desired iterations are readily implemented, either manually or with a computer system or graphing calculator. Then we list the indicated values of $y(\frac{1}{2})$ rounded off accurate to 3 decimal places.

1. For the differential equation $y' = f(x, y)$ with $f(x, y) = -y$, the iterative formula of Euler's method is $y_{n+1} = y_n + h(-y_n)$. The TI-83 screen on the left shows a graphing calculator implementation of this iterative formula.

```
0.1→H:0→X:2→Y
               2.0000
X+H→X:Y+H*(-Y)→Y

               1.8000
               1.6200
               1.4580
```

```
               1.8000
               1.6200
               1.4580
               1.3122
               1.1810
X
                .5000
```

After the variables are initialized (in the first line), and the formula is entered, each press of the enter key carries out an additional step. The screen on the right shows the results of 5 steps from $x = 0$ to $x = 0.5$ with step size $h = 0.1$ — winding up with $y(0.5) \approx 1.181$.

Approximate values 1.125 and 1.181; true value $y(\frac{1}{2}) \approx 1.213$

The following *Mathematica* instructions produce precisely this line of data.

```
f[x_,y_] = -y;
g[x_] = 2 Exp[-x];

h = 0.25;    x = 0;    y1 = y0;

Do[  k = f[x,y1];           (* the left-hand slope     *)
```

```
        y1 = y1 + h*k;           (* Euler step to update y  *)
        x = x + h,               (* update x                *)
        {i,1,2} ]

h = 0.1;    x = 0;    y2 = y0;
Do[  k = f[x,y2];                (* the left-hand slope     *)
     y2 = y2 + h*k;              (* Euler step to update y  *)
     x = x + h,                  (* update x                *)
     {i,1,5} ]

Print[x,"      ",y1,"      ",y2,"      ",g[0.5]]
0.5      1.125      1.18098      1.21306
```

2. Iterative formula: $y_{n+1} = y_n + h(2y_n)$

Approximate values 1.125 and 1.244; true value $y(\frac{1}{2}) \approx 1.359$

3. Iterative formula: $y_{n+1} = y_n + h(y_n + 1)$

Approximate values 2.125 and 2.221; true value $y(\frac{1}{2}) \approx 2.297$

4. Iterative formula: $y_{n+1} = y_n + h(x_n - y_n)$

Approximate values 0.625 and 0.681; true value $y(\frac{1}{2}) \approx 0.713$

5. Iterative formula: $y_{n+1} = y_n + h(y_n - x_n - 1)$

Approximate values 0.938 and 0.889; true value $y(\frac{1}{2}) \approx 0.851$

6. Iterative formula: $y_{n+1} = y_n + h(-2x_n y_n)$

Approximate values 1.750 and 1.627; true value $y(\frac{1}{2}) \approx 1.558$

7. Iterative formula: $y_{n+1} = y_n + h(-3x_n^2 y_n)$

Approximate values 2.859 and 2.737; true value $y(\frac{1}{2}) \approx 2.647$

8. Iterative formula: $y_{n+1} = y_n + h\exp(-y_n)$

Approximate values 0.445 and 0.420; true value $y(\frac{1}{2}) \approx 0.405$

9. Iterative formula: $y_{n+1} = y_n + h(1 + y_n^2)/4$

Approximate values 1.267 and 1.278; true value $y(\frac{1}{2}) \approx 1.287$

10. Iterative formula: $y_{n+1} = y_n + h(2x_n y_n^2)$

Approximate values 1.125 and 1.231; true value $y(\frac{1}{2}) \approx 1.333$

The tables of approximate and actual values called for in Problems 11–16 were produced using the following MATLAB script (appropriately altered for each problem).

```
% Section 6.1, Problems 11-16
x0 = 0;     y0 = 1;
% first run:
h = 0.01;
x = x0;  y = y0;  y1 = y0;
for  n = 1:100
   y = y + h*(y-2);
   y1 = [y1,y];
   x = x + h;
   end
% second run:
h = 0.005;
x = x0;  y = y0;  y2 = y0;
for  n = 1:200
   y = y + h*(y-2);
   y2 = [y2,y];
   x = x + h;
   end
% exact values
x = x0 : 0.2 : x0+1;
ye = 2 - exp(x);
% display table
ya = y2(1:40:201);
err = 100*(ye-ya)./ye;
[x; y1(1:20:101); ya; ye; err]
```

11. The iterative formula of Euler's method is $y_{n+1} = y_n + h(y_n - 2)$, and the exact solution is $y(x) = 2 - e^x$. The resulting table of approximate and actual values is

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	1.0000	0.7798	0.5111	0.1833	–0.2167	–0.7048
y (h=0.005)	1.0000	0.7792	0.5097	0.1806	–0.2211	–0.7115
y actual	1.0000	0.7786	0.5082	0.1779	–0.2255	–0.7183
error	0%	–0.08%	–0.29%	–1.53%	1.97%	0.94%

12. Iterative formula: $y_{n+1} = y_n + h(y_n - 1)^2/2$

Exact solution: $y(x) = 1 + 2/(2 - x)$

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	2.0000	2.1105	2.2483	2.4250	2.6597	2.9864
y (h=0.005)	2.0000	2.1108	2.2491	2.4268	2.6597	2.9931
y actual	2.0000	2.1111	2.2500	2.4286	2.6597	3.0000
error	0%	0.02%	0.04%	0.07%	0.13%	0.23%

13. Iterative formula: $y_{n+1} = y_n + 2hx_n^3/y_n$

Exact solution: $y(x) = (8 + x^4)^{1/2}$

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	3.0000	3.1718	3.4368	3.8084	4.2924	4.8890
y (h=0.005)	3.0000	3.1729	3.4390	3.8117	4.2967	4.8940
y actual	3.0000	3.1739	3.4412	3.8149	4.3009	4.8990
error	0%	0.03%	0.06%	0.09%	0.10%	0.10%

14. Iterative formula: $y_{n+1} = y_n + hy_n^2/x_n$

Exact solution: $y(x) = 1/(1 - \ln x)$

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	1.0000	1.2215	1.5026	1.8761	2.4020	3.2031
y (h=0.005)	1.0000	1.2222	1.5048	1.8814	2.4138	3.2304
y actual	1.0000	1.2230	1.5071	1.8868	2.4259	3.2589
error	0%	0.06%	0.15%	0.29%	0.50%	0.87%

15. Iterative formula: $y_{n+1} = y_n + h(3 - 2y_n/x_n)$

Exact solution: $y(x) = x + 4/x^2$

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.0000	3.0253	3.0927	3.1897	3.3080	3.4422
y (h=0.005)	3.0000	3.0259	3.0936	3.1907	3.3091	3.4433
y actual	3.0000	3.0264	3.0944	3.1917	3.3102	3.4444
error	0%	0.019%	0.028%	0.032%	0.033%	0.032%

16. Iterative formula: $y_{n+1} = y_n + 2hx_n^5/y_n^2$

Exact solution: $y(x) = (x^6 - 37)^{1/3}$

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.0000	4.2476	5.3650	6.4805	7.6343	8.8440
y (h=0.005)	3.0000	4.2452	5.3631	6.4795	7.6341	8.8445
y actual	3.0000	4.2429	5.3613	6.4786	7.6340	8.8451
error	0%	–0.056%	–0.034%	–0.015%	0.002%	0.006%

The tables of approximate values called for in Problems 17–24 were produced using a MATLAB script similar to the one listed preceding the Problem 11 solution above.

17.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.1)	0.0000	0.0010	0.0140	0.0551	0.1413	0.2925
y (h=0.02)	0.0000	0.0023	0.0198	0.0688	0.1672	0.3379
y (h=0.004)	0.0000	0.0026	0.0210	0.0717	0.1727	0.3477
y (h=0.0008)	0.0000	0.0027	0.0213	0.0723	0.1738	0.3497

These data indicate that $y(1) \approx 0.35$, in contrast with Example 5 in the text, where the initial condition is $y(0) = 1$.

In Problems 18–24 we give only the final approximate values of y obtained using Euler's method with step sizes $h = 0.1$, $h = 0.02$, $h = 0.004$, and $h = 0.0008$.

18. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	1.6680	1.6771	1.6790	1.6794

19. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	6.1831	6.3653	6.4022	6.4096

20. With $x_0 = 0$ and $y_0 = -1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	−1.3792	−1.2843	−1.2649	−1.2610

21. With $x_0 = 1$ and $y_0 = 2$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	2.8508	2.8681	2.8716	2.8723

22. With $x_0 = 0$ and $y_0 = 1$, the approximate values of $y(2)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	6.9879	7.2601	7.3154	7.3264

23. With $x_0 = 0$ and $y_0 = 0$, the approximate values of $y(1)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	1.2262	1.2300	1.2306	1.2307

24. With $x_0 = -1$ and $y_0 = 1$, the approximate values of $y(1)$ obtained are:

h	0.1	0.02	0.004	0.0008
y	0.9585	0.9918	0.9984	0.9997

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0,\ v_0 = 0$.

With $h = 0.01$, 100 iterations of $v_{n+1} = v_n + h\,f(t_n, v_n)$ yield $v(1) \approx 16.014$, and 200 iterations with $h = 0.005$ yield $v(1) \approx 15.998$. Thus we observe an approximate velocity of 16.0 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.01$, 200 iterations yield $v(2) \approx 19.2056$, and 400 iterations with $h = 0.005$ yield $v(2) \approx 19.1952$. Thus we observe an approximate velocity of 19.2 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0,\ P_0 = 25$.

With $h = 1$, 60 iterations of $P_{n+1} = P_n + h\,f(t_n, P_n)$ yield $P(60) \approx 49.3888$, and 120 iterations with $h = 0.5$ yield $P(60) \approx 49.3903$. Thus we observe a population of 49 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 1$, 120 iterations yield $P(120) \approx 66.1803$, and 240 iterations with $h = 0.5$ yield $P(60) \approx 66.1469$. Thus we observe a population of 66 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x,y) = x^2 + y^2 - 1$ and $x_0 = 0,\ y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.00$ rounded off accurate to 2 decimal places.

h	0.1	0.01	0.001	0.0001	0.00001
n	20	200	2000	20000	200000
$y(2)$	0.7772	0.9777	1.0017	1.0042	1.0044

28. Here $f(x,y) = x + \frac{1}{2}y^2$ and $x_0 = -2,\ y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.46$ rounded off accurate to 2 decimal places.

h	0.1	0.01	0.001	0.0001	0.00001
n	40	400	4000	40000	400000
$y(2)$	1.2900	1.4435	1.4613	1.4631	1.4633

29. With step sizes $h = 0.15$, $h = 0.03$, and $h = 0.006$ we get the following results:

x	y with h=0.15	y with h=0.03	y with h=0.006
−1.0	1.0000	1.0000	1.0000
−0.7	1.0472	1.0512	1.0521
−0.4	1.1213	1.1358	1.1390
−0.1	1.2826	1.3612	1.3835
+0.2	0.8900	1.4711	0.8210
+0.5	0.7460	1.2808	0.7192

While the values for $h = 0.15$ alone are not conclusive, a comparison of the values of y for all three step sizes with $x > 0$ suggests some anomaly in the transition from negative to positive values of x.

30. With step sizes $h = 0.1$ and $h = 0.01$ we get the following results:

x	y with $h = 0.1$	y with $h = 0.01$
0.0	0.0000	0.0000
0.1	0.0000	0.0003
0.2	0.0010	0.0025
0.3	0.0050	0.0086
.	.	.
.	.	.
.	.	.
1.8	2.8200	4.3308
1.9	3.9393	7.9425
2.0	5.8521	28.3926

Clearly there is some difficulty near $x = 2$.

31. With step sizes $h = 0.1$ and $h = 0.01$ we get the following results:

x	y with $h = 0.1$	y with $h = 0.01$
0.0	1.0000	1.0000
0.1	1.2000	1.2200
0.2	1.4428	1.4967
.	.	.
.	.	.
.	.	.

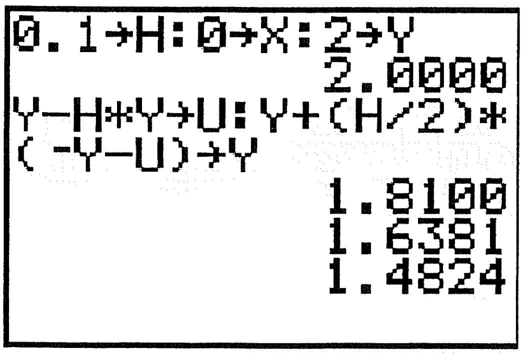
0.7	4.3460	6.4643
0.8	5.8670	11.8425
0.9	8.3349	39.5010

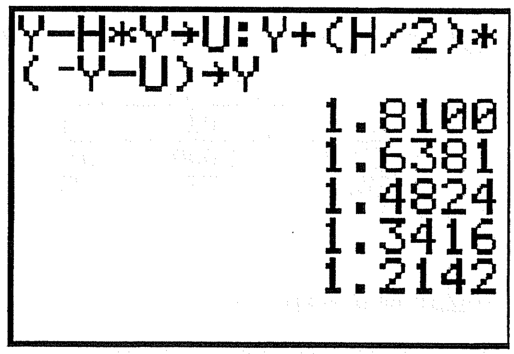
Clearly there is some difficulty near $x = 0.9$.

SECTION 6.2

A CLOSER LOOK AT THE EULER METHOD

In each of Problems 1–10 we give first the predictor formula for u_{n+1} and then the improved Euler corrector for y_{n+1}. These predictor-corrector iterations are readily implemented, either manually or with a computer system or graphing calculator (as we illustrate in Problem 1). We give in each problem a table showing the approximate values obtained, as well as the corresponding values of the exact solution.





1. $u_{n+1} = y_n + h(-y_n)$

$y_{n+1} = y_n + (h/2)[-y_n - u_{n+1}]$

The TI-83 screen on the left above shows a graphing calculator implementation of this iteration. After the variables are initialized (in the first line), and the formulas are entered, each press of the enter key carries out an additional step. The screen on the right shows the results of 5 steps from $x = 0$ to $x = 0.5$ with step size $h = 0.1$ — winding up with $y(0.5) \approx 1.2142$ — and we see the approximate values shown in the second row of the table below.

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	2.0000	1.8100	1.6381	1.4824	1.3416	1.2142
y actual	2.0000	1.8097	1.6375	1.4816	1.3406	1.2131

2. $u_{n+1} = y_n + 2hy_n$

$y_{n+1} = y_n + (h/2)[2y_n + 2u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	0.5000	0.6100	0.7422	0.9079	1.1077	1.3514
y actual	0.5000	0.6107	0.7459	0.9111	1.1128	1.3591

3. $u_{n+1} = y_n + h(y_n + 1)$

$y_{n+1} = y_n + (h/2)[(y_n + 1) + (u_{n+1} + 1)]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.2100	1.4421	1.6985	1.9818	2.2949
y actual	1.0000	1.2103	1.4428	1.6997	1.9837	2.2974

4. $u_{n+1} = y_n + h(x_n - y_n)$

$y_{n+1} = y_n + (h/2)[(x_n - y_n) + (x_n + h - u_{n+1})]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	0.9100	0.8381	0.7824	0.7416	0.7142
y actual	1.0000	0.9097	0.8375	0.7816	0.7406	0.7131

5. $u_{n+1} = y_n + h(y_n - x_n - 1)$

$y_{n+1} = y_n + (h/2)[(y_n - x_n - 1) + (u_{n+1} - x_n - h - 1)]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	0.9950	0.9790	0.9508	0.9091	0.8526
y actual	1.0000	0.9948	0.9786	0.9501	0.9082	0.8513

6. $u_{n+1} = y_n - 2x_n y_n h$

$y_{n+1} = y_n - (h/2)[2x_n y_n + 2(x_n + h)u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	2.0000	1.9800	1.9214	1.8276	1.7041	1.5575
y actual	2.0000	1.9801	1.9216	1.8279	1.7043	1.5576

7. $u_{n+1} = y_n - 3x_n^2 y_n h$

$y_{n+1} = y_n - (h/2)[3x_n^2 y_n + 3(x_n + h)^2 u_{n+1}]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	3.0000	2.9955	2.9731	2.9156	2.8082	2.6405
y actual	3.0000	2.9970	2.9761	2.9201	2.8140	2.6475

8. $u_{n+1} = y_n + h\exp(-y_n)$

$y_{n+1} = y_n + (h/2)[\exp(-y_n) + \exp(-u_{n+1})]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	0.0000	0.0952	0.1822	0.2622	0.3363	0.4053
y actual	0.0000	0.0953	0.1823	0.2624	0.3365	0.4055

9. $u_{n+1} = y_n + h(1 + y_n^2)/4$

$y_{n+1} = y_n + h[1 + y_n^2 + 1 + (u_{n+1})^2]/8$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.0513	1.1053	1.1625	1.2230	1.2873
y actual	1.0000	1.0513	1.1054	1.1625	1.2231	1.2874

10. $u_{n+1} = y_n + 2x_n y_n^2 h$

$y_{n+1} = y_n + h[x_n y_n^2 + (x_n + h)(u_{n+1})^2]$

x	0.0	0.1	0.2	0.3	0.4	0.5
y with h=0.1	1.0000	1.0100	1.0414	1.0984	1.1895	1.3309
y actual	1.0000	1.0101	1.0417	1.0989	1.1905	1.3333

The results given below for Problems 11–16 were computed using the following MATLAB script.

```
% Section 6.2, Problems 11-16
x0 = 0;  y0 = 1;
% first run:
h = 0.01;
x = x0;  y = y0;  y1 = y0;
for  n = 1:100
   u = y + h*f(x,y);                  %predictor
   y = y + (h/2)*(f(x,y)+f(x+h,u));   %corrector
   y1 = [y1,y];
   x = x + h;
   end
```

```
% second run:
h = 0.005;
x = x0;   y = y0;   y2 = y0;

for  n = 1:200
   u = y + h*f(x,y);                        %predictor
   y = y + (h/2)*(f(x,y)+f(x+h,u));         %corrector
   y2 = [y2,y];
   x = x + h;
end

% exact values
x = x0 : 0.2 : x0+1;
ye = g(x);

% display table
ya = y2(1:40:201);
err = 100*(ye-ya)./ye;
x = sprintf('%10.5f',x), sprintf('\n');
y1 = sprintf('%10.5f',y1(1:20:101)), sprintf('\n');
ya = sprintf('%10.5f',ya), sprintf('\n');
ye = sprintf('%10.5f',ye), sprintf('\n');
err = sprintf('%10.5f',err), sprintf('\n');
table = [x; y1; ya; ye; err]
```

For each problem the differential equation $y' = f(x, y)$ and the known exact solution $y = g(x)$ are stored in the files `f.m` and `g.m` — for instance, the files

```
function  yp = f(x,y)
yp = y-2;

function ye = g(x,y)
ye = 2-exp(x);
```

for Problem 11. (The exact solutions for Problems 11–16 here are given in the solutions for Problems 11–16 in Section 6.1)

11.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	1.00000	0.77860	0.50819	0.17790	–0.22551	–0.71824
y (h=0.005)	1.00000	0.77860	0.50818	0.17789	–0.22553	–0.71827
y actual	1.00000	0.77860	0.50818	0.17788	–0.22554	–0.71828
error	0.000%	–0.000%	–0.001%	–0.003%	0.003%	0.002%

12.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.01)	2.00000	2.11111	2.25000	2.42856	2.66664	2.99995
y (h=0.005)	2.00000	2.11111	2.25000	2.42857	2.66666	2.99999
y actual	2.00000	2.11111	2.25000	2.42857	2.66667	3.00000
error	0.0000%	0.0000%	0.0001%	0.0001%	0.0002%	0.0004%

13.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	3.00000	3.17390	3.44118	3.81494	4.30091	4.89901
y (h=0.005)	3.00000	3.17390	3.44117	3.81492	4.30089	4.89899
y actual	3.00000	3.17389	3.44116	3.81492	4.30088	4.89898
error	0.0000%	–0.0001%	–0.0001%	0.0001%	–0.0002%	–0.0002%

14.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.01)	1.00000	1.22296	1.50707	1.88673	2.42576	3.25847
y (h=0.005)	1.00000	1.22297	1.50709	1.88679	2.42589	3.25878
y actual	1.00000	1.22297	1.50710	1.88681	2.42593	3.25889
error	0.0000%	0.0002%	0.0005%	0.0010%	0.0018%	0.0033%

15.

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.000000	3.026448	3.094447	3.191719	3.310207	3.444448
y (h=0.005)	3.000000	3.026447	3.094445	3.191717	3.310205	3.444445
y actual	3.000000	3.026446	3.094444	3.191716	3.310204	3.444444
error	0.00000%	–0.00002%	–0.00002%	–0.00002%	–0.00002%	–0.00002%

16.

x	2.0	2.2	2.4	2.6	2.8	3.0
y (h=0.01)	3.000000	4.242859	5.361304	6.478567	7.633999	8.845112
y (h=0.005)	3.000000	4.242867	5.361303	6.478558	7.633984	8.845092
y actual	3.000000	4.242870	5.361303	6.478555	7.633979	8.845085
error	0.00000%	0.00006%	–0.00001%	–0.00005%	–0.00007%	–0.00007%

17. With h = 0.1: $y(1) \approx 0.35183$
With h = 0.02: $y(1) \approx 0.35030$
With h = 0.004: $y(1) \approx 0.35023$
With h = 0.0008: $y(1) \approx 0.35023$

The table of numerical results is

x	y with $h = 0.1$	y with $h = 0.02$	y with $h = 0.004$	y with $h = 0.0008$
0.0	0.00000	0.00000	0.00000	0.00000
0.2	0.00300	0.00268	0.00267	0.00267
0.4	0.02202	0.02139	0.02136	0.02136
0.6	0.07344	0.07249	0.07245	0.07245
0.8	0.17540	0.17413	0.17408	0.17408
1.0	0.35183	0.35030	0.35023	0.35023

In Problems 18–24 we give only the final approximate values of y obtained using the improved Euler method with step sizes $h = 0.1$, $h = 0.02$, $h = 0.004$, and $h = 0.0008$.

18. With $h = 0.1$: $y(2) \approx 1.68043$
With $h = 0.02$: $y(2) \approx 1.67949$
With $h = 0.004$: $y(2) \approx 1.67946$
With $h = 0.0008$: $y(2) \approx 1.67946$

19. With $h = 0.1$: $y(2) \approx 6.40834$
With $h = 0.02$: $y(2) \approx 6.41134$
With $h = 0.004$: $y(2) \approx 6.41147$
With $h = 0.0008$: $y(2) \approx 6.41147$

20. With $h = 0.1$: $y(2) \approx -1.26092$
With $h = 0.02$: $y(2) \approx -1.26003$
With $h = 0.004$: $y(2) \approx -1.25999$
With $h = 0.0008$: $y(2) \approx -1.25999$

21. With $h = 0.1$: $y(2) \approx 2.87204$
With $h = 0.02$: $y(2) \approx 2.87245$
With $h = 0.004$: $y(2) \approx 2.87247$
With $h = 0.0008$: $y(2) \approx 2.87247$

22. With $h = 0.1$: $y(2) \approx 7.31578$
With $h = 0.02$: $y(2) \approx 7.32841$
With $h = 0.004$: $y(2) \approx 7.32916$
With $h = 0.0008$: $y(2) \approx 7.32920$

23. With $h = 0.1$: $y(1) \approx 1.22967$
With $h = 0.02$: $y(1) \approx 1.23069$
With $h = 0.004$: $y(1) \approx 1.23073$
With $h = 0.0008$: $y(1) \approx 1.23073$

24. With $h = 0.1$: $y(1) \approx 1.00006$
With $h = 0.02$: $y(1) \approx 1.00000$
With $h = 0.004$: $y(1) \approx 1.00000$
With $h = 0.0008$: $y(1) \approx 1.00000$

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0$, $v_0 = 0$.

With $h = 0.01$, 100 iterations of

$$k_1 = f(t, v_n), \quad k_2 = f(t+h, v_n + hk_1), \quad v_{n+1} = v_n + \frac{h}{2}(k_1 + k_2)$$

yield $v(1) \approx 15.9618$, and 200 iterations with $h = 0.005$ yield $v(1) \approx 15.9620$. Thus we observe an approximate velocity of 15.962 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.01$, 200 iterations yield $v(2) \approx 19.1846$, and 400 iterations with $h = 0.005$ yield $v(2) \approx 19.1847$. Thus we observe an approximate velocity of 19.185 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0$, $P_0 = 25$.

With $h = 1$, 60 iterations of

$$k_1 = f(t,P_n), \quad k_2 = f(t+h, P_n + hk_1), \quad P_{n+1} = P_n + \frac{h}{2}(k_1 + k_2)$$

yield $P(60) \approx 49.3909$, and 120 iterations with $h = 0.5$ yield $P(60) \approx 49.3913$. Thus we observe an approximate population of 49.391 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 1$, 120 iterations yield $P(120) \approx 66.1129$, and 240 iterations with $h = 0.5$ yield $P(60) \approx 66.1134$. Thus we observe an approximate population of 66.113 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x,y) = x^2 + y^2 - 1$ and $x_0 = 0$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.0045$ rounded off accurate to 4 decimal places.

h	0.1	0.01	0.001	0.0001
n	20	200	2000	20000
$y(2)$	1.01087	1.00452	1.00445	1.00445

28. Here $f(x,y) = x + \frac{1}{2}y^2$ and $x_0 = -2$, $y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.4633$ rounded off accurate to 4 decimal places.

h	0.1	0.01	0.001	0.0001
n	40	400	4000	40000
$y(2)$	1.46620	1.46335	1.46332	1.46331

In the solutions for Problems 29 and 30 we illustrate the following general MATLAB ode solver.

```
function   [t,y] = ode(method, yp, t0,b, y0, n)
%  [t,y] = ode(method, yp, t0,b, y0, n)
```

```
%   calls the method described by 'method' for the
%   ODE 'yp' with function header
%
%              y' = yp(t,y)
%
%   on the interval  [t0,b]  with initial (column)
%   vector  y0.  Choices for method are 'euler',
%   'impeuler', 'rk' (Runge-Kutta), 'ode23', 'ode45'.
%   Results are saved at the endPoints of n subintervals,
%   that is, in steps of length  h = (b - t0)/n.  The
%   result  t  is an (n+1)-column vector from b to t1,
%   while  y  is a matrix with  n+1  rows (one for each
%   t-value) and one column for each dependent variable.

h = (b - t0)/n;                 % step size
t = t0 : h : b;
t = t';                               % col. vector of t-values
y = y0';                        % 1st row of result matrix
for  i = 2 : n+1                % for i=2 to i=n+1
   t0 = t(i-1);                 % old t
   t1 = t(i);                         % new t
   y0 = y(i-1,:)';                    % old y-row-vector
   [T,Y] = feval(method, yp, t0,t1, y0);
   y = [y;Y'];                        % adjoin new y-row-vector
end
```

To use the improved Euler method, we call as '**method**' the following function.

```
function [t,y] = impeuler(yp, t0,t1, y0)
%
%  [t,y] = impeuler(yp, t0,t1, y0)
%  Takes one improved Euler step for
%
%       y' = yprime( t,y ),
%
%  from t0  to  t1  with initial value  the
%  column vector  y0.

h = t1 - t0;
k1 = feval( yp, t0, y0        );
k2 = feval( yp, t1, y0 + h*k1 );
k  = (k1 + k2)/2;
t = t1;
y = y0 + h*k;
```

29. Here our differential equation is described by the MATLAB function

```
function  vp = vpbolt1(t,v)
vp = -0.04*v - 9.8;
```

Then the commands

```
n = 50;
[t1,v1] = ode('impeuler','vpbolt1',0,10,49,n);
n = 100;
```

```
[t2,v2] = ode('impeuler','vpbolt1',0,10,49,n);
t = (0:10)';
ve = 294*exp(-t/25)-245;
[t, v1(1:5:51), v2(1:10:101), ve]
```

generate the table

t	with $n = 50$	with $n = 100$	actual v
0	49.0000	49.0000	49.0000
1	37.4722	37.4721	37.4721
2	26.3964	26.3963	26.3962
3	15.7549	15.7547	15.7546
4	5.5307	5.5304	5.5303
5	-4.2926	-4.2930	-4.2932
6	-13.7308	-13.7313	-13.7314
7	-22.7989	-22.7994	-22.7996
8	-31.5115	-31.5120	-31.5122
9	-39.8824	-39.8830	-39.8832
10	-47.9251	-47.9257	-47.9259

We notice first that the final two columns agree to 3 decimal places (each difference being than 0.0005). Scanning the $n = 100$ column for sign changes, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.5$ sec. Then interpolation between $t = 4.5$ and $t = 4.6$ in the table

```
[t2(40:51),v2(40:51)]

    3.9000    6.5345
    4.0000    5.5304
    4.1000    4.5303
    4.2000    3.5341
    4.3000    2.5420
    4.4000    1.5538
    4.5000    0.5696
    4.6000   -0.4108
    4.7000   -1.3872
    4.8000   -2.3597
    4.9000   -3.3283
    5.0000   -4.2930
```

indicates that $t = 4.56$ at the bolt's apex. Finally, interpolation in

```
[t2(95:96),v2(95:96)]

    9.4000  -43.1387
    9.5000  -43.9445
```

gives the impact velocity $v(9.41) \approx -43.22$ m/s.

30. Now our differential equation is described by the MATLAB function

```
function  vp = vpbolt2(t,v)
vp = -0.0011*v.*abs(v) - 9.8;
```

Then the commands

```
n = 100;
[t1,v1] = ode('impeuler','vpbolt2',0,10,49,n);
n = 200;
[t2,v2] = ode('impeuler','vpbolt2',0,10,49,n);
t = (0:10)';
[t, v1(1:10:101), v2(1:20:201)]
```

generate the table

t	with $n = 100$	with $n = 200$
0	49.0000	49.0000
1	37.1547	37.1547
2	26.2428	26.2429
3.	15.9453	15.9455
4	6.0041	6.0044
5	-3.8020	-3.8016
6	-13.5105	-13.5102
7	-22.9356	-22.9355
8	-31.8984	-31.8985
9	-40.2557	-40.2559
10	-47.9066	-47.9070

We notice first that the final two columns agree to 2 decimal places (each difference being less than 0.005). Scanning the $n = 200$ column for sign changes, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.6$ sec. Then interpolation between $t = 4.60$ and $t = 4.65$ in the table

```
[t2(91:101),v2(91:101)]
```

4.5000	1.0964
4.5500	0.6063
4.6000	0.1163
4.6500	-0.3737
4.7000	-0.8636
4.7500	-1.3536
4.8000	-1.8434
4.8500	-2.3332
4.9000	-2.8228
4.9500	-3.3123
5.0000	-3.8016

indicates that $t = 4.61$ at the bolt's apex. Finally, interpolation in

```
[t2(189:190),v2(189:190)]

    9.4000   -43.4052
    9.4500   -43.7907
```

gives the impact velocity $v(9.41) \approx -43.48$ m/s.

SECTION 6.3

THE RUNGE-KUTTA METHOD

Each problem can be solved with a "template" of computations like those listed in Problem 1. We include a table showing the slope values $k_1,\ k_2,\ k_3,\ k_4$ and the xy-values at the ends of two successive steps of size $h = 0.25$.

1. To make the first step of size $h = 0.25$ we start with the function defined by

```
f[x_, y_] := -y
```

and the initial values

```
x = 0;      y = 2;     h = 0.25;
```

and then perform the calculations

```
k1 = f[x, y]
k2 = f[x + h/2, y + h*k1/2]
k3 = f[x + h/2, y + h*k2/2]
k4 = f[x + h, y + h*k3]
y  = y + h/6*(k1 + 2*k2 + 2*k3 + k4)
x  = x + h
```

in turn. Here we are using Mathematica notation that translates transparently to standard mathematical notation describing the corresponding manual computations. A repetition of this same block of calculations carries out a second step of size $h = 0.25$. The following table lists the intermediate and final results obtained in these two steps.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
–2	–1/75	–1.78125	–1.55469	0.25	1.55762	1.55760
–1.55762	–1.36292	–1.38725	–1.2108	0.5	1.21309	1.21306

2.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
1	1.25	1.3125	1.65625	0.25	0.82422	0.82436
1.64844	2.06055	2.16357	2.73022	0.5	1.35867	1.35914

3.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
2	2.25	2.28125	2.57031	0.25	1.56803	1.56805
2.56803	2.88904	2.92916	3.30032	0.5	2.29740	2.29744

4.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
–1	–0.75	–0.78128	–55469	0.25	0.80762	0.80760
–0.55762	–0.36292	–0.38725	–0.21080	0.5	0.71309	0.71306

5.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.125	–0.14063	–0.28516	0.25	0.96598	0.96597
–28402	–0.44452	–0.46458	–0.65016	0.5	0.85130	0.85128

6.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.5	–0.48438	–0.93945	0.25	1.87882	1.87883
–0.93941	–1.32105	–1.28527	–1.55751	0.5	1.55759	1.55760

7.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	–0.14063	–0.13980	–0.55595	0.25	2.95347	2.95349
–0.55378	–1.21679	–1.18183	–1.99351	0.5	2.6475	2.64749

8.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
1	0.88250	0.89556	0.79940	0.25	0.22315	0.22314
0.80000	0.72387	0.73079	0.66641	0.5	0.40547	0.40547

9.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0.5	0.53223	0.53437	0.57126	0.25	1.13352	1.13352
0.57122	0.61296	0.61611	0.66444	0.5	1.28743	1.28743

10.

k_1	k_2	k_3	k_4	x	Approx. y	Actual y
0	0.25	0.26587	0.56868	0.25	1.06668	1.06667
0.56891	0.97094	1.05860	1.77245	0.5	1.33337	1.33333

The results given below for Problems 11–16 were computed using the following MATLAB script.

```
% Section 6.3, Problems 11-16
x0 = 0;   y0 = 1;

% first run:
h = 0.2;
x = x0;   y = y0;   y1 = y0;
for  n = 1:5
   k1 = f(x,y);
   k2 = f(x+h/2,y+h*k1/2);
   k3 = f(x+h/2,y+h*k2/2);
   k4 = f(x+h,y+h*k3);
   y = y +(h/6)*(k1+2*k2+2*k3+k4);
   y1 = [y1,y];
   x = x + h;
   end

% second run:
h = 0.1;
x = x0;   y = y0;   y2 = y0;
for  n = 1:10
   k1 = f(x,y);
   k2 = f(x+h/2,y+h*k1/2);
   k3 = f(x+h/2,y+h*k2/2);
   k4 = f(x+h,y+h*k3);
   y = y +(h/6)*(k1+2*k2+2*k3+k4);
   y2 = [y2,y];
   x = x + h;
end

% exact values
x = x0 : 0.2 : x0+1;
ye = g(x);

% display table
y2 = y2(1:2:11);
err = 100*(ye-y2)./ye;
x = sprintf('%10.6f',x), sprintf('\n');
y1 = sprintf('%10.6f',y1), sprintf('\n');
y2 = sprintf('%10.6f',y2), sprintf('\n');
ye = sprintf('%10.6f',ye), sprintf('\n');
err = sprintf('%10.6f',err), sprintf('\n');
table = [x;y1;y2;ye;err]
```

For each problem the differential equation $y' = f(x, y)$ and the known exact solution $y = g(x)$ are stored in the files **f.m** and **g.m** — for instance, the files

```
function  yp = f(x,y)
yp = y-2;
```

and

```
function ye = g(x,y)
ye = 2-exp(x);
```

for Problem 11.

11.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.2)	1.000000	0.778600	0.508182	0.177894	–0.225521	–0.718251
y (h=0.1)	1.000000	0.778597	0.508176	0.177882	–0.225540	–0.718280
y actual	1.000000	0.778597	0.508175	0.177881	–0.225541	–0.718282
error	0.00000%	–0.00002%	–0.00009%	–0.00047%	–0.00061%	–0.00029%

12.

x	0.0	0.2	0.4	0.6	0.8	1.0
y (h=0.2)	2.000000	2.111110	2.249998	2.428566	2.666653	2.999963
y (h=0.1)	2.000000	2.111111	2.250000	2.428571	2.666666	2.999998
y actual	2.000000	2.111111	2.250000	2.428571	2.666667	3.000000
error	0.000000%	0.000002%	0.000006%	0.000014%	0.000032%	0.000080%

13.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.2)	3.000000	3.173896	3.441170	3.814932	4.300904	4.899004
y (h=0.1)	3.000000	3.173894	3.441163	3.814919	4.300885	4.898981
y actual	3.000000	3.173894	3.441163	3.814918	4.300884	4.898979
error	0.00000%	–0.00001%	–0.00001%	–0.00002%	–0.00003%	–0.00003%

14.

x	1.0	1.2	1.4	1.6	1.8	2.0
y (h=0.2)	1.000000	1.222957	1.507040	1.886667	2.425586	3.257946
y (h=0.1)	1.000000	1.222973	1.507092	1.886795	2.425903	3.258821
y actual	1.000000	1.222975	1.507096	1.886805	2.425928	3.258891
error	0.0000%	0.0001%	0.0003%	0.0005%	0.0010%	0.0021%

15.

x	2.0	2.2	2.4	2.6	2.9	3.0
y (h=0.2)	3.000000	3.026448	3.094447	3.191719	3.310207	3.444447
y (h=0.1)	3.000000	3.026446	3.094445	3.191716	3.310204	3.444445
y actual	3.000000	3.026446	3.094444	3.191716	3.310204	3.444444
error	0.000000%	–0.000004%	–0.000005%	–0.000005%	–0.000005%	–0.000004%

16.

x	2.0	2.2	2.4	2.6	2.9	3.0
y (h=0.2)	3.000000	4.243067	5.361409	6.478634	7.634049	8.845150
y (h=0.1)	3.000000	4.242879	5.361308	6.478559	7.633983	8.845089
y actual	3.000000	4.242870	5.361303	6.478555	7.633979	8.845085
error	0.000000%	–0.000221%	–0.000094%	–0.000061%	–0.000047%	–0.000039%

17. With $h = 0.2$: $y(1) \approx 0.350258$
With $h = 0.1$: $y(1) \approx 0.350234$
With $h = 0.05$: $y(1) \approx 0.350232$
With $h = 0.025$: $y(1) \approx 0.350232$

The table of numerical results is

x	y with $h = 0.2$	y with $h = 0.1$	y with $h = 0.05$	y with $h = 0.025$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.002667	0.002667	0.002667	0.002667
0.4	0.021360	0.021359	0.021359	0.021359
0.6	0.072451	0.072448	0.072448	0.072448
0.8	0.174090	0.174081	0.174080	0.174080
1.0	0.350258	0.350234	0.350232	0.350232

In Problems 18–24 we give only the final approximate values of y obtained using the Runge-Kutta method with step sizes $h = 0.2$, $h = 0.1$, $h = 0.05$, and $h = 0.025$.

18. With $h = 0.2$: $y(2) \approx 1.679513$
With $h = 0.1$: $y(2) \approx 1.679461$
With $h = 0.05$: $y(2) \approx 1.679459$
With $h = 0.025$: $y(2) \approx 1.679459$

19. With $h = 0.2$: $y(2) \approx 6.411464$
With $h = 0.1$: $y(2) \approx 6.411474$
With $h = 0.05$: $y(2) \approx 6.411474$
With $h = 0.025$: $y(2) \approx 6.411474$

20. With $h = 0.2$: $y(2) \approx -1.259990$
With $h = 0.1$: $y(2) \approx -1.259992$
With $h = 0.05$: $y(2) \approx -1.259993$
With $h = 0.025$: $y(2) \approx -1.259993$

21. With $h = 0.2$: $y(2) \approx 2.872467$
With $h = 0.1$: $y(2) \approx 2.872468$
With $h = 0.05$: $y(2) \approx 2.872468$
With $h = 0.025$: $y(2) \approx 2.872468$

22. With $h = 0.2$: $y(2) \approx 7.326761$
With $h = 0.1$: $y(2) \approx 7.328452$
With $h = 0.05$: $y(2) \approx 7.328971$
With $h = 0.025$: $y(2) \approx 7.329134$

23. With $h = 0.2$: $y(1) \approx 1.230735$
With $h = 0.1$: $y(1) \approx 1.230731$
With $h = 0.05$: $y(1) \approx 1.230731$
With $h = 0.025$: $y(1) \approx 1.230731$

24. With $h = 0.2$: $y(1) \approx 1.000000$
With $h = 0.1$: $y(1) \approx 1.000000$
With $h = 0.05$: $y(1) \approx 1.000000$
With $h = 0.025$: $y(1) \approx 1.000000$

25. Here $f(t,v) = 32 - 1.6v$ and $t_0 = 0,\ v_0 = 0$.

With $h = 0.1$, 10 iterations of

$$k_1 = f(t_n, v_n), \qquad k_2 = f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_1),$$
$$k_3 = f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_2), \qquad k_4 = f(t_n + h, v_n + hk_3),$$
$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad v_{n+1} = v_n + hk$$

yield $v(1) \approx 15.9620$, and 20 iterations with $h = 0.05$ yield $v(1) \approx 15.9621$. Thus we observe an approximate velocity of 15.962 ft/sec after 1 second — 80% of the limiting velocity of 20 ft/sec.

With $h = 0.1$, 20 iterations yield $v(2) \approx 19.1847$, and 40 iterations with $h = 0.05$ yield $v(2) \approx 19.1848$. Thus we observe an approximate velocity of 19.185 ft/sec after 2 seconds — 96% of the limiting velocity of 20 ft/sec.

26. Here $f(t,P) = 0.0225P - 0.003P^2$ and $t_0 = 0,\ P_0 = 25$.

With $h = 6$, 10 iterations of

$$k_1 = f(t_n, P_n), \qquad k_2 = f(t_n + \tfrac{1}{2}h, P_n + \tfrac{1}{2}hk_1),$$
$$k_3 = f(t_n + \tfrac{1}{2}h, v_n + \tfrac{1}{2}hk_2), \qquad k_4 = f(t_n + h, P_n + hk_3),$$
$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad P_{n+1} = P_n + hk$$

yield $P(60) \approx 49.3915$, as do 20 iterations with $h = 3$. Thus we observe an approximate population of 49.3915 deer after 5 years — 65% of the limiting population of 75 deer.

With $h = 6$, 20 iterations yield $P(120) \approx 66.1136$, as do 40 iterations with $h = 3$. Thus we observe an approximate population of 66.1136 deer after 10 years — 88% of the limiting population of 75 deer.

27. Here $f(x,y) = x^2 + y^2 - 1$ and $x_0 = 0,\ y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.00445$ rounded off accurate to 5 decimal places.

h	1	0.1	0.01	0.001
n	2	20	200	2000
$y(2)$	1.05722	1.00447	1.00445	1.00445

28. Here $f(x,y) = x + \frac{1}{2}y^2$ and $x_0 = -2,\ y_0 = 0$. The following table gives the approximate values for the successive step sizes h and corresponding numbers n of steps. It appears likely that $y(2) = 1.46331$ rounded off accurate to 5 decimal places.

h	1	0.1	0.01	0.001
n	4	40	00	40000
$y(2)$	1.48990	1.46332	1.46331	1.46331

In the solutions for Problems 29 and 30 we use the general MATLAB solver **ode** that was listed prior to the Problem 29 solution in Section 6.2. To use the Runge-Kutta method, we call as **'method'** the following function.

```
function [t,y] = rk(yp, t0,t1, y0)

%   [t, y] = rk(yp, t0, t1, y0)
%   Takes one Runge-Kutta step for
%
%        y' = yp( t,y ),
%
%   from t0  to  t1  with initial value  the
%   column vector  y0.

h = t1 - t0;
k1 = feval(yp, t0      , y0           );
k2 = feval(yp, t0 + h/2, y0 + (h/2)*k1 );
k3 = feval(yp, t0 + h/2, y0 + (h/2)*k2 );
k4 = feval(yp, t0 + h   ,y0 +     h *k3 );
k  = (1/6)*(k1 + 2*k2 + 2*k3 + k4);
t = t1;
y = y0 + h*k;
```

29. Here our differential equation is described by the MATLAB function

```
function  vp = vpbolt1(t,v)
vp = -0.04*v - 9.8;
```

Then the commands

```
n = 100;
[t1,v1] = ode('rk','vpbolt1',0,10,49,n);
n = 200;
[t2,v] = ode('rk','vpbolt1',0,10,49,n);
t = (0:10)';
ve = 294*exp(-t/25)-245;
[t, v1(1:n/20:1+n/2), v(1:n/10:n+1), ve]
```

generate the table

t	with $n = 100$	with $n = 200$	actual v
0	49.0000	49.0000	49.0000
1	37.4721	37.4721	37.4721
2	26.3962	26.3962	26.3962
3	15.7546	15.7546	15.7546
4	5.5303	5.5303	5.5303
5	-4.2932	-4.2932	-4.2932
6	-13.7314	-13.7314	-13.7314
7	-22.7996	-22.7996	-22.7996
8	-31.5122	-31.5122	-31.5122
9	-39.8832	-39.8832	-39.8832
10	-47.9259	-47.9259	-47.9259

We notice first that the final three columns agree to the 4 displayed decimal places. Scanning the last column for sign changes in v, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.5$ sec. Then interpolation between $t = 4.55$ and $t = 4.60$ in the table

```
[t2(91:95),v(91:95)]

   4.5000    0.5694
   4.5500    0.0788
   4.6000   -0.4109
   4.6500   -0.8996
   4.7000   -1.3873
```

indicates that $t = 4.56$ at the bolt's apex. Now the commands

```
y = zeros(n+1,1);
h = 10/n;

for j = 2:n+1
   y(j) = y(j-1) + v(j-1)*h +
                   0.5*(-.04*v(j-1) - 9.8)*h^2;
end
ye = 7350*(1 - exp(-t/25)) - 245*t;
[t, y(1:n/10:n+1), ye]
```

generate the table

t	Approx y	Actual y
0	0	0
1	43.1974	43.1976
2	75.0945	75.0949
3	96.1342	96.1348
4	106.7424	106.7432
5	107.3281	107.3290
6	98.2842	98.2852
7	79.9883	79.9895
8	52.8032	52.8046
9	17.0775	17.0790
10	-26.8540	-26.8523

We see at least 2-decimal place agreement between approximate and actual values of y. Finally, interpolation between $t = 9$ and $t = 10$ here suggests that $y = 0$ just after $t = 9.4$. Then interpolation between $t = 9.40$ and $t = 9.45$ in the table

```
[t2(187:191),y(187:191)]

   9.3000    4.7448
   9.3500    2.6182
   9.4000    0.4713
   9.4500   -1.6957
   9.5000   -3.8829
```

indicates that the bolt is aloft for about 9.41 seconds.

30. Now our differential equation is described by the MATLAB function

```
function  vp = vpbolt2(t,v)
vp = -0.0011*v.*abs(v) - 9.8;
```

Then the commands

```
n = 200;
[t1,v1] = ode('rk','vpbolt2',0,10,49,n);
n = 2*n;
[t2,v] = ode('rk','vpbolt2',0,10,49,n);
t = (0:10)';
ve = zeros(size(t));
ve(1:5)= 94.388*tan(0.478837 - 0.103827*t(1:5));
ve(6:11)= -94.388*tanh(0.103827*(t(6:11)-4.6119));

[t, v1(1:n/20:1+n/2), v(1:n/10:n+1), ve]
```

generate the table

t	with $n = 200$	with $n = 400$	actual v
0	49.0000	49.0000	49.0000
1	37.1548	37.1548	37.1547
2	26.2430	26.2430	26.2429
3	15.9456	15.9456	15.9455
4	6.0046	6.0046	6.0045
5	-3.8015	-3.8015	-3.8013
6	13.5101	-13.5101	-13.5100
7	-22.9354	-22.9354	-22.9353
8	-31.8985	-31.8985	-31.8984
9	-40.2559	-40.2559	-40.2559
10	-47.9071	-47.9071	-47.9071

We notice first that the final three columns almost agree to the 4 displayed decimal places. Scanning the last colmun for sign changes in v, we suspect that $v = 0$ (at the bolt's apex) occurs just after $t = 4.6$ sec. Then interpolation between $t = 4.600$ and $t = 4.625$ in the table

```
[t2(185:189),v(185:189)]

    4.6000     0.1165
    4.6250    -0.1285
    4.6500    -0.3735
    4.6750    -0.6185
    4.7000    -0.8635
```

indicates that $t = 4.61$ at the bolt's apex. Now the commands

```
y = zeros(n+1,1);
h = 10/n;
for j = 2:n+1
   y(j) = y(j-1) + v(j-1)*h + 0.5*(-.04*v(j-1) - 9.8)*h^2;
end
ye = zeros(size(t));
ye(1:5)= 108.465+909.091*log(cos(0.478837 - 0.103827*t(1:5)));
ye(6:11)= 108.465-909.091*log(cosh(0.103827
                          *(t(6:11)-4.6119)));
[t, y(1:n/10:n+1), ye]
```

generate the table

t	Approx y	Actual y
0	0	0.0001
1	42.9881	42.9841
2	74.6217	74.6197
3	95.6719	95.6742
4	106.6232	106.6292
5	107.7206	107.7272
6	99.0526	99.0560

```
7       80.8027      80.8018
8       53.3439      53.3398
9       17.2113      17.2072
10     -26.9369    -26.9363
```

We see almost 2-decimal place agreement between approximate and actual values of y. Finally, interpolation between $t = 9$ and $t = 10$ here suggests that $y = 0$ just after $t = 9.4$. Then interpolation between $t = 9.400$ and $t = 9.425$ in the table

```
[t2(377:381),y(377:381)]

    9.4000     0.4740
    9.4250    -0.6137
    9.4500    -1.7062
    9.4750    -2.8035
    9.5000    -3.9055
```

indicates that the bolt is aloft for about 9.41 seconds.

SECTION 6.4

NUMERICAL METHODS FOR SYSTEMS

In Problems 1–8 we first write the given system in the form $x' = f(t,x,y), \quad y' = g(t,x,y)$. Then we use the template

$$\begin{aligned} h &= 0.1; \quad t_1 = t_0 + h \\ x_1 &= x_0 + h\,f(t_0,x_0,y_0); \quad y_1 = y_0 + h\,g(t_0,x_0,y_0) \\ x_2 &= x_1 + h\,f(t_1,x_1,y_1); \quad y_2 = y_1 + h\,g(t_1,x_1,y_1) \end{aligned}$$

(with the given values of t_0, x_0, and y_0) to calculate the Euler approximations $x_1 \approx x(0.1)$, $y_1 \approx y(0.1)$ and $x_2 \approx x(0.2)$, $y_2 \approx y(0.2)$ in part (a). We give these approximations and the actual values $x_{\text{act}} = x(0.2)$, $y_{\text{act}} = y(0.2)$ in tabular form. We use the template

$$\begin{aligned} h &= 0.2; \quad t_1 = t_0 + h \\ u_1 &= x_0 + h\,f(t_0,x_0,y_0); \quad v_1 = y_0 + h\,g(t_0,x_0,y_0) \\ x_1 &= x_0 + \tfrac{1}{2}h\left[f(t_0,x_0,y_0) + f(t_1,u_1,v_1)\right] \\ y_1 &= y_0 + \tfrac{1}{2}h\left[g(t_0,x_0,y_0) + g(t_1,u_1,v_1)\right] \end{aligned}$$

to calculate the improved Euler approximations $u_1 \approx x(0.2)$, $u_1 \approx y(0.2)$ and $x_1 \approx x(0.2)$, $y_1 \approx y(0.2)$ in part (b). We give these approximations and the actual values $x_{\text{act}} = x(0.2)$, $y_{\text{act}} = y(0.2)$ in tabular form. We use the template

$$\begin{aligned}
&h = 0.2; \\
&F_1 = f(t_0, x_0, y_0); \quad G_1 = g(t_0, x_0, y_0) \\
&F_2 = f(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}h F_1, y_0 + \tfrac{1}{2}h G_1); \quad G_2 = g(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}h F_1, y_0 + \tfrac{1}{2}h G_1) \\
&F_3 = f(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}h F_2, y_0 + \tfrac{1}{2}h G_2); \quad G_3 = g(t_0 + \tfrac{1}{2}h, x_0 + \tfrac{1}{2}h F_2, y_0 + \tfrac{1}{2}h G_2) \\
&F_4 = f(t_0 + h, x_0 + h F_3, y_0 + h G_3); \quad G_4 = g(t_0 + h, x_0 + h F_3, y_0 + h G_3) \\
&x_1 = x_0 + \frac{h}{6}(F_1 + 2F_2 + 2F_3 + F_4); \quad y_1 = y_0 + \frac{h}{6}(G_1 + 2G_2 + 2G_3 + G_4)
\end{aligned}$$

to calculate the intermediate slopes and Runge-Kutta approximations $x_1 \approx x(0.2)$, $y_1 \approx y(0.2)$ for part (c). Again, we give the results in tabular form.

1. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.4	2.2	0.88	2.5	1.0034	2.6408

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
0.8	2.4	0.96	2.6	1.0034	2.6408

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
4	2	4.8	3	5.08	3.26	6.32	4.684
x_1	y_1	x_{act}	y_{act}				
1.0027	2.6401	1.0034	2.6408				

2. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	–0.9	0.81	–0.81	0.8187	–0.8187

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
0.8	–0.8	0.82	–0.82	0.8187	–0.8187

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–1	1	–0.9	0.9	–0.91	0.91	–0.818	0.818
x_1	y_1	x_{act}	y_{act}				
0.8187	–0.8187	0.8187	–0.8187				

3. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
1.7	1.5	2.81	2.31	3.6775	2.9628

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
2.4	2	3.22	2.62	3.6775	2.9628

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
7	5	11.1	8.1	13.57	9.95	23.102	17.122
x_1	y_1	x_{act}	y_{act}				
3.6481	2.9407	3.6775	2.9628				

4. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
1.9	–0.6	3.31	–1.62	4.2427	–2.4205

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
2.8	–1.2	3.82	–2.04	4.2427	–2.4205

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
9	–6	14.1	–10.2	16.59	–12.42	26.442	–20.94
x_1	y_1	x_{act}	y_{act}				
4.2274	–2.4060	4.2427	–2.4205				

5. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	3.2	–0.52	2.92	–0.5793	2.4488

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
–0.2	3.4	–0.84	2.44	–0.5793	2.4488

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–11	2	–14.2	–2.8	–12.44	–3.12	–12.856	–6.704
x_1	y_1	x_{act}	y_{act}				
–0.5712	2.4485	–0.5793	2.4488				

6. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
–0.8	4.4	–1.76	4.68	–1.9025	4.4999

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
–1.6	4.8	–1.92	4.56	–1.9025	4.4999

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
–8	4	–9.6	2.8	–9.52	2.36	–10.848	0.664
x_1	y_1	x_{act}	y_{act}				
–1.9029	4.4995	–1.9025	4.4999				

7. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
2.5	1.3	3.12	1.68	3.2820	1.7902

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
3	1.6	3.24	1.76	3.2820	1.7902

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
5	3	6.2	3.8	6.48	4	8.088	5.096
x_1	y_1	x_{act}	y_{act}				
3.2816	1.7899	3.2820	1.7902				

8. **(a)**

x_1	y_1	x_2	y_2	x_{act}	y_{act}
0.9	–0.9	2.16	–0.63	2.5270	–0.3889

(b)

u_1	v_1	x_1	y_1	x_{act}	y_{act}
1.8	–0.8	2.52	–0.46	2.5270	–0.3889

(c)

F_1	G_1	F_2	G_2	F_3	G_3	F_4	G_4
9	1	12.6	2.7	12.87	3.25	16.02	5.498
x_1	y_1	x_{act}	y_{act}				
2.5320	–0.3867	2.5270	–0.3889				

In Problems 9–11 we use the same Runge-Kutta template as in part (c) of Problems 1–8 above, and give both the Runge-Kutta approximate values with step sizes $h = 0.1$ and $h = 0.05$, and also the actual values.

9. With $h = 0.1$: $x(1) \approx 3.99261$, $y(1) \approx 6.21770$
With $h = 0.05$: $x(1) \approx 3.99234$, $y(1) \approx 6.21768$
Actual values: $x(1) \approx 3.99232$, $y(1) \approx 6.21768$

10. With $h = 0.1$: $x(1) \approx 1.31498$, $y(1) \approx 1.02537$
With $h = 0.05$: $x(1) \approx 1.31501$, $y(1) \approx 1.02538$
Actual values: $x(1) \approx 1.31501$, $y(1) \approx 1.02538$

11. With $h = 0.1$: $x(1) \approx -0.05832$, $y(1) \approx 0.56664$
With $h = 0.05$: $x(1) \approx -0.05832$, $y(1) \approx 0.56665$
Actual values: $x(1) \approx -0.05832$, $y(1) \approx 0.56665$

12. We first convert the given initial value problem to the two-dimensional problem

$$x' = y, \qquad x(0) = 0,$$
$$y' = -x + \sin t, \qquad y(0) = 0.$$

Then with both step sizes $h = 0.1$ and $h = 0.05$ we get the actual value $x(1) \approx 0.15058$ accurate to 5 decimal places.

13. With $y = x'$ we want to solve numerically the initial value problem

$$x' = y, \qquad x(0) = 0$$
$$y' = -32 - 0.04y, \qquad y(0) = 288.$$

When we run Program RK2DIM with step size $h = 0.1$ we find that the change of sign in the velocity v occurs as follows:

t	x	v
7.6	1050.2	+2.8
7.7	1050.3	-0.4

Thus the bolt attains a maximum height of about 1050 feet in about 7.7 seconds.

14. Now we want to solve numerically the initial value problem

$$x' = y, \qquad x(0) = 0,$$
$$y' = -32 - 0.0002y^2, \qquad y(0) = 288.$$

Running Program RK2DIM with step size $h = 0.1$, we find that the bolt attains a maximum height of about 1044 ft in about 7.8 sec. Note that these values are comparable to those found in Problem 13.

15. With $y = x'$, and with x in miles and t in seconds, we want to solve numerically the initial value problem

$$x' = y$$

$$y' = -95485.5/(x^2 + 7920x + 15681600)$$

$$x(0) = 0, \qquad y(0) = 1.$$

We find (running RK2DIM with $h = 1$) that the projectile reaches a maximum height of about 83.83 miles in about 168 sec = 2 min 48 sec.

16. We first defined the MATLAB function

```
function  xp  =  fnball(t,x)
%  Defines the baseball system
%      x1'  =  x'  =  x3,  x3'  =  -cvx'
%      x2'  =  y'  =  x4,  x4'  =  -cvy'- g
%  with air resistance coefficient c.

g   =  32;
c   =  0.0025;
xp  =  x;
v  =  sqrt(x(3).^2) + x(4).^2);
xp(1)  =   x(3);
xp(2)  =   x(4);
xp(3)  =  -c*v*x(3);
xp(4)  =  -c*v*x(4)  - g;
```

Then, using the n-dimensional program **rkn** with step size 0.1 and initial data corresponding to the indicated initial inclination angles, we got the following results:

Angle	Time	Range
40	5.0	352.9
45	5.4	347.2
50	5.8	334.2

We have listed the time to the nearest tenth of a second, but have interpolated to find the range in feet.

17. The data in Problem 16 indicate that the range increases when the initial angle is decreased below 45°. The further data

Angle	Range
41.0	352.1
40.5	352.6
40.0	352.9
39.5	352.8
39.0	352.7
35.0	350.8

indicate that a maximum range of about 353 ft is attained with $\alpha \approx 40°$.

18. We "shoot" for the proper inclination angle by running program **rkn** (with $h = 0.1$) as follows:

Angle	Range
60	287.1
58	298.5
57.5	301.1

Thus we get a range of 300 ft with an initial angle just under 57.5°.

19. First we run program **rkn** (with $h = 0.1$) with $v_0 = 250$ ft/sec and obtain the following results:

t	x	y
5.0	457.43	103.90
6.0	503.73	36.36

Interpolation gives $x = 494.4$ when $y = 50$. Then a run with $v_0 = 255$ ft/sec gives the following results:

t	x	y
5.5	486.75	77.46
6.0	508.86	41.62

Finally a run with $v_0 = 253$ ft/sec gives these results:

t	x	y
5.5	484.77	75.44
6.0	506.82	39.53

Now $x \approx 500$ ft when $y = 50$ ft. Thus Babe Ruth's home run ball had an initial velocity of 253 ft/sec.

20. A run of program `rkn` with $h = 0.1$ and with the given data yields the following results:

t	x	y	v	α
5.5	989	539	162	+0.95
5.6	1005	539	161	−0.18
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
11.5	1868	16	214	−52
11.6	1881	−1	216	−53

The first two lines of data above indicate that the crossbow bolt attains a maximum height of about 1005 ft in about 5.6 sec. About 6 sec later (total time 11.6 sec) it hits the ground, having traveled about 1880 ft horizontally.

21. A run with $h = 0.1$ indicates that the projectile has a range of about 21,400 ft $\approx$ 4.05 mi and a flight time of about 46 sec. It attains a maximum height of about 8970 ft in about 17.5 sec. At time $t \approx 23$ sec it has its minimum velocity of about 368 ft/sec. It hits the ground ($t \approx 46$ sec) at an angle of about 77° with a velocity of about 518 ft/sec.

CHAPTER 7

NONLINEAR SYSTEMS AND PHENOMENA

SECTION 7.1

EQUILIBRIUM SOLUTIONS AND STABILITY

In Problems 1–12 we identify the stable and unstable critical points as well as the funnels and spouts along the equilibrium solutions. In each problem the indicated solution satisfying $x(0) = x_0$ is derived fairly routinely by separation of variables. In some cases, various signs in the solution depend on the initial value, and we give a typical solution. For each problem we show typical solution curves corresponding to different values of x_0.

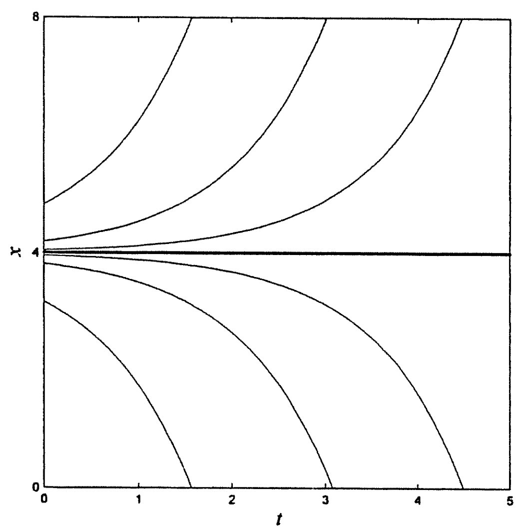
1. Unstable critical point: $x = 4$
Spout: Along the equilibrium solution $x(t) = 4$

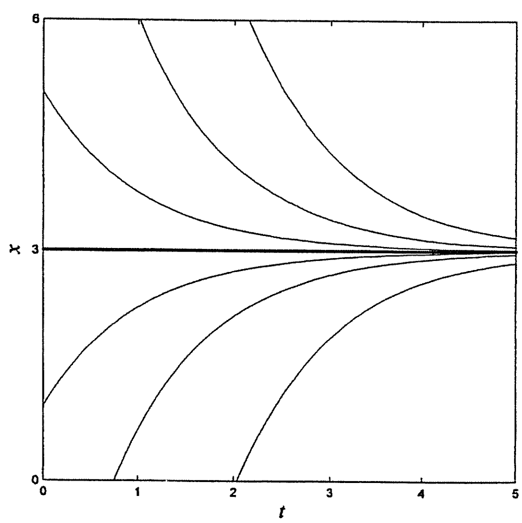
Solution: If $x_0 > 4$ then

$$\int \frac{dx}{x-4} = \int dt; \quad \ln(x-4) = t + C; \quad C = \ln(x_0 - 4)$$

$$x - 4 = (x_0 - 4)e^t; \quad x(t) = 4 + (x_0 - 4)e^t.$$

Typical solution curves are shown in the figure on the left below.





2. Stable critical point: $x = 3$
Funnel: Along the equilibrium solution $x(t) = 3$

Solution: If $x_0 > 3$ then

$$\int \frac{dx}{x-3} = \int (-1)\,dt; \quad \ln(x-3) = -t + C; \quad C = \ln(x_0 - 3)$$

$$x - 3 = (x_0 - 3)e^{-t}; \quad x(t) = 3 + (x_0 - 3)e^{-t}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

3. Stable critical point: $x = 0$
Unstable critical point: $x = 4$
Funnel: Along the equilibrium solution $x(t) = 0$
Spout: Along the equilibrium solution $x(t) = 4$

Solution: If $x_0 > 4$ then

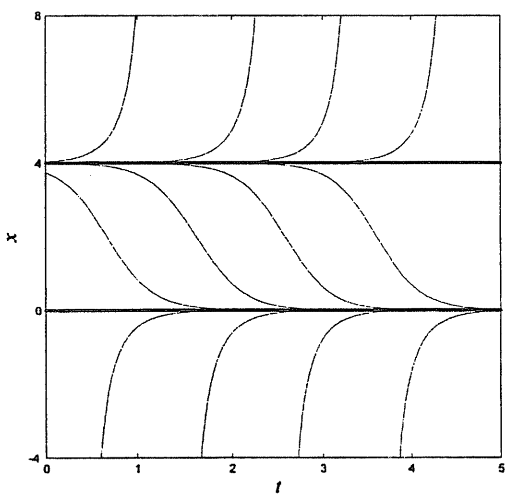
$$\int 4\,dt = \int \frac{4\,dx}{x(x-4)} = \int \left(\frac{1}{x-4} - \frac{1}{x}\right) dx$$

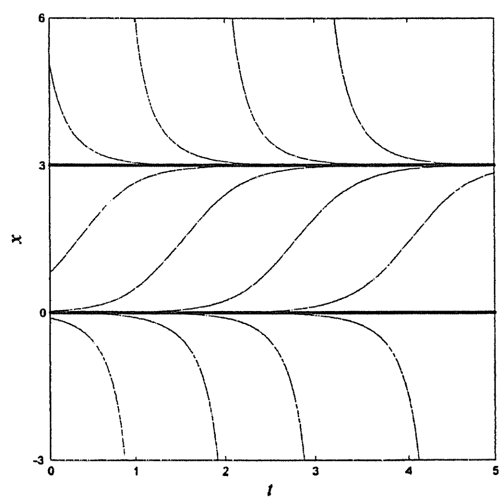
$$4t + C = \ln\frac{x-4}{x}; \quad C = \ln\frac{x_0 - 4}{x_0}$$

$$4t = \ln\frac{x_0(x-4)}{x(x_0-4)}; \quad e^{4t} = \frac{x_0(x-4)}{x(x_0-4)}$$

$$x(t) = \frac{4x_0}{x_0 + (4 - x_0)e^{4t}}.$$

Typical solution curves are shown in the figure on the left below.





4. Stable critical point: $x = 3$
Unstable critical point: $x = 0$
Funnel: Along the equilibrium solution $x(t) = 3$
Spout: Along the equilibrium solution $x(t) = 0$

Solution: If $x_0 > 3$ then

$$\int(-3)\,dt = \int \frac{3\,dx}{x(x-3)} = \int\left(\frac{1}{x-3} - \frac{1}{x}\right)dx$$

$$-3t + C = \ln\frac{x-3}{x}; \quad C = \ln\frac{x_0 - 3}{x_0}$$

$$-3t = \ln\frac{x_0(x-3)}{x(x_0-3)}; \quad e^{-3t} = \frac{x_0(x-3)}{x(x_0-3)}$$

$$x(t) = \frac{3x_0}{x_0 + (3 - x_0)e^{-3t}}.$$

Typical solution curves are shown in the figure on the right at the bottom of the preceding page.

5. Stable critical point: $x = -2$
Unstable critical point: $x = 2$
Funnel: Along the equilibrium solution $x(t) = -2$
Spout: Along the equilibrium solution $x(t) = 2$

Solution: If $x_0 > 2$ then

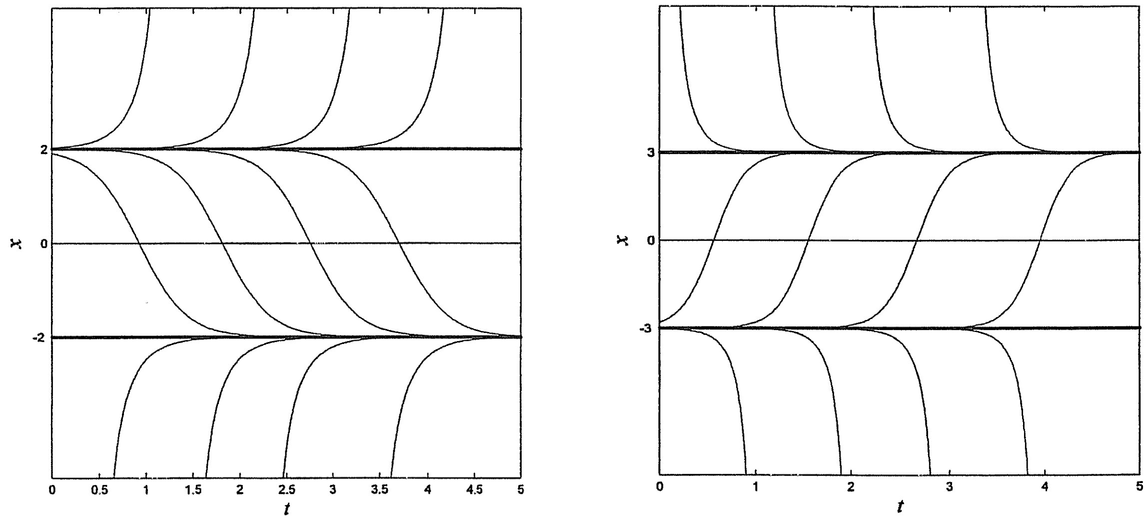
$$\int 4\,dt = \int \frac{4\,dx}{x^2 - 4} = \int\left(\frac{1}{x-2} - \frac{1}{x+2}\right)dx$$

$$4t + C = \ln\frac{x-2}{x+2}; \quad C = \ln\frac{x_0 - 2}{x_0 + 2}$$

$$4t = \ln\frac{(x-2)(x_0+2)}{(x+2)(x_0-2)}; \quad e^{4t} = \frac{(x-2)(x_0+2)}{(x+2)(x_0-2)}$$

$$x(t) = \frac{2\left[(x_0+2) + (x_0-2)e^{4t}\right]}{(x_0+2) - (x_0-2)e^{4t}}.$$

Typical solution curves are shown in the figure on the left at the top of the next page.



6. Stable critical point: $x = 3$
Unstable critical point: $x = -3$
Funnel: Along the equilibrium solution $x(t) = 3$
Spout: Along the equilibrium solution $x(t) = -3$

Solution: If $x_0 > 3$ then

$$\int 6\,dt = \int \frac{6\,dx}{9-x^2} = \int \left(\frac{1}{3+x} + \frac{1}{3-x} \right) dx$$

$$6t + C = \ln\frac{x+3}{x-3}; \quad C = \ln\frac{x_0+3}{x_0-3}$$

$$6t = \ln\frac{(x+3)(x_0-3)}{(x_0+3)(x-3)}; \quad e^{6t} = \frac{(x+3)(x_0-3)}{(x_0+3)(x-3)}$$

$$x(t) = \frac{3\left[(x_0-3)+(x_0+3)e^{6t}\right]}{(3-x_0)+(x_0+3)e^{6t}}.$$

Typical solution curves are shown in the figure on the right above.

7. Critical point: $x = 2$

This single critical point is *semi-stable*, meaning that solutions with $x_0 > 2$ go to infinity as t increases, while solutions with $x_0 < 2$ approach 2.

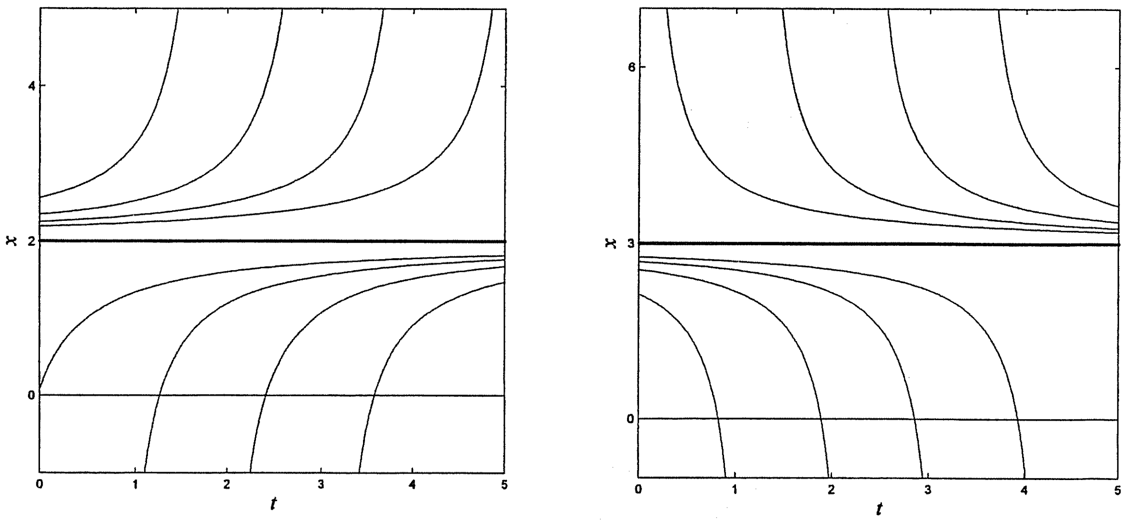
Solution: If $x_0 > 2$ then

$$\int \frac{-dx}{(x-2)^2} = \int (-1)\,dt; \quad \frac{1}{x-2} = -t+C; \quad C = \frac{1}{x_0-2}$$

$$\frac{1}{x-2} = -t+\frac{1}{x_0-2} = \frac{1-t(x_0-2)}{x_0-2}$$

$$x(t) = 2+\frac{x_0-2}{1-t(x_0-2)} = \frac{x_0(2t-1)-4t}{t\,x_0-2t-1}.$$

Typical solution curves are shown in the figure on the left below.



8. Critical point: $x = 3$

This single critical point is *semi-stable*, meaning that solutions with $x_0 < 3$ go to minus infinity as t increases, while solutions with $x_0 > 3$ approach 3.

Solution: If $x_0 > 3$ then

$$\int \frac{-dx}{(x-3)^2} = \int dt; \quad \frac{1}{x-3} = t+C; \quad C = \frac{1}{x_0-3}$$

$$\frac{1}{x-3} = t+\frac{1}{x_0-3} = \frac{1+t(x_0-3)}{x_0-3}$$

$$x(t) = 3+\frac{x_0-3}{1+t(x_0-3)} = \frac{x_0(3t+1)-9t}{t\,x_0-3t+1}.$$

Typical solution curves are shown in the figure on the right above.

9. Stable critical point: $x = 1$
Unstable critical point: $x = 4$
Funnel: Along the equilibrium solution $x(t) = 1$
Spout: Along the equilibrium solution $x(t) = 4$

Solution: If $x_0 > 4$ then

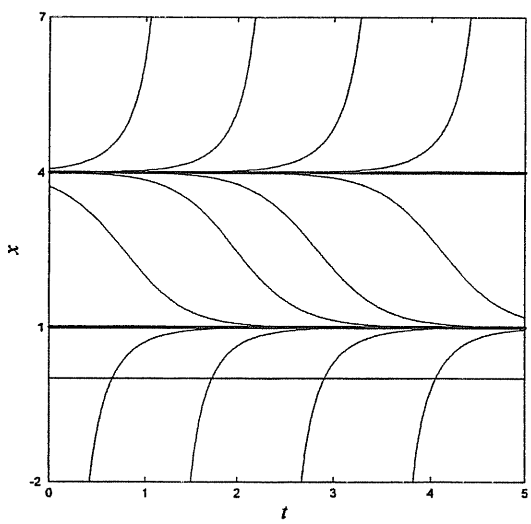
$$\int 3\,dt = \int \frac{3\,dx}{(x-4)(x-1)} = \int\left(\frac{1}{x-4} - \frac{1}{x-1}\right)dx$$

$$3t + C = \ln\frac{x-4}{x-1}; \qquad C = \ln\frac{x_0-4}{x_0-1}$$

$$3t = \ln\frac{(x-4)(x_0-1)}{(x-1)(x_0-4)}; \quad e^{3t} = \frac{(x-4)(x_0-1)}{(x-1)(x_0-4)}$$

$$x(t) = \frac{4(1-x_0)+(x_0-4)e^{3t}}{(1-x_0)+(x_0-4)e^{3t}}.$$

Typical solution curves are shown in the figure below.



10. Stable critical point: $x = 5$
Unstable critical point: $x = 2$
Funnel: Along the equilibrium solution $x(t) = 5$
Spout: Along the equilibrium solution $x(t) = 2$

Solution: If $x_0 > 5$ then

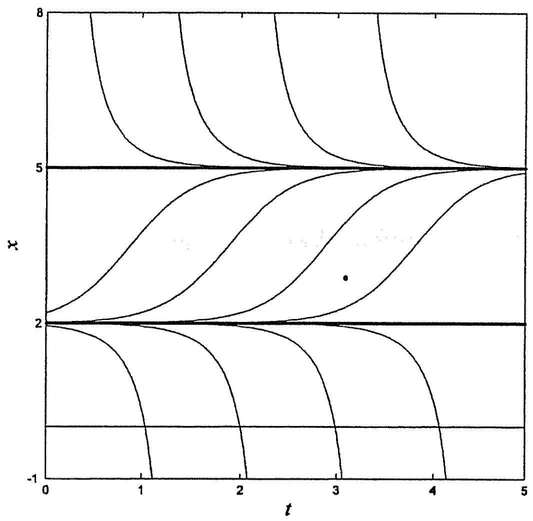
$$\int 3\,dt = \int \frac{(-3)\,dx}{(x-5)(x-2)} = \int\left(\frac{1}{x-2} - \frac{1}{x-5}\right)dx$$

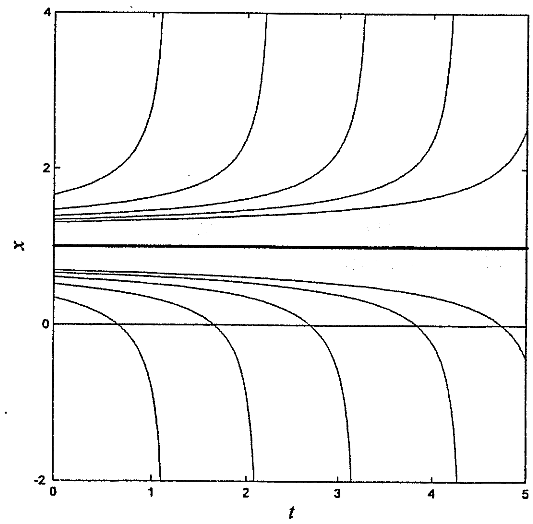
$$3t + C = \ln\frac{x-2}{x-5}; \qquad C = \ln\frac{x_0-2}{x_0-5}$$

$$3t = \ln\frac{(x-2)(x_0-5)}{(x-5)(x_0-2)}; \qquad e^{3t} = \frac{(x-2)(x_0-5)}{(x-5)(x_0-2)}$$

$$x(t) = \frac{2(5-x_0)+5(x_0-2)e^{3t}}{(5-x_0)+(x_0-2)e^{3t}}.$$

Typical solution curves are shown in the figure on the left below.





11. Unstable critical point: $x = 1$
Spout: Along the equilibrium solution $x(t) = 1$

Solution: $\displaystyle\int \frac{-2\,dx}{(x-1)^3} = \int(-2)\,dt; \qquad \frac{1}{(x-1)^2} = -2t + \frac{1}{(x_0-1)^2}.$

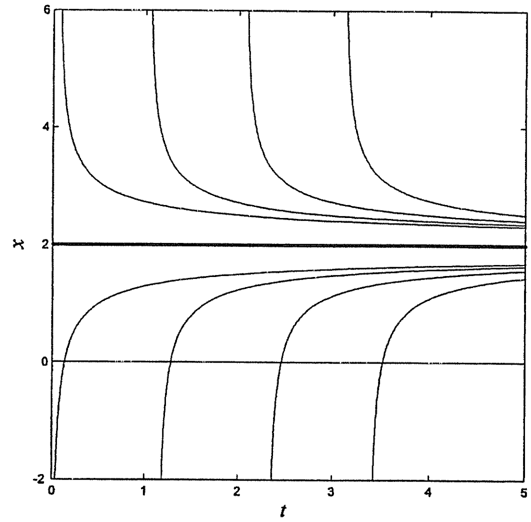
Typical solution curves are shown in the figure on the right above.

12. Stable critical point: $x = 2$

Funnel: Along the equilibrium solution $x(t) = 2$

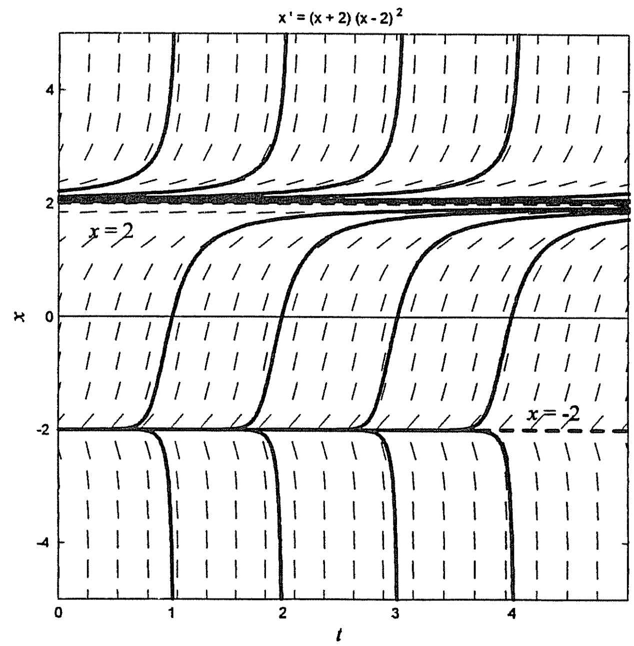
Solution: $\displaystyle\int \frac{2\,dx}{(2-x)^3} = \int 2\,dt; \qquad \frac{1}{(2-x)^2} = 2t + \frac{1}{(2-x_0)^2}.$

Typical solution curves are shown in the figure at the top of the next page.

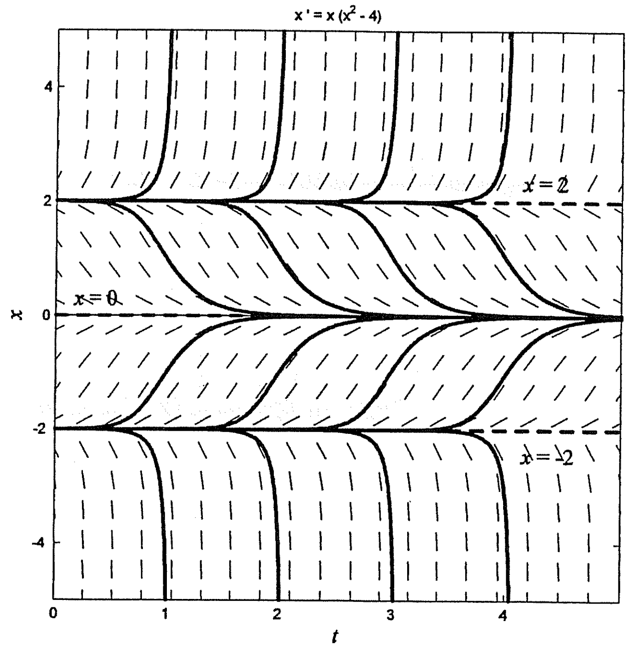


In each of Problems 13 through 18 we present the figure showing slope field and typical solution curves, and then record the visually apparent classification of critical points for the given differential equation.

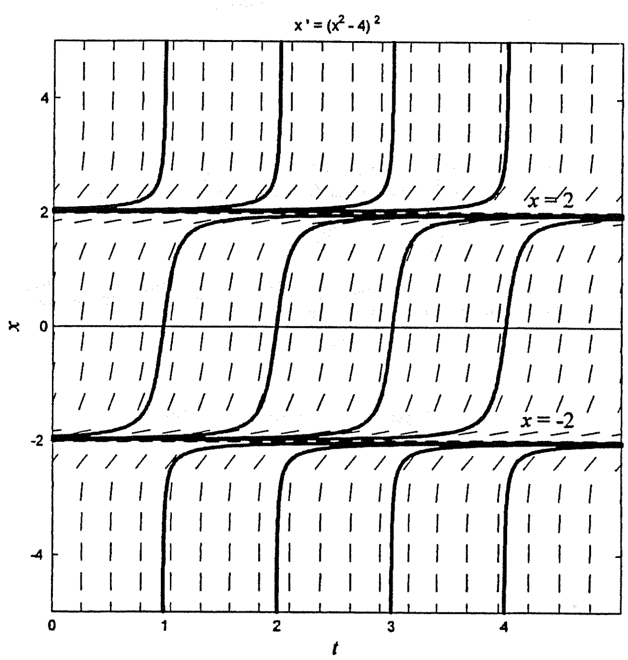
13. The critical points $x = 2$ and $x = -2$ are both unstable. A slope field and typical solution curves of the differential equation are shown below.



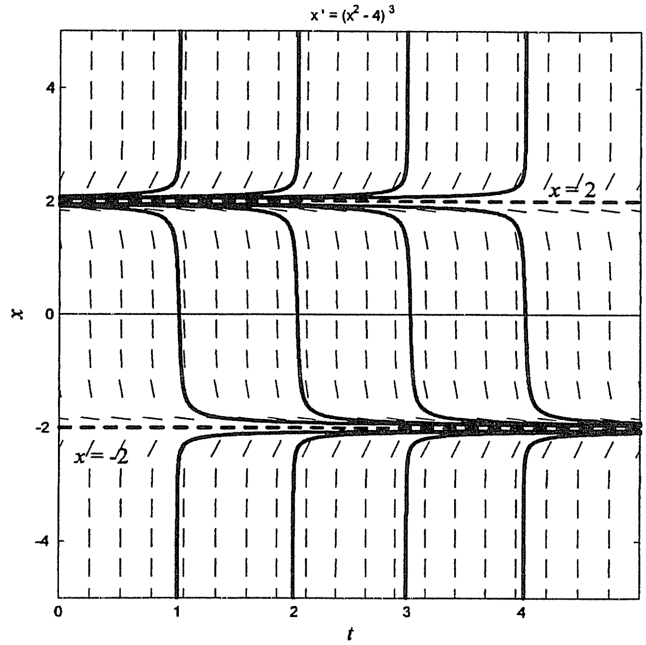
14. The critical points $x = \pm 2$ are both unstable, while the critical point $x = 0$ is stable. A slope field and typical solution curves of the differential equation are shown below.



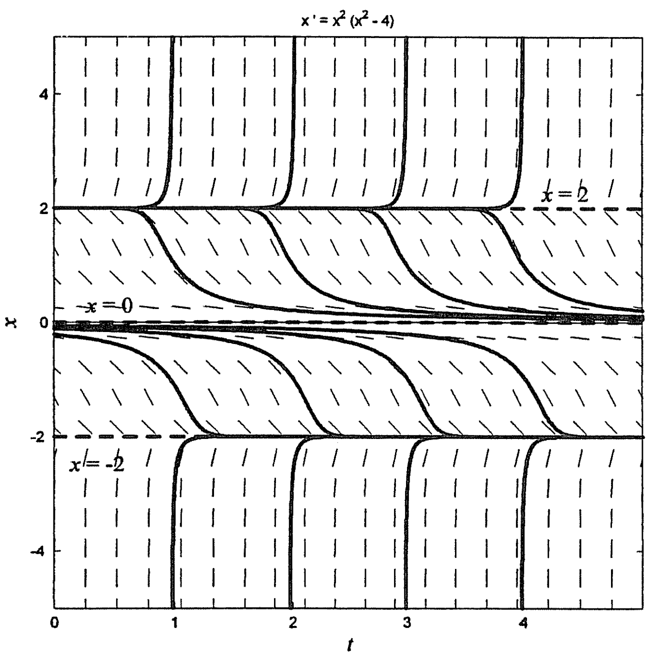
15. The critical points $x = 2$ and $x = -2$ are both unstable. A slope field and typical solution curves of the differential equation are shown below.



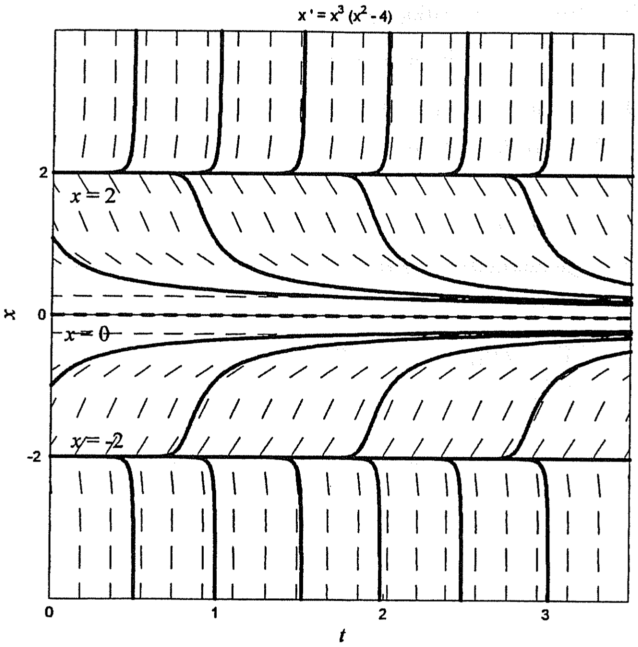
16. The critical point $x = 2$ is unstable, while the critical point $x = -2$ is stable. A slope field and typical solution curves of the differential equation are shown below.



17. The critical points $x = 2$ and $x = 0$ are unstable, while the critical point $x = -2$ is stable. A slope field and typical solution curves of the differential equation are shown below.



18. The critical points $x = 2$ and $x = -2$ are unstable, while the critical point $x = 0$ is stable. A slope field and typical solution curves of the differential equation are shown below.



19. The critical points of the given differential equation are the roots of the quadratic equation

$$\tfrac{1}{10}x(10 - x) - h = 0, \qquad \text{that is,} \qquad x^2 - 10x + 10h = 0.$$

Thus a critical point c is given in terms of h by

$$c = \frac{10 \pm \sqrt{100 - 40h}}{2} = 5 \pm \sqrt{25 - 10h}.$$

It follows that there is no critical point if $h > 2\tfrac{1}{2}$, only the single critical point $c = 0$ if $h = 2\tfrac{1}{2}$, and two distinct critical points if $h < 2\tfrac{1}{2}$ (so $10 - 25h > 0$). Hence the bifurcation diagram in the hc-plane is the parabola with the $(c - 5)^2 = 25 - 10h$ that is obtained upon squaring to eliminate the square root above.

20. The critical points of the given differential equation are the roots of the quadratic equation

$$\tfrac{1}{100}x(x - 5) + s = 0, \qquad \text{that is,} \qquad x^2 - 5x + 100s = 0.$$

Thus a critical point c is given in terms of s by

$$c = \frac{5 \pm \sqrt{25-400s}}{2} = \frac{5}{2} \pm \frac{5}{2}\sqrt{1-16s}.$$

It follows that there is no critical point if $s > \frac{1}{16}$, only the single critical point $c = 0$ if $s = \frac{1}{16}$, and two distinct critical points if $s < \frac{1}{16}$ (so $1-16s > 0$). Hence the bifurcation diagram in the sc-plane is the parabola $(2c-5)^2 = 25(1-16s)$ that is obtained upon elimination of the radical above.

21. **(a)** If $k = -a^2$ where $a \geq 0$, then $kx - x^3 = -a^2x - x^3 = -x(a^2+x^2)$ is 0 only if $x = 0$, so the only critical point is $c = 0$. If $a > 0$ then we can solve the differential equation by writing

$$\int \frac{a^2dx}{x(a^2+x^2)} = \int \left(\frac{1}{x} - \frac{x}{a^2+x^2}\right) dx = -\int a^2\, dt,$$

$$\ln x - \frac{1}{2}\ln(a^2+x^2) = -a^2t + \frac{1}{2}\ln C,$$

$$\frac{x^2}{a^2+x^2} = Ce^{-2a^2t} \quad \Rightarrow \quad x^2 = \frac{a^2Ce^{-2a^2t}}{1-Ce^{-2a^2t}}.$$

It follows that $x \to 0$ as $t \to 0$, so the critical point $c = 0$ is *stable*.

(b) If $k = +a^2$ where $a > 0$ then $kx - x^3 = +a^2x - x^3 = -x(x+a)(x-a)$ is 0 if either $x = 0$ or $x = \pm a = \pm\sqrt{k}$. Thus we have the three critical points $c = 0, \pm\sqrt{k}$, and this observation together with part (a) yields the pitchfork bifurcation diagram shown in Fig. 7.1.13 of the textbook. If $x \neq 0$ then we can solve the differential equation by writing

$$\int \frac{2a^2dx}{x(x-a)(x+a)} = \int \left(-\frac{2}{x} + \frac{1}{x-a} + \frac{1}{x+a}\right) dx = -\int 2a^2\, dt,$$

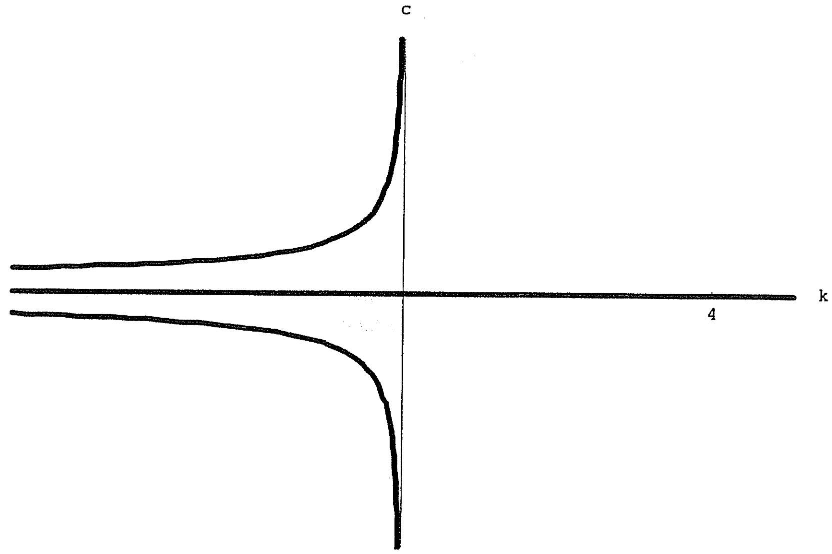
$$-2\ln x + \ln(x-a) + \ln(x-a) = -2a^2t + \ln C,$$

$$\frac{x^2-a^2}{x^2} = Ce^{-2a^2t} \quad \Rightarrow \quad x^2 = \frac{a^2}{1-Ce^{-2a^2t}} \quad \Rightarrow \quad x = \frac{\pm\sqrt{k}}{\sqrt{1-Ce^{-2a^2t}}}.$$

It follows that if $x(0) \neq 0$ then $x \to \sqrt{k}$ if $x > 0$, $x \to -\sqrt{k}$ if $x < 0$. This implies that the critical point $c = 0$ is *unstable*, while the critical points $c = \pm\sqrt{k}$ are *stable*.

22. If $k = 0$ then the only critical point $c = 0$ of the equation $x' = x$ is unstable, because the solutions $x(t) = x_0e^t$ diverge to infinity if $x_0 \neq 0$. If $k = +a^2 > 0$, then $x + a^2x^3 = x(1+a^2x^2) = 0$ only if $x = 0$, so again $c = 0$ is the only critical point. If $k = -a^2 < 0$, then $x - a^2x^3 = x(1-a^2x^2) = x(1-ax)(1+ax) = 0$ if either $x = 0$ or

$x = \pm 1/a = \pm\sqrt{-1/k}$. Hence the bifurcation diagram of the differential equation $x' = x + kx^3$ looks as pictured below:



23. **(a)** If $h < kM$ then the differential equation is $x' = kx\left((M - h/k) - x\right)$, which is a logistic equation with the *reduced* limiting population $M - h/k$.

(b) If $h > kM$ then the differential equation can be rewritten in the form $x' = -ax - bx^2$ with a and b both positive. The solution of this equation is

$$x(t) = \frac{a\,x_0}{(a + b\,x_0)e^{at} - bx_0}$$

so it is obvious that $x(t) \to 0$ as $t \to \infty$.

24. If $x_0 > N$ then

$$\int -k(N-H)\,dt = \int \frac{(N-H)\,dx}{(x-N)(x-H)} = \int \left(\frac{1}{x-N} - \frac{1}{x-H}\right)dx$$

$$-k(N-H)t + C = \ln\frac{x-N}{x-H}; \quad C = \ln\frac{x_0 - N}{x_0 - H}$$

$$-k(N-H)t = \ln\frac{(x-N)(x_0-H)}{(x-H)(x_0-M)}; \quad e^{-k(N-H)t} = \frac{(x-N)(x_0-H)}{(x-H)(x_0-M)}$$

$$x(t) = \frac{N(x_0-H) - H(x_0-N)e^{-k(N-H)t}}{(x_0-H) - (x_0-N)e^{-k(N-H)t}}$$

25. **(i)** In the first alternative form that is given, all of the coefficients within parentheses are positive if $H < x_0 < N$. Hence it is obvious that $x(t) \to N$ as $t \to \infty$.

(ii) In the second alternative form that is given, all of the coefficients within parentheses are positive if $x_0 < H$. Hence the denominator is initially equal to $N - H > 0$, but decreases as t increases, and reaches the value 0 when

$$t = \frac{1}{N-H}\ln\frac{N-x_0}{H-x_0} > 0.$$

26. If $4h = kM^2$ then Eqs. (13) and (14)in the text show that the differential equation takes the form $x' = -k(M/2-x)^2$ with the single critical point $x = M/2$. This equation is readily solved by separation of variables, but clearly x' is negative whether x is less than or greater than $M/2$.

27. Separation of variables in the differential equation $x' = -k\left((x-a)^2+b^2\right)$ yields

$$x(t) = a - b\tan\left(bkt + \tan^{-1}\frac{a-x_0}{b}\right).$$

It therefore follows that $x(t)$ goes to minus infinity in a finite period of time.

28. Aside from a change in sign, this calculation is the same as that indicated in Eqs. (13) and (14) in the text.

29. This is simply a matter of analyzing the signs of x' in the cases $x < a$, $a < x < b$, $b < x < c$, and $c > x$. Alternatively, plot slope fields and typical solution curves for the two differential equations using typical numerical values such as $a = -1, b = 1, c = 2$.

SECTION 7.2

STABILITY AND THE PHASE PLANE

1. The only solution of the homogeneous system $2x - y = 0$, $x - 3y = 0$ is the origin (0, 0). The only figure among Figs. 7.2.11 through 7.2.18 showing a single critical point at the origin is Fig. 7.2.13. Thus the only critical point of the given autonomous system is the saddle point (0, 0) shown in Figure 7.2.13 in the text.

2. The only solution of the system $x - y = 0$, $x + 3y + 4 = 0$ is the point (1, 1). The only figure among Figs. 7.2.11 through 7.2.18 showing a single critical point at (1, 1) is Fig. 7.2.15. Thus the only critical point of the given autonomous system is the node (1, 1) shown in Figure 7.2.15 in the text.

3. The only solution of the system $x-2y+3=0,\ x-y+2=0$ is the point $(-1, 1)$. The only figure among Figs. 7.2.11 through 7.2.18 showing a single critical point at (–1, 1) is Fig. 7.2.18. Thus the only critical point of the given autonomous system is the stable center (–1, 1) shown in Figure 7.2.18 in the text.

4. The only solution of the system $2x-2y-4=0,\ x+4y+3=0$ is the point $(1,-1)$. The only figure among Figs. 7.2.11 through 7.2.18 showing a single critical point at (1, –1) is Fig. 7.2.12. Thus the only critical point of the given autonomous system is the spiral point (1,–1) shown in Figure 7.2.12 in the text.

5. The first equation $1-y^2=0$ gives $y = 1$ or $y = -1$ at a critical point. Then the second equation $x+2y=0$ gives $x = -2$ or $x = 2$, respectively. The only figure among Figs. 7.2.11 through 7.2.18 showing two critical points at (–2, 1) and (2, –1) is Fig. 7.2.11. Thus the critical points of the given autonomous system are the spiral point (–2, 1) and the saddle point (2, 1) shown in Figure 7.2.11 in the text.

6. The second equation $4-x^2=0$ gives $x = 2$ or $x = -2$ at a critical point. Then the first equation $2-4x-15y=0$ gives $y = -2/5$ or $x = 2/3$, respectively. The only figure among Figs. 7.2.11 through 7.2.18 showing two critical points at (–2, 2/3) and (2, –2/5) is Fig. 7.2.17. Thus the critical points of the given autonomous system are the spiral point (–2, 2/3) and the saddle point (2,–2/5) shown in Figure 7.2.17 in the text.

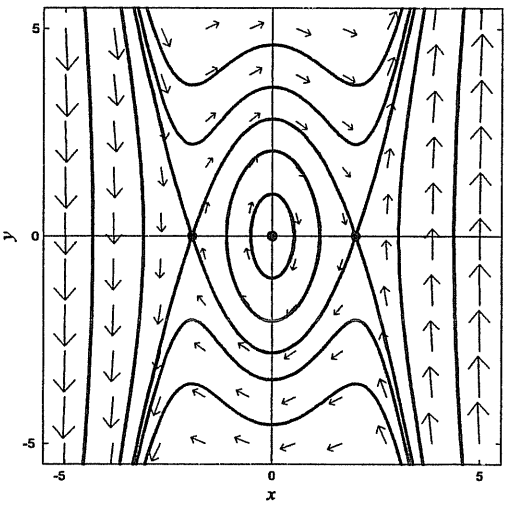
7. The first equation $4x-x^3=0$ gives $x = -2$, $x = 0$, or $x = 2$ at a critical point. Then the second equation $x-2y=0$ gives $y = -1$, $y = 0$, or $y = 1$, respectively. The only figure among Figs. 7.2.11 through 7.2.18 showing three critical points at (–2, –1), (0, 0), and (2, 1) is Fig. 7.2.14. Thus the critical points of the given autonomous system are the spiral point (0, 0) and the saddle points (–2, 1) and (2, 1) shown in Figure 7.2.14 in the text.

8. The second $-y-x^2=0$ equation gives $y = -x^2$ at a critical point. Substitution of this in the first equation $x-y-x^2+xy=0$ then gives $x - x^3 = 0$, so $x = -1$, $x = 0$, or $x = 1$. The only figure among Figs. 7.2.11 through 7.2.18 showing three critical points at
(–1, –1), (0, 0), and (1, –1) is Fig. 7.2.16. Thus the critical points of the given autonomous system are the spiral point (–1,–1), the saddle point (0, 0), and the node (1,–1) shown in Figure 7.2.16 in the text.

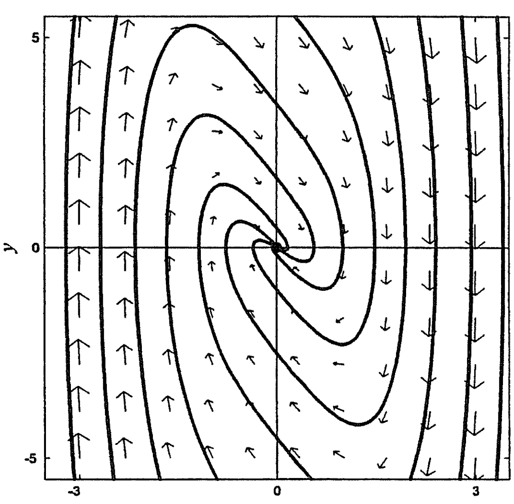
In each of Problems 9–12 we need only set $x' = x'' = 0$ and solve the resulting equation for x.

9. The equation $4x-x^3=x(4-x^2)=0$ has the three solutions $x=0,\pm 2$. This gives the three equilibrium solutions $x(t) \equiv 0$, $x(t) \equiv 2$, $x(t) \equiv -2$ of the given 2nd-order differential equation. A phase plane portrait for the equivalent 1st-order system

$x' = y,\ y' = -4x + x^3$ is shown in the figure below. We observe that the critical point $(0,0)$ in the phase plane appears to be a center, whereas the points $(\pm 2, 0)$ appear to be saddle points.

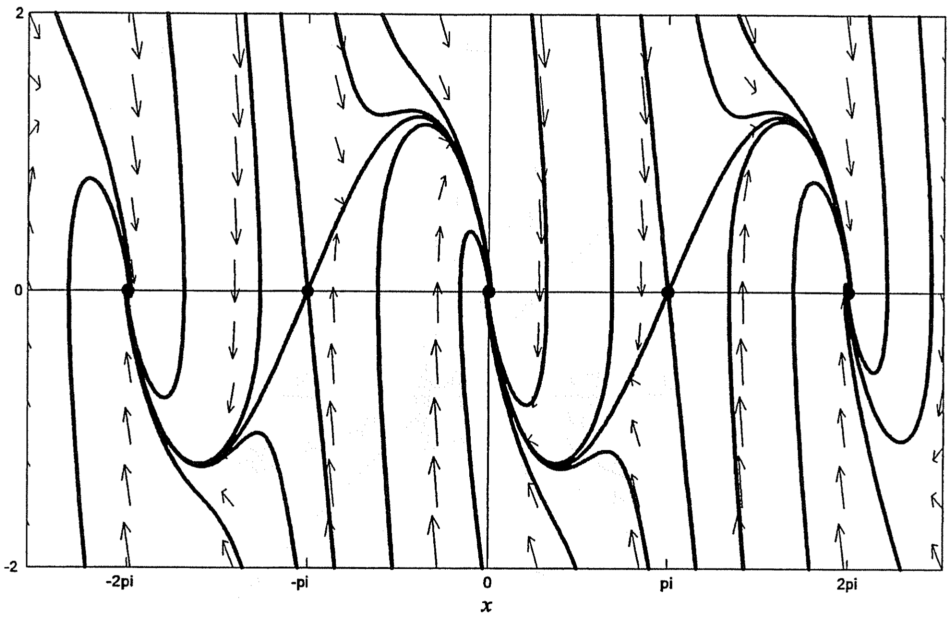


10. The equation $x + 4x^3 = x(1 + 4x^2) = 0$ has the single real solution $x = 0$. This gives the single equilibrium solution $x(t) \equiv 0$ of the given 2nd-order differential equation. A phase plane portrait for the equivalent 1st-order system $x' = y,\ y' = -2y - x - 4x^3$ is shown in the figure below. We observe that the critical point $(0,0)$ in the phase plane appears to be a spiral sink.

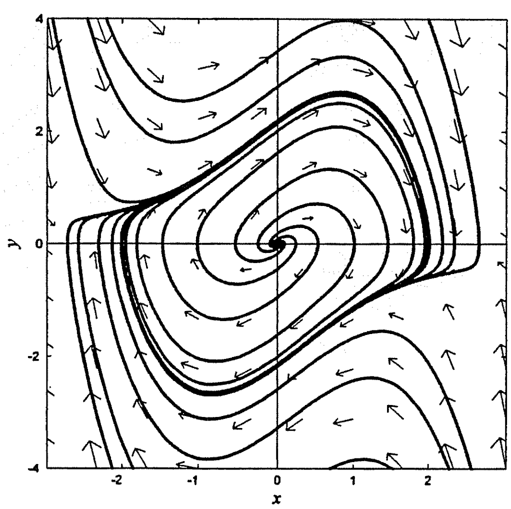


11. The equation $4\sin x = 0$ is satisfied by $x = n\pi$ for any integer n. Thus the given 2nd-order equation has infinitely many equilibrium solutions: $x(t) \equiv n\pi$ for any integer n.

A phase portrait for the equivalent 1st-order system $x' = y,\ y' = -3y - 4\sin x$ is shown below. We observe that the critical point $(n\pi, 0)$ in the phase plane looks like a spiral sink if n is even, but a saddle point if n is odd.



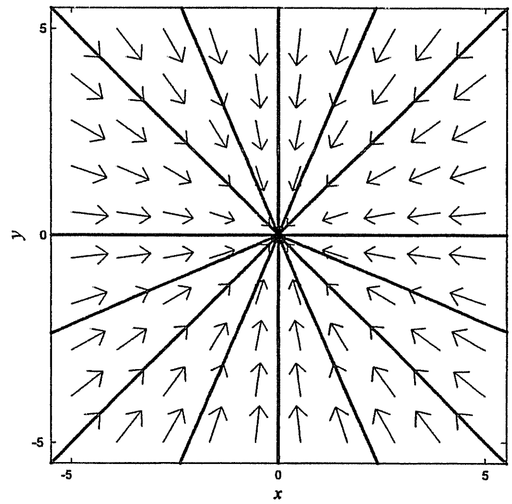
12. We immediately get the single solution $x = 0$ and thus the single equilibrium solution $x(t) \equiv 0$. A phase plane portrait for the equivalent 1st-order system $x' = y,\ y' = -(x^2 - 1)y - x$ is shown below. We observe that the critical point $(0,0)$ in the phase plane looks like a spiral source, with the solution curves emanating from this source spiraling outward toward a closed curve trajectory.



In Problems 13–16, the given x- and y-equations are independent exponential differential equations that we can solve immediately by inspection.

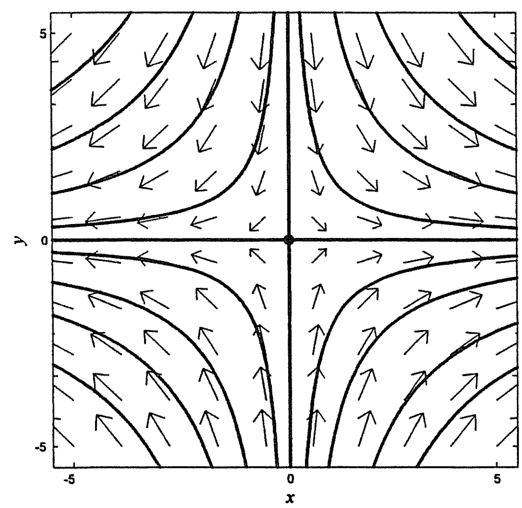
13. Solution: $x(t) = x_0e^{-2t}, \; y(t) = y_0e^{-2t}$

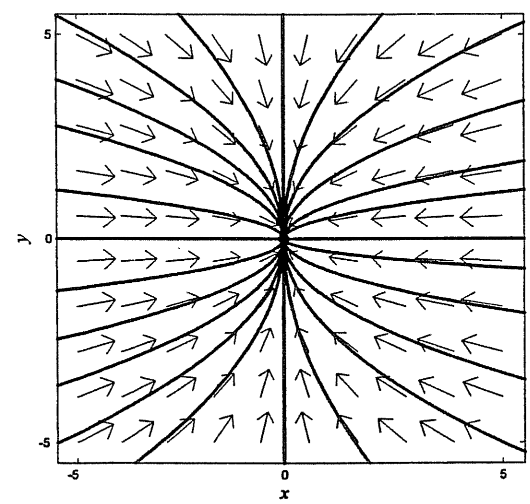
Then $y = (y_0/x_0)x = kx$, so the trajectories are straight lines through the origin. Clearly $x(t), y(t) \to 0$ as $t \to +\infty$, so the origin is a stable proper node like the one shown below.



14. Solution: $x(t) = x_0e^{2t}, \; y(t) = y_0e^{-2t}$

Then $xy = x_0y_0 = k$, so the trajectories are rectangular hyperbolas. Thus the origin is an unstable saddle point like the one in the left-hand figure below.



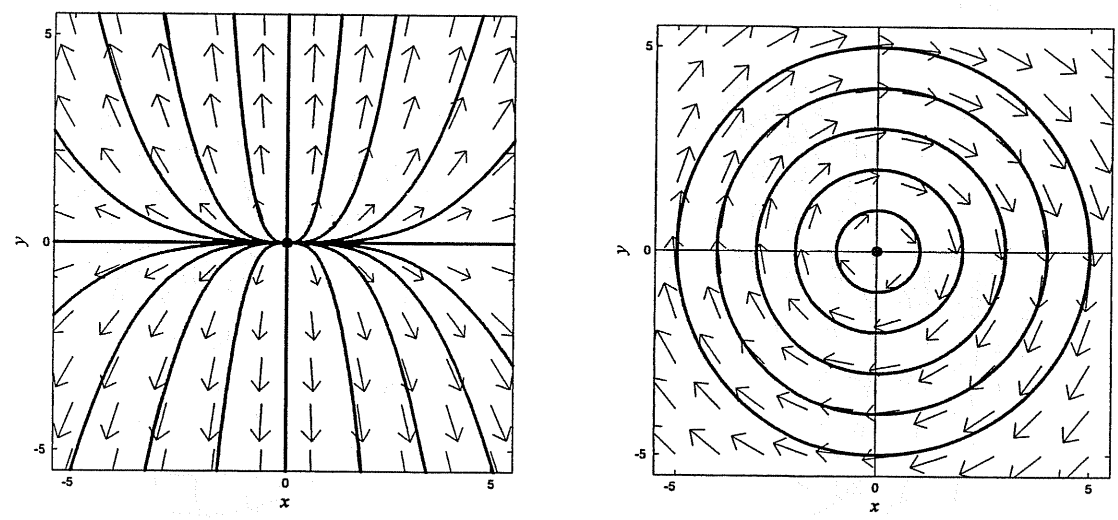


15. Solution: $x(t) = x_0 e^{-2t}, \; y(t) = y_0 e^{-t}$

Then $x = (x_0 / y_0^2)(y_0 e^{-t})^2 = ky^2$, so the trajectories are parabolas of the form $x = ky^2$, and clearly $x(t), y(t) \to 0$ as $t \to +\infty$. Thus the origin is a stable improper node like the one shown in the right-hand figure above.

16. Solution: $x(t) = x_0 e^{t}, \quad y(t) = y_0 e^{3t}$

The origin is an unstable improper node. The trajectories consist of the y-axis and curves of the form $y = kx^3$, departing from the origin as in the left-hand figure below.



17. Differentiation of the first equation and substitution using the second one gives

$$x'' = y' = x, \quad \text{so} \quad x'' + x = 0.$$

We therefore get the general solution

$$\begin{aligned} x(t) &= A \cos t + B \sin t \\ y(t) &= B \cos t - A \sin t \qquad (y = x'). \end{aligned}$$

Then

$$\begin{aligned} x^2 + y^2 &= (A\cos t + B\sin t)^2 + (B\cos t - A\sin t)^2 \\ &= (A^2 + B^2)\cos^2 t + (A^2 + B^2)\sin^2 t = A^2 + B^2. \end{aligned}$$

Therefore the trajectories are clockwise-oriented circles centered at the origin, and the origin is a stable center as in the right-hand figure above.

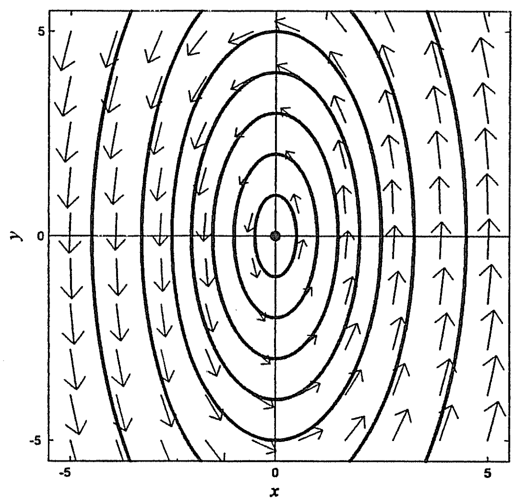
18. Elimination of y as in Problem 17 gives $x'' + 4x = 0$, so we get the general solution

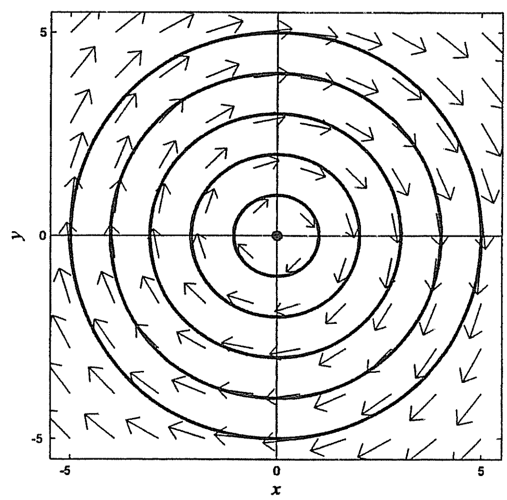
$$\begin{aligned} x(t) &= \quad A \cos 2t + \quad B \sin 2t, \\ y(t) &= -2B \cos 2t + 2A \sin 2t \qquad (y = -x'). \end{aligned}$$

It follows readily that

$$4x^2 + y^2 = 4A^2 + 4B^2, \text{ so } \frac{x^2}{(b/2)^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = 4A^2 + 4B^2$. Hence the origin is a stable center like the one illustrated in the left-hand figure below, and the vertical semiaxis of each ellipse is twice its horizontal semiaxis.





19. Elimination of y as in Problem 17 gives $x'' + 4x = 0$, so we get the general solution

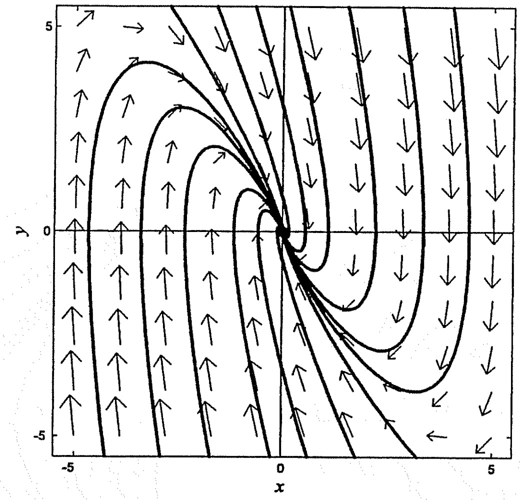
$$\begin{aligned} x(t) &= A \cos 2t + B \sin 2t, \\ y(t) &= B \cos 2t - A \sin 2t \qquad (y = \tfrac{1}{2}x'). \end{aligned}$$

Then $x^2 + y^2 = A^2 + B^2$, so the origin is a stable center, and the trajectories are clockwise-oriented circles centered at (0, 0), as in the right-hand figure above.

20. Substitution of $y' = x''$ from the first equation into the second one gives $x'' = -5x - y = -5x - 4x'$, so $x'' + 4x' + 5x = 0$. The characteristic roots of this equation are $r = -2 \pm i$, so we get the general solution

$$\begin{aligned} x(t) &= e^{-2t}(A \cos t + B \sin t), \\ y(t) &= e^{-2t}[(-2A + B)\cos t - (A + 2B)\sin t] \end{aligned}$$

(the latter because $y = x'$). Clearly $x(t), y(t) \to 0$ as $t \to +\infty$, so the origin is an asymptotically stable spiral point with trajectories approaching $(0,0)$.



21. We want to solve the system

$$\begin{aligned} -ky + x(1 - x^2 - y^2) &= 0 \\ kx + y(1 - x^2 - y^2) &= 0. \end{aligned}$$

If we multiply the first equation by $-y$ and the second one by x, then add the two results, we get $k(x^2 + y^2) = 0$. It therefore follows that $x = y = 0$.

22. After separation of variables, a partial-fractions decomposition gives

$$\begin{aligned} t &= \int \frac{dr}{r(1-r^2)} = \int \left(\frac{1}{r} - \frac{1}{2(r+1)} - \frac{1}{2(r-1)} \right) \\ &= \ln r - \frac{1}{2}\ln(r+1) - \frac{1}{2}\ln(r-1) + \frac{1}{2}, \end{aligned}$$

so

$$2t = \ln \frac{Cr^2}{r^2 - 1}$$

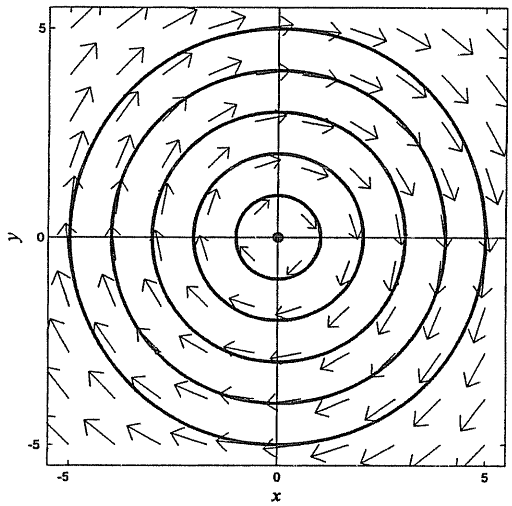
(assuming that $r > 1$, for instance). The initial condition $r(0) = r_0$ then gives

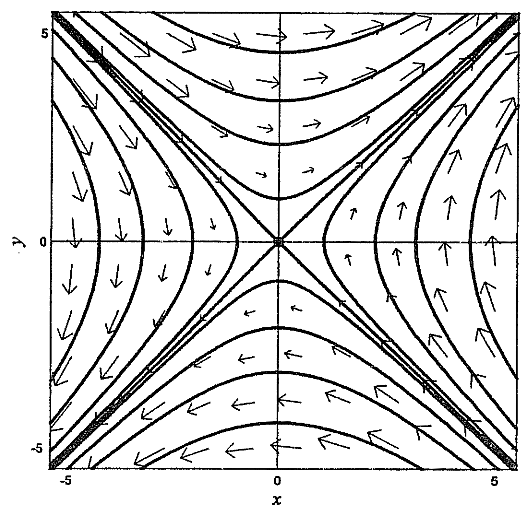
$$2t = \ln \frac{r^2(r_0^2 - 1)}{r_0^2(r^2 - 1)}, \quad \text{so} \quad e^{2t} = \frac{r^2(r_0^2 - 1)}{r_0^2(r^2 - 1)}.$$

We now solve readily for

$$r^2 = \frac{r_0^2 e^{2t}}{r_0^2 e^{2t} + (1 - r_0^2)} = \frac{r_0^2}{r_0^2 + (1 - r_0^2)e^{-2t}}.$$

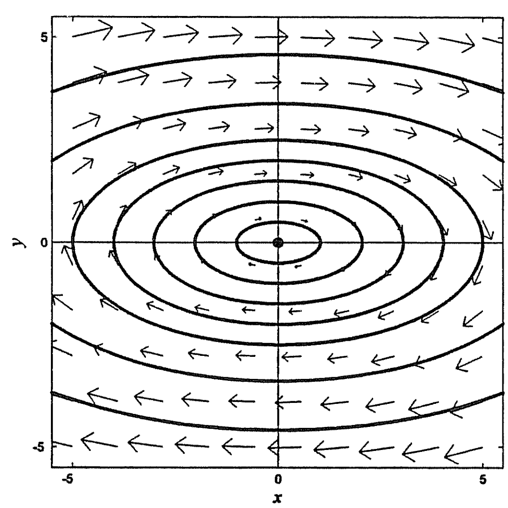
23. The equation $dy/dx = -x/y$ separates to $x\,dx + y\,dy = 0$, so $x^2 + y^2 = C$. Thus the trajectories consist of the origin $(0, 0)$ and the circles $x^2 + y^2 = C > 0$, as shown in the left-hand figure below.

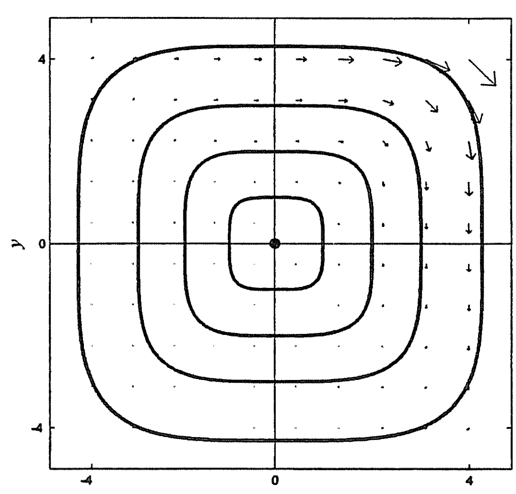




24. The equation $dy/dx = x/y$ separates to $y\,dy - x\,dx = 0$, so $y^2 - x^2 = C$. Thus the trajectories consist of the origin $(0, 0)$ and the hyperbolas $y^2 - x^2 = C$, as shown in the right-hand figure above.

25. The equation $dy/dx = -x/4y$ separates to $x\,dx + 4y\,dy = 0$, so $x^2 + 4y^2 = C$. Thus the trajectories consist of the origin $(0, 0)$ and the ellipses $x^2 + 4y^2 = C > 0$, as shown in the left-hand figure below.





26. The equation $dy/dx = -x^3/y^3$ separates to $x^3\,dx + y^3\,dy = 0$, so $x^4 + y^4 = C$. Thus the trajectories consist of the origin (0, 0) and the ovals of the form $x^4 + y^4 = C$, as illustrated in the right-hand figure at the bottom of the preceding page..

27. If $\phi(t) = x(t+\gamma)$ and $\psi(t) = y(t+\gamma)$ then

$$\phi'(t) = x'(t+\gamma) = y(t+\gamma) = \psi(t),$$

but

$$\psi(t) = y'(t+\gamma) = x(t+\gamma)\cdot(t+\gamma) = t\,\phi(t) + \gamma\,\phi(t) \neq t\,\phi(t).$$

28. If $\phi(t) = x(t+\gamma)$ and $\psi(t) = y(t+\gamma)$ then

$$\phi'(t) = x'(t+\gamma) = F(x(t+\gamma),\quad y(t+\gamma)) = F(\phi(t),\,\psi(t)),$$

and

$$\psi'(t) = G(\phi(t), \psi(t))$$

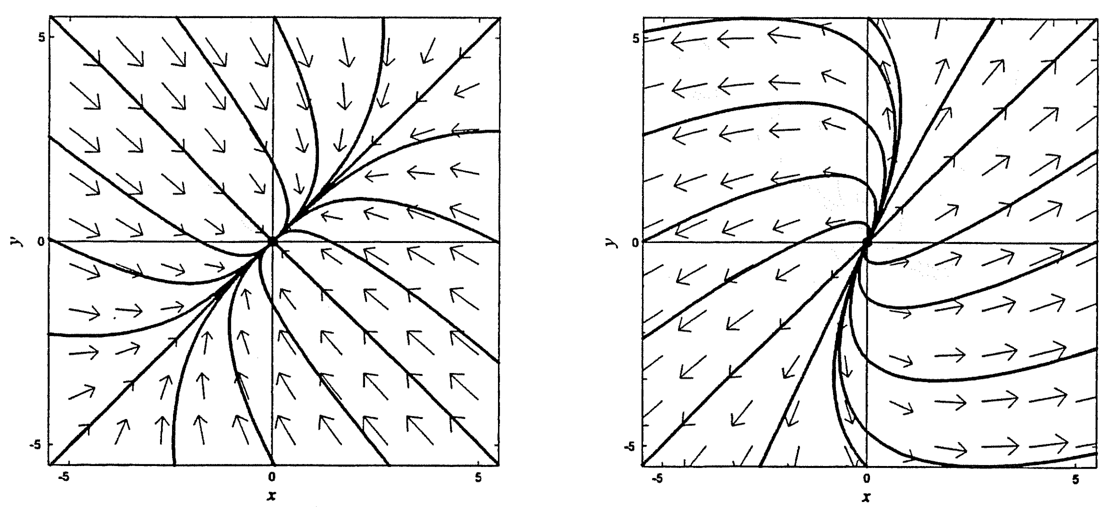
similarly. Therefore $\phi(t)$ and $\psi(t)$ satisfy the given differential equations.

SECTION 7.3

LINEAR AND ALMOST LINEAR SYSTEMS

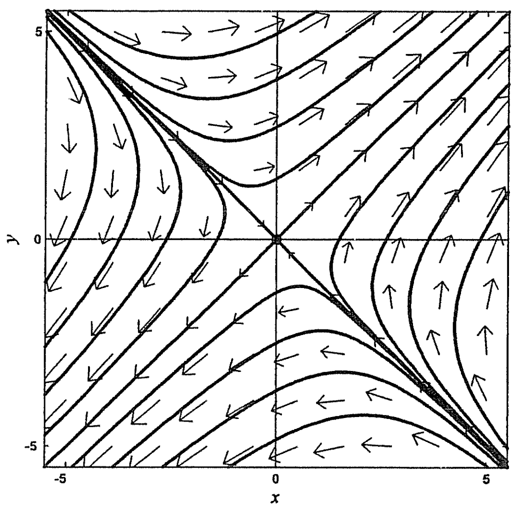
In Problems 1–10 we first find the roots λ_1 and λ_2 of the characteristic equation of the coefficient matrix of the given linear system. We can then read the type and stability of the critical point (0,0) from Theorem 1 and the table of Figure 7.3.9 in the text.

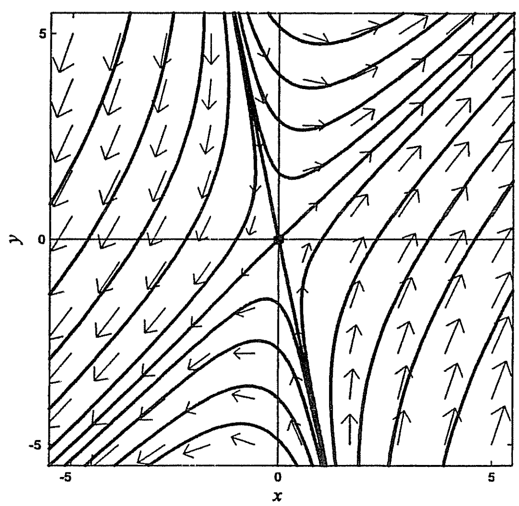
1. The roots $\lambda_1 = -1$ and $\lambda_2 = -3$ of the characteristic equation $\lambda^2 + 4\lambda + 3 = 0$ are both negative, so (0,0) is an asymptotically stable node as shown on the left below.



2. The roots $\lambda_1 = 2$ and $\lambda_2 = 3$ of the characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ are both positive, so $(0,0)$ is an unstable improper node as shown on the right at the bottom of the preceding page.

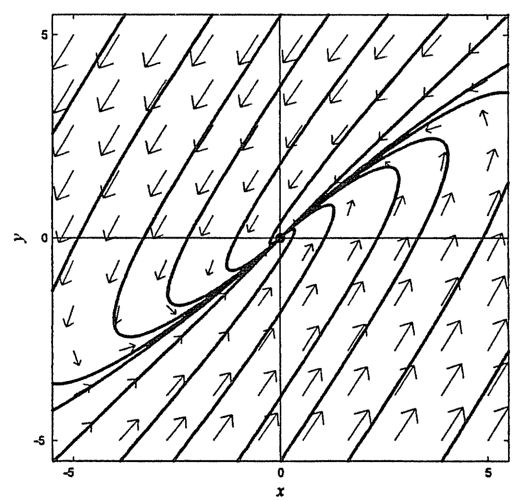
3. The roots $\lambda_1 = -1$ and $\lambda_2 = 3$ of the characteristic equation $\lambda^2 - 2\lambda - 3 = 0$ have different signs, so $(0,0)$ is an unstable saddle point as shown on the left below.

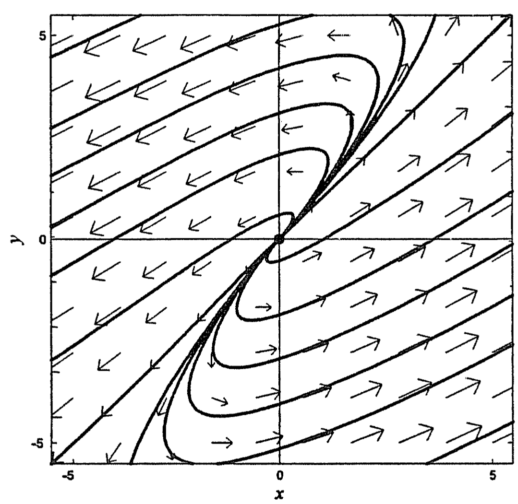




4. The roots $\lambda_1 = -2$ and $\lambda_2 = 4$ of the characteristic equation $\lambda^2 - 2\lambda - 3 = 0$ have different signs, so $(0,0)$ is an unstable saddle point as shown on the right above.

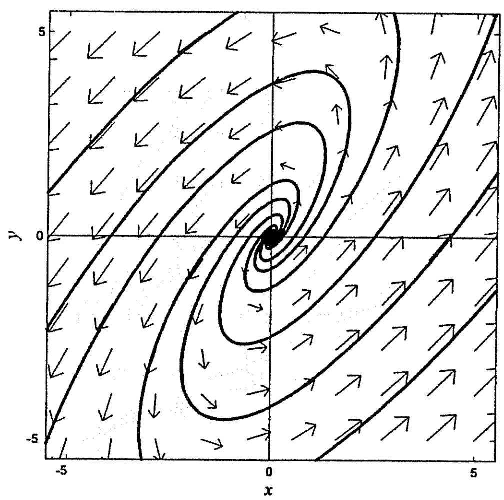
5. The roots $\lambda_1 = \lambda_2 = -1$ of the characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ are negative and equal, so $(0,0)$ is an asymptotically stable node as in the left-hand figure below.

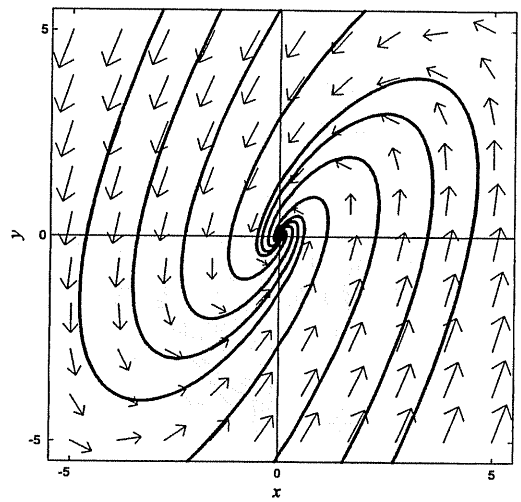




6. The roots $\lambda_1 = \lambda_2 = 2$ of the characteristic equation $\lambda^2 - 4\lambda + 4 = 0$ are positive and equal, so $(0,0)$ is an unstable node as in the right-hand figure at the bottom of the preceding page.

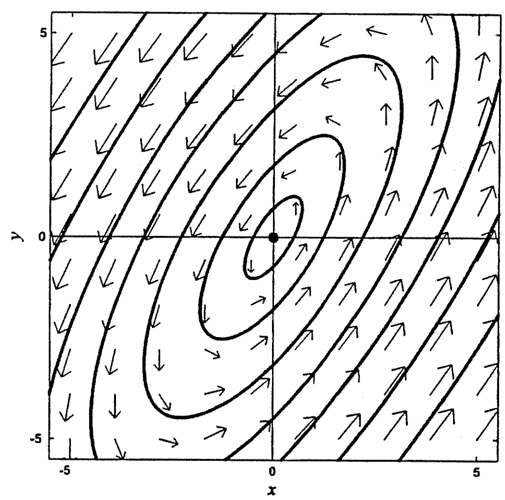
7. The roots $\lambda_1, \lambda_2 = 1 \pm 2i$ of the characteristic equation $\lambda^2 - 2\lambda + 5 = 0$ are complex conjugates with positive real part, so $(0,0)$ is an unstable spiral point as shown in the left-hand figure below.

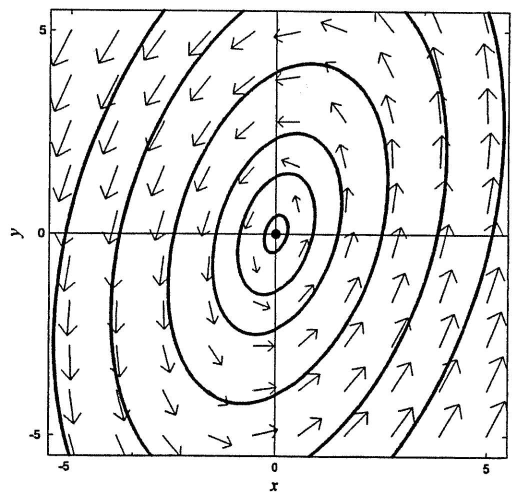




8. The roots $\lambda_1, \lambda_2 = -2 \pm 3i$ of the characteristic equation $\lambda^2 + 4\lambda + 13 = 0$ are complex conjugates with negative real part, so $(0,0)$ is an asymptotically stable spiral point as shown in the right-hand figure above.

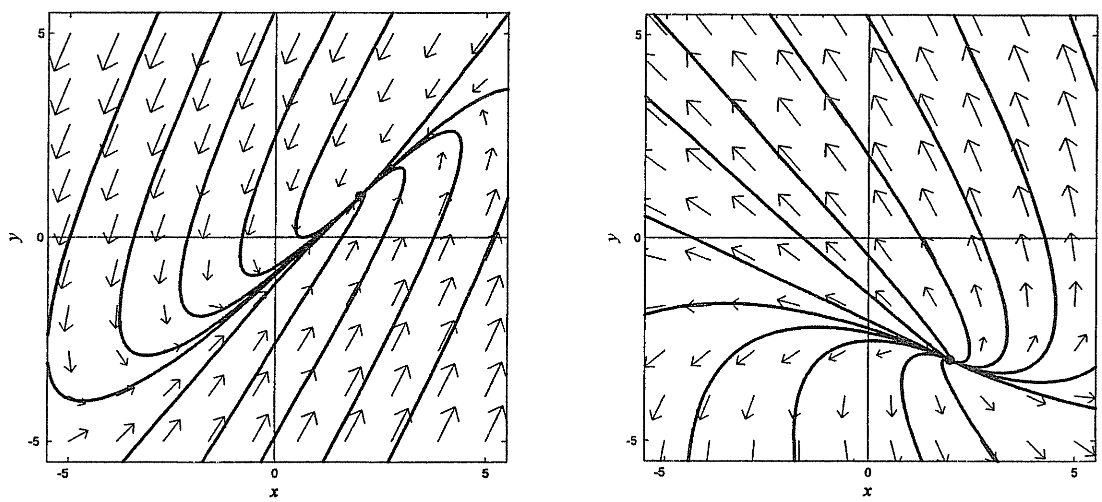
9. The roots $\lambda_1, \lambda_2 = \pm 2i$ of the characteristic equation $\lambda^2 + 4 = 0$ are pure imaginary, so $(0,0)$ is a stable (but not asymptotically stable) center as in the left-hand figure below.





10. The roots $\lambda_1, \lambda_2 = \pm 3\,i$ of the characteristic equation $\lambda^2 + 9 = 0$ are pure imaginary, so (0,0) is a stable (but not asymptotically stable) center as in the right-hand figure at the bottom of the preceding page.

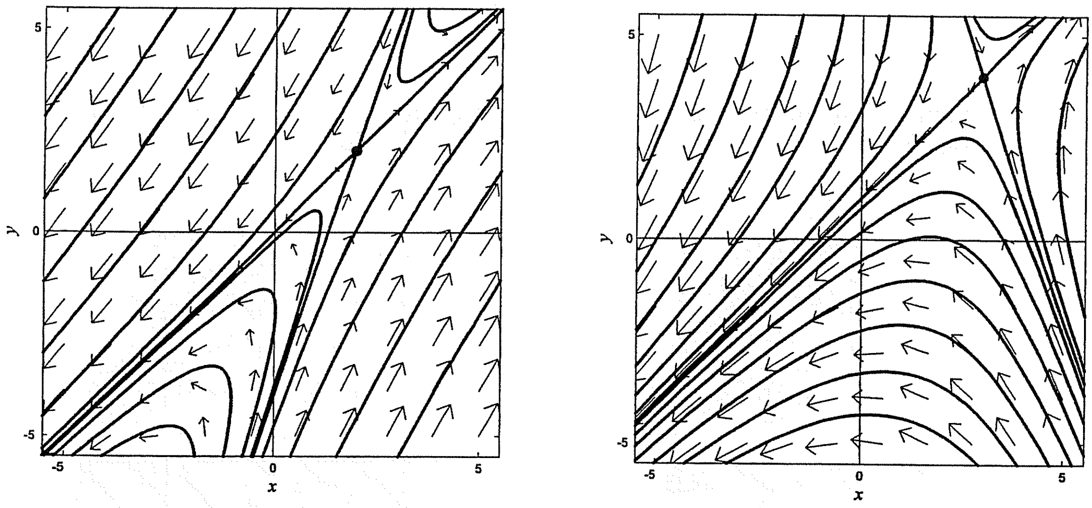
11. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ has characteristic equation $\lambda^2 + 3\lambda + 2 = 0$ and eigenvalues $\lambda_1 = -1,\ \lambda_2 = -2$ that are both negative. Hence the critical point $(2, 1)$ is an asymptotically stable node as in the left-hand figure below.



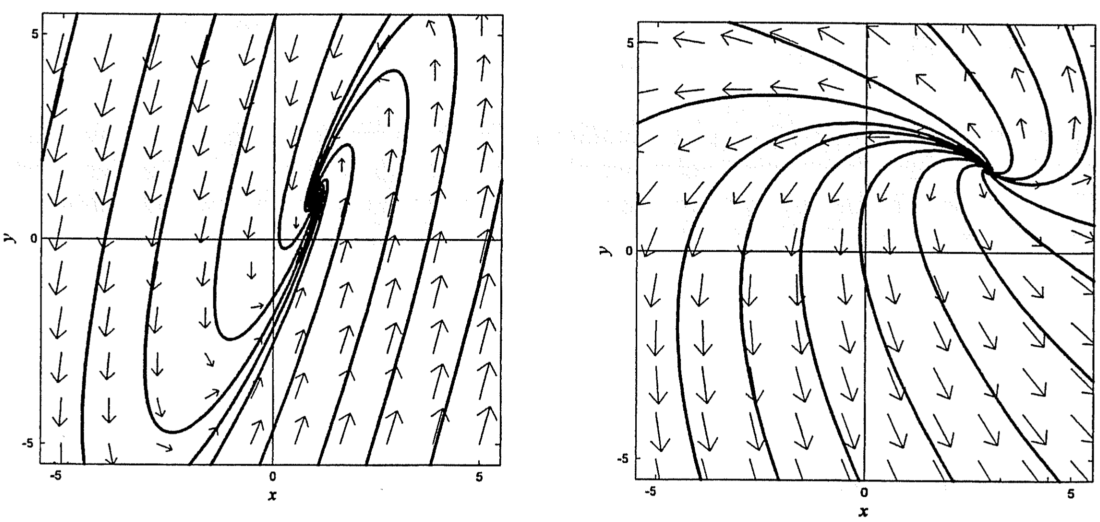
12. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ and eigenvalues $\lambda_1 = 2,\ \lambda_2 = 3$ that are both positive. Hence the critical point $(2,-3)$ is an unstable node as in the right-hand figure above.

13. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 - 1 = 0$ and eigenvalues $\lambda_1 = -1,\ \lambda_2 = +1$ having different signs. Hence the critical point $(2, 2)$ is an unstable saddle point as in the left-hand figure at the top of the next page.

14. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 4 = 0$ and eigenvalues $\lambda_1 = -2, \lambda_2 = 2$ that are real with different signs. Hence the critical point $(3, 4)$ is an unstable saddle point as in the right-hand figure at the top of the next page.

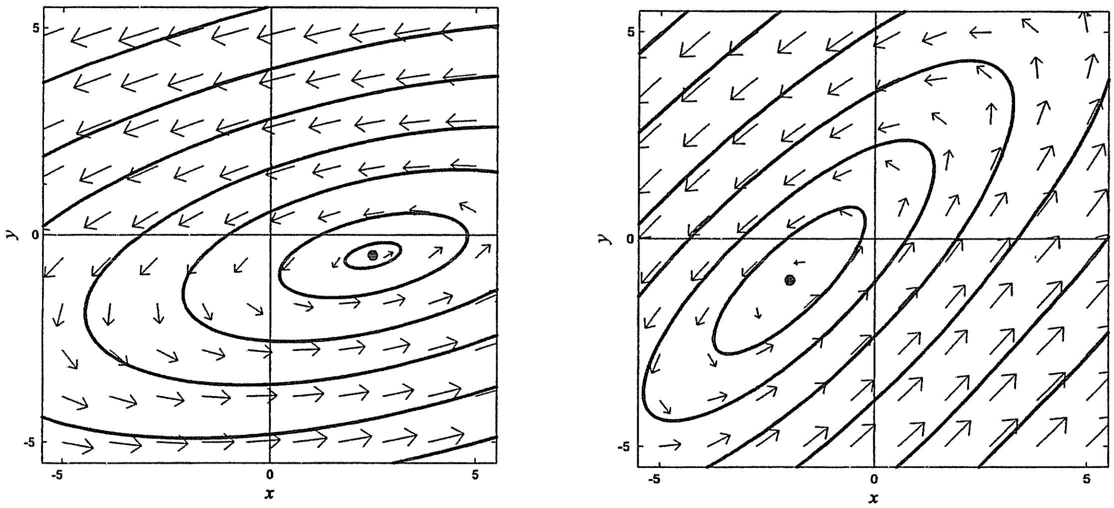


15. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 2 = 0$ and eigenvalues $\lambda_1, \lambda_2 = -1 \pm i$ that are complex conjugates with negative real part. Hence the critical point $(1, 1)$ is an asymptotically stable spiral point as shown in the figure on the left below.



16. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ has characteristic equation $\lambda^2 - 4\lambda + 5 = 0$ and eigenvalues $\lambda_1, \lambda_2 = 2 \pm i$ that are complex conjugates with positive real part. Hence the critical point $(3, 2)$ is an unstable spiral point as shown in the right-hand figure above.

17. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + 4 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 2i$. Hence $(5/2, -1/2)$ is a stable (but not asymptotically stable) center as shown on the left below.



18. The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix}$ has characteristic equation $\lambda^2 + 9 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 3i$. Hence $(-2, -1)$ is a stable (but not asymptotically stable) center as shown on the right above.

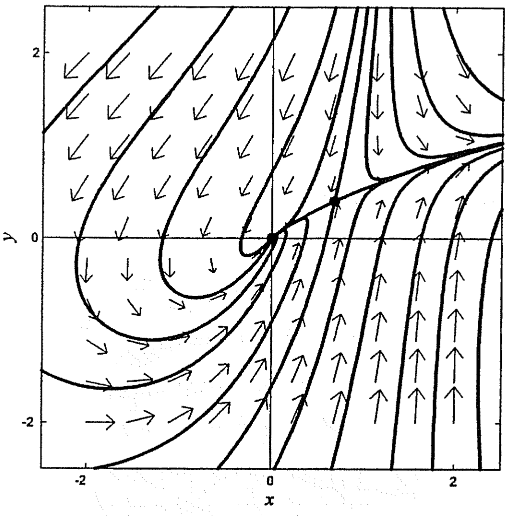
In each of Problems 19–28 we first calculate the Jacobian matrix $\mathbf{J}$ and its eigenvalues at (0,0) and at each of the other critical points we observe in our phase portrait for the given system. Then we apply Theorem 2 to determine as much as we can about the type and stability of each of these critical points of the given almost linear system. Finally we a phase portrait that

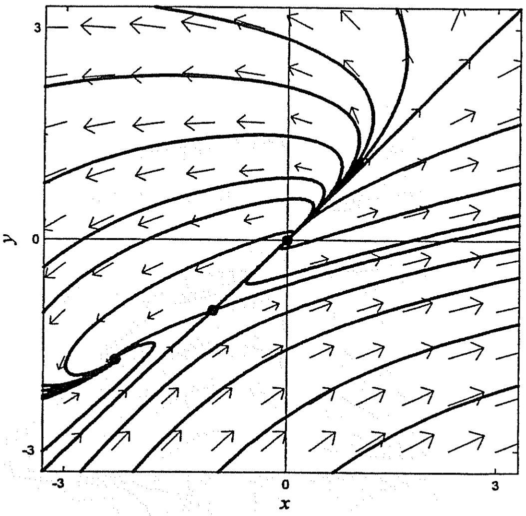
19. $\mathbf{J} = \begin{bmatrix} 1+2y & -3+2x \\ 4-y & -6-x \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -3 \\ 4 & -6 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5\lambda + 6 = 0$ and eigenvalues $\lambda_1 = -3$, $\lambda_2 = -2$ that are both negative. Hence (0,0) is an asymptotically stable node of the given almost linear system.

At (2/3, 2/5): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 9/5 & -5/3 \\ 18/5 & -20/3 \end{bmatrix}$ has characteristic equation $\lambda^2 + \frac{73}{15}\lambda - 6 = 0$ and approximate eigenvalues $\lambda_1 \approx -5.89$, $\lambda_2 \approx 1.02$ with different signs. Hence (2/3, 2/5) is a saddle point.

The left-hand figure at the top of the next page shows both these critical points.





20. $\mathbf{J} = \begin{bmatrix} 6+2x & -5 \\ 2 & -1+2y \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 4 = 0$ and eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$ that are both positive. Hence (0,0) is an unstable node of the given almost linear system.

At (–1,–1): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 - \lambda - 2 = 0$ and eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$ with different signs. Hence (–1,–1) is a saddle point.

At (–2.30,–1.70): The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 1.40 & -5 \\ 2 & -4.40 \end{bmatrix}$ has complex conjugate eigenvalues $\lambda_1 \approx -1.5 + 1.25i$, $\lambda_2 \approx -1.5 - 1.25i$ with negative real parts. Hence (–2.30,–1.70) is a spiral sink.

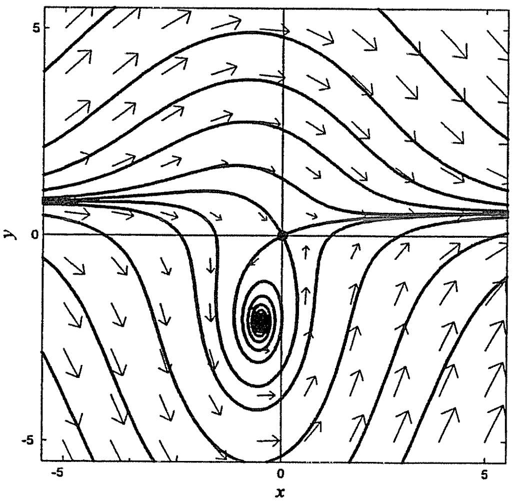
The figure on the right above shows these three critical points.

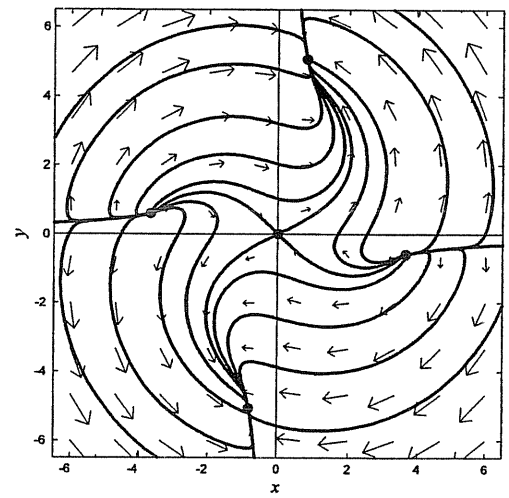
21. $\mathbf{J} = \begin{bmatrix} 1+2x & 2+2y \\ 2-3y & -2-3x \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 6 = 0$ and eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$ with different signs. Hence (0,0) is a saddle point of the given almost linear system.

At $(-0.51,-2.12)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -0.014 & -2.236 \\ 8.354 & -0.479 \end{bmatrix}$ has complex conjugate eigenvalues $\lambda_1 \approx -0.25+4.32i,\ \lambda_2 \approx -0.25+4.32i$ with negative real parts. Hence $(-0.51,-2.12)$ is a spiral sink.

The figure on the left below shows these two critical points.





22. $\mathbf{J} = \begin{bmatrix} 1-y^2 & 4-2xy \\ 2+2xy & -1+x^2 \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 9 = 0$ and eigenvalues $\lambda_1 = -3,\ \lambda_2 = 3$ that have different signs. Hence $(0,0)$ is a saddle point of the given almost linear system.

At $(\pm 3.65, \mp 0.59)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 0.649 & 8.325 \\ -2.325 & 12.325 \end{bmatrix}$ has positive real eigenvalues $\lambda_1 \approx 2.649,\ \lambda_2 \approx 10.325$. Hence these critical points are both nodal sources.

At $(\pm 0.82, \pm 5.06)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -24.649 & -4.325 \\ 10.325 & -0.325 \end{bmatrix}$ has negative real eigenvalues $\lambda_1 \approx -22.649,\ \lambda_2 \approx -2.325$. Hence these critical points are both nodal sinks.

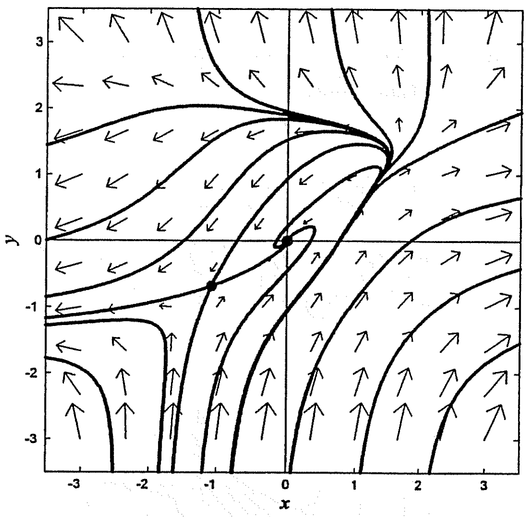
The figure on the right above shows these five critical points.

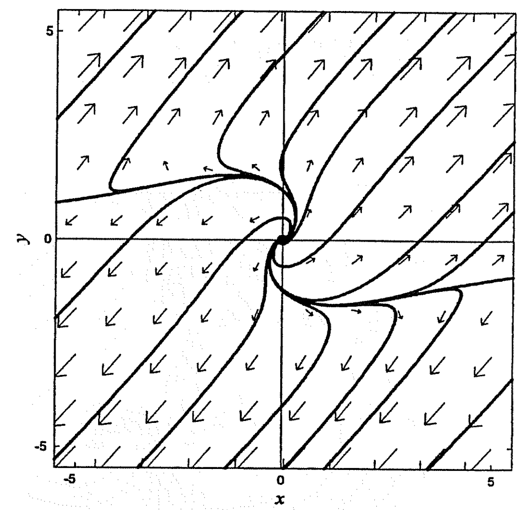
23. $\mathbf{J} = \begin{bmatrix} 2+3x^2 & -5 \\ 4 & -6+4y^3 \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & -5 \\ 4 & -6 \end{bmatrix}$ has characteristic equation $\lambda^2 + 4\lambda + 8 = 0$ and complex conjugate eigenvalues $\lambda_1 = -2 + 2i$, $\lambda_2 = -2 - 2i$ with negative real part. Hence (0,0) is a spiral sink of the given almost linear system.

At $(-1.08, -0.68)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 5.495 & -5 \\ 4 & -7.276 \end{bmatrix}$ has eigenvalues $\lambda_1 \approx -5.45$, $\lambda_2 \approx 3.67$ with different signs. Hence $(-1.08, -0.68)$ is a saddle point.

The figure on the left below shows these two critical points.





24. $\mathbf{J} = \begin{bmatrix} 5 + 2xy & -3 + x^2 + 3y^2 \\ 5 + 2xy & x^2 + 3y^2 \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 5 & -3 \\ 5 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 15 = 0$ and complex conjugate eigenvalues $\lambda_1 \approx 2.5 + 2.96i$, $\lambda_2 \approx 2.5 - 2.96i$ with positive real part. Hence (0,0) is a spiral source of the given almost linear system. The figure on the right above shows this critical point.

25. $\mathbf{J} = \begin{bmatrix} 1 + 3y & -2 + 3x \\ 2 - 2x & -3 - 2y \end{bmatrix}$

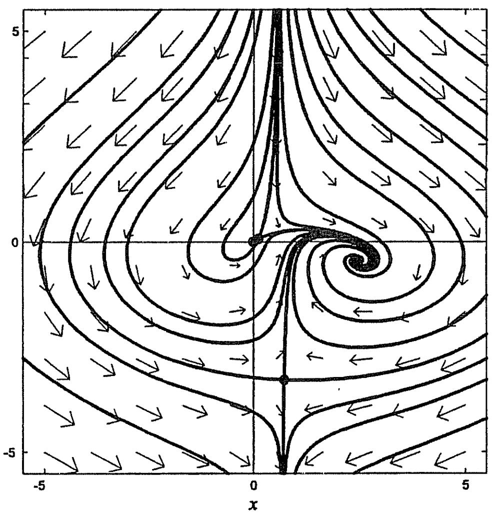
At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ and equal negative eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1$. Hence (0,0) is either a nodal sink or a spiral sink of the given almost linear system.

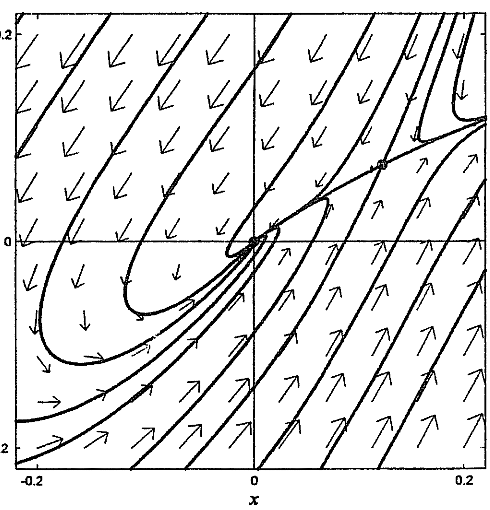
At $(0.74,-3.28)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -8.853 & 0.226 \\ 0.516 & 3.568 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -8.86,\ \lambda_2 \approx 3.58$ with different signs. Hence $(0.74,-3.28)$ is a saddle point.

At $(2.47,-0.46)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -0.370 & 5.410 \\ -2.940 & -2.087 \end{bmatrix}$ has complex conjugate eigenvalues $\lambda_1 \approx -1.23+3.89i,\ \lambda_2 \approx -1.23+3.89i$ with negative real part. Hence $(2.47,-0.46)$ is a spiral sink.

At $(0.121,0.074)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 1.222 & -1.636 \\ 1.758 & -3.148 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -2.34,\ \lambda_2 \approx 0.42$ with different signs. Hence $(0.121,0.074)$ is a saddle point.

The left-hand figure below shows clearly the first three of these critical points. The right-hand figure is a close-up near the origin with the final critical point now visible.





26. $\mathbf{J} = \begin{bmatrix} 3-2x & -2-2y \\ 2-3y & -1-3x \end{bmatrix}$

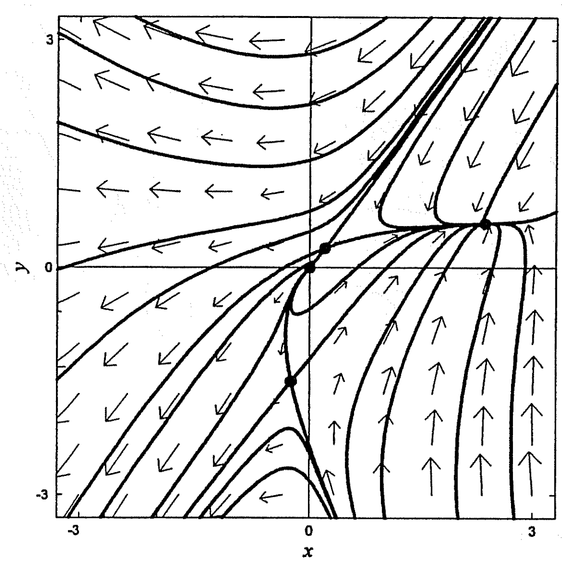
At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 2\lambda + 1 = 0$ and equal positive eigenvalues $\lambda_1 = 1,\ \lambda_2 = 1$. Hence (0,0) is either a nodal source or a spiral source of the given almost linear system.

At $(0.203,0.253)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 2.592 & -2.506 \\ 1.241 & -1.611 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -0.65,\ \lambda_2 \approx 1.63$ with different signs. Hence $(0.203,0.253)$ is a saddle point.

At $(-0.231,-1.504)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 3.462 & 1.008 \\ 6.511 & -0.307 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -1.60$, $\lambda_2 \approx 4.76$ with different signs. Hence $(-0.231,-1.504)$ is a saddle point.

At $(2.360,0.584)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -1.721 & -3.168 \\ -0.247 & -8.081 \end{bmatrix}$ has unequal negative eigenvalues $\lambda_1 \approx -7.96$, $\lambda_2 \approx -1.85$. Hence $(2.47,-0.46)$ is a nodal sink.

The figure below shows these four critical points.



27. $\mathbf{J} = \begin{bmatrix} 1+4x^3 & -1-2y \\ 2-2x & -1+4y^3 \end{bmatrix}$

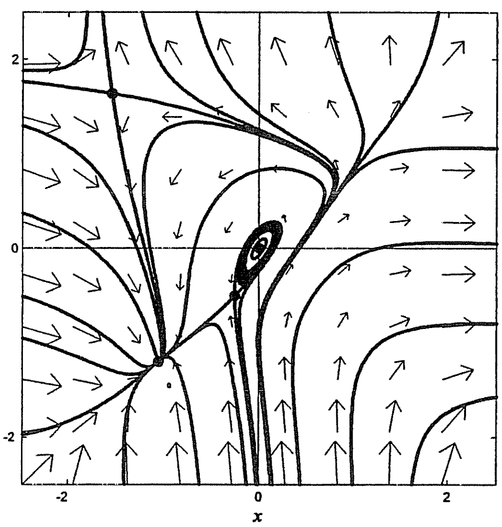
At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + 1 = 0$ and equal positive eigenvalues $\lambda_1 = -i$, $\lambda_2 = +i$. Hence (0,0) is either a center or a spiral point, but its stability is not determined by Theorem 2.

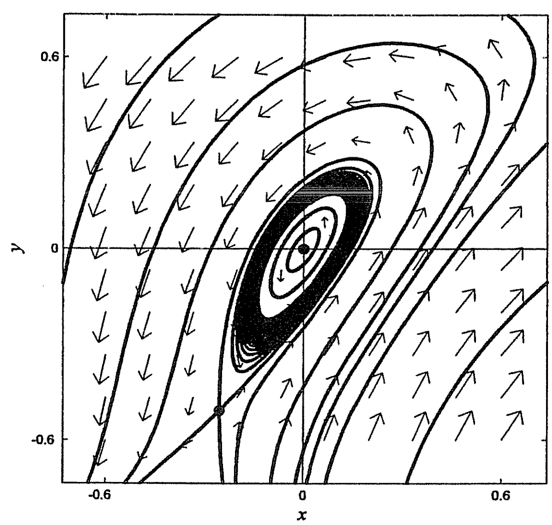
At $(-0.254,-0.507)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 0.934 & 0.014 \\ 2.508 & -1.521 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -1.53$, $\lambda_2 \approx 0.95$ with different signs. Hence $(-0.254,-0.507)$ is a saddle point.

At $(-1.557,-1.637)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -14.087 & -4.273 \\ 5.113 & 16.532 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -13.36$, $\lambda_2 \approx 15.80$ with different signs. Hence $(-1.557,-1.637)$ is a saddle point.

At $(-1.070, -1.202)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} -3.905 & 1.403 \\ 4.141 & -7.940 \end{bmatrix}$ has unequal negative eigenvalues $\lambda_1 \approx -9.07,\ \lambda_2 \approx -2.78$. Hence $(-1.070, -1.202)$ is a nodal sink.

The left-hand figure below shows these four critical points. The close-up on the right suggests that the origin may (but may not) be a stable center.





28. $\mathbf{J} = \begin{bmatrix} 3+3x^2 & -1+3y^2 \\ 13+3y & -3+3x \end{bmatrix}$

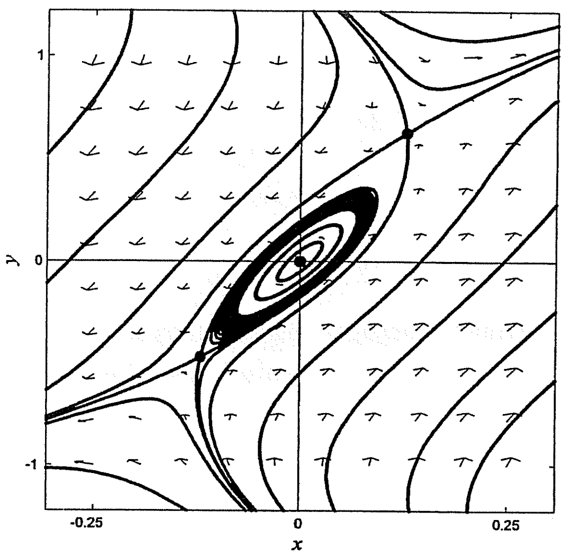
At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & -1 \\ 13 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 + 4 = 0$ and equal positive eigenvalues $\lambda_1 = -2i,\ \lambda_2 = +2i$. Hence $(0,0)$ is either a center or a spiral point, but its stability is not determined by Theorem 2.

At $(-0.121, -0.469)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 3.044 & -0.340 \\ 11.593 & -3.364 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -2.67,\ \lambda_2 \approx 2.35$ with different signs. Hence $(-0.121, -0.469)$ is a saddle point.

At $(0.126, 0.626)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 3.048 & 0.176 \\ 14.878 & -2.621 \end{bmatrix}$ has real eigenvalues $\lambda_1 \approx -3.05,\ \lambda_2 \approx 3.48$ with different signs. Hence $(0.126, 0.626)$ is a saddle point.

At $(5.132, -5.382)$: The Jacobian matrix $\mathbf{J} \approx \begin{bmatrix} 82.000 & 85.903 \\ -3.146 & 12.395 \end{bmatrix}$ has unequal positive eigenvalues $\lambda_1 \approx 16.52,\ \lambda_2 \approx 77.87$. Hence $(5.132, -5.382)$ is a nodal source.

The first three of these critical points are shown in the figure at the top of the next page.

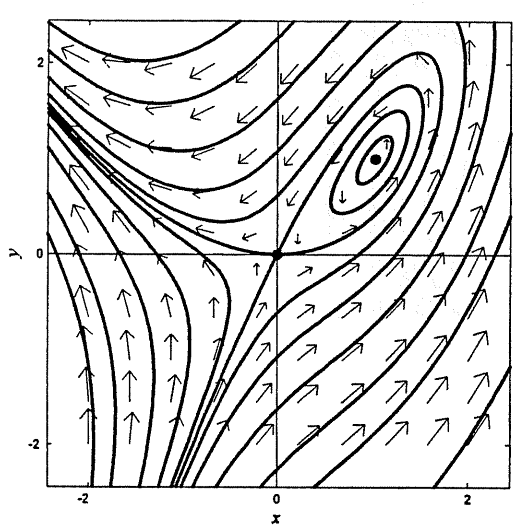


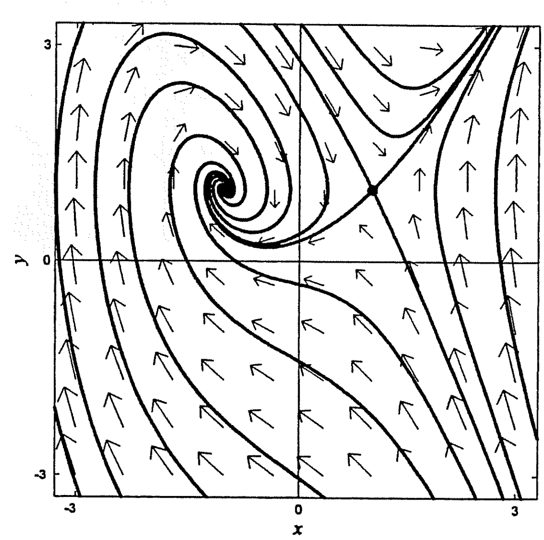
29. $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 2x & -1 \end{bmatrix}$

At (0,0): The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 1 = 0$ and real eigenvalues $\lambda_1 \approx -1,\ \lambda_2 \approx +1$ with different signs. Hence (0,0) is a saddle point.

At $(1,1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + 1 = 0$ and pure imaginary eigenvalues $\lambda_1 \approx +i,\ \lambda_2 \approx -i.$ Hence $(1,1)$ is either a center or a spiral point, but its stability is not determined by Theorem 2.

The left-hand figure below suggests that $(1,1)$ is a stable center.





30. $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 2x & -1 \end{bmatrix}$

At $(1,1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 2 = 0$ and real eigenvalues $\lambda_1 \approx -2$, $\lambda_2 \approx +1$ with different signs. Hence $(1,1)$ is a saddle point.

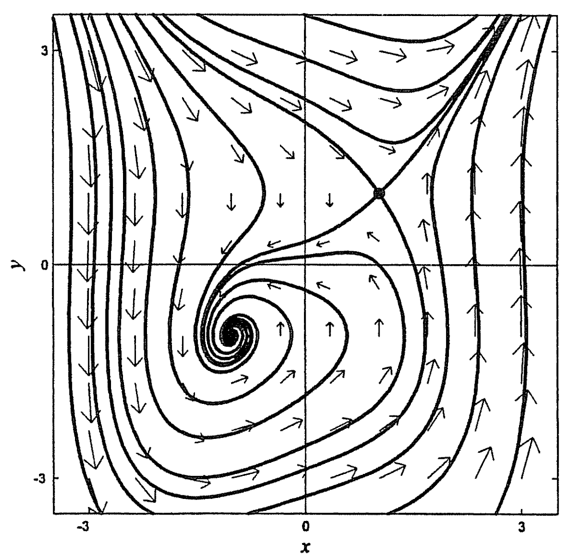
At $(1,-1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + 2 = 0$ and complex conjugate eigenvalues $\lambda_1 \approx -0.5 + 1.323i$, $\lambda_2 \approx -0.5 - 1.323i$ with negative real part. Hence $(1,-1)$ is a spiral sink as in the right-hand figure on the preceding page.

31. $\mathbf{J} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix}$

At $(1,1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 6 = 0$ and real eigenvalues $\lambda_1 = -3$, $\lambda_2 = +2$ with different signs. Hence $(1,1)$ is a saddle point.

At $(-1,-1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + 6 = 0$ and complex conjugate eigenvalues $\lambda_1 \approx -0.5 + 2.398i$, $\lambda_2 \approx -0.5 - 2.398i$ with negative real part. Hence $(-1,-1)$ is a spiral sink.

These two critical points are shown in the figure below.

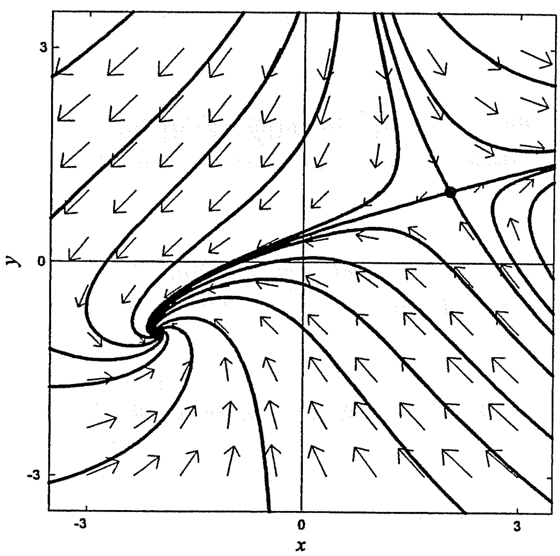


32. $\mathbf{J} = \begin{bmatrix} y & x \\ 1 & -2 \end{bmatrix}$

At $(2,1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 4 = 0$ and real eigenvalues $\lambda_1 \approx -2.56$, $\lambda_2 \approx +1.56$ with different signs. Hence $(1,1)$ is a saddle point.

At $(-2,-1)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & -2 \\ 1 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 3\lambda + 4 = 0$ and complex conjugate eigenvalues $\lambda_1 \approx -1.5 + 1.323i$, $\lambda_2 \approx -1.5 - 1.323i$ with negative real part. Hence $(-2,-1)$ is a spiral sink.

These two critical points are shown in the figure below.



33. The characteristic equation of the given linear system is

$$(\lambda - \varepsilon)^2 + 1 = 0$$

with characteristic roots $\lambda_1, \lambda_2 = \varepsilon \pm i$.

(a) So if $\varepsilon < 0$ then λ_1, λ_2 are complex conjugates with negative real part, and hence $(0, 0)$ is an asymptotically stable spiral point.

(b) If $\varepsilon = 0$ then $\lambda_1, \lambda_2 = \pm i$ (pure imaginary), so $(0,0)$ is a stable center.

(c) If $\varepsilon > 0$, the situation is the same as in (a) except that the real part is positive, so $(0, 0)$ is an unstable spiral point.

34. The characteristic equation of the given linear system is

$$(\lambda + 1)^2 - \varepsilon = 0.$$

(a) If $\varepsilon < 0$ then $\lambda_1, \lambda_2 = -1 \pm i\sqrt{-\varepsilon}$. Thus the characteristic roots are complex conjugates with negative real part, so it follows that $(0,0)$ is an asymptotically stable spiral point.

(b) If $\varepsilon = 0$ then the characteristic roots $\lambda_1 = \lambda_2 = -1$ are equal and negative, so $(0,0)$ is an asymptotically stable node. If $0 < \varepsilon < 1$ then $\lambda_1, \lambda_2 = -1 \pm \sqrt{\varepsilon}$ are both negative, so $(0,0)$ is an asymptotically stable improper node.

35. **(a)** If $h = 0$ we have the familiar system $x' = y$, $y' = -x$ with circular trajectories about the origin, which is therefore a center.

(b) The change to polar coordinates as in Example 6 of Section 7.2 is routine, yielding $r' = hr^3$ and $\theta' = -1$.

(c) If $h = -1$, then $r' = -r^3$ integrates to give $2r^2 = 1/(t + C)$ where C is a positive constant, so clearly $r \to 0$ as $t \to +\infty$, and thus the origin is a stable spiral point.

(d) If $h = +1$, then $r' = r^3$ integrates to give $2r^2 = -1/(t + C)$ where $C = -B$ is a positive constant. It follows that $2r^2 = 1/(B - t)$, so now r increases as t starts at 0 and increases.

36. **(a)** Again, the change of variables is essentially the same as in Example 6 of Section 7.2.

(b) If $\varepsilon = -a^2$ then the equation $r' = -r(a^2 + r^2)$ integrates to give the equation

$$t + C = -\frac{\ln r}{a^2} + \frac{\ln(a^2 + r^2)}{2a^2}$$

that (after exponentiating) we readily solve for

$$r^2 = \frac{a^2 \exp(-2ta^2 - 2Ca^2)}{1 - \exp(-2ta^2 - 2Ca^2)}.$$

This makes it clear that $r \to 0$ as $t \to +\infty$, so the origin is an asymptotically stable spiral point in this case.

(c) If $\varepsilon = a^2$ then the equation $r' = r(a^2 - r^2)$ integrates to give the equation

$$t+C = \frac{2\ln r - \ln(a-r) - \ln(a+r)}{2a^2}$$

that (after exponentiating) we solve for

$$r^2 = \frac{a^2}{1+\exp(-2ta^2 - 2Ca^2)}.$$

It therefore follows that $r \to a$ as $t \to +\infty$.

37. The substitution $y = vx$ in the homogeneous first-order equation

$$\frac{dy}{dx} = \frac{y(2x^3 - y^3)}{x(x^3 - 2y^3)}$$

yields

$$x\frac{dv}{dx} = -\frac{v^4 + v}{2v^3 - 1}.$$

Separating the variables and integrating by partial fractions, we get

$$\int\left(-\frac{1}{v} + \frac{1}{v+1} + \frac{2v-1}{v^2 - v + 1}\right)dv = -\int\frac{dx}{x}$$

$$\ln((v+1)(v^2 - v + 1)) = \ln v - \ln x + \ln C$$

$$(v+1)(v^2 - v + 1) = \frac{Cv}{x}$$

$$v^3 + 1 = \frac{Cv}{x}.$$

Finally, the replacement $v = y/x$ yields $x^3 + y^3 = Cxy$.

38. The roots of the characteristic equation $\lambda^2 - T\lambda + D = 0$ are given by

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

We examine the various possibilities individually.

- If the point (T, D) lies above the parabola $T^2 = 4D$ in the trace-determinant plane but off the D-axis, so the radicand $T^2 - 4D$ is negative, then λ_1 and λ_2 have nonzero imaginary part and nonzero real part $T/2$. Hence we have a spiral source if $T > 0$, a spiral sink if $T < 0$.

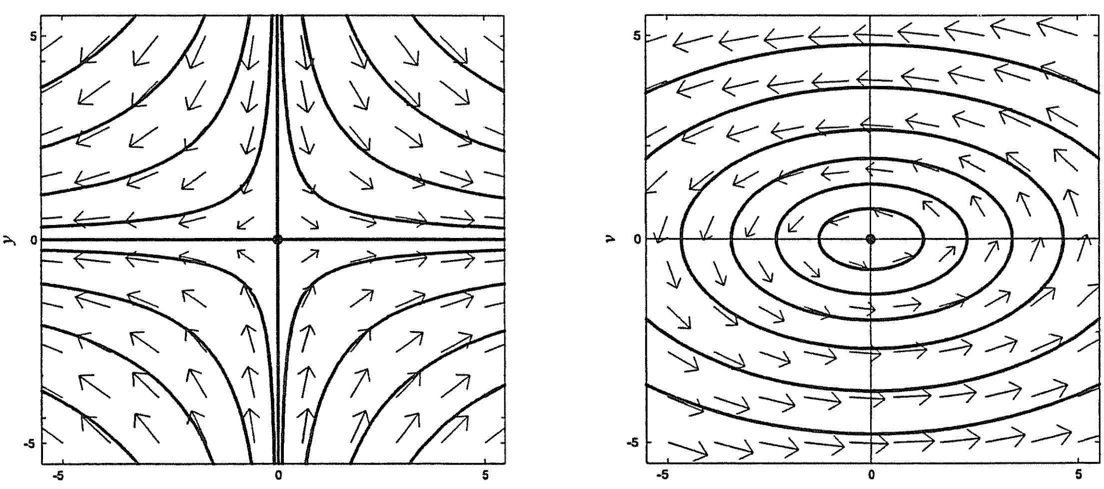
- If the point (T, D) lies on the positive D-axis, so $T = 0$ but $D > 0$, then $\lambda_1 = \lambda_2 = i\sqrt{D}$, pure imaginary, so we have a stable center.
- If the point (T, D) lies beneath the T-axis, then $\lambda_1, \lambda_2 = \frac{1}{2}\left(T \pm \sqrt{T^2 + 4|D|}\right)$ because $D < 0$. It follows that λ_1 and λ_2 are real with different signs, so we have a saddle point.
- If the point (T, D) lies between the T-axis and the parabola $T^2 = 4D$, then the radicand $T^2 - 4D$ is positive but less than T^2. It follows that λ_1 and λ_2 are real and both have the same sign as T, so we have a nodal source if $T > 0$, a nodal sink if $T < 0$.

SECTION 7.4

ECOLOGICAL APPLICATIONS: PREDATORS AND COMPETITORS

1. $\mathbf{J} = \begin{bmatrix} 200 - 4y & -4x \\ 2y & -150 + 2x \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 200 & 0 \\ 0 & -150 \end{bmatrix}$ has characteristic equation $(200 - \lambda)(-150 - \lambda) = 0$ and real eigenvalues $\lambda_1 = -150$, $\lambda_2 = 200$ with different signs. Hence $(0,0)$ is a saddle point of the linearized system $x' = 200x$, $y' = -150y$. See the left-hand figure below.



At $(75,50)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & -300 \\ 100 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 30000 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 100i\sqrt{3}$. Hence $(75,50)$ is a stable center of the linearization $u' = -300v,\ v' = 100u$. See the right-hand figure at the bottom of the preceding page.

2. Upon separation of variables, the equation

$$\frac{dy}{dx} = \frac{-150y + 2xy}{200x - 4xy} = \frac{y(-150 + 2x)}{x(200 - 4y)}$$

yields

$$\int \left(\frac{200}{y} - 4 \right) dy = \int \left(2 - \frac{150}{x} \right) dx,$$

$$200 \ln y - 4y = 2x - 150 \ln x + C$$

assuming that $x, y > 0$.

3. The effect of using the insecticide is to replace b by $b + f$ and a by $a - f$ in the predator-prey equations, while leaving p and q unchanged. Hence the new harmful population is $(b + f)/q > b/q = x_E$, and the new benign population is $(a - f)/p < a/p = y_E$.

Problems 4–7 deal with the competition system

$$x' = 60x - 4x^2 - 3xy, \qquad y' = 42y - 2y^2 - 3xy \tag{2}$$

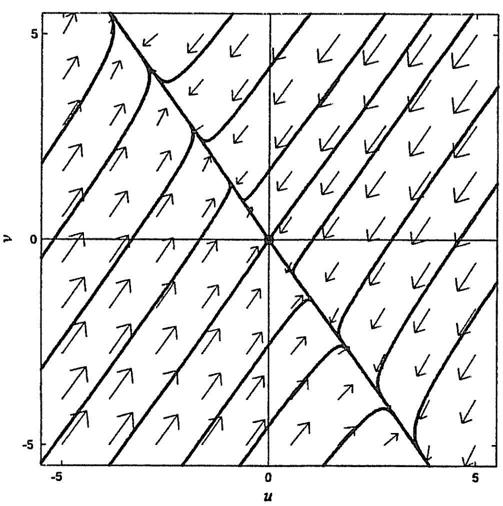
that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 60 - 8x - 3y & -3x \\ -3y & 42 - 4y - 3x \end{bmatrix}$.

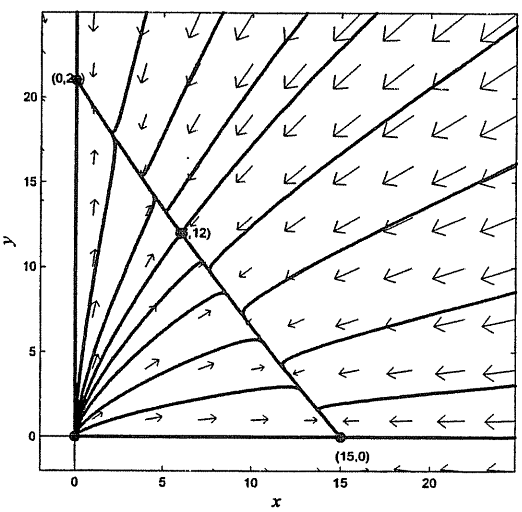
4. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 60 & 0 \\ 0 & 42 \end{bmatrix}$ has characteristic equation $(60 - \lambda)(42 - \lambda) = 0$ and positive real eigenvalues $\lambda_1 = 42,\ \lambda_2 = 60$. Hence $(0,0)$ is a nodal source of the linearized system $x' = 60x,\ y' = 42y$.

5. At $(0,21)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & 0 \\ -63 & -42 \end{bmatrix}$ has characteristic equation $(-3 - \lambda)(-42 - \lambda) = 0$ and negative real eigenvalues $\lambda_1 = -42,\ \lambda_2 = -3$. Hence $(0,21)$ is a nodal sink of the linearized system $u' = -3u,\ v' = -63u - 42v$.

6. At $(15,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -60 & -45 \\ 0 & -3 \end{bmatrix}$ has characteristic equation $(-60-\lambda)(-3-\lambda)=0$ and negative real eigenvalues $\lambda_1 = -60,\ \lambda_2 = -3$. Hence $(15,0)$ is a nodal sink of the linearized system $u' = -60u - 45v,\ v' = -3v$.

7. At $(6,12)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -24 & -18 \\ -36 & -24 \end{bmatrix}$ has characteristic equation $(-24-\lambda)^2 - (-36)(-18) = 0$ and real eigenvalues $\lambda_1 = -24 + 18\sqrt{2} > 0$, $\lambda_2 = -24 - 18\sqrt{2} < 0$ with different signs. Hence $(6,12)$ is a saddle point of the linearized system $u' = -24u - 18v,\ v' = -36u - 24v$. The figure on the left below illustrates this saddle point. The figure on the right shows all four critical points of the system.





Problems 8–10 deal with the competition system

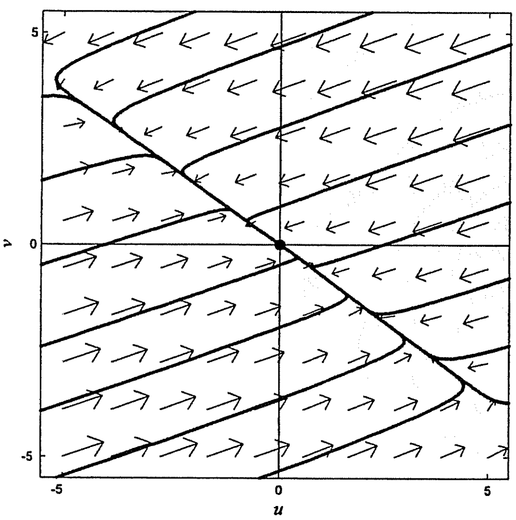
$$x' = 60x - 3x^2 - 4xy, \quad y' = 42y - 3y^2 - 2xy \tag{3}$$

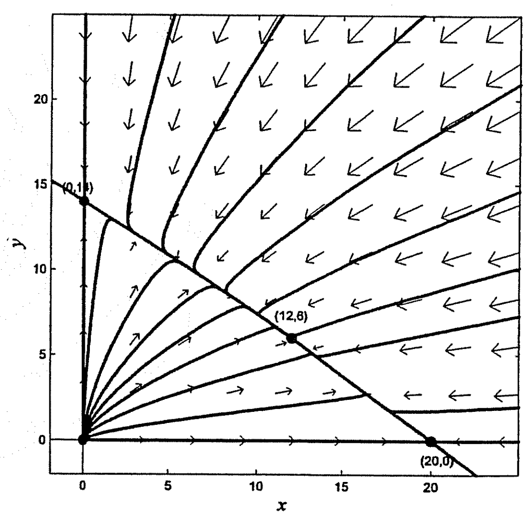
that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 60-6x-4y & -4x \\ -2y & 42-6y-2x \end{bmatrix}$.

8. At $(0,14)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 4 & 0 \\ -28 & -42 \end{bmatrix}$ has characteristic equation $(4-\lambda)(-42-\lambda)=0$ and real eigenvalues $\lambda_1 = -42,\ \lambda_2 = 4$ with different signs. Hence $(0,14)$ is a saddle point of the linearized system $u' = 4u,\ v' = -28u - 42v$.

9. At $(20,0)$ the Jacobian matrix $\mathbf{J}=\begin{bmatrix}-60 & -80\\ 0 & 2\end{bmatrix}$ has characteristic equation $(-60-\lambda)(2-\lambda)=0$ and real eigenvalues $\lambda_1=-60,\ \lambda_2=2$ with different signs. Hence $(20,0)$ is a saddle point of the linearized system $u'=-60u-80v,\ v'=2v$.

10. At $(12,6)$ the Jacobian matrix $\mathbf{J}=\begin{bmatrix}-36 & -48\\ -12 & -18\end{bmatrix}$ has characteristic equation $(-36-\lambda)(-18-\lambda)-(-12)(-48)=0$ and negative real eigenvalues $\lambda_1,\lambda_2=-27\pm 3\sqrt{73}$. Hence $(12,6)$ is a nodal sink of the linearized system $u'=-36u-48v,\ v'=-12u-18v$. The figure on the left below illustrates this sink. The figure on the right shows all four critical points of the system.





Problems 11–13 deal with the predator-prey system

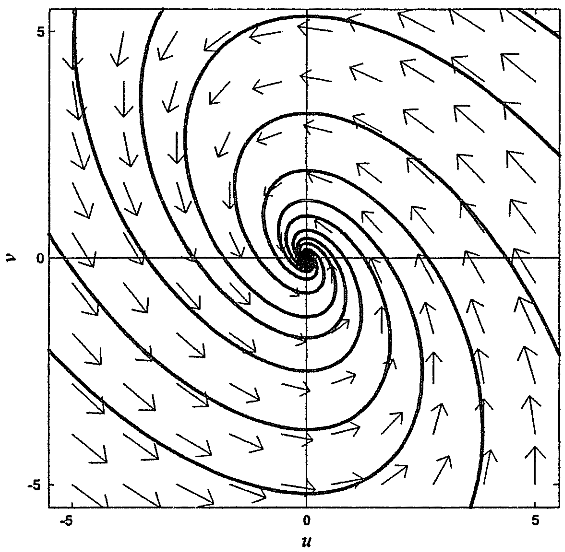
$$x' = 5x - x^2 - xy, \qquad y' = -2y + xy \tag{4}$$

that has Jacobian matrix $\mathbf{J}=\begin{bmatrix}5-2x-y & -x\\ y & -2+x\end{bmatrix}$.

11. At $(0,0)$ the Jacobian matrix $\mathbf{J}=\begin{bmatrix}5 & 0\\ 0 & -2\end{bmatrix}$ has characteristic equation $(5-\lambda)(-2-\lambda)=0$ and real eigenvalues $\lambda_1=-2,\ \lambda_2=5$ with different signs. Hence $(0,0)$ is a saddle point of the linearized system $x'=5x,\ y'=-2y$.

12. At $(5,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -5 & -5 \\ 0 & 3 \end{bmatrix}$ has characteristic equation $(-5-\lambda)(3-\lambda) = 0$ and real eigenvalues $\lambda_1 = -5,\ \lambda_2 = 3$ with different signs. Hence $(5,0)$ is a saddle point of the linearized system $u' = -5u - 5v,\ v' = 3v$.

13. At $(2,3)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -2 & -2 \\ 3 & 0 \end{bmatrix}$ has characteristic equation $(-2-\lambda)(-\lambda) - (3)(-2) = \lambda^2 + 2\lambda + 6 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm i\sqrt{5}$ with negative real part. Hence $(2,3)$ is a spiral sink of the linearized system $u' = -2u - 2v,\ v' = 3u$ (illustrated below).



Problems 14–17 deal with the predator-prey system

$$x' = x^2 - 2x - xy, \quad y' = y^2 - 4y + xy \tag{5}$$

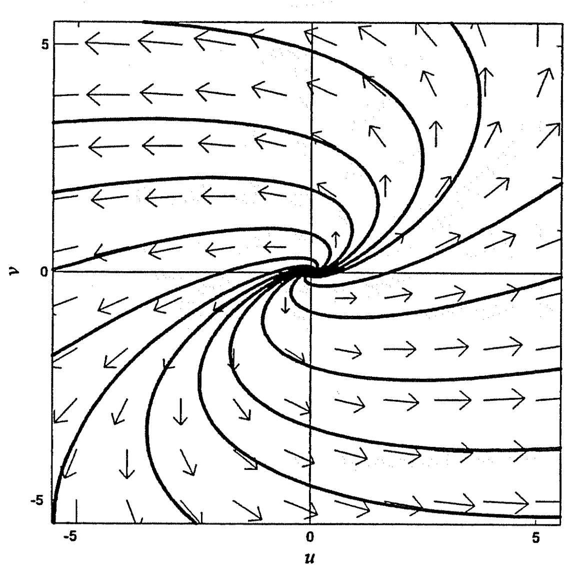
that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2x-2-y & -x \\ y & 2y-4+x \end{bmatrix}$.

14. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$ has characteristic equation $(-2-\lambda)(-4-\lambda) = 0$ and negative real eigenvalues $\lambda_1 = -4,\ \lambda_2 = -2$. Hence $(0,0)$ is a nodal sink of the linearized system $x' = -2x,\ y' = -4y$.

15. At $(0,4)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix}$ has characteristic equation $(-6-\lambda)(4-\lambda)=0$ and real eigenvalues $\lambda_1 = -6,\ \lambda_2 = 4$ with different signs. Hence $(0,4)$ is a saddle point of the linearized system $u' = -6u,\ v' = 4u+4v$.

16. At $(2,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix}$ has characteristic equation $(2-\lambda)(-2-\lambda)=0$ and real eigenvalues $\lambda_1 = -2,\ \lambda_2 = 2$ with different signs. Hence $(2,0)$ is a saddle point of the linearized system $u' = 2u-2v,\ v' = -2v$.

17. At $(3,1)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$ has characteristic equation $(3-\lambda)(1-\lambda)-(1)(-3) = \lambda^2 - 4\lambda + 6 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = 2 \pm i\sqrt{2}$ with positive real part. Hence $(3,1)$ is a spiral source of the linearized system $u' = 3u-3v,\ v' = u+v$ (illustrated below).



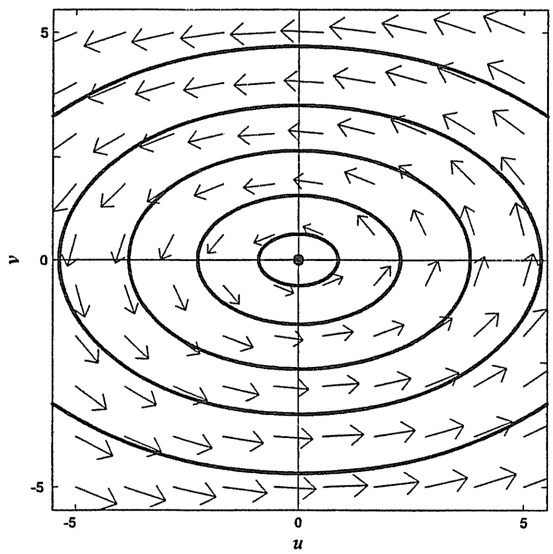
Problems 18 and 19 deal with the predator-prey system

$$x' = 2x - xy, \qquad y' = -5y + xy \tag{7}$$

that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2-y & -x \\ y & -5+x \end{bmatrix}$.

18. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ has characteristic equation $(2-\lambda)(-5-\lambda)=0$ and real eigenvalues $\lambda_1 = -5,\ \lambda_2 = 2$ with different signs. Hence $(0,0)$ is a saddle point of the linearized system $x' = 2x,\ y' = -5y$.

19. At $(5,2)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & -5 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $(-\lambda)(-\lambda)-(2)(-5) = \lambda^2+10=0$ and pure imaginary roots $\lambda = \pm i\sqrt{10}$, so the origin is a stable center for the linearized system $u' = -5v,\ v' = 2u$. This is the indeterminate case, but the figure below suggests that $(5,2)$ is also a stable center for the original system in (7).



Problems 20–22 deal with the predator-prey system

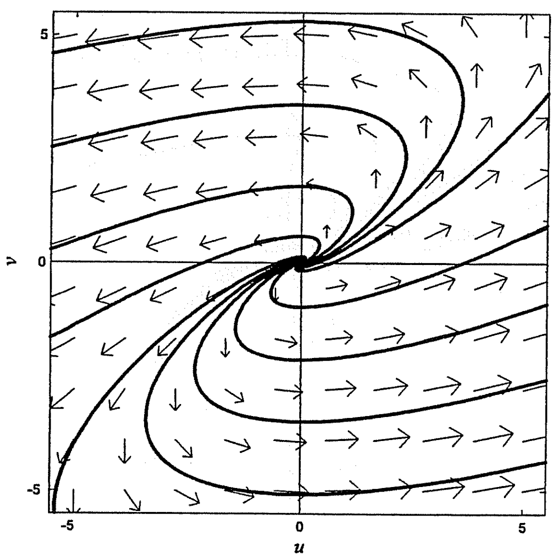
$$x' = -3x + x^2 - xy, \qquad y' = -5y + xy \tag{8}$$

that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3+2x-y & -x \\ y & -5+x \end{bmatrix}$.

20. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & 0 \\ 0 & -5 \end{bmatrix}$ has characteristic equation $(-3-\lambda)(-5-\lambda)=0$ and negative real eigenvalues $\lambda_1 = -5,\ \lambda_2 = -3$. Hence $(0,0)$ is a nodal sink of the linearized system $x' = -3x,\ y' = -5y$.

21. At $(3,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & -3 \\ 0 & -2 \end{bmatrix}$ has characteristic equation $(3-\lambda)(-2-\lambda) = 0$ and real eigenvalues $\lambda_1 = -2,\ \lambda_2 = 3$ with different signs. Hence $(3,0)$ is a saddle point of the linearized system $u' = 3u - 3v,\ v' = -2v$.

22. At $(5,2)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 5 & -5 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $(5-\lambda)(-\lambda) - (2)(-5) = \lambda^2 - 5\lambda + 10 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = \frac{1}{2}\left(5 \pm i\sqrt{15}\right)$ with positive real part. Hence $(5,2)$ is a spiral source of the linearized system $u' = 5u - 5v,\ v' = 2u$ (illustrated below).



Problems 23–25 deal with the predator-prey system

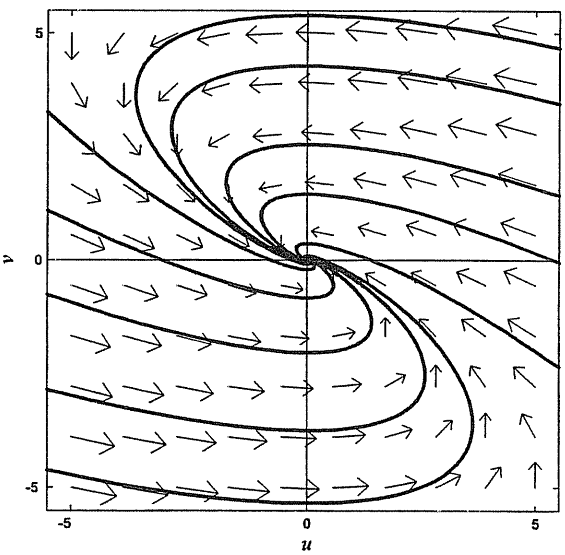
$$x' = 7x - x^2 - xy, \quad y' = -5y + xy \tag{9}$$

that has Jacobian matrix $\mathbf{J} = \begin{bmatrix} 7-2x-y & -x \\ y & -5+x \end{bmatrix}$.

23. At $(0,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} 7 & 0 \\ 0 & -5 \end{bmatrix}$ has characteristic equation $(7-\lambda)(-5-\lambda) = 0$ and real eigenvalues $\lambda_1 = -5,\ \lambda_2 = 7$ with different signs. Hence $(0,0)$ is a saddle point of the linearized system $x' = 7x,\ y' = -5y$.

24. At $(7,0)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -7 & -7 \\ 0 & 2 \end{bmatrix}$ has characteristic equation $(-7-\lambda)(2-\lambda) = 0$ and real eigenvalues $\lambda_1 = -7,\ \lambda_2 = 2$ with different signs. Hence $(7,0)$ is a saddle point of the linearized system $u' = -7u - 7v,\ v' = 2v$.

25. At $(5,2)$ the Jacobian matrix $\mathbf{J} = \begin{bmatrix} -5 & -5 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $(-5-\lambda)(-\lambda) - (2)(-5) = \lambda^2 + 5\lambda + 10 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = \frac{1}{2}\left(-5 \pm i\sqrt{15}\right)$ with negative real part. Hence $(5,2)$ is a spiral sink of the linearized system $u' = -5u - 5v,\ v' = 2u$ (illustrated below).

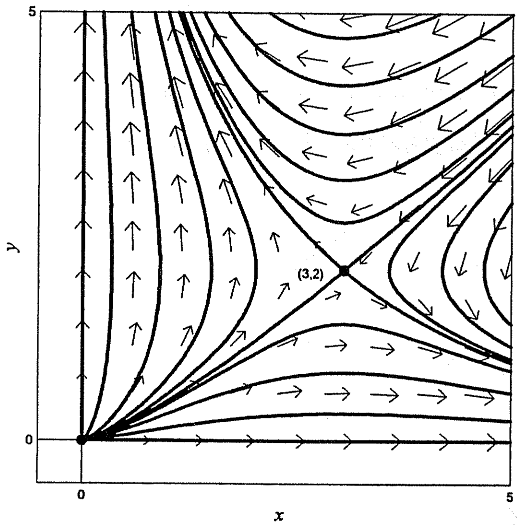


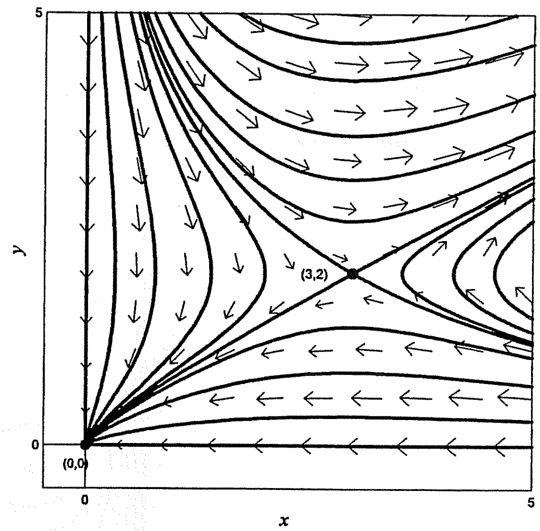
26. $\mathbf{J} = \begin{bmatrix} 2-y & -x \\ -y & 3-x \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ has characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ and positive real eigenvalues $\lambda_1 = 2,\ \lambda_2 = 3$. Hence $(0,0)$ is a nodal source.

At $(3,2)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & -3 \\ -2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 6 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{6}$ with different signs. Hence $(3,2)$ is a saddle point.

If the initial point (x_0, y_0) lies above the southwest-northeast separatrix through $(3,2)$, then $(x(t), y(t)) \to (0, \infty)$ as $t \to \infty$. But if (x_0, y_0) lies below this separatrix, then $(x(t), y(t)) \to (\infty, 0)$ as $t \to \infty$. See the left-hand figure below.





27. $\mathbf{J} = \begin{bmatrix} 2y-4 & 2x \\ y & x-3 \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}$ has characteristic equation $\lambda^2 + 7\lambda + 12 = 0$ and negative real eigenvalues $\lambda_1 = -4,\ \lambda_2 = -3$. Hence $(0,0)$ is a nodal sink.

At $(3,2)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 6 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 12 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm 2\sqrt{3}$ with different signs. Hence $(3,2)$ is a saddle point.

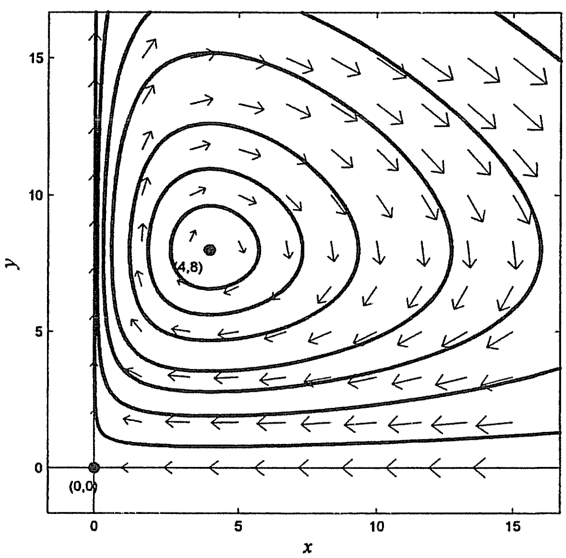
If the initial point (x_0, y_0) lies below the northwest-southeast separatrix through $(3,2)$, then $(x(t), y(t)) \to (0,0)$ as $t \to \infty$. But if (x_0, y_0) lies above this separatrix, then $(x(t), y(t)) \to (\infty, \infty)$ as $t \to \infty$. See the right-hand figure above.

28. $\mathbf{J} = \begin{bmatrix} 2y-16 & 2x \\ -y & 4-x \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -16 & 0 \\ 0 & 4 \end{bmatrix}$ has characteristic equation $\lambda^2 + 12\lambda - 64 = 0$ and real eigenvalues $\lambda_1 = -16,\ \lambda_2 = 4$ with opposite signs. Hence $(0,0)$ is a saddle point.

At $(4,8)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 64 = 0$ and conjugate imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 8i$. This is the indeterminate case, but the figure in the answers section of the textbook indicates that $(4,8)$ is a stable center for the original nonlinear system.

As $t \to \infty$, each solution point $(x(t), y(t))$ with nonzero initial conditions encircles the stable center $(4,8)$ periodically in a clockwise direction. See the figure below.



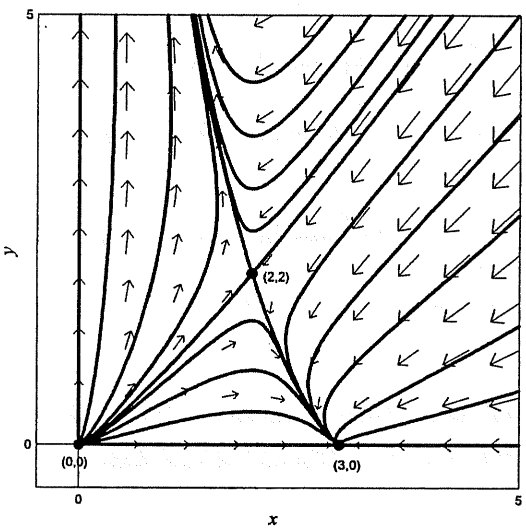
29. $\mathbf{J} = \begin{bmatrix} -2x - \frac{1}{2}y + 3 & -\frac{1}{2}x \\ -2y & 4 - 2x \end{bmatrix}$

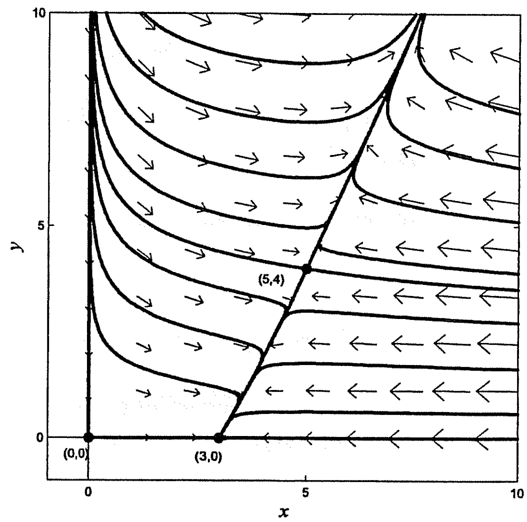
At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ has characteristic equation $\lambda^2 - 7\lambda + 12 = 0$ and positive real eigenvalues $\lambda_1 = 3,\ \lambda_2 = 4$. Hence $(0,0)$ is a nodal source.

At $(3,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & -\frac{3}{2} \\ 0 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5\lambda + 6 = 0$ and negative real eigenvalues $\lambda_1 = -3,\ \lambda_2 = -2$. Hence $(3,0)$ is a nodal sink.

At $(2,2)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -2 & -1 \\ -4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda - 4 = 0$ and real eigenvalues $\lambda_1 \approx -3.2361, \lambda_2 = 1.2361$ with different signs. Hence $(2,2)$ is a saddle point.

If the initial point (x_0, y_0) lies above the southwest-northeast separatrix through $(2,2)$, then $(x(t), y(t)) \to (0, \infty)$ as $t \to \infty$. But if (x_0, y_0) lies below this separatrix, then $(x(t), y(t)) \to (3,0)$ as $t \to \infty$. See the left-hand figure below.





30. $\mathbf{J} = \begin{bmatrix} -2x + \frac{1}{2}y + 3 & \frac{1}{2}x \\ \frac{1}{5}y & \frac{1}{5}x - 1 \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 - 2\lambda - 3 = 0$ and real eigenvalues $\lambda_1 = -1,\ \lambda_2 = 3$ of opposite sign. Hence $(0,0)$ is a saddle point.

At $(3,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & -\frac{3}{2} \\ 0 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + \frac{17}{5}\lambda + \frac{17}{5} = 0$ and negative real eigenvalues $\lambda_1 = -3,\ \lambda_2 = -\frac{17}{5}$. Hence $(3,0)$ is a nodal sink.

At $(5,4)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -5 & \frac{5}{2} \\ \frac{4}{5} & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5\lambda - 2 = 0$ and real eigenvalues $\lambda_1 \approx -5.3723, \lambda_2 = 0.3723$ with different signs. Hence $(5,4)$ is a saddle point.

If the initial point (x_0, y_0) lies above the northwest-southeast separatrix through $(5,4)$, then $(x(t), y(t)) \to (\infty, \infty)$ as $t \to \infty$. But if (x_0, y_0) lies below this separatrix, then $(x(t), y(t)) \to (3,0)$ as $t \to \infty$. See the right-hand figure above.

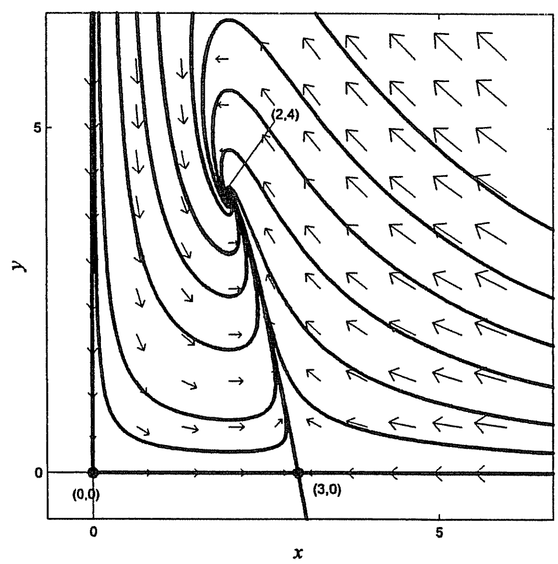
31. $\mathbf{J} = \begin{bmatrix} -2x - \frac{1}{4}y + 3 & -\frac{1}{4}x \\ y & x-2 \end{bmatrix}$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 - \lambda - 6 = 0$ and real eigenvalues $\lambda_1 = -2,\ \lambda_2 = 3$ of opposite sign. Hence $(0,0)$ is a saddle point.

At $(3,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -3 & -\frac{3}{4} \\ 0 & 1 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda - 3 = 0$ and real eigenvalues $\lambda_1 = -3,\ \lambda_2 = 1$ of opposite sign. Hence $(3,0)$ is a saddle point.

At $(2,4)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -2 & -\frac{1}{2} \\ 4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 2 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm i$ with negative real part. Hence $(2,4)$ is a spiral sink.

As $t \to \infty$, each solution point $(x(t), y(t))$ with nonzero initial conditions approaches the spiral sink $(2,4)$, as indicated by the direction arrows in the figure below.



32. $\mathbf{J} = \begin{bmatrix} -6x + y + 30 & x \\ 4y & 4x - 6y + 60 \end{bmatrix}$

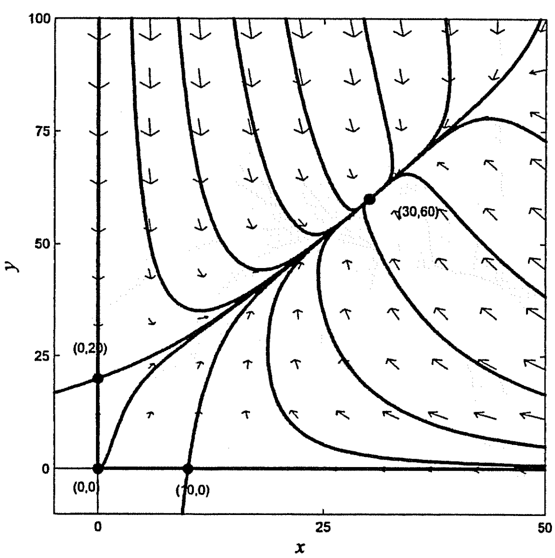
At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 30 & 0 \\ 0 & 60 \end{bmatrix}$ has characteristic equation $\lambda^2 - 90\lambda + 1800 = 0$ and positive real eigenvalues $\lambda_1 = 30,\ \lambda_2 = 60$. Hence $(0,0)$ is a nodal source.

At $(0,20)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 50 & 0 \\ 80 & -60 \end{bmatrix}$ has characteristic equation $\lambda^2 + 10\lambda - 3000 = 0$ and real eigenvalues $\lambda_1 = -60,\ \lambda_2 = 50$ of opposite sign. Hence $(0,20)$ is a saddle point.

At $(10,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -30 & 10 \\ 0 & 100 \end{bmatrix}$ has characteristic equation $\lambda^2 - 70\lambda - 3000 = 0$ and real eigenvalues $\lambda_1 = -30,\ \lambda_2 = 100$ of opposite sign. Hence $(10,0)$ is a saddle point.

At $(30,60)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -90 & 30 \\ 240 & -180 \end{bmatrix}$ has characteristic equation $\lambda^2 + 240\lambda + 9000 = 0$ and negative real eigenvalues $\lambda_1 \approx -231.05, \lambda_2 = -38.95$. Hence $(30,60)$ is a nodal sink.

As $t \to \infty$, each solution point $(x(t), y(t))$ with nonzero initial conditions approaches the nodal sink $(30,60)$. See the figure below.



33. $\mathbf{J} = \begin{bmatrix} -6x + y + 30 & x \\ 4y & 4x - 6y + 60 \end{bmatrix}$

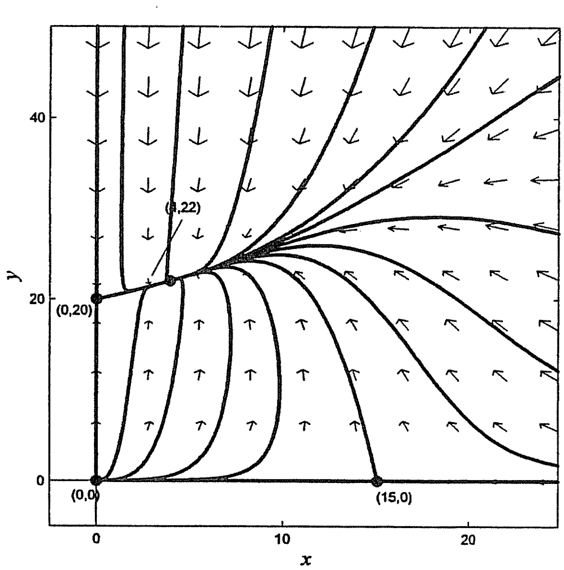
At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 30 & 0 \\ 0 & 80 \end{bmatrix}$ has characteristic equation $\lambda^2 - 110\lambda + 2400 = 0$ and positive real eigenvalues $\lambda_1 = 30,\ \lambda_2 = 80$. Hence $(0,0)$ is a nodal source.

At $(0,20)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}10 & 0\\ 40 & -80\end{bmatrix}$ has characteristic equation $\lambda^2+70\lambda-800=0$ and real eigenvalues $\lambda_1=-80,\ \lambda_2=10$ of opposite sign. Hence $(0,20)$ is a saddle point.

At $(15,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}-30 & 15\\ 0 & 110\end{bmatrix}$ has characteristic equation $\lambda^2-80\lambda-3300=0$ and real eigenvalues $\lambda_1=-30,\ \lambda_2=110$ of opposite sign. Hence $(15,0)$ is a saddle point.

At $(4,22)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}-8 & -4\\ 44 & -88\end{bmatrix}$ has characteristic equation $\lambda^2+96\lambda+880=0$ and negative real eigenvalues $\lambda_1\approx-85.736, \lambda_2=-10.264$. Hence $(4,22)$ is a nodal sink.

As $t\to\infty$, each solution point $(x(t),y(t))$ with nonzero initial conditions approaches the nodal sink $(4,22)$. See the figure below.



34. $\mathbf{J}=\begin{bmatrix}-4x-y+30 & -x\\ 2y & 2x-8y+20\end{bmatrix}$

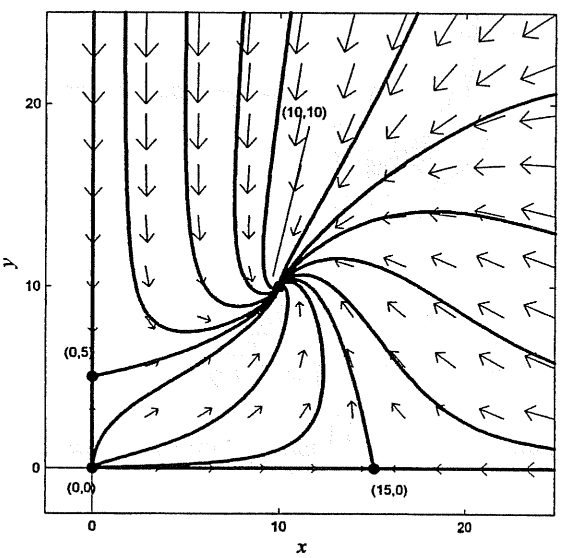
At $(0,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}30 & 0\\ 0 & 20\end{bmatrix}$ has characteristic equation $\lambda^2-50\lambda+600=0$ and positive real eigenvalues $\lambda_1=20,\ \lambda_2=30$. Hence $(0,0)$ is a nodal source.

At $(0,5)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}25 & 0\\ 10 & -20\end{bmatrix}$ has characteristic equation $\lambda^2-5\lambda-500=0$ and real eigenvalues $\lambda_1=-20,\ \lambda_2=25$ of opposite sign. Hence $(0,5)$ is a saddle point.

At $(15,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}-30 & -15\\ 0 & 50\end{bmatrix}$ has characteristic equation $\lambda^2-20\lambda-1500=0$ and real eigenvalues $\lambda_1=-30,\ \lambda_2=50$ of opposite sign. Hence $(15,0)$ is a saddle point.

At $(10,10)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}-20 & -10\\ 20 & -40\end{bmatrix}$ has characteristic equation $\lambda^2+60\lambda+1000=0$ and complex conjugate eigenvalues $\lambda_1,\lambda_2\approx-30\pm10i$ with negative real part. Hence $(10,10)$ is a spiral sink.

As $t\to\infty$, each solution point $(x(t),y(t))$ with nonzero initial conditions approaches the nodal sink $(10,10)$. See the figure below.



SECTION 7.5

NONLINEAR MECHANICAL SYSTEMS

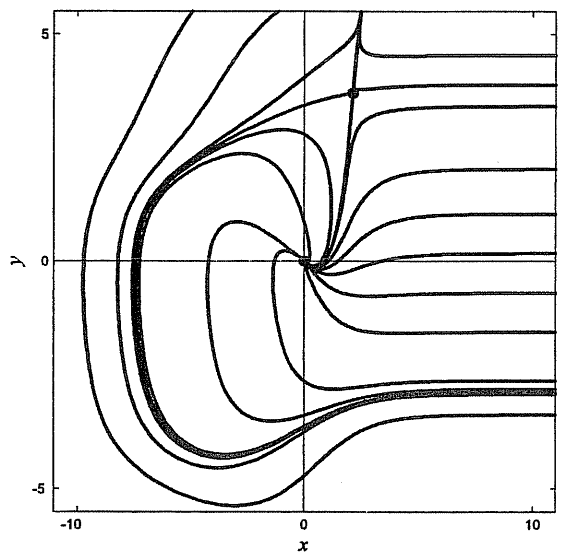
In each of Problems 1–4 we need only substitute the familiar power series for the exponential, sine, and cosine functions, and then discard all higher-order terms. For each problem we give the corresponding linear system, the eigenvalues λ_1 and λ_2, and the type of this critical point.

1. $x' = 1-\left(1+x+\frac{1}{2}x^2+\cdots\right)+2y \approx -x+2y$

$y' = -x-4\left(y-\frac{1}{6}y^3+\cdots\right) \approx -x-4y$

The coefficient matrix $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}$ has negative eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -3$ indicating a stable nodal sink as illustrated in the figure below. Alternatively, we can calculate the Jacobian matrix

$$\mathbf{J}(x,y) = \begin{bmatrix} -e^x & 2 \\ -1 & -4\cos x \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}.$$



2. $x' = 2\left(x-\frac{1}{6}x^3+\cdots\right)+\left(y-\frac{1}{6}y^3+\cdots\right) \approx 2x+y$

$y' = \left(x-\frac{1}{6}x^3+\cdots\right)+2\left(y-\frac{1}{6}y^3+\cdots\right) \approx x+2y$

The coefficient matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has positive eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$

indicating an unstable nodal source. Alternatively, we can calculate the Jacobian matrix

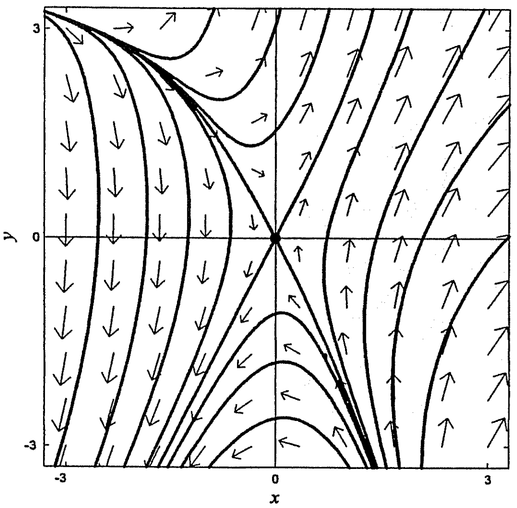
$$\mathbf{J}(x,y) = \begin{bmatrix} 2\cos x & \cos y \\ \cos x & 2\cos y \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

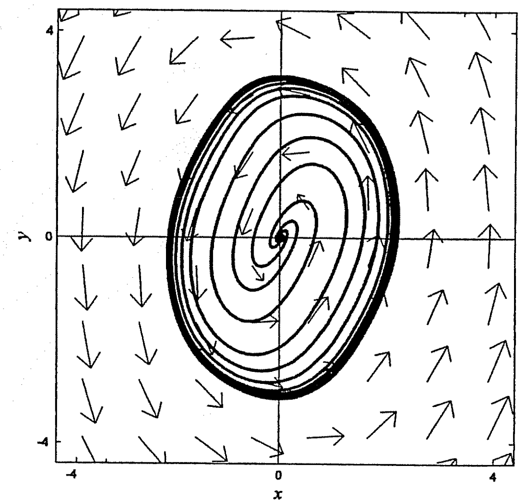
3. $x' = \left(1+x+\frac{1}{2}x^2+\cdots\right)+2y-1 \approx x+2y$

$y' = 8x+\left(1+y+\frac{1}{2}y^2+\cdots\right)-1 \approx 8x+y$

The coefficient matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$ has real eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 5$ of opposite sign, indicating an unstable saddle point as illustrated in the left-hand figure below. Alternatively, we can calculate the Jacobian matrix

$$\mathbf{J}(x,y) = \begin{bmatrix} e^x & 2 \\ 8 & e^y \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}.$$





4. The linear system is $x' = x - 2y,\ y' = 4x - 3y$ because

$$\sin x \cos y = (x - x^3/3! + \cdots)(1 - y^2/2! + \cdots) = x + \cdots,$$

and $\cos x \sin y \approx y$ similarly. The coefficient matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$ has complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm 2i$ with negative real part, indicating a stable spiral point as illustrated in the right-hand figure above. Alternatively, we can calculate the Jacobian matrix

$$\mathbf{J}(x,y) = \begin{bmatrix} \cos x \cos y & -\sin x \sin y - 2 \\ 3 \sin x \sin y + 4 & -3 \cos x \cos y \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}.$$

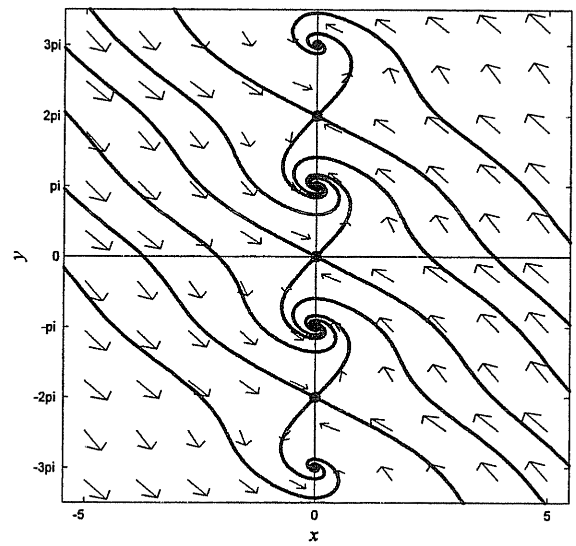
5. The critical points are of the form $(0, n\pi)$ where n is an integer, so we substitute $x = u,\ \ y = v + n\pi$. Then

$$u' = x' = -u + \sin(v + n\pi) = -u + (\cos n\pi)v = -u + (-1)^n v.$$

Hence the linearized system at $(0, n\pi)$ is

$$u' = -u \pm v, \qquad v' = 2u$$

where we take the plus sign if n is even, the minus sign if n is odd. If n is even the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, so $(0, n\pi)$ is an unstable saddle point. If n is odd the eigenvalues are $\lambda_1, \lambda_2 = (-1 \pm i\sqrt{7})/2$, so $(0, n\pi)$ is a stable spiral point.



Alternatively, we can start by calculating the Jacobian matrix $\mathbf{J}(x,y) = \begin{bmatrix} -1 & \cos y \\ 2 & 0 \end{bmatrix}$.

At $(0, n\pi)$, n even: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 2 = 0$ and real eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$ of opposite sign. Hence $(0, n\pi)$ is a saddle point if n is even, as we see in the figure above.

At $(0, n\pi)$, n odd: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + 2 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = \frac{1}{2}(-1 \pm i\sqrt{7})$ with negative real. Hence $(0, n\pi)$ is a spiral sink if n is odd, as indicated in the figure.

6. The critical points are of the form $(n, 0)$ where n is an integer, so we substitute $x = u + n$, $y = v$. Then

$$v' = y' = \sin\pi(u+n) - v = \cos n\pi \sin\pi u \approx (-1)^n \pi u - v,$$

Hence the linearized system at $(n, 0)$ is

$$u' = v, \quad v' = \pm \pi u - v$$

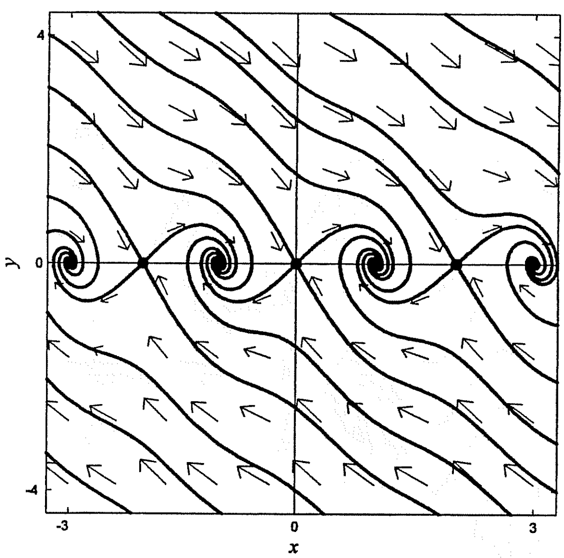
with coefficient matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ \pm\pi & -1 \end{bmatrix}$ where we take the plus sign if n is even, the minus sign if n is odd. The characteristic equation

$$\lambda^2 + \lambda - \pi = 0$$

has one positive and one negative root, so $(n, 0)$ is an unstable saddle point if n is even. The equation

$$\lambda^2 + \lambda + \pi = 0$$

has complex conjugate roots with negative real part, so $(n, 0)$ is a stable spiral point if n is odd.



Alternatively, we can start by calculating the Jacobian matrix $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ \pi\cos\pi x & -1 \end{bmatrix}$.

At $(n,0)$, n even: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ \pi & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - \pi = 0$ and real eigenvalues $\lambda_1 \approx -2.3416$, $\lambda_2 \approx 1.3416$ of opposite sign. Hence $(n,0)$ is a saddle point if n is even, as we see in the figure above.

At $(n,0)$, n odd: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -\pi & -1 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + \pi = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 \approx -0.5 \pm 1.7005i$ with negative real. Hence $(n,0)$ is a spiral sink if n is odd, as we see in the figure.

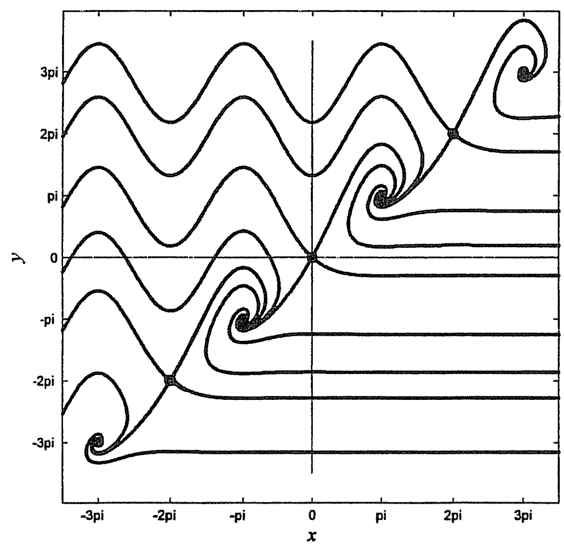
7. The critical points are of the form $(n\pi, n\pi)$ where n is an integer, so we substitute $x = u + n\pi$, $y = v + n\pi$. Then

$$u' = x' = 1 - e^{u-v} = 1 - \left(1 + (u-v) + \tfrac{1}{2}(u-v)^2 + \cdots\right) \approx -u + v,$$
$$v' = y' = 2\sin(u + n\pi) = 2\sin u \cos n\pi \approx 2(-1)^n u.$$

Hence the linearized system at $(n\pi, n\pi)$ is

$$u' = -u + v, \qquad v' = \pm 2u$$

and has coefficient matrix $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ \pm 2 & 0 \end{bmatrix}$, where we take the plus sign if n is even, the minus sign if n is odd. With n even, The characteristic equation $\lambda^2 + \lambda - 2 = 0$ has real roots $\lambda_1 = 1$ and $\lambda_2 = -2$ of opposite sign, so $(n\pi, n\pi)$ is an unstable saddle point. With n odd, the characteristic equation $\lambda^2 + \lambda + 2 = 0$ has complex conjugate eigenvalues are $\lambda_1, \lambda_2 = (-1 \pm i\sqrt{7})/2$ with negative real part, so $(n\pi, n\pi)$ is a stable spiral point.



Alternatively, we can start by calculating the Jacobian matrix $\mathbf{J}(x,y) = \begin{bmatrix} -e^{x-y} & e^{x-y} \\ 2\cos x & 0 \end{bmatrix}$.

At $(n\pi, n\pi)$, n even: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda - 2 = 0$ and real eigenvalues $\lambda_1 = -2$, $\lambda_2 = 1$ of opposite sign. Hence $(n\pi, n\pi)$ is a saddle point if n is even, as we see in the figure above.

At $(n\pi, n\pi)$, n odd: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + \lambda + 2 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 \approx -0.5 \pm 1.3229i$ with negative real. Hence $(n\pi, n\pi)$ is a spiral sink if n is odd, as we see in the figure.

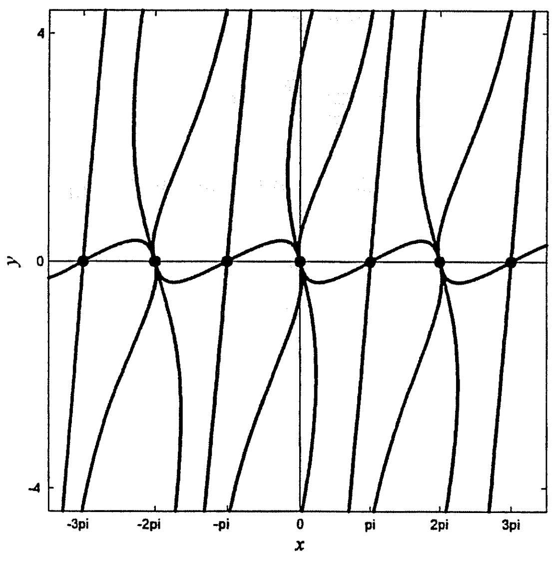
8. The critical points are of the form $(n\pi, 0)$ where n is an integer, so we substitute $x = u + n\pi$, $y = v$. Then

$$\begin{aligned} u' &= x' = 3\sin(u+n\pi)+v = 3\sin u \cos n\pi + v \approx 3(-1)^n u + v, \\ v' &= y' = \sin(u+n\pi)+2v = \sin u \cos n\pi + 2v \approx (-1)^n u + 2v, \end{aligned}$$

Hence the linearized system at $(n\pi, 0)$ is

$$u' = \pm 3u + v, \qquad v' = \pm u + 2v$$

with coefficient matrix $\mathbf{A} = \begin{bmatrix} \pm 3 & 1 \\ \pm 1 & 2 \end{bmatrix}$, where we take the plus signs if n is even, the minus signs if n is odd. If n is even then the characteristic equation $\lambda^2 - 5\lambda + 5 = 0$ has roots $\lambda_1, \lambda_2 = (5 \pm \sqrt{5})/2$ that are both positive, so $(n\pi, 0)$ is an unstable nodal source. If n is odd then the characteristic equation $\lambda^2 - 5\lambda + 5 = 0$ has real roots $\lambda_1, \lambda_2 = (-1 \pm \sqrt{21})/2$ with opposite signs, so $(n\pi, 0)$ is an unstable saddle point.



Alternatively, we can start by calculating the Jacobian matrix $\mathbf{J}(x,y) = \begin{bmatrix} 3\cos x & 1 \\ \cos x & 2 \end{bmatrix}$.

At $(n\pi,0)$, n even: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}$ has characteristic equation $\lambda^2-5\lambda+5=0$ and positive real eigenvalues $\lambda_1\approx 1.3812$, $\lambda_2=2.6180$. Hence $(n\pi,0)$ is a nodal source if n is even, as we see in the figure on the preceding page.

At $(n\pi,0)$, n odd: The Jacobian matrix $\mathbf{J}=\begin{bmatrix}-3 & 1\\ -1 & 2\end{bmatrix}$ has characteristic equation $\lambda^2+\lambda-5=0$ and real eigenvalues $\lambda_1\approx -2.7913$, $\lambda_2=1.7913$ of opposite sign. Hence $(n\pi,0)$ is a saddle point if n is odd, as we see in the figure.

As preparation for Problems 9–11, we first calculate the Jacobian matrix

$$\mathbf{J}(x,y)=\begin{bmatrix}0 & 1\\ -\omega^2\cos x & -c\end{bmatrix},$$

of the damped pendulum system in (34) in the text. At the critical point $(n\pi,0)$ we have

$$\mathbf{J}(n\pi,0) = \begin{bmatrix}0 & 1\\ -\omega^2\cos n\pi & -c\end{bmatrix} = \begin{bmatrix}0 & 1\\ \pm\omega^2 & -c\end{bmatrix},$$

where we take the plus sign if n is odd, the minus sign if n is even.

9. If n is odd then the characteristic equation $\lambda^2+c\lambda-\omega^2=0$ has real roots

$$\lambda_1,\lambda_2 = \frac{-c\pm\sqrt{c^2+4\omega^2}}{2}$$

with opposite signs, so $(n\pi, 0)$ is an unstable saddle point.

10. If n is even then the characteristic equation $\lambda^2+c\lambda+\omega^2 = 0$ has roots

$$\lambda_1,\lambda_2 = \frac{-c\pm\sqrt{c^2-4\omega^2}}{2}.$$

If $c^2>4\omega^2$ then λ_1 and λ_2 are both negative so $(n\pi, 0)$ is a stable nodal sink.

11. If n is even and $c^2<4\omega^2$ then the two eigenvalues

$$\lambda_1,\lambda_2 = \frac{-c\pm\sqrt{c^2-4\omega^2}}{2} = -\frac{c}{2}\pm\frac{i}{2}\sqrt{4\omega^2-c^2}$$

are complex conjugates with negative real part, so $(n\pi, 0)$ is a stable spiral point.

Problems 12–16 call for us to find and classify the critical points of the first order-system $x' = y,\ \ y' = -f(x,y)$ that corresponds to the given equation $x'' + f(x,x') = 0$. After finding the critical points $(x,0)$ where $f(x,0) = 0$, we first calculate the Jacobian matrix $\mathbf{J}(x,y)$.

12. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ 15x^2 - 20 & 0 \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -20 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 20 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm i\sqrt{20}$ consistent with the stable center we see at (0,0) in Fig. 7.5.4 in the textbook.

At $(\pm 2,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 40 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 40 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{40}$ of opposite sign, consistent with the saddle points we see at $(\pm 2,0)$ in Fig. 7.5.4.

13. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ 15x^2 - 20 & -2 \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -20 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 20 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm i\sqrt{19}$ consistent with the spiral node we see at (0,0) in Fig. 7.5.6 in the textbook.

At $(\pm 2,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 40 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda - 40 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = -1 \pm \sqrt{41}$ of opposite sign, consistent with the saddle points we see at $(\pm 2,0)$ in Fig. 7.5.6.

14. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ 8 - 6x^2 & 0 \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 8 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{8}$ of opposite sign, consistent with the saddle point we see at (0,0) in Fig. 7.5.12 in the textbook.

At $(\pm 2,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 16 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 4i$, consistent with the stable centers we see at $(\pm 2,0)$ in Fig. 7.5.12.

15. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ 2x-4 & 0 \end{bmatrix}$.

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 4 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 2i$, consistent with the stable center we see at (0,0) in Fig. 7.5.13 in the textbook.

At $(4,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 4 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm 2$ of opposite sign, consistent with the saddle point we see at $(4,0)$ in Fig. 7.5.13.

16. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -4+15x^2-5x^4 & 0 \end{bmatrix}$.

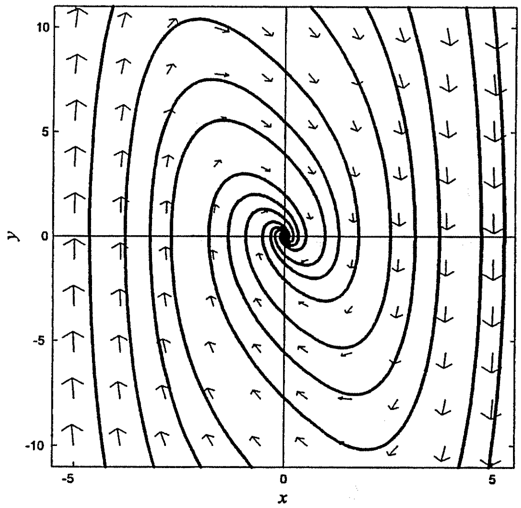
At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 4 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm 2i$, consistent with the stable center we see at (0,0) in Fig. 7.5.14 in the textbook.

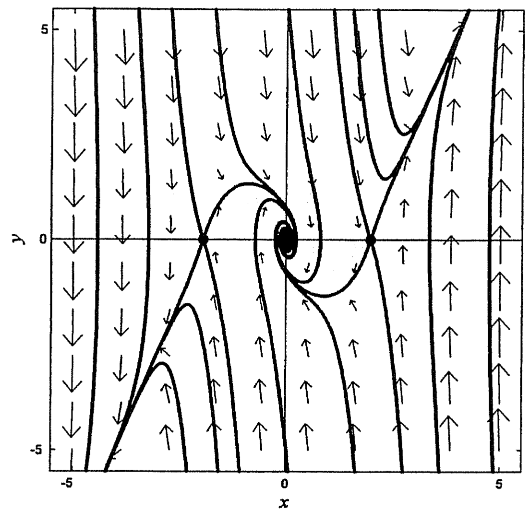
At $(\pm 1,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 6 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{6}$ of opposite sign, consistent with the saddle points we see at $(\pm 1,0)$ in Fig. 7.5.14.

At $(\pm 2,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -24 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 24 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm i\sqrt{24}$, consistent with the stable centers we see at $(\pm 2,0)$ in Fig. 7.5.14.

17. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -5-\frac{15}{4}x^2 & -2 \end{bmatrix}$.

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$ has characteristic equation $\lambda^2 + 2\lambda + 5 = 0$ and complex conjugate eigenvalues $\lambda_1, \lambda_2 = -1 \pm 2i$ with negative real part, consistent with the spiral sink we see in the left-hand figure at the top of the next page.

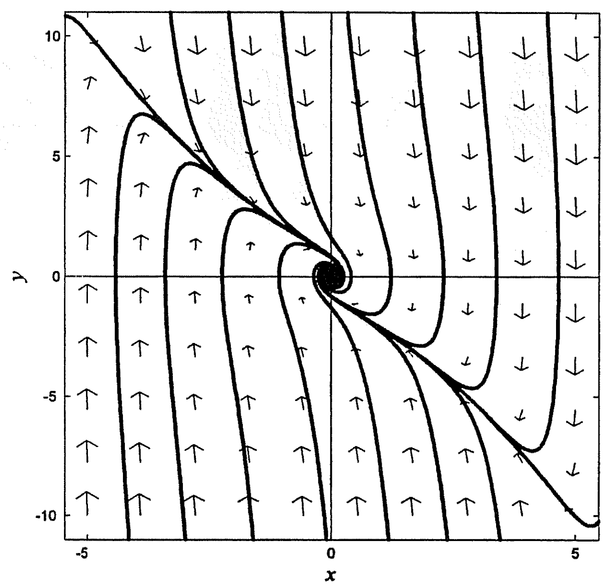




18. $\mathbf{J}(x,y) = \begin{bmatrix} 0 & 1 \\ -5 + \frac{15}{4}x^2 & -4|y| \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 + 5 = 0$ and pure imaginary eigenvalues $\lambda_1, \lambda_2 = \pm i\sqrt{5}$. This corresponds to the indeterminate case of Theorem 2 in Section 7.4, but is not inconsistent with the spiral sink we see at the origin in the figure on the right above.

At $(\pm 2,0)$: The Jacobian matrix $\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 10 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2 - 10 = 0$ and real eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{10}$, consistent with the saddle points we see at $(\pm 2, 0)$ in the right-hand figure above.



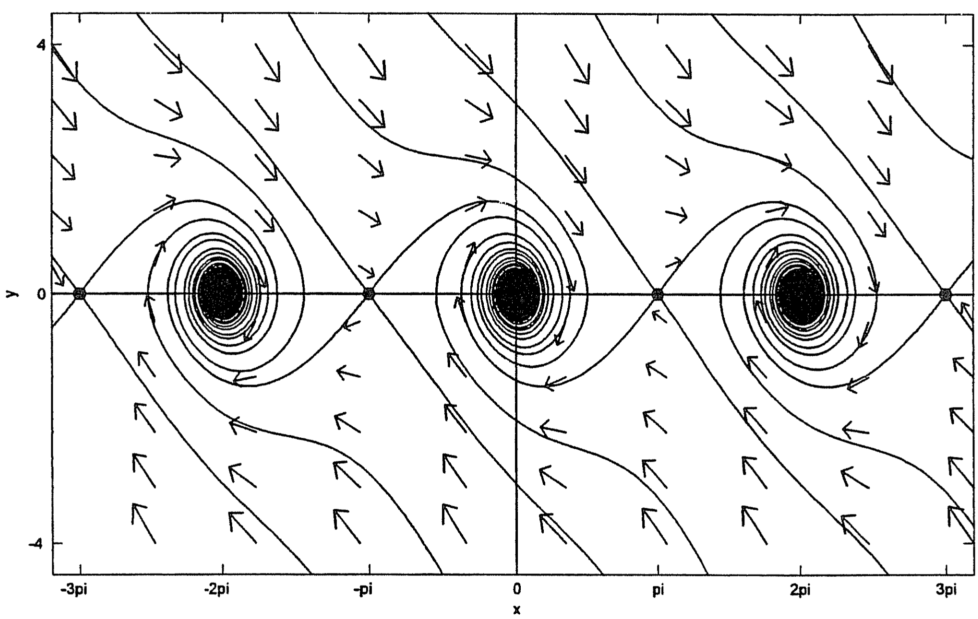
19. $\mathbf{J}(x,y)=\begin{bmatrix} 0 & 1 \\ -5-\frac{15}{4}x^2 & -4|y| \end{bmatrix}.$

At $(0,0)$: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2+5=0$ and pure imaginary eigenvalues $\lambda_1,\lambda_2=\pm i\sqrt{5}$. This corresponds to the indeterminate case of Theorem 2 in Section 7.4, but is not inconsistent with the spiral sink we see in the figure at the bottom of the preceding page.

20. $\mathbf{J}(x,y)=\begin{bmatrix} 0 & 1 \\ -\cos x & -\frac{1}{2}|y| \end{bmatrix}.$

At $(n\pi,0)$, n even: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2+1=0$ and pure imaginary eigenvalues $\lambda_1,\lambda_2=\pm i$. This corresponds to the indeterminate case of Theorem 2 in Section 7.4, but is not inconsistent with the spiral sinks we see in the figure below.

At $(n\pi,0)$, n odd: The Jacobian matrix $\mathbf{J}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has characteristic equation $\lambda^2-1=0$ and real eigenvalues $\lambda_1,\lambda_2=\pm 1$ of opposite sign, consistent with the saddle points we see in the figure.



The statements of Problems 21–26 in the text include their answers and rather fully outline their solutions, which therefore are omitted here.

SECTION 7.6

CHAOS IN DYNAMICAL SYSTEMS

We list here some programs that may be useful in the projects for this section. Further discussion of these projects can be found in the applications manual that accompanies this text.

As indicated in Fig. 7.6.1 in the text, you can use the Maple commands

```
r := 1.5:
x = array(1..200):
x[1] := 0.5:
for n from 2 to 200 do
   z := x[n-1]:
   x[n] := r*z*(1-z):
   od:
```

the Mathematica commands

```
r = 1.5;
x = Table[n,{n,1,200}];
x[[1]] = 0.5;
For[n=2, n<=200,
   n=n+1,
   z = x[[n-1]];
   x[[n]] = r*z*(1-z)];
```

or the MATLAB commands

```
r = 1.5;
x = 1:200;
x(1) = 0.5;
for n = 2:200
   z = x(n-1);
   x(n) = r*z*(1-z);
   end
```

to calculate and assemble a list of the successive iterates given by $x_{n+1} = r x_n(1-x_n)$, as illustrated in Figures 7.6.2 through 7.6.7 in the text. The following BASIC program can be used to investigate periodic cycles for this iteration.

```
100 'Program PERIODS
110 '
120 'The period-doubling iteration
130 '
140 '         x   =   rx(1 - x)
150 '
160 '   r  =  2.75  :  Period  1
170 '   r  =  3.25  :  Period  2
180 '   r  =  3.50  :  Period  4
```

```
190 '  r  =  3.55  :  Period  8
200 '  r  =  3.565 :  Period 16
210 '  r  =  3.57  :  CHAOS
220 '  r  =  3.84  :  Period  3
230 '  r  =  3.845 :  Period  6
240 '  r  =  3.848 :  Period 12
250 '
260  DEFDBL R,X
270  INPUT "Value of r"; R
280  INPUT "Print in blocks of k  =  "; K
290  P$  =  "#.####    "
300  X  =  .5                    'Initial seed
310 '
320  FOR I  =  1 TO 500          '500 initial
330     X  =  R*X*(1 - X)        'iterations to
340  NEXT I                      'stabilize.
350 '
360  FOR  I  =  1  TO  K
370     X  =  R*X*(1 - X)        'Final iterations
380     PRINT USING P$; X;
390  NEXT
400  IF K <> 8 THEN PRINT
410 '
420 'Press any key but Q to continue:
430  A$  =  INKEY$
440  IF A$  =  "" THEN GOTO 430
450  IF A$  =  "q" OR A$  =  "Q" THEN END
460  GOTO 360                    'End of loop
470 '
480  END
```

The next BASIC program below can be used to plot pitchfork diagrams as in Figures 7.6.8 and 7.6.9 in the text. As written, it runs well in Borland TurboBasic (probably now obsolete), but may have to be fine-tuned to run in other dialects of BASIC.

```
100 'Program PICHFORK
110 '
120 'Exhibits the period-doubling toward chaos
130 'generated by the Verhulst iteration
140 '
150 '        x  =  rx(1 - x)
160 '
170 'as the growth parameter  r  is increased
180 'in the range from about  3  to about  4.
190 '
200  DEFDBL H,K,R,X
210  DEFINT I,J,M,N,P,Q
220  INPUT "Rmin,Rmax"; RMIN, RMAX   'Try 2.8 and 4.0
230  INPUT "Xmin,Xmax"; XMIN, XMAX   'Try  0  and  1
240 '
```

```
250  KEY OFF   : CLS
260 'SCREEN 1   :  N  =  319        'For med resolution
270  SCREEN 2   :  N  =  639        'For hi  resolution
280  M  =  200                      'Hor rows for either
290  H  =  (RMAX - RMIN)/N
300  K  =  (XMAX - XMIN)/M
310 '
320  LINE (0,0) - (N,0)             'Draws a box
330  LINE - (N,199)                 'around
340  LINE - (0,199)                 'the screen
350  LINE - (  0,0)
360 '
370  FOR P  =  1 TO 9               'Tick marks on
380      Q  =  (P*(N+1)/10) - 1     'top and bottom
390      LINE (Q,0) - (Q,5)         'of box
400      LINE (Q,195) - (Q,199)
410  NEXT P
420 '
430  FOR J  =  0 TO  N              'Jth vertical column
440      R  =  RMIN + J*H           'of pixels on screen
450      X  =  .5
460      FOR P  =  0 TO 1000        'These iterations
470          X  =  R*X*(1-X)        'to settle down.
480      NEXT P
490      FOR Q  =  0 TO 250         'These iterations
500          X  =  R*X*(1-X)        'are recorded.
510          I  =  INT((X - XMIN)/K)
520          I  =  200 - I
530          IF (0< = I) AND (I<200) THEN PSET (J,I)
540      NEXT Q
550  NEXT J
560 '
570  WHILE INKEY$  =  ""            'Press a key when
580  WEND                           'finished looking.
590  SCREEN 0   :  CLS   :  KEY ON
600  END
```

A more elaborate construction of these pitchfork diagrams is given by the following Mathematica program, a slight elaboration of one found on page 102 of T. Gray and J. Glynn, *Exploring Mathematics with Mathematica*, Addison-Wesley, 1991.

```
g[x_] : =  r x (1 - x);
Clear[r];
a  =  2.8;  b  =  4.0.      (* r-range for Fig 7.6.8 *)
c  =  0;    d  =  1;        (* x-range *)
m  =  250;                      (* no of x-points *)
n  =  500;                      (* no of r-values *)
ListPlot[
   Flatten[Table[
```

```
         Transpose[{
              Table[r, {m+1}],
                   NestList[g, Nest[q, 0.5, 2m], m] }],
                   {r, a, b, (b-a)/n} ],
                   1],
        PlotStyle -> PointSize[0.001],
        PlotRange -> {{a,b},{c,d}},
        AspectRatio -> 0.75,
        Frame -> True,
        AxesLabel -> {"r","x"} ]
```

This Mathematica program runs slowly, and requires a fast machine with plenty of memory to finish within a reasonable waiting time. The following MATLAB program (which was actually used to construct Figs 7.6.8 and 7.6.9) runs much faster on a comparable computer, and may be easier to understand.

```
% pitchfork diagram script
% for Figures 7.6.8 and 7.6.9

hold off
m  =  400;                          % no of r-subintervals
n  =  400;                          % no of x-subintervals

 a  =  2.8;   b  =  4.0;            % r-range for 7.6.9
dr  =  (b - a)/m;
R  =  a+dr/2 : dr : b;              % vector of r-values
c  =  0;      d  =  1;              % x-range
dx  =  (d - c)/n
X  =  c+dx/2 : dx : d;              % vector of x-values
[rr,xx]  =  meshgrid(R,X);          % matrices of r- and x-coords
                                    % of grid points in rx-rect

C = zeros(m,n);
for j  =  1 : m                     % Cycle through r-values
   r  =  a - dr/2 + j*dr;
   x  =  0.5;                       % Initialize x-value
   for k  =  1:1000                 % 1000 iterations to stabilize
      x  =  r*x*(1-x);
   end
   for k  =  1:1000                 % 1000 more iterations
      x  =  r*x*(1-x);
   i = ceil(x/dx);
      C(i,j) = 1;                   % lattice point to plot
      end
   end

C = C + 1;
C = flipud(C);                      % matrix of points for image
image(R,X,C)
```

```
colormap([1 1 1; 0 0 0])        % color them black or white
axis square
```

The following MATLAB function defines the forced Duffing equation for Figures 7.6.13 through 7.6.16.

```
function  yp = ypduffing(t,x)
F0 = 0.80;
yp = x;
y = x(2);    x = x(1);
yp(1) = y;
yp(2) = F0*cos(t)-y+x-x.^3;
```

Then the following MATLAB script can be used to construct Fig. 7.6.16.

```
% fig7_6_16.m script
options = odeset('RelTol',1e-8,'AbsTol',1e-8);
[t,y] = ode45('ypduffing', [0 100], [1;0],options);
n = length(t);
y100 = y(n,:)';
[t,y] = ode45('ypduffing', [100 300], y100,options);
hold off
plot(y(:,1),y(:,2),'b')          % Fig. 7.6.16(a)
axis([-1.5 1.5 -1.5 1.5])
axis square
hold on
plot([-1.5 1.5],[0 0],'k')
plot([0 0],[-1.5 1.5],'k')
pause
hold off
plot(t,y(:,1),'b')               % Fig. 7.6.16(b)
axis([100 300 -1.5 1.5])
axis square
hold on
plot([100 300],[0 0],'k')
plot([0 0],[-1.5 1.5],'k')
```

CHAPTER 8

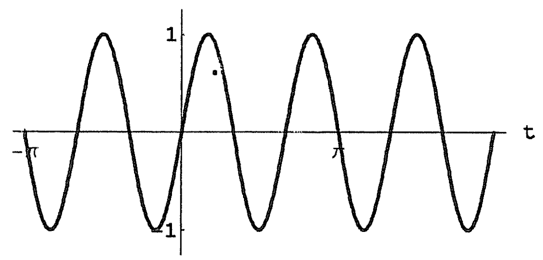
FOURIER SERIES METHODS

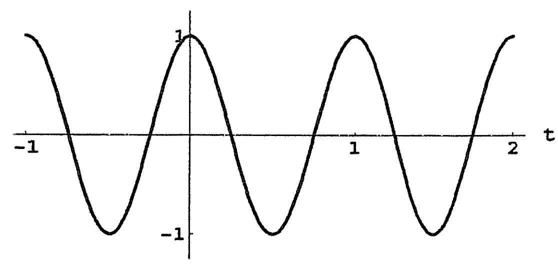
SECTION 8.1

PERIODIC FUNCTIONS AND TRIGONOMETRIC SERIES

The basic trigonometric functions cos(t) and sin(t) have period $P = 2\pi$, so the sine or cosine of ωt (as in Problems 1–4) completes its first period when $\omega t = 2\pi$; hence $P = 2\pi / \omega$.

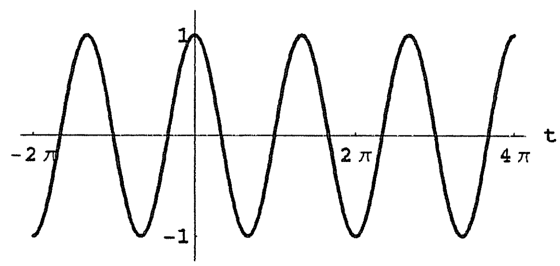
1. Smallest period $P = 2\pi/3$ (left-hand figure below)

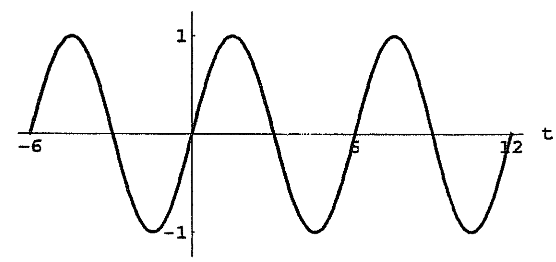




2. Smallest period $P = 1$ (right-hand figure above)

3. Smallest period $P = 4\pi/3$ (left-hand figure below)



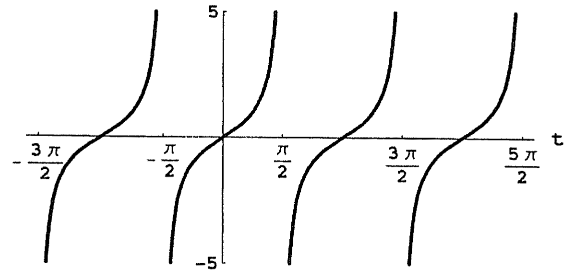


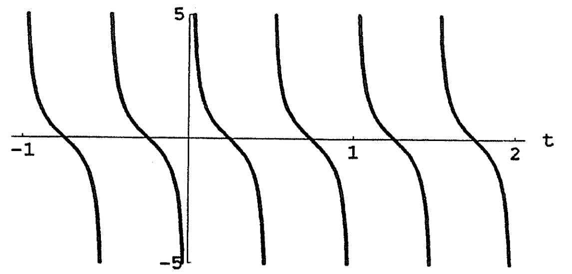
4. Smallest period $P = 6$ (right-hand figure above)

However, the basic tangent and cotangent functions have period π (instead of 2π), so $P = \pi / \omega$ in Problems 5 and 6.

5. Smallest period $P = \pi$; see the left-hand figure at the top of the next page.

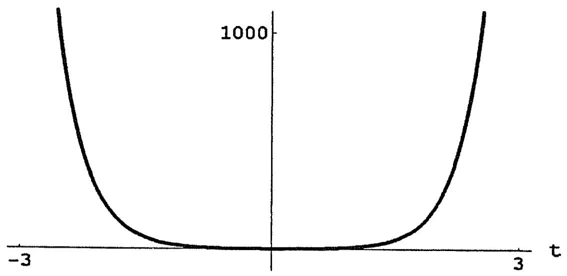
6. Smallest period $P = 1/2$; see the right-hand figure at the top of the next page.

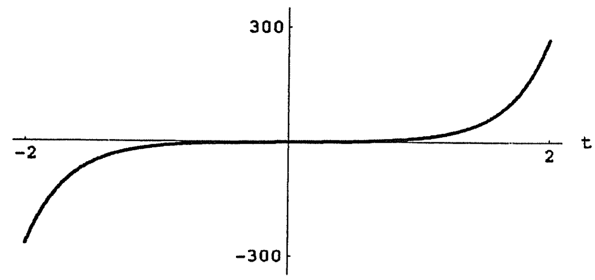




The hyperbolic sine and cosine functions of Problems 7 and 8 are steadily increasing (for $t > 0$), and hence are not periodic.

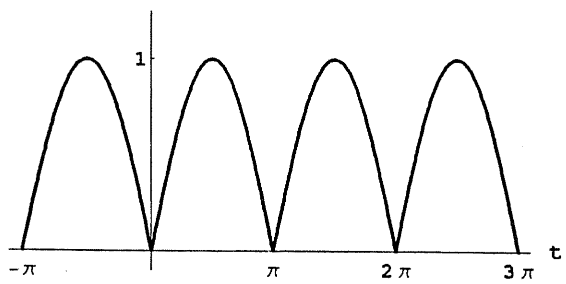
7. Not periodic (left-hand figure below)

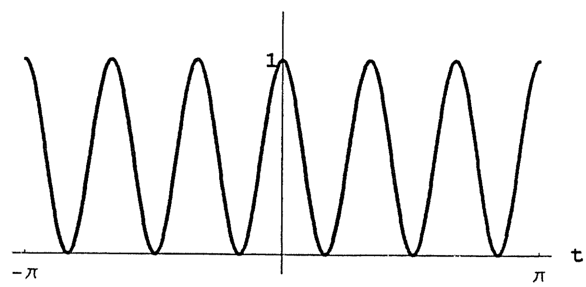




8. Not periodic (right-hand figure above)

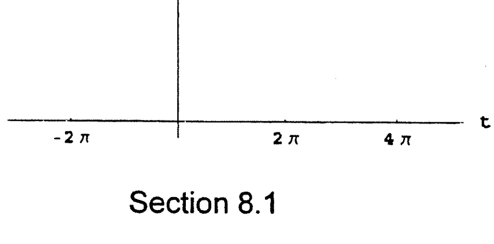
9. Smallest period $P = \pi$ (left-hand figure below)





10. Smallest period $P = \pi/3$ (right-hand figure above)

11. With $f(t) = 1$ the integral formulas of Eqs. (16) and (17) in the text give $a_0 = 2$ and $a_n = b_n = 0$ for $n \geq 0$. Thus the Fourier series of f is the single term series $f(t) = 1$.



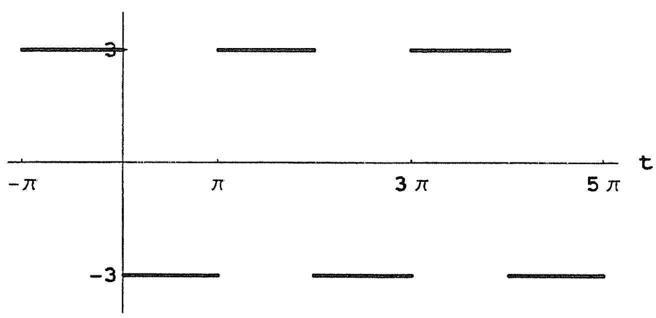
In Problems 12–13, 18–19, 23, and 26 the function $f(t)$ is defined by one formula on the interval $(-\pi, 0)$ and by another formula on $(0, \pi)$. The coefficient integrals must therefore be split accordingly, and the appropriate formula substituted in each integral:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\,dt, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos nt\,dt, \qquad (n > 0) \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin nt\,dt. \end{aligned}$$

12.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} (+3)\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-3)\,dt = 0 \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{0} (+3)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-3)\cos nt\,dt = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{0} (+3)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (-3)\sin nt\,dt = \\ &= \frac{6}{n\pi}\left[\cos n\pi - 1\right] = \begin{cases} 0 & \text{for } n \text{ even} \\ -12/n\pi & \text{for } n \text{ odd} \end{cases} \end{aligned}$$

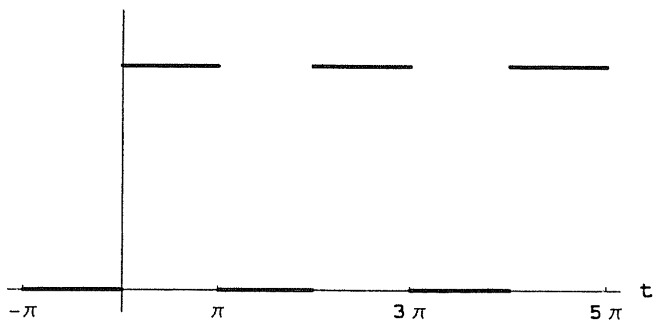
$$f(t) \sim -\frac{12}{\pi}\left[\frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \cdots\right]$$



13.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} (0)\,dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\,dt = 1 \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{0} (0)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\cos nt\,dt = \frac{\sin n\pi}{n\pi} = 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{0} (0)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\sin nt\,dt = \\ &= \frac{1-\cos n\pi}{n\pi} = \begin{cases} 0 & \text{for } n \text{ even} \\ 2/n\pi & \text{for } n \text{ odd} \end{cases} \end{aligned}$$

$$f(t) \sim \frac{1}{2}+\frac{2}{\pi}\left[\frac{\sin t}{1}+\frac{\sin 3t}{3}+\frac{\sin 5t}{5}+\frac{\sin 7t}{7}+\cdots\right] \qquad \text{(figure below)}$$



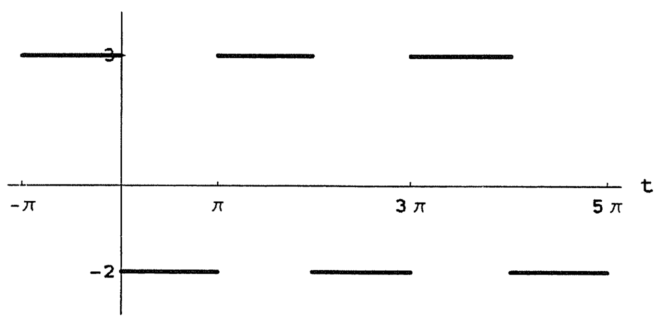
14. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(3)\,dt+\frac{1}{\pi}\int_{0}^{\pi}(-2)\,dt = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(3)\cos nt\,dt+\frac{1}{\pi}\int_{0}^{\pi}(-2)\cos nt\,dt = \frac{\sin n\pi}{n\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(3)\sin nt\,dt+\frac{1}{\pi}\int_{0}^{\pi}(-2)\sin nt\,dt =$$

$$= \frac{5(\cos n\pi-1)}{n\pi} = \begin{cases} 0 & \text{for } n \text{ even} \\ -10/n\pi & \text{for } n \text{ odd}\end{cases}$$

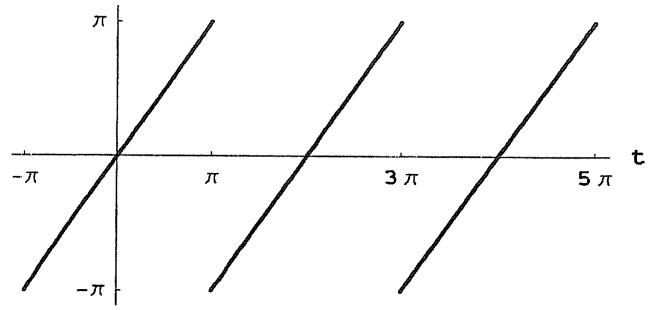
$$f(t) \sim \frac{1}{2}-\frac{10}{\pi}\left[\frac{\sin t}{1}+\frac{\sin 3t}{3}+\frac{\sin 5t}{5}+\frac{\sin 7t}{7}+\cdots\right] \qquad \text{(figure below)}$$



15. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = 0, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{2\sin n\pi-2n\pi\cos n\pi}{n^2\pi} = \begin{cases} -2/n & \text{for } n \text{ even} \\ +2/n & \text{for } n \text{ odd}\end{cases}$$

$$f(t) \sim 2\left[\frac{\sin t}{1}-\frac{\sin 2t}{2}+\frac{\sin 3t}{3}-\frac{\sin 4t}{4}+\cdots\right] \qquad \text{(figure at top of next page)}$$

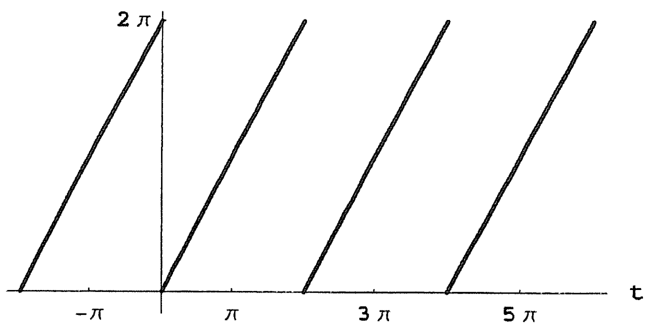


16. $$a_0 = \frac{1}{\pi}\int_0^{2\pi} t\,dt = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos nt\,dt = \frac{\cos 2n\pi - 2n\pi\sin 2n\pi - 1}{n^2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin nt\,dt = \frac{2\sin 2n\pi - 2n\pi\cos 2n\pi}{n^2\pi} = -\frac{2}{n}$$

$$f(t) \sim \pi - 2\left[\frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right]$$ (figure below)



17. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-t)\,dt + \frac{1}{\pi}\int_0^{\pi}(t)\,dt = \pi$$

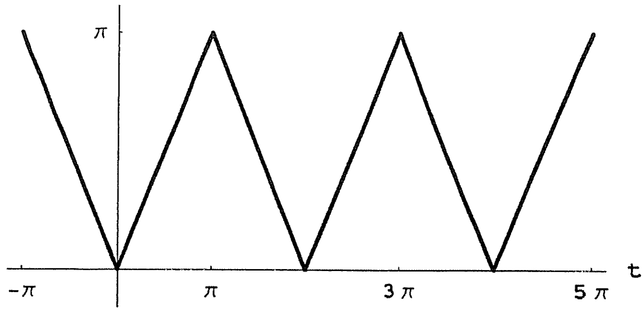
$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-t)\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}(t)\cos nt\,dt$$

$$= \frac{2(\cos n\pi + n\pi\sin n\pi - 1)}{n^2\pi} = \begin{cases} 0 \text{ for } n \text{ even} \\ -4/n^2\pi \text{ for } n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-t)\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi}(t)\sin nt\,dt = 0$$

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos t}{1} + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \frac{\cos 7t}{49} + \cdots\right]$$

See the figure at the top of the next page.



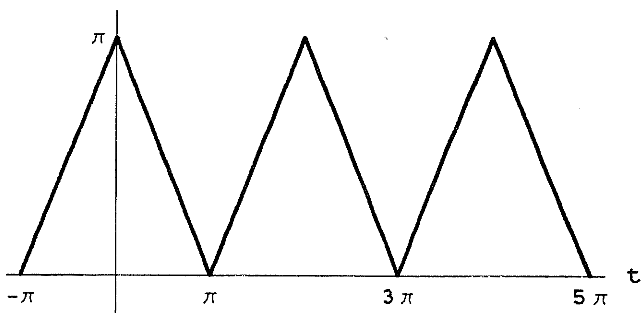
18. $a_0 = \dfrac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\,dt + \dfrac{1}{\pi}\int_0^{\pi}(\pi-t)\,dt = \pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}(\pi-t)\cos nt\,dt$$

$$= \frac{2(1-\cos n\pi)}{n^2\pi} = \begin{cases} 0 \text{ for } n \text{ even} \\ 4/n^2\pi \text{ for } n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi}(\pi-t)\sin nt\,dt = 0$$

$$f(t) \sim \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos t}{1} + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \frac{\cos 7t}{49} + \cdots\right]$$ (figure below)



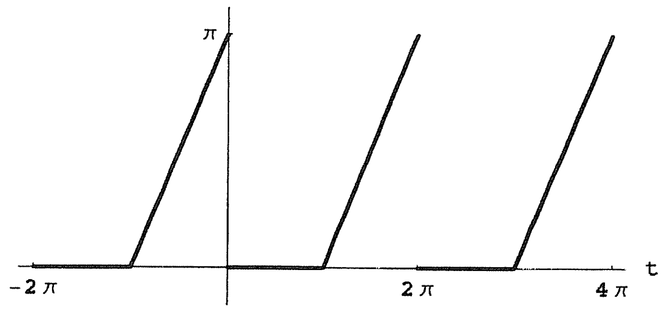
19. $a_0 = \dfrac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\,dt + \dfrac{1}{\pi}\int_0^{\pi}(0)\,dt = \dfrac{\pi}{2}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}(0)\cos nt\,dt$$

$$= \frac{1-\cos n\pi}{n^2\pi} = \begin{cases} 0 \text{ for } n \text{ even} \\ 2/n^2\pi \text{ for } n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi}(0)\sin nt\,dt = \frac{\sin n\pi - n\pi}{n^2\pi} = -\frac{1}{n}$$

$$f(t) \sim \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos t}{1} + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \frac{\cos 7t}{49} + \cdots\right] - \left[\frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right]$$

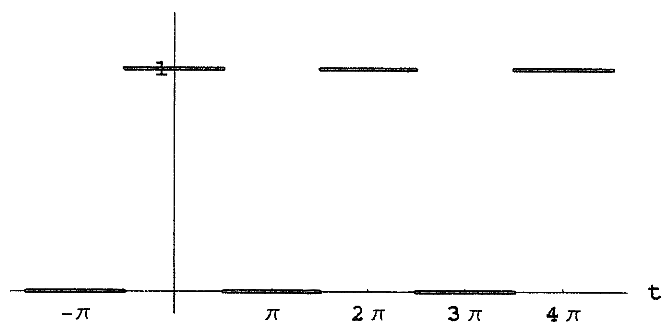


20. $$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dt = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nt\,dt = \frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0 \text{ for } n \text{ even} \\ +1 \text{ for } n = 1, 5, \cdots \\ -1 \text{ for } n = 3, 7, \cdots \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nt\,dt = 0$$

$$f(t) \sim \frac{1}{2} + \frac{2}{\pi}\left[\frac{\cos t}{1} - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \frac{\cos 7t}{7} + \cdots\right]$$ (figure below)

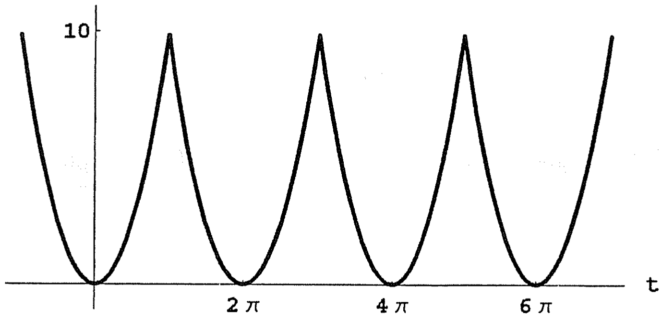


21. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\,dt = \frac{4n\pi\cos n\pi - 2(n^2\pi^2 - 2)\sin n\pi}{n^3\pi} = \begin{cases} +4/n^2 \text{ for } n \text{ even} \\ -4/n^2 \text{ for } n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin nt\,dt = 0$$

$$f(t) \sim \frac{\pi^2}{3} - 4\left[\frac{\cos t}{1} - \frac{\cos 2t}{4} + \frac{\cos 3t}{9} - \frac{\cos 4t}{16} + \cdots\right] \qquad \text{(figure below)}$$

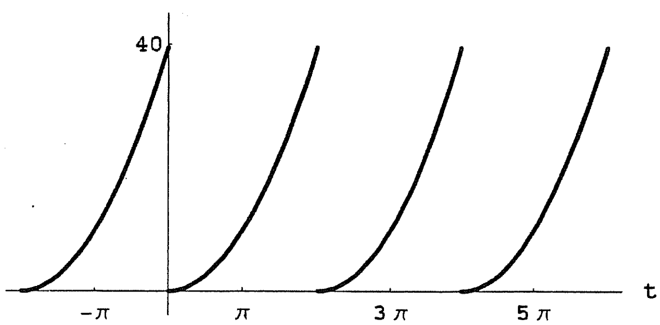


22. $$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt\,dt = \frac{4n\pi\cos 2n\pi + 2(n^2\pi^2-1)\sin 2n\pi}{n^3\pi} = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt = \frac{(2-4n^2\pi^2)\cos 2n\pi + 4n\pi\sin 2n\pi - 2}{n^3\pi} = -\frac{4\pi}{n}$$

$$f(t) \sim \frac{4\pi^2}{3} + 4\left[\frac{\cos t}{1} + \frac{\cos 2t}{4} + \frac{\cos 3t}{9} + \frac{\cos 4t}{16} + \cdots\right] - 4\pi\left[\frac{\sin t}{1} + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right] \qquad \text{(figure below)}$$



23. $$a_0 = \frac{1}{\pi}\int_0^{\pi} t^2\,dt = \frac{\pi^2}{3}$$

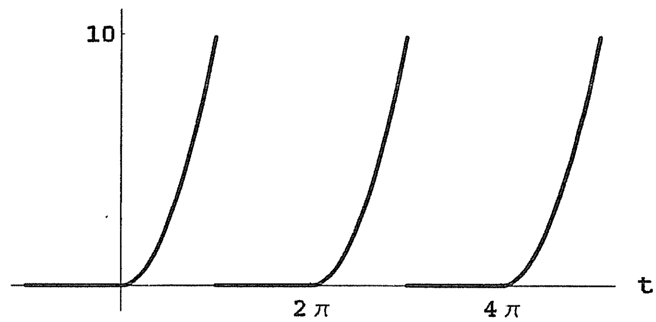
$$a_n = \frac{1}{\pi}\int_0^{\pi} t^2 \cos nt\,dt = \frac{2n\pi\cos n\pi + (n^2\pi^2-2)\sin n\pi}{n^3\pi} = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\, dt$$

$$= \frac{(2-n^2\pi^2)\cos n\pi + 2n\pi \sin n\pi - 2}{n^3\pi} = \begin{cases} -\pi/n \text{ for } n \text{ even} \\ (n^2\pi^2 - 4)/\pi n^3 \text{ for } n \text{ odd} \end{cases}$$

$$f(t) \sim \frac{\pi^2}{6} - 2\left[\frac{\cos t}{1} - \frac{\cos 2t}{4} + \frac{\cos 3t}{9} - \frac{\cos 4t}{16} + \cdots\right] \qquad \text{(figure below)}$$

$$+\pi\left[\frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \cdots\right] - \frac{4}{\pi}\left[\frac{\sin t}{1} + \frac{\sin 3t}{27} + \frac{\sin 5t}{125} + \frac{\sin 7t}{343} + \cdots\right]$$



The trigonometric identities

$$\begin{aligned} 2\cos A \cos B &= \cos(A+B) + \cos(A-B) \\ 2\sin A \cos B &= \sin(A+B) + \sin(A-B) \\ 2\sin A \sin B &= \cos(A-B) - \cos(A+B) \end{aligned}$$

are needed to evaluate the integrals that appear in Problems 24–26.

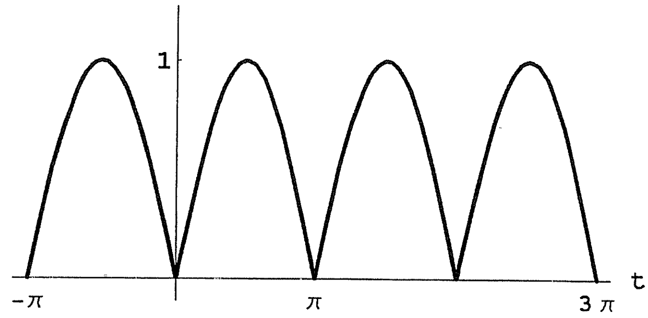
24. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin t)\,dt + \frac{1}{\pi}\int_0^{\pi}(\sin t)\,dt = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin t)\cos nt\,dt + \frac{1}{\pi}\int_0^{\pi}(\sin t)\cos nt\,dt$$

$$= \frac{2(1+\cos n\pi)}{\pi(1-n^2)} = \begin{cases} -4/\pi(n^2-1) \text{ for } n \text{ even} \\ 0 \text{ for } n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-\sin t)\sin nt\,dt + \frac{1}{\pi}\int_0^{\pi}(\sin t)\sin nt\,dt = 0$$

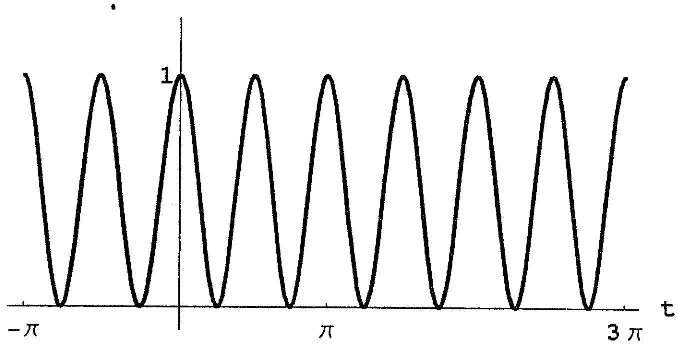
$$f(t) \sim \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2t}{1} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \frac{\cos 8t}{63} + \cdots\right] \qquad \text{(figure at top of next page)}$$



25. In order to evaluate the coefficient integrals in Eqs. (16) and (17) of the text we would need the trigonometric identity

$$\cos^2 2t = \frac{1}{2}(1+\cos 4t)$$

which, however, tells us in advance that the coefficients in the Fourier series of $f(t) = \cos^2 2t$ are given by $a_0 = 1$, $a_4 = 1/2$, $a_n = 0$ otherwise, and $b_n = 0$ for all $n \geq 1$.



26.

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(\sin t)\,dt = \frac{2}{\pi}$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin t\cos t\,dt = 0$$

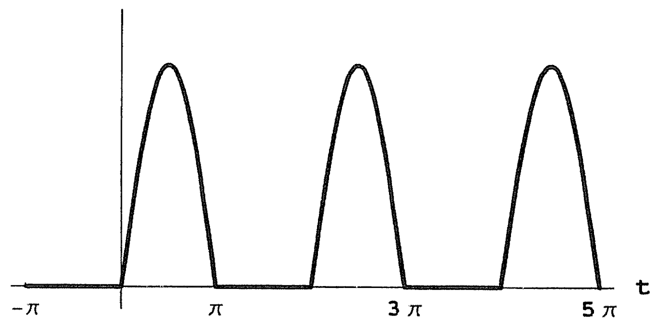
$$a_n = \frac{1}{\pi}\int_0^{\pi}(\sin t)\cos nt\,dt = \frac{1+\cos n\pi}{\pi(1-n^2)} = \begin{cases} -2/\pi(n^2-1) & \text{for } n \text{ even} \\ 0 & \text{for } n>1 \text{ odd} \end{cases}$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin^2 t\,dt = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\sin t)\sin nt\,dt = \frac{\sin n\pi}{\pi(1-n^2)} = 0 \text{ for } n>1$$

$$f(t) \sim \frac{1}{\pi}+\frac{1}{2}\sin t-\frac{2}{\pi}\left[\frac{\cos 2t}{1}+\frac{\cos 4t}{15}+\frac{\cos 6t}{35}+\frac{\cos 8t}{63}+\cdots\right]$$

Note that $f(t) = (\sin t + |\sin t|)/2$, so this answer agrees with the answer to Problem 24.



SECTION 8.2

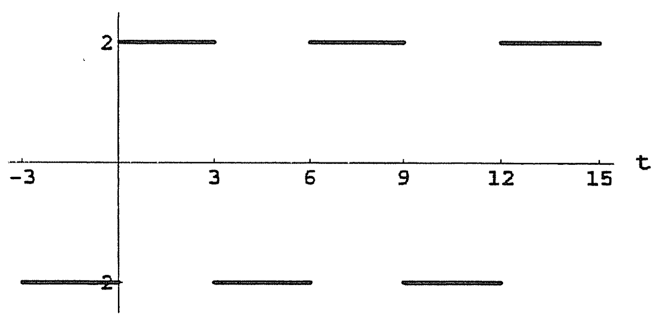
GENERAL FOURIER SERIES AND CONVERGENCE

1. $$a_0 = \frac{1}{3}\int_{-3}^{0}(-2)\,dt + \frac{1}{3}\int_{0}^{3}(2)\,dt = 0$$

$$a_n = \frac{1}{3}\int_{-3}^{0}(-2)\cos\frac{n\pi t}{3}\,dt + \frac{1}{3}\int_{0}^{3}(2)\cos\frac{n\pi t}{3}\,dt = 0$$

$$b_n = \frac{1}{3}\int_{-3}^{0}(-2)\sin\frac{n\pi t}{3}\,dt + \frac{1}{3}\int_{0}^{3}(2)\sin\frac{n\pi t}{3}\,dt = \frac{4(1-\cos n\pi)}{n\pi} = \frac{4}{n\pi}\left[1-(-1)^n\right]$$

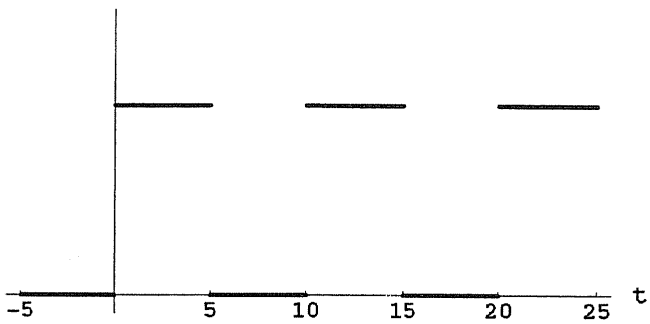
$$f(t) = \frac{8}{\pi}\left[\sin\frac{\pi t}{3} + \frac{1}{3}\sin\frac{3\pi t}{3} + \frac{1}{5}\sin\frac{5\pi t}{3} + \frac{1}{7}\sin\frac{7\pi t}{3} + \cdots\right]$$ (figure below)



2. $$a_0 = \frac{1}{5}\int_{0}^{5}(1)\,dt = 1, \qquad a_n = \frac{1}{5}\int_{0}^{5}(1)\cos\frac{n\pi t}{5}\,dt = \frac{\sin n\pi}{n\pi} = 0$$

$$b_n = \frac{1}{5}\int_{0}^{5}(1)\sin\frac{n\pi t}{5}\,dt = \frac{1-\cos n\pi}{n\pi} = \frac{1-(-1)^n}{n\pi}$$

$$f(t) = \frac{1}{2}+\frac{2}{\pi}\left[\sin\frac{\pi t}{5}+\frac{1}{3}\sin\frac{3\pi t}{5}+\frac{1}{5}\sin\frac{5\pi t}{5}+\frac{1}{7}\sin\frac{7\pi t}{5}+\cdots\right] \qquad \text{(figure below)}$$



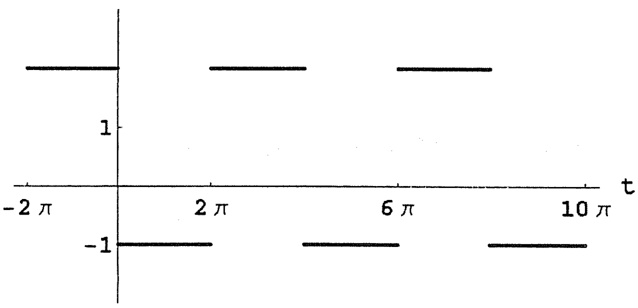
3.

$$a_0 = \frac{1}{2\pi}\int_{-2\pi}^{0}(2)\,dt+\frac{1}{2\pi}\int_0^{2\pi}(-1)\,dt = 1$$

$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{0}(2)\cos\frac{nt}{2}\,dt+\frac{1}{2\pi}\int_0^{2\pi}(-1)\cos\frac{nt}{2}\,dt = \frac{\sin n\pi}{n\pi} = 0$$

$$b_n = \frac{1}{2\pi}\int_{-2\pi}^{0}(2)\sin\frac{nt}{2}\,dt+\frac{1}{2\pi}\int_0^{2\pi}(-1)\sin\frac{nt}{2}\,dt = \frac{3(\cos n\pi-1)}{n\pi} = \frac{3}{n\pi}\left[(-1)^n-1\right]$$

$$f(t) = \frac{1}{2}-\frac{6}{\pi}\left[\sin\frac{t}{2}+\frac{1}{3}\sin\frac{3t}{2}+\frac{1}{5}\sin\frac{5t}{2}+\frac{1}{7}\sin\frac{7t}{2}+\cdots\right] \qquad \text{(figure below)}$$



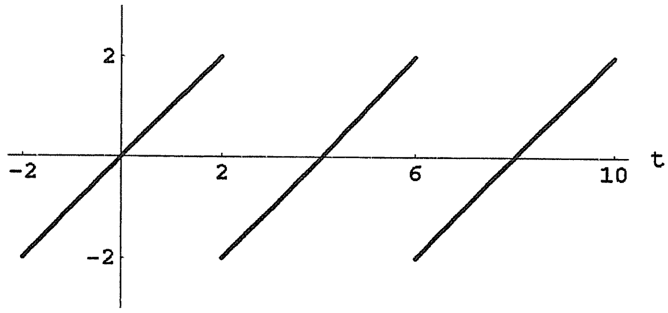
4.

$$a_0 = \frac{1}{2}\int_{-2}^{2}t\,dt = 0, \qquad a_n = \frac{1}{2}\int_{-2}^{2}t\cos\frac{n\pi t}{2}\,dt = 0$$

$$b_n = \frac{1}{2}\int_{-2}^{2}t\sin\frac{n\pi t}{2}\,dt = \frac{4(\sin n\pi - n\pi\cos n\pi)}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n\pi}$$

$$f(t) = \frac{4}{\pi}\left[\sin\frac{\pi t}{2}-\frac{1}{2}\sin\frac{2\pi t}{2}+\frac{1}{3}\sin\frac{3\pi t}{2}-\frac{1}{4}\sin\frac{4\pi t}{2}+\cdots\right]$$

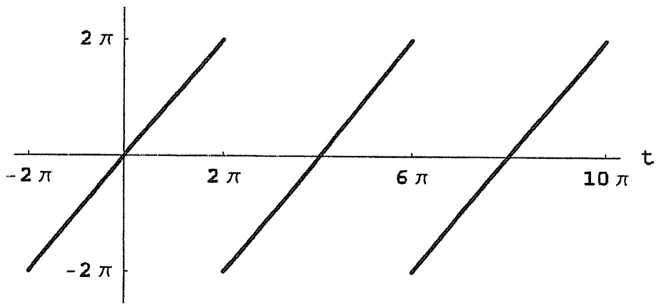
See figure at top of next page.



5. $a_0 = \dfrac{1}{2\pi}\int_{-2\pi}^{2\pi} t\,dt = 0\,,\qquad a_n = \dfrac{1}{2\pi}\int_{-2\pi}^{2\pi} t\cos\dfrac{nt}{2}dt = 0$

$$b_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} t\sin\frac{nt}{2}dt = \frac{4(\sin n\pi - n\pi\cos n\pi)}{n^2\pi} = \frac{4(-1)^{n+1}}{n}$$

$$f(t) = 4\left[\sin\frac{t}{2} - \frac{1}{2}\sin\frac{2t}{2} + \frac{1}{3}\sin\frac{3t}{2} - \frac{1}{4}\sin\frac{4t}{2} + \cdots\right]$$ (figure below)

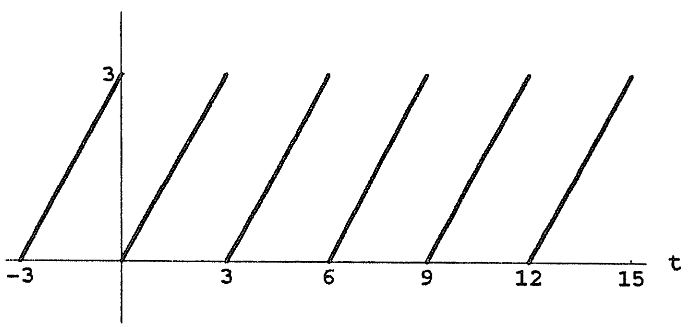


6. $a_0 = \dfrac{2}{3}\int_0^3 t\,dt = 3$

$$a_n = \frac{2}{3}\int_0^3 t\cos\frac{2n\pi t}{3}dt = \frac{3[\cos 2n\pi + 2n\pi\sin 2n\pi - 1]}{2n^2\pi^2} = 0$$

$$b_n = \frac{2}{3}\int_0^3 t\sin\frac{2n\pi t}{3}dt = \frac{3\sin 2n\pi - 6n\pi\cos 2n\pi}{2n^2\pi^2} = \frac{3(-1)^{n+1}}{n\pi}$$

$$f(t) = \frac{3}{2} - \frac{3}{\pi}\left[\sin\frac{2\pi t}{3} + \frac{1}{2}\sin\frac{4\pi t}{3} + \frac{1}{3}\sin\frac{6\pi t}{3} + \frac{1}{4}\sin\frac{8\pi t}{3} + \cdots\right]$$ (figure below)

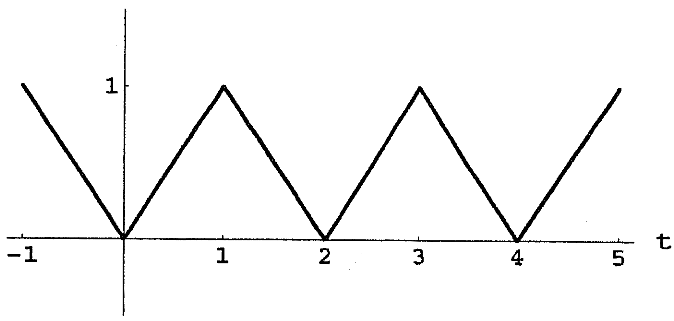


7. $$a_0 = \int_{-1}^{0}(-t)\,dt + \int_{0}(t)\,dt = 1$$

$$a_n = \int_{-1}^{0}(-t)\cos n\pi t\,dt + \int_{0}^{1} t\cos n\pi t\,dt = \frac{2[\cos n\pi + n\pi\sin n\pi - 1]}{n^2\pi^2} = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \int_{-1}^{0}(-t)\sin n\pi t\,dt + \int_{0}^{1} t\sin n\pi t\,dt = 0$$

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\left[\cos\pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \frac{1}{49}\cos 7\pi t + \cdots\right] \qquad \text{(figure below)}$$



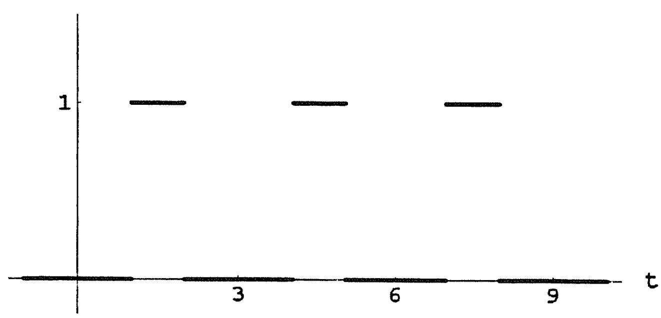
8. $$a_0 = \frac{2}{3}\int_{1}^{2} 1\,dt = \frac{2}{3}$$

$$a_n = \frac{2}{3}\int_{1}^{2} \cos\frac{2n\pi t}{3}\,dt = \frac{1}{n\pi}\left[\sin\frac{4n\pi}{3} - \sin\frac{2n\pi}{3}\right]$$

$$b_n = \frac{2}{3}\int_{1}^{2} \sin\frac{2n\pi t}{3}\,dt = \frac{1}{n\pi}\left[\cos\frac{2n\pi}{3} - \cos\frac{4n\pi}{3}\right]$$

Analyzing separately the cases $n = 3k$, $n = 3k+1$, and $n = 3k+2$, we find that $a_{3k} = 0$, $a_{3k+1} = -\sqrt{3}/\pi n$, $a_{3k+2} = +\sqrt{3}/\pi n$, and that $b_n = 0$ for all n. Hence the Fourier series of $f(t)$ is

$$f(t) = \frac{1}{3} - \frac{\sqrt{3}}{\pi}\left[\cos\frac{2\pi t}{3} - \frac{1}{2}\cos\frac{4\pi t}{3} + \frac{1}{4}\cos\frac{8\pi t}{3} - \frac{1}{5}\cos\frac{10\pi t}{3} + \frac{1}{7}\cos\frac{14\pi t}{3} - \cdots\right].$$

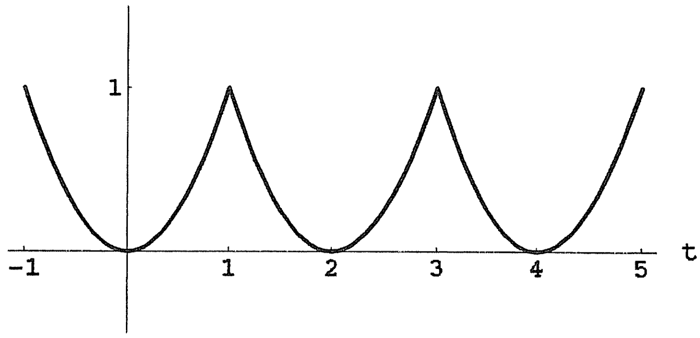


9. $a_0 = \int_{-1}^{1} t^2\,dt = \dfrac{2}{3}$

$$a_n = \int_{-1}^{1} t^2 \cos n\pi t\,dt = \frac{4n\pi\cos n\pi + 2(n^2\pi^2 - 2)\sin n\pi}{n^3\pi^3} = \frac{4(-1)^n}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} t^2 \sin n\pi t\,dt = 0$$

$$f(t) = \frac{1}{3} - \frac{4}{\pi^2}\left[\cos\pi t - \frac{1}{4}\cos 2\pi t + \frac{1}{9}\cos 3\pi t - \frac{1}{16}\cos 4\pi t + \cdots\right]$$ (figure below)



10. $a_0 = \dfrac{1}{2}\int_0^2 t^2\,dt = \dfrac{4}{3}$

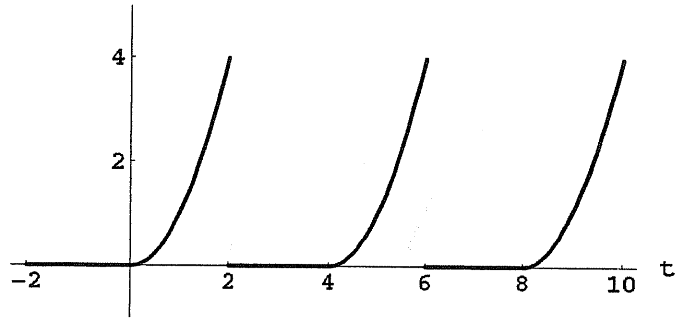
$$a_n = \frac{1}{2}\int_0^2 t^2\cos\frac{n\pi t}{2}\,dt = \frac{8n\pi\cos n\pi + (n^2\pi^2 - 2)\sin n\pi}{n^3\pi^3} = \frac{8(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{2}\int_0^2 t^2\sin\frac{n\pi t}{2}\,dt$$

$$= -\frac{4\left[(n^2\pi^2 - 2)\cos n\pi - 2n\pi\sin n\pi + 2\right]}{n^3\pi^3} = \begin{cases} -4/n\pi & \text{for } n \text{ even} \\ +4/n\pi - 16/n^3\pi^3 & \text{for } n \text{ odd} \end{cases}$$

$$f(t) = \frac{2}{3} - \frac{8}{\pi^2}\left[\cos\frac{\pi t}{2} - \frac{1}{4}\cos\frac{2\pi t}{2} + \frac{1}{9}\cos\frac{3\pi t}{2} - \frac{1}{16}\cos\frac{4\pi t}{2} + \cdots\right]$$
$$+ \frac{4}{\pi}\left[\sin\frac{\pi t}{2} - \frac{1}{2}\sin\frac{2\pi t}{2} + \frac{1}{3}\sin\frac{3\pi t}{2} - \cdots\right]$$
$$- \frac{16}{\pi^3}\left[\sin\frac{\pi t}{2} + \frac{1}{27}\sin\frac{3\pi t}{2} + \frac{1}{125}\sin\frac{5\pi t}{2} + \cdots\right]$$

See the figure at the top of the next page.



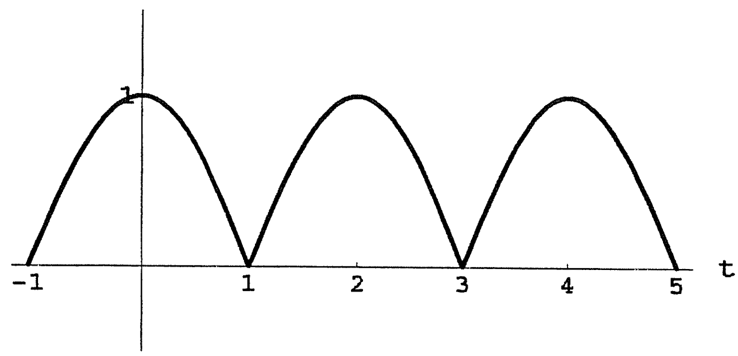
To calculate the Fourier coefficients in Problems 11–14 we use the trigonometric identities for $\sin A \cos B$ and $\sin A \sin B$ that are listed above in Section 8.1 (prior to Problems 24–26 there).

11. $$a_0 = \int_{-1}^{1} \cos\frac{\pi t}{2}\,dt = \frac{4}{\pi}$$

$$a_n = \int_{-1}^{1} \cos\frac{\pi t}{2}\cos n\pi t\,dt = -\frac{4\cos n\pi}{\pi(4n^2-1)} = \frac{4(-1)^{n+1}}{\pi(4n^2-1)}$$

$$b_n = \int_{-1}^{1} \cos\frac{\pi t}{2}\sin n\pi t\,dt = 0$$

$$f(t) = \frac{2}{\pi}+\frac{4}{\pi}\left[\frac{1}{3}\cos\pi t-\frac{1}{15}\cos 2\pi t+\frac{1}{35}\cos 3\pi t-\frac{1}{63}\cos 4\pi t+\cdots\right]$$ (figure below)

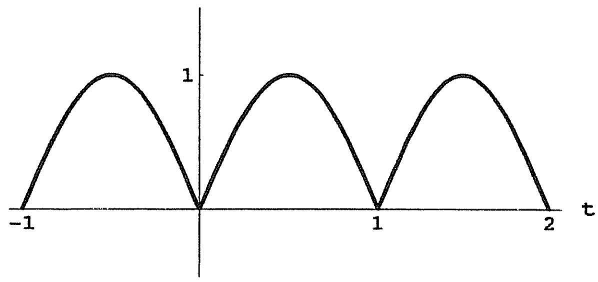


12. $$a_0 = 2\int_0^1 \sin\pi t\,dt = \frac{4}{\pi}$$

$$a_n = 2\int_0^1 \sin\pi t\cos 2n\pi t\,dt = -\frac{4\cos^2 n\pi}{\pi(4n^2-1)} = -\frac{4}{\pi(4n^2-1)}$$

$$b_n = 2\int_0^1 \sin\pi t\sin 2n\pi t\,dt = -\frac{4\cos n\pi\sin n\pi}{\pi(4n^2-1)} = 0$$

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2\pi t + \frac{1}{15}\cos 4\pi t + \frac{1}{35}\cos 6\pi t + \frac{1}{63}\cos 8\pi t + \cdots\right]$$



13. $$a_0 = \int_0^1 \sin \pi t \, dt = \frac{2}{\pi}$$

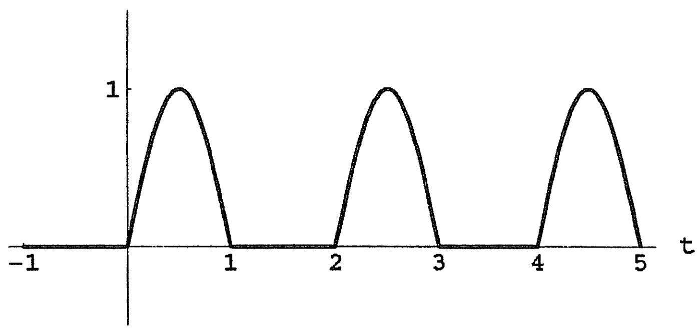
$$a_n = \int_0^1 \sin \pi t \cos n\pi t \, dt = -\frac{1+\cos n\pi}{\pi(n^2-1)} = -\frac{1+(-1)^n}{\pi(n^2-1)} \quad \text{for } n > 1$$

$$a_1 = \int_0^1 \sin \pi t \cos \pi t \, dt = 0$$

$$b_n = \int_0^1 \sin \pi t \sin n\pi t \, dt = -\frac{\sin n\pi}{\pi(n^2-1)} = 0 \quad \text{for } n > 1$$

$$b_1 = \int_0^1 \sin^2 \pi t \, dt = \frac{1}{2}$$

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin \pi t - \frac{2}{\pi}\left[\frac{1}{3}\cos 2\pi t + \frac{1}{15}\cos 4\pi t + \frac{1}{35}\cos 6\pi t + \frac{1}{63}\cos 8\pi t + \cdots\right]$$



14. $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \sin t \, dt = 0$$

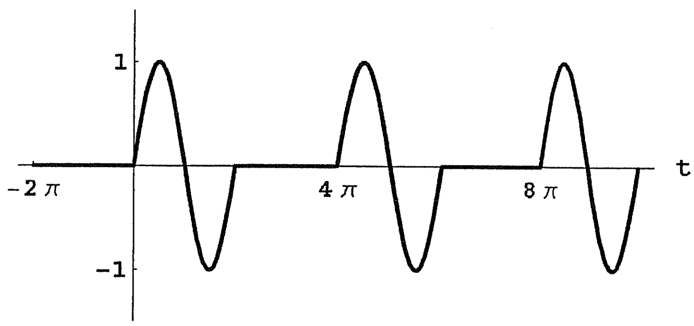
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \sin t \cos \frac{nt}{2} \, dt = \frac{2(\cos n\pi - 1)}{\pi(n^2-4)} = \frac{2\left[(-1)^n - 1\right]}{\pi(n^2-4)} \quad \text{for } n \neq 2$$

$$a_2 = \frac{1}{2\pi}\int_0^{2\pi} \sin t \cos t \, dt = 0$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \sin t \sin\frac{nt}{2}\,dt = \frac{2\sin n\pi}{\pi(n^2-4)} = 0 \text{ for } n \neq 2$$

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 t\,dt = \frac{1}{2}$$

$$f(t) = \frac{1}{2}\sin t + \frac{4}{\pi}\left[\frac{1}{3}\cos\frac{t}{2} - \frac{1}{5}\cos\frac{3t}{2} - \frac{1}{21}\cos\frac{5t}{2} - \frac{1}{45}\cos\frac{7t}{2} - \cdots\right] \quad \text{(figure below)}$$

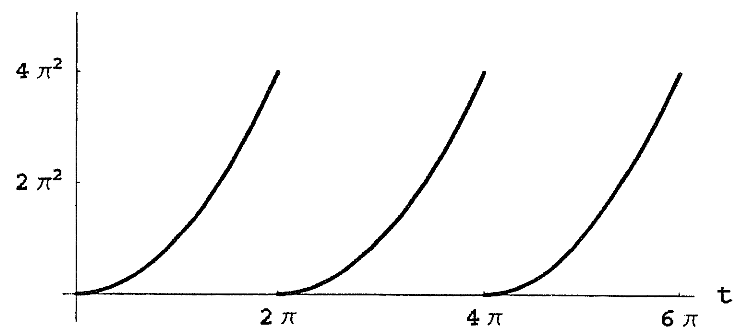


15. **(a)** $$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt\,dt = \frac{4n\pi\cos 2n\pi + 2(2n^2\pi^2-1)\sin 2n\pi}{\pi n^3} = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt = \frac{(2-4n^2\pi^2)\cos 2n\pi + 4n\pi\sin 2n\pi - 2}{\pi n^3} = -\frac{4\pi}{n}$$

$$f(t) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nt}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nt}{n} \quad \text{(figure below)}$$



(b) If we substitute $t = 0$ in the Fourier series of part (a) and note that $f(0) = \frac{1}{2}[f(0-) + f(0+)] = \frac{1}{2}\left[(2\pi)^2 + (0)^2\right] = 2\pi^2$, we get

$$2\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}, \quad \text{so} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

When we substitute $t = \pi$ and $f(\pi) = \pi^2$ in the series of part (a) we get

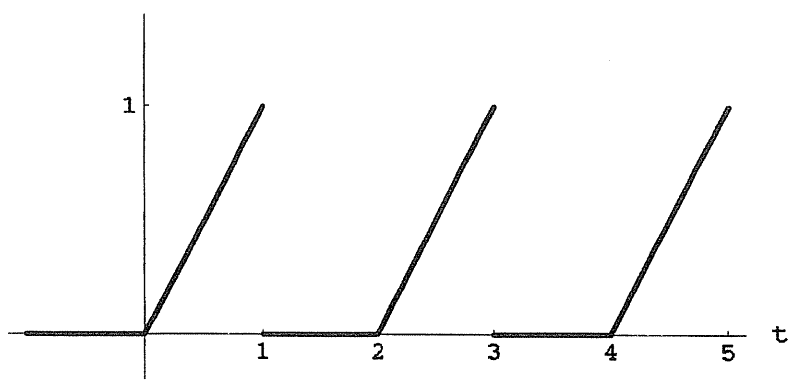
$$\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}, \quad \text{so} \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

16. **(a)** $a_0 = \int_0^1 t\,dt = \frac{1}{2}$

$$a_n = \int_0^1 t\cos n\pi t\,dt = \frac{\cos n\pi + n\pi\sin n\pi - 1}{n^2\pi^2} = \frac{(-1)^n - 1}{n^2\pi^2}$$

$$b_n = \int_0^1 t\sin n\pi t\,dt = \frac{\sin n\pi - n\pi\cos n\pi}{n^2\pi^2} = \frac{(-1)^{n+1}}{n\pi}$$

$$f(t) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{n\text{ odd}}\frac{\cos n\pi t}{n^2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi t}{n}$$ (figure below)



(b) Substitution of $t = 0$, $f(t) = 0$ in this series immediately gives $\sum_{n\text{ odd}}\frac{1}{n^2} = \frac{\pi^2}{8}$.

17. **(a)** $a_0 = \int_0^2 t\,dt = 2$

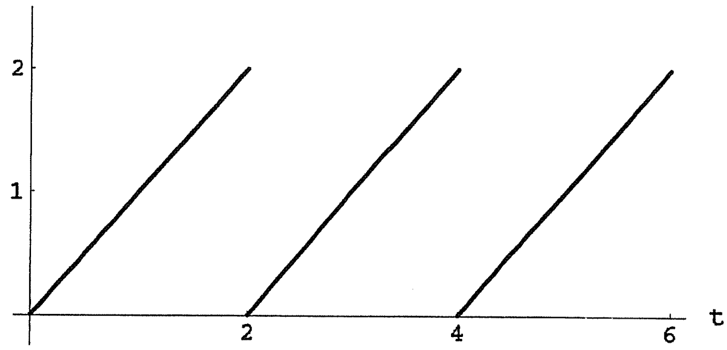
$$a_n = \int_0^2 t\cos n\pi t\,dt = \frac{\cos 2n\pi + 2n\pi\sin 2n\pi - 1}{n^2\pi^2} = 0$$

$$b_n = \int_0^2 t\sin n\pi t\,dt = \frac{\sin 2n\pi - 2n\pi\cos 2n\pi}{n^2\pi^2} = -\frac{2}{n\pi}$$

$$f(t) = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi t}{n}$$ (see figure on next page)

(b) Substitution of $t = 1/2$, $f(t) = 1/2$ in this series gives

$$\frac{1}{2} = 1 - \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right), \quad \text{so} \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$



The most efficient approach to Problems 18 and 20 is to derive first the expansions

$$t = \pi - 2\left[\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right],$$

$$t^2 = \frac{4\pi^2}{3} + 4\left[\cos t + \frac{\cos 2t}{4} + \frac{\cos 3t}{9} + \frac{\cos 4t}{16} + \cdots\right]$$
$$-4\pi\left[\sin t + \frac{\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 4t}{4} + \cdots\right].$$

for $0 < t < 2\pi$, as the Fourier series of the functions $f(t)$ and $g(t)$ of period 2π defined for $0 < t < 2\pi$ by $f(t) = t$ and $g(t) = t^2$. The first series above yields the series in Problem 18, and a combination of the two yields the series in Problem 20.

The expansions in Problems 19 and 21 are valid on the interval $-\pi < t < \pi$ rather than the interval $0 < t < 2\pi$. When we calculate the Fourier series of the functions $f(t)$ and $g(t)$ of period 2π defined for $-\pi < t < \pi$ by $f(t) = t$ and $g(t) = t^2$, we find that

$$t = 2\left[\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \cdots\right],$$

$$t^2 = \frac{\pi^2}{3} - 4\left[\cos t - \frac{\cos 2t}{4} + \frac{\cos 3t}{9} - \frac{\cos 4t}{16} + \cdots\right]$$

if $-\pi < t < \pi$.

18. First we derive the the Fourier series of the function $f(t)$ of period 2π defined for $0 < t < 2\pi$ by $f(t) = t$.

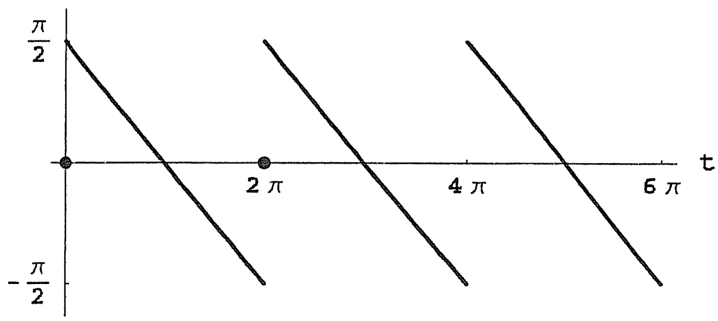
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t\,dt = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t\cos nt\,dt = \frac{\cos 2n\pi + 2n\pi\sin 2n\pi - 1}{n^2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t\sin nt\,dt = \frac{\sin 2n\pi - 2n\pi\cos 2n\pi}{n^2\pi} = -\frac{2}{n}$$

$$f(t) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

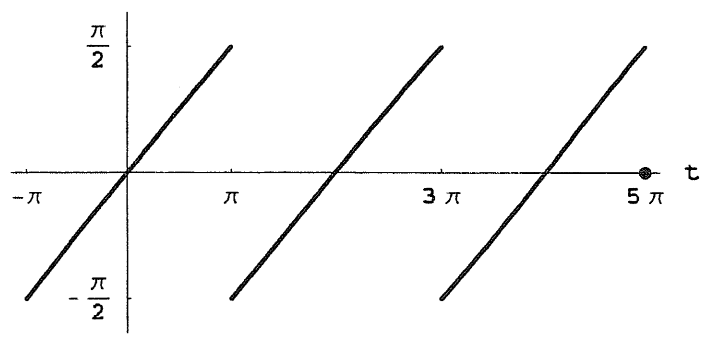
Hence $\dfrac{\pi - t}{2} = \dfrac{1}{2}[\pi - f(t)] = \displaystyle\sum_{n=1}^{\infty}\frac{\sin nt}{n}$ for $0 < t < 2\pi$ (figure below).



19. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\frac{t}{2}\,dt = 0,\qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t}{2}\cos nt\,dt = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t}{2}\sin nt\,dt = \frac{\sin n\pi - n\pi\cos n\pi}{n^2\pi} = \frac{(-1)^{n+1}}{n}$$

$$\frac{t}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nt}{n} \qquad (-\pi < t < \pi) \qquad \text{(figure below)}$$



20. First we derive the the Fourier series of the function $g(t)$ of period 2π defined for $0 < t < 2\pi$ by $g(t) = t^2$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2\cos nt\,dt = \frac{4n\pi\cos 2n\pi + 2(2n^2\pi^2-1)\sin 2n\pi}{n^3\pi} = \frac{4}{n^2}$$

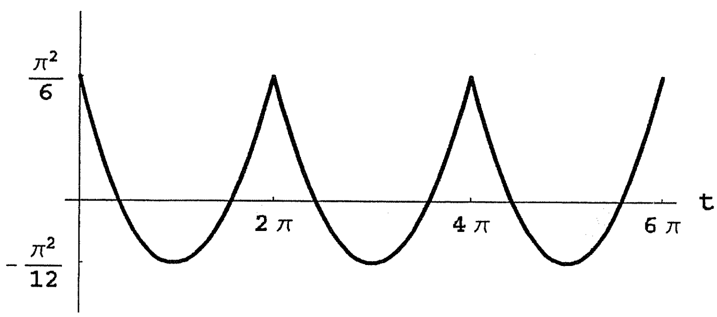
$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2\sin nt\,dt = \frac{(2-4n^2\pi^2)\cos 2n\pi + 4n\pi\sin 2n\pi - 2}{n^3\pi} = -\frac{4\pi}{n}$$

$$g(t) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nt}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nt}{n}$$

If $f(t)$ is the function of Problem 18, then for $0 < t < 2\pi$ we have

$$\frac{3t^2 - 6\pi t + 2\pi^2}{12} = \frac{1}{4}g(t) - \frac{\pi}{2}f(t) + \frac{\pi^2}{6}$$

$$= \frac{1}{4}\left(\frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nt}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nt}{n}\right) - \frac{\pi}{2}\left(\pi - 2\sum_{n=1}^{\infty}\frac{\sin nt}{n}\right) + \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{\cos nt}{n^2}.$$



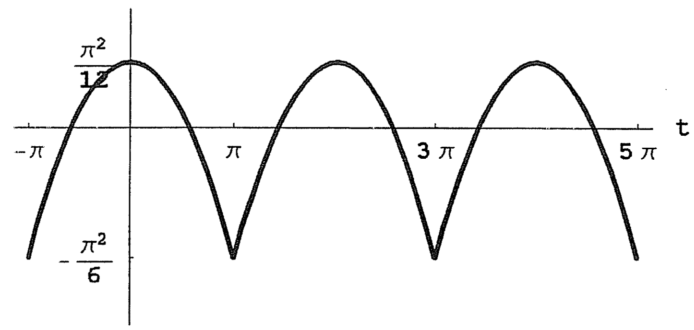
21. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{\pi^2}{3},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos nt\,dt = \frac{4n\pi\cos n\pi + 2(n^2\pi^2-1)\sin n\pi}{n^3\pi} = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\sin nt\,dt = 0$$

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nt}{n^2} \qquad (-\pi < t < \pi)$$

$$\frac{\pi^2 - 3t^2}{12} = \frac{\pi^2}{12} - \frac{1}{4}\left(\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nt}{n^2}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nt}{n^2}$$ (figure on next page)

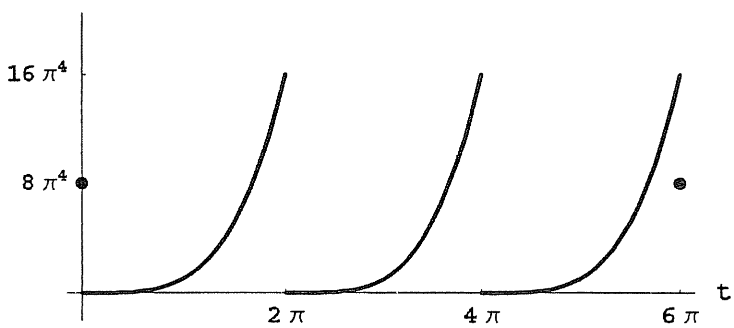


24. $$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^4\,dt = \frac{32\pi^4}{5}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^4 \cos nt\,dt$$
$$= \frac{8\left[2n\pi\left(2n^2\pi^2-3\right)\cos 2n\pi + (2n^4\pi^4 - 6n^2\pi^2+3)\sin 2n\pi\right]}{n^5\pi} = 16\left(\frac{2\pi^2}{n^2}-\frac{3}{n^4}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^4 \sin nt\,dt$$
$$= -\frac{8\left[(2n^4\pi^4 - 6n^2\pi^2+3)\cos 2n\pi + (6n\pi - 4n^3\pi^3)\sin 2n\pi - 3\right]}{n^5\pi} = 16\pi\left(\frac{3}{n^2}-\frac{\pi^2}{n}\right)$$

$$f(t) = \frac{16\pi^4}{5} + 16\sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n^2}-\frac{3}{n^4}\right)\cos nt + 16\pi\sum_{n=1}^{\infty}\left(\frac{3}{n^2}-\frac{\pi^2}{n}\right)\sin nt \qquad \text{(figure below)}$$



(b) When we substitute $t = 0, f(0) = 8\pi^4$ in the series of part (a) we get

$$8\pi^4 = \frac{16\pi^4}{5} + 32\pi^2\sum_{n=1}^{\infty}\frac{1}{n^2} - 48\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{16\pi^4}{5} + 32\pi^2\left(\frac{\pi^2}{6}\right) - 48\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

We now solve readily for $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$. Similarly, we find that $\sum_{n=1}^{\infty} (-1)^{n+1}/n^4 = 7\pi^4/720$ by substituting $t = \pi$, $f(\pi) = \pi^4$ in the series of part (a). Finally, addition of the first two series stated in part (b) yields the third one.

25. Now we want to sum the alternating series

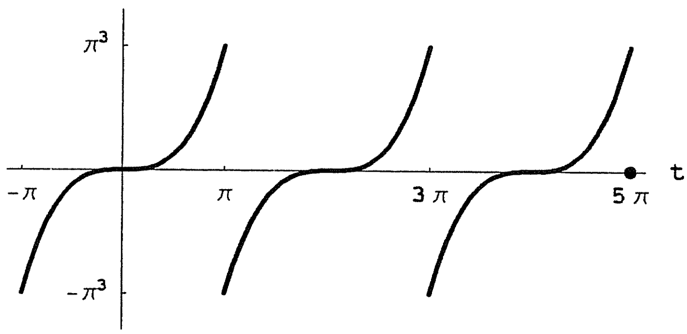
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \cdots$$

of reciprocals of odd cubes. Having used a Fourier series of t^4 in Problem 24 to evaluate $\Sigma(1/n^4)$, it is natural to look at a Fourier series of t^3. Let $f(t)$ be the period 2π function with $f(t) = t^3$ if $-\pi < t < \pi$. We calculate the Fourier coefficients of $f(t)$, and get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3\,dt = 0, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3 \cos nt\,dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3 \sin nt\,dt = -\frac{2n\pi\left(n^2\pi^2 - 6\right)\cos n\pi - 6(n^2\pi^2 - 2)\sin n\pi}{n^4\pi} = 2\left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)$$

$$t^3 = 2\pi^2 \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin nt}{n} - 12\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin nt}{n^3}.$$ (figure below)



If we substitute $t = \pi/2$ and use Leibniz's series $\Sigma(-1)^{n+1}/n = \pi/4$ of Problem 17 we find that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \cdots = \frac{\pi^3}{32}.$$

There is *no* value of t whose substitution in the Fourier series of $f(t) = t^3$ yields the series $\Sigma(1/n^3)$ containing the reciprocal cubes of *both* the odd and even integers. Indeed, the summation in "closed form" of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots$$

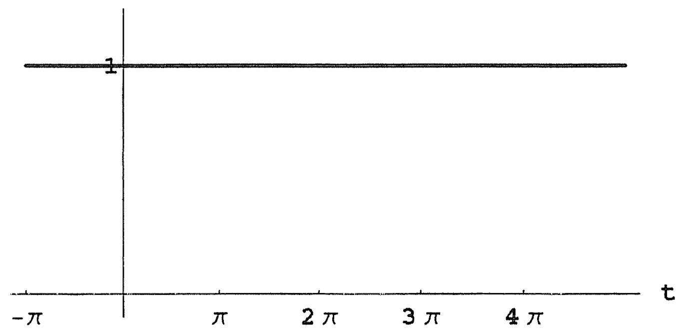
is a problem that has challenged many fine mathematicians since the time of Euler. Only in modern times (by R. Apery in 1978) has it been shown that this sum is an irrational number. For a delightful account of this work, see the article "A Proof that Euler Missed . . . An Informal Report" by Alfred van der Poorten in the *The Mathematical Intelligencer*, Volume 1 (1979), pages 195–203.

SECTION 8.3

FOURIER SINE AND COSINE SERIES

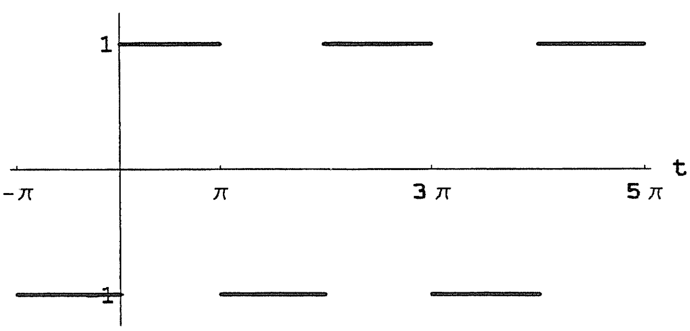
1. $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} 1\,dt = 2, \qquad a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \cos nt\,dt = \dfrac{2\sin n\pi}{n\pi} = 0$

Cosine series: $f(t) = 1$



$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nt\,dt = \frac{2(1-\cos n\pi)}{n\pi} = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

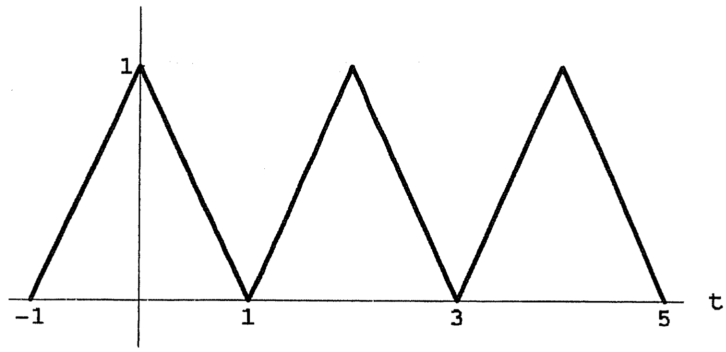
Sine series: $f(t) = \dfrac{4}{\pi}\left(\sin t + \dfrac{1}{3}\sin 3t + \dfrac{1}{5}\sin 5t + \dfrac{1}{7}\sin 7t + \cdots\right)$



2. $a_0 = 2\int_0^1 (1-t)\,dt = 1$

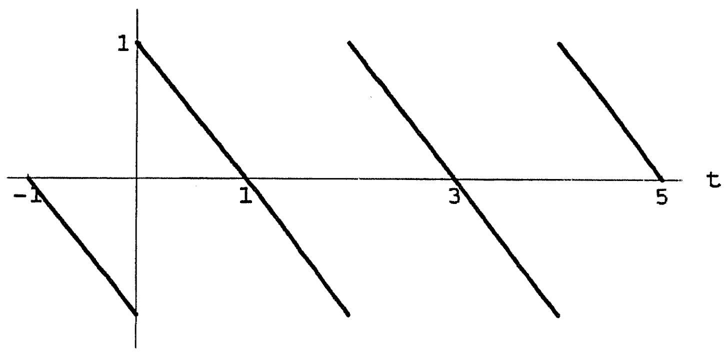
$$a_n = 2\int_0^1 (1-t)\cos n\pi t\,dt = \frac{2(1-\cos n\pi)}{n^2\pi^2} = \frac{2}{n^2\pi^2}\left[1-(-1)^2\right]$$

Cosine series: $f(t) = \dfrac{1}{2} + \dfrac{4}{\pi^2}\left(\cos \pi t + \dfrac{\cos 3\pi t}{3^2} + \dfrac{\cos 5\pi t}{5^2} + \dfrac{\cos 7\pi t}{7^2} + \cdots\right)$



$$b_n = 2\int_0^1 (1-t)\sin n\pi t\,dt = \frac{2(n\pi - \sin n\pi)}{n^2\pi^2} = \frac{2}{n\pi}$$

Sine series: $f(t) = \dfrac{2}{\pi}\left(\sin \pi t + \dfrac{\sin 2\pi t}{2} + \dfrac{\sin 3\pi t}{3} + \dfrac{\sin 4\pi t}{4} + \cdots\right)$

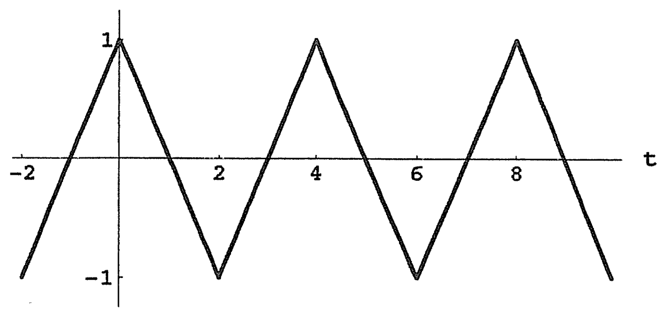


3. $a_0 = \int_0^2 (1-t)\,dt = 0$

$$a_n = \int_0^2 (1-t)\cos\frac{n\pi t}{2}\,dt = \frac{4 - 4\cos n\pi - 2n\pi \sin n\pi}{n^2\pi^2} = \frac{4}{n^2\pi^2}\left[1-(-1)^2\right]$$

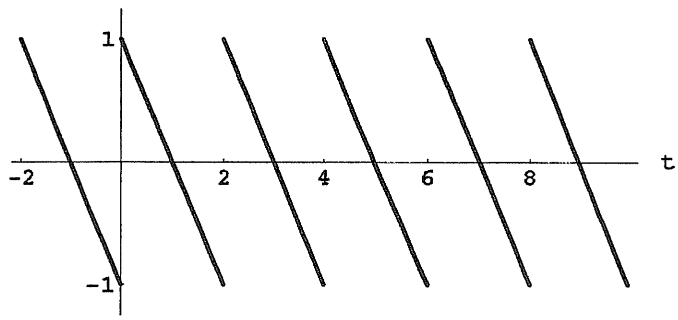
Cosine series: $f(t) = \dfrac{8}{\pi^2}\left(\cos\dfrac{\pi t}{2} + \dfrac{1}{3^2}\cos\dfrac{3\pi t}{2} + \dfrac{1}{5^2}\cos\dfrac{5\pi t}{2} + \dfrac{1}{7^2}\cos\dfrac{7\pi t}{2} + \cdots\right)$

See figure at top of next page.



$$b_n = \int_0^2 (1-t)\sin\frac{n\pi t}{2}\,dt = \frac{2n\pi(1+\cos n\pi)-2\sin n\pi}{n^2\pi^2} = \frac{2}{n\pi}\left[1+(-1)^n\right]$$

Sine series: $f(t) = \dfrac{4}{\pi}\left(\dfrac{\sin\pi t}{2}+\dfrac{\sin 2\pi t}{4}+\dfrac{\sin 3\pi t}{6}+\dfrac{\sin 4\pi t}{8}+\cdots\right)$

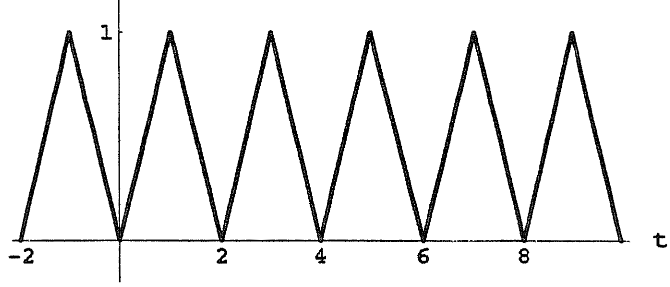


4. $a_0 = \int_0^1 t\,dt + \int_1^2 (2-t)\,dt = 1$

$$a_n = \int_0^1 t\cos\frac{n\pi t}{2}\,dt + \int_1^2 (2-t)\cos\frac{n\pi t}{2}\,dt$$

$$= \frac{16}{n^2\pi^2}\cos\frac{n\pi}{2}\sin^2\frac{n\pi}{4} = \begin{cases} 0 \text{ for } n \text{ odd} \\ 0 \text{ if } n = 4, 8, 12, \cdots \\ -16/n^2\pi^2 \text{ if } n = 2, 6, 10, \cdots \end{cases}$$

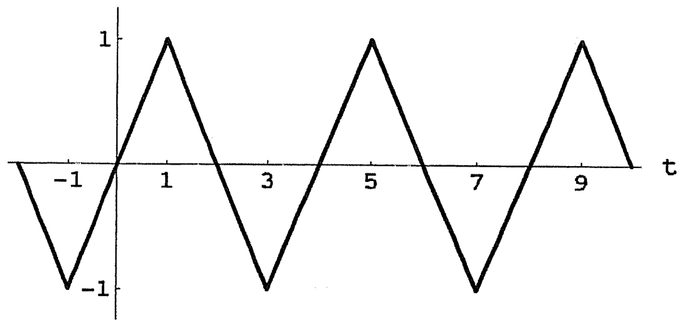
Cosine series: $f(t) = 1 - \dfrac{16}{\pi^2}\left(\dfrac{\cos\pi t}{2^2}+\dfrac{\cos 3\pi t}{6^2}+\dfrac{\cos 5\pi t}{10^2}+\dfrac{\cos 7\pi t}{14^2}+\cdots\right)$



$$b_n = \int_0^1 t\sin\frac{n\pi t}{2}dt + \int_1^2 (2-t)\sin\frac{n\pi t}{2}dt$$

$$= \frac{32}{n^2\pi^2}\cos\frac{n\pi}{4}\sin^3\frac{n\pi}{4} = \begin{cases} 0 \text{ for } n \text{ even} \\ +8/n^2\pi^2 \quad \text{if } n = 1, 5, 9, \cdots \\ -8/n^2\pi^2 \quad \text{if } n = 3, 7, 11, \cdots \end{cases}$$

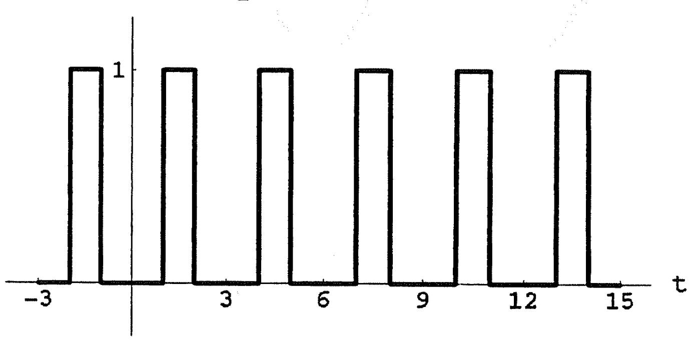
Sine series: $$f(t) = \frac{8}{\pi^2}\left(\sin\frac{\pi t}{2} - \frac{1}{3^2}\sin\frac{3\pi t}{2} + \frac{1}{5^2}\sin\frac{5\pi t}{2} - \frac{1}{7^2}\sin\frac{7\pi t}{2} + \cdots\right)$$



5. $$a_0 = \frac{2}{3}\int_1^2 1\,dt = \frac{2}{3}$$

$$a_n = \frac{2}{3}\int_1^2 \cos\frac{n\pi t}{3}dt = \frac{2}{n\pi}\left(\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3}\right) = \begin{cases} -2\sqrt{3}/n\pi \quad \text{if } n = 2, 8, 14, \cdots \\ +2\sqrt{3}/n\pi \quad \text{if } n = 4, 10, 16, \cdots \\ 0 \quad \text{otherwise} \end{cases}$$

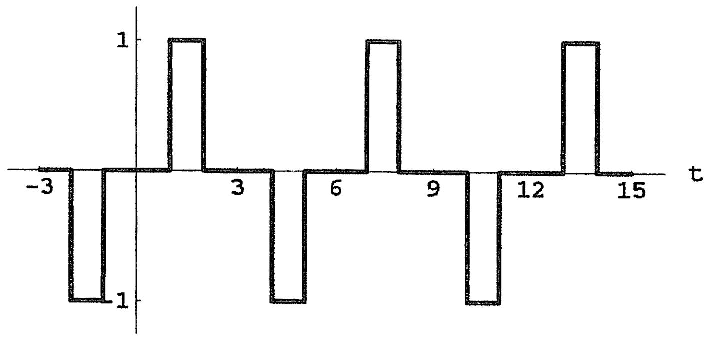
Cosine series: $$f(t) = \frac{1}{3} - \frac{2\sqrt{3}}{\pi}\left[\frac{1}{2}\cos\frac{2\pi t}{3} - \frac{1}{4}\cos\frac{4\pi t}{3} + \frac{1}{8}\cos\frac{8\pi t}{3} - \frac{1}{10}\cos\frac{10\pi t}{3} + \cdots\right]$$



$$b_n = \frac{2}{3}\int_1^2 \sin\frac{n\pi t}{3}dt = \frac{2}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right) = \begin{cases} 0 \text{ for } n \text{ even} \\ +2/n\pi \quad \text{if } n = 1, 7, 13, \cdots \\ -4/n\pi \quad \text{if } n = 3, 9, 15, \cdots \\ +2/n\pi \quad \text{if } n = 5, 11, 17, \cdots \end{cases}$$

Sine series:

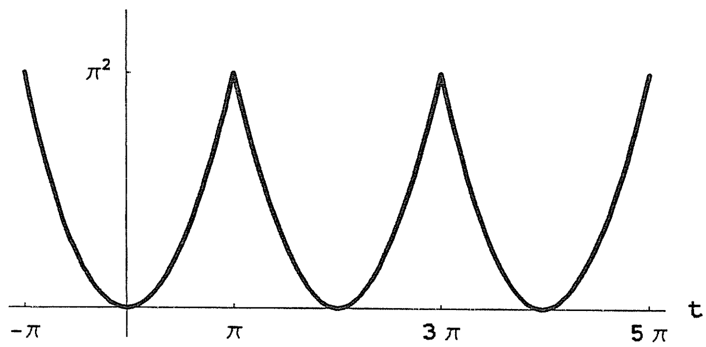
$$f(t) = \frac{2}{\pi}\left[\sin\frac{\pi t}{3} - \frac{2}{3}\sin\frac{3\pi t}{3} + \frac{1}{5}\sin\frac{5\pi t}{3} + \frac{1}{7}\sin\frac{7\pi t}{3} - \frac{2}{9}\sin\frac{9\pi t}{3} + \frac{1}{11}\sin\frac{11\pi t}{3} + \cdots\right]$$



6. $$a_0 = \frac{2}{\pi}\int_0^{\pi} t^2\,dt = \frac{2\pi^2}{3},$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t^2\cos nt\,dt = \frac{2\left[2n\pi\cos n\pi + (n^2\pi^2 - 2)\sin n\pi\right]}{n^3\pi} = \frac{4(-1)^n}{n^2}$$

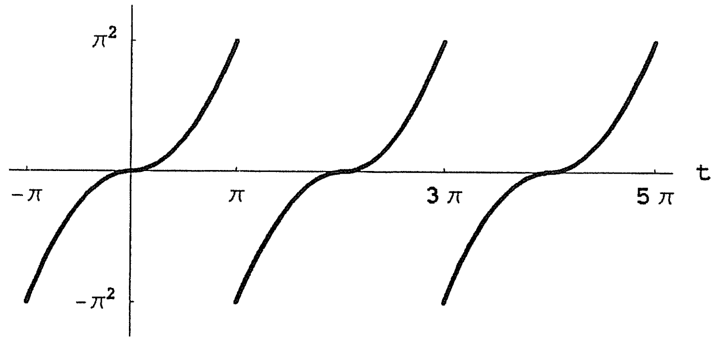
Cosine series: $f(t) = \dfrac{\pi^2}{3} - 4\left(\cos t - \dfrac{1}{2^2}\cos 2t + \dfrac{1}{3^2}\cos 3t + \dfrac{1}{4^2}\cos 4t + \cdots\right)$



$$b_n = \frac{2}{\pi}\int_0^{\pi} t^2\sin nt\,dt$$

$$= -\frac{2\left[(n^2\pi^2 - 2)\cos n\pi - 2n\pi\sin n\pi + 2\right]}{n^3\pi} = \begin{cases} +2\pi/n - 8/n^3\pi & \text{for } n \text{ odd} \\ -2\pi/n & \text{for } n \text{ even} \end{cases}$$

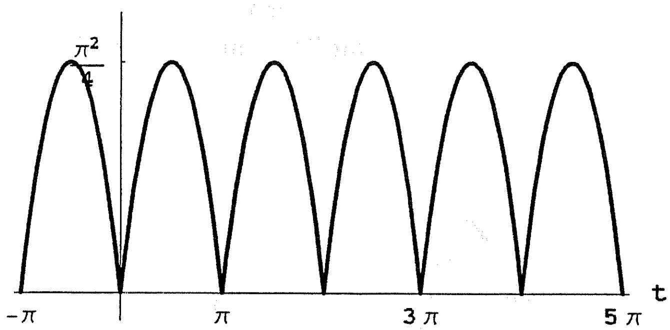
Sine series: $$f(t) = 2\pi\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \cdots\right)$$
$$-\frac{8}{\pi}\left(\sin t + \frac{1}{3^3}\sin 3t + \frac{1}{5^3}\sin 5t + \frac{1}{7^3}\sin 7t + \cdots\right)$$



7. $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} t(\pi - t)\,dt = \dfrac{\pi^2}{3},$

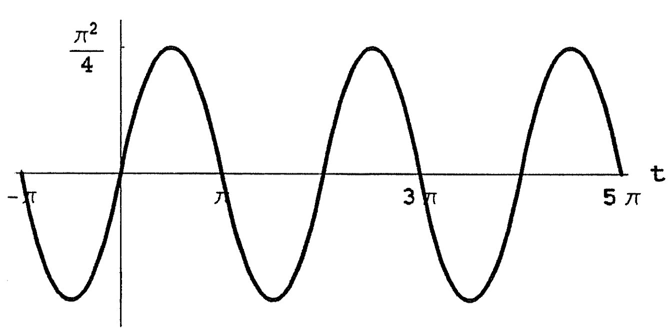
$$a_n = \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\cos nt\,dt = -\frac{2[n\pi\cos n\pi + n\pi - 2\sin n\pi]}{n^3\pi} = -\frac{2}{n^2}\left[1 + (-1)^n\right]$$

Cosine series: $f(t) = \dfrac{\pi^2}{6} - 4\left(\dfrac{\cos 2t}{2^2} + \dfrac{\cos 4t}{4^2} + \dfrac{\cos 6t}{6^2} + \dfrac{\cos 8t}{8^2} + \cdots\right)$



$$b_n = \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\sin nt\,dt = \frac{2[2 - 2\cos n\pi - 2n\pi\sin n\pi]}{n^3\pi} = \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$$

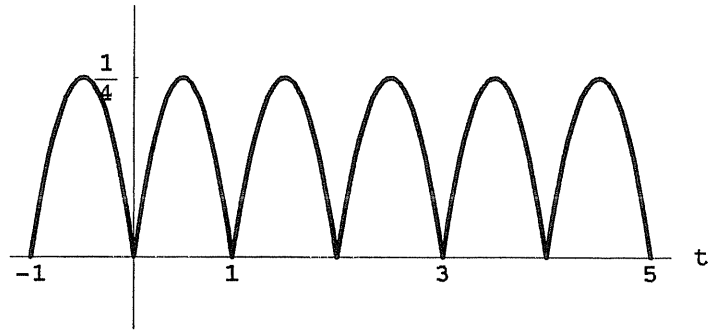
Sine series: $f(t) = \dfrac{8}{\pi}\left(\sin t + \dfrac{\sin 3t}{3^3} + \dfrac{\sin 5t}{5^3} + \dfrac{\sin 7t}{7^3} + \cdots\right)$



8. $a_0 = 2\int_0^1 (t - t^2)\, dt = \frac{1}{3},$

$$a_n = 2\int_0^1 (t-t^2)\cos n\pi t\, dt = -\frac{2\left[n\pi\cos n\pi + n\pi - 2\sin n\pi\right]}{n^3\pi^3} = -\frac{2}{n^2\pi^2}\left[1+(-1)^n\right]$$

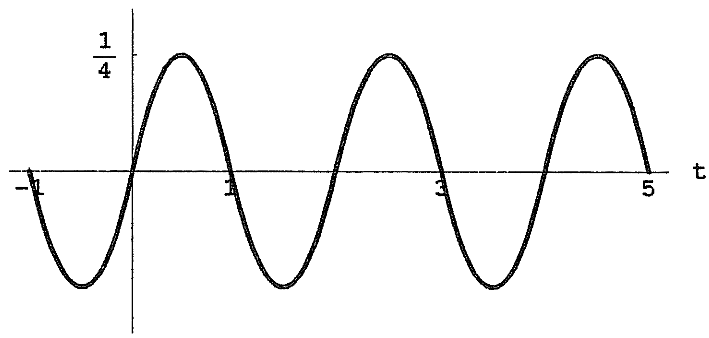
Cosine series: $f(t) = \frac{1}{6} - \frac{4}{\pi^2}\left(\frac{\cos 2\pi t}{2^2} + \frac{\cos 4\pi t}{4^2} + \frac{\cos 6\pi t}{6^2} + \frac{\cos 8\pi t}{8^2} + \cdots\right)$



$$b_n = 2\int_0^1 (t-t^2)\sin n\pi t\, dt = \frac{2\left[2 - 2\cos n\pi - n\pi\sin n\pi\right]}{n^3\pi^3} = \frac{4}{n^3\pi^3}\left[1-(-1)^n\right]$$

Sine series: $f(t) = \frac{8}{\pi^3}\left(\sin \pi t + \frac{\sin 3\pi t}{3^3} + \frac{\sin 5\pi t}{5^3} + \frac{\sin 7\pi t}{7^3} + \cdots\right)$

for $0 < t < 1$. Note that $t = 1/2$ yields the summation of Problem 25 in Section 8.2.

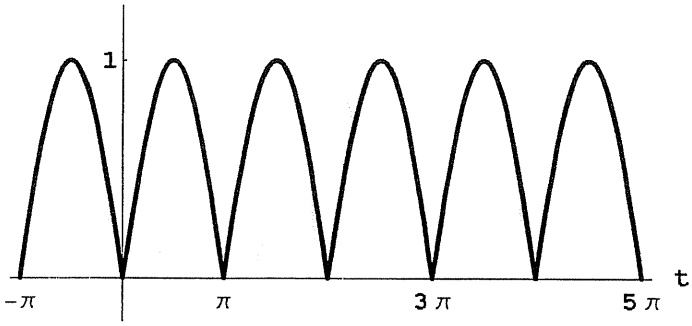


9. $a_0 = \frac{2}{\pi}\int_0^\pi \sin t\, dt = \frac{4}{\pi},$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin t\cos nt\, dt = \frac{2[1+\cos n\pi]}{\pi(1-n^2)} = \frac{2\left[1+(-1)^n\right]}{\pi(1-n^2)} \quad \text{if } n > 1$$

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin t\cos t\, dt = 0$$

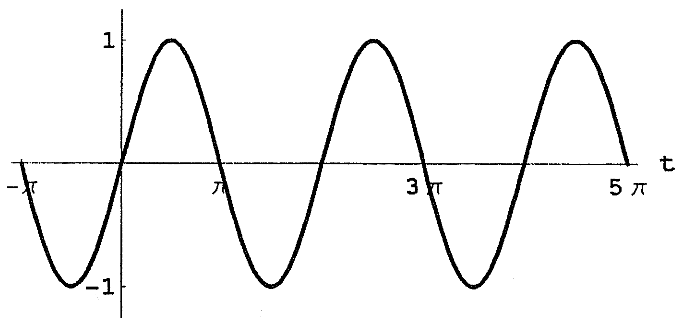
Cosine series: $f(t) = \frac{2}{\pi} - \frac{4}{\pi^2}\left(\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \frac{\cos 8t}{63} + \cdots\right)$



$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin t \sin nt\, dt = \frac{2\sin n\pi}{\pi(1-n^2)} = 0 \text{ if } n > 1$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 t\, dt = 1$$

Sine series: $\quad f(t) = \sin t$



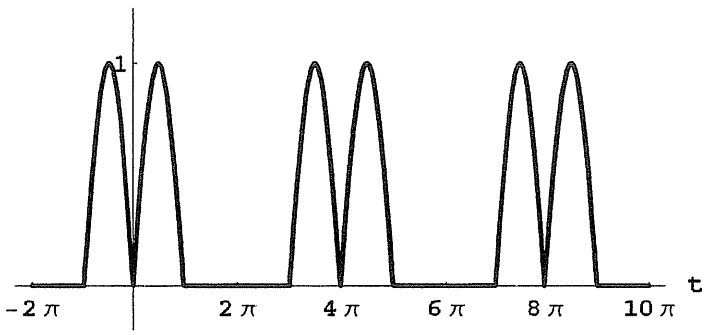
10. $$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin t\, dt = \frac{2}{\pi},$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin t \cos\frac{nt}{2}\, dt = -\frac{4\left[1+\cos\frac{n\pi}{2}\right]}{\pi(n^2-4)} = \begin{cases} -4/\pi(n^2-4) \text{ for } n \text{ odd} \\ -8/\pi(n^2-4) \text{ if } n = 4, 8, 12, \cdots \\ 0 \text{ if } n = 6, 10, 14, \cdots \end{cases}$$

$$a_2 = \frac{1}{\pi}\int_0^{\pi} \sin t \cos t\, dt = 0$$

Cosine series: $$f(t) = \frac{1}{\pi} - \frac{4}{\pi}\left(-\frac{1}{3}\cos\frac{t}{2} + \frac{1}{5}\cos\frac{3t}{2} + \frac{2}{12}\cos\frac{4t}{2} + \frac{1}{21}\cos\frac{5t}{2} + \frac{1}{45}\cos\frac{7t}{2} + \frac{2}{60}\cos\frac{8t}{2} + \cdots\right)$$

See the figure at the top of the next page.

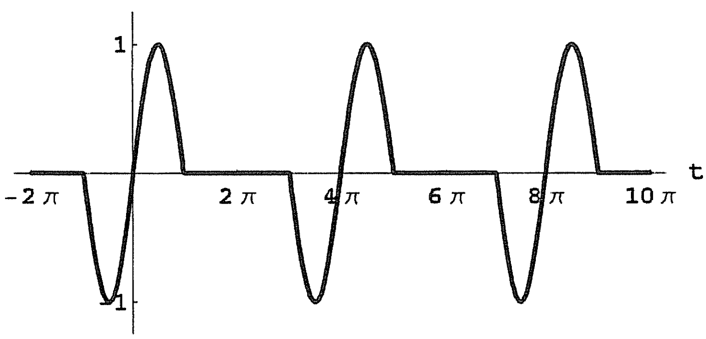


$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin t \sin\frac{nt}{2}\,dt = -\frac{4\sin\frac{n\pi}{2}}{\pi(n^2-4)} = \begin{cases} 0 \text{ for } n>2 \text{ even} \\ -4/\pi(n^2-4) \text{ if } n=1,5,9,\cdots \\ +4/\pi(n^2-4) \text{ if } n=3,7,11,\cdots \end{cases}$$

$$b_2 = \frac{1}{\pi}\int_0^{\pi} \sin^2 t\,dt = \frac{1}{2}$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 t\,dt = 1$$

Sine series: $f(t) = \frac{1}{2}\sin t - \frac{4}{\pi}\left(\frac{1}{3}\sin\frac{t}{2} + \frac{1}{5}\sin\frac{3t}{2} - \frac{1}{21}\sin\frac{5t}{2} + \frac{1}{45}\sin\frac{7t}{2} - \frac{1}{77}\sin\frac{9t}{2} + \cdots\right)$



11. In order to satisfy the endpoint conditions $x(0) = x(\pi) = 0$ we substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin nt$ and $1 = \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin nt}{n}$ (from Example 1 in Section 8.1) into the differential equation $x'' + 2x = 1$. This gives

$$-\sum_{n=1}^{\infty} n^2 b_n \sin nt + 2\sum_{n=1}^{\infty} b_n \sin nt = \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\sin nt}{n}.$$

We therefore choose $b_n = 4/\pi n(2-n^2)$ for n odd, $b_n = 0$ for n even. This gives the formal series solution

$$x(t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n(2-n^2)} = \frac{4}{\pi}\left(\sin t - \frac{\sin 3t}{21} - \frac{\sin 5t}{115} - \frac{\sin 7t}{329} - \cdots\right).$$

12. In order to satisfy the endpoint conditions $x(0) = x(\pi) = 0$ we substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin nt$ and $1 = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}$ (from Example 1 in Section 8.1) into the differential equation $x'' - 4x = 1$. This gives

$$-\sum_{n=1}^{\infty} n^2 b_n \sin nt - 4\sum_{n=1}^{\infty} b_n \sin nt = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}.$$

We therefore choose $b_n = -4/\pi n(n^2+4)$ for n odd, $b_n = 0$ for n even. This gives the formal series solution

$$x(t) = -\frac{4}{\pi}\left[\frac{\sin t}{5} + \frac{\sin 3t}{39} + \frac{\sin 5t}{145} + \frac{\sin 7t}{371} + \cdots\right].$$

13. In order to satisfy the endpoint conditions $x(0) = x(1) = 0$ we substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$ and $t = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin n\pi t}{n}$ (from Example 1 in Section 8.3, with $L = 1$) into the differential equation $x'' + x = t$. This gives

$$-\sum_{n=1}^{\infty} n^2\pi^2 b_n \sin n\pi t + \sum_{n=1}^{\infty} b_n \sin n\pi t = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin n\pi t}{n}.$$

We therefore choose $b_n = 2(-1)^{n+1}/\pi n(1-n^2\pi^2)$. This gives the formal series solution $x(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi t}{n(n^2\pi^2-1)}$ of our endpoint value problem.

14. In order to satisfy the endpoint conditions $x(0) = x(2) = 0$ we substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{2}$ and $t = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{2}$ (from Example 1 in Section 8.3 ,with $L = 2$) into the differential equation $x'' + 2x = t$. This gives

$$-\sum_{n=1}^{\infty} \frac{n^2\pi^2}{4} b_n \sin n\pi t + 2\sum_{n=1}^{\infty} b_n \sin n\pi t = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\frac{n\pi t}{2}.$$

We therefore choose

$$b_n = \frac{4(-1)^{n+1}/\pi n}{2 - n^2\pi^2/4} = \frac{16(-1)^{n+1}}{\pi n(8 - n^2\pi^2)}$$

for n odd, $b_n = 0$ for n even. This gives the formal series solution

$$x(t) = \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin(n\pi t/2)}{n(n^2\pi^2 - 8)} \quad \text{of our endpoint value problem.}$$

15. In order to satisfy the endpoint conditions $x'(0) = x'(2) = 0$ we substitute the cosine series $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$ and $t = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\cos nt}{n^2}$ (from Example 1 in Section 8.3, with $L = \pi$) into the differential equation $x'' + 2x = t$. This gives

$$-\sum_{n=1}^{\infty} n^2 a_n \cos nt + a_0 + 2\sum_{n=1}^{\infty} a_n \cos nt = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\cos nt}{n^2}.$$

We therefore choose $a_0 = \pi/2$, $a_n = 0$ for $n > 0$ even, and $a_n = 4/\pi n^2(n^2 - 2)$ for n odd. This gives the formal series solution

$$x(t) = \frac{\pi}{4} + \frac{4}{\pi}\sum_{n\text{ odd}}\frac{\cos nt}{n^2(n^2-2)} = \frac{\pi}{4} + \frac{4}{\pi}\left(-\cos t + \frac{\cos 3t}{63} + \frac{\cos 5t}{575} + \frac{\cos 7t}{2303} + \cdots\right)$$

of our endpoint value problem.

16. **(a)** Obviously $x_p(t) = t$ is a particular solution of $x'' + 4x = 4t$, so a general solution is given by $x(t) = A\cos 2t + B\sin 2t + t$. We satisfy the endpoint conditions $x(0) = x(2) = 0$ by choosing $A = 0$ and $B = -1/\sin 2$.

(b) The point is simply that the series in (31) is the Fourier sine series of the period 2 function defined by $f(t) = t - (\sin 2t)/(\sin 2)$ for $0 < t < 1$:

$$2\int_0^1 t\sin n\pi t\, dt = \frac{2[\sin n\pi - n\pi\cos n\pi]}{n^2\pi^2} = -\frac{2(-1)^n}{n\pi}$$

$$2\int_0^1 \sin 2t \sin n\pi t\, dt = \frac{2[2(\cos 2)\sin n\pi - n\pi(\sin 2)\cos n\pi]}{n^2\pi^2 - 4} = -\frac{2n\pi(-1)^n(\sin 2)}{n^2\pi^2 - 4}$$

$$b_n = -\frac{2(-1)^n}{n\pi} + \frac{2n\pi(-1)^n}{n^2\pi^2 - 4} = \frac{8(-1)^n}{n\pi(n^2\pi^2 - 4)}$$

$$t - \frac{\sin 2t}{\sin 2} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi t}{n(n^2\pi^2 - 4)} \quad \text{for } 0 < t < 1$$

17. *Suggestion*: Substitute $u = -t$ in the left-hand integral.

18. The termwise derivative of the given Fourier series is

$$-(4/\pi)\,\Sigma\,(\sin n\pi t)/n - 4\,\Sigma\cos n\pi t.$$

But the series $\Sigma \cos n\pi t$ diverges at $t = 0$ (for instance). Hence the derived series does not converge to any function at all, let alone to $f'(t)$.

19. The first termwise integration yields

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n \cos nt}{n^2} + C_1,$$

and substitution of $t=0$ gives $C_1 = 2\sum_{n=1}^{\infty}(-1)^{n+1}/n^2 = \pi^2/6$, so

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^n \cos nt}{n^2} + \frac{\pi^2}{6}.$$

A second termwise integration gives

$$\frac{t^3}{6} = 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin nt}{n^3} + \frac{\pi^2 t}{6} + C_2,$$

and substitution of $t=0$ gives $C_2 = 0$. The final termwise integration gives

$$\frac{t^4}{24} = -2\sum_{n=1}^{\infty}\frac{(-1)^n \cos nt}{n^4} + \frac{\pi^2 t^2}{12} + C_3,$$

and substitution of $t=0$ yields $C_3 = 2\sum_{n=1}^{\infty}(-1)^n/n^4$.

20. Substitution of $t=\pi$ in the formula of Problem 19 above gives

$$\frac{\pi^4}{24} = -2\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{\pi^4}{12} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4},$$

$$\frac{\pi^4}{24} = 2\sum_{n=1}^{\infty}\frac{1}{n^4} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = 4\sum_{n\text{ odd}}\frac{1}{n^4}$$

which gives $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$. Then

$$S = \sum_{n=1}^{\infty}\frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \cdots$$

$$= \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \cdots\right)$$

$$= \left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right)+\frac{1}{2^4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)$$

$$S = \left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right)+\frac{1}{15}S.$$

Solution of this last equation for S now gives

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{16}{15}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right) = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90}.$$

21. We want to calculate the coefficients in the period $4L$ Fourier sine series

$$F(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{2L}$$

which agrees with $f(t)$ if $0 < t < L$. Then

$$b_n = \frac{2}{2L}\int_0^L f(t)\sin\frac{n\pi t}{2L}dt+\frac{2}{2L}\int_L^{2L} f(2L-t)\sin\frac{n\pi t}{2L}dt.$$

The substitution $u = 2L-t$ yields

$$b_n = \frac{1}{L}\int_0^L f(t)\sin\frac{n\pi t}{2L}dt-\frac{1}{L}\int_L^0 f(u)\sin\frac{n\pi(2L-u)}{2L}du$$

$$= \frac{1}{L}\int_0^L f(t)\sin\frac{n\pi t}{2L}dt-\frac{(-1)^n}{L}\int_0^L f(u)\sin\frac{n\pi u}{2L}du.$$

Now it is clear that

$$b_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{2L}dt$$

if n is odd, whereas $b_n = 0$ if n is even.

22. We want to calculate the coefficients in the period $4L$ Fourier cosine series

$$G(t) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2L}$$

which agrees with $f(t)$ if $0 < t < L$. Then

$$a_n = \frac{2}{2L}\int_0^L f(t)\cos\frac{n\pi t}{2L}dt+\frac{2}{2L}\int_L^{2L} f(2L-t)\cos\frac{n\pi t}{2L}dt.$$

The substitution $u = 2L-t$ yields

$$a_n = \frac{1}{L}\int_0^L f(t)\cos\frac{n\pi t}{2L}dt - \frac{1}{L}\int_L^0 f(u)\cos\frac{n\pi(2L-u)}{2L}du$$
$$= \frac{1}{L}\int_0^L f(t)\cos\frac{n\pi t}{2L}dt - \frac{(-1)^n}{L}\int_0^L f(u)\cos\frac{n\pi u}{2L}du.$$

Now it is clear that

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{2L}dt$$

if n is odd, whereas $a_n = 0$ if n is even (including $n = 0$).

23. $$b_n = \frac{2}{\pi}\int_0^\pi t\sin\frac{nt}{2}dt = \frac{4}{\pi n^2}\left(2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right) = \frac{8(-1)^{(n-1)/2}}{\pi n^2} \quad \text{for } n \text{ odd}$$
$$f(t) = \frac{8}{\pi^2}\left(\sin\frac{t}{2} - \frac{1}{3^2}\sin\frac{3t}{2} + \frac{1}{5^2}\sin\frac{5t}{2} - \frac{1}{7^2}\sin\frac{7t}{2} + \cdots\right)$$

24. In order to satisfy the endpoint conditions $x(0) = x'(\pi) = 0$ we substitute the odd half-multiple sine series $x(t) = \sum_{n\text{ odd}} b_n \sin\frac{nt}{2}$ and $t = \frac{8}{\pi}\sum_{n\text{ odd}}\frac{(-1)^{(n-1)/2}}{n^2}\sin\frac{nt}{2}$ (from Problem 21) into the differential equation $x'' - x = t$. This gives

$$-\sum_{n\text{ odd}}\frac{n^2 b_n}{4}\sin\frac{nt}{2} + \sum_{n\text{ odd}} b_n \sin\frac{nt}{2} = \frac{8}{\pi}\sum_{n\text{ odd}}\frac{(-1)^{(n-1)/2}}{n^2}\sin\frac{nt}{2}.$$

We therefore choose

$$b_n = \frac{8(-1)^{(n-1)/2}/\pi n^2}{1-n^2/4} = \frac{32(-1)^{(n+1)/2}}{\pi n^2(n^2-4)}$$

for n odd. This gives the formal series solution

$$x(t) = \frac{32}{\pi}\sum_{n\text{ odd}}\frac{(-1)^{(n+1)/2}}{n^2(n^2-4)}\sin\frac{nt}{2}$$
$$= \frac{32}{\pi}\left(\frac{1}{3}\sin\frac{nt}{2} + \frac{1}{45}\sin\frac{3nt}{2} - \frac{1}{525}\sin\frac{5nt}{2} + \frac{1}{2205}\sin\frac{7nt}{2} - \cdots\right)$$

of our endpoint value problem.

SECTION 8.4

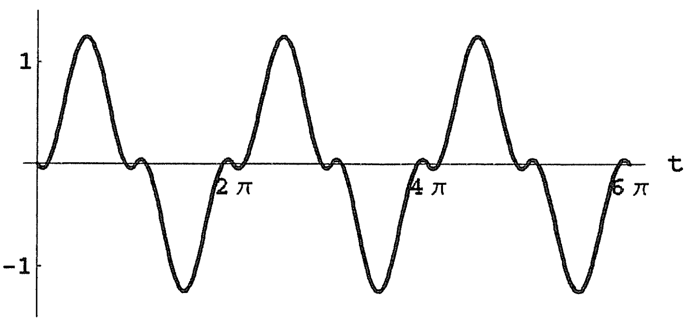
APPLICATIONS OF FOURIER SERIES

1. We substitute the sine series $x(t)=\sum_{n=1}^{\infty} b_n \sin nt$ and $F(t)=\frac{12}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}$ (from Example 1 in Section 8.1) into the differential equation $x''+5x=F(t)$. This gives

$$-\sum_{n=1}^{\infty} n^2 b_n \sin nt + 5\sum_{n=1}^{\infty} b_n \sin nt \;=\; \frac{12}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}.$$

We therefore choose $b_n=0$ for $n>0$ even, and $b_n = 12/\pi n(5-n^2)$ for n odd. This gives the formal series solution

$$x_{sp}(t) \;=\; \frac{12}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n(5-n^2)} \;=\; \frac{12}{\pi}\left(\frac{\sin t}{4}-\frac{\sin 3t}{12}-\frac{\sin 5t}{100}-\frac{\sin 7t}{308}-\cdots\right).$$



2. We substitute the cosine series $x(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2}$ and

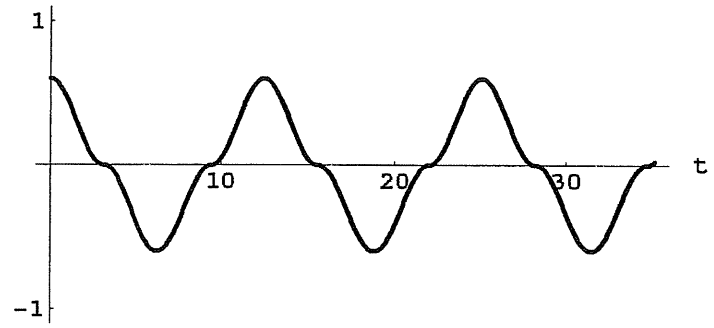
$F(t)=\frac{12}{\pi}\sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n}\cos\frac{n\pi t}{2}$ into the differential equation $x''+10x=F(t)$. This gives

$$-\sum_{n=1}^{\infty} \frac{n^2\pi^2}{4} a_n \cos\frac{n\pi t}{2} + 5a_0 + 10\sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2} \;=\; \frac{12}{\pi}\sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n}\cos\frac{n\pi t}{2}.$$

We therefore choose $a_0=0$ and $a_n=0$ for $n>0$ even, and

$$a_n \;=\; \frac{12(-1)^{(n-1)/2}/\pi n}{10-\pi^2 n^2/4} \;=\; \frac{48(-1)^{(n-1)/2}}{\pi n(40-\pi^2 n^2)}$$

for n odd. This gives the formal series solution $x_{sp}(t) = \frac{48}{\pi}\sum_{n\text{ odd}} \frac{(-1)^{(n-1)/2}}{n(40-\pi^2 n^2)} \cos\frac{n\pi t}{2}$.

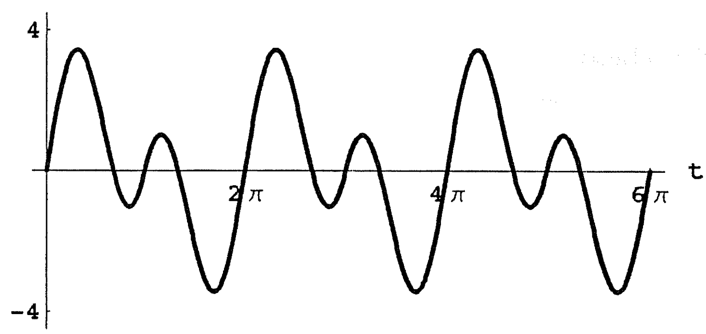


3. We substitute the sine series $x(t) = \sum_{n=1}^{\infty} b_n \sin nt$ and $F(t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nt}{n}$ (from Example 1 in Section 8.3, with $L = \pi$) into the differential equation $x'' + 3x = F(t)$. This gives

$$-\sum_{n=1}^{\infty} n^2 b_n \sin nt + 3\sum_{n=1}^{\infty} b_n \sin nt \;=\; 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nt}{n}.$$

We therefore choose $b_n = 4(-1)^{n-1}/n(3-n^2)$. This gives the formal series solution

$$x_{sp}(t) \;=\; 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nt}{n(3-n^2)} \;=\; 4\left(\frac{\sin t}{2} + \frac{\sin 2t}{2} - \frac{\sin 3t}{18} + \frac{\sin 4t}{52} - \cdots\right).$$



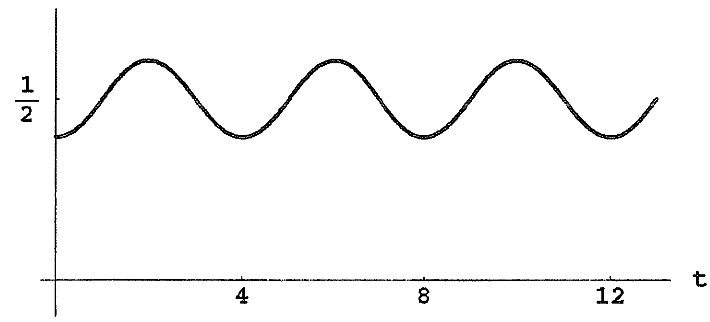
4. We substitute the cosine series $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2}$ and $F(t) = 2 - \frac{16}{\pi^2}\sum_{n\text{ odd}} \frac{1}{n^2}\cos\frac{n\pi t}{2}$ (from Example 1 in Section 8.3, with $L = 2$) into the differential equation $x'' + 4x = F(t)$. This gives

$$-\sum_{n=1}^{\infty}\frac{n^2\pi^2}{4}a_n\cos\frac{n\pi t}{2}+2a_0+4\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{2} \;=\; 2-\frac{16}{\pi^2}\sum_{n\text{ odd}}\frac{1}{n^2}\cos\frac{n\pi t}{2}.$$

We therefore choose $a_0 = 1$ and $a_n = 0$ for $n > 0$ even, and

$$a_n \;=\; \frac{-16/\pi^2 n^2}{4-\pi^2 n^2/4} \;=\; -\frac{64}{\pi^2 n^2(16-\pi^2 n^2)}$$

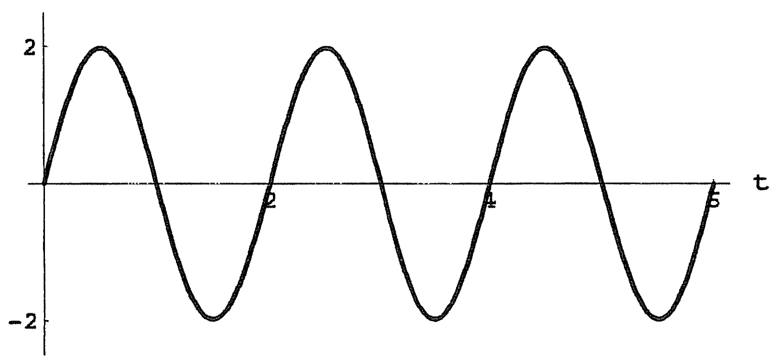
for n odd. This gives the formal series solution $x_{sp}(t) \;=\; \dfrac{1}{2}-\dfrac{64}{\pi^2}\displaystyle\sum_{n\text{ odd}}\frac{\cos n\pi t/2}{\pi^2 n^2(16-\pi^2 n^2)}.$



5. We substitute the sine series $x(t)=\sum_{n=1}^{\infty}b_n\sin n\pi t$ and $F(t)=\dfrac{8}{\pi^3}\displaystyle\sum_{n\text{ odd}}\frac{\sin n\pi t}{n^3}$ into the differential equation $x''+10x = F(t)$. This gives

$$-\sum_{n=1}^{\infty}n^2\pi^2 b_n\sin n\pi t+10\sum_{n=1}^{\infty}b_n\sin n\pi t \;=\; \frac{8}{\pi^3}\sum_{n\text{ odd}}\frac{\sin n\pi t}{n^3}.$$

We therefore choose $b_n = 8/n^3\pi^3(10-n^2\pi^2)$. This gives the formal series solution $x_{sp}(t) \;=\; \dfrac{8}{\pi^3}\displaystyle\sum_{n\text{ odd}}\frac{\sin n\pi t}{n^3(10-n^2\pi^2)}.$



6. We substitute the cosine series $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$ and $F(t) = \frac{4}{\pi} - \frac{4}{\pi} \sum_{n \text{ even}} \frac{\cos nt}{n^2 - 1}$ into the differential equation $x'' + 2x = F(t)$. This gives

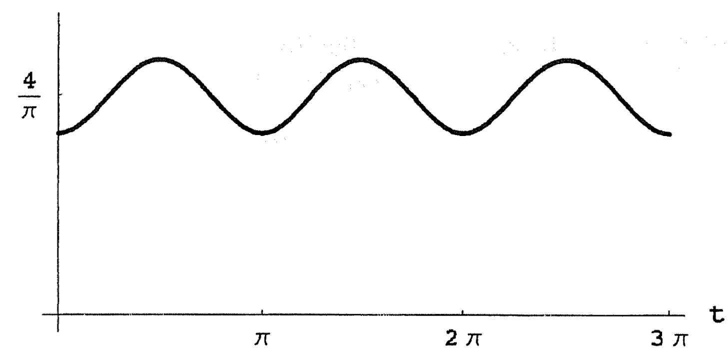
$$-\sum_{n=1}^{\infty} n^2 a_n \cos nt + a_0 + 2\sum_{n=1}^{\infty} a_n \cos nt = \frac{4}{\pi} - \frac{4}{\pi} \sum_{n \text{ even}} \frac{\cos nt}{n^2 - 1}.$$

We therefore choose $a_0 = 4/\pi$ and $a_n = 0$ for n odd, and

$$a_n = \frac{-4/\pi(n^2-1)}{2-n^2} = \frac{4}{\pi(n^2-1)(n^2-2)}$$

for n even. This gives the formal series solution

$$x_{\text{sp}}(t) = \frac{4}{\pi} - \frac{4}{\pi} \sum_{n \text{ even}} \frac{\cos nt}{(n^2-1)(n^2-2)} = \frac{4}{\pi}\left(1 - \frac{\cos 2t}{6} - \frac{\cos 4t}{210} - \frac{\cos 6t}{1190} - \cdots\right).$$



In Problems 7–12 we are dealing with the equation $mx'' + kx = F(t)$ where $F(t)$ is the external periodic force. The natural frequency is $\omega_0 = \sqrt{k/m}$. If the Fourier series of $F(t)$ contains a term of the form $\cos(N\pi t/L)$ or $\sin(N\pi t/L)$ with $\omega_0 = N\pi/L$, then pure resonance occurs. Otherwise, it does not.

7. The natural frequency is $\omega_0 = 3$, and

$$F(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \cdots\right).$$

Thus the Fourier series of $F(t)$ contains a $\sin 3t$ term, so resonance does occur.

8. The natural frequency is $\omega_0 = \sqrt{5}$, and $F(t) = \Sigma\, b_n \sin n\pi t$. Since $n\pi \neq \sqrt{5}$ for any integer n, pure resonance does not occur.

9. The natural frequency is $\omega_0 = 2$, and

$$F(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \cdots\right).$$

Because the $\sin 2t$ term is missing from the Fourier series of $F(t)$, resonance will not occur.

10. The natural frequency is $\omega_0 = 2\pi$. From Equation (16) in Section 8.3 of the text we see that the Fourier series of $F(t)$ contains a $\sin 2\pi t$ term. Hence pure resonance occurs.

11. The natural frequency is $\omega_0 = 4$. From Equation (15) in Section 8.3 we see that

$$F(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \cdots\right).$$

Because the $\cos 4t$ term is missing, we see that resonance will not occur.

12. The natural frequency is $\omega_0 = 5$, and the Fourier series of $F(t)$ is of the form $F(t) = \Sigma\, b_n \sin nt$. We calculate b_5, and find that

$$b_5 = \frac{2}{\pi}\int_0^\pi (\pi t - t^2)\sin 5t\, dt = \frac{4 - 4\cos 5\pi - 10\pi \sin 5\pi}{125\pi} = \frac{8}{125\pi} \neq 0,$$

Thus the term $\sin 5t$ is present in $F(t)$, and so pure resonance occurs.

Problems 13–18 are based on Equations (14)–(16) in the text, according to which the steady periodic solution of

$$mx'' + cx' + kx = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi t}{L}$$

is given by

$$x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin(\omega_n t - \alpha_n),$$

where

$$\omega_n = \frac{n\pi}{L},$$

$$\alpha_n = \tan^{-1}\frac{c\omega_n}{k - m\omega_n^2} \quad \text{in the interval } [0, \pi],$$

$$b_n = \frac{B_n}{\sqrt{\left(k - m\omega_n^2\right)^2 + \left(c\omega_n\right)^2}}.$$

This calculation is readily automated. The following MATLAB script was written to calculate the coefficients $\{b_n\}$ for Problem 13. Only the values of m, c, k, L and the calculation of the force function coefficients $\{B_n\}$ need to be changed for Problems 14–18.

```
m = 1;    c = 0.1;    k = 4;
L = pi;
results = ones(0,4);
for n = 1:9
   w = n*pi/L;
   alpha = atan(c*w/(k-m*w^2));
   if k-m*w^2<0
      alpha = pi + alpha;
   end
   B = 12/(pi*n);           % force function coeffs
   if floor(n/2)==n/2       % are nonzero if n is odd,
      B = 0;                % zero if n is even
   end
   b = B/sqrt((k-m*w^2)^2+(c*w)^2);
   results = [results; n, b, w, alpha];
end
results
```

13. $B_n = 12/\pi n$ for n odd, $B_n = 0$ for n even

$$x_{sp}(t) \approx 1.2725\sin(t-0.0333)+0.2542\sin(3t-3.0817)+0.0364\sin(5t-3.1178)+\cdots$$

14. $B_n = 4(-1)^{n+1}/n$ for $n = 1, 2, 3, \cdots$

$$x_{sp}(t) \approx 0.2500\sin(t-0.0063)-0.2000\sin(2t-0.0200)$$
$$+4.444\sin(3t-1.5708)-0.0714\sin(4t-3.1130)+\cdots$$

Note the dominance of the $n = 3$ term.

15. $B_n = 8/n^3\pi^3$ for n odd, $B_n = 0$ for n even

$$x_{sp}(t) \approx 0.08150\sin(\pi t-1.44692)+0.00004\sin(3\pi t-3.10176)+\cdots$$

16. $F(t) = A_0 + \Sigma A_n\cos(n\pi t/2)$ where $A_0 = 2$, $A_n = -16/\pi^2 n^2$ for n odd, $A_n = 0$ for n even and positive.

$$x_{sp}(t) \approx 0.5000+1.0577\cos(\pi t/2-0.0103)$$
$$-0.0099\cos(3\pi t/2-3.1390)-0.0011\cos(5\pi t/2-3.1402)\cdots$$

17. $B_n = 60/n\pi$ for n odd, $B_n = 0$ for n even

$$x_{sp}(t) \approx 0.5687 \sin(\pi t - 0.0562) + 0.4271 \sin(3\pi t - 0.3891) + 0.1396 \sin(5\pi t - 2.7899) + 0.0318 \sin(7\pi t - 2.9874) + \cdots$$

$x_{sp}(5) \approx 0.248$ ft ≈ 2.98 in.

18. $B_n = (4/\pi n^2)\sin(n\pi/2)$

$$x_{sp}(t) \approx 0.0531 \sin(t - 0.0004) - 0.0088 \sin(3t - 0.0019) + 1.0186 \sin(5t - 1.5708) - 0.0011 \sin(7t - 3.1387) + \cdots$$

Note the dominance of the $n = 5$ term.

SECTION 8.5

HEAT CONDUCTION AND SEPARATION OF VARIABLES

1. From Equation (31) in the text, with $L = \pi$ and $k = 3$, we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp(-3n^2 t) \sin nx .$$

With $b_2 = 4$ and $b_n = 0$ otherwise we get the solution

$$u(x, t) = 4e^{-12t} \sin 2x.$$

2. From Equation (40) in the text, with $k = 10$ and $L = 5$ we get,

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 t}{25}\right) \cos\frac{n\pi x}{5}.$$

With $a_0 = 14$ and $a_n = 0$ for $n > 0$ we get the solution $u(x, t) = 7$ (constant).

3. With $L = 1$ and $k = 2$ in Equation (31), we take $b_1 = 5$, $b_3 = -1/5$, and $b_n = 0$ otherwise. The result is the solution

$$u(x,t) = 5e^{-2\pi^2 t} \sin \pi x - \frac{1}{5} e^{-18\pi^2 t} \sin 3\pi x.$$

4. From Equation (31) with $k = 1$ and $L = \pi$ we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp(-n^2 t)\sin nx.$$

But the $\sin A \cos B$ identity yields

$$4 \sin 4x \cos 2x = 2 \sin 2x + 2 \sin 6x.$$

Hence we choose $b_2 = b_6 = 2$ and $b_n = 0$ for $n \neq 2, 6$. Thus

$$u(x, t) = 2e^{-4t}\sin 2x + 2e^{-36t}\sin 6x.$$

5. From Equation (40) in the text, with $k = 2$ and $L = 3$ we get,

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{2n^2\pi^2 t}{9}\right)\cos\frac{n\pi x}{3}.$$

With $a_0 = 0$, $a_2 = 4$, $a_4 = -2$, and $a_n = 0$ otherwise, and $a_n = 0$ we get the solution

$$u(x,t) = 4\exp\left(-\frac{8\pi^2 t}{9}\right)\cos\frac{2\pi x}{3} - 2\exp\left(-\frac{32\pi^2 t}{9}\right)\cos\frac{4\pi x}{3}.$$

6. From Equation (31) with $k = 1/2$ and $L = 1$ we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 t}{2}\right)\sin n\pi x.$$

Trigonometric identities yield

$$\begin{aligned} 4 \sin \pi x \cos^3 \pi x &= (2 \sin \pi x \cos \pi x)(2 \cos^2 \pi x) \\ &= (\sin 2\pi x)(1 + \cos 2\pi x) = \sin 2\pi x + (1/2)\sin 4\pi x. \end{aligned}$$

Hence we choose $b_2 = 1$, $b_4 = 1/2$, and $b_n = 0$ otherwise to get

$$u(x, t) = \exp(-2\pi^2 t) \sin 2\pi x + (1/2)\exp(-8\pi^2 t) \sin 4\pi x.$$

7. From Equation (40) in the text, with $k = 1/3$ and $L = 2$ we get,

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 t}{12}\right)\cos\frac{n\pi x}{2}.$$

Because of the identity $\cos^2 2\pi x = (1 + \cos 4\pi x)/2$, we choose $a_0 = 1$, $a_8 = 1/2$, and $a_n = 0$ otherwise. This gives the solution

$$u(x,t) = \frac{1}{2}+\frac{1}{2}\exp\left(-\frac{16\pi^2 t}{3}\right)\cos 4\pi x.$$

8. From Equation (40) with $k = 1$ and $L = 2$ we get

$$u(x,t) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 t}{4}\right)\cos\frac{n\pi x}{2}.$$

But

$$10 \cos \pi x \cos 3\pi x = 5 \cos 2\pi x + 5 \cos 4\pi x.$$

Hence we choose $b_4 = b_8 = 5$ and $b_n = 0$ otherwise to get the solution

$$u(x, t) = 5 \exp(-4\pi^2 t) \cos 2\pi x + 5 \exp(-16\pi^2 t) \cos 4\pi x.$$

9. Because of the zero endpoint conditions $u(0,t) = u(5,t) = 0$, we use the Fourier sine series expansion

$$u(x,0) = \frac{100}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\sin\frac{n\pi x}{5}$$

of $u(x,0) = 25$ on the interval $0 < x < 5$. When we supply the exponential factors in Eq. (31) with $k = 1/10$ and $L = 5$, we get the solution

$$u(x,t) = \frac{100}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\exp\left(\frac{-n^2\pi^2 t}{250}\right)\sin\frac{n\pi x}{5}$$

10. Because of the zero endpoint conditions $u(0,t) = u(10,t) = 0$, we use the Fourier sine series expansion

$$u(x,0) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{10}$$

of $u(x,0) = 4x$ on the interval $0 < x < 10$ (from Eq. (16) in Section 8.3). When we supply the exponential factors in Eq. (31) here with $k = 1/5$ and $L = 10$, we get

$$u(x,t) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp\left(-\frac{n^2\pi^2 t}{500}\right)\sin\frac{n\pi x}{10}.$$

11. Because of the zero-derivative endpoint conditions $u_x(0,t) = u_x(10,t) = 0$, we use the Fourier cosine series expansion

$$u(x,0) = 20-\frac{160}{\pi^2}\sum_{n\text{ odd}}\frac{1}{n^2}\cos\frac{n\pi x}{10}$$

of $u(x,0) = 4x$ on the interval $0 < x < 10$ (from Eq. (15) in Section 8.3). When we supply the exponential factors in Eq. (40) here with $k = 1/5$ and $L = 10$, we get

$$u(x,t) = 20 - \frac{160}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \exp\left(\frac{-n^2\pi^2 t}{500} \right) \cos\frac{n\pi x}{10}$$

12. From Equation (31) with $k = 1$ and $L = 100$ we get

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 t}{10000} \right) \sin\frac{n\pi x}{100}.$$

Because of the zero endpoint conditions $u(0,t) = u(100,t) = 0$, the $\{b_n\}$ should be the Fourier sine coefficients of $f(x) = x(100 - x)$ on $[0, 100]$, given by

$$\begin{aligned} b_n &= \frac{2}{100} \int_0^{100} x(100-x) \sin\frac{n\pi x}{100}\, dx \\ &= \frac{20000(2 - 2\cos n\pi - n\pi \sin n\pi)}{n^3\pi^3} = \begin{cases} 80000/n^3\pi^3 & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ odd}. \end{cases} \end{aligned}$$

This gives the solution

$$u(x,t) = \frac{80000}{\pi^3} \sum_{n \text{ odd}} \frac{1}{n^3} \exp\left(-\frac{n^2\pi^2 t}{10000} \right) \sin\frac{n\pi x}{100}.$$

13. **(a)** The boundary value problem is

$$\begin{aligned} u_t &= k u_{xx} \quad (0 < x < 40), \\ u_x(0, t) &= u_x(40, t) = 0, \\ u(x, 0) &= 100. \end{aligned}$$

By Equation (31) in the text (with $L = 40$) the solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 kt}{1600} \right) \sin\frac{n\pi x}{40}.$$

We use the Fourier sine coefficients $b_n = 400/\pi n$ for n odd, $b_n = 0$ otherwise, of the initial value function $f(x) = 100$ on the interval $0 < x < 100$. This gives

$$u(x,t) = \frac{400}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \exp\left(\frac{-n^2\pi^2 kt}{1600} \right) \sin\frac{n\pi x}{40}.$$

(b) With $k = 1.15$ for copper we find that

$$u(20,300) \approx 15.1591 - 0.000000204 + \cdots \approx 15.16°C.$$

(c) With $k = 0.005$ for concrete, the first term of the series gives

$$u(20,t) = \frac{400}{\pi}\exp\left(-\frac{0.00\pi^2 t}{1600}\right) = 15,$$

and we solve for $t \approx 66{,}342$ sec ≈ 19 hr 15 min 42 sec. As a check that the first term suffices for this computation, we find that the next term in the series is then approximately 0.00000019.

14. (a) The boundary value problem is

$$u_t = ku_{xx} \quad (0 < x < 50)$$
$$u_x(0, t) = u_x(50, t) = 0$$
$$u(x, 0) = 2x$$

with $k = 1.15$ cm^2/sec for copper. By Equation (40) in the text (with $L = 50$) the solution is of the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 kt}{2500}\right)\cos\frac{n\pi x}{50}.$$

Consulting the Fourier series given in Equation (15) of Section 8.2, we satisfy the initial condition $u(x, 0) = 2x$ by choosing $a_0 = 100$, $a_n = -400/n^2\pi^2$ for n odd, and $a_n = 0$ for n even. Thus

$$u(x,t) = 50 - \frac{400}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}\exp\left(-\frac{n^2\pi^2 kt}{2500}\right)\cos\frac{n\pi x}{50}.$$

(b) With $k = 1.15$ for copper we find that

$$u(10,60) \approx 50 - 24.9698 + 0.1199 + 0.0018 + 0.0000 - \cdots \approx 25.15°C.$$

(c) To find out how long it takes the temperature to reach 45°C at the point $x = 10$, we solve the equation

$$50 - \frac{400}{\pi^2}\exp\left(-\frac{1.15\pi^2 t}{2500}\right)\cos\frac{\pi}{5} = 45$$

that we get upon retaining only the first two terms of the series above (with $x = 10$). Using logarithms (for instance) we find that $t \approx 414.23$ sec ≈ 6 min 54 sec. To confirm that two terms suffice for this calculation, we retain 3 terms and use a computer or calculator to solve the equation

$$50-\frac{400}{\pi^2}\exp\left(-\frac{1.15\pi^2 t}{2500}\right)\cos\frac{\pi}{5}-\frac{400}{9\pi^2}\exp\left(-\frac{9\times 1.15\pi^2 t}{2500}\right)\cos\frac{3\pi}{5} = 45$$

for (again) $t \approx 414.23$ sec.

15. We need only calculate the coefficients in the usual zero-endpoint series

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)\sin\frac{n\pi x}{L}.$$

For the function $f(x) \equiv A$ for $0 < x < L/2$, $f(x) \equiv 0$ for $L/2 < x < L$ we calculate the Fourier sine coefficient

$$b_n = \frac{2}{L}\int_0^{L/2} A\sin\frac{n\pi x}{L}\,dx = \frac{4A}{n\pi}\sin^2\frac{n\pi}{4} = \frac{4A}{n\pi}\times\begin{cases} 1/2 & \text{for } n \text{ odd,} \\ 1 & \text{for } n = 2, 6, 10, \cdots, \\ 0 & \text{for } n = 4, 8, 12, \cdots. \end{cases}$$

16. **(a)** Summing numerically the series in Problem 15 with the values $k = 0.15$ for iron, $L = 50$, $A = 100$, $x = 25$, and $t = 1800$, we find that

$$u(25,1800) \approx 21.9259 - 0.0014 + 0.0000 - \cdots \approx 21.9245 \approx 22\,°\text{C}.$$

(b) Because for x fixed the temperature is a function of the ***product*** kt, in the case of concrete slabs with $k = 0.005$ the same temperature will be attained when

$$(0.005)(t) = (0.15)(1800),$$

that is, when $t = 54000 \text{ sec} = 15 \text{ hr}.$

SECTION 8.6

VIBRATING STRINGS AND THE ONE-DIMENSIONAL WAVE EQUATION

In Problems 1–10 we use the general solution

$$y(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi at}{L} + B_n\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L} \qquad (*)$$

of the string equation $y_{tt} = a^2 y_{xx}$ with endpoint conditions $y(0, t) = y(L, t) = 0$. This form of the solution is obtained by superposition of the solutions in Equations (23) and (33) of Problems A and B in this section. It remains only to choose the coefficients $\{A_n\}$ and $\{B_n\}$ so as to satisfy given initial conditions

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x), \text{ thus, } A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx; \text{ and}$$

$$y_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} B_n \sin\frac{n\pi x}{L} = g(x), \text{thus, } B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

1. Here $a = 2$ and $L = \pi$. To satisfy the condition $y(x, 0) = (1/10)\sin 2x$ we choose $A_2 = 1/10$ in Eq. (*) above, and $A_n = 0$ otherwise. To satisfy the condition $y_t(x, 0) = 0$ we choose $B_n = 0$ for all n. Thus

$$y(x, t) = \frac{1}{10}\cos 4t \sin 2x.$$

2. Here $a = L = 1$. To satisfy the condition

$$y(x, 0) = \frac{1}{10}\sin \pi x - \frac{1}{20}\sin 3\pi x$$

we choose $A_1 = 1/10$ and $A_3 = -1/20$ in Eq. (*) above, and $A_n = 0$ otherwise. To satisfy the condition $y_t(x, 0) = 0$ we choose $B_n = 0$ for all n. Thus

$$y(x, t) = \frac{1}{10}\cos \pi t \sin \pi x - \frac{1}{20}\cos 3\pi t \sin 3\pi x.$$

3. Here $a = 1/2$ and $L = \pi$. Choosing $A_1 = 1/10$ and $A_n = 0$ otherwise, $B_1 = 1/5$ and $B_n = 0$ otherwise, we get

$$y(x, t) = \frac{1}{10}\left(\cos\frac{t}{2} + 2\sin\frac{t}{2}\right)\sin x.$$

4. Here $a = 1/2$ and $L = 2$, so $n\pi x/L = n\pi x/2$ and $n\pi at/L = n\pi t/4$. To satisfy the condition

$$y(x, 0) = \frac{1}{5}\sin \pi x \cos \pi x = \frac{1}{10}\sin 2\pi x = \frac{1}{10}\sin\frac{4\pi x}{2},$$

we choose $A_4 = 1/10$ and $A_n = 0$ for $n \neq 4$. To satisfy the condition $y_t(x, 0) = 0$ we choose $B_n = 0$ for all n. Thus

$$y(x, t) = \frac{1}{10}\cos \pi t \sin 2\pi x.$$

5. Here $a = 5$ and $L = 3$. Choosing $A_3 = 1/4$ and $A_n = 0$ for $n \neq 3$, $B_6 = 1/\pi$ and $B_n = 0$ for $n \neq 6$, we get

$$y(x,\,t) \;=\; \frac{1}{4}\cos 5\pi t\,\sin \pi x + \frac{1}{\pi}\sin 10\pi t\,\sin 2\pi x.$$

6. Here $a = 10$ and $L = \pi$. To satisfy the condition $y_t(x, 0) = 0$ we choose $B_n = 0$ for all n, so

$$y(x,\,t) \;=\; \sum_{n=1}^{\infty} A_n \cos 10nt \sin nx.$$

To satisfy the condition $y(x, 0) = x(\pi - x)$ we choose

$$A_n \;=\; \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx \;=\; \frac{4 - 4\cos n\pi - 2n\pi\sin n\pi}{n^3\pi} \;=\; \begin{cases} 8/n^3\pi & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

This gives the solution

$$y(x,t) \;=\; \frac{8}{\pi}\sum_{n \text{ odd}} \frac{\cos 10nt \sin nx}{n^3}.$$

7. Here $a = 10$ and $L = 1$. To satisfy the condition $y(x, 0) = 0$ we choose $A_n = 0$ for all n, so

$$y(x,\,t) \;=\; \sum_{n=1}^{\infty} B_n \sin 10n\pi t \sin n\pi x.$$

To satisfy the condition $y_t(x, 0) = x$ we choose

$$B_n \;=\; \frac{1}{10n\pi}\cdot\frac{2(-1)^{n+1}}{n\pi} \;=\; \frac{(-1)^{n+1}}{5n^2\pi^2}$$

for $n \geq 1$ (see Equation (16) in Section 8.3). This gives

$$y(x,\,t) \;=\; \frac{1}{5\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sin 10n\pi t \sin n\pi x.$$

8. Here $a = 2$ and $L = \pi$. To satisfy the condition $y(x, 0) = \sin x$ we choose $A_1 = 1$ and $A_n = 0$ for $n > 1$, so

$$y(x,\,t) \;=\; \cos 2t \sin x + \sum_{n=1}^{\infty} B_n \sin 2nt \sin nx, \text{ so}$$

$$y_t(x,\,t) \;=\; -2\sin 2t \sin x + \sum_{n=1}^{\infty} 2nB_n \cos 2nt \sin nx.$$

The condition $y_t(x, 0) = 1$ will be satisfied if $2nB_n = 4/\pi n$ for n odd and $b_n = 0$ for n even. We therefore choose $B_n = 2/\pi n^2$ for n odd and $B_n = 0$ for n even. so

$$y(x,t) = \cos 2t \sin x + \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin 2nt \sin nx}{n}.$$

9. Here $a = 2$ and $L = 1$. To satisfy the condition $y(x, 0) = 0$ we choose $A_n = 0$ for all n, so

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin 2n\pi t \sin n\pi x.$$

To satisfy the condition $y_t(x, 0) = x(1 - x)$ we choose

$$B_n = \frac{1}{n\pi}\int_0^1 x(1-x)\sin n\pi x\,dx = \frac{2 - 2\cos n\pi - n\pi \sin n\pi}{n^4\pi^4}.$$

Hence

$$y(x,t) = \frac{4}{\pi^4}\sum_{n \text{ odd}} \frac{\sin 2n\pi t \sin n\pi x}{n^4}.$$

10. Here $a = 5$ and $L = \pi$ so

$$y(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos 5nt + B_n \sin 5nt\right)\sin nx.$$

We first compute the Fourier sine series $\sin^2 x = \sum_{n=1}^{\infty} b_n \sin nx$ and find that $b_n = 0$ if n is even whereas

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin nx\,dx = \frac{4(\cos n\pi - 1)}{\pi n(n^2 - 4)} = \frac{8}{\pi n(4 - n^2)}$$

if n is odd. To satisfy the condition $y(x, t) = \sin^2 x$ we choose $A_n = b_n$, and to satisfy the condition $y_t(x, t) = \sin^2 x$ we choose $B_n = b_n/5n$. Then

$$y(x,t) = \frac{8}{5\pi}\sum_{n \text{ odd}} \frac{\left(5n\cos 5nt + \sin 5nt\right)\sin nx}{n^2(4 - n^2)}.$$

11. Substitution of $L = 2$ ft, $T = 32$ lb, and the *linear* density

$$\rho = \frac{1/32 \text{ oz}}{2 \text{ ft}} = \frac{1 \text{ oz}}{64 \text{ ft}}\cdot\frac{1 \text{ lb}}{16 \text{ oz}}\cdot\frac{1 \text{ slug}}{32 \text{ lb}} = \frac{1}{32^3}\frac{\text{slug}}{\text{ft}}$$

in Eqs. (2) and (26) in the text yields the velocity $a = \sqrt{T/\rho} = \sqrt{32^4} = 1024$ ft/sec with which waves move along the string, and its fundamental frequency

$$\nu_1 = \frac{1}{2L}\sqrt{\frac{T}{\rho}} = \frac{a}{2L} = 256 \text{ Hz},$$

which is approximately middle C.

12. The value of

$$y(x,t) = \frac{4v_0L}{\pi^2 a}\sum_{n\text{ odd}}\frac{1}{n^2}\sin\frac{n\pi at}{L}\sin\frac{n\pi x}{L}$$

is maximal when each of the sine products is 1. This happens when $x = L/2,\ t = L/2a$:

$$y\left(\frac{L}{2},\frac{L}{2a}\right) = \frac{4v_0L}{\pi^2 a}\sum_{n\text{ odd}}\frac{1}{n^2}\sin^2\frac{n\pi}{2} = \frac{4v_0L}{\pi^2 a}\sum_{n\text{ odd}}\frac{1}{n^2} = \frac{4v_0L}{\pi^2 a}\cdot\frac{\pi^2}{8} = \frac{v_0L}{2a}.$$

Using fps units with the string of Problem 11 where $L = 2$ ft, $a = 1024$ ft/sec, and $v_0 = 60$ mph $= 88$ ft/sec, we get

$$y_{\max} = \frac{88\times 2}{2\times 1024} \approx 0.0859\,\text{ft} \approx 1\ \text{inch}.$$

13. If $y(x,t) = F(x+at) = F(u)$ with $u = x+at$, then the chain rule gives

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{dF}{du}\frac{\partial u}{\partial x} = F'(u)\cdot 1 = F'(x+at);\\
\frac{\partial y}{\partial t} &= \frac{dF}{du}\frac{\partial u}{\partial t} = F'(u)\cdot a = a\,F'(x+at) = a\frac{\partial y}{\partial x};\\
\frac{\partial^2 y}{\partial x^2} &= \frac{dF'}{du}\frac{\partial u}{\partial x} = F''(u)\cdot 1 = F''(x+at);\\
\frac{\partial^2 y}{\partial t^2} &= a\frac{dF'}{du}\frac{\partial u}{\partial t} = a\cdot F''(u)\cdot a = a^2F''(x+at) = a^2\frac{\partial^2 y}{\partial x^2}.
\end{aligned}$$

14.

$$\begin{aligned}
y(0,t) &= \tfrac{1}{2}[F(at)+F(-at)] = \tfrac{1}{2}[F(at)-F(at)] = 0\\
y(L,t) &= \tfrac{1}{2}[F(L+at)+F(L-at)]\\
&= \tfrac{1}{2}[F(L+at)-F(-L+at)] = \tfrac{1}{2}[F(2L+(-L+at))-F(-L+at)] = 0\\
y(x,0) &= \tfrac{1}{2}[F(x)+F(x)] = F(x)\\
y_t(x,t) &= \tfrac{1}{2}[aF'(x+at)-aF'(x-at)]\\
y_t(x,0) &= \tfrac{1}{2}[aF'(x)-aF'(x)] = 0
\end{aligned}$$

15. If $y(x,0) = 0$ then the fundamental theorem of calculus gives

$$y(x,t) = y(x,t)-y(x,0) = \int_0^t y_t(x,\tau)\,d\tau = \int_0^t \frac{1}{2}\left[G(x+a\tau)+G(x-a\tau)\right]d\tau.$$

16. If $u = x + at,\ v = x - at$ then we solve readily for $x = \frac{1}{2}(u+v), t = \frac{1}{2a}(u-v)$. Hence

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left[y\left(\tfrac{1}{2}(u+v),\tfrac{1}{2a}(u-v)\right)\right] = \frac{\partial y}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial u} = \frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t};$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left[y\left(\tfrac{1}{2}(u+v),\tfrac{1}{2a}(u-v)\right)\right] = \frac{\partial y}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial v} = \frac{1}{2}\frac{\partial y}{\partial x} - \frac{1}{2a}\frac{\partial y}{\partial t};$$

$$\frac{\partial^2 y}{\partial v \partial u} = \frac{\partial}{\partial v}\left(\frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}\right) - \frac{1}{2a}\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\partial y}{\partial x} + \frac{1}{2a}\frac{\partial y}{\partial t}\right)$$

$$= \frac{1}{4}\frac{\partial^2 y}{\partial x^2} + \frac{1}{4a}\frac{\partial^2 y}{\partial x \partial t} - \frac{1}{4a}\frac{\partial^2 y}{\partial t \partial x} - \frac{1}{4a^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{4a^2}\left(\frac{\partial^2 y}{\partial x^2} - a^2\frac{\partial^2 y}{\partial t^2}\right) = 0.$$

Now if $\dfrac{\partial^2 y}{\partial v \partial u} = \dfrac{\partial}{\partial v}\left(\dfrac{\partial y}{\partial u}\right) = 0$ then antidifferentiation with respect to v gives $\partial y / \partial u = G(v)$, an arbitrary function of v. Finally, antidifferentiation with respect to u gives $y = F(u) + G(v) = F(x + at) + G(x - at)$.

18. When we separate variables as in Equations (8)–(12) in this section, we find that $X(x)$ must satisfy the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(L) = 0.$$

In Example 4 of Section 2.8 we found that the eigenvalues and eigenfunctions of this problem are

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \qquad X_n(x) = \sin\frac{(2n-1)\pi x}{2L}$$

for $n = 1, 2, 3, \cdots$. The function $T_n(t)$ must satisfy the conditions

$$T_n'' + \lambda_n a^2 T_n = 0, \qquad T_n'(0) = 0,$$

so it follows that

$$T_n(t) = \cos\frac{(2n-1)\pi at}{2L}.$$

Thus the form of $y(t)$ is

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi at}{2L}\sin\frac{(2n-1)\pi x}{2L}.$$

Finally, in order to satisfy the initial condition $y(x, 0) = f(x)$ we use the odd half-multiple sine series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{(2n-1)\pi x}{2L}$$

discussed in Problem 21 of Section 8.3.

19. The general solution of the second-order ordinary differential equation $a^2 y'' = g$ is a second-order polynomial in x with leading coefficient $g/2a^2$. But the polynomial $\phi(x) = gx(x-L)/2a^2$ has this leading coefficient and satisfies the endpoint conditions $y(0) = y(L) = 0$.

20. If $y(x,t) = v(x,t) + \phi(x)$, then $y_{tt} = v_{tt}$ and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \phi''(x) = \frac{\partial^2 v}{\partial x^2} + \frac{g}{a^2}, \quad \text{so} \quad a^2\frac{\partial^2 y}{\partial x^2} - g = a^2\frac{\partial^2 v}{\partial x^2}.$$

The transformation of the boundary conditions is straightforward.

22. The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X(L) = 0$$

has the usual eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad X_n(x) = \sin\frac{n\pi x}{L}$$

for $n = 1, 2, 3, \cdots$. The function $T_n(t)$ satisfies the equation

$$T_n'' + \omega_n^2 R\, T_n = 0, \qquad \omega_n^2 = \frac{n^2\pi^2 a^2}{L^2} - h^2 > 0,$$

so

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t.$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\omega_n t + B_n \sin\omega_n t\right)\sin\frac{n\pi x}{L}.$$

To satisfy the conditions $v(x, 0) = f(x)$ and $v_t(x, 0) = h f(x)$ we choose $A_n = b_n$ and $B_n = hb_n/\omega_n$ where

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

Then

$$v(x,t) = \sum_{n=1}^{\infty} \frac{b_n}{\omega_n}(\omega_n \cos\omega_n t + h \sin\omega_n t) \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} c_n \cos(\omega_n t - \alpha_n) \sin\frac{n\pi x}{L},$$

where

$$c_n = \frac{b_n}{\cos\alpha_n} \quad \text{and} \quad \alpha_n = \tan^{-1}\frac{h}{\omega_n}.$$

Finally,

$$y(x,t) = e^{-ht} v(x,t) = e^{-ht} \sum_{n=1}^{\infty} c_n \cos(\omega_n t - \alpha_n) \sin\frac{n\pi x}{L}.$$

23. If $\pi/4 \leq x \leq 3\pi/4$ then

$$\frac{\pi}{2} \leq x + \frac{\pi}{4} \leq \pi \quad \text{and} \quad 0 \leq x - \frac{\pi}{4} \leq \frac{\pi}{2},$$

so

$$y(x, \pi/4) = \frac{1}{2}[F(x + \pi/4) + F(x - \pi/4)]$$

$$= \frac{1}{2}[1 - \cos 2(x + \pi/4) + 1 - \cos 2(x - \pi/4)]$$

$$= \frac{1}{2}[1 - \cos (2x + \pi/2) + 1 - \cos (2x - \pi/2)]$$

$$= \frac{1}{2}[2 + \sin 2x - \sin 2x]$$

$$y(x, \pi/4) = 1$$

24. **(a)** $f''(x) = 4 \cos 2x = 0$ if $x = \pi/4$ or $x = 3\pi/4$.

(b) If $0 \leq t \leq \pi/4$ then $0 \leq \pi/4 \pm t \leq \pi/2$ so

$$y(\pi/4, t) = \frac{1}{2}[F(\pi/4 + t) + F(\pi/4 - t)]$$

$$= \frac{1}{2}[1 - \cos 2(\pi/4 + t) + 1 - \cos 2(\pi/4 - t)]$$

$$= \frac{1}{2}[1 - \cos (\pi/2 + 2t) + 1 - \cos (\pi/2 - 2t)]$$

$$= \frac{1}{2}[2 + \sin 2t - \sin 2t]$$

$$y(\pi/4, t) = 1$$

SECTION 8.7

STEADY-STATE TEMPERATURE AND LAPLACE'S EQUATION

1. Because $Y(0) = Y(b) = 0$ we take our separation of variables in the form

$$X'' - \lambda X = 0 = Y'' + \lambda Y$$

with $\lambda > 0$. Then it follows that

$$Y_n(y) = \sin\frac{n\pi y}{b}, \qquad \lambda_n = \frac{n^2\pi^2}{b^2}$$

and thence that

$$X_n(x) = A_n \cosh\frac{n\pi x}{b} + B_n \sinh\frac{n\pi x}{b}.$$

The condition that $X(0) = 0$ implies that $A_n = 0$ so $X_n(x) = B_n \cosh n\pi x / b$, and hence

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}.$$

Finally we satisfy the condition $u(a,y) = g(y)$ by choosing $C_n = b_n / (\sinh n\pi a / b)$, where the $\{b_n\}$ are the Fourier sine coefficients of $g(y)$ on $0 \le y \le b$.

2. Because $Y(0) = Y(b) = 0$ we take our separation of variables in the form

$$X'' - \lambda X = 0 = Y'' + \lambda Y$$

with $\lambda > 0$. Then it follows that

$$Y_n(y) = \sin\frac{n\pi y}{b}, \qquad \lambda_n = \frac{n^2\pi^2}{b^2}$$

and thence that

$$X_n(x) = A_n \cosh\frac{n\pi x}{b} + B_n \sinh\frac{n\pi x}{b}.$$

The condition $X(a) = 0$ implies that

$$B_n = -\frac{A_n \cosh n\pi a / b}{\sinh n\pi a / b}.$$

It now follows as in Equation (12) in the text that

$$X_n(x) = C_n \sinh\frac{n\pi(a-x)}{b},$$

so

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi(a-x)}{b}\sin\frac{n\pi y}{b}.$$

Finally we satisfy the condition $u(0,y) = g(y)$ by choosing $C_n = b_n/(\sinh n\pi a/b)$, where the $\{b_n\}$ are the Fourier sine coefficients of $g(y)$ on $0 \le y \le b$.

3. Just as in Example 1 of Section 8.7 we have $X_n(x) = \sin n\pi x/a$ and

$$Y_n(y) = A_n\cosh\frac{n\pi y}{a} + B_n\sinh\frac{n\pi y}{a}.$$

The condition $Y(0) = 0$ now yields $A_n = 0$ so $Y_n(y) = B_n \sinh n\pi y/a$, and hence

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}.$$

Finally we satisfy the condition $u(x,b) = f(x)$ by choosing $C_n = b_n/(\sinh n\pi b/a)$, where the $\{b_n\}$ are the Fourier sine coefficients of $f(x)$ on $0 \le x \le a$.

4. Because $X'(0) = X'(a) = 0$, we work with the separation of variables

$$X'' + \lambda X = 0 = Y'' - \lambda Y.$$

The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X'(0) = X'(a) = 0$$

has eigenvalues and eigenfunctions $\lambda_0 = 0$, $X_0(x) = 1$ and

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \qquad X_n(x) = \cos\frac{n\pi x}{a}$$

for $n = 1, 2, 3, \cdots$. When $n = 0$, $Y_0'' = 0$ yields $Y_0(y) = Ay + B$. Then $Y_0(0) = 0$ gives $B = 0$, so we take $Y_0(y) = y$. For $n > 0$ we have

$$Y_n(y) = A_n\cosh\frac{n\pi y}{a} + B_n\sinh\frac{n\pi y}{a},$$

and $Y_n(0) = 0$ gives $A_n = 0$, so

$$u(x,y) = B_0 y + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}.$$

Finally

$$u(x,b) = B_0 b + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi b}{a},$$

so we satisfy the condition $u(x, b) = f(x)$ by taking $B_0 = a_0/2b$ and $B_n = a_n / \left(\sinh \frac{n\pi b}{a} \right)$, where $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a}$.

5. Now $Y'(0) = Y'(b) = 0$, so we work with the separation of variables

$$X'' - \lambda X = 0 = Y'' + \lambda Y.$$

The eigenvalue problem

$$Y'' + \lambda Y = 0, \qquad Y'(0) = Y'(b) = 0,$$

has eigenvalues and eigenfunctions $\lambda_0 = 0$, $Y_0(y) = 1$ and

$$\lambda_n = \frac{n^2\pi^2}{b^2}, \qquad Y_n(y) = \cos \frac{n\pi y}{b}$$

for $n = 1, 2, 3, \cdots$. When $n = 0$, $X_0''(x) \equiv 0$ yields $X_0(x) = Ax + B$. Then $X_0(a) = 0$ is satisfied by $X_0(x) = a - x$. For $n > 0$ we have

$$X_n(x) = A_n \cosh \frac{n\pi x}{b} + B_n \sinh \frac{n\pi x}{b},$$

and $X_n(a) = 0$ is satisfied by the particular linear combination

$$X_n(x) = C_n \sinh \frac{n\pi(a-x)}{b},$$

of $\cosh n\pi x/b$ and $\sinh n\pi x/b$. Therefore

$$u(x, y) = C_0(a-x) + \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi(a-x)}{b} \cos \frac{n\pi y}{b}.$$

Finally we satisy the condition $u(0, y) = g(y)$ by choosing

$$C_0 = \frac{a_0}{2a} \quad \text{and} \quad C_n = \frac{b_n}{\sinh n\pi a/b},$$

where the $\{a_n\}$ are the Fourier cosine coefficients of $g(y)$ on $0 \le y \le b$.

6. This is the same as Problem 4 except that $Y'(0) = 0$ instead of $Y(0) = 0$, so $Y_0(y) = 1$ and $Y_n(y) = A_n \cosh n\pi y/a$ for $n > 0$. Then

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{a} \cosh \frac{n\pi y}{a},$$

so we satisfy the condition $u(x, b) = f(x)$ by choosing $A_0 = a_0/2$ and $A_n =$

$a_n/(\cosh n\pi b/a)$, where $\{a_n\}$ are the Fourier cosine coefficients of $f(x)$ on $[0, a]$.

7. The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X(a) = 0$$

yields the eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \qquad X_n(x) = \sin\frac{n\pi x}{a}$$

for $n = 1, 2, 3, \cdots$. Then

$$Y_n'' + \lambda_n Y_n = 0$$

yields

$$Y_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}.$$

In order that $Y(y) \to 0$ as $y \to \infty$ we take $A_n = 0$, so

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-n\pi y/a} \sin\frac{n\pi x}{a}.$$

Finally we satisfy the condition $u(x,0) = f(x)$ by choosing the constants $\{B_n\}$ as the Fourier sine coefficients of $f(x)$ on $0 \leq x \leq a$.

8. The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X'(0) = X'(a) = 0$$

yields $\lambda_0 = 0$, $X_0(x) = 1$ and

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \qquad X_n(x) = \cos\frac{n\pi x}{a}$$

for $n > 0$. Then

$$Y_n'' + \lambda_n Y_n = 0$$

yields $Y_0(y) = A_0 y + B_0$ and

$$Y_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}.$$

In order that $Y(y)$ be bounded as $y \to \infty$, we take $A_0 = 0$ and $A_n = 0$ for $n > 0$, so

$$u(x,y) = B_0 + \sum_{n=1}^{\infty} B_n e^{-n\pi y/a} \cos\frac{n\pi x}{a}.$$

Finally we satisfy the condition $u(x, 0) = f(x)$ by choosing $B_0 = a_0/2$ and $B_n = a_n$ where the $\{a_n\}$ are the Fourier cosine coefficients of $f(x)$ on $[0, a]$.

9. If in Problem 8 we have $f(x) = 10x$ on $0 < x < 10$, then

$$a_0 = \frac{2}{10}\int_0^{10} 10x\,dx = 100,$$

$$a_n = \frac{2}{10}\int_0^{10} 10x\cos\frac{n\pi x}{10}\,dx = \frac{200(\cos n\pi - 1 + n\pi\sin\pi)}{n^2\pi^2},$$

so

$$u(x,y) = 50 - \frac{400}{\pi^2}\sum_{n\ \mathrm{odd}}\frac{1}{n^2}e^{-n\pi y/10}\cos\frac{n\pi x}{10}.$$

Then

$$u(0,5) \approx 50 - 8.4250 - 0.0405 - 0.0006 - 0.0000 - \cdots \approx 41.53,$$

$$u(5,5) = 50 - 0 - 0 - 0 - 0 - \cdots = 50,$$

$$u(0,5) \approx 50 + 8.4250 + 0.0405 + 0.0006 + 0.0000 + \cdots \approx 58.47.$$

10. The boundary value problem is

$$u_{xx} + u_{yy} = 0 \qquad (0 < x < a,\ 0 < y < b)$$

$$u(0, y) = u_x(a, y) = u(x, 0) = 0,$$

$$u(x, b) = f(x).$$

The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(a) = 0$$

yields (by Example 4 in Section 3.8)

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4a^2}, \qquad X_n(x) = \sin\frac{(2n-1)\pi x}{2a}$$

for $n = 1, 2, 3, \cdots$. Then

$$Y_n'' - \lambda_n Y_n = 0$$

yields

$$Y_n(y) = A_n\cosh\frac{(2n-1)\pi y}{2a} + B_n\sinh\frac{(2n-1)\pi y}{2a}.$$

Because $Y(0) = 0$, we choose $A_n = 0$, so

$$u(x,y) = \sum_{n=1}^{\infty} B_n\sin\frac{(2n-1)\pi x}{2a}\sinh\frac{(2n-1)\pi y}{2a}.$$

Finally we satisfy the condition $u(x, b) = f(x)$ by choosing

$$B_n = \frac{b_{2n-1}}{\sinh[(2n-1)\pi b/2a]},$$

where the $\{b_{2n-1}\}$ are the odd half-multiple sine coefficients of $f(x)$ on $[0, a]$, as given by Problem 21 in Section 8.3.

11. Now the boundary value problem is

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad (0 < x < a,\ 0 < y < b) \\ u(a, y) &= u_y(x, 0) = u(x, b) = 0, \\ u(0, y) &= g(y). \end{aligned}$$

The eigenvalue problem

$$Y'' + \lambda Y = 0, \qquad Y'(0) = Y(b) = 0$$

yields (similar to Example 4 in Section 2.8)

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4b^2}, \quad Y_n(y) = \cos\frac{(2n-1)\pi y}{2b}$$

for $n = 1, 2, 3, \cdots$. Then

$$X_n'' - \lambda_n X_n = 0$$

yields

$$X_n(x) = A_n \cosh\frac{(2n-1)\pi x}{2b} + B_n \sin\frac{(2n-1)\pi x}{2b}.$$

Now $X_n(a) = 0$ is satisfied by the particular linear combination

$$X_n(x) = C_n \sinh\frac{(2n-1)\pi(a-x)}{2b}$$

of $\cosh(2n-1)\pi x/2b$ and $\sinh(2n-1)\pi x/2b$. Hence

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} C_n \sinh\frac{(2n-1)\pi(a-x)}{2b}\cos\frac{(2n-1)\pi y}{2b} \\ &= \sum_{n \text{ odd}} A_n \sinh\frac{n\pi(a-x)}{2b}\cos\frac{n\pi y}{2b}. \end{aligned}$$

Finally we satisfy the condition $u(0, y) = g(y)$ by choosing

$$A_n = \frac{a_n}{\sinh n\pi a/2b},$$

where the $\{a_n\}$ are the odd half-multiple cosine coefficients of $g(y)$ on $[0, b]$, as given by Problem 22 in Section 8.3.

12. The boundary value problem is

$$u_{xx} + u_{yy} = 0 \qquad (0 < x < 30,\ y > 0)$$
$$u(0, y) = u_x(30, y) = 0$$
$$u(x, y) \text{ bounded as } y \to \infty$$
$$u(x, 0) = 25$$

The eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(30) = 0$$

yields (by Example 4 in Section 2.8)

$$\lambda_n = \frac{(2n-1)^2\pi^2}{3600}, \qquad X_n(x) = \sin\frac{(2n-1)\pi x}{60}$$

for $n = 1, 2, 3, \cdots$. Then

$$Y_n'' - \lambda_n Y_n = 0$$

yields

$$Y_n(y) = A_n \exp\left[\frac{(2n-1)\pi y}{60}\right] + B_n \exp\left[-\frac{(2n-1)\pi y}{60}\right].$$

and we take $A_n = 0$ in order that $Y_n(y)$ be bounded as $y \to \infty$. Hence

$$u(x, y) = \sum_{n\text{ odd}} b_n e^{-n\pi y/60} \sin\frac{n\pi x}{60}.$$

Finally, by Problem 21 in Section 8.3, the odd half-multiple Fourier sine coefficients of $u(x, 0) = 25$ on $[0, 30]$ are given by

$$b_n = \frac{2}{30}\int_0^{30} 25\sin\frac{n\pi x}{60}\,dx = \frac{200}{n\pi}\sin^2\frac{n\pi}{4} = \frac{100}{n\pi}$$

for n odd. Thus

$$u(x, y) = \frac{100}{\pi}\sum_{n\text{ odd}} \frac{1}{n} e^{-n\pi y/60} \sin\frac{n\pi x}{60}.$$

13. We start with the periodic polar-coordinate solution

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

and choose $a_n \equiv 0$ in order to satisfy the conditions $u(r,0) = u(r,\pi) = 0$. Then

$$u(r,\theta) = \sum_{n=1}^{\infty} r^n c_n \sin n\theta$$

satisfies the nonhomogeneous boundary condition $u(a,\theta) = f(\theta)$ provided that $a^n c_n$ is is the nth Fourier sine coefficient of $f(\theta)$ on the interval $0 < \theta < \pi$, that is,

$$c_n = \frac{2}{\pi a^n} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta.$$

14. We start with the periodic polar-coordinate solution

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

and choose $b_n \equiv 0$ in order to satisfy the conditions $u_\theta(r,0) = u_\theta(r,\pi) = 0$. Then

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n c_n \cos n\theta$$

satisfies the nonhomogeneous boundary condition $u(a,\theta) = f(\theta)$ provided that $a^n c_n$ is is the nth Fourier sine coefficient of $f(\theta)$ on the interval $0 < \theta < \pi$, that is,

$$c_n = \frac{2}{\pi a^n} \int_0^{\pi} f(\theta) \cos n\theta \, d\theta.$$

15. As in the textbook discussion of the polar-coordinate Dirichlet problem, the substitution $u(r,\theta) = R(r)\Theta(\theta)$ in Laplace's equation yields the separated ordinary differential equations

$$r^2 R'' + rR' - \lambda R = 0 \tag{25}$$

and

$$\Theta'' + \lambda \Theta = 0. \tag{26}$$

With $\lambda = \alpha^2$ the general solution of (26) is

$$\Theta(\theta) = A \cos \alpha\theta + B \sin \alpha\theta,$$

and the endpoint condition $\Theta(0) = \Theta'(0) = 0$ yields $A = 0$ and $\theta = (2n-1)/2$, so the nth eigenvalue and eigenfunction are given by

$$\lambda_n = \frac{(2n-1)^2}{4}, \qquad \Theta_n(\theta) = \sin \frac{(2n-1)\theta}{2}.$$

As in the discussion of Eqs. (29) and (30) in the text, the bounded solution of

$$r^2 R_n'' + rR_n' - \frac{(2n-1)^2}{4} R_n = 0$$

is

$$R_n(r) = r^{(2n-1)/2}$$

for $n = 1, 2, 3, \cdots$. We thereby obtain the formal series solution

$$u(r,\theta) = \sum_{n \text{ odd}} c_n r^{n/2} \sin\frac{n\theta}{2}.$$

It remains only to satisfy the nonhomogeneous boundary condition $u(a,\theta) = f(\theta)$ by choosing

$$c_n = \frac{2}{\pi a^{n/2}} \int_0^{\pi} f(\theta) \sin\frac{n\theta}{2}\, d\theta,$$

so that (for n odd) $c_n a^{n/2}$ equals the nth odd half-multiple sine coefficient of $f(\theta)$.

16. The only difference between the exterior problem here and the interior problem in the text is that in

$$R_n(r) = C_n r^n + D_n r^{-n}$$

we must choose $C_n = 0$ in order that $R_n(r)$ be bounded as $r \to \infty$.

17. The substitution $u(r,\theta) = R(r)\Theta(\theta)$ in Laplace's equation yields the same separated solution functions

$$\Theta_0(\theta) = 1, \qquad R_0(r) = C_0 + D_0 \ln r$$

and

$$\Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, \qquad R_n(r) = C_n r^n + \frac{D_n}{r^n}$$

as in Eqs. (28)-(30) in the text. We choose $B_n \equiv 0$ to satisfy the boundary condition $u(r,\theta) = u(r,-\theta)$, and $n = 1$ with $C_1 = U_0$ to satisfy the given limit condition as $r \to \infty$. Then the condition that $u_r(a,\theta) = 0$ requires that $D_1 = U_0 a^2$, so

$$u(r,\theta) = \frac{U_0}{r}\left(r^2 + a^2\right)\cos\theta.$$

20. When we substitute $v(r, t) = r\,u(r, t)$ we get the boundary value problem

$$v_t = k v_{rr} \qquad (r < a, \quad t > 0)$$

$$v(0, t) = v(a, t) = 0$$

$$v(r, 0) = T_0 r$$

that corresponds to a heated rod along the interval $0 \le r \le a$. It therefore follows from

Equation (31) in Section 8.5 that

$$v(r,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2 kt}{a^2}\right)\sin\frac{n\pi x}{a}.$$

To get the formula given in the text it remains only to calculate the Fourier sine coefficients $\{b_n\}$ of $f(r) = T_0 r$ on $0 < r < a$, and finally to divide $v(r, t)$ by r to get $u(r, t)$.

21. **(a)** Since we cannot simply substitute $r = 0$, we apply continuity of $u(r, t)$ at $r = 0$ and calculate

$$u(0,t) = \lim_{r\to 0} u(r,t)$$

noting that

$$\lim_{r\to 0}\frac{\sin n\pi r/a}{r} = \frac{n\pi}{a}\lim_{r\to 0}\frac{\sin n\pi r/a}{n\pi r/a} = \frac{n\pi}{a}\lim_{r\to 0}\frac{\sin\theta}{\theta} = \frac{n\pi}{a}$$

by the elementary fact that $(\sin\theta)/\theta \to 1$ as $\theta \to 0$.

(b) With $a = 30$ and $T_0 = 100$ we have

$$u(0,t) = 200\sum_{n=1}^{\infty}(-1)^{n+1}\exp\left(-\frac{n^2\pi^2 kt}{900}\right).$$

If $k = 0.15$ for iron then after 15 minutes = 900 seconds the center temperature is

$$u(0,900) \approx 45.5075 - 0.5361 + 0.0003 - 0.0000 + \cdots \approx 44.97.$$

If $k = 0.005$ for iron then after 15 minutes the center temperature is

$$\begin{aligned} u(0,900) &\approx 190.37 - 164.174 + 128.276 - 90.8081 + 58.2426 \\ &\quad - 33.8449 + 17.8190 - 8.4998 + 3.6734 - 1.4384 \\ &\quad + 0.5103 - 0.1640 + 0.0478 - 0.0126 + 0.0030 \\ &\quad - 0.0007 + 0.0001 - 0.00002 + 0.00000 - \cdots \\ u(0,900) &\approx 100.00 \end{aligned}$$

Thus the center of the ball has not yet begun to cool. For the center of this concrete ball to reach 45° (as with the iron ball after 15 minutes) would require $(0.15/0.005)\times 15$ = 450 minutes, that is, seven and a half hours!

CHAPTER 9

EIGENVALUES AND BOUNDARY VALUE PROBLEMS

SECTION 9.1

STURM-LIOUVILLE PROBLEMS AND EIGENFUNCTION EXPANSIONS

1. In the notation of Equation (9) in Section 9.1 of the text we have $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$, so Theorem 1 implies that the eigenvalues are all nonnegative. If $\lambda = 0$, then $y'' = 0$ implies that $y(x) = Ax + B$. Then $y'(x) = A$, so the endpoint conditions yield $A = 0$, but B remains arbitrary. Hence $\lambda_0 = 0$ is an eigenvalue with eigenfunction

$$y_0(x) = 1.$$

If $\lambda = \alpha^2 > 0$, then the equation $y'' + \alpha^2 y = 0$ has general solution

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

with

$$y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

Then $y'(0) = 0$ yields $B = 0$ so $A \neq 0$, and then

$$y'(L) = -A\alpha\sin\alpha L = 0,$$

so αL must be an integral multiple of π. Thus the nth positive eigenvalue is

$$\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{L^2},$$

and the associated eigenfunction is

$$y_n(x) = \cos\frac{n\pi x}{L}.$$

2. In the notation of Equation (9) in this section we have $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$, so Theorem 1 implies that the eigenvalues are all nonnegative. If $\lambda = 0$, then $y'' = 0$ implies $y(x) = Ax + B$. But then $y(0) = B = 0$ and $y'(L) = A = 0$, so it follows that 0 is not an eigenvalue. We may therefore write $\lambda = \alpha^2 > 0$, so our equation is

$y'' + \alpha^2 y = 0$ with general solution

$$y(x) = A \cos \alpha x + B \sin \alpha x.$$

Now $y(0) = A = 0$, so $y(x) = B \sin \alpha x$ and

$$y'(x) = B\alpha \cos \alpha x.$$

Hence

$$y'(L) = B\alpha \cos \alpha L = 0,$$

so it follows that αL must be an odd multiple of $\pi/2$. Thus

$$\alpha_n = \frac{(2n-1)\pi}{2L}, \quad \lambda_n = \alpha_n^2, \quad y_n(x) = \sin \alpha_n x.$$

3. If $\lambda = 0$ then $y'' = 0$ yields $y(x) = Ax + B$ as usual. But $y'(0) = A = 0$, and then $hy(L) + y'(L) = h(B) + 0 = 0$, so $B = 0$ also. Thus $\lambda = 0$ is not an eigenvalue. If $\lambda = \alpha^2 > 0$ so our equation is $y'' + \alpha^2 y = 0$, then

$$y(x) = A \cos \alpha x + B \sin \alpha x,$$

so

$$y'(x) = -A\alpha \sin \alpha x + B\alpha \cos \alpha x.$$

Now $y'(0) = 0$ yields $B = 0$, so we may write

$$y(x) = \cos \alpha x, \qquad y'(x) = -\alpha \sin \alpha x.$$

The equation

$$hy(L) + y'(L) = h \cos \alpha L - \alpha \sin \alpha L = 0$$

then gives

$$\tan \alpha L = \frac{h}{\alpha} = \frac{hL}{\alpha L},$$

so $\beta_n = \alpha_n L$ is the nth positive root of the equation

$$\tan x = \frac{hL}{x}.$$

Thus

$$\lambda_n = \alpha_n^2 = \frac{\beta_n^2}{L^2}, \qquad y_n(x) = \cos \frac{\beta_n x}{L}.$$

Finally, a sketch of the graphs $y = \tan x$ and $y = hL/x$ indicates that $\beta_n \approx (n - 1)\pi$ for n large.

4. Here $\alpha_1 = h > 0$, $\alpha_2 = \beta_1 = 1$, and $\beta_2 = 0$, so by Theorem 1 in Section 9.1 there are no negative eigenvalues. If $\lambda = 0$ and $y(x) = Ax + B$, then the equations

$$hy(0) - y'(0) = hB - A = 0, \qquad y(L) = AL + B = 0$$

imply $h = A/B = -1/L < 0$. Thus 0 is not an eigenvalue. If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

then the condition $hy(0) = y'(0)$ yields $B = hA/\alpha$, so

$$\begin{aligned} y(x) &= \frac{A}{\alpha}(\alpha\cos\alpha x + h\sin\alpha x) \\ &= \frac{A}{\beta}\left(\beta\cos\frac{\beta x}{L} + hL\sin\frac{\beta x}{L}\right) \end{aligned}$$

where $\beta = \alpha L$. Then the condition

$$y(L) = \frac{A}{\beta}(\beta\cos\beta + hL\sin\beta) = 0$$

reduces to $\tan\beta = -\dfrac{\beta}{hL}$.

6. $y_n(x) = \sin\dfrac{(2n-1)\pi x}{2L}$ so Equation (25) in Section 9.1 — with $r(x) \equiv 1$ — yields

$$c_n = \frac{\int_0^L f(x)\sin\dfrac{(2n-1)\pi x}{2L}dx}{\int_0^L \sin^2\dfrac{(2n-1)\pi x}{2L}dx} = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}dx,$$

because the denominator integral here evaluates — by use of the trigonometric identity $\sin^2 A = \frac{1}{2}(1-\cos 2A)$ — to $L/2$.

7. The coefficient c_n in Eq. (23) of this section is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = L$, and $y_n(x) = \sin\dfrac{\beta_n x}{L}$. Using the fact that $\tan\beta_n = -\dfrac{\beta_n}{hL}$, so $\dfrac{\sin\beta_n}{\beta_n} = -\dfrac{\cos\beta_n}{hL}$, we find that

$$\int_0^L \sin^2\frac{\beta_n x}{L}dx = \int_0^L \frac{1}{2}\left(1-\cos\frac{2\beta_n x}{L}\right)dx = \frac{1}{2}\left[x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right]_0^L$$

$$= \frac{1}{2}\left(L - L\frac{\sin\beta_n}{\beta_n}\cos\beta_n\right) = \frac{1}{2}\left(L + L\frac{\cos\beta_n}{hL}\cos\beta_n\right) = \frac{hL+\cos^2\beta_n}{2h}$$

and $\int_0^L \sin\frac{\beta_n x}{L}\,dx = \frac{L(1-\cos\beta_n)}{\beta_n}$. Hence the desired eigenfunction expansion is

$$1 = 2hL\sum_{n=1}^{\infty}\frac{1-\cos\beta_n}{\beta_n\left(hL+\cos^2\beta_n\right)}\sin\frac{\beta_n x}{L}.$$

for $0 < x < L$.

8. The coefficient c_n in (23) is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = L$, and $y_n(x) = \cos\beta_n x/L$:

$$c_n = \frac{\int_0^L \cos\frac{\beta_n x}{L}\,dx}{\int_0^L \cos^2\frac{\beta_n x}{L}\,dx} = \frac{\int_0^L \cos\frac{\beta_n x}{L}\,dx}{\int_0^L \frac{1}{2}\left(1+\cos\frac{2\beta_n x}{L}\right)dx} = \frac{\left[\frac{L}{\beta_n}\sin\frac{\beta_n x}{L}\right]_0^L}{\left[\frac{1}{2}\left(x+\frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right)\right]}$$

$$= \frac{\frac{L}{\beta_n}\sin\beta_n}{\frac{1}{2}\left(L+\frac{L}{2\beta_n}\sin 2\beta_n\right)} = \frac{4\sin\beta_n}{2\beta_n+\sin 2\beta_n}.$$

Hence the desired eigenfunction expansion is

$$1 = \sum_{n=1}^{\infty}\frac{4\sin\beta_n}{2\beta_n+\sin 2\beta_n}\cos\frac{\beta_n x}{L}.$$

9. The coefficient c_n in (23) is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = 1$, and $y_n(x) = \sin\beta_n x$. Using the fact that $\tan\beta_n = -\beta_n/h$, so $h\sin\beta_n = -\beta_n\cos\beta_n$, we find that

$$\int_0^1 \sin^2\beta_n x\,dx = \int_0^1 \frac{1}{2}(1-\cos 2\beta_n x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2\beta_n x}{2\beta_n}\right]_0^1$$

$$= \frac{1}{2}\left(1-\frac{\sin\beta_n}{\beta_n}\cos\beta_n\right) = \frac{1}{2}\left(1+\frac{\cos^2\beta_n}{h}\right) = \frac{h+\cos^2\beta_n}{2h}$$

and

$$\int_0^1 x\sin\beta_n x\,dx = \frac{1}{\beta_n^2}\int_0^1 \beta_n x\sin\beta_n x\cdot\beta_n dx = \frac{1}{\beta_n^2}\int_0^{\beta_n} u\sin u\,du$$

$$= \frac{1}{\beta_n^2}\left[\sin u - u\cos u\right]_0^{\beta_n} = \frac{\sin\beta_n - \beta_n\cos\beta_n}{\beta_n^2}$$

$$= \frac{\sin\beta_n - \beta_n\cos\beta_n}{\beta_n^2} = \frac{(1+h)\sin\beta_n}{\beta_n^2}.$$

It follows that the desired expansion is given by

$$x = 2h(1+h)\sum_{n=1}^{\infty}\frac{\sin\beta_n \sin\beta_n x}{\beta_n^2\left(h+\cos^2\beta_n\right)}$$

for $0 < x < 1$.

10. The coefficient c_n in (23) is given by Formula (25) with $f(x) = x$, $r(x) = 1$, $a = 0$, $b = 1$, and $y_n(x) = \cos\beta_n x$. Integrations similar to those in Problems 8 and 9 give

$$c_n = \frac{\int_0^1 x\cos\beta_n x\,dx}{\int_0^1 \cos^2\beta_n x\,dx} = \frac{4\left(\beta_n\sin\beta_n + \cos\beta_n - 1\right)}{\beta_n\left(2\beta_n + \sin 2\beta_n\right)}.$$

With this value of c_n for $n = 1, 2, 3, \cdots$, the desired eigenfunction expansion is

$$x = \sum_{n=1}^{\infty} c_n\cos\beta_n x.$$

11. If $\lambda = 0$ then $y'' = 0$ implies that $y(x) = Ax + B$. Then $y(0) = 0$ gives $B = 0$, so $y(x) = Ax$. Hence

$$hy(L) - y'(L) = h(AL) - A = A(hL - 1) = 0$$

if and only if $hL = 1$, in which case $\lambda_0 = 0$ has associated eigenfunction $y_0(x) = x$.

12. If $\lambda = -\alpha^2 < 0$, then the general solution of $y'' - \alpha^2 y = 0$ is

$$y(x) = A\cosh\alpha x + B\sinh\alpha x.$$

But then $y(0) = A = 0$, so we may take $y(x) = \sinh\alpha x$. Now the condition $hy(L) = y'(L)$ yields

$$h\sinh\alpha L = \alpha\cosh\alpha L.$$

It follows that $\beta = \alpha L$ must be a root of the equation

$$\tanh x = \frac{x}{hL}.$$

The curve $y = \tanh x$ passes through the origin with slope 1, and is concave upward for $x < 0$, concave downward for $x > 0$. Hence this curve and the straight line $y = x/hL$ intersect other than at the origin if and only if the slope of the line is less than 1 — that is, if and only if $hL > 1$. In this case, with β_0 the positive root of $\tanh x = x/hL$, we have $\lambda_0 = -\beta_0^2$ and $y_0(x) = \sinh \beta_0 x$.

13. If $\lambda = +\alpha^2 > 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A \cos \alpha x + B \sin \alpha x.$$

But then $y(0) = A = 0$, so we may take $y(x) = \sin \alpha x$. Now the condition $hy(L) = y'(L)$ yields

$$h \sin \alpha L = \alpha \cos \alpha L.$$

It follows that $\beta = \alpha L$ must be a root of the equation

$$\tan x = \frac{x}{hL}.$$

So if β_n is the nth positive root of this equation, then $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and the corresponding eigenfunction is $y_n(x) = \sin \beta_n x / L$.

14. With $\lambda = 0$, $y'' = 0$, and hence $y(x) = Ax + B$, we have $y(0) = B = 0$, so $y(x) = Ax$. Then the condition $hy(L) = y'(L)$ reduces to the equation $hL = A$, which is satisfied because $hL = 1$. Thus $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = x$. Together with the positive eigenvalues and associated eigenfunctions provided by Problem 13, this gives the eigenfunction expansion

$$f(x) = c_0 x + \sum_{n=1}^{\infty} c_n \sin \frac{\beta_n x}{L}$$

where $\tan \beta_n = \beta_n$. The coefficients are given by

$$c_0 = \frac{\int_0^L f(x)\, x\, dx}{\int_0^L x^2\, dx} = \frac{3}{L^3} \int_0^L x f(x)\, dx,$$

$$c_0 = \frac{\int_0^L f(x) \sin \beta_n x / L\, dx}{\int_0^L \sin^2 \beta_n x / L\, dx} = \frac{2}{L \sin^2 \beta_n} \int_0^L f(x) \sin \beta_n x / L\, dx,$$

the latter because

$$\int_0^L \sin^2 \beta_n x/L\,dx = \frac{1}{2}\int_0^L (1-\cos 2\beta_n x/L)\,dx = \frac{1}{2}\left[x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right]_0^L$$

$$= \frac{1}{2}\left(L - L\cdot\frac{\sin\beta_n}{\beta_n}\cdot\cos\beta_n\right) = \frac{L}{2}\left(1-\cos^2\beta_n\right) = \frac{L\sin^2\beta_n}{2}.$$

15. If $\lambda_0 = 0$, then a general solution of $y'' = 0$ is $y(x) = Ax + B$. The conditions

$$y(0) + y'(0) = B + A = 0, \qquad y(1) = A + B = 0$$

both say that $B = -A$, so we may take $y_0(x) = x - 1$ as the eigenfunction associated with $\lambda_0 = 0$. If $\lambda = +\alpha^2 < 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A\cos\alpha x + B\sin\alpha x.$$

But $y(0) + y'(0) = A + B\alpha = 0$, so $A = -B\alpha$, and then

$$y(1) = A\cos\alpha + B\sin\alpha = -B(\alpha\cos\alpha - \sin\alpha) = 0.$$

Thus the possible values of α are the positive roots $\{\beta_n\}$ of the equation $\tan x = x$, and the nth eigenfunction is $y_n(x) = \beta_n\cos\beta_n x - \sin\beta_n x$,

17. The Fourier sine series of the constant function $f(x) \equiv w$ for $0 < x < L$ is

$$w = \frac{4w}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\sin\frac{n\pi x}{L}.$$

If $y = \sum b_n \sin n\pi x/L$, then

$$EI\,y^{(4)} = EI\sum_{n=1}^{\infty}\frac{n^4\pi^4 b_n}{L^4}\sin\frac{n\pi x}{L}.$$

Upon equating coefficients in these two series and solving for b_n, we see that

$$y(x) = \frac{4wL^4}{EI\pi^5}\sum_{n\text{ odd}}\frac{1}{n^5}\sin\frac{n\pi x}{L}.$$

18. By Equation (16) in Section 9.3, the Fourier sine series of $f(x) = bx$ for $0 < x < L$ is

$$bx = \frac{2bL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}.$$

If $y = \sum b_n \sin n\pi x/L$, then

$$EI\,y^{(4)} = EI\sum_{n=1}^{\infty}\frac{n^4\pi^4 b_n}{L^4}\sin\frac{n\pi x}{L}.$$

Upon equating coefficients in these two series and solving for b_n, we see that

$$y(x) = \frac{2bL^5}{EI\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi x}{L}.$$

19. With $\lambda = \alpha^4$, the general solution of $y^{(4)} - \alpha^4 y = 0$ is

$$y(x) = A\cosh\alpha x + B\sinh\alpha x + C\cos\alpha x + D\sin\alpha x,$$

and then

$$y'(x) = \alpha(A\sinh\alpha x + B\cosh\alpha x - C\sin\alpha x + D\cos\alpha x).$$

The conditions $y(0) = 0$ and $y'(0) = 0$ yield $C = -A$ and $D = -B$, so now

$$y(x) = A(\cosh\alpha x - \cos\alpha x) + B(\sinh\alpha x - \sin\alpha x).$$

The conditions $y(L) = 0$ and $y'(L) = 0$ yield the two linear equations

$$A(\cosh\alpha L - \cos\alpha L) + B(\sinh\alpha L - \sin\alpha L) = 0,$$
$$A(\sinh\alpha L + \sin\alpha L) + B(\cosh\alpha L - \cos\alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh\alpha L - \cos\alpha L)^2 - (\sinh^2\alpha L - \sin^2\alpha L) = 0.$$

Using the facts that $\cosh^2 A - \sinh^2 A = 1$ and $\cos^2 A + \sin^2 A = 1$, this equation simplifies to

$$\cosh\alpha L\cos\alpha L - 1 = 0,$$

so $\beta = \alpha L = x$ satisfies the equation

$$\cosh x\cos x = 1.$$

The eigenvalue corresponding to the nth positive root β_n is

$$\lambda_n = \alpha_n^4 = \left(\frac{\beta_n}{L}\right)^4.$$

Finally the first equation in the pair above yields

$$B = -\frac{\cosh\alpha L - \cos\alpha L}{\sinh\alpha L - \sin\alpha L},$$

so we may take

$$y_n(x) = \left(\sinh\beta_n - \sin\beta_n\right)\left(\cosh\frac{\beta_n x}{L} - \cos\frac{\beta_n x}{L}\right)$$
$$-\left(\cosh\beta_n - \cos\beta_n\right)\left(\sinh\frac{\beta_n x}{L} - \sin\frac{\beta_n x}{L}\right)$$

as the eigenfunction associated with the eigenvalue λ_n.

20. As in Problem 19, the solution of $y^{(4)} - \alpha^4 y = 0$ satisfying the left-endpoint conditions $y(0) = 0$ and $y'(0) = 0$ is given by

$$y(x) = A(\cosh \alpha x - \cos \alpha x) + B(\sinh \alpha x - \sin \alpha x).$$

The right-endpoint conditions $y''(L) = 0$ and $y^{(3)}(L) = 0$ now yield the two linear equations

$$A(\cosh \alpha L + \cos \alpha L) + B(\sinh \alpha L + \sin \alpha L) = 0,$$
$$A(\sinh \alpha L - \sin \alpha L) + B(\cosh \alpha L + \cos \alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh \alpha L + \cos \alpha L)^2 - (\sinh^2 \alpha L - \sin^2 \alpha L) = 0.$$

This equation simplifies to

$$\cosh \alpha L \cos \alpha L + 1 = 0,$$

so $\beta = \alpha L = x$ satisfies the equation

$$\cosh x \cos x = -1.$$

The eigenvalue corresponding to the nth root β_n is

$$\lambda_n = \alpha_n^4 = \left(\frac{\beta_n}{L}\right)^4.$$

Finally the first equation in the pair above yields

$$B = -\frac{\cosh \alpha L + \cos \alpha L}{\sinh \alpha L + \sin \alpha L},$$

so we may take

$$y_n(x) = \left(\sinh\beta_n + \sin\beta_n\right)\left(\cosh\frac{\beta_n x}{L} - \cos\frac{\beta_n x}{L}\right)$$
$$-\left(\cosh\beta_n + \cos\beta_n\right)\left(\sinh\frac{\beta_n x}{L} - \sin\frac{\beta_n x}{L}\right)$$

as the eigenfunction associated with the eigenvalue λ_n.

21. As in Problem 19, the solution of $y^{(4)} - \alpha^4 y = 0$ satisfying the left-endpoint conditions $y(0) = 0$ and $y'(0) = 0$ is given by

$$y(x) = A(\cosh\alpha x - \cos\alpha x) + B(\sinh\alpha x - \sin\alpha x).$$

The right-endpoint conditions $y(L) = 0$ and $y''(L) = 0$ yield the two linear equations

$$A(\cosh\alpha L - \cos\alpha L) + B(\sinh\alpha L - \sin\alpha L) = 0,$$
$$A(\cosh\alpha L + \cos\alpha L) + B(\sinh\alpha L + \sin\alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh\alpha L - \cos\alpha L)(\sinh\alpha L + \sin\alpha L)$$
$$-(\cosh\alpha L + \cos\alpha L)(\sinh\alpha L - \sin\alpha L) = 0.$$

This equation simplifies to $2\cosh\alpha L\sin\alpha L - 2\cos\alpha L\sinh\alpha L = 0$, which is equivalent to $\tanh\alpha L = \tan\alpha L$. Hence $\beta = \alpha L = x$ satisfies the equation $\tanh x = \tan x$, and the eigenvalue corresponding to the nth positive root β_n is $\lambda_n = \alpha_n^4 = (\beta_n / L)^4$.

SECTION 9.2

APPLICATIONS OF EIGENFUNCTION SERIES

1. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 3 of Section 9.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X'(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \cos\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = hL/x$. The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\cos\frac{\beta_n x}{L}.$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx}{\int_0^L \cos^2\frac{\beta_n x}{L}\,dx} = \frac{2h}{hL+\sin^2\beta_n}\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx,$$

because

$$\begin{aligned}
\int_0^L \cos^2\frac{\beta_n x}{L}\,dx &= \int_0^L \frac{1}{2}\left(1+\cos\frac{2\beta_n x}{L}\right)dx = \left[\frac{1}{2}\left(x+\frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right)\right]_0^L \\
&= \frac{1}{2}\left(L+\frac{L}{2\beta_n}\sin 2\beta_n\right) = \frac{1}{2h}\left(hL+\sin\beta_n\cdot\frac{hL\cos\beta_n}{\beta_n}\right) \\
&= \frac{hL+\sin^2\beta_n}{2h}.
\end{aligned}$$

In the final step here we use the fact that $(hL\cos\beta_n)/\beta_n = \sin\beta_n$ because $\tan\beta_n = hL/\beta_n$.

2. The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad Y'' - \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Example 5 of Section 9.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = -x/hL$. The solution of

$$Y_n'' - \frac{\beta_n^2}{L^2}Y_n = 0, \qquad Y(L) = 0$$

is

$$Y_n(y) = \sinh\frac{\beta_n(L-y)}{L},$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{\beta_n x}{L}\sinh\frac{\beta_n(L-y)}{L}.$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\sin\frac{\beta_n x}{L}\,dx}{(\sinh\beta_n)\int_0^L \sin^2\frac{\beta_n x}{L}\,dx} = \frac{4\beta_n}{L(\sinh\beta_n)(2\beta_n - \sin 2\beta_n)}\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx,$$

because

$$\int_0^L \sin^2\frac{\beta_n x}{L}\,dx = \int_0^L \frac{1}{2}\left(1-\cos\frac{2\beta_n x}{L}\right)dx = \left[\frac{1}{2}\left(x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right)\right]_0^L$$

$$= \frac{1}{2}\left(L - \frac{L}{2\beta_n}\sin 2\beta_n\right) = \frac{L(2\beta_n - \sin 2\beta_n)}{4\beta_n}.$$

3. The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X'' - \alpha^2 X = 0 \quad \text{and} \quad Y'' + \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. Problem 3 of Section 9.1 we saw that the Sturm-Liouville problem

$$Y'' + \alpha^2 Y = 0, \qquad Y'(0) = hY(L) + Y'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2/L^2$ and eigenfunctions

$$Y_n(y) = \cos\frac{\beta_n y}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = hL/x$. The solution of

$$X_n'' - \frac{\beta_n^2}{L^2} X_n = 0, \qquad X(L) = 0$$

is

$$X_n(x) = \sinh \frac{\beta_n(L-x)}{L},$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh \frac{\beta_n(L-x)}{L} \cos \frac{\beta_n y}{L}.$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L g(y) \cos \frac{\beta_n y}{L}\, dy}{(\sinh \beta_n) \int_0^L \cos^2 \frac{\beta_n y}{L}\, dy} = \frac{2h}{(\sinh \beta_n)\left(hL + \sin^2 \beta_n\right)} \int_0^L g(y) \cos \frac{\beta_n y}{L}\, dy,$$

because

$$\int_0^L \cos^2 \frac{\beta_n y}{L}\, dy = \int_0^L \frac{1}{2}\left(1 + \cos \frac{2\beta_n y}{L}\right) dy = \left[\frac{1}{2}\left(y + \frac{L}{2\beta_n} \sin \frac{2\beta_n y}{L}\right)\right]_0^L$$

$$= \frac{1}{2}\left(L + \frac{L}{2\beta_n} \sin 2\beta_n\right) = \frac{hL + \sin^2 \beta_n}{2h}.$$

The final step here is the same as in Problem 1, using the fact that $(hL \cos \beta_n)/\beta_n = \sin \beta_n$ because $\tan \beta_n = hL/\beta_n$.

4. The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \quad \text{and} \quad Y'' - \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Example 5 of Section 9.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \sin \frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = -x/hL$. The bounded solution of

$$Y_n'' - \frac{\beta_n^2}{L^2} Y_n = 0$$

is

$$Y_n(y) = \exp\left(-\frac{\beta_n y}{L}\right),$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{\beta_n x}{L}\exp\left(-\frac{\beta_n y}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\sin\frac{\beta_n x}{L}\,dx}{\int_0^L \sin^2\frac{\beta_n x}{L}\,dx} = \frac{4\beta_n}{L(2\beta_n - \sin 2\beta_n)}\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx,$$

the calculation of the denominator integral here being the same as in Problem 2.

5. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 4 of Section 9.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad hX(0) - X'(0) = X(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = -x/hL$. The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)dx}{\int_0^L \left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)^2 dx}.$$

The evaluation of the denominator integral here is elementary, but there seems little point in carrying it out explicitly.

6. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 5 of Section 9.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad hX(0) - X'(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation

$$\tan x = \frac{2hLx}{x^2 - h^2L^2}.$$

The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)dx}{\int_0^L \left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)^2 dx}.$$

7. The boundary value problem here is

$$\begin{aligned} u_{xx}+u_{yy} &= 0 \quad (0<x<1, \quad y>0) \\ u_x(0,y)= u(1,y)+u_x(1,t) &= 0, \\ u(x,0) &= 100. \end{aligned}$$

The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X''+\alpha^2 X = 0 \qquad \text{and} \qquad Y''-\alpha^2 Y = 0$$

with separation constant $\lambda=\alpha^2$. In Problem 3 of Section 9.1 we saw (taking $h=L=1$) that the Sturm-Liouville problem

$$X''+\alpha^2 X = 0, \qquad X'(0) = X(1)+X'(1) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2$ and eigenfunctions

$$X_n(x) = \cos\alpha_n x$$

for $n = 1, 2, 3, \cdots$, with $\{\alpha_n\}$ being the positive roots of the equation $\tan x = 1/x$. The bounded solution of $Y_n''-\alpha_n^2 Y_n = 0$ is then

$$Y_n(y) = \exp(-\alpha_n y),$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \cos\alpha_n x \exp(-\alpha_n y).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L 100\cos\alpha_n x\,dx}{\int_0^L \cos^2\alpha_n x\,dx} = \frac{\left[\dfrac{100}{\alpha_n}\sin\alpha_n x\right]_0^1}{\left[\dfrac{1}{2}\left(x+\dfrac{1}{2\alpha_n}\sin 2\alpha_n x\right)\right]_0^1} = \frac{200\sin\alpha_n}{\alpha_n+\sin\alpha_n\cos\alpha_n},$$

so

$$u(x,y) = 200\sum_{n=1}^{\infty}\frac{\sin\alpha_n\cos\alpha_n x\exp(-\alpha_n y)}{\alpha_n+\sin\alpha_n\cos\alpha_n}.$$

The first five positive solutions of $\tan x = 1/x$ are 0.8603, 3.4256, 7.4373, 9.5293, and 12.6453, and we find that

$$u(1,1) \approx 30.8755+0.4737+0.0074+0.0002+0.0000+\cdots \approx 31.4\,^\circ\text{C}.$$

8. With $m = 0$ the boundary value problem in Example 2 is

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \qquad (0 < x < L, \quad t > 0), \\ u(0,t) &= u_x(L,t) = 0, \\ u_t(x,0) &= 0, \\ u(x,0) &= bx. \end{aligned}$$

The substitution $u(x,t) = X(x)T(t)$ gives the separated equations

$$X'' + \lambda X = T'' + \lambda a^2 T = 0.$$

and the eigenfunctions of the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(L) = 0$$

are of the form

$$X_n(x) = \sin\frac{n\pi x}{2L}$$

with n odd, with corresponding eigenvalue $\lambda_n = n^2\pi^2/4L^2$. This leads readily to the solution

$$u(x,t) = \sum_{n \text{ odd}} c_n \sin\frac{n\pi x}{2L}\cos\frac{n\pi at}{2L},$$

where c_n is the odd half-multiple sine coefficient (of Problem 21 in Section 9.3) given by

$$c_{2n-1} = \frac{2}{L}\int_0^L bx\sin\frac{(2n-1)\pi x}{L}\,dx = \frac{8bL\sin\frac{(2n-1)\pi}{2}}{(2n-1)^2\pi^2} = \frac{8bL(-1)^{n+1}}{(2n-1)^2\pi^2}.$$

9. **(a)** With $\lambda = 0$, the endpoint-value problem in (19) is $X'' = 0$, $X(0) = X'(0) = 0$, which has only the trivial solution $X(x) \equiv 0$. Thus $\lambda = 0$ is not an eigenvalue.

(b) With $\lambda = -\alpha^2 < 0$, the endpoint-value problem in (19) is

$$X'' - \alpha^2 X = 0, \qquad X(0) = 0, \quad -m\alpha^2 X(L) = A\delta X'(L).$$

The differential equation and the left-endpoint condition here give $X(x) = \sinh\alpha x$, and substitution in the right-endpoint condition gives

$$-m\alpha^2\sinh\alpha L = A\delta\alpha\cosh\alpha L, \text{ that is, } \tanh\alpha L = -\frac{k}{\alpha L}$$

with $k = A\delta L/m > 0$. But the graph $y = \tanh x$ lies (aside from the origin) in the first and third quadrants, while the graph $y = -k/x$ lies interior to the second and fourth quadrants. Hence the two cannot intersect, and it follows that there cannot be an eigenvalue of the assumed form $\lambda = -\alpha^2 < 0$,.

10. **(a)** With $\delta = 7.75 \text{ gm/cm}^3$ and $E = 2{\cdot}10^{12}$ in Equation (16), the speed of sound in steel is

$$a = \sqrt{\frac{E}{\delta}} \approx 5.08 \, x \, 10^5 \text{ cm/sec} \approx 11364 \text{ mph.}$$

(b) With $\delta = 1 \text{ gm/cm}^3$ and $K = 2.25{\cdot}10^{10}$ in Equation (16), the speed of sound in water is

$$a = \sqrt{\frac{K}{\delta}} \approx 1.50 \, x \, 10^5 \text{ cm/sec} \approx 3355 \text{ mph.}$$

11. **(a)** $$a = \sqrt{\frac{K}{\delta}} = \sqrt{\frac{\lambda p}{m/V}} = \sqrt{\frac{\gamma p V}{m}} = \sqrt{\frac{\gamma n R T_K}{n m_0}} = \sqrt{\frac{\gamma R T_K}{m_0}}$$

(b) $$a = \sqrt{\frac{\gamma R T_K}{m_0}} = \sqrt{\frac{1.4 \times 8314(273 + T_C)}{29}} = \sqrt{\frac{1.4 \times 8314 \times 273}{29}\left(1 + \frac{T_C}{273}\right)}$$

$$\approx 331.02\sqrt{1 + \frac{T_C}{273}} \, \frac{\text{m}}{\text{sec}} \approx 740.47\sqrt{1 + \frac{T_C}{273}} \, \frac{\text{miles}}{\text{hour}}$$

$$\approx 740.47\left[1 + \frac{1}{2}\left(\frac{T_C}{273}\right) + \cdots\right] \approx 740.47 + 1.356 T_C$$

12. The boundary value problem is

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \quad (0 < x < L, \; t > 0) \\ u(0, t) &= ku(L, t) + AEu_x(L, t) = 0 \\ u(x, 0) &= f(x), \\ u_t(x, 0) &= 0. \end{aligned}$$

Starting with the general solution

$$X(x) = A \cos \alpha x + B \sin \alpha x$$

of $X'' + \alpha^2 X = 0$, the condition $X(0) = 0$ gives $A = 0$, so

$$X(x) = \sin \alpha x, \qquad X'(x) = \alpha \cos \alpha x.$$

Then the condition $kX(L) + AEX'(L) = 0$ yields

$$k \sin \alpha L + AE\alpha \cos \alpha L = 0,$$

which is equivalent to the equation

$$\tan x = -\frac{AEx}{kL}$$

with $x = \alpha L$, $\alpha = x/L$. If $\{\beta_n\}$ are the positive roots of this equation, then the nth eigenvalue is $\lambda_n = \alpha_n^2 = (\beta_n/L)^2$ with associated eigenfunction

$$X_n(x) = \sin\frac{\beta_n x}{L}.$$

The associated function of t is

$$T_n(t) = A_n \cos\frac{\beta_n at}{L} + B_n \sin\frac{\beta_n at}{L},$$

but the condition $T'(0) = 0$ yields $B_n = 0$. Hence we obtain a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{\beta_n x}{L}\cos\frac{\beta_n at}{L}.$$

15. $$\int_0^L \sin\frac{\beta_m x}{L}\sin\frac{\beta_n x}{L}\,dx = \frac{L}{2}\left[\frac{\sin(\beta_m-\beta_n)}{\beta_m-\beta_n} - \frac{\sin(\beta_m+\beta_n)}{\beta_m+\beta_n}\right]$$

$$= \frac{L}{2\left(\beta_m^2-\beta_n^2\right)}\left[\begin{array}{l}(\beta_m+\beta_n)(\sin\beta_m\cos\beta_n - \sin\beta_n\cos\beta_m) \\ \qquad -(\beta_m-\beta_n)(\sin\beta_m\cos\beta_n + \sin\beta_n\cos\beta_m)\end{array}\right]$$

$$= \frac{L}{\beta_m^2-\beta_n^2}\left[\beta_n\sin\beta_m\cos\beta_n - \beta_m\sin\beta_n\cos\beta_m\right]$$

$$= \frac{L}{\beta_m^2-\beta_n^2}\left[\beta_n\cdot\frac{M\cos\beta_m}{m\beta_m}\cdot\cos\beta_n - \beta_m\cdot\frac{M\cos\beta_n}{m\beta_n}\cdot\cos\beta_m\right]$$

$$= \frac{LM}{m\left(\beta_m^2-\beta_n^2\right)}\cos\beta_m\cos\beta_n\left(\frac{\beta_n}{\beta_m}-\frac{\beta_m}{\beta_n}\right) = -\frac{LM\cos\beta_m\cos\beta_n}{m\beta_m\beta_n} \neq 0$$

16. When we substitute $v(r, t) = ru_r(r, t)$ we get the boundary value problem

$$v_t = kv_{rr}$$

$$v(0, t) = v(a, t) - av_r(a, t) = 0$$

$$v(r, 0) = r\,f(r).$$

Then $v(r, t) = R(r)T(t)$ yields the equations

$$R'' + \lambda R = 0, \qquad T' = -\lambda kT.$$

If $\lambda_0 = 0$ then $R(r) = Ar + B$. The condition $R(0) = 0$ gives $B = 0$, and $R(r) = Ar$ satisfies the condition $R(a) - aR'(a) = 0$. Thus $\lambda_0 = 0$ is an eigenvalue with eigenfunction

$$R_0(r) = r; \qquad T_0(t) = 1.$$

If $\lambda = \alpha^2 > 0$ then

$$R(r) = A\cos\alpha r + B\sin\alpha r$$

and $R(0) = 0$ gives $A = 0$, so

$$R(r) = \sin\alpha r, \qquad R'(r) = \alpha\cos\alpha r.$$

The condition $R(a) = aR'(a)$ yields $\sin\alpha a = a\alpha\cos\alpha a$, that is,

$$\tan x = x$$

where $x = \alpha a$. If $\{\beta_n\}$ are the roots of this equation, then $\lambda_n = (\beta_n/a)^2$ is an eigenvalue with associated eigenfunction

$$R_n(r) = \sin\frac{\beta_n r}{a}, \qquad \text{and} \qquad T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{a^2}\right).$$

We therefore obtain a solution of the form

$$v(r,t) = c_0 r + \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{a^2}\right)\sin\frac{\beta_n r}{a}.$$

The coefficient formulas given in the textbook follow immediately from Problem 14 in Section 9.1, and finally we obtain $u(r, t)$ upon division of $v(r, t)$ by r.

18. The only difference from Example 3 in the text is that the solution of Equation (37) with $T_n'(0) = 0$ is $T_n(t) = \sin\dfrac{n^2\pi^2 a^2 t}{L^2}$.

19. With the given initial velocity function $g(x)$ with constant value $P/2\rho\varepsilon$ concentrated in the interval $L/2 - \varepsilon < x < L/2 + \varepsilon$, the coefficient formula of Problem 18 gives

$$c_n = \frac{2L}{n^2\pi^2a^2}\int_{L/2-\varepsilon}^{L/2+\varepsilon}\frac{P}{2\rho\varepsilon}\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{L^2P}{n^3\pi^3a^2\rho\varepsilon}\left[\cos\left(\frac{n\pi}{2}-\frac{n\pi\varepsilon}{L}\right)-\cos\left(\frac{n\pi}{2}+\frac{n\pi\varepsilon}{L}\right)\right] = \frac{2L^2P}{n^3\pi^3a^2\rho\varepsilon}\sin\frac{n\pi}{2}\sin\frac{n\pi\varepsilon}{L}.$$

This gives the ε-dependent solution

$$y(x,t,\varepsilon) = \frac{2L^2P}{\pi^3a^2\rho\varepsilon}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi\varepsilon}{L}\sin\frac{n^2\pi^2a^2t}{L^2}\sin\frac{n\pi x}{L}.$$

Because

$$\frac{L}{n\pi\varepsilon}\sin\frac{n\pi\varepsilon}{L} = \frac{\sin(n\pi\varepsilon/L)}{n\pi\varepsilon/L} \to 1 \quad \text{as} \quad \varepsilon\to 0,$$

the limit $y(x,t)=\lim_{\varepsilon\to 0}y(x,t,\varepsilon)$ has the expansion

$$y(x,t) = \frac{2LP}{\pi^2a^2\rho}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n^2\pi^2a^2t}{L^2}\sin\frac{n\pi x}{L}.$$

20. $$c_n = \frac{2L}{n^2\pi^2a^2}\int_0^L v_0\sin\frac{n\pi x}{L}\,dx = \frac{2v_0L^2}{n^3\pi^3a^2}(1-\cos n\pi)$$

The fundamental frequency is

$$\omega_1 = \frac{\pi^2a^2}{L^2} = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}\,\omega_1.$$

With

$$E = 2\cdot 10^{12}\ \text{dyne/cm}^2,$$

$$I = (2.54)^4/12 \approx 3.47\ \text{cm}^4,$$

$$\rho = (7.75)(2.54)^2 \approx 50.00\ \text{gm/cm},$$

$$L = (19)(2.54) \approx 48.26\ \text{cm},$$

we calculate

$$\omega_1 \approx 1578\ \text{rad/sec} \approx 251\ \text{cycles/sec}.$$

Thus we hear middle C (approximately).

SECTION 9.3

STEADY PERIODIC SOLUTIONS AND NATURAL FREQUENCIES

In Problems 1–6 we substitute $u(x,t) = X(x)\cos\omega t$ in

$$u_{tt} = a^2 u_{xx} \qquad (a^2 = E/\delta)$$

and then cancel the factor $\cos\omega t$ to obtain the ordinary differential equation

$$a^2X'' + \omega^2 X = 0$$

with general solution

$$X(x) = A\cos\frac{\omega x}{a} + B\sin\frac{\omega x}{a}. \qquad (*)$$

It then remains only to apply the given endpoint conditions to determine the natural (circular) frequencies — the values of ω for which a non-trivial solution exists.

1. Endpoint conditions: $X(0) = X(L) = 0$

With conditions $X(0) = 0$ in (*) implies that $A = 0$, so $X(x) = \sin(\omega x/a)$. Then $X(L) = \sin(\omega L/a) = 0$ implies that $\omega L/a = n\pi$, an integral multiple of π. Hence the nth natural frequency is $\omega_n = \dfrac{n\pi a}{L} = \dfrac{n\pi}{L}\sqrt{\dfrac{E}{\delta}}$.

2. Endpoint conditions: $X'(0) = X'(L) = 0$

The condition $X'(0) = 0$ gives $B = 0$ in (*), so we have

$$X(x) = \cos\frac{\omega x}{a}, \qquad \text{so} \qquad X'(x) = -\frac{\omega}{a}\sin\frac{\omega x}{a}.$$

Hence the condition $X'(L) = 0$ implies that $\omega L/a$ is an integral multiple of π. Thus the nth natural frequency is $\omega_n = \dfrac{n\pi a}{L} = \dfrac{n\pi}{L}\sqrt{\dfrac{E}{\delta}}$.

3. Endpoint conditions: $X(0) = X'(L) = 0$

The condition $X(0) = 0$ gives $A = 0$ in (*), so we have

$$X(x) = \sin\frac{\omega x}{a}, \qquad \text{so} \qquad X'(x) = \frac{\omega}{a}\cos\frac{\omega x}{a}.$$

Hence the condition $X'(L) = 0$ implies that $\omega L/a$ is an *odd* integral multiple of $\pi/2$. Thus the nth natural frequency is $\omega_n = \dfrac{(2n-1)\pi a}{2L} = \dfrac{(2n-1)\pi}{2L}\sqrt{\dfrac{E}{\delta}}$.

4. Endpoint conditions: $u(0,t) = m u_{tt}(L,t) + AE u_x(L,t) = 0$

The condition $X(0) = 0$ gives $A = 0$ in (*), so we have

$$X(x) = \sin\frac{\omega x}{a}, \qquad \text{so} \qquad u(x,t) = \sin\frac{\omega x}{a}\cos\omega t.$$

Then

$$u_{tt}(x,t) = -\omega^2 \sin\frac{\omega x}{a}\cos\omega t, \qquad u_x(x,t) = \frac{\omega}{a}\cos\frac{\omega x}{a}\cos\omega t,$$

so the other endpoint condition is

$$-m\omega^2 \sin\frac{\omega L}{a}\cos\omega t + AE\frac{\omega}{a}\cos\frac{\omega L}{a}\cos\omega t = 0.$$

Upon canceling the $\cos\omega t$ factor, we find that

$$\tan\frac{\omega L}{a} = \frac{AE}{ma\omega} = \frac{AEL}{ma^2\cdot\omega L/a} = \frac{AEL}{mE/\rho\cdot\omega L/a} = \frac{AE\rho}{m\cdot\omega L/a} = \frac{M}{m\cdot\omega L/a}.$$

Thus $\beta = \omega L/a$ is a positive root of the equation $\tan x = \dfrac{M/m}{x}$, and the nth natural frequency is given by

$$\omega_n = \frac{\beta_n a}{L} = \frac{\beta_n}{L}\sqrt{\frac{E}{\delta}}$$

where β_n is the nth positive root of this equation. This is the special case $k = 0$ of Problem 7 below.

5. Endpoint conditions: $u_x(0,t) = ku(L,t) + AE u_x(L,t) = 0$

The condition $X'(0) = 0$ gives $B = 0$ in (*), so we have

$$X(x) = \cos\frac{\omega x}{a}, \qquad \text{so} \qquad u(x,t) = \cos\frac{\omega x}{a}\cos\omega t.$$

Then

$$u_x(x,t) = -\frac{\omega}{a}\sin\frac{\omega x}{a}\cos\omega t,$$

so the other endpoint condition is

$$k\cos\frac{\omega L}{a}\cos\omega t - AE\frac{\omega}{a}\sin\frac{\omega L}{a}\cos\omega t = 0.$$

Upon canceling the $\cos\omega t$ factor, we find that

$$AE\frac{\omega L}{a}\tan\frac{\omega L}{a} = kL.$$

Thus $\beta = \omega L/a$ is a positive root of the equation $AEx\tan x = kL$, and the nth natural frequency is given by

$$\omega_n = \frac{\beta_n a}{L} = \frac{\beta_n}{L}\sqrt{\frac{E}{\delta}}$$

where β_n is the nth positive root of this equation.

6. Endpoint conditions:

$$m_0 u_{tt}(0, t) - AEu_x(0, t) = 0,$$
$$m_1 u_{tt}(L, t) + AEu_x(L, t) = 0$$

When we substitute $u(x, t) = X(t)\cos\omega t$ in the two endpoint conditions and then cancel the $\cos\omega t$ factor, we get the equations

$$m_0\omega^2 X(0) + KX'(0) = 0$$
$$m_1\omega^2 X(L) - KX'(L) = 0$$

where we write $K = AE$ to avoid confusion with the coefficient of $\cos\omega x/a$ in

$$X(x) = A\cos\frac{\omega x}{a} + B\sin\frac{\omega x}{a}.$$

Then

$$X(0) = A, \qquad X'(0) = \frac{B\omega}{a}$$
$$X(L) = A\cos\frac{\omega L}{a} + B\sin\frac{\omega L}{a}$$
$$X'(L) = \frac{\omega}{a}\left(-A\sin\frac{\omega L}{a} + \cos\frac{\omega L}{a}\right).$$

If we write $z = \omega L/a$, then

$$X(0) = A, \qquad X'(0) = \frac{Bz}{L}$$
$$X(L) = A\cos z + B\sin z$$

$$X'(L) = \frac{z}{L}(-A \sin z + B \cos z).$$

When we substitute these values and $\omega = az/L$ in the two endpoint conditions above and collect coefficients of A and B, we get the equations

$$m_0a^2zA + KLB = 0,$$
$$A(m_1a^2z \cos z + KL \sin z) + B(m_1a^2z \sin z - KL \cos z) = 0.$$

In order for this system to have a non-trivial solution for A and B, its determinant of coefficients must vanish,

$$m_0a^2z(m_1a^2z \sin z - KL \cos z) - KL(m_1a^2z \cos z + KL \sin z) = 0.$$

When we substitute $a^2 = E/\delta$, $M = \delta AL$, and $K = AE$, this last equation simplifies finally to the frequency equation

$$(m_0m_1z^2 - M^2) \sin z = M(m_0 + m_1)\, z \cos z.$$

If β_n is the nth positive root, then the nth natural frequency is

$$\omega_n = \frac{\beta_n a}{L} = \frac{\beta_n}{L}\sqrt{\frac{E}{\delta}}.$$

7. Endpoint conditions:

$$u(0, t) = mu_{tt}(L, t) + AEu_x(L, t) + ku(L, t) = 0$$

The condition $u(0, t) = 0$ implies that

$$X(x) = \sin\frac{\omega x}{a}, \qquad \text{so} \qquad X'(x) = \frac{\omega}{a}\cos\frac{\omega x}{a}.$$

When we substitute $u(x, t) = X(x)\cos \omega t$ in the endpoint condition at $x = L$ and cancel the $\cos \omega t$ factor we get

$$-m\omega^2X(L) + AEX'(L) + kX(L) = 0.$$

Next we substitute

$$z = \omega L/a, \qquad \omega = az/L, \qquad a^2 = E/\delta,$$
$$X(L) = \sin z, \qquad X'(L) = (z/L)\cos z.$$

The result simplifies readily to the frequency equation

$$(mEz^2 - k\delta L^2)\sin z = MEz\cos z.$$

If β_n is the nth positive root, then the nth natural frequency is $\omega_n = \dfrac{\beta_n a}{L} = \dfrac{\beta_n}{L}\sqrt{\dfrac{E}{\delta}}$.

In Problems 8–14 we substitute $y(x, t) = X(x)\cos\omega t$ in

$$y_{tt} + a^4 y_{xxxx} = 0 \qquad (a^4 = EI/\rho)$$

and then cancel the factor $\cos\omega t$ to obtain the ordinary differential equation

$$a^4 X^{(4)} - \omega^2 X = 0$$

with general solution

$$X(x) = A\cosh\frac{\theta x}{a} + B\sinh\frac{\theta x}{a} + C\cos\frac{\theta x}{a} + D\sin\frac{\theta x}{a} \qquad (**)$$

where $\theta = \sqrt{\omega}$. We then get the natural frequencies of vibration by applying the given endpoint conditions.

8. Endpoint conditions: $y(0, t) = y_{xx}(0, t) = 0, \quad y(L, t) = y_{xx}(L, t) = 0$

Just as in Example 3 of Section 9.2, the conditions $X(0) = X''(0) = 0$ imply that $A = C = 0$ in (**), so

$$X(L) = B\sinh\frac{\theta L}{a} + D\sin\frac{\theta L}{a} = 0,$$

$$X''(L) = \frac{\theta^2}{a^2}\left(B\sinh\frac{\theta L}{a} - D\sin\frac{\theta L}{a}\right) = 0.$$

It follows that

$$B\sinh\frac{\theta L}{a} = D\sin\frac{\theta L}{a} = 0.$$

But $\sinh\theta L/a \neq 0$ so $B = 0$. Hence $D \neq 0$ so $\sin\theta L/a = 0$. Thus $\theta L/a = n\pi$, an integral multiple of π. Therefore the nth natural frequency $\omega_n = \theta_n^2$ is given by

$$\omega_n = \frac{n^2\pi^2 a^2}{L^2} = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

9. Endpoint conditions: $y(0, t) = y_x(0, t) = 0, \quad y(L, t) = y_{xx}(L, t) = 0$

Just as in Problem 21 of Section 9.1, the endpoint conditions $X(0) = X'(0) = 0$ and

$X(L) = X''(L) = 0$ imply that

$$\lambda_n = \frac{\omega_n^2}{a^4} = \left(\frac{\beta_n}{L}\right)^4$$

where β_n is the nth positive zero of the frequency equation

$$\tanh x = \tan x.$$

Therefore the nth natural frequency ω_n is given by

$$\omega_n = \left(\frac{\beta_n}{L}\right)^2 a^2 = \frac{\beta_n^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

10. Endpoint conditions: $y(0, t) = y_x(0, t) = 0, \quad y_{xx}(L, t) = y_{xxx}(L, t) = 0$

Here we have the equation

$$X^{(4)} - \lambda X = 0$$

with $\lambda = \omega^2/a^4$ and endpoint conditions

$$X(0) = X'(0) = X''(L) = X^{(3)}(L) = 0.$$

According to Problem 20 in Section 9.1 the nth eigenvalue is

$$\lambda_n = \left(\frac{\omega_n}{a^2}\right)^2 = \left(\frac{\beta_n}{L}\right)^4$$

where the $\{\beta_n\}$ are the positive roots of the equation

$$\cosh x \cos x = -1.$$

Thus the nth natural frequency is

$$\omega_n = a^2\sqrt{\lambda_n} = \left(\frac{\beta_n}{L}\right)^2 a^2 = \left(\frac{\beta_n}{L}\right)^2\sqrt{\frac{EI}{\rho}}.$$

11. Endpoint conditions: $y(0, t) = y_x(0, t) = 0, \quad y_x(L, t) = y_{xxx}(L, t) = 0$

Here we have the equation

$$X^{(4)} - \lambda X = 0$$

with $\lambda = \omega^2/a^4 = \theta^4/a^4 = \alpha^4$ and endpoint conditions

$$X(0) = X'(0) = X'(L) = X^{(3)}(L) = 0.$$

The left-endpoint conditions readily give $C = -A$ and $D = -B$ in (**), so

$$\begin{aligned} X(x) &= A\cosh\alpha x + B\sinh\alpha x - A\cos\alpha x - B\sin\alpha x. \\ &= A(\cosh\alpha x - \cos\alpha x) + B(\sinh\alpha x - \sin\alpha x). \end{aligned}$$

Then the right-endpoint conditions give

$$\begin{aligned} A(\sinh\alpha L + \sin\alpha L) + B(\cosh\alpha L - \cos\alpha L) &= 0, \\ A(\sinh\alpha L - \sin\alpha L) + B(\cosh\alpha L + \cos\alpha L) &= 0. \end{aligned}$$

The determinant of coefficients of A and B must vanish if there is to be a nontrivial solution, so

$$\begin{aligned} &(\sinh\alpha L + \sin\alpha L)(\cosh\alpha L + \cos\alpha L) \\ &\qquad -(\sinh\alpha L - \sin\alpha L)(\cosh\alpha L - \cos\alpha L) = 0. \end{aligned}$$

This equation simplifies to $2\sinh\alpha L\cos\alpha L + 2\cosh\alpha L\sin\alpha L = 0,$ which upon division by $\cosh\alpha L\cos\alpha L$ gives the frequency equation

$$\tanh x + \tan x = 0$$

for $\beta = \alpha L$. Then the nth frequency is given as usual by

$$\omega_n = \alpha_n^2 a^2 = \frac{\beta_n^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

12. This problem is the special case $k = 0$ of Problem 14 below.

13. This problem is the special case $m = 0$ of Problem 14 below.

14. Endpoint conditions:

$$\begin{aligned} y(0, t) &= y_x(0, t) = y_{xxx}(L, t) = 0 \\ my_{tt}(L, t) &= EIy_{xxx}(L, t) - ky(L, t) \end{aligned}$$

With $p = \theta/a$, $\theta = \sqrt{\omega}$ we may write

$$X(x) = A\cosh px + B\sinh px + C\cos px + D\sin px.$$

The conditions $X(0) = X'(0) = 0$ readily imply that $C = -A$ and $D = -B$, so

$$\begin{aligned} X &= A(\cosh px - \cos px) + B(\sinh px - \sin px), \\ X' &= pA(\sinh px + \sin px) + pB(\cosh px - \cos px), \\ X'' &= p^2A(\cosh px + \cos px) + p^2B(\sinh px + \sin px), \\ X^{(3)} &= p^3A(\sinh px - \sin px) + p^3B(\cosh px + \cos px). \end{aligned}$$

The endpoint conditions at $x = L$ are

$$\begin{aligned} &X''(L) = 0, \\ &(k - m\omega^2)X(L) - EIX^{(3)}(L) = 0. \end{aligned}$$

When we substitute the derivatives above and write $z = pL$ we get

$$A(\cosh z + \cos z) + B(\sinh z + \sin z) = 0,$$

$$\begin{aligned} &A[(k - m\omega^2)(\cosh z - \cos z) - EIp^3(\sinh z - \sin z)] \\ &+ B[(k - m\omega^2)(\sinh z - \sin z) - EIp^3(\cosh z + \cos z)] = 0. \end{aligned}$$

If Δ denotes the coefficient determinant of these two linear equations in A and B, then the necessary condition $\Delta = 0$ for a non-trivial solution reduces eventually to the equation

$$EIp^3(1 + \cosh z \cos z) - (k - m\omega^2)(\sinh z \cos z - \cosh z \sin z) = 0.$$

Finally we substitute $p = z/L$, $M = \rho L$, and

$$\omega^2 = p^4 a^4 = (z^4/L^4)(EI/\rho)$$

to get the frequency equation

$$MEIz^3(1 + \cosh z \cos z) = (kML^3 - mEIz^4)(\sinh z \cos z - \cosh z \sin z).$$

We may divide by $\cosh z \cos z$ to write this equation in the form

$$MEIz^3(1 + \operatorname{sech} z \sec z) = (kML^3 - mEIz^4)(\tanh z - \tan z).$$

If β_n denotes the nth positive root of this equation, then as usual the nth natural frequency is

$$\omega_n = \frac{\beta_n^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

15. We want to calculate the fundamental frequency of transverse vibration of a cantilever with the numerical parameters

$$L = 400\text{ cm}$$
$$E = 2\cdot 10^{12}\text{ gm/cm-sec}^2$$
$$I = (1/12)(30\text{ cm})(2\text{ cm})^3 = 20\text{ cm}^4$$
$$\rho = (7.75\text{ gm/cm}^3)(60\text{ cm}^2) = 465\text{ gm/cm}.$$

When we substitute these values and $\beta_1 = 1.8751$ in the frequency formula

$$\omega_1 = \frac{\beta_1^2}{L^2}\sqrt{\frac{EI}{\rho}},$$

we find that $\omega_1 \approx 6.45$ rad/sec, so the fundamental frequency is $\omega_1/2\pi \approx 1.03$ cycles/sec. Thus the diver should bounce up and down on the end of the diving board about once every second.

16. When we substitute $y(x, t) = X(x)\cos\omega t$ in the given partial differential equation

$$\rho\frac{\partial^2 y}{\partial t^2} + P\frac{\partial^2 y}{\partial x^2} + EI\frac{\partial^4 y}{\partial x^4} = 0$$

and cancel the factor $\cos\omega t$, we get the ordinary differential equation

$$EIX^{(4)} + PX'' - \lambda X = 0$$

where $\lambda = \rho\omega^2$. By solving the characteristic equation

$$EI\,r^4 + Pr^2 - \lambda = EI(r^2-\alpha^2)(r^2+\beta^2) = 0$$

we find the general solution

$$X(x) = A\cosh\alpha x + B\sinh\alpha x + C\cos\beta x + D\sin\beta x$$

where

$$\alpha^2 = \frac{-P+\sqrt{P^2+4\lambda EI}}{2EI}, \qquad \beta^2 = -\frac{-P-\sqrt{P^2+4\lambda EI}}{2EI}.$$

The endpoint conditions $X(0) = X''(0) = 0$ imply that $A = C = 0$, so

$$X(x) = B\sinh\alpha x + D\sin\beta x.$$

Then the conditions $X(L) = X''(L) = 0$ yield the equations

$$B \sinh \alpha L + \quad D \sin \beta L = 0,$$
$$\alpha^2 B \sinh \alpha L - \beta^2 D \sin \beta L = 0.$$

The determinant of these two linear equations in B and D must vanish in order that a nontrivial solution exist, so

$$(\alpha^2 + \beta^2)\sinh \alpha L \sin \beta L = 0.$$

It follows that $\sin \beta L = 0$, so βL must be an integral multiple of π. The definitions of α^2 and β^2 imply that

$$\beta^2 - \alpha^2 = \frac{P}{EI}, \qquad \alpha^2\beta^2 = \frac{\lambda}{EI}.$$

Hence if $\beta_n = n\pi/L$, the corresponding value of α_n is

$$\alpha_n = \sqrt{\frac{n^2\pi^2}{L^2} - \frac{P}{EI}}.$$

Then the corresponding value of λ is

$$\lambda_n = EI\alpha_n^2\beta_n^2 = EI\left(\frac{n^4\pi^4}{L^4}\right)\left(1 - \frac{PL^2}{n^2\pi^2 EI}\right).$$

Finally, the nth natural frequency is given by

$$\omega_n = \sqrt{\frac{\lambda_n}{\rho}} = \frac{n^2\pi^2}{L^2}\left(1 - \frac{PL^2}{n^2\pi^2 EI}\right)^{1/2}\sqrt{\frac{EI}{\rho}}.$$

17. When we substitute $y(x, t) = X(x)\cos \omega t$ in the given partial differential equation

$$\rho\frac{\partial^2 y}{\partial t^2} - \frac{I}{A}\frac{\partial^4 y}{\partial x^2 \partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = 0$$

and cancel the factor $\cos \omega t$, we get the ordinary differential equation

$$EIX^{(4)} + PX'' - \lambda X = 0$$

where $P = \lambda I / \rho A$ and $\lambda = \rho\omega^2$. By solving the characteristic equation

$$EI\, r^4 + Pr^2 - \lambda = EI(r^2 - \alpha^2)(r^2 + \beta^2) = 0$$

we find the general solution

$$X(x) = A \cosh \alpha x + B \sinh \alpha x + C \cos \beta x + D \sin \beta x$$

where

$$\alpha^2 = \frac{-P+\sqrt{P^2+4\lambda EI}}{2EI}, \qquad \beta^2 = -\frac{-P-\sqrt{P^2+4\lambda EI}}{2EI}.$$

The endpoint conditions $X(0) = X''(0) = 0$ imply that $A = C = 0$, so

$$X(x) = B \sinh \alpha x + D \sin \beta x.$$

Then the conditions $X(L) = X''(L) = 0$ yield the equations

$$\begin{aligned} B \sinh \alpha L + \quad D \sin \beta L &= 0, \\ \alpha^2 B \sinh \alpha L - \beta^2 D \sin \beta L &= 0. \end{aligned}$$

The determinant of these two linear equations in B and D must vanish in order that a nontrivial solution exist, so

$$(\alpha^2 + \beta^2)\sinh \alpha L \sin \beta L = 0.$$

It follows that $\sin \beta L = 0$, so βL must be an integral multiple of π. The definitions of α^2 and β^2 imply that

$$\beta^2 - \alpha^2 = \frac{P}{EI} = \frac{\lambda}{\rho AE}, \qquad \alpha^2\beta^2 = \frac{\lambda}{EI}.$$

Hence if $\beta_n = n\pi/L$, the corresponding value of α_n is given by

$$\alpha_n^2 = \frac{n^2\pi^2}{L^2} - \frac{\lambda_n}{\rho AE}.$$

Then $\alpha_n^2\beta_n^2 = \lambda_n / EI$ gives the equation

$$\left(\frac{n^2\pi^2}{L^2} - \frac{\lambda_n}{\rho AE}\right)\frac{n^2\pi^2}{L^2} = \frac{\lambda_n}{EI}$$

that we readily solve for λ_n. The resulting value of the nth natural frequency is

$$\omega_n = \sqrt{\frac{\lambda_n}{\rho}} = \frac{n^2\pi^2}{L^2}\left(1+\frac{n^2\pi^2 I}{\rho AL^2}\right)^{-1/2}\sqrt{\frac{EI}{\rho}}.$$

18. Substitution of $u(x,t) = X(x)\sin \omega t$ in the longitudinal bar problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad \left(a^2 = \frac{E}{\rho}\right)$$

$$u(0,t) = 0, \quad AEu_x(L,t) = F_0 \sin \omega t$$

yields the endpoint problem

$$X'' + \frac{\omega^2}{a^2} X = 0, \quad X(0) = 0, \quad AE\, X'(L) = F_0.$$

Because of the right-endpoint condition, we try $X(x) = B \sin \omega x / a$ and get

$$AE \cdot B \cdot \frac{\omega}{a} \cos \frac{\omega L}{a} = F_0, \quad \text{so} \quad B = \frac{F_0 a}{AE\omega \cos(\omega L / A)}.$$

The resulting steady periodic solution is

$$u(x,t) = \frac{F_0 a \sin(\omega x / a) \sin \omega t}{AE\omega \cos(\omega L / A)}.$$

19. Substitution of $y(x,t) = X(x) \sin \omega t$ in the transverse bar problem

$$\frac{\partial^2 y}{\partial t^2} + a^4 \frac{\partial^4 y}{\partial x^4} = 0 \qquad \left(a^4 = \frac{EI}{\rho}\right)$$

$$y(0,t) = y_x(0,t) = 0,$$

$$y_{xx}(L,t) = EI\, y_{xxx}(L,t) + F_0 \sin \omega t = 0$$

yields the endpoint problem

$$X^{(4)} - p^4 X = 0 \qquad (\text{where } p^2 = \omega / a^2),$$

$$X(0) = X'(0) = 0,$$

$$X''(L) = EI\, X'''(L) + F_0 = 0.$$

When we impose the fixed-end conditions $X(0) = X'(0) = 0$ on the general solution

$$X(x) = A \cosh px + B \sinh px + C \cos px + D \sin px$$

we find readily that $C = -A$ and $D = -B$, so

$$X(x) = A(\cosh px - \cos px) + B(\sinh px - \sin px).$$

It remains only to find A and B. But the free-end conditions yield the linear equations

$$A(\cosh pL + \cos pL) + B(\sinh pL + \sin pL) = 0$$
$$A(\sinh pL - \sin pL) + B(\cosh pL + \cos pL) = -F_0/p^3EI$$

that can be solved for

$$A = K(\sinh pL + \sin pL), \qquad B = -K(\cosh pL + \cos pL)$$

where

$$K = \frac{F_0}{2EIp^3(1+\cosh pL \cos pL)}.$$

22. When we substitute

$$e(x,t) = E(x)e^{i\omega t}$$

in the partial differential equation

$$e_{xx} = LCe_{tt} + (LG + RC)e_t + RGe$$

and cancel the factor $e^{i\omega t}$, the result is the ordinary differential equation

$$E''(x) - \gamma E(x) = 0$$

where

$$\gamma = (RG - LC\omega^2) + i\,\omega(LG + RC).$$

If $(\alpha + \beta i)^2 = \gamma$, then the general solution is

$$E(x) = Ae^{-\alpha x}e^{-i\beta x} + Be^{\alpha x}e^{i\beta x}.$$

In order that $e(x,t)$ be bounded as $x \to \infty$ we choose $B = 0$, and in order that $e(0,t) = E_0\cos\omega t$ we choose $A = E_0$. Then our steady periodic solution is the real part

$$\mathrm{Re}[E(x)e^{i\omega t}] = \mathrm{Re}[E_0e^{-\alpha x}e^{-i\beta x}e^{i\omega t}] = E_0e^{-\alpha x}\cos(\omega t - \beta x).$$

SECTION 9.4

CYLINDRICAL COORDINATE PROBLEMS

1. Substitution of $u(r,t) = R(r)T(t)$ in the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

yields the separation

$$\frac{T''}{a^2T} = \frac{R''+\frac{1}{r}R'}{R} = \lambda = -\alpha^2.$$

The t-equation has general solution

$$T(t) = A\cos\alpha at + B\sin\alpha at,$$

and we choose $B = 0$, so that $T'(0) = 0$ (because the membrane is initially at rest). The r-equation can be written in the form

$$r^2R'' + rR' + \alpha^2r^2R = 0,$$

which is the parametric Bessel equation of order zero, with continuous solution $R(r) = J_0(\alpha r)$. In order that the fixed boundary condition $R(c) = 0$ be satisfied, we choose $\alpha = \gamma_n / c$, where γ_n is the nth positive solution of $J_0(x) = 0$. At this point we have product functions of the form $J_0(\gamma_n r / c)\cos(\gamma_n at / c)$ that satisfy the wave equation and the homogeneous boundary conditions, so we form the formal series solution

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\cos\frac{\gamma_n at}{c}.$$

In order to satisfy the initial position condition $u(r,0) = f(x)$ is suffices that the $\{c_n\}$ be the Fourier-Bessel coefficients of the function $f(x)$ given by

$$c_n = \frac{2}{c^2\left[J_1(\gamma_n)\right]^2}\int_0^c rf(r)J_0\left(\frac{\gamma_n r}{c}\right)dr.$$

2. This is the same as Problem 1, except that the membrane has initial position $u(r,0) = 0$, so in the t-factor $T(t) = A\cos\alpha at + B\sin\alpha at$ we choose $A = 0$ so that $T(0) = 0$. We then get product functions of the form $J_0(\gamma_n r / c)\sin(\gamma_n at / c)$ that satisfy the wave equation and the homogeneous boundary conditions, so we form the formal series solution

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\sin\frac{\gamma_n at}{c}.$$

In order that the initial velocity condition $u_t(r,0) \equiv v_0$ we satisfied, we want

$$v_0 = \sum_{n=1}^{\infty}\frac{\gamma_n a}{c}\cdot c_n J_0\left(\frac{\gamma_n r}{c}\right),$$

and hence

$$\begin{aligned} c_n &= \frac{c}{\gamma_n a}\cdot\frac{2}{c^2 J_1(\gamma_n)^2}\int_0^c r\, v_0 J_0\left(\frac{\gamma_n r}{c}\right)dr \\ &= \frac{2cv_0}{a\gamma_n^3 J_1(\gamma_n)^2}\int_0^{\gamma_n} x J_0(x)\,dx \qquad (\text{with } x = \gamma_n r/c) \\ &= \frac{2cv_0}{a\gamma_n^3 J_1(\gamma_n)^2}\Big[x J_1(x)\Big]_0^{\gamma_n} = \frac{2cv_0}{a\gamma_n^2 J_1(\gamma_n)}. \end{aligned}$$

This gives the desired solution

$$u(r,t) = \frac{2cv_0}{a}\sum_{n=1}^{\infty}\frac{J_0(\gamma_n r/c)\sin(\gamma_n a t/c)}{\gamma_n^2 J_1(\gamma_n)}.$$

3. **(a)** As in Problem 2,

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\sin\frac{\gamma_n a t}{c}.$$

In order to satisfy the given initial condition we must choose

$$\begin{aligned} c_n &= \frac{c}{\gamma_n a}\cdot\frac{2}{c^2 J_1(\gamma_n)^2}\int_0^{\varepsilon}\left(\frac{P_0}{\rho\pi\varepsilon^2}\right) r J_0\left(\frac{\gamma_n r}{c}\right)dr \\ &= \frac{2P_0 c}{\rho\pi\varepsilon^2\gamma_n^3 a J_1(\gamma_n)^2}\int_0^{\gamma_n\varepsilon/c} x J_0(x)\,dx \qquad (\text{with } x = \gamma_n r/c) \\ &= \frac{2P_0 c}{\rho\pi\varepsilon^2\gamma_n^3 a J_1(\gamma_n)^2}\cdot\frac{\gamma_n\varepsilon}{c} J_1\left(\frac{\gamma_n\varepsilon}{c}\right). \\ c_n &= \frac{2aP_0}{\pi c\rho a^2\gamma_n J_1(\gamma_n)^2}\cdot\frac{J_1(\gamma_n\varepsilon/c)}{\gamma_n\varepsilon/c}. \end{aligned}$$

(b) The final formula given in the text for $u(r, t)$ now follows because $\rho a^2 = T$ and $J_1(x)/x \to 1/2$ as $x \to 0$.

4. **(a)** Just as in Example 1 of the text we derive first a formal series solution of the form

$$u(r,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\gamma_n^2 kt}{c^2}\right)J_0\left(\frac{\gamma_n r}{c}\right).$$

In order to satisfy the given initial condition we calculate the Fourier-Bessel coefficient

$$c_n = \frac{2}{c^2 J_1(\gamma_n)^2}\int_0^{\varepsilon}\left(\frac{q_0}{s\pi\varepsilon^2}\right) r J_0\left(\frac{\gamma_n r}{c}\right)dr$$

$$= \frac{2q_0}{s\pi\varepsilon^2\gamma_n^2 J_1(\gamma_n)^2} \int_0^{\gamma_n\varepsilon/c} x J_0(x)\, dx \qquad (\text{with } x = \gamma_n r/c)$$

$$= \frac{2q_0}{s\pi\varepsilon^2\gamma_n^2 J_1(\gamma_n)^2} \cdot \frac{\gamma_n\varepsilon}{c} J_1\left(\frac{\gamma_n\varepsilon}{c}\right).$$

$$c_n = \frac{2q_0}{s\pi c^2 J_1(\gamma_n)^2} \cdot \frac{J_1(\gamma_n\varepsilon/c)}{\gamma_n\varepsilon/c}.$$

(b) The final formula given in the text for $u(r, t)$ now follows because $J_1(x)/x \to 1/2$ as $x \to 0$.

5. **(a)** We start with the steady-state boundary value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (r < c, \quad 0 < z < L)$$

$$u(c, z) = 0$$

$$u(r, 0) = 0,$$

$$u(r, L) = u_0.$$

The substitution $u(r, z) = R(r)Z(z)$ yields the equations

$$rR'' + R' + \alpha^2 rR = 0, \qquad Z'' - \alpha^2 Z = 0$$

with separation constant $\lambda = \alpha^2$. The homogeneous endpoint conditions are

$$R(c) = Z(0) = 0.$$

If $\lambda = \alpha^2 = 0$ then $rR'' + R = 0$ implies

$$R(r) = A + B \ln r.$$

We choose $B = 0$ for continuity at $r = 0$, so $R(r) = A$. Then $R(c) = 0$, so $A = 0$ also, and hence 0 is not an eigenvalue.

If $\lambda = \alpha^2 > 0$ then we have the parametric Bessel equation with general solution

$$R(r) = AJ_0(\alpha r) + BY(\alpha r).$$

In order that $R(r)$ be continuous at $r = 0$ we choose $B = 0$, so $R(r) = AJ_0(\alpha r)$. Then

$$R(c) = \alpha AJ_0(\alpha c) = 0$$

requires that $\gamma = \alpha c$ be a root of the equation

$$J_0(x) = 0.$$

If $\alpha_n = \gamma_n/c$ where γ_n is the nth positive root of this equation, then

$$R_n(r) = J_0\left(\frac{\gamma_n r}{c}\right).$$

The corresponding function $Z(z)$ of z is

$$Z_n(z) = A_n \cosh\frac{\gamma_n z}{c} + B_n \sinh\frac{\gamma_n z}{c},$$

and we choose $A_n = 0$ because $Z(0) = 0$. Thus we get the formal series solution

$$u(r,z) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\sinh\frac{\gamma_n z}{c}$$

where $J_0(\gamma_n) = 0$. To satisfy the condition $u(r, L) = u_0$, we need (by Eq. (22) in the text)

$$\begin{aligned}
c_n &= \frac{1}{\sinh(\gamma_n L/c)}\cdot\frac{2}{c^2 J_1(\gamma_n)^2}\int_0^c r\,u_0 J_0\left(\frac{\gamma_n r}{c}\right)dr \\
&= \frac{2u_0}{\gamma_n^2 J_1(\gamma_n)^2 \sinh(\gamma_n L/c)}\int_0^{\gamma_n} x\,J_0(x)\,dx \qquad (\text{with } x = \gamma_n r/c) \\
&= \frac{2u_0}{\gamma_n^2 J_1(\gamma_n)^2 \sinh(\gamma_n L/c)}\Big[x\,J_1(x)\Big]_0^{\gamma_n} = \frac{2u_0}{\gamma_n J_1(\gamma_n)}.
\end{aligned}$$

This gives the desired solution

$$u(r,t) = 2u_0 \sum_{n=1}^{\infty} \frac{J_0(\gamma_n r/c)\sinh(\gamma_n z/c)}{\gamma_n J_1(\gamma_n)\sinh(\gamma_n L/c)}.$$

6. **(a)** Here we have the same separation of variables

$$rR'' + R' + \alpha^2 rR = 0, \qquad Z'' - \alpha^2 Z = 0$$

as in Problem 5, but the insulation condition $u_r(c,z) = 0$ implies

If $\lambda = \alpha^2 = 0$ then $rR'' + R = 0$ implies

$$R(r) = A + B\ln r.$$

We choose $B = 0$ for continuity at $r = 0$, so $R(r) = A$. Then $R'(c) = 0$, so $\lambda_0 = 0$ is an eigenvalue, and we may take $R_0(r) = 1$. The equation $Z''(z) = 0$ implies $Z(z) = Az + B$, but $Z(0) = 0$ implies $B = 0$, so we take $Z_0(z) = z$.

If $\lambda = \alpha^2 > 0$ then we have the parametric Bessel equation with continuous solution $R(r) = AJ_0(\alpha r)$. Then

$$R'(c) = \alpha A J_0'(\alpha c) = 0$$

requires that $\gamma = \alpha c$ be a root of the equation

$$J_0'(x) = 0.$$

If $\alpha_n = \gamma_n / c$ where γ_n is the nth positive root of this equation, then

$$R_n(r) = J_0\left(\frac{\gamma_n r}{c}\right).$$

The corresponding function $Z(z)$ of z is

$$Z_n(z) = A_n \cosh\frac{\gamma_n z}{c} + B_n \sinh\frac{\gamma_n z}{c},$$

and we choose $A_n = 0$ because $Z(0) = 0$. Thus we get the solution

$$u(r,z) = c_0 z + \sum_{n=1}^{\infty} c_n J_0\left(\frac{\gamma_n r}{c}\right)\sinh\frac{\gamma_n z}{c}$$

where $J_0'(\gamma_n) = 0$. To satisfy the condition $u(r, L) = f(r)$ we apply the formulas in (24) in the text and choose

$$c_0 = \frac{2}{Lc^2}\int_0^c r\, f(r)\, dr,$$

$$c_n = \frac{2}{c^2 \sinh(\gamma_n L / c) J_0(\gamma_n)^2}\int_0^c r\, f(r) J_0\left(\frac{\gamma_n r}{c}\right) dr.$$

(b) If $f(r) = u_0$ (constant), then the coefficient formulas above readily yield $c_0 = u_0 / L$ and $c_n = 0$ for $n > 0$, the latter because

$$\int x J_0(x)\, dx = x J_1(x) + C = -x J_0'(x) + C.$$

Hence the series reduces to the solution $u(r, z) = u_0 z / L$ that one might well guess without all these computations.

7. We want to solve the boundary value problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2} = 0 \qquad (r < 1,\ z > 0)$$

$$hu(1,z)+u_r(1,z) = 0$$

$$u(r,z) \text{ bounded as } z \to \infty$$

$$u(r,0) = u_0.$$

We start with the separation of variables in Problem 5,

$$rR''+R'+\alpha^2 rR = 0, \qquad Z''-\alpha^2 Z = 0$$

and readily see that $\alpha = 0$ is not an eigenvalue. When we impose the condition

$$hR(1)+R'(1) = 0$$

on $R(r) = J_0(\alpha r)$, we find that α must satisfy the equation

$$hJ_0(x)+xJ_0'(x) = 0$$

that corresponds to Case 2 with $n = 0$ in Figure 9.4.2 of the text. If $\{\gamma_n\}$ are the positive roots of this equation then

$$R_n(r) = J_0(\gamma_n r).$$

The general solution of $Z'' = \gamma_n^2 Z$ is

$$Z_n(z) = A_n\exp(-\gamma_n z)+B_n\exp(\gamma_n z),$$

and we choose $B_n = 0$ so that $Z_n(z)$ will be bounded as $z \to \infty$. Thus we obtain a solution of the form

$$u(r,z) = \sum_{n=1}^{\infty} c_n \exp(-\gamma_n z)J_0(\gamma_n r)$$

where

$$hJ_0(\gamma_n)+\gamma_n J_0'(\gamma_n) = 0,$$

so $\gamma_n J_1(\gamma_n) = hJ_0(\gamma_n)$ because $J_0' = -J_1$. Finally, Eq. (23) in the text gives

$$c_n = \frac{2\gamma_n^2}{c^2\left(\gamma_n^2+h^2\right)J_0(\gamma_n)^2}\int_0^c r\,u_0 J_0\left(\frac{\gamma_n r}{c}\right)dr$$

$$= \frac{2u_0}{(\gamma_n^2+h^2)J_0(\gamma_n)^2}\int_0^{\gamma_n} x\,J_0(x)\,dx \qquad (\text{with } x = \gamma_n r/c)$$

$$= \frac{2u_0}{(\gamma_n^2+h^2)J_0(\gamma_n)^2}\left[x\,J_1(x)\right]_0^{\gamma_n}$$

$$= \frac{2u_0\gamma_n J_1(\gamma_n)}{(\gamma_n^2+h^2)J_0(\gamma_n)^2} = \frac{2hu_0}{(\gamma_n^2+h^2)J_0(\gamma_n)},$$

so

$$u(r,z) = 2hu_0\sum_{n=1}^{\infty}\frac{\exp(-\gamma_n z)\,J_0(\gamma_n r)}{(\gamma_n^2+h^2)J_0(\gamma_n)}.$$

11. When we substitute $u(r, t) = R(r)\sin\omega t$ in the given partial differential equation and cancel the factor $\sin\omega t$, we get the ordinary differential equation

$$R'' + \frac{1}{r}R' + \left(\frac{\omega}{a}\right)^2 R = -\frac{F_0}{a^2}$$

The associated homogeneous equation is the Bessel equation of order zero with parameter ω/a. Hence it follows readily that the solution that is continuous at $r = 0$ is

$$R(r) = A\,J_0\!\left(\frac{\omega r}{a}\right) - \frac{F_0}{\omega^2}.$$

The condition $R(b) = 0$ yields $A = F_0/\omega^2 J_0(\omega b/a)$, so it follows that the desired steady periodic solution is

$$u(r,t) = \frac{F_0}{\omega^2 J_0(\omega b/a)}\left[J_0\!\left(\frac{\omega r}{a}\right) - J_0\!\left(\frac{\omega b}{a}\right)\right]\sin\omega t.$$

12. When we substitute $y(x, t) = X(x)\sin\omega t$ in the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{g}{w}\frac{\partial}{\partial x}\left(wx\frac{\partial y}{\partial x}\right) = g\left(\frac{\partial y}{\partial x} + x\frac{\partial^2 y}{\partial x^2}\right)$$

and then cancel the $\sin\omega t$ factor, we get the ordinary differential equation

$$x^2X'' + xX' + \frac{\omega^2 x}{g}X = 0.$$

This is of the form of Equation (3) in Section 3.6 with $A = 1$, $B = 0$, $C = \omega^2/g$, and $q = 1$, so its general solution is given by

$$X(x) = A J_0\left(2\omega\sqrt{\frac{x}{g}}\right) + B Y_0\left(2\omega\sqrt{\frac{x}{g}}\right).$$

We choose $B = 0$ for continuity at $x = 0$, and the condition $X(L) = 0$ then requires that $2\omega\sqrt{L/g} = \gamma_n$, one of the roots $\{\gamma_n\}$ of the equation $J_0(x) = 0$. Hence the nth natural frequency of vibration of the hanging cable is

$$\omega_n = \frac{\gamma_n}{2}\sqrt{\frac{x}{g}}.$$

13. With $w(x) = wx$ and $h(x) = h$ (where w and h on the right are constants) the given partial differential equation

$$\frac{w(x)}{g}\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(w(x)h(x)\frac{\partial y}{\partial x}\right) \qquad (*)$$

reduces to

$$x\frac{\partial^2 y}{\partial t^2} = gh\left(\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\right).$$

When we substitute $y(x, t) = X(x)\cos \omega t$ we get the parametric Bessel equation

$$x^2 X'' + xX' + \frac{\omega^2 x^2}{gh} X = 0$$

with bounded solution

$$X(x) = A J_0\left(\frac{\omega x}{\sqrt{gh}}\right).$$

The condition $X(L) = y_0$ implies that $A = y_0 / J_0\left(\omega L/\sqrt{gh}\right)$, so

$$y(x,t) = y_0 \frac{J_0\left(\omega x/\sqrt{gh}\right)}{J_0\left(\omega L/\sqrt{gh}\right)}\cos \omega t.$$

14. With $w(x) = w$ and $h(x) = hx$ (with w and h being constants on the right) the partial differential equation in (*) above reduces to

$$\frac{\partial^2 y}{\partial t^2} = gh\left(\frac{\partial y}{\partial x} + x\frac{\partial^2 y}{\partial x^2}\right).$$

When we substitute $y(x, t) = X(x)\cos \omega t$ we get the ordinary differential equation

$$x^2X'' + xX' + \frac{\omega^2 x}{gh}X = 0.$$

This has the form of Equation (3) in Section 3.6 with $A = 1$, $B = 0$, $C = \omega^2/gh$, and $q = 1$, so its (bounded) solution is given by

$$X(x) = AJ_0\left(2\omega\sqrt{\frac{x}{gh}}\right).$$

The condition $X(L) = y_0$ now implies that $A = y_0/J_0\left(2\omega\sqrt{L/gh}\right)$, so

$$y(x,t) = y_0\frac{J_0\left(2\omega\sqrt{x/gh}\right)}{J_0\left(2\omega\sqrt{L/gh}\right)}\cos\omega t.$$

15. With $w(x) = wx$ and $h(x) = hx$ (with w and h being constants on the right) the partial differential equation in (*) above reduces to

$$\frac{\partial^2 y}{\partial t^2} = gh\left(2\frac{\partial y}{\partial x} + x\frac{\partial^2 y}{\partial x^2}\right).$$

When we substitute $y(x, t) = X(x)\cos\omega t$ we get the ordinary differential equation

$$x^2X'' + 2xX' + \frac{\omega^2 x}{gh}X = 0.$$

This has the form of Equation (3) in Section 3.6 with $A = 2$, $B = 0$, $C = \omega^2/gh$, and $q = 1$, so its (bounded) solution is given by

$$X(x) = \frac{A}{\sqrt{x}}J_1\left(2\omega\sqrt{\frac{x}{gh}}\right).$$

The condition $X(L) = y_0$ now implies that $A = y_0\sqrt{L}/J_1\left(2\omega\sqrt{L/gh}\right)$, so

$$y(x,t) = y_0\sqrt{\frac{L}{x}}\frac{J_1\left(2\omega\sqrt{x/gh}\right)}{J_1\left(2\omega\sqrt{L/gh}\right)}\cos\omega t.$$

16. With $\lambda = \alpha^2$ the general solution of the parametric Bessel equation of order 0 is

$$y(x) = AJ_0(\alpha x) + BY_0(\alpha x).$$

The endpoint conditions $y(a) = y(b) = 0$ yield the linear equations

$$A\ J_0(\alpha a) + B\ Y_0(\alpha a) = 0,$$
$$A\ J_0(\alpha b) + B\ Y_0(\alpha b) = 0$$

in A and B. In order for there to exist a non-trivial solution for A and B the coefficient determinant must vanish. Hence α must be one of the solutions $\{\gamma_n\}$ of the equation

$$J_0(ax)Y_0(bx) - J_0(bx)Y_0(ax) = 0. \qquad (\#)$$

With $\alpha = \gamma_n$, $A = Y_0(\gamma_n a)$ and $B = -J_0(\gamma_n a)$, both conditions above are satisfied and we have the eigenfunction

$$R_n(x) = Y_0(\gamma_n a)J_0(\gamma_n x) - J_0(\gamma_n a)Y_0(\gamma_n x).$$

17. Just as in Problem 1 above, substitution of $u(r,t) = R(r)T(t)$ in the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

yields the separation

$$\frac{T''}{a^2 T} = \frac{R'' + \frac{1}{r}R'}{R} = \lambda = -\alpha^2.$$

The t-equation has general solution

$$T(t) = A\cos\alpha at + B\sin\alpha at,$$

and we choose $B = 0$, so that $T'(0) = 0$ (assuming, for instance, that the membrane is initially at rest). The r-equation can be written in the form

$$r^2 R'' + rR' + \alpha^2 r^2 R = 0,$$

which is the parametric Bessel equation of order zero. By Problem 16, its solutions satisfying $R(a) = R(b) = 0$ are of the form $R_n(x) = Y_0(\gamma_n a)J_0(\gamma_n x) - J_0(\gamma_n a)Y_0(\gamma_n x)$ with $\alpha = \gamma_n$ being one of the positive roots of Equation (#) there. This leads to a formal series solution of the form

$$u(r,t) = \sum_{n=1}^{\infty} R_n(x)\left(A_n \cos\gamma_n at + B_n \sin\gamma_n at\right),$$

where the frequency of the nth term is $\omega_n = \gamma_n a = \gamma_n\sqrt{T/\rho}$.

18. We start with the substitution $u(r, t) = R(r)T(t)$ in the heat equation. The result is given in Equations (25) and (26):

$$r^2R'' + rR' + \alpha^2r^2R = 0, \quad T' = -\alpha^2kT.$$

The first of these equations, together with the endpoint conditions

$$R(a) = R(b) = 0,$$

comprise the regular Sturm-Liouville problem of Problem 16. Hence its eigenvalues are given by $\alpha_n = \gamma_n$ where $\{\gamma_n\}$ are the positive roots of the equation in Eq.(41) in the text. The nth eigenfunction is the function $R_n(r)$ defined in (42). Finally the solution of $T_n' = -\gamma_n^2kT_n$ is

$$T_n(t) = \exp(-\gamma_n^2kt),$$

so we get a solution of the form

$$u(r,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\gamma_n^2kt\right) R_n(r).$$

19. We want to solve the boundary value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (a < r < 1,\ z > 0)$$

$$u(a,z) = u(b,z) = 0$$

$$u(r,z) \text{ bounded as } z \to \infty$$

$$u(r,0) = f(r).$$

Just as in Problem 5, the substitution $u(r, z) = R(r)Z(z)$ yields the equations

$$rR'' + R' + \alpha^2rR = 0, \qquad Z'' - \alpha^2Z = 0$$

with separation constant $\lambda = \alpha^2$. When we impose the conditions $R(a) = R(b) = 0$ on the r-equation here, we have the Sturm-Liouville problem of Problem 16, so $R(r)$ must be one of the eigenfunctions $\{R_n(r)\}$ corresponding to the positive roots $\{\gamma_n\}$ of Equation (#) there. The general solution of $Z'' = \gamma_n^2Z$ is

$$Z_n(z) = A_n\exp(-\gamma_nz) + B_n\exp(\gamma_nz),$$

and we choose $B_n = 0$ so that $Z_n(z)$ will be bounded as $z \to \infty$. Thus we obtain a solution of the form

$$u(r,z) = \sum_{n=1}^{\infty} c_n \exp(-\gamma_n z) R_n(r)$$

with the coefficients $\{c_n\}$ calculated as in Problem 18.

SECTION 9.5

HIGHER-DIMENSIONAL PHENOMENA

This section provides the interested student with an opportunity to study several applications at greater depth than is afforded by the usual textbook exercises. The problem sets outlined in Section 9.5 can serve as the basis for several fairly substantial computational projects. Because these problem sets and projects are rather heavily annotated in the text, further outlines of solutions are not included in this manual. However, additional discussion — particularly regarding computer implementations — may be found in the applications manual that accompanies the text.

APPENDIX

EXISTENCE AND UNIQUENESS OF SOLUTIONS

In Problems 1–12 we apply the iterative formula

$$y_{n+1} = b + \int_a^x f(t, y_n(t))\, dt$$

to compute successive approximations $\{y_n(x)\}$ to the solution of the initial value problem

$$y' = f(x, y), \qquad\qquad y(a) = b.$$

starting with $y_0(x) = b$.

1.
$$\begin{aligned}
y_0(x) &= 3 \\
y_1(x) &= 3 + 3x \\
y_2(x) &= 3 + 3x + 3x^2/2 \\
y_3(x) &= 3 + 3x + 3x^2/2 + x^3/2 \\
y_4(x) &= 3 + 3x + 3x^2/2 + x^3/2 + x^4/8 \\
y(x) &= 3 - 3x + 3x^2/2 + x^3/2 + x^4/8 + \cdots = 3e^x
\end{aligned}$$

2.
$$\begin{aligned}
y_0(x) &= 4 \\
y_1(x) &= 4 - 8x \\
y_2(x) &= 4 - 8x + 8x^2 \\
y_3(x) &= 4 - 8x + 8x^2 - (16/3)x^3 \\
y_4(x) &= 4 - 8x + 8x^2 - (16/3)x^3 + (8/3)x^4 \\
y(x) &= 4 - 8x + 8x^2 - (16/3)x^3 + (8/3)x^4 - \cdots = 4e^{-2x}
\end{aligned}$$

3.
$$\begin{aligned}
y_0(x) &= 1 \\
y_1(x) &= 1 - x^2 \\
y_2(x) &= 1 - x^2 + x^4/2 \\
y_3(x) &= 1 - x^2 + x^4/2 - x^6/6
\end{aligned}$$

$$y_4(x) = 1 - x^2 + x^4/2 - x^6/6 + x^8/24$$
$$y(x) = 1 - x^2 + x^4/2 - x^6/6 + x^8/24 - \cdots = \exp(-x^2)$$

4.
$$y_0(x) = 2$$
$$y_1(x) = 2 + 2x^3$$
$$y_2(x) = 2 + 2x^3 + x^6$$
$$y_3(x) = 2 + 2x^3 + x^6 + (1/3)x^9$$
$$y_4(x) = 2 + 2x^3 + x^6 + (1/3)x^9 + (1/12)x^{12}$$
$$y(x) = 2 + 2x^3 + x^6 + (1/3)x^9 + (1/12)x^{12} + \cdots = 2\exp(x^3)$$

5.
$$y_0(x) = 0$$
$$y_1(x) = 2x$$
$$y_2(x) = 2x + 2x^2$$
$$y_3(x) = 2x + 2x^2 + 4x^3/3$$
$$y_4(x) = 2x + 2x^2 + 4x^3/3 + 2x^4/3$$
$$y(x) = 2x + 2x^2 + 4x^3/3 + 2x^4/3 + \cdots = e^{2x} - 1$$

6.
$$y_0(x) = 0$$
$$y_1(x) = (1/2)x^2$$
$$y_2(x) = (1/2)x^2 + (1/6)x^3$$
$$y_3(x) = (1/2)x^2 + (1/6)x^3 + (1/24)x^4$$
$$y_4(x) = (1/2)x^2 + (1/6)x^3 + (1/24)x^4 + (1/120)x^5$$
$$y(x) = (1/2!)x^2 + (1/3!)x^3 + (1/4!)x^4 + (1/5!)x^5 + \cdots = e^x - x - 1$$

7.
$$y_0(x) = 0$$
$$y_1(x) = x^2$$
$$y_2(x) = x^2 + x^4/2$$
$$y_3(x) = x^2 + x^4/2 + x^6/6$$
$$y_4(x) = x^2 + x^4/2 + x^6/6 + x^8/24$$
$$y(x) = x^2 + x^4/2 + x^6/6 + x^8/24 + \cdots = \exp(x^2) - 1$$

8.
$$\begin{aligned}
y_0(x) &= 0\\
y_1(x) &= 2x^4\\
y_2(x) &= 2x^4 + (4/3)x^6\\
y_3(x) &= 2x^4 + (4/3)x^6 + (2/3)x^8\\
y_4(x) &= 2x^4 + (4/3)x^6 + (2/3)x^8 + (4/15)x^{10}\\
y(x) &= 2x^4 + (4/3)x^6 + (2/3)x^8 + (4/15)x^{10} + \cdots = \exp(2x^2) - 2x^2 - 1
\end{aligned}$$

9.
$$\begin{aligned}
y_0(x) &= 1\\
y_1(x) &= (1 + x) + x^2/2\\
y_2(x) &= (1 + x + x^2) + x^3/6\\
y_3(x) &= (1 + x + x^2 + x^3/3) + x^4/24\\
y(x) &= 1 + x + x^2 + x^3/3 + x^4/12 + \cdots = 2e^x - 1 - x
\end{aligned}$$

10.
$$\begin{aligned}
y_0(x) &= 0\\
y_1(x) &= x + (1/2)x^2 + (1/6)x^3 + (1/24)x^4 + \cdots = e^x - 1\\
y_2(x) &= x + \quad x^2 + (1/3)x^3 + (1/12)x^4 + \cdots = 2e^x - x - 2\\
y_3(x) &= x + \quad x^2 + (1/2)x^3 + (1/8)x^4 + \cdots = 3e^x - (1/2)x^2 - 2x - 3\\
y(x) &= x + \quad x^2 + (1/2)x^3 + (1/6)x^4 + \cdots = xe^x
\end{aligned}$$

11.
$$\begin{aligned}
y_0(x) &= 1\\
y_1(x) &= 1 + x\\
y_2(x) &= (1 + x + x^2) + x^3/3\\
y_3(x) &= (1 + x + x^2 + x^3) + 2x^4/3 + x^5/3 + x^6/9 + x^7/63\\
y(x) &= 1 + x + x^2 + x^3 + x^4 + \cdots = 1/(1 - x)
\end{aligned}$$

12.
$$\begin{aligned}
y_0(x) &= 1\\
y_1(x) &= 1 + (1/2)x\\
y_2(x) &= 1 + (1/2)x + (3/8)x^2 + (1/8)x^3 + (1/64)x^4\\
y_3(x) &= 1 + (1/2)x + (3/8)x^2 + (5/16)x^3 + (13/64)x^4 + \cdots\\
y(x) &= 1 + (1/2)x + (3/8)x^2 + (5/16)x^3 + (35/128)x^4 + \cdots = (1 - x)^{-1/2}
\end{aligned}$$

13. $$\begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 1+3t \\ -1+5t \end{bmatrix}$$

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1+3t+\frac{1}{2}t^2 \\ -1+5t-\frac{1}{2}t^2 \end{bmatrix}$$

$$\begin{bmatrix} x_3(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 1+3t+\frac{1}{2}t^2+\frac{1}{3}t^3 \\ -1+5t-\frac{1}{2}t^2+\frac{5}{6}t^3 \end{bmatrix}$$

14. $$\mathbf{x}(t) = \left[\sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} t^n \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!} & \displaystyle\sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} \\ 0 & \displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} (1+t)e^t \\ e^t \end{bmatrix}$$

16. $$y_0(x) = 0$$
$$y_1(x) = (1/3)x^3$$
$$y_2(x) = (1/3)x^3 + (1/63)x^7$$
$$y_3(x) = (1/3)x^3 + (1/63)x^7 + (2/2079)x^{11} + (1/59535)x^{15}$$

Then $y_3(1) \approx 0.350185$, which differs by only 0.0134% from the Runge-Kutta approximation $y(1) \approx 0.350232$. As a denouement we may recall from the result of Problem 16 in Section 3.6 that the exact solution of our initial value problem here is

$$y(x) = x \cdot \frac{J_{3/4}\left(\frac{1}{2}x^2\right)}{J_{-1/4}\left(\frac{1}{2}x^2\right)}$$

so the exact value at $x = 1$ is

$$y(1) = \frac{J_{3/4}\left(\frac{1}{2}\right)}{J_{-1/4}\left(\frac{1}{2}\right)} \approx 0.35023\ 18443.$$