The Entanglement Feature Formalism of Entanglement Dynamics

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June 2020

Structure of My Undergrad Research Thesis

- Basic Concepts of Quantum Information
- Intriguing Dynamical & Thermal Phenomenons in Many-body Physics
  - Area Law and Volume Law
  - Thermalization, Eigenstate thermalization hypothesis
  - Information Scrambling, Out-of-time Correlators, Tripartite Mutual Information
- Review on Entanglement Transitions, Entanglement Feature Formalism
- Numerical Simulations
- Appendix: Replica Method and Weingarten Function

Prerequisites[Thesis Page 1-4]

Entanglement [NRT09]

- The Hilbert Space of a bipartite system: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- Reduced Density Matrix:

$$\rho_A = \mathrm{Tr}_B\, \rho_{AB} = \sum_{i=1}^{N} \lambda_i |\xi\rangle_A \langle\xi| \tag{1}$$

- Consider:

$$|\Psi\rangle = \sum_{i,\mu} c_{i,\mu} |i\rangle_A \otimes |\mu\rangle_B \tag{2}$$

$c_{i,\mu} = c_{i_A} c_{\mu_B}$: Product State; $c_{i,\mu} \neq c_{i_A} c_{\mu_B}$: Entangled State.

Prerequisites [Thesis Page 1-4]

Entropy [NRT09]

- Entropy: a measure for the amount of information we gain on average about a system by triggering and observing one random event of the distribution.
- Von Neumann Entropy[VN27]:

$$S_A(\rho_A) = -\operatorname{Tr}(\rho_A \log \rho_A) \tag{3}$$

- Rényi Entropy[R$^+$61]:

$$S^{(n)}(A) = \frac{1}{1-n} \log \operatorname{Tr}_A \rho_A^n \tag{4}$$

Prerequisites [Thesis Page 8]

Area Law, Volume Law[Eis13]

- Volume Law: $S(X) \sim O(A)$
  EE grows proportionally with the volume of the partitions.
- Area Law: $S(X) \sim O(\partial A)$
  - EE grows proportionally with the area of the partitions.
  - Indicates the correlation length[Has04a][Has04b]
  - Ground State Ansatz: works in $1D$ system, gapped or local[Has07]; $1D$ Matrix Product State[VWPGC06];

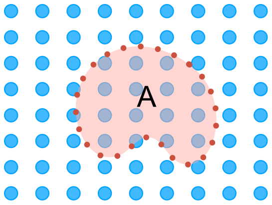

Figure: Area Law

Prerequisites [Thesis Page 11]

Information Scrambling [SBSSH16][Swi18]

- As Entanglement spreading...
  - No local measurement could fully reconstruct the original quantum state $\Longleftrightarrow$ information of initial state lost locally
  - Need to know the signaling between subsystems $\Longrightarrow$ two way to diagnose: OTOC and TMI

Prerequisites [Thesis Page 11-16]

Out-of-time Correlator, OTOC[LO69][MSS16][SBSSH16][Swi18]

- Classical chaos:

$$\frac{\partial q_t}{\partial q_0} \sim e^{\lambda t} \tag{5}$$

- In Quantum chaos the OTOC as eq.6: $\hat{W}(t) = e^{i\hat{H}t}\hat{W}e^{i\hat{H}t}$; and $\langle ... \rangle_\beta$ represents the ensemble at temperature $K_B T$.

$$F(t) = \langle \hat{W}(t)^\dagger \hat{V}(0)^\dagger \hat{W}(t)\hat{V}(0)\rangle_\beta := \mathrm{Tr}[\rho \hat{W}(t)^\dagger \hat{V}(0)^\dagger \hat{W}(t)\hat{V}(0)] \tag{6}$$

- Indicates the commutator's (an operator under pertubation and another operator) growth.
- local perturbation will destroy the correlation of local operators, and the region destroyed would grow.

Prerequisites [Thesis Page 17]

Mutual Information[Pre98]

- Quantifies the correlation between information sources
- Bipartite (between regions X, Y):

$$\begin{aligned} \mathcal{I}_2(X:Y) &\equiv H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y) \\ &= S(X) + S(Y) - S(X \cup Y) \end{aligned} \tag{7}$$

- Tripatite (between regions X, Y, and Z):

$$\begin{aligned} \mathcal{I}_3(X:Y:Z) &= \mathcal{I}_2(X:Y) + \mathcal{I}_2(Y:Z) - \mathcal{I}_2(X:YZ) \\ &= S(X) + S(Y) + S(Z) - S(X \cup Y) \\ &\quad - S(X \cup Z) - S(Y \cup Z) + S(X \cup Y \cup Z) \end{aligned} \tag{8}$$

- Negative $\mathcal{I}_3(X:Y:Z)$ indicates information scrambling [HQRY16]

Prerequisites [Thesis Page 7]

Random Unitary Circuit(RUC)[NVH18]

- Consists an $1D$ array of N qubits
- Evolve from product state, with independent Haar ramdom $q^2 \times q^2$ gates acting on neighbor qudits:

$$U_t = \begin{cases} \bigotimes_x U_{t;2x-1,2x} & \text{odd } t \\ \bigotimes_x U_{t;2x,2x+1} & \text{even } t \end{cases} \tag{9}$$

- The wave function at t:

$$|\psi'\rangle = U_t|\psi\rangle \tag{10}$$

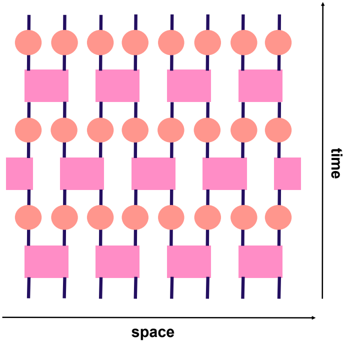

Figure: RUC. Pink brick: Random Unitary Gate; Orange dot: Measurement.

Entanglement Transitions [Thesis Page 19-22]

Driven/Induced by measurement in RUC [LCF18][SRS19][BCA19][CBQA19][SRN19][GH19][LCF19][CDLC18][CNPS19][SRS20][ZGW⁺20]

- From Dynamical Perspective:
  Wave Function Collapse! How much local measurement would destroy entanglements?
- From Thermal Perspective:
  How Eigenstate Thermalization Hypothesis was violated?
- From Information Perspective:
  How information was scrambled? How to retrieve information?

(Projective Measurement rate p or Weak Measurement Strength are proved to be equivalent [BCA19][JYVL20].)

Figure: Entanglement Transition, redrawn from[SRN19]

Entanglement Transitions

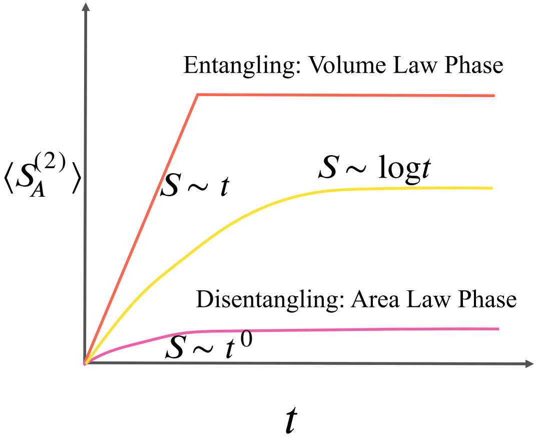

Figure: $t - \langle S_A^{(2)} \rangle$, Redrawn from [SRN19]

Entanglement Entropy Dynamics in RUC

Generic Solutions: the keys of analytical methods [YG18][KAAY19][FZSZ17][CDLC18][ZN19][HJ19][BCA19][JYVL20][SRN19]

- $S^{(2)}(A)$ has two replica $\Longrightarrow$ Use Replica Method, SWAP operator
- Compute the nth moments (ensemble average on Haar Unitary): $\Longrightarrow$ Use Weingarten Formula[Wei78][CŚ06][CM09][CN10]

$$\int_{U_d} U_{i_1 j_1} \dots U_{i_q j_q} U^{\dagger}_{j'_1 i'_1} \dots U^{\dagger}_{j'_q i'_q} dU = \sum_{\sigma,\tau \in q} \delta_{i_1 i'_{\sigma(1)}} \dots \delta_{i_q i'_{\sigma(q)}} \delta_{j_1 j'_{\tau(1)}} \dots \delta_{j_q j'_{\tau(q)}} \mathrm{Wg}(\sigma\tau^{-1}, d) \quad (11)$$

- The behavior of Operators in Random Unitary Circuit $\Longrightarrow$ Biased Random Walk[NVH18][VKRPS18][HJ18]
- Measurement $\Longrightarrow$ Kraus Operator and Quantum Channel

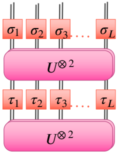

Figure: Redrawn from[KAAY19]

Entanglement Feature Operator [Thesis Page 22-29]

[YG18][KAAY19][VPYL19][FVVY20]

- Label all bipartition subsystems in a total size of 2^n by classical Ising Variables set, $\sigma = [\sigma_1 ... \sigma_n]$. (eq. 12)

$$\sigma_i = \begin{cases} \uparrow & i \in \bar{A} \\ \downarrow & i \in A \end{cases} \tag{12}$$

- Define the Entanglement Feature Operator (eq. 13) by $\sigma, \tau \in S^{\times N}$. $W_U[\sigma, \tau]$ can be considered as the correlation function. $X_\sigma(t) = U^{\bigotimes n} X_\sigma (U^\dagger)^{\bigotimes n}$: Evolution under Heisenberg Picture.

$$W_U[\sigma, \tau] = \mathrm{Tr}[U^{\bigotimes n} X_\sigma (U^\dagger)^{\bigotimes n} X_\tau] \tag{13}$$

- The basis of Entanglement Feature State:

$$|W_\Psi\rangle = \sum_A e^{-S_\Psi(A)} |A\rangle \tag{14}$$

$$\langle A | W_\Psi \rangle = e^{-S_\Psi(A)} = \mathrm{Tr}\, \rho_A^2 \tag{15}$$

Entanglement Feature Operator [Thesis Page 22-29]

Local Information Scrambling [KAAY19]

- Consider: $U = \prod_t U_t$. The unitary operator drawn from the ensemble at t remains invariant under local basis transformation, $P(U_t) = P\left(VU_tV^\dagger\right)$. A local scrambler can be written as:

$$U = V_t^\dagger U_t' V_t V_{t-1}^\dagger U_{t-1}' V_{t-1}... \tag{16}$$

- The process is markovian. we can write the Transfer Matrix $\hat{T}_t$ of each step. $\hat{W}_{U_t}$, $\hat{W}_1^{-1}$ are EF Operators of Unitary Gate and the Inverse of Identity.

$$|W_{\Psi_{t+1}}\rangle = \hat{T}_t |W_{\Psi_t}\rangle = \hat{T}_t = \hat{W}_{U_t} \hat{W}_1^{-1} |W_{\Psi_t}\rangle \tag{17}$$

- Use the continuum limit, we can pack the entanglement into H.

$$U_t = V_t^\dagger e^{-\mathrm{i}\epsilon H} V_t \tag{18}$$

Numerical Simulation Results [Thesis Page 29-36]

2nd Renyi Entropy

The averaged steady-state value of the entanglement entropy for a system of size $L = 16$ as a function of subsystem size L_A

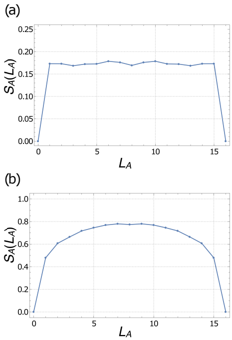

Figure: $L_A - \langle S_A^{(2)} \rangle$ from [LCF18]

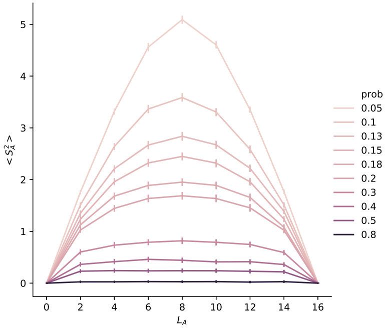

Figure: $L_A - \langle S_A^{(2)} \rangle$, $L = 16$, $t = 30$, 500 realizations

Numerical Simulation Results [Thesis Page 29-36]

2nd Renyi Entropy

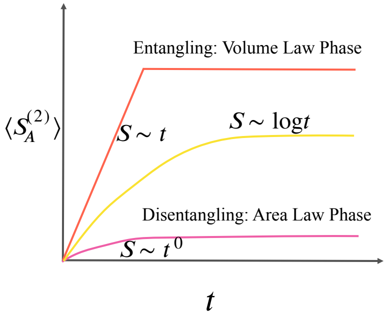

Figure: $t - \langle S_A^{(2)} \rangle$, Redrawn from [SRN19]

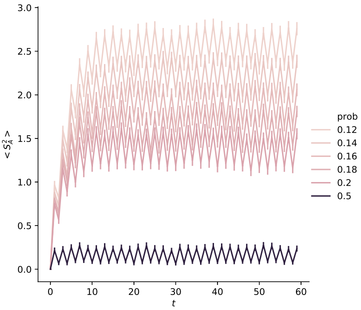

Figure: $t - \langle S_A^{(2)} \rangle$, $L = 14$, $t = 60$, 500 realizations. Tooth shape means Odd-even effect

Numerical Simulation Results [Thesis Page 29-36]

2nd Renyi Entropy

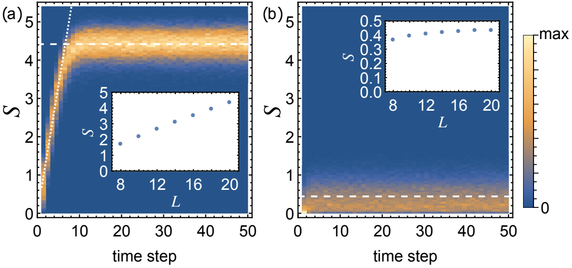

Figure: $t - \langle S_A^{(2)} \rangle$ from [SRS19]

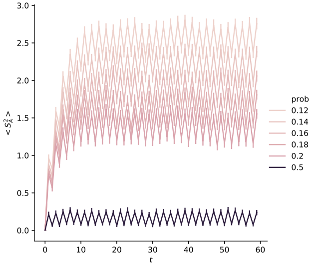

Figure: $t - \langle S_A^{(2)} \rangle$, $L = 14$, $t = 60$, 500 realizations

Numerical Simulation Results [Thesis Page 29-36]

Bipartite Mutual Information

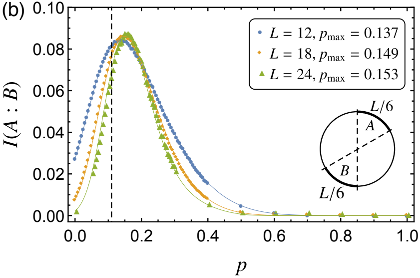

Figure: $p - \mathcal{I}_2$, from[SRS19]

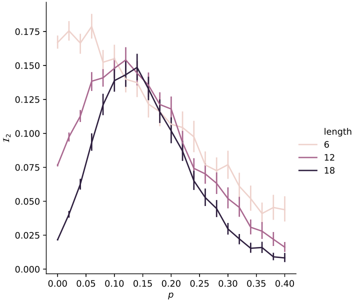

Figure: $p - \langle \mathcal{I}_2 \rangle$, $\Delta p = 0.02$, $L_{sub} = L/6$, $t = 40$, 1000 realizations

Numerical Simulation Results [Thesis Page 29-36]

Bipartite Mutual Information

System partition size sensitive. Need future analysis.

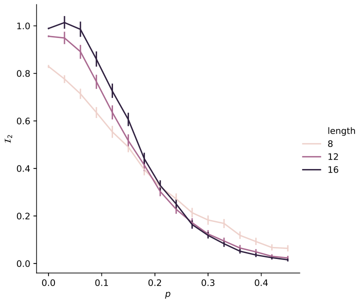

Figure: $p - \langle \mathcal{I}_2 \rangle$, $\Delta p = 0.03$, $L_{sub} = L/4$, $t = 40$, 1000 realizations

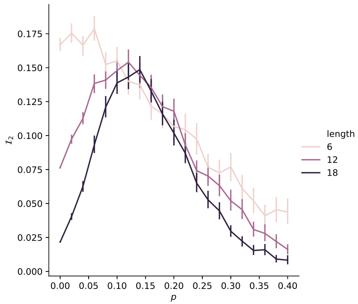

Figure: $p - \langle \mathcal{I}_2 \rangle$, $\Delta p = 0.02$, $L_{sub} = L/6$, $t = 40$, 1000 realizations

Numerical Simulation Results [Thesis Page 29-36]

Mutual Information between Ancillae

Use Ancillae to entangle a system qubit: like we make a copy.

Remember that we cannot define the mutual information of two qubits at different time steps.

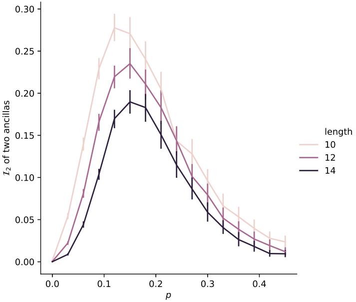

Figure: Maximal Entangled State, $p - \langle \mathcal{I}_2 \rangle$, $\Delta p = 0.03$, $L_{sub} = L/4$, $t = 40$, 1000 realizations

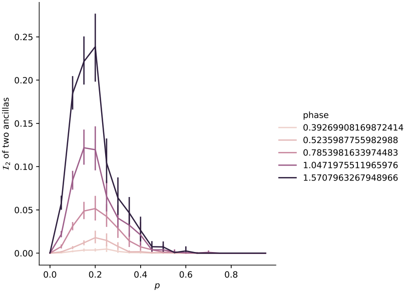

Figure: Weak Entangling, $p - \langle \mathcal{I}_2 \rangle$, $\Delta p = 0.03$, $L_{sub} = L/4$, $t = 40$, 1000 realizations

Thank you for your listening and review!

Acknowledgement: Prof. Yizhuang You (UC San Diego), Ruihua Fan(Harvard), Yiming Chen(Princeton), Dr. Marcin Szyniszewski(Lancaster), Dr. Nicholas Hunter-Jones(Perimeter Institute), Dr. Xin Dai(Ohio State U), Aidan Zabalo(Rutgers), Dr. Tianci Zhou(KITP), Tingfung Lau, Man Luo(Tsinghua), Yida Li(Tsinghua)

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