Sufficient-Statistic Formula for Optimal Stimulus Spending

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Implicit formula for optimal stimulus spending:

$$\frac{g/c - g/c^*}{g/c^*} = \underbrace{2\varepsilon}_{} \times \underbrace{m}_{} \times \frac{u - u^*}{u^*}$$

function of g/c

→ We want an explicit formula for optimal stimulus spending → formula involving initial unemployment gap $\frac{u_0 - u^*}{u^*}$ and other sufficient statistics.

Q. we are at $\frac{u_0 - u^*}{u^*}$ & spending is g/c^*:

↑ recession

how much should spending increase/decrease?

To make formula explicit, we express $\frac{u - u^*}{u^*}$ as a function of $\frac{u_0 - u^*}{u^*}$ & $\frac{g/c - g/c^*}{g/c^*}$.

First-order approximation of $\frac{u - u^*}{u^*}$ around initial situation.

$$\frac{u - u^*}{u^*} = \frac{u_0 - u^*}{u^*} + \frac{1}{u^*} \times \underbrace{\frac{du}{dg/c} \times g/c^*}_{\frac{du}{d\ln g/c}} \times \left[\frac{g/c - g/c^*}{g/c^*} \right]$$

$$\frac{u - u^*}{u^*} = \frac{u_0 - u^*}{u^*} + \frac{1}{u^*} \times \underbrace{\frac{du}{d\ln g/c}}_{\text{need to rework}} \times \left[\frac{g/c - g/c^*}{g/c^*}\right]$$

Compute $du / d\ln g/c$: evaluated at g/c^*, u^*

$$\frac{du}{d\ln g/c} = \frac{du}{dg} \times \frac{dg}{d\ln g/c} = -m \times \frac{dg}{d\ln g/c}$$

Compute $dg / d\ln g/c$.

$$c = 1 - (u + v) - g$$

$$\frac{dc}{dg} = \frac{-d(u+v)}{dg} - 1$$

at $u^*, g/c^*$: $\frac{d(u+v)}{dg}$ at $u^* = 0$

$\hookrightarrow \underbrace{\frac{d(u+v)}{du}}_{0} \times \frac{du}{dg}$

$$\frac{dc}{dg} = -1 \quad \text{at } u^*$$

at g/c^*

$$\frac{d\ln g/c}{dg} = \frac{1}{g/c} \times \frac{d\,g/c}{dg} = \frac{1}{g/c} \times \left[\frac{1}{c} - g/c^2 \times \underbrace{\frac{dc}{dg}}_{-1}\right]$$

$$\frac{d\ln g/c}{dg} = \frac{c^*}{g^*} \times \left[\frac{1}{c^*} + g^*/c^{*2}\right] = \frac{1}{g^*} + \frac{1}{c^*}$$

$$\Rightarrow \frac{du}{d\ln g/c} = -m \times \left(\frac{1}{\frac{1}{g^*}+\frac{1}{c^*}}\right) = -\frac{m}{2} \times \left[\underbrace{\frac{2}{\frac{1}{g^*}+\frac{1}{c^*}}}_{\text{harmonic mean of } g^*,\, c^*}\right]$$

$$\Rightarrow \underline{\frac{u-u^*}{u^*}} = \underbrace{\frac{u_0-u^*}{u^*}}_{\text{initial unemployment gap}} - \underbrace{\frac{m}{2u^*}\left(\frac{2}{\frac{1}{g^*}+\frac{1}{c^*}}\right)}_{\text{suff. stat.}}\underbrace{\left(\frac{g/c-g/c^*}{g/c^*}\right)}_{\text{stimulus spending}}$$

$\Rightarrow$ plug this into implicit formula to make it explicit.

$$\underline{\frac{g/c-g/c^*}{g/c^*}} = 2\varepsilon m \times \left(\frac{u_0-u^*}{u^*}\right) - \varepsilon m^2 \times \underbrace{\frac{1}{u^*} \times \frac{2}{\frac{1}{g^*}+\frac{1}{c^*}}}_{\equiv z} \times \underline{\frac{g/c-g/c^*}{g/c^*}}$$

$$\left[1+\varepsilon \cdot m^2 \cdot z\right] \times \frac{g/c-g/c^*}{g/c^*} = 2\varepsilon m \left(\frac{u_0-u^*}{u^*}\right)$$

<u>Explicit formula for optimal stimulus spending</u>

$$\boxed{\frac{g/c - g/c^*}{g/c^*} = \frac{2\,\varepsilon\, m}{1 + z\,\varepsilon\, m^2} \times \frac{u_0 - u^*}{u^*}}$$

stimulus spending

initial unemployment gap

ε = elasticity of substitution b/w g & c

m = unemployment multiplier

$$z = \frac{1}{u^*} \times \frac{2}{\frac{1}{g^*} + \frac{1}{c^*}}$$