Constant-Elasticity-of-Substitution Matching Function

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Cobb Douglas matching function

$$m(S, B) = \omega \, S^{\eta} \, B^{1-\eta}$$

$$m(S, B) > \min(S, B) \quad \text{if } S \text{ a } B \text{ large}$$

CES matching function

$$\boxed{m(S, B) = \left[S^{-\gamma} + B^{-\gamma} \right]^{-1/\gamma}}$$

$\gamma > 0$

- $m(0, B) = m(S, 0) = 0$
- $\dfrac{\partial m}{\partial S} > 0 \qquad \dfrac{\partial m}{\partial B} > 0$
- CRS

$$m(\lambda S, \lambda B) = \left[(\lambda S)^{-\gamma} + (\lambda B)^{-\gamma} \right]^{-1/\gamma}$$

$$= \left[(\lambda^{-\gamma})(S^{-\gamma} + B^{-\gamma}) \right]^{-1/\gamma}$$

$$= \lambda \left[S^{-\gamma} + B^{-\gamma} \right]^{-1/\gamma}$$

$$= \lambda \, m(S, B)$$

- Check that $m(S, B) < \min(S, B)$

$$S^{-\gamma} + B^{-\gamma} > S^{-\gamma} \qquad (B^{-\gamma} > 0)$$

$$\left[S^{-\gamma} + B^{-\gamma}\right]^{-1/\gamma} < \left(S^{-\gamma}\right)^{-1/\gamma}$$

$$\left.\begin{aligned} &\boxed{m(S, B) < S} \\ &\boxed{m(S, B) < B} \end{aligned}\right\} m(S, B) \leq \min(S, B)$$

- Matching elasticity: $\boxed{\eta \equiv \dfrac{\partial \ln m}{\partial \ln S}}$

$$\eta = \frac{\partial \ln m}{\partial \ln S} = \frac{\partial \ln \left\{\left[S^{-\gamma} + B^{-\gamma}\right]^{-1/\gamma}\right\}}{\partial \ln S}$$

$$= -\frac{1}{\gamma} \frac{\partial \ln\left(S^{-\gamma} + B^{-\gamma}\right)}{\partial \ln S}$$

$$= -\frac{1}{\gamma} \cdot \left[\frac{\partial \ln S^{-\gamma}}{\partial \ln S} \times \frac{S^{-\gamma}}{S^{-\gamma} + B^{-\gamma}}\right]$$

$$= -\frac{1}{\gamma} \frac{S^{-\gamma}}{S^{-\gamma} + B^{-\gamma}} \cdot (-\gamma)$$

$\theta = B/S$

$$\eta = \frac{S^{-\gamma}}{S^{-\gamma} + B^{-\gamma}} = \frac{1}{1 + \left(\frac{B}{S}\right)^{-\gamma}}$$

$$\boxed{\eta(\theta) = \frac{1}{1 + \theta^{-\gamma}}}$$

- $\eta(0) = 0$
  $\eta(\theta) = 1$
  $\eta'(\theta) > 0$