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# DIAGRAMMATIC EXPOSITION OF A THEORY OF PUBLIC EXPENDITURE

Paul A. Samuelson

IN the November 1954 issue of this REVIEW my paper on "The Pure Theory of Public Expenditure" presented a mathematical exposition of a public expenditure theory that goes back to Italian, Austrian, and Scandinavian writers of the last 75 years. After providing that theory with its needed logically-complete optimal conditions, I went on to demonstrate the fatal inability of any decentralized market or voting mechanism to attain or compute this optimum. The present note presents in terms of two-dimensional diagrams an essentially equivalent formulation of the theory's optimum conditions and briefly discusses some criticisms.

## *A polar-case model of government*

Doctrinal history shows that theoretical insight often comes from considering strong or extreme cases. The grand Walrasian model of competitive general equilibrium is one such extreme polar case. We can formulate it so stringently as to leave no economic role for government. What strong polar case shall the student of public expenditure set alongside this pure private economy?

One possibility is the model of a group-mind. Such a model, which has been extensively used by nationalists and by Romantic critics of classical economics, can justify any, and every, configuration of government. So there is perhaps little that an economic theorist can usefully say about it.

My alternative is a slightly more sophisticated one, but still — intentionally — an extreme polar case. It is consistent with individualism, yet at the same time it explicitly introduces the vital external interdependencies that no theory of government can do without. Its basic assumption is an oversharpest distinction between the following two kinds of goods:

(i) A *private* consumption good, like bread, whose total can be parcelled out among two or more persons, with one man having a loaf less if another gets a loaf more. Thus if  $X_1$  is total bread, and  $X_1^1$  and  $X_1^2$  are the respective pri-

vate consumptions of Man 1 and Man 2, we can say that the total equals the sum of the separate consumptions — or  $X_1 = X_1^1 + X_1^2$ .

(ii) A *public* consumption good, like an outdoor circus or national defense, which is provided for each person to enjoy or not, according to his tastes. I assume the public good can be varied in total quantity, and write  $X_2$  for its magnitude. It differs from a private consumption good in that each man's consumption of it,  $X_2^1$  and  $X_2^2$  respectively, is related to the total  $X_2$  by a condition of *equality* rather than of summation. Thus, by definition,  $X_2^1 = X_2$ , and  $X_2^2 = X_2$ .

Obviously, I am introducing a strong polar case. We could easily lighten the stringency of our assumptions. But on reflection, I think most economists will see that this is a natural antipodal case to the admittedly extreme polar case of traditional individualistic general equilibrium. The careful empiricist will recognize that many — though not all — of the realistic cases of government activity can be fruitfully analyzed as some kind of a blend of these two extreme polar cases.

## *Graphical depiction of tastes and technology*

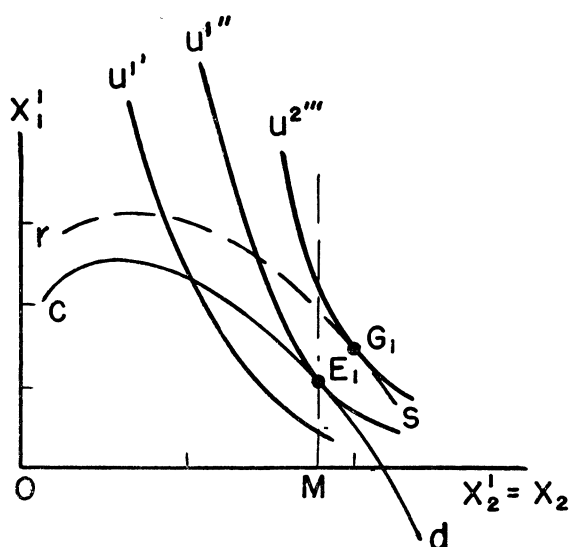
The first three charts summarize our assumptions about tastes and technology. Each diagram has a private good, such as bread, on its vertical axis; each has a public good on its horizontal axis. The heavy indifference curves of Chart 1 summarize Man 1's preferences between public and private goods. Chart 2's indifference curves do the same for Man 2; and the relative flatness of the contour shows that, in a sense, he has less liking for the public good.

The heavy production-possibility or opportunity-cost curve  $AB$  in Chart 3 relates the total productions of public and private goods in the usual familiar manner: the curve is convex from above to reflect the usual assumption of increasing relative marginal costs (or generalized diminishing returns).<sup>1</sup>

<sup>1</sup> Even though a public good is being compared with a

Because of our special definition of a public good, the three diagrams are not independent. Each must be lined up with *exactly the same horizontal scale*. Because increasing a public good for society simultaneously increases it for each and every man, we must always be simultaneously at exactly the same longitude in all three figures. Moving an inch east in one diagram moves us the same amount east in all.

CHART 1. — INDIFFERENCE CONTOURS RELATING MAN 1'S CONSUMPTION OF PUBLIC AND PRIVATE GOODS



The private good on the vertical axis is subject to no new and unusual restrictions. Each man can be moved north or south on his indifference diagram independently. But, of course, the third diagram does list the total of bread summed over the private individuals; so it must have a larger vertical axis, and our momentary northward position on it must correspond to the sum of the independent northward positions of the separate individuals.

#### *Tangency conditions for Pareto optima*

What is the best or ideal state of the world for such a simple system? That is, what three

private good, the indifference curves are drawn with the usual convexity to the origin. This assumption, as well as the one about diminishing returns, could be relaxed without hurting the theory. Indeed, we could recognize the possible case where one man's circus is another man's poison, by permitting indifference curves to bend forward. This would not affect the analysis but would answer a critic's minor objection. Mathematically, we could without loss of generality set  $X_2^1 = \text{a function of } X_2$ , relaxing strict equality.

vertically-aligned points corresponding to a determination of a given total of both goods and a determinate parcelling out of them among all separate individuals will be the ethically preferred final configuration?

To answer this ethical, normative question we must be given a set of norms in the form of a *social welfare function* that renders interpersonal judgments. For expository convenience, let us suppose that this will be supplied later and that we know in advance it will have the follow-

CHART 2. — INDIFFERENCE CONTOURS RELATING MAN 2'S CONSUMPTION OF PUBLIC AND PRIVATE GOODS

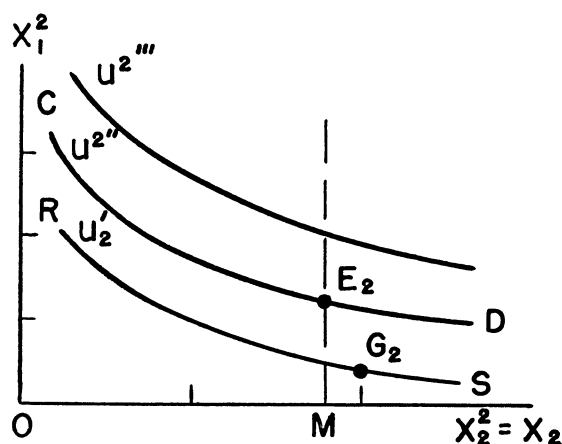
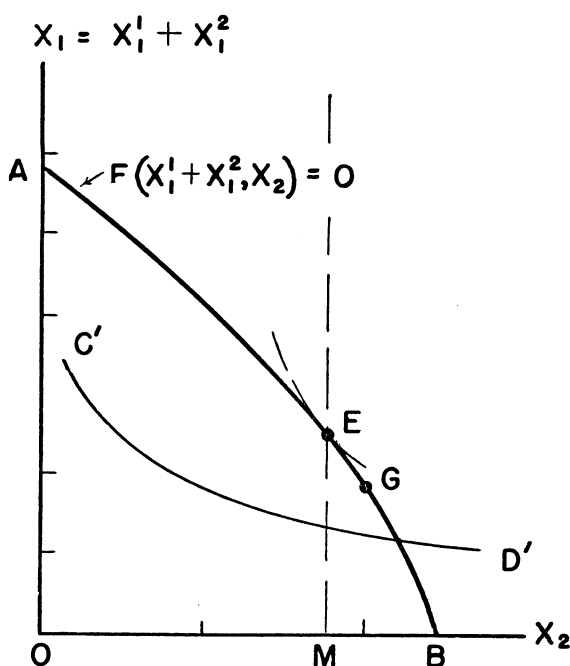


CHART 3. — TRANSFORMATION SCHEDULE RELATING TOTALS OF PUBLIC AND PRIVATE GOODS



ing special individualistic property: leaving each person on his same indifference level will leave social welfare unchanged; at any point, a move of each man to a higher indifference curve can be found that will increase social welfare.

Given this rather weak assurance about the forthcoming social welfare function, we can proceed to determine tangency conditions of an "efficiency" type that are at least necessary, though definitely not sufficient. We do this by setting up a preliminary maximum problem which will eventually necessarily have to be satisfied.

Holding all but one man at specified levels of indifference, how can we be sure that the remaining man reaches his highest indifference level?

Concretely, this is how we define such a tangency optimum: Set Man 2 on a specified indifference curve, say his middle one  $CD$ . Paying attention to Mother Nature's scarcity, as summarized in Chart 3's  $AB$  curve, and following Man 1's tastes as given by Chart 1's indifference curves, how high on those indifference curves can we move Man 1?

The answer is given by the tangency point  $E_1$ , and the corresponding aligned points  $E_2$  and  $E$ .

How is this derived? Copy  $CD$  on Chart 3 and call it  $C'D'$ . The distance between  $C'D'$  and  $AB$  represents the amounts of the two goods that are physically available to Man 1. So subtract  $C'D'$  vertically from  $AB$  and plot the algebraic result as  $cd$  in Chart 1. Now where on  $cd$  would Man 1 be best off? Obviously at the tangency point  $E_1$  where  $cd$  touches (but does not cross) his highest attainable indifference contour.<sup>2</sup>

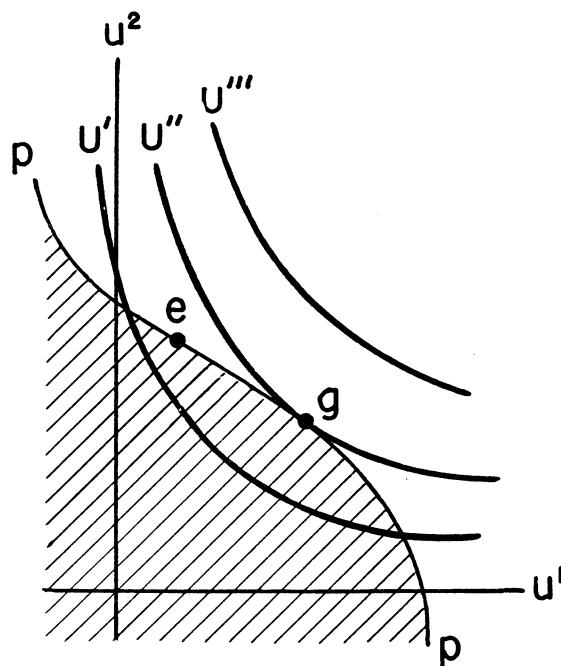
How many such Pareto-optimal points are

<sup>2</sup> The reader can easily derive  $rs$  and the tangency point  $G_1$  corresponding to an original specification of Man 2's indifference level at the lower level  $RS$  rather than at  $AB$ . He can also interchange the roles of the two men, thereby deriving the point  $E_2$  by a tangency condition. As a third approach, he can *vertically add* Man 2's specified indifference curve to each and every indifference curve of Man 1; the resulting family of contours can be conveniently plotted on Chart 3, and the final optimum can be read off from the tangency of  $AB$  to that family at the point  $E$ —as shown by the short broken-line indifference curve at  $E$ . It is easy to show that any of these tangencies are, in the two-good case, equivalent to Equation (2) of my cited paper; with a single private good my Equation (1) becomes redundant.

there? Obviously, for each of the infinite possible initial indifference curves to put Man 2 on, we can derive a new highest attainable tangency level for Man 1. So there are an infinity of such optimal points—as many in number as there are points on the usual contract curve. All of these Pareto-optimal points have the property that from them there exists no physically-feasible movement that will make every man better off. Of course we cannot compare two different Pareto points until we are given a social welfare function. For a move from one Pareto point to another must always hurt one man while it is helping another, and an interpersonal way of comparing these changes must be supplied.

Chart 4 indicates these utility possibilities on an ordinal diagram. Each axis provides an

CHART 4.—UTILITY FRONTIER OF PARETO-OPTIMAL EFFICIENCY POINTS AND ITS TANGENCY TO HIGHEST ATTAINABLE SOCIAL WELFARE CONTOUR



indicator of the two men's respective indifference curve levels. The utility frontier of Pareto-optimal points is given by  $pp$ : the double fold infinity of "inefficient," non-Pareto-optimal points is given by the shaded area; the  $pp$  frontier passes from northwest to southeast to reflect the inevitable conflict of interests char-

acterizing any contract locus; the curvature of the  $pp$  locus is of no particular type since we have no need to put unique cardinal numbers along the indifference contours and can content ourselves with east-west and north-south relationships in Chart 4 without regard to numerical degree and to uneven stretchings of either utility axis.

### *The optimum of all the Pareto optima*

Now we can answer the fundamental question: what is the best configuration for this society?

Use of the word "best" indicates we are in the ascientific area of "welfare economics" and must be provided with a set of norms. Economic science cannot deduce a social welfare function; what it can do is neutrally interpret any arbitrarily specified welfare function.

The heavy contours labelled  $U'$ ,  $U''$ , and  $U'''$  summarize all that is relevant in the provided social welfare function (they provide the needed ordinal scoring of every state of the world, involving different levels of indifference for the separate individuals).<sup>3</sup>

Obviously society cannot be best off inside the utility frontier. Where then on the utility frontier will the "best obtainable bliss point" be? We will move along the utility frontier  $pp$  until we touch the highest social indifference curve: this will be at  $g$  where  $pp$  tangentially touches, without crossing, the highest obtainable social welfare level  $U'''$ . In words, we can interpret this final tangency condition<sup>4</sup> in the following terms:

(i) The social welfare significance of a unit of any private good allocated to private indi-

viduals must at the margin be the same for each and every person.

(ii) The Pareto-optimal condition, which makes relative marginal social cost equal to the sum of all persons' marginal rates of substitution, is already assured by virtue of the fact that bliss lies on the utility frontier.<sup>5</sup>

### *Relations with earlier theories*

This completes the graphical interpretation of my mathematical model. There remains the pleasant task of relating this graphical treatment to earlier work of Bowen<sup>6</sup> and others.

To do this, look at Chart 5, which gives an alternative depiction of the optimal tangency condition at a point like  $E$ . I use the private good  $X_1$  as numeraire, measuring all values in terms of it. The  $MC$  curve is derived from the  $AB$  curve of Chart 3: it is nothing but the absolute slope of that production-possibility schedule plotted against varying amounts of the public good; it is therefore a marginal cost curve, with  $MC$  measured in terms of the numeraire good.

The marginal rate of substitution curves  $MRS^1$  and  $MRS^2$  are derived in a similar fashion from the respective indifference curves of Man 1 and Man 2: thus,  $MRS^1$  is the absolute slope of the  $u'''$  indifference curve plotted against varying amounts of the public good;  $MRS^2$  is the similar slope function derived from Man 2's indifference curve  $CD$ . (All three are "marginal" curves, bearing the usual relationship to their respective "total" curves.)

These schedules look like demand curves. We are accustomed to adding horizontally or laterally the separate demand curves of individ-

<sup>3</sup> These social welfare or social indifference contours are given no particular curvature. Why? Again because we are permitting any arbitrary ordinal indicator of utility to be used on the axes of Chart 4.

An ethical postulate ruling out all "dog-in-the-manger phenomena" will make all partial derivatives of the social welfare function  $U(u^1, u^2, \dots)$  always positive. This will assure the usual negative slopes to the  $U$  contours of Chart 4. However, without hurting the Pareto part of the new welfare economics, we can relax this assumption a little and let the contours bend forward. If, at every point there can be found at least one positive partial derivative, this will be sufficient to rule out satiation points and will imply the necessity of the Pareto-optimal tangency condition of the earlier diagrams.

<sup>4</sup> This tangency condition would have to be expressed mathematically in terms of numerical indicators of utility that are not invariant under a monotonic renumbering.

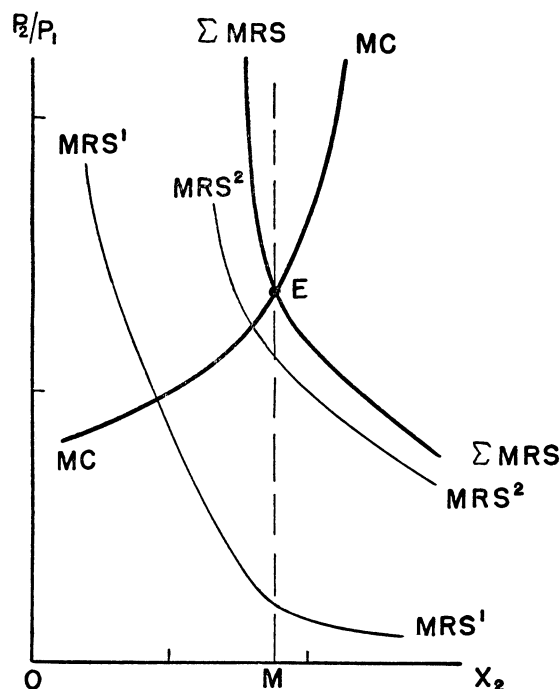
However, it is easy to combine this tangency with the earlier Pareto-type tangency to get the formulation (3) of my cited paper, which is independent of the choice of numerical indicators of  $U$ ,  $u^1$ , or  $u^2$ .

<sup>5</sup> A remarkable duality property of private and public goods should be noted. Private goods whose totals add—such as  $X_1 = X_1^1 + X_1^2$ —lead ultimately to marginal conditions of simultaneous equality—such as  $MC = MRS^1 = MRS^2$ . Public goods whose totals satisfy a relation of simultaneous equality—such as  $X_1 = X_1^1 = X_1^2$ —lead ultimately to marginal conditions that add—such as  $MC = MRS^1 + MRS^2$ .

<sup>6</sup> Howard R. Bowen, "The Interpretation of Voting in the Allocation of Economic Resources," *Quarterly Journal of Economics*, LVIII (November 1943), 27–49. Much of this is also in Bowen's *Toward Social Economy* (New York, 1948), ch. 18.



CHART 5. — INTERSECTION OF PUBLIC GOOD'S MARGINAL COST SCHEDULE AND THE VERTICALLY-SUMMED INDIVIDUAL'S MARGINAL RATES OF SUBSTITUTION, AS ENVISAGED BY LINDAHL AND BOWEN



uals to arrive at total market demand. But this is valid only for private goods. As Bowen rightly says, *we must in the case of public goods add different individuals' curves vertically.*

This gives us the heavy  $\Sigma MRS$  curve for the whole community. Where is equilibrium? It is at  $E$ , where the community  $MC$  curve intersects the community  $\Sigma MRS$  curve. Upon reflection the reader will realize that the equality  $MC = \Sigma MRS = MRS^1 + MRS^2$  is the precise equivalent of my mathematical equation (2) and of our Pareto-type tangency condition at  $E_1$ ,  $E_2$ , or  $E$ . Why? Because of the stipulated requirement that Chart 5's curves are to depict the absolute slopes of the curves of Charts 1–3.

Except for minor details of notation and assumption, Chart 5 is identical with the figure shown on page 31 of the first Bowen reference, and duplicated on page 177 of the second reference. I am happy to acknowledge this priority. Indeed anyone familiar with Musgrave's valuable summary of the literature bearing on this area<sup>7</sup> will be struck with the similarity between

<sup>7</sup> Richard A. Musgrave, "The Voluntary Exchange Theory of Public Economy," *Quarterly Journal of Economics*,

this Bowen type of diagram and the Lindahl 100-per-cent diagram reproduced by Musgrave.<sup>8</sup>

Once the economic theorist has related my graphical and mathematical analysis to the Lindahl and Bowen diagrams, he is in a position, I believe, to discern the logical advantage of the present formulation. For there is something circular and unsatisfactory about both the Bowen and Lindahl constructions: they show what the final equilibrium looks like, but by themselves they are not generally able to find the desired equilibrium. To see this, note that whereas we might know  $MC$  in Chart 5, we would not know the appropriate  $MRS$  schedules for *all* men until we already were familiar with the final  $E$  intersection point. (We might know  $MRS^2$  from the specification that Man 2 is to be on the  $AB$  level; but then we wouldn't know  $MRS^1$  until Chart 1's tangency had given us Man 1's highest attainable level,  $u^{1''}$ .) Under conditions of general equilibrium, Charts 1–3 logically contain Chart 5 inside them, but not vice versa. Moreover, Charts 1–3 explicitly call attention to the fact that there is an infinite number of different diagrams of the Lindahl-Bowen type, one for each specified level of relative interpersonal well-being.<sup>9</sup>

### Concluding reflections

I hope that the analytic model outlined here may help make a small and modest step toward understanding the complicated realities of polit-

LIII (February 1939), 213–17. This gives citations to the relevant works of Sax, De Viti de Marco, Wicksell, and Lindahl. I have greatly benefited from preliminary study of Professor Musgrave's forthcoming treatise on public finance, which I am sure will constitute a landmark in this area.

<sup>8</sup> Musgrave, *op. cit.*, 216, which is an acknowledged adaption from Erik Lindahl, *Die Gerechtigkeit in der Besteuerung* (Lund, 1919), 89. I have not had access to this important work. This diagram plots instead of the functions of Chart 5 the exact same functions after each has been divided by the  $MC$  function. The equilibrium intersection corresponding to  $E$  now shows up as the point at which all persons will together voluntarily provide 100 per cent of the full (unit? marginal?) cost of the public service. (If  $MC$  is not constant, some modifications in the Musgrave diagram may be required.)

<sup>9</sup> The earlier writers from Wicksell on were well aware of this. They explicitly introduce the assumption that there is to have been a *prior* optimal interpersonal distribution of income, so what I have labelled  $E$  might better be labelled  $G$ . But the general equilibrium analyst asks: how can the appropriate distribution of income be decided on

ical economy. Much remains to be done. This is not the place to discuss the wider implications and difficulties of the presented economic theory.<sup>10</sup> However, I should like to comment briefly on some of the questions about this theory that have been raised in this REVIEW.<sup>11</sup>

(i) On the deductive side, the theory presented here is, I believe, a logically coherent one. This is true whether expressed in my original mathematical notation or in the present diagrammatic form. Admittedly, the latter widens the circle of economists who can understand and follow what is being said. The present version, with its tangencies of methodologically the same type as characterize Cournot-Marshall marginal theory and Bergson-Pigou welfare theory, should from its easily recognized equivalence with the mathematical version make clear my refusal to agree with Dr. Enke's view that my use of mathematics was limited "to notation."

(ii) In terms of the history of similar theories, I hope the present paper will make clear relationships to earlier writers. (In particular, see the above discussion relating my early diagrams and equations to the Bowen-Lindahl formulation.) I shall not bore the reader with irrelevant details of independent rediscoveries of doctrine that my ignorance of the available literature may have made necessary. Yet is it presumptuous to suggest that there does not exist in the present economic literature very much in the way of "conclusions and reasoning" that are, in Dr. Margolis' words, "familiar"? Except for the writers I have cited, and

a prior basis *before* the significant problems of public consumptions have been determined? A satisfactory general analysis can resist the temptation to assume (i) the level of government expenditure must be so small as not to affect appreciably the marginal social significance of money to the different individuals; (ii) each man's indifference curves run parallel to each other in a vertical direction so that every and all indifference curves in Chart 1 (or in Chart 2) give rise to the same  $MRS^1$  (or  $MRS^2$ ) curve in Chart 5. The modern theorist is anxious to free his analysis from the incubus of unnecessarily restrictive partial equilibrium assumptions.

<sup>10</sup> At the 1955 Christmas Meetings of the American Economic Association and Econometric Society, I hope to present some further developments and qualifications of this approach.

<sup>11</sup> Stephen Enke, "More on the Misuse of Mathematics in Economics: A Rejoinder," this REVIEW, xxxvii (May 1955), 131-33; Julius Margolis, "On Samuelson on the Pure Theory of Public Expenditure," this issue, p. 347.

the important unpublished thoughts of Dr. Musgrave, there is much opaqueness in the literature. Much of what goes by the name of the "voluntary exchange theory of public finance" seems pure obfuscation.<sup>12</sup>

(iii) Far from my formulation's being, as some correspondents have thought, a revival of the voluntary exchange theory — it is in fact an attempt to demonstrate how right Wicksell was to worry about the inherent political difficulty of ever getting men to reveal their tastes so as to attain the definable optimum. This intrinsic "game theory" problem has been sufficiently stressed in my early paper so that it has not been emphasized here. I may put the point most clearly in terms of the familiar tools of modern literary economics as follows:

Government supplies products jointly to many people. In ordinary market economics as you increase the number of sellers of a homogeneous product indefinitely, you pass from monopoly through indeterminate oligopoly and can hope to reach a determinate competitive equilibrium in the limit. It is sometimes thought that increasing the number of citizens who are jointly supplied public goods leads to a similar determinate result. This is reasoning from an incorrect analogy. A truer analogy in private economics would be the case of a bilateral-monopoly supplier of joint products whose number of joint products — meat, horn, hide, and so on — is allowed to increase without number: such a process does not lead to a determinate equilibrium of the harmonistic type praised in the literature. My simple model is able to demonstrate this point — which does have "policy implications."

(iv) I regret using "the" in the title of my earlier paper and have accordingly changed the present title. Admittedly, public expenditure and regulation proceed from considerations other than those emphasized in my models. Here are a few:

a. Taxes and expenditure aim at redistrib-

<sup>12</sup> See Gerhard Colm, "The Theory of Public Expenditure," *Annals of the American Academy of Political and Social Sciences*, CLXXXIII (January 1936), 1-11, reprinted in his *Essays in Public Finance and Fiscal Policy* (New York, 1955), 27-43 for an admirable criticism of the Graziani statement, "We know that the tax tends to take away from each and all that quantity of wealth which they would each have voluntarily yielded to the state for the satisfaction of their purely collective wants" (page 32).

uting incomes. I am anxious to clear myself from Dr. Margolis' understandable suspicion that I am the type of liberal who would insist that all redistributions take place through tax policies and transfer expenditures: much public expenditure on education, hospitals, and so on, can be justified by the feasibility consideration that, even if these are not 100 per cent efficient in avoiding avoidable dead-weight loss, they may be better than the attainable imperfect tax alternatives.<sup>13</sup>

b. Paternalistic policies are voted upon themselves by a democratic people because they do not regard the results from spontaneous market action as optimal. Education and forced paces of economic development are good examples of this.

c. Governments provide or regulate services that are incapable of being produced under the strict conditions of constant returns that go to characterize optimal self-regulating atomistic competition.

d. Myriad "generalized external economy and diseconomy" situations, where private pecuniary interest can be expected to deviate from social interests, provide obvious needs for government activity.

I am sure this list of basic considerations underlying government expenditure could be extended farther, including even areas where government probably ought not to operate from almost anyone's viewpoint.

(v) This brief list can end with the most important criticism that the various commentators on my paper have put forth. They all ask: "Is it factually true that most — or any! — of the functions of government can be properly fitted into your extreme category of a public good? Can education, the courts, public defense, highway programs, police and fire protection be put into this rigid category of a 'public good available to all'? In practically every one

of these cases isn't there an element of variability in the benefit that can go to one citizen *at the expense* of some other citizens?"

To this criticism, I fully agree. And that is why in the present formulation I have insisted upon the polar nature of my category. However, to say that a thing is not located at the South Pole does not logically place it at the North Pole. To deny that most public functions fit into my extreme definition of a public good is not to grant that they satisfy the logically equally-extreme category of a private good. To say that your absence at a concert may contribute to my enjoyment is not to say that the elements of public services can be put into homogeneous additive packages capable of being optimally handled by the ordinary market calculus.

Indeed, I am rash enough to think that in almost every one of the legitimate functions of government that critics put forward there is to be found a blending of the extreme antipodal models. One might even venture the tentative suspicion that any function of government not possessing any trace of the defined public good (and no one of the related earlier described characteristics) ought to be carefully scrutinized to see whether it is truly a legitimate function of government.

(vi) Whether or not I have overstated the applicability of this one theoretical model to actual governmental functions, I believe I did not go far enough in claiming for it relevance to the vast area of decreasing costs that constitutes an important part of economic reality and of the welfare economics of monopolistic competition. I must leave to future research discussions of these vital issues.

Economic theory should add what it can to our understanding of governmental activity. I join with critics in hoping that its pretentious claims will not discourage other economic approaches, other contributions from neighboring disciplines, and concrete empirical investigations.

<sup>13</sup> See my "Evaluation of Real National Income," *Oxford Economic Papers*, N.S. II (January 1950), 18 ff. for analytic discussion of this important truth.