3.2 Operations that preserve convexity

In this section we describe some operations that preserve convexity or concavity of functions, or allow us to construct new convex and concave functions. We start with some simple operations such as addition, scaling, and pointwise supremum, and then describe some more sophisticated operations (some of which include the simple operations as special cases).

3.2.1 Nonnegative weighted sums

Evidently if f is a convex function and $\alpha \geq 0$, then the function αf is convex. If f_1 and f_2 are both convex functions, then so is their sum $f_1 + f_2$. Combining nonnegative scaling and addition, we see that the set of convex functions is itself a convex cone: a nonnegative weighted sum of convex functions,

$$f = w_1 f_1 + \cdots + w_m f_m,$$

is convex. Similarly, a nonnegative weighted sum of concave functions is concave. A nonnegative, nonzero weighted sum of strictly convex (concave) functions is strictly convex (concave).

These properties extend to infinite sums and integrals. For example if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, and $w(y) \geq 0$ for each $y \in \mathcal{A}$, then the function g defined as

$$g(x) = \int_{\mathcal{A}} w(y) f(x, y)\, dy$$

is convex in x (provided the integral exists).

The fact that convexity is preserved under nonnegative scaling and addition is easily verified directly, or can be seen in terms of the associated epigraphs. For example, if $w \geq 0$ and f is convex, we have

$$\mathbf{epi}(wf) = \begin{bmatrix} I & 0 \\ 0 & w \end{bmatrix} \mathbf{epi}\, f,$$

which is convex because the image of a convex set under a linear mapping is convex.

3.2.2 Composition with an affine mapping

Suppose $f : \mathbf{R}^n \to \mathbf{R}$, $A \in \mathbf{R}^{n \times m}$, and $b \in \mathbf{R}^n$. Define $g : \mathbf{R}^m \to \mathbf{R}$ by

$$g(x) = f(Ax + b),$$

with $\mathbf{dom}\, g = \{x \mid Ax + b \in \mathbf{dom}\, f\}$. Then if f is convex, so is g; if f is concave, so is g.

Scalar composition

We first consider the case $k = 1$, so $h : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$. We can restrict ourselves to the case $n = 1$ (since convexity is determined by the behavior of a function on arbitrary lines that intersect its domain).

To discover the composition rules, we start by assuming that h and g are twice differentiable, with $\mathbf{dom}\, g = \mathbf{dom}\, h = \mathbf{R}$. In this case, convexity of f reduces to $f'' \geq 0$ (meaning, $f''(x) \geq 0$ for all $x \in \mathbf{R}$).

The second derivative of the composition function $f = h \circ g$ is given by

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x). \tag{3.9}$$

Now suppose, for example, that g is convex (so $g'' \geq 0$) and h is convex and nondecreasing (so $h'' \geq 0$ and $h' \geq 0$). It follows from (3.9) that $f'' \geq 0$, *i.e.*, f is convex. In a similar way, the expression (3.9) gives the results:

$$\begin{array}{l} f \text{ is convex if } h \text{ is convex and nondecreasing, and } g \text{ is convex,} \\ f \text{ is convex if } h \text{ is convex and nonincreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave and nondecreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave and nonincreasing, and } g \text{ is convex.} \end{array} \tag{3.10}$$

These statements are valid when the functions g and h are twice differentiable and have domains that are all of $\mathbf{R}$. It turns out that very similar composition rules hold in the general case $n > 1$, without assuming differentiability of h and g, or that $\mathbf{dom}\, g = \mathbf{R}^n$ and $\mathbf{dom}\, h = \mathbf{R}$:

$$\begin{array}{l} f \text{ is convex if } h \text{ is convex, } \tilde{h} \text{ is nondecreasing, and } g \text{ is convex,} \\ f \text{ is convex if } h \text{ is convex, } \tilde{h} \text{ is nonincreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave, } \tilde{h} \text{ is nondecreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave, } \tilde{h} \text{ is nonincreasing, and } g \text{ is convex.} \end{array} \tag{3.11}$$

Here $\tilde{h}$ denotes the extended-value extension of the function h, which assigns the value ∞ ($-\infty$) to points not in $\mathbf{dom}\, h$ for h convex (concave). The only difference between these results, and the results in (3.10), is that we require that the *extended-value extension* function $\tilde{h}$ be nonincreasing or nondecreasing, on all of $\mathbf{R}$.

To understand what this means, suppose h is convex, so $\tilde{h}$ takes on the value ∞ outside $\mathbf{dom}\, h$. To say that $\tilde{h}$ is nondecreasing means that for *any* $x,\ y \in \mathbf{R}$, with $x < y$, we have $\tilde{h}(x) \leq \tilde{h}(y)$. In particular, this means that if $y \in \mathbf{dom}\, h$, then $x \in \mathbf{dom}\, h$. In other words, the domain of h extends infinitely in the negative direction; it is either $\mathbf{R}$, or an interval of the form $(-\infty, a)$ or $(-\infty, a]$. In a similar way, to say that h is convex and $\tilde{h}$ is nonincreasing means that h is nonincreasing and $\mathbf{dom}\, h$ extends infinitely in the positive direction. This is illustrated in figure 3.7.

Example 3.12 Some simple examples will illustrate the conditions on h that appear in the composition theorems.

- The function $h(x) = \log x$, with $\mathbf{dom}\, h = \mathbf{R}_{++}$, is concave and satisfies $\tilde{h}$ nondecreasing.

Putting (3.13) and (3.14) together, we have

$$h(g(\theta x + (1-\theta)y)) \leq \theta h(g(x)) + (1-\theta)h(g(y)).$$

which proves the composition theorem.

Example 3.13 *Simple composition results.*

- If g is convex then $\exp g(x)$ is convex.
- If g is concave and positive, then $\log g(x)$ is concave.
- If g is concave and positive, then $1/g(x)$ is convex.
- If g is convex and nonnegative and $p \geq 1$, then $g(x)^p$ is convex.
- If g is convex then $-\log(-g(x))$ is convex on $\{x \mid g(x) < 0\}$.

Remark 3.3 The requirement that monotonicity hold for the extended-value extension $\tilde{h}$, and not just the function h, cannot be removed. For example, consider the function $g(x) = x^2$, with $\mathbf{dom}\, g = \mathbf{R}$, and $h(x) = 0$, with $\mathbf{dom}\, h = [1, 2]$. Here g is convex, and h is convex and nondecreasing. But the function $f = h \circ g$, given by

$$f(x) = 0, \qquad \mathbf{dom}\, f = [-\sqrt{2}, -1] \;\cup\; [1, \sqrt{2}],$$

is not convex, since its domain is not convex. Here, of course, the function $\tilde{h}$ is *not* nondecreasing.

Vector composition

We now turn to the more complicated case when $k \geq 1$. Suppose

$$f(x) = h(g(x)) = h(g_1(x), \ldots, g_k(x)),$$

with $h : \mathbf{R}^k \to \mathbf{R}$, $g_i : \mathbf{R}^n \to \mathbf{R}$. Again without loss of generality we can assume $n = 1$. As in the case $k = 1$, we start by assuming the functions are twice differentiable, with $\mathbf{dom}\, g = \mathbf{R}$ and $\mathbf{dom}\, h = \mathbf{R}^k$, in order to discover the composition rules. We have

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x), \tag{3.15}$$

which is the vector analog of (3.9). Again the issue is to determine conditions under which $f(x)'' \geq 0$ for all x (or $f(x)'' \leq 0$ for all x for concavity). From (3.15) we can derive many rules, for example:

f is convex if h is convex, h is nondecreasing in each argument, and g_i are convex,

f is convex if h is convex, h is nonincreasing in each argument, and g_i are concave,

f is concave if h is concave, h is nondecreasing in each argument, and g_i are concave.

As in the scalar case, similar composition results hold in general, with $n > 1$, no assumption of differentiability of h or g, and general domains. For the general results, the monotonicity condition on h must hold for the extended-value extension $\tilde{h}$.

To understand the meaning of the condition that the extended-value extension $\tilde{h}$ be monotonic, we consider the case where $h : \mathbf{R}^k \to \mathbf{R}$ is convex, and $\tilde{h}$ nondecreasing, *i.e.*, whenever $u \preceq v$, we have $\tilde{h}(u) \leq \tilde{h}(v)$. This implies that if $v \in \mathbf{dom}\, h$, then so is u: the domain of h must extend infinitely in the $-\mathbf{R}^k_+$ directions. We can express this compactly as $\mathbf{dom}\, h - \mathbf{R}^k_+ = \mathbf{dom}\, h$.

Example 3.14 *Vector composition examples.*

- Let $h(z) = z_{[1]} + \cdots + z_{[r]}$, the sum of the r largest components of $z \in \mathbf{R}^k$. Then h is convex and nondecreasing in each argument. Suppose $g_1, \dots, g_k$ are convex functions on $\mathbf{R}^n$. Then the composition function $f = h \circ g$, *i.e.*, the pointwise sum of the r largest g_i's, is convex.
- The function $h(z) = \log(\sum_{i=1}^k e^{z_i})$ is convex and nondecreasing in each argument, so $\log(\sum_{i=1}^k e^{g_i})$ is convex whenever g_i are.
- For $0 < p \leq 1$, the function $h(z) = (\sum_{i=1}^k z_i^p)^{1/p}$ on $\mathbf{R}^k_+$ is concave, and its extension (which has the value $-\infty$ for $z \not\succeq 0$) is nondecreasing in each component. So if g_i are concave and nonnegative, we conclude that $f(x) = (\sum_{i=1}^k g_i(x)^p)^{1/p}$ is concave.
- Suppose $p \geq 1$, and $g_1, \dots, g_k$ are convex and nonnegative. Then the function $(\sum_{i=1}^k g_i(x)^p)^{1/p}$ is convex.

  To show this, we consider the function $h : \mathbf{R}^k \to \mathbf{R}$ defined as

$$h(z) = \left(\sum_{i=1}^k \max\{z_i, 0\}^p \right)^{1/p},$$

  with $\mathbf{dom}\, h = \mathbf{R}^k$, so $h = \tilde{h}$. This function is convex, and nondecreasing, so we conclude $h(g(x))$ is a convex function of x. For $z \succeq 0$, we have $h(z) = (\sum_{i=1}^k z_i^p)^{1/p}$, so our conclusion is that $(\sum_{i=1}^k g_i(x)^p)^{1/p}$ is convex.
- The geometric mean $h(z) = (\prod_{i=1}^k z_i)^{1/k}$ on $\mathbf{R}^k_+$ is concave and its extension is nondecreasing in each argument. It follows that if $g_1, \dots, g_k$ are nonnegative concave functions, then so is their geometric mean, $(\prod_{i=1}^k g_i)^{1/k}$.

3.2.5 Minimization

We have seen that the maximum or supremum of an arbitrary family of convex functions is convex. It turns out that some special forms of minimization also yield convex functions. If f is convex in (x, y), and C is a convex nonempty set, then the function

$$g(x) = \inf_{y \in C} f(x, y) \tag{3.16}$$