Student Information

Name : Batuhan Karaca

ID : 2310191

Answer 1

a)

Using the *Addition Rule*, one can calculate *marginal distributions* for corresponding variables. In the textbook page 45, formula of the *Addition Rule* is given

$$P_X(x) = P\{X = x\} = \sum_y P_{(X,Y)}(x, y)$$
$$P_Y(y) = P\{Y = y\} = \sum_x P_{(X,Y)}(x, y)$$

Adding *marginal probabilities*, the *joint pmf* becomes

$P_{(X,Y)}(x,y)$		x			$P_Y(y)$
		0	1	2	
y	0	1/12	4/12	1/12	6/12
	2	2/12	2/12	2/12	6/12
$P_X(x)$		3/12	6/12	3/12	12/12

Table 1: The joint probability table of discrete random variables X and Y, including marginal probabilities.

In the textbook page 47, formula of the *expectation* for discrete variables is given

$$\mu = E(X) = \sum_x xP(x)$$

Using the *marginal probabilities* in **Table 1**, we can compute

$$E(X) = (0 * \tfrac{3}{12}) + (1 * \tfrac{6}{12}) + (2 * \tfrac{3}{12}) = 1$$

In the textbook page 50, formula of the *variance* for discrete variables is given

$$\sigma^2 = Var(X) = E(X - EX)^2 = \sum_x (x - \mu)^2 P(x)$$

Similarly, using the *marginal probabilities*

$$Var(X) = ((0 - 1)^2 * \tfrac{3}{12}) + ((1 - 1)^2 * \tfrac{6}{12}) + ((2 - 1)^2 * \tfrac{3}{12}) = 0.5$$

b)

Assume $Z = X + Y$. Looking at the **Table 1**, we see that $Z \in \{0, 1, 2, 3, 4\}$ are the possible values. We see $Y \neq 1$, $Y \neq 3$ and $X \neq 3$ in the table. Using the values in **Table 1**

$P_Z(0) = P\{X = 0 \cap Y = 0\} = P_{(X,Y)}(0, 0) = \frac{1}{12} \sim 0.083$

$P_Z(1) = P\{X = 1 \cap Y = 0\} = P_{(X,Y)}(1, 0) = \frac{4}{12} \sim 0.334$

$P_Z(2) = P\{X = 0 \cap Y = 2\} + P\{X = 2 \cap Y = 0\} = P_{(X,Y)}(0, 2) + P_{(X,Y)}(2, 0) = \frac{2}{12} + \frac{1}{12} = 0.25$

$P_Z(3) = P\{X = 1 \cap Y = 2\} = P_{(X,Y)}(1, 2) = \frac{2}{12} \sim 0.167$

$P_Z(4) = P\{X = 2 \cap Y = 2\} = P_{(X,Y)}(2, 2) = \frac{2}{12} \sim 0.167$

c)

In the textbook page 51, formula of the *covariance* is given

$Cov(X, Y) = E\{(X - EX)(Y - EY)\} = E(XY) - E(X)E(Y)$

$E(X) = 1$ from **part a**. Similarly from **part a**

$E(Y) = 0 * \frac{6}{12} + 2 * \frac{6}{12} = 1$

By the textbook page 49

$$
\begin{aligned}
E(XY) &= \sum_x \sum_y (xy) P_{(X,Y)}(x,y) \\
&= \sum_x (x*0) P_{(X,Y)}(x,0) + (x*2) P_{(X,Y)}(x,2) \\
&= \sum_x 0 + (x*2) P_{(X,Y)}(x,2) \\
&= \sum_x (x*2) P_{(X,Y)}(x,2) \\
\\
&= (0*2) P_{(X,Y)}(0,2) + (1*2) P_{(X,Y)}(1,2) + (2*2) P_{(X,Y)}(2,2) \\
\\
&= 0 + (1*2) P_{(X,Y)}(1,2) + (2*2) P_{(X,Y)}(2,2) \\
\\
&= (1*2) P_{(X,Y)}(1,2) + (2*2) P_{(X,Y)}(2,2) \\
\\
&= 2P_{(X,Y)}(1,2) + 4P_{(X,Y)}(2,2) \\
\\
&= 2(\tfrac{2}{12}) + 4(\tfrac{2}{12}) = 1
\end{aligned}
$$

Then using the *formula of the covariance*

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 1 - 1*1 = 0$$

d)

By the textbook page 49,

For independent X and Y, $E(XY) = E(X)E(Y)$

The *formula of the covariance*

$$Cov(X,Y) = E(XY) - E(X)E(Y) = E(X)E(Y) - E(X)E(Y) = 0$$

Therefore, if X and Y are independent, then $Cov(X,Y) = 0$.

e)

Assume $Cov(X,Y) = 0$ for any random variables X, Y. Then,

$$
\begin{aligned}
&Cov(X,Y) = E(XY) - E(X)E(Y) = 0 \\
&E(XY) = E(X)E(Y)
\end{aligned}
$$

In **part d**, for independent X and Y, $E(XY) = E(X)E(Y)$. Therefore X and Y are independent. We showed that for any random variables X, Y, if $Cov(X, Y) = 0$, then X and Y are independent. In **part c** we found that $Cov(X, Y) = 0$, therefore, X and Y in the question are independent.

Answer 2

a)

In the textbook page 41, formula of the ***Cumulative Distribution Function (cdf)*** is given

$$F(x) = P\{X \leq x\} = \sum_{y \leq x} P(y)$$

In the textbook page 58, formula of the *Binomial probability mass function* is given as

$$P(x) = P\{X = x\} = \binom{n}{x} p^x q^{n-x}$$

which is the probability of exactly x successes in n trials. In this example, pens are tested, each pen corresponding to a *Bernoulli trial. Binomial Distribution* is a suitable approach. We choose X as our random variable, the amount of broken pens. In this case, probability of success, probability of getting a pen broken is $p = 0.2$. There are 12 pens tested ($n = 12$ trials). *Cdf* of X corresponds to the probability that at most k pens are broken. Then

$$P\{X \geq k\} = \sum_{y \geq k} P(y)$$

$$P\{X \geq k\} = \sum_{y} P(y) - \sum_{y \leq k-1} P(y)$$

$$P\{X \geq k\} = 1 - \sum_{y \leq k-1} P(y)$$

$$P\{X \geq k\} = 1 - F(k-1)$$

In our case

$$P\{X \geq 3\} = 1 - F(2)$$

We need to find *Cdf* of *Binomial distribution* for $x = 2$. Fortunately, by the *Cdf* table of *Binomial distribution* in the textbook page 413, We find $F(2) = 0.558$, for the values $p = 0.2$, $n = 12$, $x = 2$. Substituting gives $P\{X \geq 3\} = 1 - F(2) = 1 - 0.558 = 0.442$.

b)

In the question, it is asked that

P{5 pens will have to be tested to find 2 broken pens} *(expression 1)*

In the textbook page 63, formula of the *Negative Binomial probability mass function* is given

$P(x) = P\{X = x\} = \binom{x-1}{k-1} p^k q^{x-k}, \ x = k, k+1$

x-th trial results in the k-th success. We reach $k = 2$-nd broken pen, success, in 5-th pen we test, $x = 5$ trials. *Negative Binomial Distribution* suits the problem, by the *(expression 1)*. Substituting $x = 5$, $k = 2$

$P(x) = \binom{4}{1} p^2 q^3$

$P(x) = \binom{4}{1} (0.2)^2 (0.8)^3$

$P(x) = 4(2 * 10^{-1})^2 (8 * 10^{-1})^3$
$P(x) = 4(2 * 10^{-1})^2 (2^3 * 10^{-1})^3$
$P(x) = 4 * 2^2 * 10^{-2} * 2^9 * 10^{-3}$
$P(x) = 2^2 * 2^2 * 10^{-2} * 2^9 * 10^{-3}$
$P(x) = 2^{13} * 10^{-5} = 8192 * 10^{-5} = 0.08192$

is the answer.

c)

X is a random variable, the amount of tests k to find 4 broken pens. The statement, *number of tests k needed to find fixed number of* 4 *broken pens* has the *Negative Binomial Distribution*, because the statement is equivalent to $k = 4$*-th success occurs at x-th test*. In the question, The average number of tests is asked, which is the *expectation*, or *mean* value. In the textbook page 63, *expectation* formula for *Negative Binomial Bistribution* is given

$E(X) = \frac{k}{p}$ for k successes

In our case, $k = 4$. Furthermore, $p = 0.2$, from
part a, doesn't change. Substituting gives

$E(X) = \frac{4}{0.2} = 20$

Answer 3

In the textbook page 45, formula for the *Exponential Cdf* is given

$F(X) = 1 - e^{-\lambda x}, \ (x > 0)$

λ is the *frequency parameter*, measure of how many events occured in a unit of time. Sum of independent exponential variables has *Gamma Distribution*. Mathematically, $S_n = \sum_{i=1}^{n} X_i$, such that the set $\{X_i\}$ is a set of independent exponential random variables. Then, S_n ,in our case the time until the n-th call, is a *Gamma Distributed* random variable with number of steps $\alpha = n$ (α is the shape parameter).

a)

It is asked that Bob doesn't get a phone call for at least the first two hours, is the same as Bob gets his first call at a time $t > 2\ hours$. We need to find $P\{T > 2\ hours\}$. The time until the first call has exponential distribution. We know

$P\{T > 2\ hours\} = 1 - P\{T \leq 2\ hours\}$
$P\{T > 2\ hours\} = 1 - F(2)$
Using the formula for *Exponential Cdf*
$P\{T > 2\ hours\} = 1 - (1 - e^{-2\lambda})$
$P\{T > 2\ hours\} = e^{-2\lambda}$
Substituting $\lambda = 0.25$ (In our case Bob gets a call in 4 hours, 0.25 average calls in an hour)
$P\{T > 2\ hours\} = e^{-0.5} \sim 0.607$

Alternative solution
In addition to using calculators, we could use *Gamma-Poisson* formula given in page 87

$P\{T > t\} = P\{X < \alpha\}$
$P\{T < t\} = P\{X \geq \alpha\}$

X has *Poisson Distribution* with parameter λt, α is the number of steps, or phone calls, each taking an *Exponential* amount of time. *Exponential Distribution* is a specific form of the *Gamma Distribution* with $\alpha = 1$, then

$P\{T > 2\} = P\{X < 1\}$ with a new *frequency parameter* $\lambda t = 0.25 * 2 = 0.5$
$P\{T > 2\} = F(0)$ by the definition of *cdf*

We will use the *cdf* table of *Poisson Distribution* in textbook page 415. Our *frequency parameter* has changed to $\lambda = 0.5$, and we are searching for $F(0)$ $(x = 0)$. We find $F(0) = 0.607$ which is the answer.

b)

It is asked that

P {for the first 10 hours, Bob gets at most 3 phone calls}

Since, Bob is assumed to have at most 3 calls in 10 hours, then he only can get fourth call after the 10 hours of interval. Therefore, equivalently it is asked that

P {Bob gets his fourth phone call after the first 10 hours}
$P\{T > 10\}$ for $\alpha = 4$, corresponding to the fourth call
Using the *Gamma-Poisson* formula
$P\{X < \alpha = 4\}$, X is a *Poisson* variable
$F(3)$

$\lambda = 0.25$ by **part a**, and $t = 10$ *hours*. New *frequency parameter* is $\lambda t = 0.25 * 10 = 2.5$. Looking at the *cdf* table of *Poisson Distribution* in textbook page 415, with the values $x = 3$, $\lambda = 2.5$, we see the answer is 0.758.

c)

Let Y and Z be events

$Y =$*Bob does not get more than* 3 *phone calls for the first* 16 *hours*
$Z =$*Bob does not get more than* 3 *phone calls for the first* 10 *hours*

Let us find their equivalent events

$Y =$*Bob gets fourth call after the first* 16 *hours of interval*
$Z =$*Bob gets fourth call after the first* 10 *hours of interval*

The *conditional probability*

$P(Y \mid Z)$

is asked. Which is equivalently

$\frac{P(Y \ \cap \ Z)}{P(Z)}$

By the formula of *conditional probability* in the textbook page 27. We can deduce $Y \subseteq Z$, since for any time $t > 16$ that Bob gets fourth call, also $t > 10$. Then $Y \ \cap \ Z = Y$, we have

$P(Y \mid Z) = \frac{P(Y)}{P(Z)}$

Both probabilities have *gamma distribution*, since it is required we predict α-th call, $\alpha = 4$; and each call has an *exponential distribution*, given in the question.

Mathematically

$P(Y) = P\{T > 16\}$ for $\alpha = 4$
$P(Z) = P\{T > 10\}$ for $\alpha = 4$

Using the *Gamma-Poisson* formula (again), we have

$P(Y) = P\{X < 4\}$ for parameter $\lambda t = 0.25 * 16 = 4$
$P(Y) = F(3)$ for parameter $\lambda t = 0.25 * 16 = 4$
$P(Z) = P\{X < 4\}$ for parameter $\lambda t = 0.25 * 10 = 2.5$
$P(Z) = F(3)$ for parameter $\lambda t = 0.25 * 10 = 2.5$

Using the new parameters λt and $x = 3$, and the *cdf* table of *Poisson Distribution* in textbook page 415, we need to find each *cdf* value. However, we already found $P(Z) = F(3) = 0.758$ in **part b** ($\lambda = 2.5$). For $\lambda = 4$ and $x = 3$ in the table, we find $P(Y) = F(3) = 0.433$. With the aid of a calculator, we find the answer

$P(Y \mid Z) = \frac{P(Y)}{P(Z)} = \frac{0.433}{0.758} \sim 0.571$