Student Information

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Answer 1

a)

We are given the information that the selected box is X. This means we could only choose a ball from X. This reduces, or restricts the sample space to the set of the balls in X. There are 2 *green* balls in X, 6 balls in total. Then

$|\Omega_X|$: Total number of balls in X (reduced sample space)

$|green_X|$: Total number of *green* balls in X

$P(green \mid X) = \frac{|green_X|}{|\Omega_X|} = \frac{2}{6} = \frac{1}{3} \sim \underline{\underline{0.33}}$

b)

In the textbook page 30, ***Law of Total Probability*** is given

> *Consider some partition of the sample space Ω with mutually exclusive and exhaustive events $B_1, ..., B_k$. It means that*
>
> $B_i \cap B_j = \emptyset$ for any $i \neq j$ and $B1 \cup ... \cup B_k = \Omega$.
>
> *These events also partition the event A,*
>
> $A = (A \cap B_1) \cup ... \cup (A \cap B_k)$,
>
> *and this is also a union of mutually exclusive events [...]. Hence,*
>
> $$P(A) = \sum_{j=1}^{k} P\{A \cap B\}$$

Picking a *red* ball from either X, or Y are disjoint events , since set of *red* balls in X and Y are disjoint. In this case, A corresponds to the set of *red* balls, denoted as *red*. By ***Law of Total Probability***

$P(red) = P\{red \cap X\} + P\{red \cap Y\} \quad (1)$

In the textbook page 27, formula of the *conditional probability* is given

$$P\{A \mid B\} = \frac{P\{A \cap B\}}{P(B)}$$
$$P\{A \cap B\} = P(A|B) * P(B)$$

Rewriting the expression (1), in terms of *conditional probabilities*,

$$P(red) = P\{red \mid X\} * P(X) + P\{red \mid Y\} * P(Y)$$

Using the methodology in **part a**, we can calculate the *conditional probabilities* as $P\{red \mid X\} = 1/3 \sim 0.3$,$P\{red \mid Y\} = 1/5 = 0.2$. We know $P(X) = 0.4$. Choosing either X or Y, are exhaustive and disjoint events. Therefore, $P(X) + P(Y) = 1$, hence $P(Y) = 0.6$. Substituting gives $P(red) \sim \underline{\underline{0.25}}$

c)

By ***Bayes' Rule***

$$P(Y \mid blue) = \frac{P(blue \mid Y) * P(Y)}{P(blue)}$$

Using the methodology in **part a**, we can calculate $P\{blue|Y\} = 2/5 = 0.4$. . We know $P(Y) = 0.6$ from **part b**. Furthermore, using ***Law of Total Probability***, as done in **part b**, we can calculate $P(blue) = P\{blue|X\} * P(X) + P\{blue|Y\} * P(Y) = (1/3)(0.4) + (0.4)(0.6) \sim 0.37$. Then $P(Y|blue) \sim \underline{\underline{0.64}}$

Answer 2

Sample space will be denoted Ω. ***Mutually exhaustive*** events are also called ***disjoint*** events.

Events $A_1, A_2...A_k$ are disjoint <u>if and only if</u> the sets $A_i \cap A_j = \emptyset$ for each and every pair
Events $A_1, A_2...A_k$ are exhaustive <u>if and only if</u> the sets $A_1 \cup A_2 \cup ... \cup A_k = \Omega$

a)

By de Morgan's laws we have $\overline{A} \cup \overline{B} = \overline{(A \cap B)}$. We also know $\overline{\emptyset} = \Omega$ and $\overline{\overline{\emptyset}} = \emptyset = \overline{\Omega}$.

1)If A and B are disjoint, then

$$A \cap B = \emptyset$$
$$\overline{A} \cup \overline{B} = \overline{(A \cap B)} = \overline{\emptyset} = \Omega$$

The sets $\overline{A}$ and $\overline{B}$ are exhaustive.

2) If $\overline{A}$ and $\overline{B}$ are exhaustive, then

$$\overline{A} \cup \overline{B} = \Omega$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B} = \Omega$$
$$A \cap B = \overline{(\overline{A} \cup \overline{B})} = \emptyset$$

the sets A and B are disjoint completing the proof.

b)

By de Morgan's laws we have $\overline{A} \cup \overline{B} \cup \overline{C} = \overline{(A \cap B \cap C)}$. We also know $\overline{\emptyset} = \Omega$ and $\overline{\overline{\emptyset}} = \emptyset = \overline{\Omega}$.

1) If A,B and C are disjoint, then

$$A \cap B \cap C = \emptyset$$
$$\overline{A} \cup \overline{B} \cup \overline{C} = \overline{(A \cap B \cap C)} = \overline{\emptyset} = \Omega$$

The sets $\overline{A}$,$\overline{B}$ and $\overline{C}$ are exhaustive.

2) If $\overline{A}$,$\overline{B}$ and $\overline{C}$ are exhaustive, then

$$\overline{A} \cup \overline{B} \cup \overline{C} = \Omega$$
$$\overline{(A \cap B \cap C)} = \overline{A} \cup \overline{B} \cup \overline{C} = \Omega$$
$$A \cap B \cap C = \overline{(\overline{A} \cup \overline{B} \cup \overline{C})} = \emptyset$$

the sets A, B and C are disjoint completing the proof.

Answer 3

a)

In the textbook page 58, formula of the *Binomial probability mass function* is given as

$$P(x) = P\{X = x\} = \tbinom{n}{x} p^x q^{n-x}$$

which is the probability of exactly x successes in n trials. In this example, there are 5 dice thrown, each die is being a ***Bernoulli trial***. This is a sequence of **Bernoulli trials**. Therefore, we can use ***Binomial Distribution***. In this case, the probability of success p is ,2 outcomes (5 and 6) out of 6 outcomes in total, $\frac{1}{3}$. q is the complement of p, as both events(getting success and failure) are disjoint and exhaustive. $q = 1 - p = \frac{2}{3}$. There are 5 trials ($n = 5$) and exactly 2 successful outcomes ($x = 2$). Subsituting gives

$$P(x) = P\{X = x\} = \tbinom{5}{2} (\tfrac{1}{3})^2 (\tfrac{2}{3})^3 \sim \underline{\underline{0.33}}$$

b)

In the textbook page 41, formula of the ***Cumulative Distribution Function (cdf)*** is given

$$F(x) = P\{X \le x\} = \sum_{y \le x} P(y)$$

Probability of having exactly k succesful dice is $P(k)$, having less than,or equal to k succesful dice is $P\{X \le k\} = \sum_{y \le k} P(y)$, having at least k succesful dice is $P\{X \ge k\} = \sum_{y \ge k} P(y)$. Since for exhaustive events, probability sum of the events in the sample space is $\sum_{y} P(y) = 1$, we have

$$P\{X \ge k\} = \sum_{y \ge k} P(y)$$

$$P\{X \ge k\} = \sum_{y} P(y) - \sum_{y \le k-1} P(y)$$

$$P\{X \ge k\} = 1 - \sum_{y \le k-1} P(y)$$

$$P\{X \ge k\} = 1 - F(k-1)$$

By the definition of *cdf*. Substituting gives

$$P\{X \ge 2\} = 1 - F(1) = 1 - \sum_{y \le 1} P(y) = 1 - (P(0) + P(1))$$

$P(0) = \binom{5}{0}(\frac{1}{3})^0(\frac{2}{3})^5 \sim 0.13$ and $P(1) = \binom{5}{1}(\frac{1}{3})^1(\frac{2}{3})^4 \sim 0.33$ as they can be calculated by using the *Binomial Distribution.*

$$P\{X \ge 2\} = 1 - (P(0) + P(1)) = 1 - ((\tfrac{1}{3})^0(\tfrac{2}{3})^6 + \binom{6}{1}(\tfrac{1}{3})^1(\tfrac{2}{3})^5) \sim \underline{\underline{0.54}}$$

is the answer.

Answer 4

a)

In the textbook page 45, formula of the *Addition Rule* is given

$$P_X(x) = P\{X = x\} = \sum_{y} P_{(X,Y)}(x, y)$$
$$P_Y(y) = P\{Y = y\} = \sum_{x} P_{(X,Y)}(x, y)$$

In this case, we have 3 variables. Similarly,

$$P\{X = x, Y = y\} = \sum_{z} P_{(X,Y,Z)}(x, y, z) \quad (2)$$

For any random variable X, Y and Z, numbers x and y. Then

$$P\{A=1, C=0\} = \sum_b P_{(A,B,C)}(1,b,0) = P_{(A,B,C)}(1,0,0) + P_{(A,B,C)}(1,1,0) = 0.06 + 0.09 = \underline{\underline{0.15}}$$

b)

Similar to the previus part (*part a*), however in this case, 2 variables are unknown in 3 variables

$$P\{X=x\} = \sum_y \sum_z P_{(X,Y,Z)}(x,y,z)$$

We have $B = 1$

$$\begin{aligned}
P\{B=1\} &= \sum_a \sum_c P_{(A,B,C)}(a,1,c) \\
&= \sum_a (\ P_{(A,B,C)}(a,1,0) + P_{(A,B,C)}(a,1,1)\ ) \\
&= P_{(A,B,C)}(0,1,0) + P_{(A,B,C)}(1,1,0) + P_{(A,B,C)}(0,1,1) + P_{(A,B,C)}(1,1,1) \\
&= 0.21 + 0.09 + 0.02 + 0.08 \\
&= 0.4
\end{aligned}$$

c)

In the textbook page 45, ***Independence of Random Variables*** is defined

Random variables X and Y are independent if
$P_{(X,Y)}(x,y) = P_X(x)P_Y(y)$

Marginal Distribution of A and B

$$
\begin{aligned}
P_A(a) = P\{A = a\} &= \sum_b \sum_c P_{(A,B,C)}(a, b, c) \\
P_A(0) = P\{A = 0\} &= \sum_b \sum_c P_{(A,B,C)}(0, b, c) \\
&= \sum_b (\ P_{(A,B,C)}(0, b, 0) + P_{(A,B,C)}(0, b, 1)\ ) \\
&= P_{(A,B,C)}(0, 0, 0) + P_{(A,B,C)}(0, 1, 0) + P_{(A,B,C)}(0, 0, 1) + P_{(A,B,C)}(0, 1, 1) \\
&= 0.14 + 0.21 + 0.08 + 0.02 \\
&= 0.45 \\
P_B(b) = P\{B = b\} &= \sum_a \sum_c P_{(A,B,C)}(a, b, c) \\
P_B(1) = P\{B = 1\} &= \sum_a \sum_c P_{(A,B,C)}(a, 1, c) \\
&= \sum_a (\ P_{(A,B,C)}(a, 1, 0) + P_{(A,B,C)}(a, 1, 1)\ ) \\
&= P_{(A,B,C)}(0, 1, 0) + P_{(A,B,C)}(1, 1, 0) + P_{(A,B,C)}(0, 1, 1) + P_{(A,B,C)}(1, 1, 1) \\
&= 0.21 + 0.09 + 0.02 + 0.08 \\
&= 0.40
\end{aligned}
$$

Using (2) in ***part a***,

$$
\begin{aligned}
P_{(A,B)}(0, 1) = P\{A = 0, B = 1\} &= \sum_c P_{(A,B,C)}(0, 1, c) \\
&= P_{(A,B,C)}(0, 1, 0) + P_{(A,B,C)}(0, 1, 1) \\
&= 0.21 + 0.02 \\
&= 0.23
\end{aligned}
$$

We see that $P_{(A,B)}(0, 1) \neq P_A(0) * P_B(1)$, which contradicts the ***Independence of Random Variables*** argument. Hence by contradiction, A and B are not independent.

d)

Definition of the ***Conditional Independence*** (from Wikipedia, refer here)

In the standard notation of probability theory, A and B are conditionally independent given C if and only if
$P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$

Expanding both sides seperately, Using the formula of *conditional probability*

$$P(A \cap B \mid C) = \frac{P((A \cap B) \cap C)}{P(C)}$$

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$$

$$P(B \mid C) = \frac{P(B \cap C)}{P(C)}$$

Then, A and B are conditionally independent given C if and only if

$$P((A \cap B) \cap C) = \frac{P(A \cap C)P(B \cap C)}{P(C)}$$

In this case, $C = 1$ given

$$P\{(A \cap B) \cap (C = 1)\} = \frac{P\{A \cap (C=1)\}P\{B \cap (C=1)\}}{P\{C=1\}}$$

By the *Addition Rule*

$$\begin{aligned} P\{C = 1\} \quad &= \sum_a \sum_b P_{(A,B,C)}(a, b, 1) \\ &= \sum_a P_{(A,B,C)}(a, 0, 1) + P_{(A,B,C)}(a, 1, 1) \\ &= P_{(A,B,C)}(0, 0, 1) + P_{(A,B,C)}(1, 0, 1) + P_{(A,B,C)}(0, 1, 1) + P_{(A,B,C)}(1, 1, 1) \\ &= 0.08 + 0.32 + 0.02 + 0.08 \\ &= 0.50 \end{aligned}$$

$$P\{(A \cap B) \cap (C = 1)\} = \frac{P\{A \cap (C=1)\}P\{B \cap (C=1)\}}{0.50}$$

Since we have a joint distribution, we can alter the expression

$$P_{(A,B,C)}(a, b, 1) = \frac{P_{(A,C)}(a,1)P_{(B,C)}(b,1)}{0.50} \quad (3)$$

For all $a, b \in \{0, 1\}$. Using the *Addition Rule*

$P_{(A,C)}(0,1) = 0.08 + 0.02 = 0.10$
$P_{(A,C)}(1,1) = 0.32 + 0.08 = 0.40$
$P_{(B,C)}(0,1) = 0.08 + 0.32 = 0.40$
$P_{(B,C)}(1,1) = 0.02 + 0.08 = 0.10$

Left hand sides (corresponding values from the table in the question)

$P_{(A,B,C)}(0,0,1) = 0.08$
$P_{(A,B,C)}(0,1,1) = 0.02$
$P_{(A,B,C)}(1,0,1) = 0.32$
$P_{(A,B,C)}(1,1,1) = 0.08$

Right hand sides

$\frac{P_{(A,C)}(0,1)P_{(B,C)}(0,1)}{0.50} = 0.08$

$\frac{P_{(A,C)}(0,1)P_{(B,C)}(1,1)}{0.50} = 0.02$

$\frac{P_{(A,C)}(1,1)P_{(B,C)}(0,1)}{0.50} = 0.32$

$\frac{P_{(A,C)}(1,1)P_{(B,C)}(1,1)}{0.50} = 0.08$

Checking the equality(3)

$P_{(A,B,C)}(0,0,1) = \frac{P_{(A,C)}(0,1)P_{(B,C)}(0,1)}{0.50} = 0.08$

$P_{(A,B,C)}(0,1,1) = \frac{P_{(A,C)}(0,1)P_{(B,C)}(1,1)}{0.50} = 0.02$

$P_{(A,B,C)}(1,0,1) = \frac{P_{(A,C)}(1,1)P_{(B,C)}(0,1)}{0.50} = 0.32$

$P_{(A,B,C)}(1,1,1) = \frac{P_{(A,C)}(1,1)P_{(B,C)}(1,1)}{0.50} = 0.08$

As seen, both sides are equal for all permutations of a and b. The equality (3) holds for all $a, b \in \{0, 1\}$. Therefore given $C = 1$, A and B are conditionally independent.