Exam 1 Answers: Logic and Proof

September 17, 2012

Instructions:

Please answer each question completely, and show all of your work. Partial credit will be awarded where appropriate. Justify all of your decisions as clearly as possible. The more work you show the easier it will be to assign partial credit.

If you have any questions you may raise your hand and I will come talk to you. I may not be able to answer all questions. I cannot tell you whether your answer is correct or in the right direction.

After you are done, please hand in all pages of the exam. This is closed book, closed notes test. No calculators, cell phones, PDA's or laptops allowed during the test. No collaborations with your peers.

All the best.

Name:

Double negation	$\neg(\neg p) \equiv p$
Excluded middle	$p \vee \neg p \equiv True$
Contradiction	$p \wedge \neg p \equiv False$
Identity laws	$True \wedge p \equiv p$ $False \vee p \equiv p$
Idempotent laws	$p \wedge p \equiv p$ $p \vee p \equiv p$
Commutative laws	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
DeMorgan's laws	$\neg(p \wedge q) \equiv (\neg p) \vee (\wedge q)$ $\neg(p \vee q) \equiv (\neg p) \wedge (\wedge q)$
Conjunction introduction.	If you know A, and you know B, you can conclude $A \wedge B$.
Conjunction elimination	If you know $A \wedge B$, then A.
Negation elimination	If $\neg(\neg A)$ is true, then so is A.
Negation introduction	If you assumed A, and you arrived at a contradiction, then you can say $\neg A$ is true.
Disjunction introduction	If A is true, then $A \vee B$ is true.
Disjunction elimination	If you know $A \vee B$, and you know B is false, then A is true.
Implication introduction	If you assume A, and arrive at B, you can conclude $A \rightarrow B$.
Modus Ponens	If you know $A \rightarrow B$ is true, and you know A is true, you can conclude B.
Modus Tollens	If you know $A \rightarrow B$, and you know $\neg B$, you can conclude $\neg A$.
Law of syllogism	If you know $A \rightarrow B$, and you know $B \rightarrow C$, you can conclude $A \rightarrow C$.
Universal elimination	If you know $\forall x P(x)$, you can conclude $P(c)$ for an arbitrary c in the universe of discourse.
Universal introduction	If you know $P(c)$ for an arbitrary c in the universe of discourse, you can conclude $\forall x P(x)$.
Existential introduction	If you know $P(c)$ for a specific c in the universe of discourse, you can conclude $\exists x P(x)$.
Existential elimination	If you know $\exists x P(x)$, you can conclude $P(c)$ for a specific c in the universe of discourse.

Table 0.1: Some logical equivalences and inference rules.

1. (8 pts total, 4 each) Let S = it is sunny, C = camping is fun, H = the homework is done, and M = mathematics is easy.

a. Translate the following proposition into the most natural equivalent statement in English. Try to make the sentence as simple and as natural as possible.

$(M \to H) \land (S \to C)$

Any of the following solutions (and similar) are equally valid: "The homework is done when mathematics is easy, and camping is fun when it is sunny." "If mathematics is easy then the homework is done, and if it is sunny then camping is fun."

b. Translate the following statement into propositional logic.

"Mathematics is easy or camping is fun, as long as it is sunny and the homework is done."

Either of these two solutions (and logically equivalents) are valid:

$(S \land H) \to (M \lor C)$

and

$(S \land H) \leftrightarrow (M \lor C)$

The following solution received a one point deduction:

$(M \lor C) \to (S \land H)$

All other answers received 2 or more point deductions.

2. (12 pts total, 6 each) For the following proposition, indicate whether it is a tautology, a contradiction, or neither. Use a truth table to decide.

Here the points were distributed as follows: 1 point for setting up the rows of the table appropriately (all combinations of true and false for A and B); 1 point for setting up the columns of the table appropriately (dividing the complex proposition into identifiable units; 2 points for getting the logic definitions for the operators correctly; 2 points for knowing what a tautology or a contradiction are.

a. $((A \to B) \wedge (B \to \neg A)) \to A$

A	B	$(A \to B)$	$\neg A$	$(B \to \neg A)$	$((A \to B) \wedge (B \to \neg A))$	$((A \to B) \wedge (B \to \neg A)) \to A$
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	F
F	F	T	T	T	T	F

Neither: when A is true and B is true, the statement is true; when A and B are both false, the statement is false.

b. $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

A	B	$\neg B$	$\neg A$	$(\neg B \to \neg A)$	$(\neg B \to A)$	$((\neg B \to A) \to B)$	$(\neg B \to \neg A) \to ((\neg B \to A) \to B)$
T	T	F	F	T	T	T	T
T	F	T	F	F	T	F	T
F	T	F	T	T	T	T	T
F	F	T	T	T	F	T	T

Tautology: for all possibilities of A and B, the statement is true.

3. (6 pts total, 3 each) Give the converse and the contrapositive of the following sentence.

Here the points were distributed as follows: 1 point for translating the sentence into logic correctly; 2 points for showing knowledge of the concept of converse and contradiction, regardless of the translation.

"I drive if it is too dangerous or too far to bike."

The key to translating this sentence was to realize it says: $p \to q$, where p represents 'it is too dangerous or too far to bike,' and q represents 'I drive.'

a. Converse:

The converse is $q \to p$. In English, that is: "If I drive, then it is too dangerous or too far to bike." It could also be said as: "It is too dangerous or too far to bike, if I drive."

b. Contrapositive:

The contrapositive is $\neg q \to \neg p$. In English, that is: "If I don't drive, then it is not too dangerous and not too far to bike." It could also be said as: "If I don't drive, then it is not too dangerous or too far to bike.' One final way it could be translated as: "It is not too dangerous or too far to bike, if I don't drive.'

4. (20 pts total, 10 each) For each of the following pairs of propositions, show that the two propositions are logically equivalent by finding a chain of equivalences from one to the other. State which definition or law of logic justifies each equivalence in the chain.

Full points were given to those who showed the two propositions were logically equivalent. Within that context, full proofs skipping a step, misusing a law, or labeling a law incorrectly were deducted only one point, sometimes none. Leaving substantial holes in the proof led to 2 or more points being deducted. Solutions that attempted to infer one proposition from the other one (which is only half of what is being ask for in a logical equivalence) were evaluated out of a total of 6 points maximum. Similarly, mistakes within the proof of inference were deducted from the total maximum.

a. $(p \wedge q) \rightarrow r$, $p \rightarrow (q \rightarrow r)$

$$
\begin{aligned}
(p \wedge q) \rightarrow r &\equiv \neg(p \wedge q) \rightarrow r && \text{Definition of the conditional} \\
&\equiv (\neg p \vee \neg q) \vee r && \text{DeMorgan's law} \\
&\equiv \neg p \vee (\neg q \vee r) && \text{Associative law} \\
&\equiv \neg p \vee (q \rightarrow r) && \text{Definition of the conditional} \\
&\equiv p \rightarrow (q \rightarrow r) && \text{Definition of the conditional}
\end{aligned}
$$

b. $(q \vee r) \rightarrow p$, $(q \rightarrow p) \wedge (r \rightarrow p)$

$$
\begin{aligned}
(q \vee r) \rightarrow p &\equiv \neg(q \vee r) \vee p && \text{Definition of the conditional} \\
&\equiv (\neg q \wedge \neg r) \vee p && \text{DeMorgan's law} \\
&\equiv (\neg q \vee p) \wedge (\neg r \vee p) && \text{Distributive law} \\
&\equiv (q \rightarrow p) \wedge (\neg r \vee p) && \text{Definition of the conditional} \\
&\equiv (q \rightarrow p) \wedge (r \rightarrow p) && \text{Definition of the conditional}
\end{aligned}
$$

5. (18 pts total, 6 each) Let Loves(x,y) mean "x loves y," Traveler(x) mean "x is a traveler," City(x) mean "x is a city," Lives(x,y) mean "x lives in y."

Half of the points were given for any attempted translation. One point was deducted when the proposition in logic was not well constructed (e.g., quantifiers missing). One point was deducted when the meaning of the translation was only slightly off. Two points where deducted when the translation was more significantly off.

a. Translate the following proposition into the most natural equivalent statement in English. Try to make the sentence as simple and as natural as possible.

$$\exists x \forall y \forall z (City(x) \land Traveler(y) \land Lives(z,x)) \rightarrow (Loves(y,x) \land \neg Loves(z,x))$$

A few of the translations that work here: "There is a city that all travelers love but everyone who lives there doesn't love." Also: "Some cities are loved by all travelers but no person who lives there." Many other translations within that context work well.

b. Translate the following statement into predicate logic.

"No traveler loves the city they live in."

Either of these two translations worked well:

$\forall x \forall y ((Traveler(x) \land City(y) \land Live(x,y)) \rightarrow \neg Love(x,y))$

$\forall x \exists y ((Traveler(x) \land City(y) \land Live(x,y)) \rightarrow \neg Love(x,y))$

c. Translate the following statement into predicate logic.

"Travelers love at most two cities."

The important thing in this translation is to make sure that the traveler can love 0, 1, or 2 cities, but no more. Given that no solution accounted for 0, we are not penalizing solutions that just state 1 or 2, but not 3. Solutions that translated something other than that, but within the ballpark of 2 cities were given 5 points. Solutions not quantifying the amount of cities were given 4 or less.

This is one way to allow for 0, or 1 or 2 and not 3. We can say that for anyone, if they are a traveler, then one of two things must be true. First, it could be the case that the traveler loves no cities. Second, if there are two cities that the traveler loves (and there's no restriction on whether those two cities are the same or not, so they could be the same or different - hence accounting for one or two), then for all other cities, if he loves them, then that city must be one of the first two. I simplify the predicates to T(x) for "x is a traveler," C(x) for "x is a city," and L(x,y) for "x loves y."

$$\begin{aligned}
&\forall x (T(x) \rightarrow \\
&\quad (\forall d (C(d) \rightarrow \neg L(x,d))\lor \\
&\quad (\exists a \exists b \forall c (C(a) \land L(x,a) \land C(b) \land L(x,b) \land C(c) \land L(x,c)) \rightarrow ((a = c) \lor (b = c)))))
\end{aligned}$$

6. (20 pts total, 10 each) Decide whether the following argument is valid. If it is valid argument, give a formal proof (i.e., justify which laws of logic need to be applied to each of the premises, in a sequence of arguments, to arrive at the conclusion). If the argument is invalid (i.e., you cannot find a sequence of laws that leads from the premises to the conclusion), show that it is invalid by finding an appropriate assignment of truth values to the propositional variables.

Full points for arriving at the conclusion from the premises using valid rules of inference. Full proofs skipping a step or mislabeling a law were not deducted any points. Misusing a rule of inference led to a point deduction. Not arriving at the conclusion, but still showing correct understanding of the method of deduction and the use of rules of inference, led to 3 point deduction. Entirely arbitrary methods of proof were evaluated out of 6 points.

a. Premises: $p \to (q \wedge r)$, $s \to r$, $r \to p$. Prove: $s \to q$.

1.	$p \to (q \wedge r)$	Premise
2.	$s \to r$	Premise
3.	$r \to p$	Premise
4.	s	Assumption
5.	r	Modus Ponens, 2,4
6.	p	Modus Ponens, 3,5
7.	$(q \wedge r)$	Modus Ponens, 1,6
8.	q	Conjunction elimination, 7
9.	$s \to q$	Implication introduction, 4-8

b. Premises: $\neg(r \vee s)$, $\neg p \to s$, $p \to q$. Prove: q.

1.	$\neg(r \vee s)$	Premise
2.	$\neg p \to s$	Premise
3.	$p \to q$	Premise
4.	$\neg r \wedge \neg s$	DeMorgan's law, 1
5.	$\neg s$	Conjunction elimination, 4
6.	$\neg(\neg p)$	Modus Tollens, 5,2
7.	p	Double negation, 6
8.	q	Modus Ponens, 7,3

7. (16 pts total) Transform the informal argument below into predicate logic (6 pts). Then give a formal proof (10 pts).

For this exercise, we are dividing the points into four sections: 3 points for defining the predicates appropriately, 3 points for translating the sentences into predicate logic correctly, 5 points for eliminating and re-introducing the quantifiers correctly, and 5 points for the proof. Well-thought proofs using unique variables instead of predicates were evaluated out of 10 points.

> "All cats are liked by some dogs. No dog likes a socialist. Therefore, no cat is a socialist."

Definitions of the predicates:
C(x) = "x is a cat."
D(x) = "x is a dog."
S(x) = "x is a socialist."
L(x,y) = "x likes y."

Translations of the sentences:
All cats are liked by some dogs: $\forall x \exists y (C(x) \rightarrow (D(y) \wedge L(y,x)))$
No dog likes a socialist: $\forall x \forall y ((D(x) \wedge S(y)) \rightarrow \neg L(x,y)$
Therefore, no cat is a socialist: $\forall x (C(x) \rightarrow \neg S(x))$

Proof:

1.	$\forall x \exists y (C(x) \rightarrow (D(y) \wedge L(y,x)))$	Premise
2.	$\forall x \forall y ((D(x) \wedge S(y)) \rightarrow \neg L(x,y)$	Premise
3.	$\forall x (C(x) \rightarrow (D(a) \wedge L(a,x)))$	Existential elimination, 1
4.	$C(b) \rightarrow (D(a) \wedge L(a,b))$	Universal elimination, 1
5.	$\forall y ((D(a) \wedge S(y)) \rightarrow \neg L(a,y))$	Universal elimination, 2
6.	$(D(a) \wedge S(b)) \rightarrow \neg L(a,b)$	Universal elimination, 2
7.	$C(b)$	Assumption
8.	$D(a) \wedge L(a,b)$	Modus Ponens, 4,7
9.	$L(a,b)$	Conjunction elimination, 8
10.	$\neg(D(a) \wedge S(b))$	Modus Tollens, 6,9
11.	$\neg D(a) \vee \neg S(b)$	DeMorgan's law, 10
12.	$D(a)$	Conjunction elimination, 8
13.	$\neg S(b)$	Disjunction elimination, 11,12
14.	$C(b) \rightarrow \neg S(b)$	Implication introduction, 7-13
15.	$\forall x (C(x) \rightarrow \neg S(x))$	Universal introduction, 14

Therefore, the argument is valid.