Project 2: Ocean Volume

MATH 3315.803 Scientific Computing

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[redacted]

November 25, 2019

$$L_{ij} = A + B(x - x_i) + C(y - y_j) + D(x - x_i)(y - y_j)$$

The linear interpolating polynomial $L(x, y)$ approximates $f(x, y)$ between explicitly calculated values $f(x_i, y_j)$. We can solve for the coefficients of L_{ij} within a certain rectangular region by substituting L_{ij}, x, y for our calculated $f(x_i, y_j), x_i, y_j$.

$$f(x_i, y_j) = A + B(x_i - x_i) + C(y_j - y_j) + D(x_i - x_i)(y_j - y_j)$$

$$f(x_{i+1}, y_j) = A + B(x_{i+1} - x_i) + C(y_j - y_j) + D(x_{i+1} - x_i)(y_j - y_j)$$

$$f(x_i, y_{j+1}) = A + B(x_i - x_i) + C(y_{j+1} - y_j) + D(x_i - x_i)(y_{j+1} - y_j)$$

$$f(x_{i+1}, y_{j+1}) = A + B(x_{i+1} - x_i) + C(y_{j+1} - y_j) + D(x_{i+1} - x_i)(y_{j+1} - y_j)$$

We now have equations at the corners of the rectangle which can be used to solve for the unknown coefficients A, B, C, D via a system of equations. First, we can simplify and substitute $h = x_{i+1} - x_i$ and $k = y_{j+1} - y_j$.

$$f(x_i, y_j) = A$$

$$f(x_{i+1}, y_j) = A + Bh$$

$$f(x_i, y_{j+1}) = A + Ck$$

$$f(x_{i+1}, y_{j+1}) = A + Bh + Ck + Dhk$$

Then use substitution and solve.

$$A = f(x_i, y_j)$$

$$B = \frac{1}{h}\left(f(x_{i+1}, y_j) - f(x_i, y_j)\right)$$

$$C = \frac{1}{k}\left(f(x_i, y_{j+1}) - f(x_i, y_j)\right)$$

$$D = \frac{1}{hk}\left(f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)\right)$$

And thus we can unambiguously represent L_{ij} for any given rectangular area. Double integrating L_{ij} over the length and width gives the volume spanned by the rectangle.

$$V_{ij} = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} L_{ij}\, dxdy = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} \left(A + B(x - x_i) + C(y - y_j) + D(x - x_i)(y - y_j)\right) dxdy$$

$$= \iint A dxdy + \iint B(x - x_i) dxdy + \iint C(y - y_j) dxdy + \iint D(x - x_i)(y - y_j) dxdy$$

$$= Ahk + \frac{1}{2}Bh^2k + \frac{1}{2}Chk^2 + \frac{1}{4}Dh^2k^2$$

Substitute for our solved coefficients A, B, C, D.

$$= f(x_i, y_j)hk + \frac{h^2k}{2h}\Big(f(x_{i+1}, y_j) - f(x_i, y_j)\Big) + \frac{hk^2}{2k}\Big(f(x_i, y_{j+1}) - f(x_i, y_j)\Big) + \frac{1}{4}hk\Big(f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j)\Big)$$

Simplify to acquire a 2D variant of the trapezoid rule.

$$V_{ij} = \frac{1}{4}hk\Big(f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_j)\Big)$$

A global equation for summing the rectangular volumes across a plane can be found by iterating through values of i and j. Note that this assumes h and k are a constant spacing between every point, which is not always the case for real-world data.

$$V = \frac{1}{4}hk\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\Big(f(x_{i+1}, y_{j+1}) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_i, y_j)\Big)$$

The application of this equation approximates the volume of water contained by a set of points mapping the depth of the ocean in a rectangular region. The region's opposing corners are defined by the points 19.874192690735587°S 80.5693359375°E and 24.134722990415206°S 88.48828125°E. The calculation yields an estimate of 1,699,000 cubic kilometers of water. A sanity check to ensure this answer is of the right magnitude is done by performing a 2D trapezoid rule on the region as a whole, with h and k equal to the length width of the entire region. We take the convert the degree differences in longitude and latitude to find values of h and k to work with.

$$h = 88.48828125°\text{E} - 80.5693359375°\text{E} \approx 824\text{ km}$$

$$k = 24.134722990415206°\text{S} - 19.874192690735587°\text{S} \approx 473\text{ km}$$

$$f(80.5693359375°\text{E}, 19.874192690735587°\text{S}) = -4.92842822265625\text{ km}$$

$$f(88.48828125°\text{E}, 19.874192690735587°\text{S}) = -3.2452099609375\text{ km}$$

$$f(80.5693359375°\text{E}, 24.12670195868167°\text{S}) = -4.87695703125\text{ km}$$

$$f(88.48828125°\text{E}, 24.12670195868167°\text{S}) = -3.841930419921875\text{ km}$$

$$V = -1{,}645{,}973.912800293\text{ km}^3$$

Our back-of-the-envelope calculation shows there is a volume 1,646,000 cubic kilometers between the seafloor of our region and sea level, which implies there is also an equivalent volume of water in the region. Sharing the same first two significant figures, we can be relatively sure that our computer calculated volume is an accurate approximation.

```
x = [];
y = [];
z = [];
[x,y,z] = parse("grid.xyz");
volume = 0;
volume = ocean(x,y,z)

%%iteratively calculates V_ij for each rectangular area of the grid and
%%sums them to approximate total volume
function volume = ocean(x,y,z)
    volume = 0;
    for j = 1:523
        for i = 1:901
            h = x(i+1) - x(i);
            k = y(j+1) - y(j);
            V_ij = .25 * h * k * ( z(i,j) + z(i+1,j) + z(i,j+1) + z(i+1,j+1) );
            volume = volume + V_ij;
        end
    end
end

%%function to read x, y, and z values from a file into matrices and
%%convert to kilometers
function [x,y,z] = parse(filename)
    M = dlmread(filename);
    x = M(1:902,1) .* 104; %at 20 degrees south of the equator, longitude lines are approximately
104 km apart
    y = [];
    for i = 1:524
        y = [y M(i * 902,2).*111]; %latitude lines are approximately 111 km apart
    end
    z = reshape(M(:,3),[902,524]) ./ 1000;
end
```

```
volume =

   1.6990e+06
```

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