Project 1 – Errors in Scientific Computing

MATH 3315.003 Scientific Computing

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The forward difference method for approximating a second order derivative of a function f(x) has a convergence order of 1. This can be shown with algebraic manipulation of a truncated Taylor series.

(1) $$\mathrm{f}(x+h) = \mathrm{f}(x) + \mathrm{h}\cdot f^{'}(x) + h^2\frac{f^{''}(x)}{2!} + \mathrm{O}(h^2)$$

(2) $$2\mathrm{f}(x+h) = 2\mathrm{f}(x) + 2\mathrm{h}\cdot f^{'}(x) + h^2 f^{''}(x) + \mathrm{O}(h^2)$$

(3) $$\mathrm{f}(x+2h) = \mathrm{f}(x) + 2\mathrm{h}\cdot f^{'}(x) + 4h^2\frac{f^{''}(x)}{2!} + \mathrm{O}(h^2)$$

Subtraction of equations 3 and 2 yields:

(4) $$\mathrm{f}(x+2h) - 2\mathrm{f}(x+h) = -\mathrm{f}(x) + h^2 f^{''}(x) + \mathrm{O}(h^2)$$

(5) $$h^2 f^{''}(x) = \mathrm{f}(x+2h) - 2\mathrm{f}(x+h) + \mathrm{f}(x) + \mathrm{O}(h^2)$$

(6) $$f^{''}(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} + \mathrm{O}(h)$$

And thus it is shown that the second order forward difference approximation has convergence order 1. The centered difference method for approximating second order derivatives can be similarly shown to have convergence order 2 using truncated Taylor series.

(7) $$\mathrm{f}(x+h) = \mathrm{f}(x) + \mathrm{h}\cdot f^{'}(x) + h^2\frac{f^{''}(x)}{2!} + \mathrm{O}(h^2)$$

(8) $$\frac{h^2}{2} f^{''}(x) = \mathrm{f}(x+h) - \mathrm{f}(x) - \mathrm{h}\cdot f(x) + \mathrm{O}(h^2)$$

(9) $$\mathrm{f}(x-h) = \mathrm{f}(x) - \mathrm{h}\cdot f^{'}(x) + h^2\frac{f^{''}(x)}{2!} + \mathrm{O}(h^2)$$

(10) $$\frac{h^2}{2} f^{''}(x) = \mathrm{f}(x-h) - \mathrm{f}(x) + \mathrm{h}\cdot f(x) + \mathrm{O}(h^2)$$

Addition of equations 8 and 10 yields:

(11) $$h^2 f^{''}(x) = \mathrm{f}(x+h) + f(x-h) - 2\mathrm{f}(x) + \mathrm{O}(h^3)$$

(12) $$f''(x) = \frac{f(x+h) - f(x) - h\cdot f(x)}{h^2} + O(h^2)$$

And thus it is shown using Taylor series that the centered difference approximation method has convergence order 2.