## Chapter 2

# Basic theory

This chapter mainly introduces the basic law of flow in porous media: Darcy's law, and other concepts such as flow velocity, flow resistance and non-linear flow. The composition and establishment of the mathematical model to describe the flow process will be introduced, which provides the basis to solve the flow problem quantitatively.

## 2.1 Basic law of flow

### 2.1.1 *Introduction*

Fluid flow in the porous medium is affected by various forces, and its essence is to consume energy and produce fluid through the wellbore. The relationship between energy and flow rate becomes the most important problem in flow mechanics through porous media. Darcy's law is the most basic law that describes this relationship. This law and the rules extended from it become the main subject of flow mechanics in porous media.

### 2.1.2 *Concepts*

**Darcy's law:** *Darcy's law describes the linear relationship between the flow rate $Q$ and pressure difference $\Delta p$ when the fluid flows in porous media, which can also be named as linear flow law.*

**Linear relationship:** *if the relationship between two variables can be expressed by a linear function, that is, $y = ax + b$, then it is also called linear relationship.*

**Non-linear relationship:** *the relationship between the two variable quantities cannot be expressed by a linear function.*

**Darcy Flow:** *Darcy Flow is the flow that satisfies the linear relationship described by Darcy's law.*

**Non-Darcy Flow:** *non-Darcy Flow is the flow that has a non-linear relationship between flow rate and pressure difference and cannot be described by Darcy's law.*

**Flow area:** *the flow area is the cross-sectional area of fluid flowing through porous media, including the rock grains area and the pore area on the cross section. The symbol is $A$. The unit is $m^2$. The flow area of linear flow regime is $Bh$ and that of radical flow regime is $2\pi rh$.*

**Specific weight:** *the specific weight is the weight of unit volume fluid. The symbol is $\gamma$. The formula is $\gamma = \rho g$. The unit is $N/m^3$.*

**Productivity index:** *the productivity index is the oil recovery per unit time under the unit pressure difference. The symbol is $J$. The common unit is $m^3/d/MPa$.*

**Darcy velocity:** *the Darcy velocity is the flow rate passing through a unit cross sectional area of the porous medium, which is an imaginary apparent flow velocity.*

**Real flow velocity:** *the real flow velocity is the flow rate passing through a unit pore area.*

**Reynolds number:** *Reynolds number is the ratio of inertia force to viscous force when the fluid flows in the porous medium. The symbol is $R_e$. The Reynolds number for air flowing through an airplane is 5,000,000, for air flowing through a seagull's wing it is 100,000, for flow through a circular pipe it is 2320, for blood flowing in the brain it is 100 and for oil and water flowing in the reservoirs it is less than 0.2.*

**Laminar:** *the fluid is said to be in a laminar state when it flows with low velocity in tube, where any material point moves smoothly in a straight line that is parallel to the axis of the tube. The velocity* *of fluid is maximum at the center of the tube and minimum near the tube wall. The Reynolds number for laminar flow is generally less than 2000. And it is larger than 4000 for turbulent flow. The Reynolds number of fluid between 2000 and 4000 is transition flow.*

### 2.1.3 *Analysis*

#### 2.1.3.1 *Darcy's law*

Darcy's law was proposed by French engineer Henry Darcy through experiments in 1856. A schematic diagram of the experimental device is shown in Figure 2.1.

If there is a stable flow rate at outlet $c$, then the total hydraulic head at the cross section 1-1 in Figure 2.1 is:

$$H_1 = Z_1 + \frac{p_1}{\gamma} \tag{2.1}$$

The total head at the cross section 2-2 is:

$$H_2 = Z_2 + \frac{p_2}{\gamma} \tag{2.2}$$

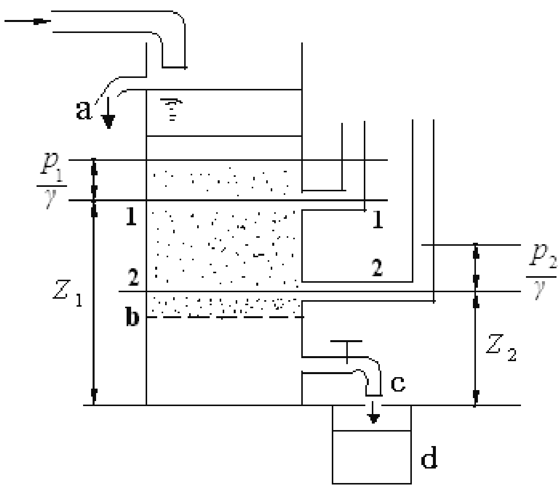


Figure 2.1. Schematic diagram of experimental device of Darcy's law

The head difference between the two cross sections is:

$$\Delta H = \left(Z_1 + \frac{p_1}{\gamma}\right) - \left(Z_2 + \frac{p_2}{\gamma}\right) \tag{2.3}$$

The pressure difference is:

$$\Delta p = \gamma \Delta H \tag{2.4}$$

The relationship between flow rate and pressure difference can be measured by adjusting the valve at the outlet $c$, and it is shown in Figure 2.2.

The first part of the curve from the origin shows that the relationship is linear. When the flow rate is large, the non-linear relationship starts to show up.

For the linear relationship segment, Darcy studied the influencing factors of flow. If the experimental sand box is placed horizontally, Darcy's law can be obtained as:

$$Q = \frac{K}{\mu} A \frac{\Delta p}{L} \tag{2.5}$$

where $Q$ is the flow rate, $\mathrm{cm^3/s}$; $K$ is the permeability, D; $A$ is the cross sectional area of flow, $\mathrm{cm^3}$; $\Delta p$ is the pressure difference between the two sections, $10^{-1}$ MPa; $\mu$ is fluid viscosity, $\mathrm{mPa \cdot s}$; and $L$ is the distance between the two sections, cm.

In this equation, the permeability $K$ was introduced as a fitting coefficient to describe the characteristics of porosity and permeability

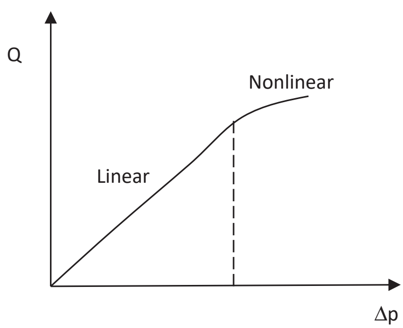


Figure 2.2. Relationship between flow rate and pressure difference

when Darcy analyzed the experiment. To honor his important contribution, the unit of permeability $K$ is named as *Darcy*. The symbol is D, which is also $\mu m^2$.

For oil and gas reservoirs, the unit of permeability, D, is too large, so the common unit is mD or $10^{-3}\,\mu m^2$.

In Darcy's equation, the units of the parameters have various forms; the main forms are experimental unit, engineering unit and international unit. They are shown in Table 2.1.

The flow resistance is defined as:

$$R = \frac{\mu L}{KA} \tag{2.6}$$

Then Darcy's equation can be written as the ratio of the driving force of flow to flow resistance:

$$Q = \frac{\text{Driving force}}{\text{Flow resistance}} = \frac{\Delta p}{R} \tag{2.7}$$

The productivity index is defined as:

$$\mathrm{J} = \frac{Q}{\Delta p} = \frac{\mathrm{KA}}{\mu \mathrm{L}} \tag{2.8}$$

If $Q$ is the production rate of oil, then $J$ is the oil productivity index. If $Q$ is the production rate of liquid, then $J$ is the fluid productivity index. If Darcy's law is expressed in the form of productivity index, then it can be written as:

$$Q = J \cdot \Delta p \tag{2.9}$$

Table 2.1. Units of parameters in Darcy's law

| Parameter | Experimental unit | Engineering unit | International unit |
|---|---|---|---|
| K | D | mD | $m^2$ |
| $\mu$ | $mPa \cdot s$ | $mPa \cdot s$ | $Pa \cdot s$ |
| A | $cm^2$ | $m^2$ | $m^2$ |
| $\Delta p$ | atm | MPa | Pa |
| L | cm | m | m |
| Q | $cm^3/s$ | $cm^3/s$ | $m^3/s$ |

### 2.1.3.2 *Darcy velocity*

Generally, in flow mechanics through porous media, the flow velocity refers to imaginary flow velocity, that is, velocity of fluid passing through per unit cross sectional area of rock.

According to the definition, Darcy velocity can be written as:

$$v = \frac{Q}{A} = \frac{K}{\mu}\frac{\Delta p}{L} \tag{2.10}$$

Define $x$ direction as the one along the direction of fluid flow, and write $\Delta p/L$ as the changing rate along $x$ direction, i.e., pressure gradient. It can be expressed by using the differential form:

$$grad(p) = -\frac{dp}{dx} \tag{2.11}$$

The Darcy velocity is:

$$v = -\frac{K}{\mu}\frac{dp}{dx} \tag{2.12}$$

This is the differential form of Darcy's law, which is also the equation of motion to establish the mathematical model of flow in future chapters.

In the pores of reservoir rocks, the flow velocity of material point, $v_\varphi$, is the real velocity, and the relationship between the real and Darcy velocity is:

$$v_\varphi = \frac{v}{\varphi} \tag{2.13}$$

### 2.1.3.3 *Non-linear flow*

(1) High-speed non-linear flow

In Figure 2.2, the non-linear segment occurs due to the higher flow velocity caused by the increased inertia force, which makes the flow not satisfy Darcy's Law and the non-linear relationship show up.

The flow mechanics through porous media is a branch of flow mechanics. In this book we mainly study the characteristics of fluid flow with low velocity in underground reservoirs.

In flow mechanics, Reynold number ($R_e$) is an important physical quantity judging the state of fluid flow. The Reynold number in porous media is:

$$R_e = \frac{v_\varphi \sqrt{K} \rho}{1750 \mu \varphi^{\frac{3}{2}}} \tag{2.14}$$

where $v_\varphi$ is the real flow velocity, cm/s; $K$ is the permeability, $\mu$m$^2$; $\rho$ is the density, g/cm$^3$; $\mu$ is the viscosity, mPa · s; and $\varphi$ is the porosity.

Darcy flow is in the laminar flow segment, where the Reynold number is generally less than 0.2 or 0.3.

If the flow velocity is large and Reynold number is larger than the critical value, it is named as high-speed non-linearity. In this case, the flow law can be expressed in exponential and binomial forms.

For the exponential form, it can be written as:

$$Q = C(\Delta p)^n \tag{2.15}$$

For the binomial form, it can be written as:

$$\Delta p = aQ + bQ^2 \tag{2.16}$$

(2) Low-speed non-linear flow

There is an immobile layer on the interface between the fluid and rocks. When the pores are small and the permeability is low, the flow velocity is small, the immobile layer becomes thicker and the additional resistance becomes larger. The essence is that the flow channel becomes smaller, the permeability also becomes smaller, and thus the flow shows a non-linear relationship. In low-speed non-linear flow, apparent permeability can be introduced to replace actual permeability or the threshold pressure gradient can be introduced. Since there is no consensus on how to deal with this issue and is still an ongoing research topic, we will not discuss it in this book.

#### 2.1.3.4 *Example*

(1) In Darcy's experiment, if the rock core is placed horizontally, the diameter of core is 25 mm, the length is 40 cm, the permeability is

250 mD, the viscosity of liquid is 5 mPa · s, and the pressure difference at two ends is 3 atm. Please calculate the flow rate $Q$ (cm$^3$/min) and Darcy velocity $v(\mu\text{m/s})$.

Solution:

$$\begin{aligned} Q &= \frac{K}{\mu} A \frac{\Delta p}{L} \\ &= \frac{250 \times 10^{-15}}{5 \times 10^{-3}} \times 3.14 \times (25 \div 2 \times 10^{-3})^2 \times \frac{3 \times 10^5}{40 \times 10^{-2}} \times 60 \times 10^6 \\ &= 1.104\,\text{cm}^3/\,\text{min} \end{aligned}$$

$$v = \frac{K}{\mu}\frac{\Delta p}{L} = \frac{250 \times 10^{-15}}{5 \times 10^{-3}} \times \frac{3 \times 10^5}{40 \times 10^{-2}} \times 10^6 = 37.5\,\mu\text{m/s}$$

Or:

$$v = \frac{Q}{A} = \frac{1.104 \times 10^{-6}/60}{3.14 \times (25 \div 2 \times 10^{-3})^2} \times 10^6 = 37.5\,\mu\text{m/s}$$

(2) In Darcy's experiment, if the core is placed horizontally, the diameter of core is 25 mm, the length is 40 cm, the permeability is 250 mD, the viscosity of liquid is 5 mPa · s, the pressure difference at two ends is 3 atm, the relative density of liquid is 0.9, and the porosity is 0.2. Please calculate the Reynold number.

Solution:

The real flow velocity in the pores is:

$$\begin{aligned} v_\phi &= \frac{v}{\phi} = \frac{37.5}{0.2} = 187.5\,\mu\text{m/s} \\ R_e &= \frac{v_\phi \sqrt{K}\rho}{1750\mu\phi^{\frac{3}{2}}} = \frac{187.5 \times 10^{-4} \times \sqrt{250 \times 10^{-3}} \times 0.9 \times 1}{1750 \times 5 \times 0.2^{1.5}} \\ &= 1.08 \times 10^{-5} \end{aligned}$$

Re < 0.1, the flow is laminar flow, which satisfies Darcy's law.

## 2.2 Basic mathematical models

### 2.2.1 *Introduction*

In the process of oil-gas flow, what we are most concerned about are the magnitude of energy, the distribution of pressure and the amount of production. The mathematical model of flow is a crucial way to conduct theoretical research on how to calculate the change characteristics and rules of these physical quantities, which is also the theoretical basis of other subjects, such as production engineering, reservoir engineering, reservoir numerical simulation, and modern well testing.

A complete mathematical model of flow generally consists of two parts: basic differential equations of flow and boundary (and initial) conditions. The detailed compositions of a mathematical model are shown in Figure 2.3.

The governing equation describes the flow behaviors in the differential form, and it is the combination of the equation of motion, the equation of state and the equation of continuity. The equation of

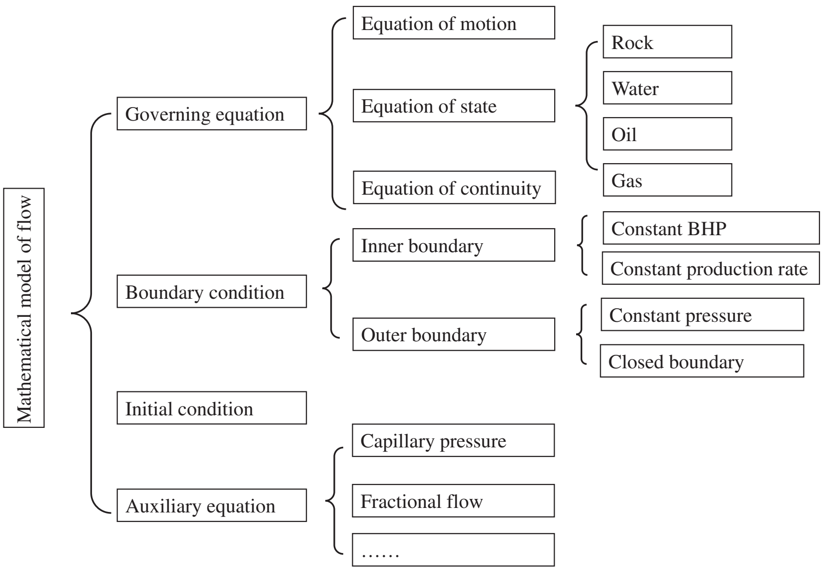


Figure 2.3. Compositions of a mathematical model of flow

state indicates the relationship of pressure, volume and temperature (PVT) for rock, water, oil and gas. The continuity equation or mass conservation equation is the mathematical expression connecting the physical quantities in the reservoir. Usually, the continuity equation is written in terms of pressure and saturation. The boundary condition and initial condition are used to obtain the solution of the differential equation governing the flow system. The auxiliary equations are not always in the mathematical model. They are used for some special occasions to help find the solutions of the flow model, such as capillary pressure equation and fractional flow equation for multiphase flow, which will be introduced in later chapters.

### 2.2.2 *Concepts*

**Mathematical model of flow:** *the mathematical model of flow is the equation or equation set used to describe all the mechanical phenomena and motion laws in the processes of reservoir fluid flow.*

**Equation of state:** *the equation of state is the equation that describes the changes of fluid (oil, gas and water) density or rock porosity with pressure for the petroleum reservoir system.*

**Representative elementary volume (REV):** *REV is also called the representative volume element (RVE) or the unit cell. It is the smallest volume that can be representative of the entire system, and is important to study the physical properties of complex media.*

**Equation of continuity:** *The equation of continuity or the mass conservation equation describes a relationship between the amount of reservoir fluids entering and leaving the REV and the accumulation of the reservoir fluids remaining in the REV.*

**Mass flow velocity:** *mass flow velocity is the product of flow velocity and density of fluid in the porous medium.*

**Ordinary differential equation:** *ordinary differential equation is a differential equation that involves one independent variable, its functions and its ordinary derivatives. Such as* $\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} = 0$.

**Partial differential equation:** *partial differential equation is an equation that involves several independent variables, their functions and partial derivatives. Such as* $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\eta}\frac{\partial p}{\partial t}$.

### 2.2.3 *Analysis*

The analytical solution of a differential equation refers to a solution that can be expressed by a mathematical expression directly. This book focuses on analytical solutions to analyze the law of flow in porous media quantitatively. Analytical solutions can be classified into general solution and specific solution. There are arbitrary constants in the general solution, and the number of arbitrary constants is the same as the order of the differential equation. Specific solution is one special case of the general solution. There is no arbitrary constant in the special solution. The constants can be determined by a set of initial and boundary conditions.

Numerical solution is the solution obtained by finite difference method, finite element method, or numerical interpolation method. When an analytical solution cannot be obtained directly by using calculus, the numerical solution can be helpful to obtain the solution to simulate the flow system by means of numerical analysis. Reservoir numerical simulation mainly demands the procedures to replace the differential equation with a difference equation, discretize the petroleum reservoir by using a numerical grid, build the equation for each grid point, solve the equation set of difference equations by using linear algebra and obtain the solution for pressure and saturation on each grid point at different times.

#### 2.2.3.1 *Establishment of the governing differential equations*

Here, we take the single-phase flow as an example to derive the differential equations that govern the flow in porous media.

(1) Equation of continuity

As shown in Figure 2.4, we take out a cubic shape REV from the reservoir. The mass flow velocity at the center point $M$ of the REV is:

$$\rho v \tag{2.17}$$

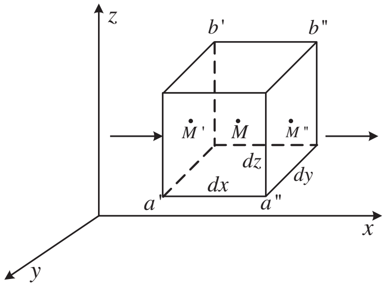


Figure 2.4. Representative elementary volume (REV) from the reservoir

The component of mass flow velocity along different directions are:

$$\rho v_x, \rho v_y, \rho v_z \tag{2.18}$$

The change rate of mass flow velocity along $x$, $y$ and $z$ directions are:

$$\frac{\partial(\rho v_x)}{\partial x}, \frac{\partial(\rho v_y)}{\partial y}, \frac{\partial(\rho v_z)}{\partial z} \tag{2.19}$$

The component of mass flow velocity along $x$ direction at the center point $M'$ on the plane $a'b'$ is:

$$\rho v_x - \frac{\partial(\rho v_x)}{\partial x}\frac{dx}{2} \tag{2.20}$$

The component of mass flow velocity along $x$ direction at the center point $M''$ on the plane $a''b''$ is:

$$\rho v_x + \frac{\partial(\rho v_x)}{\partial x}\frac{dx}{2} \tag{2.21}$$

The mass inflow of REV along $x$ direction for a period of time $dt$ is:

$$\left[\rho v_x - \frac{\partial(\rho v_x)}{\partial x}\frac{dx}{2}\right] dydzdt \tag{2.22}$$

The mass outflow of REV along $x$ direction for a period of time $dt$ is:

$$\left[\rho v_x + \frac{\partial(\rho v_x)}{\partial z}\frac{dx}{2}\right] dydzdt \tag{2.23}$$

The mass difference of inflow and outflow of the REV along $x$ direction for a period of time $dt$ is:

$$-\frac{\partial(\rho v_x)}{\partial x} dxdydzdt \tag{2.24}$$

For the same reason, the mass difference of inflow and outflow of the REV along directions $y$ and $z$ for a period of time $dt$ are:

$$-\frac{\partial(\rho v_y)}{\partial y} dxdydzdt, -\frac{\partial(\rho v_z)}{\partial z} dxdydzdt \tag{2.25}$$

The mass difference of inflow and outflow of the entire REV is:

$$-\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] dxdydzdt \tag{2.26}$$

The difference in the mass of inflow and outflow through the REV is due to the effect of elastic energy of rocks and fluids in the REV, which results in the accumulation of mass in the REV. The mass accumulation because of this process is shown as follows.

The pore volume in the REV is:

$$\varphi dxdydz \tag{2.27}$$

The fluid mass in the REV is:

$$\rho\varphi dxdydz \tag{2.28}$$

The accumulation rate of fluid mass per unit time is:

$$\frac{\partial(\rho\phi)}{\partial t} dxdydz \tag{2.29}$$

The total accumulation of fluid mass in $dt$ time is:

$$\frac{\partial(\rho\phi)}{\partial t} dxdydzdt \tag{2.30}$$

According to the law of mass conservation:

$$-\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] dxdydzdt = \frac{\partial(\rho\phi)}{\partial t} dxdydzdt \tag{2.31}$$

It can be simplified to:

$$-\left[\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}\right]=\frac{\partial(\rho\phi)}{\partial t} \tag{2.32}$$

This equation is the continuity equation of single-phase fluid, which is also named as the mass conservation equation.

It can be also expressed as:

$$-div(\rho\overrightarrow{v})=\frac{\partial(\rho\varphi)}{\partial t} \tag{2.33}$$

Or:

$$-\nabla\bullet(\rho\overrightarrow{v})=\frac{\partial(\rho\varphi)}{\partial t} \tag{2.34}$$

where div( ) is the divergence symbol; and $\nabla$ is the Hamiltonian operator.

Because the flow velocity is a vector with direction, the product of flow velocity and Hamiltonian operator in Equation (2.34) is $\cdot$ (dot product). If the parameters in brackets are scalars, such as the pressure in each direction, it is recorded as $\nabla p$.

(2) Equation of motion

In the flow mechanics of porous media, the equation of motion is generally expressed by the flow velocity in the differential form of Darcy's law. The differential forms of the equation of motion in three directions are:

$$v_x=-\frac{K}{\mu}\frac{\partial p}{\partial x}, v_y=-\frac{K}{\mu}\frac{\partial p}{\partial y}, v_z=-\frac{K}{\mu}\frac{\partial p}{\partial z} \tag{2.35}$$

It can also be written as:

$$\overrightarrow{v}=-\frac{K}{\mu}grad(p) \quad \text{or}\, \overrightarrow{v}=-\frac{K}{\mu}\nabla p \tag{2.36}$$

where *grad* represents gradient.

(3) Equation of state for fluid

The definition of the elastic compressibility coefficient of liquid is:

$$C_L=-\frac{1}{V_L}\frac{dV_L}{dp} \tag{2.37}$$

According to the equation of mass conservation, the liquid mass $M$ doesn't change when the fluid is shrinking or swelling, that is:

$$M = \rho V_L \tag{2.38}$$

Take the derivative to the above equation:

$$dV_L = -\frac{M}{\rho^2} d\rho \tag{2.39}$$

Substitute the equation into the elastic compression coefficient equation:

$$C_L = \frac{1}{\rho}\frac{d\rho}{dp} \tag{2.40}$$

By using the integration method of separated variables, the following results are obtained:

$$\int_{p_0}^{p} C_L dp = \int_{\rho_0}^{\rho} \frac{1}{\rho} d\rho \tag{2.41}$$

$$C_L(p - p_0) = \ln \frac{\rho}{\rho_0} \tag{2.42}$$

$$\rho = \rho_0 e^{C_L(p-p_0)} \tag{2.43}$$

By using the Maclaurin Series expansion and only keep the first two items (with enough accuracy), the equation of state for liquid can be obtained as:

$$\rho = \rho_0[1 + C_L(p - p_0)] \tag{2.44}$$

where $\rho_0$ is the initial density, kg/m$^3$; $p$ is the current pressure, MPa; and $p_0$ is the initial pressure, MPa.

It can be indicated from the equation of state for liquid that the larger the pressure, the larger the liquid density, and vice versa.

(4) Equation of state for rock

The rock elastic compressibility coefficient is defined as:

$$C_f = \frac{1}{V_f}\frac{dV_p}{dp} \tag{2.45}$$

where $V_f$ is the apparent volume (containing pores and grains), and $V_p$ is the pore volume.

Due to the definition of porosity $\phi = \frac{V_p}{V_f}$, it can be obtained by taking the derivatives for both sides:

$$d\varphi = \frac{dV_p}{V_f} \tag{2.46}$$

Substituting it into the equation of the rock elastic compressibility coefficient:

$$C_f = \frac{d\varphi}{dp} \tag{2.47}$$

$$d\varphi = C_f dp \tag{2.48}$$

By using the integration method of separated variables, it can be obtained that:

$$\int_{\varphi_0}^{\varphi} d\varphi = \int_{p_0}^{p} C_f dp \tag{2.49}$$

$$\varphi = \varphi_0 + C_f(p - p_0) \tag{2.50}$$

where $\varphi_0$ is the initial porosity.

This porosity formula is the state equation for rocks. It is shown that the porosity decreases with decrease in pressure, and vice versa.

(5) Basic differential equation

As has been derived, the continuity equation is:

$$-\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] = \frac{\partial(\rho\phi)}{\partial t} \tag{2.51}$$

Substitute the equation of motion into the equation of continuity:

$$\frac{\partial}{\partial x}\left(\rho\frac{K}{\mu}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho\frac{K}{\mu}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho\frac{K}{\mu}\frac{\partial p}{\partial z}\right) = \frac{\partial(\rho\phi)}{\partial t} \tag{2.52}$$

This is another important expression of continuity equation.

In the flow mechanics of porous media, there are two types of basic differential equations for single-phase flow.

The first is rigid steady state flow. In the steady state flow of the rigid reservoir, $C_L = 0$, $C_f = 0$, and the equation of state indicates that $\rho = \rho_0$, $\phi = \phi_0$. Then the basic differential equation for steady state flow in the rigid reservoir is:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \tag{2.53}$$

It is also expressed as:

$$\nabla^2(p) = 0 \tag{2.54}$$

where $\nabla^2$ is the Laplacian operator, sometimes it is also expressed by $\Delta$.

This equation is named as Laplacian equation, it is also called harmonic equation and elliptic equation.

For two-dimensional flow, the basic differential equations in Cartesian coordinates and polar coordinates are:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0, \quad \frac{d^2 p}{dr^2} + \frac{1}{r}\frac{dp}{dr} = 0 \tag{2.55}$$

The second is elastic transient flow. In the transient flow of the elastic reservoir, the basic differential equation can be obtained:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{\mu C_t}{K}\frac{\partial p}{\partial t} \tag{2.56}$$

It can also be written as:

$$\nabla^2(p) = \frac{\mu C_t}{K}\frac{\partial p}{\partial t} \tag{2.57}$$

where $C_t$ is the total compressibility coefficient, $\mathrm{MPa}^{-1}$. $C_t = \varphi C_L + C_f$.

This equation is named as Fourier equation, and it is also called as diffusion equation, heat conduction equation and parabolic equation.

Sometimes for convenience, the basic differential equation is often expressed in the form of cylindrical coordinates $(r, \theta, z)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\rho K}{\mu}\frac{\partial p}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{\rho K}{\mu}\frac{\partial p}{\partial z}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\rho K}{\mu}\frac{\partial p}{\partial \theta}\right)=\frac{\partial(\varphi\rho)}{\partial t} \tag{2.58}$$

In two-dimensional radial flow, the basic differential equation in Cartesian coordinates $(x, y)$ and polar coordinates $(r, \theta)$ is:

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=\frac{\mu C_t}{K}\frac{\partial p}{\partial t}, \quad \frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=\frac{\mu C_t}{K}\frac{\partial p}{\partial t} \tag{2.59}$$

#### 2.2.3.2 *Definite conditions*

(1) Boundary conditions

There are two types of outer boundary conditions: supply boundary and closed boundary.

There are two types of inner boundary conditions: constant bottom hole pressure (BHP) and constant production rate.

The above classifications are the boundary conditions commonly used in this book. Strictly speaking, there is a third boundary condition: the mixed boundary of the first and the second.

(2) Initial condition

The initial condition is the original state of physical parameters before reservoir production, which usually refers to the distribution of pressure and saturation.

#### 2.2.3.3 *Basic mathematical model*

The establishment of the flow mathematical model includes the basic governing differential equations, boundary conditions and initial conditions. Here, let us take an actual reservoir shown in Figure 2.5 as an example to explain the establishment of a basic flow mathematical model.

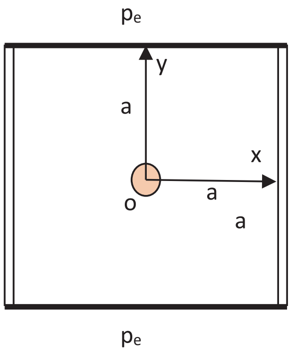


Figure 2.5. Schematic diagram of a rectangular reservoir

There is a production well in the center of the square reservoir and the length of the boundary is $a$. The fluid in the reservoir is the single-phase Newtonian fluid. The horizontal outer boundaries are supply boundaries, and the pressure of supply boundary is $p_e$. The vertical outer boundaries are the closed boundaries, the initial formation pressure is $p_i$, the permeability is $K$, the fluid viscosity is $\mu$, the porosity is $\varphi$, the total elastic compressibility coefficient of rock is $C_t$, the thickness of reservoir is $h$, the wellbore radius is $R_w$, and the production rate is $Q$. Take the center of well as the origin of the Cartesian coordinate system. The flow mathematical model of this problem is:

$$\begin{cases} \dfrac{\partial^2 p}{\partial x^2} + \dfrac{\partial^2 p}{\partial y^2} = \dfrac{\mu C_t}{K} \dfrac{\partial p}{\partial t} & \text{Governing equation} \\ p|_{y=\pm a} = p_e & \text{Supply outer boundary} \\ \left. \dfrac{\partial \mathrm{p}}{\partial \mathrm{x}} \right|_{x=\pm a} = 0 & \text{Closed outer boundary} \\ \left. r\dfrac{\partial p}{\partial r} \right|_{\mathrm{r}=\sqrt{x^2+y^2}=R_w} = \dfrac{Q\mu}{2\pi K h} & \begin{array}{l}\text{Constant production rate} \\ \text{inner boundary}\end{array} \\ p|_{\mathrm{t}=0} = p_i & \text{Initial boundary} \end{cases} \tag{2.60}$$

2.2.3.4 *Example*

1. There is a production well at the center of the circular supply boundary, which produces with the constant production rate Q. The known parameters are $R_e$, $R_w$, $K$, $h$, $\mu$, $p_e$. Write out the mathematical model of the single-phase steady state flow.

$$\begin{cases} \dfrac{d^2p}{dr^2} + \dfrac{1}{r}\dfrac{dp}{dr} = 0 & \text{governing equation} \\ p|_{r=R_e} = p_e & \text{supply outer boundary condition} \\ r\dfrac{\partial p}{\partial r}\Big|_{\mathrm{r}=R_w} = \dfrac{Q\mu}{2\pi Kh} & \text{constant production rate inner boundary condition} \end{cases}$$

Because the steady state flow has nothing to do with time, there is no initial condition in the flow mathematical model.

2. There is an injection well at the center of the circular closed boundary with constant injection rate Q. The known parameters are $R_e$, $R_w$, $K$, $h$, $\mu$, $p_i$. Write out the mathematical model of the single-phase steady state flow.

$$\begin{cases} \dfrac{\partial^2p}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial p}{\partial r} = \dfrac{\mu C_t}{K}\dfrac{\partial p}{\partial t} & \text{governing equation} \\ \dfrac{\partial p}{\partial r}\Big|_{\mathrm{r}=R_e} = 0 & \text{closed outer boundary condition} \\ r\dfrac{\partial p}{\partial r}\Big|_{\mathrm{r}=R_w} = \dfrac{-Q\mu}{2\pi Kh} & \text{constant production rate inner boundary condition} \\ p|_{t=0} = p_i & \text{initial condition} \end{cases}$$

For the constant rate of the injection well, the negative sign with $Q$ is used in the inner boundary condition of the flow mathematical model.

3. There is a production well at the center of the circular closed boundary. The change of outer boundary pressure with time is $p_e(t)$, and the change of the production rate of the well with time is $Q(t)$. The known parameters are $R_e$, $R_w$, $K$, $h$, $\mu$, $p_i$. Write out

the mathematical model of the single-phase transient flow.

$$\begin{cases} \dfrac{\partial^2 p}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial p}{\partial r} = \dfrac{\mu C_t}{K}\dfrac{\partial p}{\partial t} & \text{governing equation} \\ \mathrm{p}\big|_{\mathrm{r}=R_e} = p_e(t) & \text{outer boundary condition} \\ r\dfrac{\partial p}{\partial r}\bigg|_{\mathrm{r}=R_w,t} = \dfrac{-Q(t)\mu}{2\pi K h} & \text{inner boundary condition} \\ p\big|_{t=0} = p_i & \text{initial condition} \end{cases}$$

This shows that the flow mathematical model can take account of the variable boundary conditions.