Empirical Industrialisation Organisation
Homework 4

Problem 1:

(1) Each consumer l obtains utility from product i given by
$u_{li} = \varepsilon_{li} - p_i + y_l$.
Where the $\varepsilon_{l,i}$ are i.i.d Uniform $[0,1]$. Since y_l is the same for all products, it cancels when comparing products. Focusing on products 1 and 2, a consumer chooses product 1 if:
$\varepsilon_{l,1} - p_1 \geq \varepsilon_{l,2} - p_2$.
In a symmetric setting we set $p_2 = p^*$.
This inequality can be rearranged as:
$\varepsilon_{l,1} - \varepsilon_{l,2} \geq p_1 - p^*$.
Since $\varepsilon_{l,1}$ and ε_{l2} are independently uniformly distributed on $[0,1]$ the difference $d = \varepsilon_{l,1} - \varepsilon_{l,2}$ has a triangular distribution on $[-1,1]$ with density.
$f_D(z) = 1 - |z|$, $z \in [-1,1]$.
Thus, the probability that a consumer chooses product 1 is
$P_1(p_1, p^*) = \Pr\{d \geq p_1 - p^*\} = \int_{p_1-p^*}^{1} (1-z)\, dz$.
For $0 \leq p_1 - p^* \leq 1$,
$\int_{p_1-p^*}^{1} (1-z)\, dz = [1-(p_1-p^*)]^2/2$.
Thus, the aggregate demand for product 1 (with L consumers) is
$D_1(p_1 p^*) = L \cdot [1-(p_1-p^*)]^2/2$.

(2) Each firm maximizes its profit:
$\pi_1(p_1, p^*) = (p_1 - c) D_1(p_1, p^*)$
$= (p_1 - c)\, L \cdot [1-(p_1-p^*)]^2/2$
In symmetric equilibrium we have $p_1 = p^*$. So the profit simplifies to:
$\pi_1(p^*, p^*) = (p^* - c) \cdot L/2$.
To find the equilibrium, we differentiate the profit function with respect to p_1.
$(1/2) - (p^* - c) = 0$
$\Leftrightarrow p^* = c + 1/2$

③ Now suppose the outside option (not buying any product) gives utility y_e. Then the consumer buys product i only if:

$$\varepsilon_{e,i} - p_i + y_e \geq y_e \Rightarrow \varepsilon_{e,i} \geq p_i.$$

For product 1, a consummer will choose it if:

ⓐ $\varepsilon_{e,1} \geq p_1$ (so that the utility exceeds that from the outside option), and

ⓑ $\varepsilon_{e,1} - p_1 \geq \varepsilon_{e,2} - p^*$ (product 1 is preferred to product 2)

Thus, the aggregate demand for product 1 becomes:

$$D_1(p_1, p^*) = L \Pr\{\varepsilon_{e,1} \geq p_1 \text{ and } \varepsilon_{e,1} - p_1 \geq \varepsilon_{e,2} - p^*\}.$$

This can be written as a double integral. For example, one way is to integrate over $\varepsilon_{e,1}$ from p_1 to 1, and for each $\varepsilon_{e,1}$ the other shock must be less than $\varepsilon_{e,1} - p_1 + p^*$:

$$D_1(p_1, p^*) = L \int_{p_1}^{1} \left(\int_0^{\varepsilon_1 - p_1 + p^*} d\varepsilon_2 \right) d\varepsilon_1.$$

With the change of variable $u = \varepsilon_1 - p_1$ (so $u \in [0, 1-p_1]$), becomes:

$$D_1(p_1, p^*) = L \int_0^{1-p_1} (u + p^*)\, du.$$

Then, following the same procedure for profit maximization and setting $p_1 = p^*$ in equilibrium, one obtains (after differentiating and simplifying) a quadratic condition. The first order condition eventually becomes (after algebraic manipulation):

$$\frac{1}{2} - \frac{p^{*2}}{2} = p^* - c \Rightarrow (p^*)^2 + 2p^* - (2c+1) = 0$$

$$\Rightarrow p^* = \sqrt{2(c+1)} - 1$$

Problem 2:

① The probability that a consumer chooses product i is given by:

$$P_i = \Pr\{u_{e,i} \geq u_{ej} \;\forall j \in \{1,2,3\}\}.$$

In the general discrete choice framework, the demand for product i can be written as an integral over the distribution of shocks:

$$D_i(p_1, p_2, p_3) = L \int_{-\infty}^{\infty} \prod_{j \neq i} F(\varepsilon + v_i - v_j)\, f(\varepsilon)\, d\varepsilon.$$

Where $f(\varepsilon)$ is the Gumbel density:

$$f(x) = \frac{1}{\mu} \exp\left\{-\frac{x+\gamma}{\mu}\right\} \exp\left\{-\exp\left(-\frac{x+\gamma}{\mu}\right)\right\}.$$

A common special case is the multinomial logit (when $\mu = 1$), in which case the demand (or choice probability) simplifies to:

$$P_i = \frac{\exp(v_i)}{\sum_{j=1}^{3} \exp(v_j)}$$

so that the demand is

$$D_i(p_1, p_2, p_3) = L \frac{\exp(x_i\beta - p_i)}{\exp(x_1\beta - p_1) + \exp(x_2\beta - p_2) + \exp(x_3\beta - p_3)}$$

This expression is consistent with the utility specification and the assumed Gumbel distribution

(2) In the logit model, the choice probability for product i is:

$$P_i = \frac{\exp(v_i)}{\sum_{j=1}^{3} \exp(v_j)}.$$

Taking the derivative with respect to v_j for $j \neq i$ yields:

$$\frac{\partial P_i}{\partial v_j} = -P_i P_j.$$

Since $v_j = x_j\beta - p_j$ implies that $\partial v_j / \partial p_j = -1$, by the chain rule the derivative of P_i with respect to p_j (for $j \neq i$) is:

$$\frac{\partial P_i}{\partial p_j} = \frac{\partial P_i}{\partial v_j}\frac{\partial v_j}{\partial p_j} = P_i P_j$$

Now, if aggregate demand is $D_i = LP_i$, then for $i = 1, 2$ and $j = 3$

$$\frac{\partial D_1}{\partial p_3} = LP_1P_3 \quad \text{and} \quad \frac{\partial D_2}{\partial p_3} = LP_2P_3$$

Assume initially that consumers choose product 1 with probability 0,4 and products 2 and 3 with probability 0.3 each. Then an increase in p_3 (which makes product 3 less attractive) will lead to a shift in demand toward products 1 and 2. According to the derivative expressions, the increase in demand for product 1 will be proportional to $P_1P_3 = 0{,}4 \times 0{,}3 = 0{,}12$, and similarly for product 2 it is $0.3 \times 0{,}3 = 0{,}09$. That is, the reduction in product 3's share is reallocated proportionally to the other products. While this proportional reallocation (the IIA property) is mathematically consistent with the logit model, it's sometimes considered unrealistic if some product are more "similar" (closer substitutes) than others.