Empirical Industrial Organisation
Homework 2

① Question 1

Each buyer's valuation follows the distribution
$$F(x) = x^2 \text{ for } x \in [0,1].$$
For two independent random variables X_1 and X_2, the cumulative (cdf) of the maximum $M = \max(X_1, X_2)$ is given by:
$$G(m) = P(M \leq m) = P(X_1 \leq m \text{ and } X_2 \leq m) = F(m) \times F(m)$$
$$G(m) = (m^2)^2 = m^4. \quad \text{for } m \in [0,1].$$

② Question 2

a) Bidder 1's Problem

Let a bidder have a value v. If he submits a bid $= \beta(z)$ (potentially deviating from his equilibrium strategy by bidding as if his value were z) his profit if he wins is:
$$u(z; v) = (v - \beta(z)) \times \% \text{ of winning}$$
Since β is strictly increasing, winning means that his bid is higher than the bids of the other two. The probability that another buyer's valuation is less or equal to z is $F(z) = z^2$. Therefore, the probability of winning against the two others is:
$$P(\text{win} \mid z) = (F(z))^2 = (z^2)^2 = z^4$$
Thus the profit function becomes:
$$u(z, v) = (v - \beta(z))z^4.$$
In a symmetric equilibrium, the bidder chooses $z = v$, and the bidding function must satisfy the optimal condition at $z = v$.

b) First Order Condition

To maximize $u(z;v)$ with respect to z:
$$\frac{\partial u}{\partial z} = 4z^3(v - \beta(z)) - z^4\beta'(z)$$

Evaluating at $z = v$ and setting the derivative equal to zero (FOC), we obtain:

$$4v^3(v - \beta(v)) - v^4\beta'(v) = 0$$

Dividing by v^3 (for $v > 0$):

$$4(v - \beta(v)) - v\beta'(v) = 0 \Rightarrow \beta'(v) = \frac{4}{v}(v - \beta(v)).$$

c) Solving the differential Equation

It can be written as:

$$\beta'(v) + \frac{4}{v}\beta(v) = 4$$

Its integrating factor is:

$$\mu(v) = \exp\left(\int \frac{4}{v}\,dv\right) = \exp(4\ln v) = v^4$$

Multiplying the equation by v^4:

$$v^4\beta'(v) + 4v^3\beta(v) = 4v^4$$

The left-hand side is the derivative of $v^4\beta(v)$:

$$\frac{d}{dv}[v^4\beta(v)] = 4v^4.$$

Integrate with respect to v:

$$v^4\beta(v) = \int 4v^4\,dv = (4/5)v^5 + C.$$

Using the boundary condition $\beta(0) = 0$, we get $C = 0$.

Therefore, $\beta(v) = (4/5)v$.

The slides also offer an alternative formulation using $\beta(v) = E[U_{-i} \mid U_{-i} < v]$ and the corresponding integral which with $F(x) = x^2$ and $n = 3$ leads to the same result.

③ Question 3.

For a sealed-bid second-price auction with a reserve price r, the dominant strategy is $b(v) = v$ and the seller's revenue is maximized by choosing an optimal reserve price. According to the slide the optimal reserve price is determined by:

$$r - \frac{1 - F(r)}{f(r)} = c$$

where c is the seller's opportunity cost. For $c = 0$, the equation simplifies to

$$r = \frac{1 - F(r)}{f(r)}$$

Since $F(x) = x^2$ and the density $f(x) = F'(x) = 2x$, substituting in gives:

$$r = \frac{1 - r^2}{2r} \Rightarrow 2r^2 = 1 - r^2.$$

Thus, $3r^2 = 1 \Rightarrow r^2 = 1/3 \Rightarrow r = 1/\sqrt{3}$.