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a) $$f(Y|\theta) = \prod_{i=1}^{n} \theta e^{-\theta y_i}$$
$$= \theta^n \prod_{i=1}^{n} e^{-\theta y_i}$$
$$= n \log(\theta) + \sum_{i=1}^{n} -\theta y_i$$
$$= n \log(\theta) - \theta \sum_{i=1}^{n} y_i$$

$$f'(Y|\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} y_i = 0$$
$$\frac{n}{\theta} - n\bar{y} = 0$$
$$\frac{n}{\theta} = n\bar{y}$$
$$\theta = \frac{1}{\bar{y}}$$

b) $$f(\theta|Y) \propto f(\theta) f(Y|\theta)$$
$$= \theta^{\alpha-1} e^{-\beta\theta} \prod_{i=1}^{n} \theta e^{-\theta y_i}$$
$$= \theta^{\alpha-1} e^{-\beta\theta} \theta^n e^{-n\theta\bar{y}}$$
$$= \theta^{\alpha+n-1} e^{-\theta(\beta+n\bar{y})}$$

$$\Rightarrow \text{Gamma}(\alpha+n,\ \beta+n\bar{y})$$

2.

a) $$f(\tilde{y}|y) = \int_{\Omega} f(\tilde{y}|\theta) f(\theta|y)\, d\theta$$
$$= \int \theta e^{-\theta\tilde{y}} \frac{(\beta+n\bar{y})^{\alpha+n}}{\Gamma(\alpha+n)} \theta^{\alpha+n-1} e^{-\theta(\beta+n\bar{y})}\, d\theta$$
$$= \frac{(\beta+n\bar{y})^{\alpha+n}}{\Gamma(\alpha+n)} \int \theta^{\alpha+n} e^{-\theta(\beta+n\bar{y}+\tilde{y})}\, d\theta$$
$$= \frac{(\beta+n\bar{y})^{\alpha+n}}{\Gamma(\alpha+n)} \frac{\Gamma(\alpha+n+1)}{(\beta+n\bar{y}+\tilde{y})^{\alpha+n+1}} \cancel{\int \frac{(\beta+n\bar{y}+\tilde{y})}{\Gamma(\alpha+n+1)} \theta^{\alpha+n} e^{-\theta(\beta+n\bar{y}+\tilde{y})}}$$
$$= \frac{(\beta+n\bar{y})^{\alpha+n} (\alpha+n)}{(\beta+n\bar{y}+\tilde{y})^{\alpha+n+1}} \quad \Big|\ a = \alpha+n,\ \lambda = \beta+\bar{y}n$$
$$= \frac{\lambda^a a}{(\tilde{y}+\lambda)^{a+1}}$$

3. $f(\theta|Y) \propto f(\theta) f(Y|\theta)$

$$\binom{n}{y_1 \dots y_d} \prod_{i=1}^{d} \theta_i^{y_i} \theta_i^{\alpha_i - 1}$$

$$= \prod_{i=1}^{d} \theta_i^{y_i + \alpha_i - 1}$$

$\Rightarrow$ Dirichlet$(\alpha_1 + y_1 \dots \alpha_d + y_d)$