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$$\begin{aligned}
 a) \quad f(y|\theta) &= \prod_{i=1}^n \theta e^{-\theta y_i} \\
 &= \theta^n \prod_{i=1}^n e^{-\theta y_i} \\
 &= n \log(\theta) + \sum_{i=1}^n -\theta y_i \\
 &= n \log(\theta) - \theta \sum_{i=1}^n y_i
 \end{aligned}$$

$$\begin{aligned}
 f'(y|\theta) &= \frac{n}{\theta} - \sum_{i=1}^n y_i = 0 \\
 \frac{n}{\theta} - n\bar{y} &= 0 \\
 \frac{n}{\theta} &= n\bar{y} \\
 \theta &= \frac{1}{\bar{y}}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad f(\theta|y) &\propto f(\theta) f(y|\theta) \\
 &= \theta^{\alpha-1} e^{-\beta\theta} \prod_{i=1}^n \theta e^{-\theta y_i} \\
 &= \theta^{\alpha-1} e^{-\beta\theta} \theta^n e^{-n\theta\bar{y}} \\
 &= \theta^{\alpha+n-1} e^{-\theta(\beta+n\bar{y})}
 \end{aligned}$$

$$\Rightarrow \text{Gamma}(\alpha+n, \beta+n\bar{y})$$

2.

$$\begin{aligned}
 a) \quad f(\tilde{y}|y) &= \int f(\tilde{y}|\theta) f(\theta|y) d\theta \\
 &= \int \theta e^{-\theta\tilde{y}} \frac{(\beta+n\bar{y})^{\alpha+n}}{\Gamma(\alpha+n)} \theta^{\alpha+n-1} e^{-\theta(\beta+n\bar{y})} d\theta \\
 &= \frac{(\beta+n\bar{y})^{\alpha+n}}{\Gamma(\alpha+n)} \int \theta^{\alpha+n} e^{-\theta(\beta+n\bar{y}+\tilde{y})} d\theta \\
 &= \frac{(\beta+n\bar{y})^{\alpha+n}}{\Gamma(\alpha+n)} \frac{\Gamma(\alpha+n+1)}{(\beta+n\bar{y}+\tilde{y})^{\alpha+n+1}} \int \frac{(\beta+n\bar{y}+\tilde{y})}{\Gamma(\alpha+n+1)} \theta^{\alpha+n} e^{-\theta(\beta+n\bar{y}+\tilde{y})} d\theta \\
 &= \frac{(\beta+n\bar{y})^{\alpha+n}}{(\beta+n\bar{y}+\tilde{y})^{\alpha+n+1}} \quad | \quad a = \alpha+n, \quad \lambda = \beta+n\bar{y} \\
 &= \frac{\lambda^a a}{(\tilde{y}+\lambda)^{a+1}}
 \end{aligned}$$

$$3. f(\theta|y) \propto f(\theta) f(y|\theta)$$

$$(y_1, \dots, y_d) \prod_{i=1}^d \theta_i^{y_i} \theta_i^{\alpha_i - 1}$$

$$= \prod_{i=1}^d \theta_i^{y_i + \alpha_i - 1}$$

\Rightarrow Dirichlet $(\alpha_1 + y_1, \dots, \alpha_d + y_d)$