1a) $p(y) = \int_0^1 p(y|\theta)\, p(\theta)$

$= \int_0^1 \binom{n}{y_1 \dots y_d} \prod_{i=1}^{d} \theta_i^{y_i} \frac{\Gamma \sum (\alpha_i)}{\prod \Gamma(\alpha_i)} \prod_{i=1}^{d} \theta_i^{\alpha_i - 1}\, d\theta$

$= \binom{n}{y_1 \dots y_d} \frac{\Gamma \sum (\alpha_i)}{\prod \Gamma(\alpha_i)} \int_0^1 \theta_i^{y_i + \alpha_i - 1}\, d\theta$

$= \binom{n}{y_1 \dots y_d} \frac{\Gamma \sum (\alpha_i)}{\prod \Gamma(\alpha_i)} \frac{\prod \Gamma(\alpha_i + y_i)}{\Gamma \sum (\alpha_i + y_i)} \int_0^1 \theta_i^{y_i \alpha_i - 1} \frac{\Gamma \sum (\alpha_i + y_i)}{\prod \Gamma(\alpha_i + y_i)}\, d\theta$

$= \frac{n!}{y_1! \dots y_d!} \frac{\Gamma \sum^d (\alpha_i) \prod^d \Gamma(\alpha_i + y_i)}{\prod^d \Gamma(\alpha_i)\, \Gamma \sum^d (\alpha_i + y_i)}$

$= \frac{n!}{y_1! \dots y_d!} \frac{\Gamma(\alpha_0)}{\Gamma \sum^d (\alpha_i + y_i)} \frac{\prod^d \Gamma(\alpha_i + y_i)}{\prod^d \Gamma(\alpha_i)}$

$= \frac{n!\, \Gamma(\alpha_0)}{\Gamma \sum^d (\alpha_i + y_i)} \prod_{i=1}^{d} \frac{\Gamma(\alpha_i + y_i)}{y_i!\, \Gamma(\alpha_i)}$

$= \frac{n!\, \Gamma(\alpha_0)}{\Gamma(\alpha_0 + n)} \prod_{i=1}^{d} \frac{\Gamma(\alpha_i + y_i)}{y_i!\, \Gamma(\alpha_i)}$

b) $p(\theta|y) = \frac{p(\theta)\, p(y|\theta)}{p(y)}$

$= \frac{\cancel{\frac{\Gamma \sum^d \alpha_i}{\prod^d \Gamma \alpha_i}} \cancel{\binom{n}{y_1 \dots y_d}} \prod_{i=1}^{d} \theta_i^{y_i} \theta^{\alpha_i - 1}}{\frac{\cancel{n!\, \Gamma(\alpha_0)}}{\Gamma(\alpha_0 + n)} \prod_{i=1}^{d} \frac{\Gamma(\alpha_i + y_i)}{\cancel{y_i!\, \Gamma(\alpha_i)}}}$

$= \frac{\prod_{i=1}^{d} \theta_i^{y_i + \alpha_i - 1}}{\frac{\prod_{i=1}^{d} \Gamma(\alpha_i + y_i)}{\Gamma(\sum \alpha_i + y_i)}}$

$= \frac{\Gamma(\sum^d \alpha_i + y_i)}{\prod_{i=1}^{d} \Gamma(y_i + \alpha_i)} \prod_{i=1}^{d} \theta_i^{y_i + \alpha_i - 1}$

c) $p(\tilde{y}|y) = \int p(\tilde{y}|\theta)\, p(y|\theta)\, d\theta$

$= \int \binom{m}{\tilde{y}_1 \dots \tilde{y}_d} \prod_{i=1}^{d} \theta_i^{\tilde{y}_i} \frac{\Gamma(\sum^d \alpha_i + y_i)}{\prod_{i=1}^{d} \Gamma(y_i + \alpha_i)} \prod_{i=1}^{d} \theta_i^{y_i + \alpha_i - 1}$

$= \frac{m!}{\tilde{y}_1! \dots \tilde{y}_d!} \frac{\Gamma(\sum^d \alpha_i + y_i)}{\prod_{i=1}^{d} \Gamma(y_i + \alpha_i)} \int \prod_{i=1}^{d} \theta^{\tilde{y}_i + y_i + \alpha_i - 1}$

$$= \frac{m!}{\tilde{y}_1! \cdots \tilde{y}_d!} \frac{\Gamma\left(\sum^d \alpha_i + y_i\right)}{\prod_{i=1}^{d} \Gamma(y_i+\alpha_i)} \frac{\prod^d \Gamma(y_i+\alpha_i+\tilde{y}_i)}{\Gamma \sum^d (y_i+\alpha_i+\tilde{y}_i)}$$

$$= \frac{m!\,\Gamma\left(\sum^d \alpha_i + y_i\right)}{\Gamma \sum^d (y_i+\alpha_i+\tilde{y}_i)} \prod_{i=1}^{d} \frac{\Gamma(y_i+\alpha_i+\tilde{y}_i)}{\Gamma(y_i+\alpha_i)\, y_i!}$$

$$= \frac{m!\,\Gamma(\alpha_0+n)}{\Gamma(n+m+\alpha_0)} \prod_{i=1}^{d} \frac{\Gamma(y_i+\alpha_i+\hat{y}_i)}{\Gamma(y_i+\alpha_i)\, y_i!}$$