First Bajes #1

1. a)

$$f_{Y|\theta}(y_i|\theta) = \theta e^{-\theta y_i}$$

Derive maximum likelihood estimate

$$l(\theta; y) = \log(f(y;\theta)) = \log(\theta^n e^{-\theta y_i})$$

$$= n\log(\theta) - \theta \sum_{i=1}^{n} y_i$$

$$l'(\theta; y) = n\log\theta - \theta \sum_{i=1}^{n} y_i \quad d\theta$$
$$= \frac{n}{\theta} - \sum_{i=1}^{n} y_i$$

Set derivative to zero

$$\frac{n}{\theta} = \sum_{i=1}^{n} y_i$$
$$\theta = \frac{n}{\Sigma y_i}$$

For MLE we get $\frac{1}{\bar{y}}$

b)

$$f_{\theta|Y}(y|\theta) \propto f_\theta(\theta)\, f_{Y|\theta}(y|\theta)$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \theta^n e^{-\theta \Sigma y_i}$$
$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha+n-1} e^{-\beta\theta - \theta\Sigma y_i}$$
$$\propto \cancel{\frac{\beta^\alpha}{\Gamma(\alpha)}} \theta^{\alpha+n-1} e^{-\theta(\beta+\Sigma y_i)}$$
$$= \theta^{\alpha+n-1} e^{-\theta(\beta+\Sigma y_i)}$$

We can see that this is a gamma distribution

$$= \text{Gamma}(\alpha+n, \beta+\Sigma y_i)$$

2 a) We need to derive the posterior predictive distribution, that is

$$f_{\tilde{Y}|Y}(\tilde{y}|y) = \int_0^\infty f_{\tilde{Y}|\Theta}(\tilde{y}|\theta)\, f_{\Theta|Y}(\theta|y)\, d\theta$$
$$= \int_0^\infty \theta e^{-\theta\tilde{y}} \frac{(n\bar{y}+\beta)^{\alpha+n}}{\Gamma(\alpha+n)} \theta^{\alpha+n-1} e^{-(\beta+n\bar{y})\theta} \, d\theta$$
$$= \frac{(n\bar{y}+\beta)^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty \theta^{\alpha+n} e^{-\theta(\beta+n\bar{y}+\tilde{y})} \, d\theta$$
$$= \frac{(n\bar{y}+\beta)^{\alpha+n}}{\Gamma(\alpha+n)} \frac{\Gamma(n+\alpha+1)}{(\beta+n\bar{y}+\tilde{y})^{(n+\alpha+1)}} \underbrace{\int_0^\infty \frac{(\beta+n\bar{y}+\tilde{y})^{(n+\alpha+1)}}{\Gamma(n+\alpha+1)} \theta^{\alpha+n} e^{-\theta(\beta+n\bar{y}+\tilde{y})}}_{\text{gamma density} = 1}$$
$$= \frac{(n\bar{y}+\beta)^{\alpha+n}}{\cancel{\Gamma(\alpha+n)}} \frac{\cancel{\Gamma(n+\alpha+1)}(n+\alpha)}{(\beta+n\bar{y}+\tilde{y})^{(n+\alpha+1)}}$$
$$= \frac{(n\bar{y}+\beta)^{\alpha+n}(n+\alpha)}{(\beta+n\bar{y}+\tilde{y})^{(n+\alpha+1)}}$$

Now substitute $a := \alpha+n$ $\lambda := \beta + \sum_{i=1}^{n} y_i$

$$= \frac{a\lambda^a}{(\lambda+\tilde{y})^{a+1}}$$

Thus $f(x) = \frac{a\lambda^a}{(\lambda+x)^{a+1}}$

3. $Y = (Y_1 \dots Y_d)$
$\theta = (\theta_1 \dots \theta_d)$
$Y|\theta \sim \text{Multin}(n, \theta)$

$$f_{\theta|Y}(\theta|y) \propto f_{Y|\theta}(y|\theta)\, f_\theta(\theta)$$
$$= \binom{n}{y_1 \dots y_d} \prod_{i=1}^{d} \theta_i^{y_i} \frac{\Gamma(\alpha_1, \dots \alpha_d)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_d)} \prod_{i=1}^{d} \theta_i^{\alpha_i-1}$$
$$\propto \prod_{i=1}^{d} \theta_i^{y_i} \prod_{j=1}^{d} \theta_i^{\alpha_i-1}$$
$$= \prod_{i=1}^{d} \theta_i^{y_i+\alpha_i-1}$$

So the posterior is a Dirichlet distribution

$\alpha = (\alpha_1 + y_1, \dots \alpha_d + y_d)$
$\text{Dir}(\alpha)$