ENGS93-Midterm

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1 Approximation

1.1 Estimate exponent

Given that:

$$ln2 = 0.6932,\ ln10 = 2.3026,\ ln2.7183 = 1$$

We could interpret them as:

$$e^{0.6932}(e^{0.7}) = 2,\ e^{2.3026}(e^{2.3}) = 10,\ e^1 = 2.7183$$

Some tricks to make the result more precise when x is no so close to 0(Taylor extension), but using too much is not good:

$$e^x = 1 + x + \frac{x^2}{2!} + ...$$

General Solution: estimate the exponent as a combination of these given exponents.

Example:

$$e^{2.8} = (e^{0.7})^4 = (e^{0.6932} \times e^{0.0068})^4 = (2 \times (1 + 0.0068))^4 = 2.0136^4 \approx 16.439659$$

1.2 Estimate root using the definition of derivative

$$f(x + \triangle x) \approx f(x) + \frac{df}{dx}\triangle x$$

Given that $\triangle$x is relatively negligible compared with x:

$$\sqrt[n]{x + \triangle x} = \sqrt[n]{x} + \frac{\triangle x}{n} x^{\frac{1}{n}-1}$$

Example:

Estimate the fourth root of 85 using the definition of the derivative.

First find the nearest number that has a fourth square root. In this case it is 81.

$$\sqrt[4]{85} = \sqrt[4]{81 + 4} = \sqrt[4]{81} + \frac{1}{4}\frac{4}{x^{\frac{3}{4}}} \approx 3.037$$

1.3 Rule of 72

If:

$$interest\ rate \times number\ of\ period = 72$$

Then the number doubles.
For example:
Deposit 100 dollars for 12 years with a 6% interest rate, then after 12 years the 100 dollars result in 200 dollars. Check with calculator:

$$100 \times (1.06)^{12} = 201.2$$

Which is close to our approximation.
Derivation of Rule of 72:
Assume:

$$(1 + 0.01x)^n = 2$$

$$nln(1 + 0.01x) = ln2 = 0.693$$

Given that,

$$ln(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$0.0097nx = 0.693$$

$$nx = \frac{0.693}{0.0097} = 72$$

1.4 Stirling's approximation

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n$$

2 Mathematical Proof

2.1 Derive Poisson Distribution from Binomial Test. State ALL Assumptions.

(1) First we should figure out what is Binomial Distribution.
N independent trials with two outcomes - success or failure, Success with probability p and failure with probability q=1-p. The probability that there are k successes is:

$$P(X = k) = \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{x!(n-x)!}$$

When the number of trials n becomes very large and the probability of success p becomes very small such that the product of np remains λ constant.

$$p = \frac{\lambda}{n}$$

$$q = 1 - p = 1 - \frac{\lambda}{n}$$

The original Binomial test becomes:

$$\lim_{n\to\infty} P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k}$$

Get the constants out of the limit:

$$\frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n!}{(n-k)!}(\frac{1}{n})^k(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-k}$$

Divide and conquer:

$$\lim_{n\to\infty}\frac{n!}{(n-k)!n^k} = \lim_{n\to\infty}\frac{n(n-1)...(n-k+1)}{n^k} = \lim_{n\to\infty}\frac{n}{n}\times\frac{n-1}{n}\times...\times\frac{n-k+1}{n} = 1$$

Then we look at this term:

$$\lim_{n\to\infty}(1-\frac{\lambda}{n})^n$$

Recall that:

$$e^x = \lim_{x\to\infty}(1+\frac{1}{x})^x$$

Let,

$$x = -\frac{n}{\lambda}$$

Then,

$$\lim_{n\to\infty}(1-\frac{\lambda}{n})^n = \lim_{n\to\infty}(1+\frac{1}{x})^{-\lambda x} = e^{-\lambda}$$

Combining these three terms together, we can get the Poisson distribution:

$$P(\lambda, x) = \frac{\lambda^k e^{-\lambda}}{k!}$$

2.2 Derive Mean and Variance of a Continuous Uniform Distribution

The uniform distribution is given as:

$$f(x) = \frac{1}{b-a}, a \le x \le b$$

The general method to get the mean of a distribution:

$$E(x) = \sum_x xf(x) = \int_0^\infty xf(x)dx = \int_a^b \frac{x}{b-a}dx = \frac{x^2}{2(b-a)}|_a^b = \frac{a+b}{2}$$

$$V(x) = E(x-\mu)^2 = \sum_x (x-\mu)^2 f(x) = \sum_x (x^2-2\mu x+\mu^2)f(x) = \sum_x x^2 f(x) - 2\mu^2+\mu^2 = \sum_x x^2 f(x) - \mu^2$$

$$V(x) = \int_a^b \frac{x^2}{b-a}dx - (\frac{a+b}{2})^2 = \frac{x^3}{3(b-a)}|_a^b - \mu^2 = \frac{(a-b)^2}{12}$$

$$b^3 - a^3 = (b-a)(a^2+ab+b^2)$$

2.3 Derive the mean and variance of a discrete uniform distribution

$$f(x) = \frac{1}{n}$$

$$\mu = \sum_{k=1}^{n} k \times \frac{1}{n} = \frac{n+1}{2}$$

$$\sigma^2 = E(x^2) - E(x)^2 = \frac{(n+1)(2n+1)}{6} - (\frac{n+1}{2})^2 = \frac{n^2-1}{12}$$

2.4 Derive Exponential Distribution from Poisson Distribution

Interpretation of Poisson distribution: the number of occurrences per interval of time.
Interpretation of Exponential distribution: length of time between two occurrences.
Consider a Poisson process, there will be an average of λt occurrences per t units of time:

$$P(X = 0) = e^{-\lambda t}$$

Another interpretation of exponential distribution is the probability that time T to the first occurrence is greater than t:

$$P(T > t) = P(x = 0|\mu = \lambda t) = e^{-\lambda t} = 1 - P(T \leq t)$$

Cumulative exponential distribution is:

$$P(T \leq t) = 1 - e^{-\lambda t}$$

Take the derivative of the cumulative exponential distribution:

$$f(t) = \lambda e^{-\lambda t}$$

2.5 Derive the Mean of the Exponential Distribution is $\frac{1}{\lambda}$

The probability distribution function of exponential distribution is:

$$f(x) = \lambda e^{-\lambda x}$$

The mean of exponential distribution is:

$$\mu = \int x\lambda e^{-\lambda x} dx = \int x d(-e^{-\lambda x}) = -xe^{-\lambda x} + \int e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$\sigma^2 = \frac{1}{\lambda^2}$$

3 Using Normal Distribution to Estimate Binomial Distribution

Only if np and n(1-p) are greater than 5. Once these two are satisfied:

$$\mu = np, \ \sigma^2 = np(1-p)$$

4 Box Plot Interpretation

What can we learn from a box plot?
The interval between the two blocks is 1.35σ. If the distribution is a perfectly symmetric normal distribution, the median is also the mean of the distribution.

5 Linear combination of random variables

Assume X_1 and X_2 are two random variables. Y is a linear combination of X_1 and X_2.

$$Y = 2X_1 + 2X_2$$
$$E(Y) = 2E(X_1) + 2E(X_2)$$
$$V(Y) = 4E(X_1) + 4E(X_2)$$

6 Add sigma in quadrature

Player I with mean of 68, standard deviation of 3.
Player II with mean of 82, standard deviation of 4.
The chance player I beats player II:

$$S_T = \sqrt{3^2 + 4^2} = 5$$

Test statistics is,

$$Z = \frac{X_1 - X_2}{S_T}$$