Optimal Stochastic Multicrop Seasonal and Intraseasonal Irrigation Control

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OPTIMAL STOCHASTIC MULTICROP SEASONAL AND INTRASEASONAL IRRIGATION CONTROL

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ABSTRACT: Optimal seasonal multicrop irrigation water allocation and optimal stochastic intraseasonal (daily) irrigation scheduling are carried out using a two-stage decomposition approach based on a stochastic dynamic programming methodology. In the first stage the optimal seasonal water and acreage allocation among several crops or fields is defined using deterministic dynamic programming with the objective of maximizing total benefits from all the crops. The optimization is based on seasonal crop production functions. Seasonal crop production functions are obtained using single-crop stochastic dynamic programming, which incorporates the physics of soil moisture depletion and the stochastic properties of precipitation. In the second stage optimal intraseasonal irrigation scheduling is performed using a single-crop stochastic dynamic programming algorithm, conditional on the optimal seasonal water allocation of stage one. Optimal daily irrigation decision functions are obtained as a function of root-zone soil moisture content and the currently available irrigation water. The methodology is applied to a case study characterized by four crops in which both the optimal irrigation applications and the optimal acreage for each crop are determined.

INTRODUCTION

The problem of irrigation scheduling in the case of limited seasonal water supply has been studied extensively for the single-crop situation [e.g., Stewart et al. (1974), Bras and Córdova (1981), Ramírez and Bras (1982, 1985)]. However, most farming situations are concerned with several crops growing in the same season, or with similar crops growing on fields with different soil characteristics. Thus, two distinct types of water and acreage allocation decisions must be made. On a seasonal level, a limited amount of seasonal irrigation water must be allocated to a given number of crops with the objective of maximizing total net benefits. At this level, the optimal acreage for each crop (optimal cropping pattern) must also be defined. Decisions made at this level may lead to not growing certain crops or, in other cases, to nonoptimal single-crop irrigation policies (deliberately underirrigating certain crops) in order to make irrigation water available for additional crops. On an intraseasonal level, optimal irrigation scheduling must be determined that is conditional on a limited amount of available irrigation water, and that takes into account the different responses crops have to water stresses during different crop growth periods.

Optimal seasonal and intraseasonal irrigation water scheduling problems have been studied in a limited context before [e.g., Dudley (1972), Dudley et al. (1971a), Matanga and Mariño (1979)]. Recent studies include those by Yaron and Dinar (1982), Rao et al. (1990), Sunantara and Ramírez (1993), and Mannocchi and Mecarelli (1994). Yaron and Dinar (1982) used a decomposition approach based on linear programming (LP) and dynamic programming (DP) iterations. The procedure consisted of (1) computing optimal cropping mix by LP based on a fixed, a priori set of irrigation schedules; (2) obtaining the shadow prices of water in intraseasonal intervals from LP results and generation of new activities by DP for each crop; and (3) incorporating these activities in the LP matrix and computing a new optimal solution. Steps 2 and 3 are iterated until convergence to an overall optimum. However, LP results, expressed as continuous variables, were not compatible with the layout of fields and plots and the actual irrigation system on farms.

More recently, Rao et al. (1990) and Sunantara and Ramírez (1993) have addressed optimal multicrop weekly irrigation scheduling using a decomposition approach based solely on DP. In the Rao et al. (1990) approach, seasonal and intraseasonal water production functions are obtained heuristically. The dimensionality problem of multicrop water allocation is reduced by solving the single-crop model several times. However, in addition to neglecting the dynamics of soil moisture depletion processes, their work assumes deterministic values for precipitation and evapotranspiration, thus disregarding the highly random nature of these two processes.

For regions of deficit irrigation, that is, regions where available irrigation water is limited and used to supplement natural water supply by precipitation, accounting for the stochasticity of precipitation is of paramount importance. The random input to the soil, defined as the cumulative infiltration from a given storm, is determined not only by the dynamics of soil moisture depletion, but also by the characteristics of the precipitation process. Thus, stochastic modeling of the precipitation process is of primary importance in regions where water is a limited resource but where rainfall plays an important role as a water supply source. Stochastic rainfall inputs to the soil-plant system are characterized by randomness in rainfall intensity, rainfall duration, interarrival time, and number of storms within a given period of time. Ramírez and Bras (1985) showed the importance of the stochasticity of rainfall inputs in the context of a single crop, using stochastic DP for optimal irrigation scheduling.

The goal of this study is the optimal seasonal allocation of a limited amount of irrigation water as well as the optimal seasonal allocation of a limited acreage for two or more fields or crops; and determination of the optimal daily irrigation scheduling policy for each field or crop, by taking into account the dynamics of the soil moisture depletion process and the stochasticity of rainfall. To solve this stochastic, multicrop, seasonal and intraseasonal optimal irrigation scheduling problem, a solution based on the decomposition ideas of Yaron and Dinar (1982) and Rao et al. (1990) has been developed by Sunantara and Ramírez (1993). In the work of Sunantara and Ramírez, the random nature of precipitation and evapotranspiration is taken into account in a linearized form by using truncated Taylor series expansions in a first-order second-moment analysis context. In addition, a simplified soil water balance model is used. In the work presented in the following

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section, the stochastic multicrop seasonal and intraseasonal optimal irrigation problem solution of Sunantara and Ramírez (1993) is generalized. A more complex, daily, physically based soil-crop-climate model is used that accounts for the dynamics of soil moisture depletion. Furthermore, the randomness of precipitation arrivals, as well as of precipitation intensity and duration, are taken into account by using the full probability density functions of the respective variables. A stochastic formulation of the problem is presented.

TWO-STAGE DYNAMIC PROGRAMMING APPROACH FOR MULTICROP IRRIGATION SCHEDULING

A two-stage, dynamic programming iterative decomposition approach is used in this work, which decomposes the problem into seasonal and intraseasonal allocation cases. The basic elements of this procedure are (1) a deterministic DP solution for the seasonal water allocation; and (2) a stochastic daily DP solution for the intraseasonal allocation. Fig. 1 presents a schematic representation of the decomposition procedure.

The first stage is the seasonal allocation, which is carried out in two steps. In the first step, seasonal crop production functions are obtained using a stochastic DP methodology that determines the maximum expected value of benefits for a single crop as a function of seasonal water availability. The DP algorithm used in this step (as well as in stage two) takes into account the physics of soil moisture depletion and the randomness of the precipitation process, as explained later. This step is performed repeatedly for several values of seasonal water availability, ranging between zero and the maximum feasible irrigation requirement for every crop under consideration. As a result of the foregoing procedure, a relationship between seasonal water availability and expected value of yields (or benefits) for each crop and for all possible initial soil moisture conditions is developed. Following Hexem and Heady (1978), this relationship is modeled as an exponential function (see Figs. 2–5). The second step of the first stage in the solution procedure defines the optimal seasonal water allocation for each crop so that total benefits from all crops are maximized. This is done in a deterministic DP context where each crop corresponds to a decision stage in the DP algorithm, and where seasonal crop benefits are obtained from the seasonal crop production functions.

The second stage is the intraseasonal water allocation. Conditional on the optimal seasonal water allocation resulting from the first-stage DP process, optimal intraseasonal daily irrigation scheduling (water allocation) is obtained through the single-crop stochastic DP approach. Optimal irrigation policies for each crop on a daily basis are obtained. The resulting optimal intraseasonal (daily) irrigation scheduling solution is conditional on the optimal seasonal water allocation.

SEASONAL MULTICROP IRRIGATION WATER ALLOCATION

The optimal seasonal allocation of irrigation water and acreage for each single crop is a resource allocation problem among multiple competing users whose solution is trivial [see,

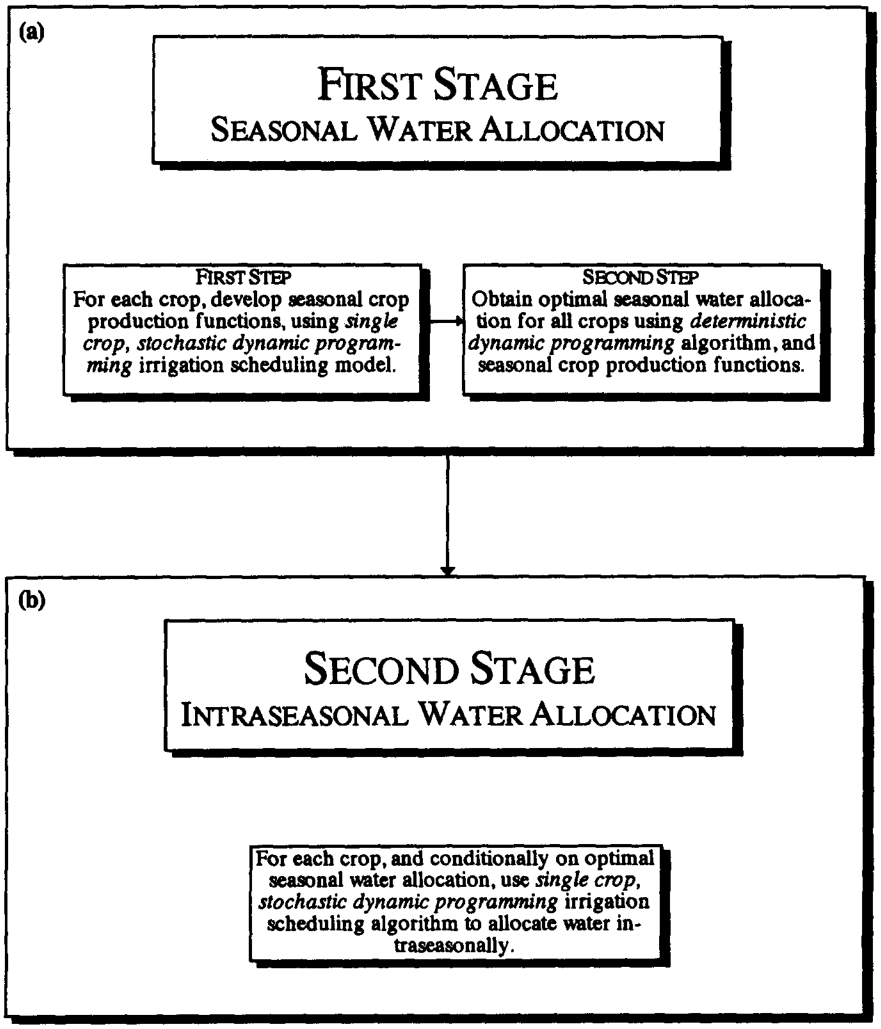

FIG. 1. Schematic of Two-Stage Decomposition Procedure

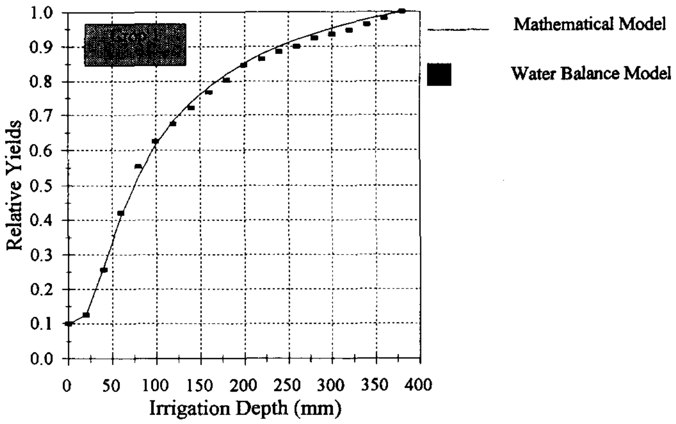

FIG. 2. **Normalized Seasonal Crop Production Function for Crop 1**

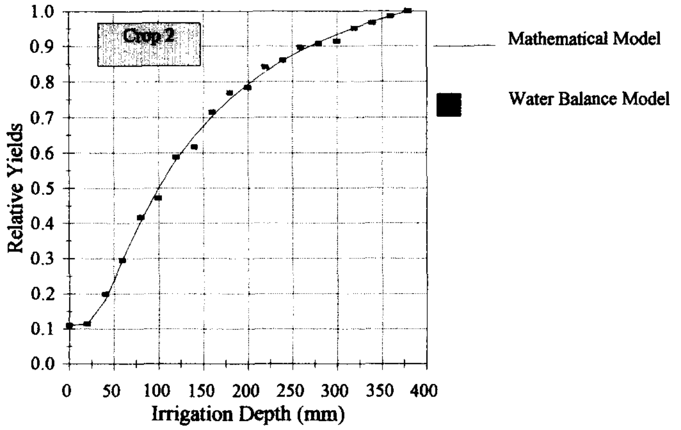

FIG. 3. **Normalized Seasonal Crop Production Function for Crop 2**

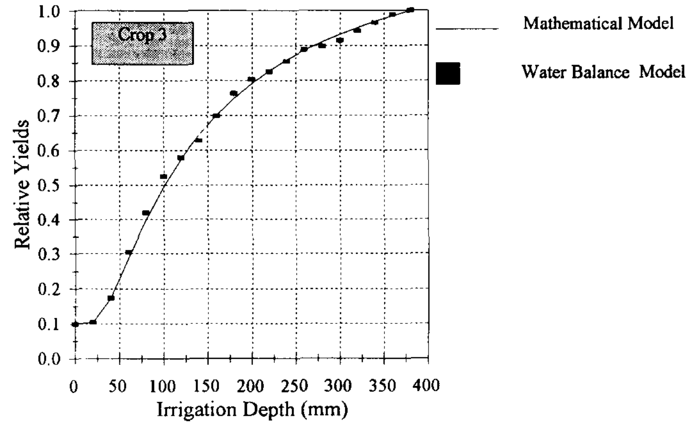

FIG. 4. **Normalized Seasonal Crop Production Function for Crop 3**

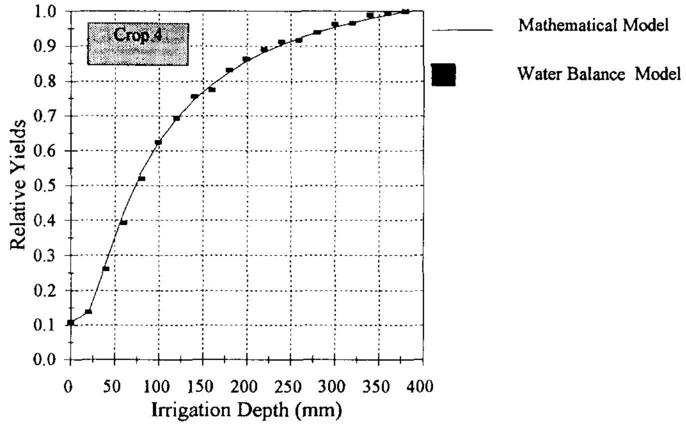

FIG. 5. **Normalized Seasonal Crop Production Function for Crop 4**

e.g., Loucks et al. (1981)]. If the seasonal water production functions are known for each crop, the solution can be accomplished using deterministic DP, as presented in the following paragraph. In that case, each stage of the DP algorithm represents a different crop. The objective function is the maximization of total seasonal net benefits from all crops as a function of seasonal irrigation water.

The mathematical relationship between crop yield and water supply, which is required for the solution of this deterministic DP problem, has been extensively studied. The implicit assumption is that crop yield can be expanded through increasing levels of any growth factor as long as it is not already in sufficient amount to produce maximum yields (Hexem and Heady 1978). Several forms of the yield-water supply function (e.g., quadratic, cubic, exponential, the so-called Mitscherlich-Spillman function, etc.) have been suggested in the literature (Yaron 1971; Hexem and Heady, 1978; Martin et al. 1984). In this study, an exponential function is used as follows:

$$y_l - y_{ol} = (y_{Ml} - y_{ol})\exp[-b_l(x_{Ml} - x_l)/x_l],$$

$$0 \le x_l \le x_{Ml} \quad l = 1, \ldots, n \tag{1}$$

where n = total number of crops; y_{ol} = relative seasonal yield for crop l when no irrigation water is supplied; and y_{Ml} = maximum attainable relative seasonal yield for the given initial conditions and irrigation system constraints. Relative seasonal yield y_l is defined as the ratio of the absolute seasonal yield Y_l (obtained when an amount x_l of seasonal irrigation water is applied) to the maximum attainable absolute seasonal yield Y_{Ml} (obtained when an amount x_{Ml} of seasonal irrigation water is applied). Seasonal production functions for each crop are obtained by using a single-crop stochastic DP model. Thus, they are derived based on an optimal intraseasonal allocation of irrigation water that maximizes an intraseasonal crop production function. For the optimal intraseasonal allocation of irrigation water, crop production functions that account for the temporal variation of crop response to water stress are needed. An additive crop yield model, function of evapotranspiration efficiencies for each crop growth period, is used as intraseasonal crop production function here, as follows:

$$Y_l = Y_{Ml} \sum_{z=1}^{NP_l} S_z^l \frac{ET_z^l}{ETP_z^l} \tag{2}$$

where S_z^l = water stress sensitivity factor in growing stage z; NP_l = number of growth periods for crop l; and ET_z^l and ETP_z^l = actual and potential evapotranspiration in growing stage z, respectively.

The recurrence DP equation to maximize total seasonal benefits from all crops can be written as follows:

$$f_l^*(x_l, A_l) = f_l(x_l^*, A_l^*) = \max_{\substack{A_l \\ x_l}}[A_l NB_l y_l(x_l) Y_{Ml}$$

$$+ f_{l-1}^*(x_{l-1}, A_{l-1})] \quad l = 1, \ldots, n \tag{3}$$

where x_l^* and A_l^* = optimal seasonal water allocation and the optimal acreage for crop l, respectively; and NB_l = net benefit of each unit of seasonal yield of crop l. The net benefit per unit of seasonal crop yield, NB_l, is itself a function of the unit price of crop yield, P, the unit cost of irrigation water, β, and the fixed costs of irrigation. This procedure results in optimal seasonal water entitlement and optimal acreage for each crop. It constitutes the first stage of the solution. Using this optimal seasonal water entitlement, the optimal intraseasonal (daily) irrigation policy for each crop can then be determined in the second stage of this solution using the single-crop optimization procedure.

SINGLE-CROP IRRIGATION WATER ALLOCATION

To develop an optimal irrigation schedule the decision makers (farmers) have to cope with a complex soil-plant-climate system. Complexities arise from soil moisture dynamics and the random nature of the volume of infiltration water from storms, in general; and from multiple crop responses and economic factors, including those related to available irrigation water, labor, and water costs, in particular. Many solutions to the single-crop irrigation scheduling problem have been obtained, such as those of Dudley et al. (1971), Stewart et al. (1974), Córdova and Bras (1979), Bras and Córdova (1981), Yaron and Dinar (1982), and Ramírez and Bras (1982, 1985). The single-crop daily irrigation scheduling solution of Ramírez and Bras (1982, 1985) is adopted in this work. Their work accounts for the stochasticity of rainfall occurrence, intensity, and duration, as well as for the dynamics of the soil-plant-climate system in a physically based manner. This solution is used in the first stage of the decomposition procedure in order to define seasonal crop production functions and, in the second stage, to perform optimal intraseasonal water allocation for each crop. The single-crop optimal irrigation decision problem contains the elements of a state space, a decision space, an output space, a state transition function, and a stochastic law of motion. These elements are briefly described in the following section. Additional details are given in Appendix I or can be found in Ramírez and Bras (1982, 1985).

The state space S, whose elements, namely the state vectors, X_k, consist of variables θ_k, ξ_k, and r_k; where θ_k is the soil moisture content at the root zone at the beginning of decision stage k; ξ_k is the number of days since the occurrence of the last rainfall, at decision stage k; and r_k is the volume of available irrigation water at the beginning of decision stage k. In this state vector both the soil moisture content and the number of days without rain are random variables. Their temporal evolution is governed by the dynamics of climate and soil moisture depletion processes.

The control space C, whose elements are the amounts of irrigation water applied at each decision stage k, U_k.

The output space Y, whose elements are the contributions to total crop yield from each crop growth period; these contributions are computed at each decision stage k and are denoted Y_k.

The state transition function F determines the temporal evolution of the state of the system. It is composed of three different dynamic equations, one for each element in the state vector X_k, as follows:

$$\theta_{k+1} = g(\theta_k, \Delta t) + V_k + U_k \tag{4}$$

$$\xi_{k+1} = \begin{cases} \xi_k + 1 & \text{if no rain during } \Delta t \\ 0 & \text{if rain during } \Delta t \end{cases} \tag{5}$$

$$r_{k+1} = r_k - U_k \quad U_k \le r_k \le r_{\max} \tag{6}$$

such that V_k = random volume of infiltration from rainfall during the given time step; $g(\theta_k, \Delta t)$ = soil moisture depletion in the root zone resulting from evapotranspiration, percolation to the aquifer, and capillary rise; and $r_{\max}$ = total seasonal available irrigation water. Details of the specific forms of these dynamic hydrologic processes can be found in Appendix I.

The stochastic law of motion P_R is a family of conditional probability distribution functions of the state vector X_{k+1} conditional on its present value and the irrigation decision. Since only two of the three elements of state vector $\mathbf{X}_{k+1}$, namely θ_{k+1} and ξ_{k+1} are random variables, the stochastic law of motion P_R only involves these two random variables. The stochastic law of motion gives rise to two probability transition matrices, one each for θ_k and ξ_k, as described in the following.

Probability Transition Matrix of θ_k

Soil moisture content (SMC) is discretized in the state space between saturation SMC and permanent wilting point (PWP). The discretization is carried out such that under the condition of no infiltration inputs, it takes only one day for the depletion process to drive the SMC from the upper bound to the lower bound of each interval, and m days to drive it from saturation to PWP. Soil moisture depletion during interstorm periods takes place through evapotranspiration, percolation, and capillary rise from the water table. The governing equations for the soil moisture depletion and the definition of soil moisture intervals are presented in Appendix I [Eqs. (27) and (28)]. Accordingly, the probability mass function (PMF) of the terminal soil moisture content for all soil moisture states is expressed in the form of a transition matrix as follows:

$$\Phi(k) = \{\phi_{ij}(k), i, j = 1, 2, \ldots, m\} \tag{7}$$

where $\phi_{ij}(k)$ = probability that at the end of decision stage k, the soil moisture content is in state j, given that at the beginning of this stage the soil moisture content is in the state i. Taking into account the probability of rainfall occurrence, $\phi_{ij}(k)$ can be expressed as follows:

$$\phi_{ij}(k) = P_{ij/W_k=0}(k)\Pr[W_k = 0/\xi_k] + P_{ij/W_k=1}(k)\Pr[W_k = 1/\xi_k] \tag{8}$$

where W_k = a binary random variable taking a value of unity if rain occurs during decision stage k or a value of zero otherwise; $P_{ij/W_k=0}(k)$ = conditional probability of the SMC at the end of the stage k being in state j, given that at the beginning of the stage it is in state i, and no rainfall occurs during the stage; and $P_{ij/W_k=1}(k)$ = conditional probability of the SMC at the end of the stage k being in state j, given that at the beginning of the stage it is in state i, and rainfall occurs during the stage.

If there is no rainfall during decision stage k ($W_k = 0$), the soil moisture content at the end of this stage only depends on the soil moisture content at the beginning of the stage. Consequently

$$P_{ij/W_k=0}(k) = \Pr\{\theta_{k+1} \in [{}_U\theta^j_{k+1}, {}_L\theta^j_{k+1}]/\theta_k \in [{}_U\theta^i_k, {}_L\theta^i_k] \text{ and } W_k = 0\}$$

$$= \begin{cases} 1, & j = i + 1 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

where U and L = upper and lower bound of the given soil moisture interval, which is denoted by a superscript. If rainfall occurs during the stage ($W_k = 1$), the conditional transition probability of the SMC state is as follows:

$$P_{ij/W_k=1}(k) = \Pr\{\theta_{k+1} \in [{}_U\theta^j_{k+1}, {}_L\theta^j_{k+1}]/\theta_k \in [{}_U\theta^i_k, {}_L\theta^i_k] \text{ and } W_k = 1\} \tag{10}$$

which, as a function of the volume of infiltration from rainfall during the given time interval can be rewritten as

$$P_{ij/W_k=1}(k) = \frac{1}{\Delta\theta^i_k} \int_{{}_L\theta^i_k}^{{}_U\theta^i_k} \Pr\{{}_LV^{ij}_k \le V^{ij}_k \le {}_UV^{ij}_k/\theta_k$$

$$\in [{}_U\theta^i_k, {}_L\theta^i_k] \text{ and } W_k = 1\}\partial\theta^i_k \tag{11}$$

which represents the conditional probability that the volume of infiltrated water is in the range of volumes required to bring the soil moisture from state $i + 1$ to state j. V^{ij}_k = volume of infiltrated water during stage k whose probability density function was derived by Córdova and Bras (1979) (see Appendix I); and ${}_LV^{ij}_k$ and ${}_UV^{ij}_k$ = lower and upper bound values required to achieve soil moisture state j from soil moisture state i over one time step, such that

$$ {}_UV_k^{ij} = {}_U\theta_k^j - \theta_k^{i+1}, \text{ and } {}_LV_k^{ij} = {}_L\theta_k^j - \theta_k^{i+1} \qquad (12a,b) $$

(observe that the soil moisture is first depleted over the given time interval and then replenished by the infiltrated volume V_k^{ij}).

The conditional probability of rain occurrence was derived by Ramírez and Bras (1985) as

$$ \Pr[W_k = 0/\xi_k = \xi] = \left(\frac{1 - qe^{-\alpha\xi}}{1 - qe^{-\alpha(\Delta t+\xi)}}\right)^{(1+\mu/\alpha)} e^{-\mu\Delta t} \qquad (13) $$

$$ \Pr[W_k = 1/\xi_k = \xi] = 1 - \Pr[W_k = 0/\xi_k = \xi] \qquad (14) $$

These expressions represent the conditional probability that the time to the next rainfall is greater than Δt days, based on the fact that the process has evolved for a period of time ξ_k without any events occurring (in the following case study, Δt equals one day). In the preceding expressions, storm arrivals are assumed to follow a Neyman-Scott cluster process in which the cluster çenter process is Poisson with parameter μ, the distribution of the cluster sizes is geometric with parameter p, the distribution of the time positions about cluster centers is exponential with parameter α, and q is $1 - p$. In the limit, these expressions reduce to the commonly used Poisson process for cases of weak temporal dependency in rainfall arrivals and/or small cluster sizes.

Probability Transition Matrix of ξ_k

The probability transition matrix of the state variable representing the number of days elapsed since the last rainfall can be written as

$$ \Psi(k) = \{\psi_{ij}(k), i, j = 1, 2, \ldots, H\} \qquad (15) $$

where $\psi_{ij}(k)$ = probability that at decision stage $k + 1$, ξ_{k+1} is in state j, given that at decision stage k, ξ_k is in state i. In this case each day is a state. H represents an upper bound on the value of ξ_k. It is defined as the time lag in days beyond which the dependence in the rainfall arrival process is negligible, that is, beyond which the probability of rainfall occurrence during the next decision stage is independent of the number of days without rain. Accounting for the probability of rainfall occurrence, this transition probability can be expressed as follows:

$$ \psi_{ij}(k) = \Pr[\xi_{k+1} = j/\xi_k = i \text{ and } W_k = 0]\Pr[W_k = 0/\xi_k] $$
$$ + \Pr[\xi_{k+1} = j/\xi_k = i \text{ and } W_k = 1]\Pr[W_k = 1/\xi_k] \qquad (16) $$

which can be simplified as

$$ \psi_{ij}(k) = \begin{cases} \Pr[W_k = 0/\xi_k = i] & \text{if } j = i + 1 \\ \Pr[W_k = 1/\xi_k = i] & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (17) $$

The joint PMF of the random variables θ_{k+1} and ξ_{k+1} can be obtained in terms of the transition matrices $\Phi(k)$ and $\Psi(k)$. Assuming that at the beginning of stage k, the SMC is in the state i, and the elapsed time is in state p, the probability that at the beginning of stage $k + 1$, the SMC is in state j and the elapsed time is in state q can be written as

$$ \Pr\{[\theta_k^i, \xi_k^p] \rightarrow [\theta_{k+1}^j, \xi_{k+1}^q]\} = \phi_{ij}(k)\psi_{pq}(k) \quad i, j = 1, \ldots, m $$
$$ p, q = 1, \ldots, H \qquad (18) $$

Stochastic Dynamic Programming Solution

Stochastic DP is formulated to maximize the expected value of net profits at the end of the growing season. The objective function is as follows:

$$ B^* = \text{Max } E\left[\sum_{k=1}^{N+1} R_k(X_k, X_{k+1}, U_k)\right] - PC \qquad (19) $$

and

$$ R_k(X_k, X_{k+1}, U_k) = PY_k - \beta U_k - \gamma C_k(U_k) \qquad (20) $$

where N = total number of days in the growing season; P = unit price of crop yield; Y_k = contribution of decision stage k to the total yield; β = unit cost of irrigation water; γ = fixed cost of irrigation; U_k = volume of irrigation water applied in stage k; PC = production costs different from irrigation costs; and $C_k(U_k)$ = binary variable taking the value of 1 if U_k is different from zero, or the value of 0 otherwise.

The stochastic DP algorithm proceeds as follows:

$$ J_{N+1}^*(X_{N+1}) = J_{N+1}^*(\theta_{N+1}^i, \xi_{N+1}^p, r_{N+1}) = 0 \quad i = 1, \ldots, m $$
$$ p = 1, \ldots, H \qquad (21) $$

And proceeding by induction

$$ J_k^*(X_k) = J_k(\theta_k^i, \xi_k^p, r_k, U_k^*) = \underset{U_k}{\text{Max}}\left\{E[R_k(\theta_k^i, \xi_k^p, U_k)]\right. $$
$$ \left. + \sum_j \sum_q \phi_{ij}(k)\psi_{pq}(k)J_{k+1}^*(\theta_{k+1}^j, \xi_{k+1}^q, r_{k+1})\right\} \quad i, j = 1, \ldots, m $$
$$ p, q = 1, \ldots, H \qquad (22) $$

where the maximum expected net benefits at the end of the season are given as

$$ J_1^*(\theta_1^i, \xi_1^p, r_1) = B^* + PC \qquad (23) $$

This algorithm yields not only the maximum expected net benefits for each decision stage and state vector, but also the optimal daily irrigation policies for each decision stage and state vector. Optimal benefits and decisions are denoted with asterisks in the previous equations. At each decision stage, the foregoing procedure yields decision functions that define the optimal amount of irrigation water to be supplied to the crop, U_k^*, as a function of the soil moisture content in the root zone, the number of days since the occurrence of the last rainfall, and the amount of irrigation water available for the remainder of the growing season.

CASE STUDY

To illustrate the methodology introduced in this work a case study analyzing potato crops in the San Luis Valley in South Central Colorado is presented. The valley is well suited for agricultural production. The main crops grown in the San Luis Valley are potatoes, barley, and various vegetables. The data used were obtained by Ramírez and Finnerty (1996) in their assessments of climate change impacts on irrigated agriculture for this region. Ramírez and Finnerty chose to study potatoes because their photosynthetic mechanisms are more sensitive to concentrations of carbon dioxide. The valley contains a vast unconfined aquifer, and a segment of the Río Grande. Prior experiments (Leib 1989) located on a potato farm near Alamosa, Colorado, in the San Luis Valley, provide detailed information on soil characteristics for the description of soil moisture depletion and water balance at the root zone (see Tables 1 and 2).

The potato crops used for this study are all assumed to have the same physiological growth periods as defined in Troolen (1988). Table 3 describes these growth periods. Two distinct rooting depths characterize these crops as presented in Table 4.

Precipitation and other hydrometerological data were collected by Ramírez and Finnerty (1996). The precipitation data

TABLE 1. Sandy Clay Loam Soil Characteristics

Parameters (1)	Values (2)
Porosity, n	0.38
Saturated hydraulic conductivity, $K(\theta_s)$	167.6 mm/d
Saturated soil matrix potential, $\psi(\theta_s)$	254 mm
Pore size distribution index,[a] m	0.35
Pore disconnectedness index[b], c	8.714

[a]Parameter of Brooks and Corey (1966) parameterization of hydraulic conductivity as a function of soil saturation.
[b]Parameter of Brooks and Corey (1966) parameterization of soil matrix potential as a function of soil saturation.

TABLE 2. Volumetric Soil Moisture Characteristics

Parameter (1)	Moisture content (2)	Plant available moisture (3)
Soil moisture at saturation, θ_s	0.38	0.26
Field capacity, θ_{FC}	0.23	0.11
Permanent wilting point, θ_{PWP}	0.12	0.0

TABLE 3. Physiological Growth Periods of Centennial Russet Potatoes

Growth period (1)	Description (2)	Timing (3)
1	Planting to stolonization[a]	May 7 to June 15
2	Stolonization to tuber initiation	June 16 to June 23
3	Tuber initiation to maximum bulking	June 24 to July 17
4	Maximum tuber bulking to maturity	July 18 to August 18

[a]Stolons are the initial "seed" form of tubers. A tuber is a potato.

TABLE 4. Rooting Depths and Plant Available Soil Moisture

Growth periods (1)	Root depth (mm) (2)	Saturation		Field Capacity	
		Volumetric (3)	Absolute (mm) (4)	Volumetric (5)	Absolute (mm) (6)
1 and 2	350	0.26	91.0	0.11	38.5
3 and 4	450	0.26	117.0	0.11	49.5

set contained 33 years of hourly precipitation measured at Alamosa, Colorado, near the study site, and spanning the years from 1951 to 1983. In addition, in order to estimate average potential evapotranspiration rates during the growing season, temperature, relative humidity, wind speed, albedo, net radiation, stomatal resistance, crop roughness height, and other measurements were also collected by Ramírez and Finnerty (1996). Average precipitation and potential evapotranspiration on a weekly basis are shown in Table 5.

The stress sensitivity factors must be determined for each specific crop. Ramírez and Finnerty, based on works by Struchtemeyer (1960), de Lis et al. (1963), and Salter and Goode (1967) used sensitivity factors for a Centennial Russetts potato crop as indicated in Table 6 for crop 1. The physiological response of crops to water stresses vary depending on the particular stage of crop growth during which the stress occurs. The crop water stress sensitivity factors encode this physiological response of the potato crops. Struchtemeyer (1960) points out that stress in the second half of the growing season reduces potato yields more than stress in the first half. de Lis et al. (1963) indicate that the most sensitive stages are stolonization (growth period 2) and tuber initiation (growth period 3). Salter and Goode (1967) point out that irrigation should be

TABLE 5. Precipitation and Potential Evapotranspiration throughout Growing Season[a]

Growth period (1)	Precipitation (mm) (2)	Potential evapotranspiration (mm) (3)
1	3.21	35.76
1	3.75	35.76
1	4.59	35.76
1	4.27	35.76
1	3.12	35.76
2	2.53	53.60
3	3.67	57.92
3	6.24	57.92
3	6.58	57.92
4	7.51	46.72
4	8.41	46.72
4	8.91	46.72
4	7.25	46.72

[a]Precipitation and evapotranspiration values are computed over eight-day intervals.

TABLE 6. Crop Water Stress Sensitivity Factors

Crops (1)	Growth period 1 (2)	Growth period 2 (3)	Growth period 3 (4)	Growth period 4 (5)
1	0.15	0.35	0.45	0.05
2	0.30	0.30	0.35	0.05
3	0.30	0.30	0.35	0.05
4	0.20	0.50	0.25	0.05

withheld in the late season to improve tuber quality. In addition, irrigation in the early vegetative stage is recommended only if crop damage is occurring due to extreme water stress. Table 6 presents the crop water stress sensitivity factors for the centennial Russetts potato crop considered in this study, as well as those of three other hypothetical varieties of potatoes. As explained, in order to simplify the numerical procedure and this illustrative case study the four crops are assumed to have the same growing stages. However, this is not a limitation of the procedure, and crops with different timing for their growth periods can be analyzed.

RESULTS

Two cases of on-farm situations are studied. In the first case there are no constraints on the system except those imposed by the available seasonal irrigation water and the size of the farm (unconstrained case). Therefore, the decision maker is free to leave one or more fields uncultivated in order to maximize net benefits. However, in most agronomic situations, there are always minimum demands to be fulfilled. Thus, in the second case, minimum and maximum acreage constraints are imposed (constrained case). Minimum and maximum acreages are taken equal for all crops and equal to 25 and 100 acres, respectively. The agroeconomic parameters used in the seasonal as well as the intraseasonal irrigation scheduling models are shown in Table 7. Observe that the differences between the four crops analyzed stem not only from physiological characteristics like water stress sensitivity factors and minimum and maximum yields, but also from agroeconomic parameters like irrigation costs, production costs, and market prices. These agroeconomic parameters depend on market dynamics as well as on the layout of the farm, accessibility to the crop fields, and efficiency of delivery systems.

The first stage in this decomposition procedure is the optimal allocation of seasonal irrigation water as well as the determination of the optimal acreage for each crop. This is ac-

TABLE 7. Agroeconomic Factors

Crops (1)	Theoretical maximum yield (100 lb/acre) (2)	Market price ($/100 lb) (3)	Irrigation cost ($/mm-acre) (4)	Production cost ($/acre) (5)
1	375	4.65	5.0	150
2	375	4.65	2.5	150
3	400	4.65	5.0	150
4	375	4.65	5.0	150

TABLE 8. Parameters of Seasonal Crop Production Functions

Crop (1)	Expected minimum yield, Y_{oi} (100 lb/acre) (2)	Expected maximum yield, Y_{Mi} (100 lb/acre) (3)	Expected maximum gross benefits, ($/acre) (4)	Minimum irrigation supply, X_{Mi} (mm) (5)	Parameter b_i (6)
1	30	301	1,400.38	380	0.199
2	33	296	1,377.76	380	0.294
3	32	317	1,476.07	380	0.294
4	31	279	1,298.70	380	0.195

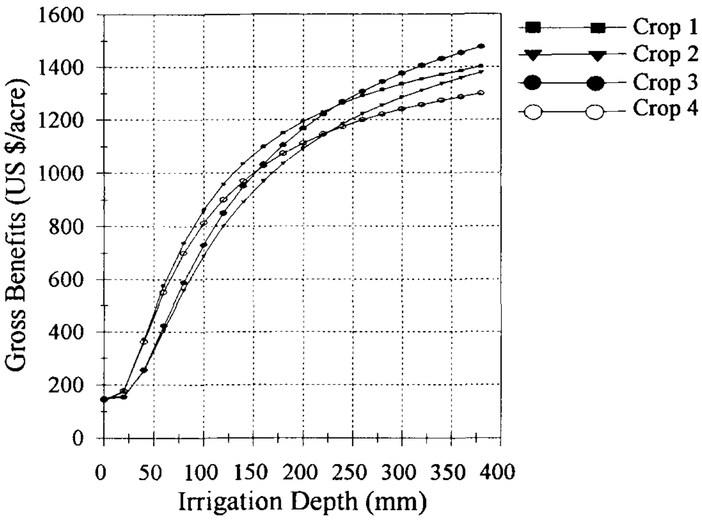

FIG. 6. Comparison of Seasonal Crop Production Functions for Case Study Crops

complished in two steps. The first step yields a seasonal crop production function for each crop, a relationship between total irrigation water supply and net seasonal yield (benefit/acre). The generation of the seasonal crop production functions is carried out using the single-crop stochastic DP. Thus, they are based on optimal intraseasonal water allocation conditional on a given value of seasonal water supply and given initial soil moisture conditions. The entire feasible region for seasonal water supply is investigated from zero to the minimum seasonal water supply required to obtain maximum crop yield. The assumption is that water is the only limiting factor for crop production. A least-squares fit to the data obtained from the soil-plant-climate system model and results from the single-crop DP algorithm yields the seasonal crop production function. Parameters of the least-squares fits of the exponential model of seasonal production functions are summarized in Table 8. The data points and the fitted functions corresponding to parameters in Table 8 are shown in Figs. 2–5. Observe that, although the resulting decay parameters (b_i) of the best-fit functions were equal in two instances, the complete seasonal crop production function is different for each of the four crops considered. The fitted functions have determination coefficients that are larger than 0.98 for all cases, indicating that the selected seasonal crop production function is able to explain more than 98% of the variance of the seasonal crop yields.

Fig. 6 presents the seasonal crop production functions for each of the four crops analyzed expressed in monetary units (benefits/acre). It can be observed that there are three distinct ranges characterizing the economic efficiency of these four crops. The first range corresponds to irrigation water supply below approximately 150 mm. Within this range crop 1 is the most efficient generator of gross benefits per unit amount of irrigation water, followed by crop 4, crop 3, and crop 2. The second range corresponds to irrigation water supply between 150 and 225 mm. Within this range, crop 1 remains as the most efficient generator of gross benefits, but crop 4 and crop 3 exchange places. The third range corresponds to irrigation water supply greater than 225 mm. Within this range, crop 3 is the most efficient generator of gross benefits per unit amount of irrigation water supply, followed by crop 1, crop 2, and crop 4. Clearly, this type of behavior induces a great deal of complexity in the optimal seasonal water allocation problem solution, particularly in the case of constrained solutions where both maximum and minimum acreages are set for each crop.

Optimal seasonal water allocation and acreage are obtained in the second step of the first stage through a deterministic DP algorithm. A water allocation problem exists only when the total seasonal water availability is less than the sum of all minimum seasonal water requirements for maximum yields from each individual crop. Thus, at the irrigation district management level, the first decision is to forecast the total seasonal water supply. Clearly, this quantity is in itself a random variable, and consequently there is an uncertainty associated with each of the forecast values. The solution presented here assumes this seasonal value is known with certainty and no attempt is made to propagate its potential uncertainty throughout the intraseasonal decision problem solution. Several solutions are presented in Tables 9 and 10 for different values of the total seasonal water availability at the beginning of the growing season, as well as for both the constrained and unconstrained cases.

As explained, the seasonal crop production functions exhibit complex behavior in three different ranges of economic effi-

TABLE 9. Optimal Multicrop Cropping Mix and Seasonal Water Allocation—Unconstrained System

Seasonal water availability (mm-acre) (1)	Crop 1			Crop 2			Crop 3			Crop 4			Total benefits ($) (14)
	A_1 (acre) (2)	U_1 (mm) (3)	Benefit ($) (4)	A_2 (acre) (5)	U_2 (mm) (6)	Benefit ($) (7)	A_3 (acre) (8)	U_3 (mm) (9)	Benefit ($) (10)	A_4 (acre) (11)	U_4 (mm) (12)	Benefit ($) (13)	
10,000	20	50	1,826	80	112.5	27,200	0	0	0	0	0	0	29,026
15,000	80	75	14,882	80	112.5	27,200	0	0	0	0	0	0	42,082
20,000	100	100	21,825	80	100.0	23,818	0	0	0	20	100	3,874	49,517
25,000	20	125	6,231	40	62.5	6,392	40	62.5	2,166	100	175	40,350	55,139
10,000	0	0	0	80	125	29,026	0	0	0	0	0	0	29,026
15,000	40	75	7,076	100	120	35,006	0	0	0	0	0	0	42,082
20,000	20	100	3,641	100	160	42,002	0	0	0	20	100	3,874	49,517
25,000	20	125	6,231	40	62.5	6,392	40	62.5	2,166	100	175	40,350	55,139

TABLE 10. Optimal Multicrop Mix and Seasonal Water Allocation—Constrained System

Seasonal water availability (mm-acre) (1)	Crop 1			Crop 2			Crop 3			Crop 4			Total benefits ($) (14)
	A_1 (acre) (2)	U_1 (mm) (3)	Benefit ($) (4)	A_2 (acre) (5)	U_2 (mm) (6)	Benefit ($) (7)	A_3 (acre) (8)	U_3 (mm) (9)	Benefit ($) (10)	A_4 (acre) (11)	U_4 (mm) (12)	Benefit ($) (13)	
10,000	25.0	50	1,854	80.0	110.0	25,357	25	0	−59	25	0	−178	26,974
15,000	80.0	75	14,882	80.0	112.5	27,200	25	0	−59	25	0	−178	41,845
20,000	75.0	100	15,900	75.0	100.0	21,742	25	100	2,021	25	100	4,060	43,723
25,000	87.5	100	18,548	57.5	100.0	25,365	25	150	2,306	25	150	2,464	48,683

ciency, leading to multiple optimal (nonunique) solutions for this irrigation scheduling problem. The algorithm developed identifies all possible optimal solutions, for both constrained and unconstrained problems. Table 9 presents examples of two alternative optimal solutions (as defined by maximum net benefits) for the case of the unconstrained irrigation problem of the case study. The existence of multiple optimal solutions (in the sense of maximum net benefits) allows the decision maker to further optimize the irrigation and cropping mix at the beginning of the season as a function of market dynamics.

Crops 1 and 2 are the most efficient generators of net benefits. Observe that even for irrigation water supply within the first region where, as indicated, crop 1 and crop 4 are the most efficient generators of gross benefits and crop 2 dominates due to its much lower associated irrigation cost. Also observe that due to the same reason, crop 2 tends to use a higher amount of water than the other three crops. This is especially important in cases of extremely low seasonal irrigation water supply. As shown in Tables 9 and 10, crops 1 and 2 are always allocated irrigation water, even at the expense of the other two crops. Also, observe that the total net benefits increases monotonically with seasonal water supply, and the rate of increase is much higher for low values of water availability. Thus the marginal cost of water is much higher for conditions of severe water limitations. Therefore, the water allocation optimization procedure is more important under this condition.

The second stage in the decomposition procedure is the optimal intraseasonal allocation of water for each crop. Conditional on the optimal seasonal acreage and irrigation water allocation resulting from the first-stage DP process, a stochastic, single-crop DP scheduling problem is solved for each crop. For each day of the growing season, the foregoing DP produces optimal decision functions that yield the optimal irrigation amount as a function of the current value of root soil moisture, and the irrigation water available at that decision stage.

Figs. 7(a), 7(b), and Table 11 are included as illustrations of the results. They correspond to crop 1 and a seasonal allocation of water of 125 mm. Figs. 7(a) and 7(b) illustrate how the optimal decisions change as a function of soil moisture content for fixed available irrigation water as a function of the decision stage. Several points are emphasized. First, the DP irrigation scheduling algorithm leads to higher amounts of irrigation during stage 85 than during stage 44, even for the same soil moisture content and availability of water. This reflects, on the one hand, the difference in relative importance of water stresses as a function of growth stage and the increase on root depth as the season progresses. On the other hand, it also reflects the fact that, for a fixed amount of irrigation water that is less than the amount required for the physiological maximum yield, water left over at the end of the season is nonoptimal. As the season progresses, the optimal decisions should lead to total water use. However, in the early season, and due to the uncertainty in future precipitation arrivals, optimal decisions should tend to save water for later stages. Second, for a fixed amount of irrigation water and a fixed decision stage, the optimal irrigation depth does not increase monotonically with decreasing soil moisture content. In addition to optimality reasons having to do with the water stress sensitivity of the crops and limited water availability, this behavior is also due to the discretization of the decision space used in the optimization. As the available soil moisture approaches a boundary between decision space discretization intervals, optimality conditions may lead to a decrease in the volume of irrigation water applied or to zero irrigation amounts, as shown in Fig. 7(a) and decision stage 60. Third, in the late stages of the growing season, at stage 85, it makes no difference whether the available water is 75 or 125 mm. However, during earlier stages in the growing season, at stage 60 for example, the optimal decisions are strongly dependent on the amount of irrigation water available.

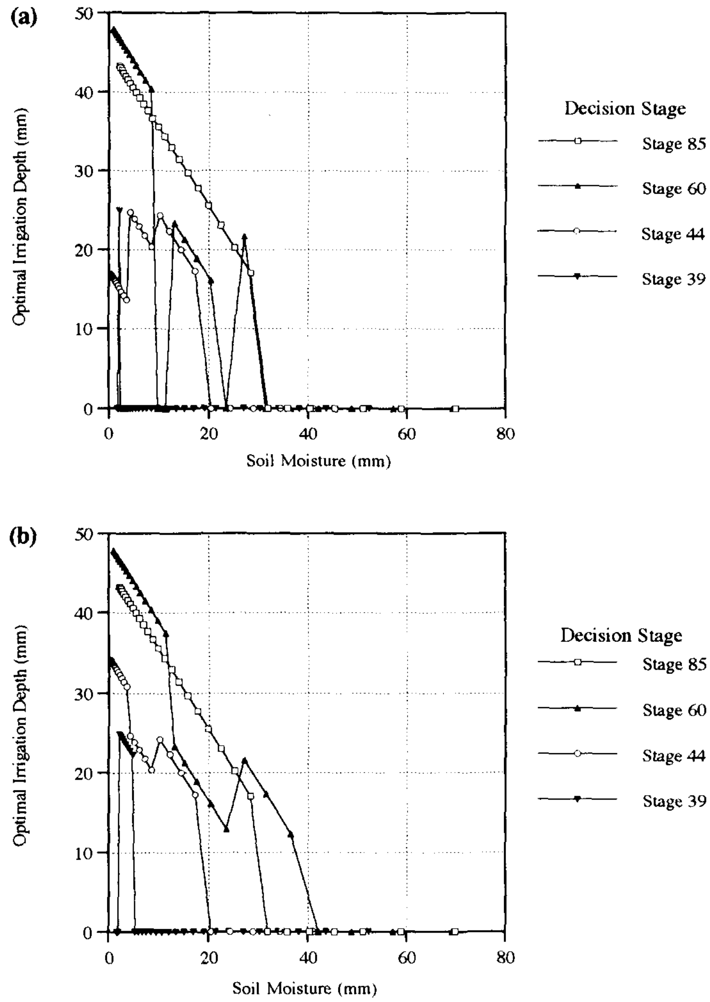

FIG. 7. Optimal Irrigation Depth as a Function of Soil Moisture Content at Different Decision Stages: (a) 75 mm of Available Irrigation Water; (b) 125 mm of Available Irrigation Water

The optimal decision functions are obtained in the form of tables for each decision stage during the growing season. To end this illustration of the results, Table 11 presents the optimal decision function of stage 44.

TABLE 11. Optimal Irrigation Decision Function for Crop 1 during Decision Stage 44

Initial soil moisture (mm) (1)	OPTIMAL IRRIGATION DEPTH (mm) Available Irrigation Water (mm) 125 (2)	100 (3)	75 (4)	50 (5)	25 (6)
0.314	34.119	34.119	16.852	16.852	16.852
0.373	34.060	34.060	16.793	16.793	16.793
0.444	33.989	33.989	16.722	16.722	16.722
0.529	33.904	33.904	16.637	16.637	16.637
0.629	33.804	33.804	16.537	16.537	16.537
0.749	33.684	33.684	16.417	16.417	16.417
0.891	33.542	33.542	16.275	16.275	16.275
1.060	33.373	33.373	16.106	16.106	16.106
1.262	33.171	33.171	15.904	15.904	15.904
1.502	32.931	32.931	15.664	15.664	15.664
1.787	32.646	32.646	15.379	15.379	15.379
2.127	32.306	32.306	15.039	15.039	15.039
2.531	31.902	31.902	14.635	14.635	14.635
3.012	31.421	31.421	14.154	14.154	14.154
3.585	30.848	30.848	13.581	13.581	13.581
4.266	24.668	24.668	24.668	24.668	24.668
5.077	23.857	23.857	23.857	23.857	23.857
6.042	22.892	22.892	22.892	22.892	22.892
7.191	21.743	21.743	21.743	21.743	21.743
8.558	20.376	20.376	20.376	20.376	20.376
10.184	24.249	24.249	24.249	24.249	24.249
12.120	22.313	22.313	22.313	22.313	0.000
14.424	20.009	20.009	20.009	20.009	0.000
17.166	17.267	17.267	17.267	0.000	0.000
20.429	0.000	0.000	0.000	0.000	0.000
24.312	0.000	0.000	0.000	0.000	0.000
28.934	0.000	0.000	0.000	0.000	0.000
34.433	0.000	0.000	0.000	0.000	0.000
41.110	0.000	0.000	0.000	0.000	0.000
51.117	0.000	0.000	0.000	0.000	0.000

CONCLUDING REMARKS

A stochastic methodology for solving the multicrop, multistage optimal irrigation scheduling problem has been presented. Optimal seasonal water allocation and optimal seasonal acreage for several crops are developed, which explicitly account for the physics of soil moisture depletion processes and the random nature of intraseasonal precipitation amounts as well as corresponding infiltration contributions to root soil moisture content. In addition to its stochastic nature, the advantage of this procedure over other studies (Rao et al. 1990, Sunantara and Ramírez 1993) is its ability to produce optimal daily irrigation policies and its ability to allow decision makers to make the optimal decision for any possible initial condition. This solution is based on a DP decomposition scheme whose primary component is a physically based, single-crop stochastic irrigation scheduling model. The results of the case study point out the strong dependence of optimal multicrop water policies on the stress sensitivity factors, the maximum yields for each crop, and the costs of irrigation and cultivation.

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APPENDIX I.

More details on the following equations for the soil moisture balance components and precipitation model can be found in Ramírez and Bras (1982, 1985).

The total volume of infiltration water from a given storm is a function of storm intensity, i, storm duration, t_r, and soil characteristics as encoded in the soil infiltration sorptivity, S_i, and soil gravitational infiltration rate, A. This volume was given by Córdova and Bras (1979) as

$$V(i, t_r) = \begin{cases} it_r & t_r \le t_o \\ At_r + S_i\sqrt{t_r/2} & \text{otherwise} \end{cases} \tag{24}$$

where t_o = time to ponding.

Assuming exponential probability density distributions for storm intensity and storm duration, the cumulative probability function of the volume infiltrated from a given storm is given by

$$F_{V(i,t_r)}(v) = \Pr[V(i, t_r) \le v] = 1 - e^{-\alpha i^* - \delta t_o^*} - \alpha \int_0^{i^*} e^{-\alpha i - \delta v/i} \partial i \tag{25}$$

$$i^* = (4vA + S_i^2 + S_i\sqrt{8vA + S_i^2})/4v \tag{26a}$$

$$t_o^* = 0.5S_i^2/(i^* - A)^2 \tag{26b}$$

where α and δ = parameters of the exponential distributions for storm intensity and storm duration, respectively.

For periods when no infiltration occurs, either from precipitation or irrigation water applications, soil moisture dynamics is governed by

$$\frac{\partial\theta}{\partial t} = -ET(\theta) - P(\theta) + w \tag{27}$$

where $ET(\theta)$, $P(\theta)$, and w = actual evapotranspiration, percolation, and capillary rise rates, respectively. Observe the explicit dependence on soil moisture content, θ. Integration of (27) over one decision stage (one day) yields the function $g(\theta, \Delta t)$, which can be expressed generically as

$$\int_{t_k}^{t_{k+1}} \partial t = -\int_{\theta_i}^{\theta_j} \frac{\partial\theta}{ET(\theta) + P(\theta) - w} \tag{28}$$

where the limits of integration depend on the initial conditions and the total actual evapotranspiration during the time step. For more details on the specific forms of the actual evapotranspiration, percolation, and capillary rise functions please refer to Ramírez and Bras (1985).

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